On the Bend-Number
of Planar and Outerplanar Graphs

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Abstract. The *bend-number* $b(G)$ of a graph G is the minimum k such that G may be represented as the edge intersection graph of a set of grid paths with at most k bends. We confirm a conjecture of Biedl and Stern showing that the maximum bend-number of outerplanar graphs is 2. Moreover we improve the formerly known lower and upper bound for the maximum bend-number of planar graphs from 2 and 5 to 3 and 4, respectively.

1 Introduction

In 2007 Golumbic, Lipshteyn and Stern defined an *EPG*[1] representation of a simple graph G as an assignment of paths in the rectangular plane grid to the vertices of G, such that two vertices are adjacent if and only if the corresponding paths intersect in at least one grid edge, see [8]. EPG representations arise from VLSI grid layout problems [4] and as generalizations of *edge-intersection graphs of paths on degree 4 trees* [7]. In the same paper Golumbic et al. show that every graph has an EPG representation and propose to restrict the number of bends per path in the representation. There has been some work related to this, see [1,2,8,12,17]. A graph is a *k-bend graph* if it has an EPG representation, where each path has at most k bends. The *bend-number* $b(G)$ of G is the minimum k, such that G is a k-bend graph.

Note that the class of 0-bend graphs coincides with the well-known class of interval graphs, i.e., intersection graphs of intervals on a real line. It is thus natural to view k-bend graphs as an extension of the concept of interval graphs and the bend-number as a measure of how far a graph is from being an interval-graph.
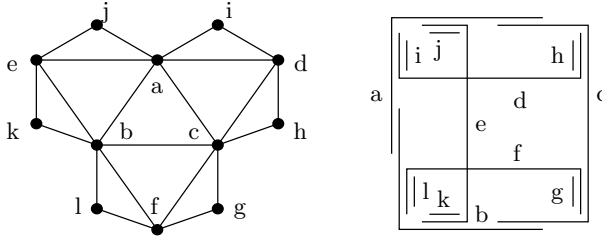
Intersection graphs of systems of intervals together with a parameter counting how many intervals are needed to represent a vertex of a given graph G have received some attention. Popular examples are the *interval-number*, see Harary and

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[1] EPG stands for *edge intersection graph of paths in the grid*.

Fig. 1. A 2-bend graph and an EPG representation. If a grid edge is shared by several paths we draw them close to each other.

Trotter [11] and the track-number, see Gyárfás and West [10]. Extremal questions for these parameters like *'What is the maximum interval-/track-number among all graphs of a particular graph class?'* have been of strong interest in the literature. Scheinermann and West [18] show that the maximum interval-number of outerplanar graphs is 2 and of planar graphs is 3. Kostochka and West [14] prove that the maximum track-number of outerplanar graphs is 2 and Gonçalves and Ochem [9] prove that the maximum track-number of planar graphs is 4.

In [2] Biedl and Stern show that outerplanar graphs are 3-bend graphs and provide an outerplanar graph which has bend-number 2, see Fig. 1. They conjecture that all outerplanar graphs are 2-bend graphs. We confirm this conjecture in Theorem 1, showing the stronger result that graphs of treewidth at most 2 are 2-bend graphs.

The major part of this paper is devoted to planar graphs. Biedl and Stern [2] show that planar graphs are 5-bend graphs but the only lower bound that they have is 2, given by the graph of Fig. 1. In Proposition 1 we provide a planar graph of treewidth 3 which has bend-number 3, thus improving the lower bound of the class of planar graphs by one. Indeed in Theorem 2 we show that every planar graph with treewidth 3 is a 3-bend graph. The main result of this article is Theorem 3. We improve the upper bound for the bend-number of general planar graphs from 5 to 4.

Some supplementary figures and proofs are provided in an extended version of this paper, see [13].

2   Preliminaries

We consider simple undirected graphs G with vertex set $V(G)$ as well as edge set $E(G)$. An EPG representation is a set of finite paths $\{P(u) \mid u \in V(G)\}$, which consist of consecutive edges of the rectangular grid in the plane. These paths are not self-intersecting (in grid edges), and $P(u) \cap P(v) \neq \emptyset$ if and only if $\{u, v\} \in E(G)$, i.e., only intersections at grid *edges* are considered. A bend of $P(u)$ is either a horizontal grid edge followed by a vertical one or a vertical edge followed by a horizontal one in $P(u)$.

Let us now introduce some terminology: The grid edges between two consecutive bends (or the first (last) bend and the start (end) of $P(u)$) are called *segments*. So a k-bend path consists of $k + 1$ segments, each of which is either horizontal or vertical. A *sub-segment* is a connected subset of a segment. In an EPG representation a set of sub-segments (eventually from different segments) is called a *part*.

Furthermore, a vertex u is *displayed*, if there is at least one grid edge, which is exclusively in $P(u)$ and in no other path. We then say that this grid edge is a *private* edge of $P(u)$. An edge $\{u, v\} \in E$ is *displayed*, if there is at least one grid edge in $P(u) \cap P(v)$, which is only contained in $P(u) \cap P(v)$ and not element of any other path. This grid edge is then called a *private* edge of $P(u) \cap P(v)$. We also say that a part (or sub-segment) *displays* the corresponding vertex or edge if it consists only of grid edges which are private edges of the respective vertex or edge.

Finally, two horizontal sub-segments in an EPG representation *see each other* if there is a vertical grid line crossing both sub-segments. Similarly, two vertical sub-segments see each other if there is a horizontal grid line crossing both.

## 3   EPG Representations of Graphs in Terms of Treewidth

In this section we consider graphs of bounded treewidth. Therefore we denote the treewidth of a graph G with $tw(G)$ and make use of the fact, that every graph G with $tw(G) \leq k$ is a subgraph of a k-tree as well as the fact, that every k-tree admits a construction sequence, starting with a $(k + 1)$-clique and iteratively stacking a new vertices into k–cliques. For further definitions and more about construction sequences of graphs of bounded treewidth we refer to [3].

Theorem 1. *For every graph G with $tw(G) \leq 2$ we have $b(G) \leq 2$.*

Proof. Let $\tilde{G}$ be the 2-tree which contains G and $(v_1, \ldots, v_n)$ be a vertex ordering of G implied by $\tilde{G}$'s construction sequence. We construct a 2-bend representation of G along the building sequence $G_2 \subset \ldots \subset G_n = G$, where we add vertex v_i to G_i, such that the two neighbors of v_i in $\tilde{G}_i$ form a 2-clique in $\tilde{G}_i$ (and not necessarily in G_i). We maintain that Γ_i is a 2-bend representation of G_i, such that every 2-clique in $\tilde{G}_i$ satisfies one of the two invariants in Fig. 2 **a)**, i.e., for $i = 2, \ldots, n$ and $\{u, v\} \in E(\tilde{G}_i)$ we have a sub-segment $p_u \subseteq P(u)$ and a part $p_v \subseteq P(v)$ such that one of the following sets of conditions holds:

(i) p_v is a sub-segment, p_u and p_v see each other, p_u displays u, and p_v displays v as depicted in the top row of Fig. 2 **a)**,

(ii) p_v consists of two consecutive sub-segments $p_{v,1}, p_{v,2}$ such that there is a bend between them, $p_{v,2}$ displays v, $p_u \setminus p_{v,1}$ displays u, and $p_u \cap p_{v,1}$ displays $\{u, v\}$ as depicted in the bottom row of Fig. 2 **a)**.

The two possible starting representations for G_2 are shown in Fig. 2 **a)**. Now for $i \geq 2$, let Γ_i be a 2-bend representation of G_i that satisfies our invariant.
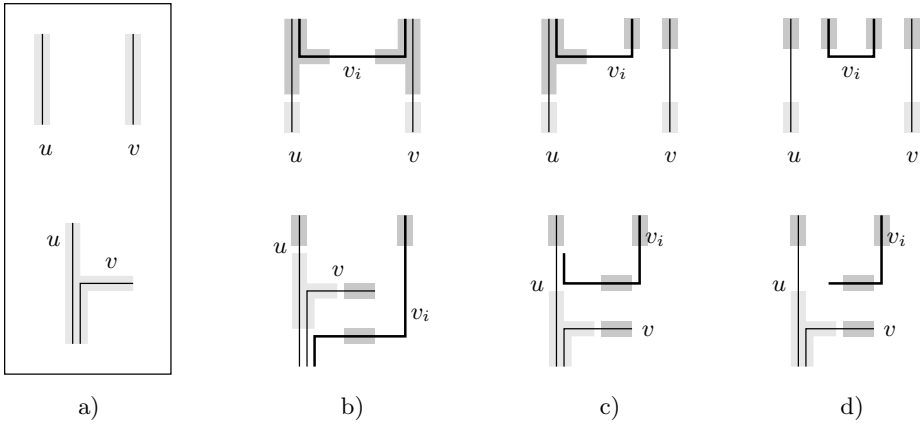
Fig. 2. Invariants for a 2-clique and insertion rules for the path for the new vertex v_i (drawn bold). a) p_u and p_v for the two types of the invariant. b) Vertex v_i has an edge with both, u and v. c): Vertex v_i has an edge with u and no edge with v. d): Vertex v_i has no edge with u or v.

Let $\{u, v\}$ be the 2-clique that is the neighborhood of v_i in $\tilde{G}_i$. If v_i has an edge in G_{i+1} with both, u and v, we introduce the path for v_i as illustrated in Fig. 2 b) depending on the type of the invariant for $\{u, v\}$. The parts, displaying the new 2-cliques $\{u, v_i\}$ and $\{v, v_i\}$ in $\tilde{G}_{i+1}$, are highlighted in dark gray and the part, which displays $\{u, v\}$, is highlighted in light gray.

If v_i has an edge in G_{i+1} with u and no edge in G_{i+1} with v, we introduce the path for v_i as illustrated in Fig. 2 c) depending on the type of invariant we have. Again the parts, which display the new 2-cliques $\{u, v_i\}$ and $\{v, v_i\}$, and the old 2-clique $\{u, v\}$ in $\tilde{G}_{i+1}$ are highlighted in dark gray, and light gray, respectively. If v_i has an edge in G_{i+1} with v and no edge in G_{i+1} with u, the roles of u and v are basically exchanged. In case, v_i neither has an edge in G_{i+1} with u nor with v, we introduce the path for v_i as illustrated in Fig. 2 d) depending on the type of type of invariant for $\{u, v\}$. $\qquad \square$

Since every outerplanar graph G has a treewidth ≤ 2 we obtain the following corollary.

Corollary 1. *For every outerplanar graph G we have $b(G) \leq 2$.*

This confirms a conjecture of Biedl and Stern [2], who show that the graph in Fig. 1 has no 1-bend representation. Therefore this bound cannot be further improved.
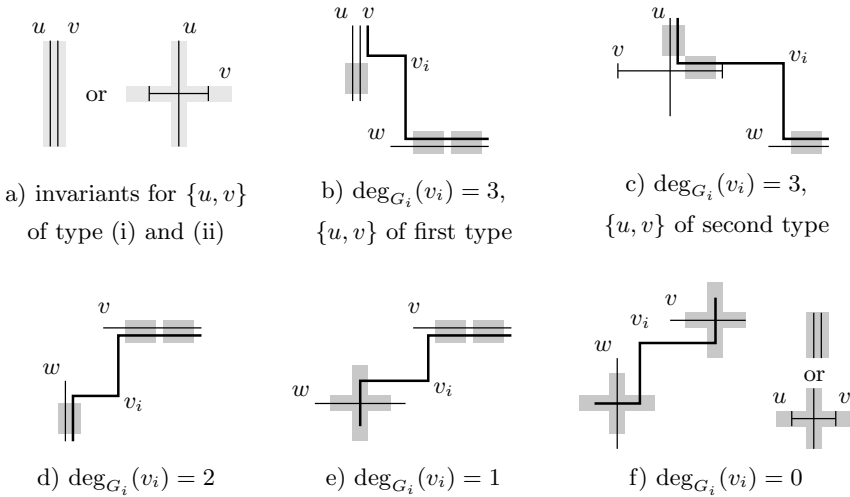
Theorem 2. *For every planar graph G with $tw(G) \leq 3$ we have $b(G) \leq 3$.*

Proof. By a result of El-Mallah and Colbourn [6] G is a subgraph of a plane 3-tree $\tilde{G}$. So there is a vertex ordering $(v_1, \ldots, v_n)$, such that $\tilde{G}_3$ is a triangle $\{v_1, v_2, v_3\}$ and $\tilde{G}_i$ is obtained from $\tilde{G}_{i-1}$ by connecting v_i to the three vertices

u, v, w of a triangle bounding an inner face of $\tilde{G}_{i-1}$. The triangle $\{u, v, w\}$ is then not bounding an inner face of $\tilde{G}_i$ anymore and hence no second vertex may be attached to it. We build a 3-bend representation of G concurrently with the building sequence $G_3 \subset \cdots \subset G_n$ of G w.r.t. the vertex ordering $(v_1, \ldots, v_n)$. We maintain the following invariant on the 3-bend representation Γ_i of G_i, for $i \geq 3$:

(a) Every vertex $u \in \Gamma_i$ has a horizontal *and* a vertical sub-segment displaying u, and
(b) every facial triangle $\{u, v, w\}$ of $\tilde{G}_i$ has two vertices, say u and v, of one of the following two types:
　(i) there is a sub-segment, which displays the edge $\{u, v\}$ as in the left example in Fig. 3 a),
　(ii) there is an *entire* segment s of $P(v)$ displaying v and a sub-segment of $P(u)$ displaying u and crossing s, see Fig. 3 a).

Moreover, we require that all the displaying parts above are pairwise disjoint and that the entire segment in every cross in (ii) *cannot* see displaying parts from (a). However, we need this assumption only in the case in Fig. 3 **c)**.

a) invariants for $\{u, v\}$ of type (i) and (ii)

b) $\deg_{G_i}(v_i) = 3$, $\{u, v\}$ of first type

c) $\deg_{G_i}(v_i) = 3$, $\{u, v\}$ of second type

d) $\deg_{G_i}(v_i) = 2$

e) $\deg_{G_i}(v_i) = 1$

f) $\deg_{G_i}(v_i) = 0$

Fig. 3. Building a 3-bend representation of a planar graph with tree-width 3, a vertex v_i is attached to the facial triangle $\{u, v, w\}$ in $\tilde{G}_{i-1}$: In a) the two types of invariant for $\{u, v\}$ are shown. In b)–f) it is shown how to insert the new vertex v_i (drawn bold) depending on its degree in G_i and the invariant of $\{u, v\}$. The invariants for the three new facial triangles $\{u, v, v_i\}$, $\{u, v_i, w\}$, and $\{v_i, v, w\}$ in $\tilde{G}_i$ are highlighted.

It is easy to find a 3-bend representation Γ_3 of the subgraph G_3 of G, which satisfies invariant (a) and (b).

For $i \geq 4$, the path for vertex v_i is introduced to Γ_{i-1} according to the degree of v_i in G_i and the type of invariant for the facial triangle $\{u, v, w\}$ in $\tilde{G}_{i-1}$ that v_i is connected to. Fig. 3 b)–f) shows all five cases and how to introduce v_i, which is illustrated by the bold path. Consider in particular the case that v_i has an edge with each of u, v, and w, and moreover the triangle $\{u, v, w\}$ has a cross, see Fig. 3 c). Here we use that the entire v-segment cannot see the partial w-segment in order to get a horizontal displaying sub-segment for the new path corresponding to v_i.

In the figure the parts from invariant (b) displaying edges of the new facial triangles $\{u, v, v_i\}$, $\{u, v_i, w\}$ and $\{v_i, v, w\}$ in $\tilde{G}_i$ are highlighted in dark gray. Additionally, every path, including the new path for v_i, has a horizontal and a vertical sub-segment, which displays them. Moreover they can be chosen such that they do not see any entire segment of a cross.

For example, consider the case that v_i is not adjacent to u, v, or w in G_i, see Fig. 3 **f)**. Here the new facial triangles $\{u, v_i, w\}$ and $\{v_i, v, w\}$ have a cross of an entire segment of $P(v_i)$ and a sub-segment displaying w and v, respectively. The new facial triangle $\{u, v, v_i\}$ has the edge $\{u, v\}$, which still satisfies both parts of the invariant. □

We conclude this section showing that the bound in Theorem 2 is tight. This confirms what Biedl and Stern strongly suspected in [2].

Proposition 1. *There is a planar graph G with $tw(G) = 3$ and $b(G) = 3$.*

The construction of a planar graph G with approximately 67.000 vertices and a proof which shows that G has bend-number and treewidth 3 are provided in [13].

4 4-Bend Representation for Planar Graphs

We show that the bend-number of every planar graph is at most 4, improving the recent upper bound of 5 due to Biedl and Stern [2]. Our proof is constructive and can indeed be seen as a linear-time algorithm to find a 4-bend representation of any given planar graph. We use the folklore fact that every plane triangulation can be constructed from a triangle by successively glueing 4-connected triangulations into inner faces. In every step our algorithm constructs a 4-bend representation Γ' of a 4-connected triangulation G' and incorporates it into the already defined representation. The construction of Γ' is based on a well-known representation of subgraphs of 4-connected triangulations by touching axis-aligned rectangles. These are the basic steps of the algorithm:

1.) Fix some plane embedding of the given planar graph and add one vertex into each face to obtain a super-graph G that is a triangulation. If we find a 4-bend representation for G, removing the paths from it that correspond to added vertices, results in a 4-bend representation of the original graph.
2.) Construct a 4-bend representation of the outer triangle of G so that invariant I, presented ahead, is satisfied.

3.) Let Γ' and G' denote the so far defined 4-bend representation and the graph that is represented, respectively. The graph G' will always be a plane triangulation, which is an induced subgraph of G. We repeat the following two steps 4.) and 5.) until we end up with a 4-bend representation Γ of the entire triangulation G.

4.) Consider a triangle Δ which is an inner face of G', but not of G. Let G_Δ be the unique 4-connected triangulated subgraph of G, which contains Δ and at least one vertex lying inside of it. (No vertex in G_Δ, except those in Δ, is represented in Γ'.)

5.) Construct a 4-bend path for every vertex in $G_\Delta \backslash \Delta$ and add it to the representation Γ' so that all edges of G_Δ are properly represented, i.e., Γ' is a 4-bend representation of $G' \cup G_\Delta$. Additionally, we ensure that our invariant is satisfied for every inner facial triangle of $G' \cup G_\Delta$.

The algorithm described above computes a 4-bend representation for every given planar graph. Moreover, steps 1.) to 4.) can easily be executed in linear time. The crucial part is step 5.), i.e., the construction of a 4-bend representation of the 4-connected triangulation G_Δ. Our construction relies on a well-known geometric representation of proper subgraphs of 4-connected plane triangulations. Actually, we need the slight strengthening in Lemma 1 below. A *plane non-separating near-triangulation, NST* for short, is a plane graph on at least four vertices without separating triangles such that all inner faces are triangles and the outer face is a quadrangle. For example, if G is a plane 4-connected triangulation and $e = (u, w)$ an outer edge of G, then $G \backslash (u, w)$ is an NST.

Lemma 1. *Let $G = (V, E)$ be an NST. Then G can be represented as follows:*

> *(a) There is an axis-aligned rectangle $R(v)$ for every $v \in V$.*
> *(b) Any two rectangles are either disjoint or intersect on a line segment of non-zero length.*
> *(c) The rectangles $R(v)$ and $R(w)$ have a non-empty intersection if and only if (v, w) is an edge in G.*

Additionally, for every $v \in V$ there is a vertical segment t_v from the bottom to the top side of $R(v)$, such that the following holds:

> *(d) The segment t_v lies to the right of t_w for every $v \in V$ and $R(w)$ touching the top side of $R(v)$.*

Proof. Items (a)–(c) are well known [15,16,19]. The proof of (d) is elementary and provided in [13]. □

We will refer to a set of rectangles satisfying (a)–(d) in Lemma 1 as a *rectangle representation of G*. Rectangle representations are also known as *rectangular duals*. It is easily seen that every inner (triangular) face $\Delta = \{u, v, w\}$ of G corresponds to a unique point in the plane given by $R(u) \cap R(v) \cap R(w)$, i.e., the common intersection of the three corresponding rectangles.

Now we describe our invariant mentioned in steps 2.) and 5.) above.

Invariant I: Let G' be a plane triangulation. A 4-bend representation Γ' of G' is said to *satisfy invariant I* if there are mutually disjoint regions, i.e., polygonal parts of the plane, associated with the inner facial triangles of G'. For each inner facial triangle $\Delta = \{u, v, w\}$ we require that inside *the region for Δ* there are segments s_u, s_v, s_w, and s_{uv} displaying u, v, w, and the edge $\{u, v\}$, respectively, such that (up to grid-symmetry) each of the following holds.

(a) Segment s_u is vertical, s_v is horizontal, lying to the top-right of s_u.
(b) Segment s_{uv} lies above s_u on the same grid line.
(c) Segment s_w lies to the right of s_u and not to the left of s_v, and is of one of the following types
   (i) s_w is vertical and sees s_u.
   (ii) s_w is horizontal and sees s_v.
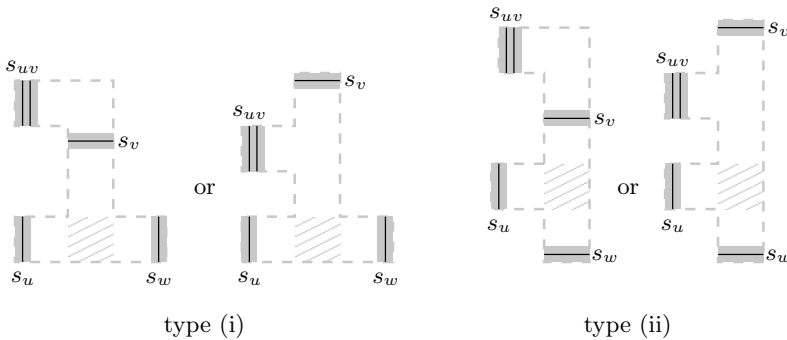
type (i)   type (ii)

Fig. 4. The regions of type (i) and type (ii) for a triangle $\Delta = \{u, v, w\}$ (up to grid-symmetry). The part of the region that can be seen by each of s_u, s_v, and s_w is highlighted.

See Fig. 4 for an illustration. The regions are illustrated by the dashed line in Fig. 4. It is important that the regions in Fig. 4 may appear reflected and/or rotated by 90 degree.

Our algorithm to find a 4-bend representation of a planar graph executes steps 1.) to 5.) as described above. It is easy to find a 4-bend representation Γ' of a triangle in step 2.) which satisfies invariant I, see [13]. Hence we only have to take care of step 5.), which boils down to the following.

Lemma 2. *Let $\{P(u), P(v), P(w)\}$ be a 4-bend representation of a triangle $\Delta = \{u, v, w\}$ which satisfies invariant I, and G be a 4-connected triangulation whose outer triangle is Δ. Then $\{P(u), P(v), P(w)\}$ can be extended to a 4-bend representation Γ of G which satisfies invariant I, such that every new 4-bend path, as well as the region for every inner face of G, lies inside the region for Δ. Moreover, such a representation can be found in linear time.*

Proof. For convenience we rotate and/or flip the entire representation such that the region for Δ is similar to one of Fig. 4, i.e., s_u and s_{uv} are the leftmost segments and s_{uv} lies above s_u. Define $G' = G \setminus \{u, w\}$ if the region for Δ is of type (i), and $G' = G \setminus \{v, w\}$ if the region for Δ is of type (ii). By Lemma 1 G' can be represented by rectangles so that conditions (a)–(d) are met. We put this rectangle representation into the part of the region for Δ which can be seen by each of s_u, s_v, and s_w. This part is highlighted in Fig. 4.

Now we explain how to replace the rectangle by a path for every vertex $i \neq u, v, w$ in G'. We start with a snake-like 4-bend path $P(i)$ within the rectangle corresponding to i. To be precise, $P(i)$ starts at the bottom-left corner of $R(i)$, goes up to the top-left corner, where it bends and goes right up to the segment t_i, which is defined in Lemma 1. Then $P(i)$ follows along t_i down to the bottom side of $R(i)$, where it bends and goes to the right up to the bottom-right corner and then up to the top-right corner. Afterwards, every path $P(i)$ is shortened at both ends by some amount small enough that no edge-intersection is lost. See Fig. 5 for an illustration. Note that if $R(j)$ touches the top side of $R(i)$, then the paths $P(i)$ and $P(j)$ do have an edge-intersection because t_i lies to the right of t_j.

Fig. 5. The 4-bend snake-like path $P(i)$ within the rectangle $R(i)$ corresponding to i

It is not difficult to see, that these snake-like paths form a 4-bend representation of $G'\setminus\{u, v, w\}$. The remaining edges in G', i.e., those incident to u, v or w, are established by extending the paths corresponding to the neighbors of u, v and w in G'. What follows is illustrated with an example in Fig. 6. Let x be the unique[2] neighbor of u and v in G. For every neighbor $i \neq x$ of v the upper horizontal segment of $P(i)$ is shifted up onto the segment s_v (extending the left and the middle vertical segment of $P(i)$). Similarly, for every neighbor $i \neq x$ of u the leftmost vertical segment of $P(i)$ is shifted to the left onto the segment s_u (extending the upper horizontal segment of $P(i)$). For the case $i = x$, the upper horizontal segment of $P(x)$ is shifted up onto some horizontal line through s_{uv}. Then the leftmost vertical segment of $P(x)$ is shifted to the left onto the segment s_{uv}, and shortened such that it is completely contained in s_{uv}. The paths of neighbors of w are extended according to the type of the region for Δ. If the region is of type (i), the rightmost vertical segment of $P(i)$ for every neighbor i of w is shifted to the right onto s_w. If the region is of type (ii), the lower horizontal segment of $P(i)$ is shifted down onto s_w. Let us again refer to Fig. 6 for an illustration.

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[2] There is no second such vertex, since G' has no separating triangles.

type (i)   type (ii)
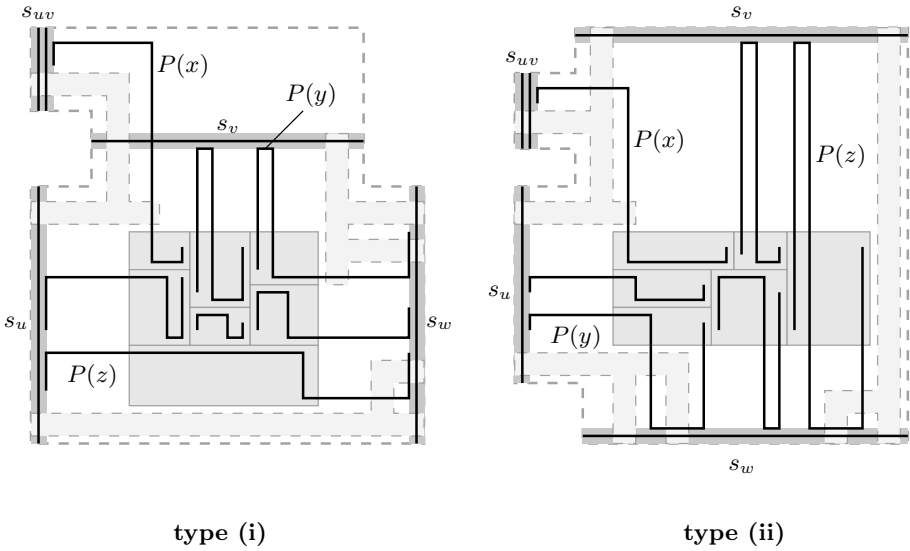
Fig. 6. Examples of the 4-bend representation Γ contained in the region for Δ. The new regions of the three facial triangles that contain two vertices from Δ are highlighted.

Including the already given paths $P(u)$, $P(v)$, and $P(w)$ gives a 4-bend representation Γ of G. It is easy to see that Γ is indeed representing the graph G and that the above construction can be done in linear time.

It remains to show that Γ satisfies invariant I. We have to identify a region for every inner facial triangle of G, such that each region lies within the region for Δ and all regions are pairwise disjoint. Therefore let $\Delta' = \{u', v', w'\}$ be such an inner facial triangle. First, assume that none of $\{u', v', w'\}$ is a vertex of Δ. This is, Δ' consists of inner vertices only. Then the intersection $R(u') \cap R(v') \cap R(w')$ of the corresponding rectangles is a single point in the region for Δ. Moreover, exactly one of the three rectangles has its bottom-left or top-right corner at this point. Hence, the paths $P(u')$, $P(v')$ and $P(w')$ locally look like in one of the cases in Fig. 7. In the figure, for each case a region for $\Delta' = \{u', v', w'\}$ is highlighted. Note that these regions can be chosen to be pairwise disjoint.

Similarly, if exactly one of $\{u, v, w\}$ is a vertex in Δ', then there is a point on the corresponding segment (s_u, s_v or s_w) where the paths $P(u')$, $P(v')$ and $P(w')$ locally look like in the previous case. Hence in this case we find a region for Δ', too.

Finally, exactly three inner facial triangles in G contain exactly two outer vertices. The regions for these three faces are defined as highlighted in Fig. 6. More formally, the region for the triangle $\{u, v, x\}$ contains the segments s_u', s_v', s_x', and s_{uv}' which are contained in s_u, s_v, the middle vertical segment of $P(x)$, and s_{uv}, respectively. The region for the triangle $\{u, w, y\}$ contains the segments s_u', s_w', s_y', and s_{wy}', which are contained in s_u, s_w, the lower horizontal (type (i)) or the middle vertical (type (ii)) segment of $P(y)$, and $P(y) \cap s_w$, respectively.
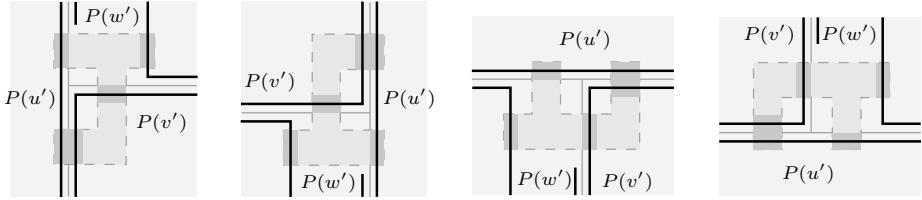
Fig. 7. The four possibilities for an inner face $\Delta' = \{u', v', w'\}$ of G with at most one vertex from Δ. The region for Δ' is highlighted.

Finally, the region for the triangle $\{v, w, z\}$ contains the segments s'_v, s'_w, s'_z, and s'_{wz}, which are contained in s_v, s_w, the lower horizontal (type (i)) or the rightmost vertical (type (ii)) segment of $P(z)$, and $P(z) \cap s_w$, respectively.

All these regions are pairwise disjoint and completely contained in the region for Δ. We remark that even in case[3] $x = y = z$ the above definitions of 4-bend paths and regions meet the required properties, see [13]. Since G does not contain separating triangles, x, y, and z either all coincide or are pairwise distinct, which completes the proof of the lemma. □

Lemma 2 gives a construction as required in step 5.) of our algorithm. Hence, we have proved our main theorem.

Theorem 3. *The bend-number of a planar graph is at most* 4. *Moreover, a 4-bend representation can be found in linear time.*

Putting Theorem 3 and Proposition 1 together we have shown the following.

Theorem 4. *In an EPG representation of a planar graph 4-bend paths are always sufficient and 3-bend paths are sometimes necessary, i.e., the maximum bend-number among all planar graphs is* 3 *or* 4.

5   Conclusions

Although we could raise the previously known lower and upper bound, it remains open to determine the maximum bend-number of planar graphs.

Conjecture 1. There is a planar graph G, such that every EPG representation of G contains at least one path with four bends.

Another interesting class of representations is as edge-intersection graphs of arbitrary polygonal paths. It is straight-forward to obtain a 4-bend representation from a representation of touching ⊤s and ⊥s [5]. What is the right answer?

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[3] This happens if and only if $G = K_4$.

References

1. Asinowski, A., Suk, A.: Edge intersection graphs of systems of paths on a grid with a bounded number of bends. Discrete Appl. Math. 157(14), 3174–3180 (2009)
2. Biedl, T., Stern, M.: On edge-intersection graphs of k-bend paths in grids. Discrete Math. Theor. Comput. Sci. 12(1), 1–12 (2010)
3. Bodlaender, H.L.: A partial k-arboretum of graphs with bounded treewidth. Theoret. Comput. Sci. 209(1-2), 1–45 (1998)
4. Brady, M.L., Sarrafzadeh, M.: Stretching a knock-knee layout for multilayer wiring. IEEE Trans. Comput. 39(1), 148–151 (1990)
5. de Fraysseix, H., Ossona de Mendez, P., Rosenstiehl, P.: On triangle contact graphs. Combin. Probab. Comput. 3(2), 233–246 (1994)
6. El-Mallah, E.S., Colbourn, C.J.: On two dual classes of planar graphs. Discrete Math. 80(1), 21–40 (1990)
7. Golumbic, M.C., Lipshteyn, M., Stern, M.: Representing edge intersection graphs of paths on degree 4 trees. Discrete Math. 308(8), 1381–1387 (2008)
8. Golumbic, M.C., Lipshteyn, M., Stern, M.: Edge intersection graphs of single bend paths on a grid. Networks 54(3), 130–138 (2009)
9. Gonçalves, D., Ochem, P.: On some arboricities in planar graphs. Electronic Notes in Discrete Mathematics 22, 427–432 (2005), 7th International Colloquium on Graph Theory
10. Gyárfás, A., West, D.: Multitrack interval graphs. In: Proceedings of the Twenty-sixth Southeastern International Conference on Combinatorics, Graph Theory and Computing, Boca Raton, FL, vol. 109, pp. 109–116 (1995)
11. Harary, F., Trotter Jr., W.T.: On double and multiple interval graphs. J. Graph Theory 3(3), 205–211 (1979)
12. Heldt, D., Knauer, K., Ueckerdt, T.: Edge-intersection graphs of grid paths – the bend-number, preprint, arXiv:1009.2861v1 (math.CO) (2010)
13. Heldt, D., Knauer, K., Ueckerdt, T.: On the bend-number of planar and outerplanar graphs, preprint, arXiv:1112.3353v1 (math.CO) (2011)
14. Kostochka, A.V., West, D.B.: Every outerplanar graph is the union of two interval graphs. In: Proceedings of the Thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and Computing, Boca Raton, FL, vol. 139, pp. 5–8 (1999)
15. Koźmiński, K., Kinnen, E.: Rectangular dual of planar graphs. Networks 15(2), 145–157 (1985)
16. Lai, Y.-T., Leinwand, S.M.: An algorithm for building rectangular floor-plans. In: 21st Design Automation Conference (DAC 1984), pp. 663–664 (1984)
17. Ries, B.: Some properties of edge intersection graphs of single bend path on a grid. Electronic Notes in Discrete Mathematics 34, 29–33 (2009); European Conference on Combinatorics, Graph Theory and Applications (EuroComb 2009)
18. Scheinerman, E.R., West, D.B.: The interval number of a planar graph: three intervals suffice. J. Combin. Theory Ser. B 35(3), 224–239 (1983)
19. Ungar, P.: On diagrams representing maps. Journal of the London Mathematical Society 28, 336–342 (1953)