Computing Minimum Geodetic Sets of Proper Interval Graphs*

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Abstract. We show that the geodetic number of proper interval graphs can be computed in polynomial time. This problem is NP-hard on chordal graphs and on bipartite weakly chordal graphs. Only an upper bound on the geodetic number of proper interval graphs has been known prior to our result.

1 Introduction

The notion of geodetic sets was introduced by Harary et al. [11], and it has applications in game theory [3,12,17]. It is closely related to convexity and convex hulls in graphs, which have applications in telephone switching centres, facility location, distributed computing, information retrieval, and communication networks [9,14,16,18,19,22]. Given a graph G and a set D of vertices of G, the *geodetic closure* of D, denoted by $I_G[D]$, is the set containing the vertices of G that lie on shortest paths between pairs of vertices from D. The set D is a *geodetic set* of G if $I_G[D]$ contains all vertices of G. Thus, a geodetic set of G is a set D of vertices of G such that every vertex of G lies on some shortest path between two vertices from D. The *geodetic number* of G, $g(G)$, is the smallest size of a geodetic set of G. Computing the geodetic number is NP-hard on chordal graphs and on bipartite weakly chordal graphs [7]. It can be done in polynomial time on cographs [7], split graphs [7], and ptolemaic graphs [8].

The main result of this paper is a polynomial-time algorithm for computing the geodetic number of proper interval graphs. Our algorithm can be implemented to also output a geodetic set of minimum size. The computational complexity of computing the geodetic number of proper interval graphs has been open since Dourado et al. [7] gave a tight upper bound on the geodetic number of proper interval graphs. Interestingly, the related notion of *hull number* has been known to be computable in polynomial time on proper interval graphs [6]. The hull number and the geodetic number problems are both defined through

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convexity, but they require quite different computation methods. The difference between these two parameters can be arbitrarily large [13].

Proper interval graphs have been subject to extensive study (see, e.g., the books [2] and [10]) since their introduction [20], and they can be recognised in linear time. In this paper, in addition to our main result on proper interval graphs, we also report on the following results: a polynomial-time algorithm to compute the geodetic number of block-cactus graphs, a polynomial-time algorithm to approximate the geodetic number of bipartite permutation graphs with an additive factor 1, and a proof of NP-hardness of computing the geodetic number of cobipartite graphs. Two variants of the geodetic number of block-cactus graphs have been studied before [23], but we are not aware of an algorithm for the exact computation of the geodetic number of such graphs.

2 Definitions and Notation

We consider simple finite undirected graphs, that have no loops. For a graph G, its *vertex set* is denoted by $V(G)$ and its *edge set* is denoted by $E(G)$. Edges of G are denoted as uv, where u and v are vertices of G, and if uv is an edge of G then u and v are *adjacent*. The *neighbourhood* of a vertex v of G, denoted by $N_G(v)$, is the set of the vertices of G that are adjacent to v. For a set S of vertices of G, $G[S]$ denotes the *subgraph of* G *induced by* S. We write $G-v$ to denote the graph $G[V(G) \setminus \{v\}]$. A *clique* of G is a set of vertices of G that are pairwise adjacent in G. A *vertex ordering* for G is an ordered tuple that corresponds to a permutation of $V(G)$. For a given vertex ordering σ, we write $u \prec_\sigma v$ if u appears before v in σ. The first position in σ will be referred to as the *left* end of σ, and the last position as the *right* end. We will use the expressions *to the left of*, *to the right of*, *leftmost* and *rightmost* accordingly.

A sequence $(y_0, \ldots, y_r)$ of distinct vertices of G is called a y_0, y_r*-path of length* r of G if $y_{i-1}y_i \in E(G)$ for every $1 \leq i \leq r$. If $(y_0, \ldots, y_r)$ is a y_0, y_r-path of G and $y_0y_r \in E(G)$ then $(y_0, \ldots, y_r)$ is a *cycle* of G. The cycle $(y_0, \ldots, y_r)$ is *chordless* if the cycle edges $y_0y_1, \ldots, y_{r-1}y_r, y_ry_0$ are exactly the edges of $G[\{y_0, \ldots, y_r\}]$. So, a cycle is chordless if no pair of non-consecutive vertices on the cycle is adjacent in G. For a vertex pair u, v of G, the *distance* between u and v in G, denoted by $\mathrm{d}_G(u, v)$, is the smallest integer k such that G has a u, v-path of length k; if no such path exists then $\mathrm{d}_G(u, v) = \infty$. G is *connected* if G has a u, v-path for every vertex pair u, v; otherwise, G is *disconnected*. A *connected component* of G is a maximal connected induced subgraph of G.

For a vertex triple u, v, x of G, $x \in I_G[\{u, v\}]$ if and only if $\mathrm{d}_G(u, v) = \mathrm{d}_G(u, x) + \mathrm{d}_G(x, v)$ [7], and for $D \subseteq V(G)$, $x \in I_G[D]$ if and only if there are vertices u, v in D with $x \in I_G[\{u, v\}]$. It directly follows that a geodetic set of a disconnected graph is the union of geodetic sets of its connected components. Hence, the geodetic number of a disconnected graph is the sum of the geodetic numbers of the connected components. It therefore suffices to study geodetic sets of connected graphs, and we will assume all input graphs to be connected in the paper. A vertex v is called *simplicial* if $N_G(v)$ is a clique of G. Since a simplicial vertex cannot lie on a shortest path between any two other vertices, it is easy to see that every geodetic set of G contains the simplicial vertices of G.

3 Minimum Geodetic Sets for Proper Interval Graphs

Proper interval graphs are equivalent to the intersection graphs of intervals of the real line where the intervals are of unit length [20]. A vertex ordering σ for a graph G is called a *proper interval ordering* if the following is true for every vertex triple u, v, w of G: $u \prec_\sigma v \prec_\sigma w$ and $uw \in E(G)$ implies $uv \in E(G)$ and $vw \in E(G)$. A graph is a proper interval graph if and only if it has a proper interval ordering [15]. The properties of proper interval orderings imply that the vertices at the left end and at the right end in a proper interval ordering are simplicial.

We construct an algorithm for computing a geodetic set of smallest size of a proper interval graph. The algorithm is based on a dynamic-programming approach and determines the minimum size of a geodetic set of very restrictive properties of a proper interval graph. The very restrictive properties are necessary to make our approach work. The underlying idea is to show that the proper interval graph can be partitioned into small pieces and the vertices of a geodetic set can be determined on the small pieces and put together to form a geodetic set of the input graph. We present the algorithm and the main correctness arguments in the second part of this section. In the first part of this section, we show the main theoretical result of the paper, namely that each proper interval graph has a geodetic set of minimum size that satisfies the very restrictive properties needed for the algorithm.

We fix some definitions, that will be valid throughout this section. We consider an arbitrary but fixed connected proper interval graph G and a proper interval ordering σ for G. Let a be the left end vertex in σ. Remember that a is a simplicial vertex of G. For $i \geq 0$, let $L_i =_{\mathrm{def}} \{x \in V(G) : \mathrm{d}_G(a, x) = i\}$ be the vertices of G at distance i to a. Let h be the largest integer such that $L_h \neq \emptyset$. We call h the *height* of a. Observe that the height of a is the maximum distance between a and the vertices of G. Let $\Lambda =_{\mathrm{def}} \langle L_0, \ldots, L_h \rangle$. We call Λ the *BFS (breadth first search) partition* of G with root vertex a. The breadth first search partition of G with root vertex a is the partition of $V(G)$ into the levels of a breadth first search of G starting from vertex a. Note that $L_0 = \{a\}$. For every vertex pair u, v of G and every index i with $1 \leq i \leq h$, if $u, v \in L_i$ and $u \prec_\sigma v$ then $N_G(v) \cap L_{i-1} \subseteq N_G(u)$, and each of $L_1, \ldots, L_h$ is a clique of G [4]. These *neighbourhood inclusion* and *clique* properties will be central throughout this section. For every $0 \leq i \leq h$, let c_i be the rightmost vertex from L_i with respect to σ; clearly, $c_0 = a$, and c_h is the vertex at the right end in σ. Note that $L_i \subseteq N_G(c_{i-1})$ for every $1 \leq i \leq h$.

3.1 Minimum Geodetic Sets with Desirable Properties

We show that G has a geodetic set of smallest size that satisfies very restrictive properties. Let u and v be two vertices of G, and assume that $u \in L_p$ and

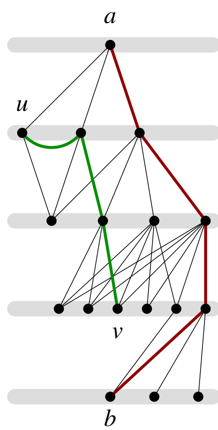

Fig. 1. The figure shows a proper interval graph and its BFS partition with root vertex a. Each level is a clique of the graph, whose edges are omitted except for one edge. Two paths of smallest length are marked, namely an a, b-path, whose length is 4, and a u, v-path of length 3.

$v \in L_q$ and $p < q$. Then, $q - p \leq \mathrm{d}_G(u, v) \leq q - p + 1$, since a u, v-path of smallest length contains a vertex from each of $L_p, \ldots, L_q$ and may contain two vertices from at most one of these partition sets. The lower bound follows from properties of breadth first search, and the upper bound follows from the neighbourhood inclusion and clique properties. Figure 1 depicts a proper interval graph and gives two examples for the structure of shortest paths.

Let D be a set of vertices of G. If D is a geodetic set of G then $a \in D$, since a is a simplicial vertex of G, and D must contain all simplicial vertices of G. We want to show that for determining the geodetic closure of D, $I_G[D]$, it is not necessary to consider the shortest paths between all vertex pairs from D but only between special vertex pairs.

We begin our analysis of the structure of geodetic sets of G. Let $D \subseteq V(G)$. We define the *range sets* $R_0(D), \ldots, R_h(D)$ and $R(D)$ of D on Λ. For every index i with $0 \leq i < h$, let

$$R_i(D) =_{\text{def}} \begin{cases} D \cap L_h & \text{, if } i = h \\ (D \cap L_i) \cup \bigcup_{v \in R_{i+1}(D)} (N_G(v) \cap L_i) & \text{, if } 0 \leq i < h \,, \end{cases}$$

and let

$$R(D) =_{\text{def}} \bigcup_{1 \leq i \leq h} R_i(D) \,.$$

For convenience, we write $R(x)$ instead of $R(\{x\})$ and, analogously, $I_G[u, v]$ instead of $I_G[\{u, v\}]$. As a simple consequence of the properties of Λ, we obtain the following lemma.

Lemma 1. *Let u, v be a vertex pair of G with $u \prec_\sigma v$.*

1) *If $u \in R(v)$ then $R(u) \subseteq R(v)$.*
2) *If $u \notin R(v)$ and $u \in L_p$ with $0 \leq p \leq h$ then $R_i(v) \subseteq R_i(u)$ for every $0 \leq i < p$.*

It follows from Lemma 1 for every vertex pair u, v of G with $u \in L_p$ and $v \in L_q$ and $u \prec_\sigma v$, that $u \in R(v)$ if and only if $\mathrm{d}_G(u, v) = q - p$. We use this result to characterise the sets $I_G[u, v]$ in the next lemma, whose proof follows from Lemma 1.

Lemma 2. *Let u, v be a vertex pair of G with $a \prec_\sigma u \prec_\sigma v$.*

1) *If $u \in R(v)$ then $I_G[u, v] \subseteq R(v)$.*
2) *If $u \notin R(v)$ then for every vertex x of G, $x \in I_G[u, v] \setminus R(v)$ if and only if $u \in R(x)$ and $x \notin R(v)$ and $u \preccurlyeq_\sigma x \prec_\sigma v$.*

Let x be a vertex of G, and let i be the index with $x \in L_i$. We say that x has a *below-neighbour* if $i < h$ and x has a neighbour in L_{i+1}, i.e., if $N_G(x) \cap L_{i+1} \neq \emptyset$. For a set $D \subseteq V(G)$, we denote by $\Upsilon^*(D)$ the set of ordered vertex pairs (u, v) from D that satisfy the following three conditions:

P1) $v \in \{c_1, \ldots, c_h\}$
P2) there is $1 \leq i \leq h$ with $a \prec_\sigma u \prec_\sigma c_i \prec_\sigma v$
P3) u has a below-neighbour and $u \notin R(v)$.

Note that the second condition, P2, requires that u and v do not belong to the same BFS partition class, since $u \prec_\sigma c_i \prec_\sigma v$ implies that $u \in L_0 \cup \cdots \cup L_i$ and $v \in L_{i+1} \cup \cdots \cup L_h$. Together with $a \prec_\sigma u$, it also follows that $1 \leq i < h$.

We show that G has a geodetic set of minimum size that satisfies very restrictive properties. One of the main properties is that it suffices to consider only special vertex pairs for computing the geodetic closure. These vertex pairs mainly satisfy the three conditions P1, P2, P3. Let $D \subseteq V(G)$ and let $\mathcal{D}$ be a set of ordered vertex pairs from D. We call $(D, \mathcal{D})$ a *geodetic pair* for G if $\mathcal{D} \subseteq \Upsilon^*(D)$. Observe that $R(D) \subseteq \bigcup_{u \in D} I_G[u, a] \subseteq I_G[D \cup \{a\}]$, and for every $(u, v) \in \mathcal{D}$, $I_G[u, v] \subseteq I_G[D]$. Thus, $R(D) \cup \bigcup_{(u,v) \in \mathcal{D}} I_G[u, v] \subseteq I_G[D \cup \{a\}]$. The following lemma shows that for determining a minimum geodetic set, it suffices to consider geodetic pairs.

Lemma 3. *There is a geodetic pair $(F, \mathcal{F})$ for G such that $F \cup \{a\}$ is a minimum geodetic set of G and $V(G) \subseteq R(F) \cup \bigcup_{(u,v) \in \mathcal{F}} I_G[u, v]$.*

Proof. Let $D \subseteq V(G)$ be a minimum geodetic set of G. Since a and c_h are simplicial vertices, D contains both a and c_h. Let $\mathcal{D}$ be the set of all ordered vertex pairs (u, v) from D with $a \prec_\sigma u \prec_\sigma v$ and $uv \notin E(G)$ and $u \notin R(v)$ and u has a below-neighbour. Using Lemma 2, it can be shown that $V(G) \subseteq R(D) \cup \bigcup_{(u,v) \in \mathcal{D}} I_G[u, v]$.

Let $\mathscr{E}$ be the set of indices i with $1 \leq i \leq h$ such that $R_i(D) \subset L_i$. Let $i \in \mathscr{E}$, and let b_i be the vertex from $L_i \setminus R_i(D)$ that is rightmost with respect to σ. Since $b_i \in I_G[D]$, there is a vertex pair (u_i, v_i) in $\mathcal{D}$ with $b_i \in I_G[u_i, v_i]$. Let

$\mathcal{D}' =_{\text{def}} \{(u_i, v_i) : i \in \mathscr{E}\}$ and $\mathscr{J} =_{\text{def}} \{j : v_i \in L_j$ for some $i \in \mathscr{E}\}$. Let ψ be the mapping: for every $i \in \mathscr{E}$ and $1 \leq j \leq h$, if $v_i \in L_j$ then $\psi(v_i) =_{\text{def}} c_j$. Let

$$F =_{\text{def}} \Big(D \setminus \{v_i : i \in \mathscr{E}\}\Big) \cup \Big\{c_j : j \in \mathscr{J}\Big\} \quad \text{and} \quad \mathcal{F} =_{\text{def}} \Big\{(u, \psi(v)) : (u, v) \in \mathcal{D}'\Big\}.$$

Observe that $|F| \leq |D|$. By carefully analysing $\mathcal{F}$, we can prove that $\mathcal{F} \subseteq \Upsilon^*(F)$, which means that $(F, \mathcal{F})$ is a geodetic pair for G, and that $V(G) \subseteq R(F) \cup \bigcup_{(u,v)\in\mathcal{F}} I_G[u, v]$. It follows that $(F, \mathcal{F})$ satisfies the claim of the lemma. □

The properties and restrictions of geodetic pairs are strong, but they are not strong enough to satisfy our algorithmic demands. We therefore define restricted geodetic pairs. Let $(D, \mathcal{D})$ be a geodetic pair for G. A vertex x from D *appears* in $\mathcal{D}$ if there is $(u, v) \in \mathcal{D}$ such that $x \in \{u, v\}$. We call $(D, \mathcal{D})$ a *normal geodetic pair* if the following two conditions are satisfied:

- N1) for every vertex u from D that does not appear in $\mathcal{D}$:
  u has no below-neighbour
- N2) for every $u, u', v, v' \in D$ with $(u, v) \in \mathcal{D}$ and $(u', v') \in \mathcal{D}$:
  if $u = u'$ then $v = v'$, and
  if $u \prec_\sigma u' \prec_\sigma v \prec_\sigma v'$ then there is $1 \leq i < h$ such that $u', v \in L_i$.

If a vertex u has a below-neighbour, say v, then $R(u) \subseteq R(v)$, as it was shown in Lemma 1. Condition N1 implies that $R(D)$ cannot be extended by simply choosing a below-neighbour of a vertex in D. Condition N2 is our most important property of normal geodetic pairs. It requires that two pairs from $\mathcal{D}$ must not overlap; if they do overlap then they meet at a common BFS partition class, as it is expressed as $u', v \in L_i$. We show that for determining a minimum geodetic set of G, it suffices to consider only normal geodetic pairs.

Theorem 1. *G has a normal geodetic pair $(F, \mathcal{F})$ such that $F \cup \{a\}$ is a minimum geodetic set of G and $V(G) \subseteq R(F) \cup \bigcup_{(u,v)\in\mathcal{F}} I_G[u, v]$.*

Proof. Suppose for a contradiction that G does not have a normal geodetic pair that satisfies the claim. Let $(D, \mathcal{D})$ be a geodetic pair for G satisfying Lemma 3 such that the number of violations of conditions N1 and N2 is a small as possible. We can assume that $(D, \mathcal{D})$ satisfies condition N1 and the uniqueness part of condition N2. Hence, there are pairs (d, c) and (d', c') in $\mathcal{D}$ with $d \prec_\sigma d' \prec_\sigma c \prec_\sigma c'$ and indices l' and m with $1 \leq l' < m < h$ and $d' \in L_{l'}$ and $c \in L_m$. Then one of the three cases below must apply. Due to the space restrictions, we only give the construction. The correctness of the arguments follows from a sequence of results about properties of $\mathcal{D}$.

Case 1: $d \in L_{l'}$
Let $\mathcal{D}' =_{\text{def}} (\mathcal{D} \setminus \{(d, c)\}) \cup \{(d', c)\}$. There is $\mathcal{F} \subseteq \mathcal{D}'$ so that $(D, \mathcal{F})$ is a geodetic pair for G satisfying Lemma 3 that has a smaller number of conflicting pairs and therefore contradicts the choice of $(D, \mathcal{D})$.

Case 2: $d \in L_1 \cup \cdots \cup L_{l'-1}$ and $d \notin R(d')$
Let $\mathcal{F} =_{\text{def}} (\mathcal{D} \setminus \{(d, c)\}) \cup \{(d, c')\}$. Then, $(D, \mathcal{F})$ is a geodetic pair for G satisfying Lemma 3 that has a smaller number of conflicting pairs and therefore contradicts the choice of $(D, \mathcal{D})$.

Case 3: $d \in L_1 \cup \cdots \cup L_{l'-1}$ and $d \in R(d')$
Let w be the rightmost vertex of G with respect to σ satisfying: $w \prec_\sigma c$ and $d' \in R(w)$. It can be shown that $I_G[d', c'] \subseteq R(\{w, c'\}) \cup I_G[d, c] \cup I_G[w, c']$. Let $F =_{\text{def}} (D \setminus \{d'\}) \cup \{w\}$ and $\mathcal{F} =_{\text{def}} (\mathcal{D} \setminus \{(d', c')\}) \cup \{(w, c')\}$. It is important to observe that $w \neq d'$, particularly since d' has a below-neighbour. It follows that $(F, \mathcal{F})$ is a geodetic pair for G satisfying Lemma 3 that has a smaller number of conflicting pairs and therefore contradicts the choice of $(D, \mathcal{D})$. □

3.2 Computing the Geodetic Number in Polynomial Time

We give a polynomial-time algorithm for computing the geodetic number of an input proper interval graph. Our algorithm can be extended to also determine a minimum geodetic set of the input graph. The algorithm is strongly based on the results from the previous subsection, namely Theorem 1. To compute the geodetic number of the input graph, it suffices to consider only normal geodetic pairs. The structural properties of normal geodetic pairs, especially the implications of condition N2, admit a dynamic-programming approach. We compute normal geodetic pairs for small induced subgraphs of the input graph and extend the small induced subgraphs and the related pairs. The challenge of the dynamic-programming approach is to give a description of the properties of normal geodetic pairs that precisely explain how a small solution can be extended to a larger solution, without knowing the actual solution.

The *size* of a geodetic pair $(D, \mathcal{D})$ is $|D|$. The *normal geodetic number* of G is the smallest size of a normal geodetic pair $(D, \mathcal{D})$ that satisfies $V(G) \subseteq R(D) \cup \bigcup_{(u,v) \in \mathcal{D}} I_G[u, v]$. Theorem 1 shows that the normal geodetic number of G plus 1 is equal to the geodetic number of G. We present an algorithm to compute the normal geodetic number of G, which also yields the geodetic number of G.

Descripts and realizers
Our algorithm to compute the normal geodetic number of G is based on the idea of incrementally computing a normal geodetic pair by extending an already covered part of $V(G)$. Such a covered part can be described by parameters. A *descript* is an extended $(9+2)$-tuple $[p, q; d, e, e'; b'; b, c, c'] + [s, t]$ where p, q are integers with $0 \leq p < q \leq h$ and:

- either $s \in V(G)$ and $t \in \{c_1, \ldots, c_h\}$ or $s = t = \times$, and
  $d, e, e' \in L_p \cup \{\times\}$ and $b' \in L_{p+1}$ and $c, c' \in L_q \cup \{\times\}$ and $b \in L_{q+1} \cup \{\times\}$
- if $d \neq \times$ then $e, e' \neq \times$ and $d \preccurlyeq_\sigma e \preccurlyeq_\sigma e'$, and
  if $s \neq \times$ then $s \prec_\sigma c_p \prec_\sigma c_q \preccurlyeq_\sigma t$, and
  if $s \neq \times$ and $d \neq \times$ then $s \prec_\sigma d$, and
  if $c \neq \times$ then $c' \neq \times$ and $c \preccurlyeq_\sigma c'$, and
  if $b \neq \times$ then $q \leq h - 1$.

We employ a special symbol $\times$, that will have the meaning of non-existing vertex: $\{\times\} =_{\text{def}} \{(\times, \times)\} =_{\text{def}} [\times \ldots \times] =_{\text{def}} \emptyset$. For a vertex pair u, v of G with $u \prec_\sigma v$ and i the integer with $u \in L_i$, let

$$[u \ldots v] =_{\text{def}} \{x : u \preccurlyeq_\sigma x \preccurlyeq_\sigma v\} \quad \text{and} \quad [[u \ldots v] =_{\text{def}} \{x : a_{i+1} \preccurlyeq_\sigma x \preccurlyeq_\sigma v\}.$$

We employ descripts to describe solutions for induced subgraphs of G. For $0 \leq i \leq h$, let a_i be the leftmost vertex from L_i with respect to σ; clearly, $a_0 = a$. Let $(D, \mathcal{D})$ be a geodetic pair for G. We call $(D, \mathcal{D})$ a *minimal geodetic pair* if the following conditions are satisfied:

M1) $R(D) \cup I_G[\mathcal{C}] \subset R(D) \cup I_G[\mathcal{D}]$ for every $\mathcal{C} \subset \mathcal{D}$
M2) for every vertex u from $D \setminus \{a_1, \ldots, a_h\}$ that does not appear in $\mathcal{D}$: $R(C) \cup I_G[\mathcal{D}] \subset R(D) \cup I_G[\mathcal{D}]$ where $C =_{\text{def}} (D \setminus \{u\}) \cup \{a_i\}$ and i is the integer with $u \in L_i$.

Condition M1 means that every pair in $\mathcal{D}$ is necessary, and condition M2 means that every vertex from D that does not appear in $\mathcal{D}$ and that is not in $\{a_1, \ldots, a_h\}$ must cover itself.

Let $(D, \mathcal{D})$ be a normal geodetic pair for G, and let $(u, v) \in \mathcal{D}$. A *(u, v)-field* of $\mathcal{D}$ is a pair (x, y) from $\mathcal{D}$ with $u \prec_\sigma x \prec_\sigma y \preccurlyeq_\sigma v$ such that there is no pair (x', y') in $\mathcal{D}$ with $u \prec_\sigma x' \prec_\sigma x \prec_\sigma y \preccurlyeq_\sigma y' \preccurlyeq_\sigma v$. We can say that a (u, v)-field is a maximal pair from $\mathcal{D}$ inside of (u, v). Let A and B be the vertices from D that respectively do and do not appear in $\mathcal{D}$. We call $(D, \mathcal{D})$ a *realizer* for $[p, q; d, e, e'; b'; b, c, c'] + [s, t]$ if $(D, \mathcal{D})$ is a minimal normal geodetic pair for G with $D \subseteq L_p \cup \cdots \cup L_q$ and $B \cap L_p = \emptyset$ and $D \cup \{b\} \neq \emptyset$ such that the following conditions are satisfied, where $C =_{\text{def}} D \cup \{b\}$ and $\mathcal{C} =_{\text{def}} \mathcal{D} \cup \{(s, t)\}$:

- d is the leftmost vertex from $A \cap L_p$ with respect to σ and $[e \ldots e'] \subseteq A$
- b' is the leftmost vertex from $R_{p+1}(C)$ with respect to σ
- $[c \ldots c'] \cap (R(C) \cup I_G[\mathcal{C}]) = \emptyset$ and $[a_{p+1} \ldots c_q] \subseteq R(C) \cup I_G[\mathcal{C}] \cup [c \ldots c']$
- if $s \neq \times$ then for every (s, t)-field (u, v) of $\mathcal{D}$: $[[u \ldots v] \subseteq R(C) \cup I_G[\mathcal{D}] \cup [c \ldots c']$.

For the condition on d, if $A \cap L_p$ is empty then $d = \times$. Note that there are descripts that have no realizer, for instance, if $b \neq \times$ and $c \in R(b)$.

The normal geodetic number of G is equal to the smallest size of a realizer for $[0, h; \times, \times, \times; a_1; \times, \times, \times] + [\times, \times]$. We give an algorithm that computes the smallest size of a realizer for an arbitrary descript by using already computed values for "smaller" descripts. We define a function Γ over the set of descripts that yields two integer values as follows: for a descript $\mathscr{A} = [p, q; d, e, e'; b'; b, c, c'] + [s, t]$ for G, $\Gamma(\mathscr{A}) =_{\text{def}} (g_1, g_2)$, where g_1 is the size of a smallest realizer for $\mathscr{A}$, and g_2 is the size of a realizer for $\mathscr{A}$ containing c_q and that is of smallest possible size. If $\mathscr{A}$ has a realizer then g_1 and g_2 exist, and $g_1 \leq g_2 \leq g_1 + 1$.

Let $\mathscr{A} = [p, q; d, e, e'; b'; b, c, c'] + [s, t]$ be a descript for G. The algorithm for computing $\Gamma(\mathscr{A}) = (g_1, g_2)$ is presented in Figure 2. The algorithm itself is simple. The difficulty lies in proving the correctness of the algorithm, that g_1 and g_2 are indeed the optimal values for the sizes of the two desired realizers. It needs to be shown that every realizer admits a reduction that yields a realizer for "smaller" descripts, and realizers for "smaller" descripts can be extended to realizers for larger descripts, and the executability of both types of operations can be determined from only considering the descripts, in particular, without the knowledge of the actual realizers. The following three lemmas show that

Algorithm. SizeOfRealizer
begin
  let $F_1 \subseteq V(G) \setminus \{a\}$ and $F_2 \subseteq V(G) \setminus \{a, c_q\}$ be of smallest size such that
    $(F_1, \emptyset)$ and $(F_2 \cup \{c_q\}, \emptyset)$ are realizers for $\mathscr{A}$;
  for every r with $p < r < q$ and $d', e'', e''' \in L_r \cup \{\times\}$ and $b'' \in L_{r+1}$ **do**
    if $\mathscr{B} =_{\text{def}} [p, r; d, e, e'; b'; b'', e'', e'''] + [s, t]$ and
      $\mathscr{C} =_{\text{def}} [r, q; d', e'', e'''; b''; b, c, c'] + [s, t]$ are descripts **then**
      let $\Gamma(\mathscr{B}) = (m, m')$ and $\Gamma(\mathscr{C}) = (n, n')$;
      let $k_{\mathscr{B},\mathscr{C}} =_{\text{def}} m + n$ and $k'_{\mathscr{B},\mathscr{C}} =_{\text{def}} m + n'$
  end if **end for**;
  if $s \neq \times$ **then**
    let $\mathscr{B} =_{\text{def}} [p, q; d, e, e'; b'; b, c, c'] + [\times, \times]$;
    let $k'_{\mathscr{B}} =_{\text{def}} m'$ where $\Gamma(\mathscr{B}) = (m, m')$
  end if;
  if $s = \times$ **and** $d \neq \times$ **then**
    for every $d', e'', e''' \in ([d \dots c_p] \setminus \{d\}) \cup \{\times\}$ **do**
      if $\mathscr{B} =_{\text{def}} [p, q; d', e'', e'''; b'; b, c, c'] + [d, c_q]$ is a descript **then**
        let $k'_{\mathscr{B}} =_{\text{def}} m' + 1$ where $\Gamma(\mathscr{B}) = (m, m')$
  end if **end for** **end if**;
  let k_1 and k'_1 be the smallest values of respectively $k_{\mathscr{B},\mathscr{C}}$ and $k'_{\mathscr{B},\mathscr{C}}$;
  let k'_2 be the smallest value of $k'_{\mathscr{B}}$;
  let $\Gamma(\mathscr{A}) =_{\text{def}} (g_1, g_2)$
    where $g_1 =_{\text{def}} \min\{|F_1|, k_1, k'_2\}$ and $g_2 =_{\text{def}} \min\{|F_2| + 1, k'_1, k'_2\})$
end.

Fig. 2. The presented algorithm takes as input a descript $\mathscr{A}$, and $\Gamma(\mathscr{A})$, that is computed, is the pair of the sizes of a smallest realizer for $\mathscr{A}$ and a smallest realizer for $\mathscr{A}$ that contains vertex c_q

the extension of a realizer is indeed possible; the three lemmas implicitly also define what we want to mean by "smaller" descript, namely $\mathscr{B}$ and $\mathscr{C}$ are smaller than $\mathscr{A}$.

Lemma 4 (Realizer extension 1). *Let* $\mathscr{A} = [p, q; d, e, e'; b'; b, c, c'] + [s, t]$ *and* $\mathscr{B} = [p, r; d, e, e'; b'; b'', c'', c'''] + [s, t]$ *and* $\mathscr{C} = [r, q; d', c'', c'''; b''; b, c, c'] + [s, t]$ *be descripts. Let* $\Gamma(\mathscr{A}) = (k, k')$ *and* $\Gamma(\mathscr{B}) = (m, m')$ *and* $\Gamma(\mathscr{C}) = (n, n')$*. Then,* $k \leq m + n$ *and* $k' \leq m + n'$*.*

Lemma 5 (Realizer extension 2). *Let* $\mathscr{A} = [p, q; d, e, e'; b'; b, c, c'] + [s, t]$ *be a descript with* $s \neq \times$ *and* $d \neq \times$ *and* $[c \dots c'] \cap I_G[s, t] = \emptyset$*. Let* $\mathscr{B} = [p, q; d, e, e'; b'; b, c, c'] + [\times, \times]$*. Let* $\Gamma(\mathscr{A}) = (k, k')$ *and* $\Gamma(\mathscr{B}) = (m, m')$*. Then,* $k' \leq m'$*.*

Lemma 6 (Realizer extension 3). *Let* $\mathscr{A} = [p, q; d, e, e'; b'; b, c, c'] + [\times, \times]$ *be a descript with* $d \neq \times$*. Let* $\mathscr{B} = [p, q; d', e'', e'''; b'; b, c, c'] + [d, c_q]$ *where* $d', e'', e''' \in ([d \dots c_p] \setminus \{d\}) \cup \{\times\}$*. Let* $\Gamma(\mathscr{A}) = (k, k')$ *and* $\Gamma(\mathscr{B}) = (m, m')$*. Then,* $k' \leq m' + 1$*.*

The proofs of Lemma 4, Lemma 5 and Lemma 6 strongly rely on the properties of realizers and the definitions of descripts. Note that the assumptions of Lemma 4 directly require $p < r < q$.

We show next that the converse results of the Lemmas 4, 5 and 6 are also true. If we say that the realizer extension provides an upper bound on the optimal sizes of realizers then we can say that the converse operation of reducing realizers provides lower bounds on the sizes of realizers. The correctness of the following results heavily relies on the properties of minimal normal geodetic pairs.

Lemma 7 (Realizer reduction 1). *Let* $\mathscr{A} = [p, q; d, e, e'; b'; b, c, c'] + [s, t]$ *be a descript with* $p \leq q - 2$*, let* $\Gamma(\mathscr{A}) = (k, k')$*, and assume that* $k < k'$*. Then, there are descripts* $\mathscr{B} = [p, r; d, e, e'; b'; b'', c'', c'''] + [s, t]$ *and* $\mathscr{C} = [r, q; d', c'', c'''; b''; b, c, c'] + [s, t]$ *such that* $k \geq m + n$*, where* $\Gamma(\mathscr{B}) = (m, m')$ *and* $\Gamma(\mathscr{C}) = (n, n')$*.*

Lemma 8 (Realizer reduction 2). *Let* $\mathscr{A} = [p, q; d, e, e'; b'; b, c, c'] + [s, t]$ *be a descript, and let* $\Gamma(\mathscr{A}) = (k, k')$*. Then, one of the three cases applies:*

1) *It holds that* $p \leq q - 2$*, and there are descripts* $\mathscr{B} = [p, r; d, e, e'; b'; b'', c'', c'''] + [s, t]$ *and* $\mathscr{C} = [r, q; d', c'', c'''; b''; b, c, c'] + [s, t]$ *such that* $k' \geq m + n'$*, where* $\Gamma(\mathscr{B}) = (m, m')$ *and* $\Gamma(\mathscr{C}) = (n, n')$*.*
2) *It holds that* $s \neq \times$*, and for* $\mathscr{B} = [p, q; d, e, e'; b'; b, c, c'] + [\times, \times]$ *and* $\Gamma(\mathscr{B}) = (m, m')$*, it holds that* $k' \geq m'$*.*
3) *It holds that* $s = \times$*, and there are* $d', e'', e''' \in ([d \ldots c_p] \setminus \{d\}) \cup \{\times\}$ *such that* $\mathscr{B} = [p, q; d', e'', e'''; b'; b, c, c'] + [d, c_q]$ *is a descript and* $k' \geq m' + 1$*, where* $\Gamma(\mathscr{B}) = (m, m')$*.*

Note that the third case of Lemma 8 implicitly assumes $d \neq \times$. We can only remark here that $d = \times$ would directly imply that the first case must be applicable.

The combination of all results established in this section leads to the main result of our paper, given as Theorem 2 below. The algorithm defines an order on the descripts and iteratively applies Algorithm SizeOfRealizer of Figure 2. The running time of a single application of the algorithm and the number of applications of the algorithm are polynomial in the number of descripts, which is a polynomial in the number of vertices of G.

Theorem 2. *There is a polynomial-time algorithm that, given a connected proper interval graph* G *and a proper interval ordering* σ *for* G*, computes the normal geodetic number of* G *with respect to* σ*.*

Corollary 1. *The geodetic number of proper interval graphs can be computed in polynomial time.*

The proofs of Lemmas 4, 5 and 6 are constructive and show how to obtain a realizer for the "bigger" descript from the "smaller" descripts. These constructions can be used to extend the algorithm of Theorem 2, and also Corollary 1, to compute a minimum geodetic set of G.

4 Concluding Remarks and Further Results

The algorithms for computing the geodetic number and a minimum geodetic set of proper interval graphs (Theorem 2 and Corollary 1) have been our major challenge and main results. In this section, we report shortly on results for other graph classes. Due to limited space, we give only the main ideas behind the results.

A graph is a *cobipartite graph* if it is the complement of a bipartite graph, i.e., the vertex set of the complement admits a partition into two independent sets. The famous NP-complete DOMINATING SET problem is known to be NP-complete on connected bipartite graphs [1,5]. Using a reduction from DOMINATING SET on bipartite graphs to GEODETIC SET on cobipartite graphs, we are able to show the following result.

Theorem 3. *Given a cobipartite graph G and an integer k, it is* NP*-complete to decide whether the geodetic number of G is at most k.*

A *cut vertex* in a graph G is a vertex whose removal disconnects G. A *block* of G is a maximal connected induced subgraph of G that itself has no cut vertex. The proof of the following theorem relies on the fact that no cut-vertex of a graph belongs to a minimum geodetic set.

Theorem 4. *Let G be a connected graph, let A be the set of cut-vertices of G, and let $B_1, \ldots, B_t$ be the blocks of G. For $1 \leq i \leq t$, let D_i be a geodetic set for B_i of smallest possible size satisfying $A \cap V(B_i) \subseteq D_i$. Then, $(D_1 \cup \cdots \cup D_t) \setminus A$ is a minimum geodetic set for G.*

A *block-cactus graph* is a graph whose blocks are either complete graphs or chordless cycles. For complete graphs and chordless cycles, the special geodetic set problem of Theorem 4 is efficiently solvable, which proves the following result.

Theorem 5. *A minimum geodetic set for a block-cactus graph can be computed in polynomial time.*

Finally, we come to bipartite permutation graphs. Let G be a bipartite graph with bi-partition (A, B), and let σ_A and σ_B be orderings for respectively A and B. We say that (σ_A, σ_B) is a *strong ordering* for G if for every vertex quadruple u, v, x, y of G with $u, v \in A$ and $x, y \in B$ and $u \prec_{\sigma_A} v$ and $x \prec_{\sigma_B} y$, $uy \in E(G)$ and $vx \in E(G)$ implies $ux \in E(G)$ and $vy \in E(G)$. G is a *bipartite permutation graph* if and only if it has a strong ordering for bi-partition (A, B) [21]. Notice the resemblance of strong orderings and proper interval orderings. Using this similarity, the results of Section 3 can be applied to bipartite permutation graphs with small modifications. The main difficulty for bipartite permutation graphs is the possible lack of simplicial vertices. In proper interval graphs, the simplicial vertices are important anchor points. In bipartite permutation graphs, we lose this property. Nevertheless, the results of Section 3, when adapted to bipartite permutation graphs, give the following result.

Theorem 6. *Let G be a bipartite permutation graph. A geodetic set for G of size at most $g(G) + 1$ can be computed in polynomial time.*

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