Chapter 8
Weil Representation and Shimura Lifting

8.1 Weil Representation

Let V be an n-dimensional real vector space and V^* be the dual space of V. Denote by B a bilinear form on $(V\times V^*)\times(V\times V^*)$ given by $B(z_1,z_2)=(v_1,v_2^*)=v_2^*(v_1)$ for $z_1=(v_1,v_1^*)$ and $z_2=(v_2,v_2^*)$. Let $A(V)$ be the Lie group with underlying manifold $V\times V^*\times T$ whose multiplication is given by

$$(z,t)(z',t')=(z+z',tt'e(B(z,z'))),\quad \forall z,z'\in V\times V^*,t,t'\in T,$$

where $T=\{z\in\mathbb{C}||z|=1\}$ and $e(z)=\mathrm{e}^{2\pi\mathrm{i}z}$.

We fix a Euclidean measure $\mathrm{d}x$ on V and denote by $\mathrm{d}x^*$ the Euclidean measure which is dual to $\mathrm{d}x$. Namely, the Fourier transformation

$$f^*(x^*)\mapsto\int_{V^*}f^*(x^*)e((x,x^*))\mathrm{d}x^*$$

gives an isometric mapping from $L^2(V^*,\mathrm{d}x^*)$ onto $L^2(V,\mathrm{d}x)$. We denote by U a unitary representation of $A(V)$ on $L^2(V)$ given by

$$\{U(z,t)f\}(x)=te((x,v^*))f(x+v),\quad \forall x\in V,z=(v,v^*)\in V\times V^*,t\in T.$$

Then U is irreducible and $\phi(V)$, the space of rapidly decreasing functions over V, is a dense invariant subspace of $L^2(V)$. A linear transformation of $V\times V^*$ is said to be sympletic if it leaves the alternating form $A(z_1,z_2)=B(z_1,z_2)-B(z_2,z_1)$ invariant. We denote by $S_p(V\times V^*)$ the group of symplectic linear transformations of $V\times V^*$. For $\sigma\in S_p(V,V^*)$ and $z=(v,v^*)\in V\times V^*$, we write

$$\sigma(z)=(v,v^*)\begin{pmatrix}a&b\\c&d\end{pmatrix}=(a(v)+c(v^*),b(v)+d(v^*)),$$

where a,b,c and d are linear mappings from V to V, from V to V^*, from V^* to V and from V^* to V^* respectively. In the following we often identity σ with the matrix $\begin{pmatrix}a&b\\c&d\end{pmatrix}$. For $\sigma\in S_p(V\times V^*)$ and $z\in V\times V^*$. Put

$$F_\sigma(z)=\exp\left(\pi\mathrm{i}B(\sigma(z),\sigma(z))\right)/\exp(\pi\mathrm{i}B(z,z)).$$

X. Wang et al., *Modular Forms with Integral and Half-Integral Weights*

It is easy to see that

$$\begin{aligned} F_\sigma(z+z') &= F_\sigma(z)F_\sigma(z')e(B(\sigma(z),\sigma(z'))-B(z,z')),\\ F_{\sigma\tau}(z) &= F_\tau(\sigma(z))F_\sigma(z). \end{aligned} \tag{8.1}$$

This shows that the group $S_p(V\times V^*)$ acts on $A(V)$ as a group of automophisms via the mapping:

$$w \mapsto w^\sigma = (\sigma(z), tF_\sigma(z)), \quad \forall w=(z,t)\in A(V).$$

Set $U^\sigma(w)=U(w^\sigma)$, then U^σ is an irreducible unitary representation of $A(V)$ which is equivalent to U. Namely, there is a unitary operator $r(\sigma)$ on $L^2(V)$ which satisfies

$$U(w^\sigma) = r(\sigma)^{-1}U(w)r(\sigma), \quad \forall w\in A(V). \tag{8.2}$$

The operator $r(\sigma)$ is unique up to a multiplication by a complex number of modulus 1. Furthermore, the mapping $\sigma\to\gamma(\sigma)$ gives rise to a projective unitary representation of $S_p(V\times V^*)$ on $L^2(V)$. In other words, for each pair $(\sigma,z)\in S_p(V\times V^*)\times S_p(V\times V^*)$, there is a constant $c(\sigma,z)$ which satisfies

$$r(\sigma z) = c(\sigma,z)r(\sigma)r(z). \tag{8.3}$$

This projective unitary representation is called the Weil representation of $S_p(V\times V^*)$. If the entry c of σ is either non-singular or zero, we may normalize $r(\sigma)$ as follows:

$$r(\sigma)f(v) = \begin{cases} |c|^{1/2}\displaystyle\int_{V^*} F_\sigma(v,v^*))f(a(v)+c(v^*))\mathrm{d}v^*, & \text{if } c \text{ is non-singular},\\ |a|^{1/2}e\left(\dfrac{1}{2}(a(v),b(v))\right)f(a(v)), & \text{if } c=0, \end{cases} \tag{8.4}$$

where $\sigma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, $\mathrm{d}(c(x^*))=|c|d^*x^*$ and $\mathrm{d}(a(x))=|a|\mathrm{d}x$.

Let L be a lattice in V and L^* be the dual lattice of L in V^*. Let M^* be a sublattice of L^* and M the dual lattice of M^* in V. Denote by $S_p(L\times M^*)$ the subgroup of $S_p(V\times V^*)$ consisting of linear transformations which leave the lattice $L\times M^*$ invariant. For a character χ of $L\times M^*$ and for a $\sigma\in S_p(L\times M^*)$, we set

$$\chi^\sigma(\lambda) = \chi(\sigma^{-1}(\lambda))F_{\sigma^{-1}}(\lambda), \quad \forall\lambda\in L\times M^*.$$

Then χ^σ is also a character of $L\times M^*$ and $\chi^{\sigma\tau}=(\chi^\sigma)^\tau$.

We denote also by χ the character of a subgroup $L\times M^*\times T$ of $A(V)$ given by

$$\chi\big((z,t)\big) = t\chi(z), \quad \forall\, z\in L\times M^*.$$

Then there exists a $(v_\chi, v_\chi^*)\in V\times V^*$ satisfying

$$\chi(\lambda,\mu^*) = e\big((v_\chi,\mu^*)-(\lambda,v_\chi^*)\big), \quad \forall\,(\lambda,\mu^*)\in L\times M^*.$$

The map $\chi \to (v_\chi, v_\chi^*)$ gives an isomorphism between the character group of $L \times M^*$ and the additive group $V/M \times V^*/L^*$. For a $\mu \in M/L$, we denote by $\chi(\mu)$ the character of $L \times L^*$ corresponding to $(v_\chi + \mu, v_\chi^*)$ of $V/L \times V^*/L^*$. Any extension of χ to a character of $L \times L^*$ coincides with $\chi(\mu)$ for a suitable $\mu \in M/L$. We denote by $T_\chi(L \times M^*)$ the unitary representation of $A(V)$ induced from the character χ of $L \times M^* \times T$ as follows: the representation space $\Theta_\chi(L \times M^*)$ is the Hilbert space of measurable functions $\theta(z)$ on $V \times V^*$ satisfying the following conditions:

$$e\big(B(\lambda, z)\big)\theta(\lambda + z) = \chi(\lambda)\theta(z), \quad \forall\, \lambda \in L \times M^*,\ z \in V \times V^*,$$
$$||\theta||^2 = \int\limits_{V/L \times V^*/M^*} |\theta(x, x^*)|^2 \mathrm{d}x\mathrm{d}x^* < +\infty$$

and $T_\chi(L \times M^*)$ is given by

$$T_\chi(L, M^*)\big((w, t)\big)\theta(z) = te\big(B(z, w)\big)\theta(z + w).$$

It is easy to see that the space $\Theta_{\chi(\mu)}(L \times L^*)$ ($\forall\, \mu \in M/L$) is a closed invariant subspace of $\Theta_\chi(L \times M^*)$ and

$$\Theta_\chi(L \times M^*) = \bigoplus_{\mu \in M/L} \Theta_{\chi(\mu)}(L \times L^*).$$

Put

$$\Theta_\chi = \Theta_\chi(L \times M^*), \quad \Theta_{\chi(\mu)} = \Theta_{\chi(\mu)}(L \times L^*), \quad T_\chi = T_\chi(L \times M^*).$$

For an $f \in \phi(V)$(where $\phi(V)$ is the space of rapidly decreasing functions on V, for the definition, please compare [**?**]), we define

$$\theta_{\chi(\mu)}(f)(x, x^*) = \Big(\sqrt{\mathrm{vol}(V^*/M^*)}\Big)^{-1} \sum_{l \in L} e\big((l + \mu + v_\chi, x^*) + (l, v_\chi^*)\big) f(x + l + \mu + v_\chi),$$

where $\mathrm{vol}(V^*/M^*) = \int\limits_{V^*/M^*} \mathrm{d}x^*$.

It is clear that $\theta_{\chi(\mu)}(f)$ depends on the choice of a representative of $(v_\chi + \mu) \in V/L$ in V. Here and after we choose representatives for $(v_\chi + \mu)$ ($\mu \in M/L$) and fix them. Then $\theta_{\chi(\mu)}(f)$ is a smooth function in $\Theta_{\chi(\mu)}$ and

$$\theta_{\chi(\mu)}\big(U(g)f\big) = T_\chi(g)\theta_{\chi(\mu)}(f), \quad \forall\, g \in A(V),$$
$$||\theta_{\chi(\mu)}(f)||^2 = ||f||^2 = \int\limits_V |f(x)|^2 \mathrm{d}x.$$

Conversely, for a smooth function $\theta \in \Theta_{\chi(\mu)}$, the following function

$$f_\theta(x) = \big(\sqrt{\mathrm{vol}(V^*/M^*)}\big)^{-1} \int\limits_{V^*/M^*} \theta(x-\mu-v_\chi, x^*) e\big(-(\mu+v_\chi, x^*)\big)\mathrm{d}x^* \tag{8.5}$$

belongs to $\phi(V)$ and $\theta_{\chi(\mu)}(f) = \theta$. Thus $\theta_{\chi(\mu)}$ gives a norm preserving linear map from $\phi(V)$ onto the space of smooth functions in $\Theta_{\chi(\mu)}$ which commutes with the action of $A(V)$. The inverse of $\theta_{\chi(\mu)}$ is given by (8.5). These show that $\theta_{\chi(\mu)}$ is extended to linear isometric map from $L^2(V)$ onto $\Theta_{\chi(\mu)}$ which gives an equivalence of two unitary representations $(U, L^2(V))$ and $(T_\chi, \Theta_{\chi(\mu)})$ for any $\mu \in M/L$. Since $(U, L^2(V))$ is irreducible and (T_χ, Θ_χ) is a direct sum of $(T_\chi, \Theta_{\chi(\mu)})$ $(\mu \in M/L)$, any bounded linear map of $L^2(V)$ into Θ_χ is a linear combination of $\theta_{\chi(\mu)}$ $(\mu \in M/L)$ if it commutes with the action of $A(V)$. Finally, put

$$\theta(f, \chi(\mu)) = \theta_{\chi(\mu)}(f)(0,0).$$

All the above results and their proofs can be found in André Weil, 1964.

Proposition 8.1(Generalized Poisson Summation Formula) (1) *Let* $r(\sigma)(\sigma \in S_p(L\times M^*))$ *be the unitary operator in* $L^2(V)$ *which satisfies* (8.2). *There exist constants* $C_\sigma^\chi(u,v)$ $(u, v \in M/L)$ *which satisfy*

$$\theta(r(\sigma)f, \chi(u)) = \sum_{v\in M/L} C_\sigma^\chi(u,v)\theta(f, \chi^\sigma(v)), \quad \forall f \in \phi(V).$$

(2) *Denote by* C_σ^χ *the matrix of size* $[M:L]$ *whose* (u,v)*-entry* $(u, v \in M/L)$ *is* $C_\sigma^\chi(u,v)$. *Then* C_σ^χ *is a unitary matrix and* $C_{\sigma\tau}^\chi = c(\sigma,\tau)C_\sigma^\chi C_\tau^{\chi^\sigma}$ *where* $c(\sigma,\tau)$ *is a complex number of modulus* 1 *defined in* (8.3).

(3) *Set* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and assume* c *is non-singular and* $r(\sigma)$ *is nomalized by the formula* (8.4). *Then the constant* $C_\sigma^\chi(u,v)$ *is given by*

$$\begin{aligned}\mathrm{vol}(V^*/M^*)|c|^{1/2}C_\sigma^\chi(u,v) = \sum_{l\in L/c^*(M^*)} & e\bigg(\frac{1}{2}(l+u'), c^{-1}a(l+u')\bigg) \\ & -(l+u', c^{-1}(v')) + \frac{1}{2}(v', dc^{-1}(v')) + (l, v_\chi^*)),\end{aligned}$$

where $u' = u + v_\chi$ *and* $v' = v + v_{\chi^\sigma}$.

Proof For the details, see T. Shintani, 1975. □

From now on, we set $V = \mathbb{R}^n$. Take a non-degenerate symmetric $n\times n$ matrix Q and identify V with its dual by setting $(x, y) = y^{\mathrm{T}}Qx$. We put $\mathrm{d}x = \mathrm{d}x_1\cdots\mathrm{d}x_n$. Then the dual measure $\mathrm{d}x^*$ is given by $\mathrm{d}x^* = |\mathrm{det}Q|\mathrm{d}x$. We denote by $r(\cdot, Q)$ the Weil representation of $S_p(V\times V^*)$ on $L^2(V)$, to emphasize its dependence on Q. Identify the group $SL_2(\mathbb{R})$ with a subgroup of $S_p(V\times V^*)$ by settings

$$\sigma(x,y) = (ax+cy, bx+dy), \quad \forall x, y \in V, \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).$$

By (8.4), we have the following expression for $r(\sigma)=r(\sigma, Q)$ $(\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}))$:

$$(r(\sigma,Q)f)(x) = \begin{cases} |c|^{-n/2}\sqrt{|\det Q|}\displaystyle\int_V e\left(\frac{a(x,x)-2(x,y)+d(y,y)}{2c}\right) f(y)\mathrm{d}y, & \text{if } c \neq 0, \\ |a|^{n/2} e\left(\dfrac{ab(x,x)}{2}\right) f(ax), & \text{if } c = 0. \end{cases}$$

The group $GL_n(\mathbb{R})$ acts on $L^2(V)$, as a group of unitary operators if we put

$$(Tf)(x) = \sqrt{|\det T|^{-1}} f(T^{-1}x). \tag{8.6}$$

It is clear to verify that

$$r(\sigma, (T^{-1})^{\mathrm{T}} Q T^{-1}) \cdot T = T \cdot r(\sigma, Q), \quad \forall \sigma \in SL_2(\mathbb{R}), T \in GL_n(\mathbb{R}). \tag{8.7}$$

We are going to determine the constant $c(\sigma, \tau)$ in (8.3) for σ, $\tau \in SL_2(\mathbb{R})$.

Denote by $\mathbb{H}$ the complex upper half plane. For $\sigma \in SL_2(\mathbb{R})$, set

$$\varepsilon(\sigma) = \begin{cases} \sqrt{\mathrm{i}}, & \text{if } c > 0, \\ \mathrm{i}^{(1-\mathrm{sgn}(d))/2}, & \text{if } c = 0, \\ \sqrt{\mathrm{i}}^{-1}, & \text{if } c < 0. \end{cases}$$

Take a positive definite symmetrix R such that $RQ^{-1}R = Q$. For $z = u + \mathrm{i}v \in \mathbb{H}$, put

$$Q_z = uQ + \mathrm{i}vR.$$

Let $P_v(x)$ be a homogeneous polynomial of degree v which has the following expression:

$$P_v(x) = \begin{cases} 1, & \text{if } v = 0, \\ (r, x), \ (r \in \mathbb{C}^n, Qr = Rr), & \text{if } v = 1, \\ \sum c_r (r,x)^v, c_r \in \mathbb{C}, r \in \mathbb{C}^n, Qr = Rr, (r,r) = 0, & \text{if } v \geqslant 2 \end{cases}$$

(if $\mathrm{rank}(Q - R) = 1$, we assume $v \leqslant 1$).

Lemma 8.1 *Assume Q has p positive and q negative eigenvalues $(p+q=n, p>0)$. Set*

$$F_z(x) = e\left(\frac{1}{2} Q_z(x)\right) P_v(x).$$

Then

$$r(\sigma, Q) F_z(x) = \varepsilon(\sigma)^{p-q} \sqrt{J(\sigma,z)}^{q-p} |J(\sigma,z)|^{-q} J(\sigma,z)^{-v} F_{\sigma(z)}(x)$$

for any $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, *and where* $J(\sigma, z) = cz + d$.

Proof There exists a $T \in GL_n(\mathbb{R})$ such that $T^{\mathrm{T}}QT = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$ and $T^{\mathrm{T}}RT = I_n$. By (8.7), it is sufficient to show the lemma under the additional assumption that

$$Q = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}, \quad R = I_n.$$

Put $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $c = 0$, the lemma is clear. If $c \neq 0$, by a direct computation, we have

$$r(\sigma, Q)F_z(x) = |c|^{-n/2}\sqrt{v - \mathrm{i}u - \mathrm{i}d/c}^{\,-p}\sqrt{v + \mathrm{i}u + \mathrm{i}d/c}^{\,-q} J(\sigma, z)^{-v} F_{\sigma(z)}(x).$$

Now the lemma follows from the definitions of $\varepsilon(\sigma)$ and $J(\sigma, z)$. This completes the proof. □

By Lemma 8.1, we have

$$\begin{aligned} c(\sigma, \tau) &= \left(\frac{\varepsilon(\sigma\tau)}{\varepsilon(\sigma)\varepsilon(\tau)}\right)^{p-q} c_0(\sigma, \tau)^{q-p}, \\ c_0(\sigma, \tau) &= \frac{\sqrt{J(\sigma\tau, i)}}{\sqrt{J(\sigma, \tau(i))}\sqrt{J(\tau, i)}}. \end{aligned} \tag{8.8}$$

For $\sigma \in SL_2(\mathbb{R})$, set

$$r_0(\sigma, Q) = \varepsilon(\sigma)^{q-p} r(\sigma, Q). \tag{8.9}$$

Let G_1 be the Lie group with the underlying manifold $SL_2(\mathbb{R}) \times T$ and the multiplication given by

$$(\sigma, t)(\sigma', t') = (\sigma\sigma', tt'c_0(\sigma, \sigma')).$$

Then the subgroup $\{(\sigma, \pm 1) | \sigma \in SL_2(\mathbb{R})\}$ of G_1 is isomorphic to the two-fold covering group of $SL_2(\mathbb{R})$. For a $\widetilde{\sigma} = (\sigma, t) \in G_1$, set $r_0(\widetilde{\sigma}, Q) = t^{p-q} r_0(\sigma, Q)$. The following lemma is now immediate to see.

Lemma 8.2 (1) *The mapping:* $\widetilde{\sigma} \mapsto r_0(\widetilde{\sigma}, Q)$ *gives a unitary representation of* G_1 *on* $L^2(V)$. *The space* $\phi(V)$ *is a dense invariant subspace.*

(2) *For any* $f \in \phi(V)$, *the mapping* $\widetilde{\sigma} \mapsto r_0(\widetilde{\sigma}, Q)f$ *is a smooth mapping from* G_1 *into* $\phi(\mathbb{R}^n)$;

It is clear that the mapping $\sigma \mapsto (\sigma, 1)$ gives a locally isomorphic imbedding of $SL_2(\mathbb{R})$ into G_1. Hence, for any element u of the universal enveloping algebra of the Lie algebra of $SL_2(\mathbb{R})$, $r_0(u, Q)$ has an obvious meaning as a differential operator on V. In particular set

$$C_Q = r_0(C, Q), \quad C = 2XY + 2YX + H^2,$$
$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8.10}$$

Then C_Q commutes with $r_0(\widetilde{\sigma}, Q)$ for any $\widetilde{\sigma} \in G_1$.

For $\theta \in \mathbb{R}$, let $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and $\Omega = \{(k_\theta, \varepsilon) | \theta \in \mathbb{R}, \varepsilon = \pm 1\}$. Put

$$\chi_m((k_\theta, \varepsilon)) = \left(\sqrt{\mathrm{e}^{-\mathrm{i}\theta}}\right)^{-m} \varepsilon^m.$$

Then χ_m is a character of Ω and for any $f \in \phi(V)$ we have

$$r_0(k, Q)f = \chi_m(k)f, \quad \forall k \in \Omega. \tag{8.11}$$

Lemma 8.3 *For $z = u + \mathrm{i}v \in \mathbb{H}$, set*

$$\sigma_z = \begin{pmatrix} \sqrt{v} & \sqrt{v}^{-1}u \\ 0 & \sqrt{v}^{-1} \end{pmatrix}.$$

Then

$$r_0(\sigma_z, Q)C_Q f = 4v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} - 2\mathrm{i}mv\frac{\partial}{\partial u}\right) r_0(\sigma_z, Q)f.$$

Proof See I. Gelfand. □

Let G be the connected component of the identity element of the group $O(Q)$ of real linear transformations which leave the quadratic form Q invariant. Then (8.6) gives a unitary representative of G on $L^2(V)$ which commutes with $r(\widetilde{\sigma}, Q)$ for any $\widetilde{\sigma} \in G_1$. Take a $T \in GL_n(\mathbb{R})$ satisfying $T^{\mathrm{T}}QT = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ and set

$$X_{ij} = T(e_{ij} - e_{ji})T^{-1}, \quad 1 \leqslant i < j \leqslant p \ \text{ or } \ p < i < j \leqslant n,$$
$$Y_{kl} = T(e_{kl} + e_{lk})T^{-1}, \quad 1 \leqslant k \leqslant p < l \leqslant n.$$

Then X_{ij} and Y_{kl} form a base of the Lie algebra of G. Put

$$L_Q = -\sum_{\substack{1 \leqslant i < j \leqslant p \text{ or} \\ p < i < j \leqslant n}} X_{ij}^2 + \sum_{1 \leqslant k \leqslant p < l \leqslant n} Y_{kl}^2. \tag{8.12}$$

Then L_Q is the Casimir operator on G. The representation (8.6) of G maps L_Q to a second order differential operator on $\mathbb{R}^n$ which is also denoted by L_Q.

Lemma 8.4 *For any $F \in \phi(V)$, we have*

$$C_Q F = (L_Q + n(n-4)/4)F.$$

Proof By (8.7), we may assume that $Q = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$. In this case, a simple computation shows that

$$\begin{aligned} r(H,Q)F &= \sum_{x=1}^{n} x_i \frac{\partial F}{\partial x_i} + \frac{n}{2}F, \\ r(X,Q)F &= \pi \mathrm{i}(x,x)F, \\ r(Y,Q)F &= \mathrm{i}(4\pi)^{-1}\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{n} \frac{\partial^2}{\partial x_j^2}\right)F, \end{aligned}$$

where $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus

$$\begin{aligned} C_Q F = &- (x,x)\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{n} \frac{\partial^2}{\partial x_j^2}\right)F + \sum_{i=1}^{n} x_i^2 \frac{\partial^2 F}{\partial x_i^2} \\ &+ (n-1)\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i} + \left(\frac{n^2}{4} - n\right)F + 2\sum_{1 \leqslant i < j \leqslant n} x_i x_j \frac{\partial^2 F}{\partial x_i \partial x_j}. \end{aligned}$$

On the other hand,

$$\begin{aligned} L_Q F &= -\sum \left(x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}\right)^2 F + \sum \left(x_i \frac{\partial}{\partial x_k} + x_k \frac{\partial}{\partial x_i}\right)^2 F \\ &= -(x,x)\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{n} \frac{\partial^2}{\partial x_j^2}\right)F + 2\sum_{1 \leqslant i < j \leqslant n} x_i x_j \frac{\partial^2 F}{\partial x_i \partial x_j} \\ &\quad + (n-1)\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} x_i^2 \frac{\partial^2 F}{\partial x_i^2}. \end{aligned}$$

Therefore, $C_Q = L_Q + n(n-4)/4$. □

Here and after, we assume Q to be a rational symmetric matrix with p (> 0) positive and q $(= n - p)$ negative eigenvalues. Let L be a lattice of V, and L^* be the dual of L in V, i.e.,

$$L^* = \{x \in V | (x,y) = x^{\mathrm{T}} Q y \in \mathbb{Z}, \forall y \in L\}.$$

We always assume $L \subset L^*$. Let $v(L)$ be the volume of the fundamental parallelotop of L in V:

$$v(L) = \int_{\mathbb{R}^n/L} \mathrm{d}x.$$

For any $f \in \phi(V)$ and $h \in L^*/L$, put $\theta(f,h) = \sum_{l \in L} f(l+h)$.

Proposition 8.2 *Let* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *satisfy the following condition*

$$ab(x,x) \equiv cd(y,y) \equiv 0 \pmod 2, \quad \forall x, y \in L. \tag{8.13}$$

Then we have

(1) $\theta(r(\sigma,Q)f,h) = \sum_{k\in L^*/L} c(h,k)_\sigma \theta(f,k), \quad \forall f \in \phi(V)$, *where*

$$c(h,k)_\sigma =$$

$$\begin{cases} \delta_{h,ak} e\left(\dfrac{ab(h,h)}{2}\right), & \text{if } c=0, \\ \sqrt{|\det Q|}^{-1} v(L)^{-1}|c|^{-n/2} \displaystyle\sum_{r\in L/cL} e\left(\dfrac{a(h+r,h+r)-2(k,h+r)+d(k,k)}{2c}\right), & \text{if } c\neq 0. \end{cases}$$

(2) *Further assume that* c *is even,* $cL^* \subset L$*,* $cd \neq 0$ *and* $c(x,x) \equiv 0 \pmod 2$ *for any* $x \in L^*$*. Let* $\{\lambda_1, \cdots, \lambda_n\}$ *be a* $\mathbb{Z}$*–base of* L *and set* $D = \det((\lambda_i, \lambda_j))$*. Then*

$$\sqrt{\mathrm{i}}^{-(p-q)\mathrm{sgn}(cd)} c(h,k)_\sigma = \begin{cases} \delta_{h,dk} e\left(\dfrac{ab(h,h)}{2}\right) \varepsilon_d^{-n} (\mathrm{sgn}(c)\mathrm{i})^n \left(\dfrac{2c}{d}\right)^n \left(\dfrac{D}{-d}\right), & \text{if } d<0, \\ \delta_{h,dk} e\left(\dfrac{ab(h,h)}{2}\right) \varepsilon_d^{n} \left(\dfrac{-2c}{d}\right)^n \left(\dfrac{D}{d}\right), & \text{if } d>0, \end{cases}$$

where $\varepsilon_d = 1$ *or* i *according to* $d \equiv 1$ *or* $3 \pmod 4$ *respectively.*

Proof (1) We note that the group $SL_2(\mathbb{Z})$ is mapped into a subgroup of $S_p(L\times L)$ by our embedding of $SL_2(\mathbb{R})$ into $S_p(V\times V^*)$. Thus, the result in (1) is an immediate consequence of Proposition 8.1.

(2) Let e_0 be the index of L in L^*. Denote by C_σ the matrix of size e_0 whose (h,k) entry is $c(h,k)_\sigma$ $(h,k \in L^*/L)$. If σ, σ' and $\sigma\sigma'$ all satisfy the condition (8.13), it follows from the second statement of Proposition 8.1 that

$$C_{\sigma\sigma'} = c(\sigma,\sigma')C_\sigma C_{\sigma'}.$$

Put $\sigma' = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}$ and $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then σ', ω and $\sigma = \sigma'\omega$ all satisfy the condition (8.13). By (8.8) we have

$$c(\sigma',\omega) = \sqrt{\mathrm{i}}^{(p-q)\mathrm{sgn}(cd)}.$$

Hence

$$\begin{aligned} c(h,k)_\sigma =& \sqrt{\mathrm{i}}^{(p-q)\mathrm{sgn}(cd)} |\det(Q)|^{-1} v(L)^{-2} |d|^{-n/2} \\ &\times \sum_{r\in L/dL} \sum_{l\in L^*/L} e\left(\frac{-b(h+r,h+r)-2(l,h+r)+c(l,l)}{-2d}\right) e(-(l,k)). \end{aligned}$$

Since $cL^* \subset L$, the map $l \mapsto dl$ induces an automorphism of L^*/L. Taking into account the assumption that $c(x,x) \in 2\mathbb{Z}$ ($\forall x \in L^*$), we have

$$\begin{aligned}&\sum_{l\in L^*/L} e\left(\frac{-b(h+r,h+r)-2(l,h+r)+c(l,l)}{-2d}\right)e(-(l,k))\\ =&e\left(\frac{b(h+r,h+r)}{2d}\right)\sum_{l\in L^*/L} e((l,h-dk))\\ =&e_0 e\left(\frac{b(h+r,h+r)}{2d}\right)\delta_{h,dk}.\end{aligned}$$

On the other hand, the Poisson summation formula implies that $|\det(Q)|^{-1}v(L)^{-2}e_0 = 1$. Furthermore,

$$\begin{aligned}\sum_{r\in L/dL} e\left(\frac{b(h+r,h+r)}{2d}\right) &= \sum_{r\in L/dL} e\left(\frac{b(adh+r,adh+r)}{2d}\right)\\ &= e\left(\frac{ab(h,h)}{2}\right)\sum_{r\in L/dL} e\left(\frac{b(r,r)}{2d}\right).\end{aligned}$$

Thus, we have

$$c(h,k)_\sigma = \delta_{h,dk}\sqrt{\mathrm{i}}^{(p-q)\mathrm{sgn}(cd)} e\left(\frac{ab(h,h)}{2}\right)|d|^{-n/2}\sum_{r\in L/dL} e\left(\frac{b(r,r)}{2d}\right).$$

Now we can use the argument in the proof of Proposition 1.1 and Proposition 1.2 with a slight modification and get

$$|d|^{-n/2}\sum_{r\in L/dL} e\left(\frac{b(r,r)}{2d}\right) = \begin{cases}\varepsilon_d^{-n}(\mathrm{sgn}(c)\mathrm{i})^n\left(\frac{2c}{d}\right)^n\left(\frac{D}{-d}\right), & \text{if } d<0,\\ \varepsilon_d^{n}\left(\frac{-2c}{d}\right)^n\left(\frac{D}{d}\right), & \text{if } d>0,\end{cases}$$

which completes the proof. □

Let G be the connected component of the identity of the real orthogonal group of Q. Let Γ be the subgroup of G of all elements which leave the lattice L invariant and leave L^*/L point-wise fixed. Then, as a function on G, $\theta(g\cdot f, h)$ ($\forall\ f \in \phi(V)$, $g \in G$, $g\cdot f$ was defined as in equality (8.6), $h \in L^*/L$) is left Γ-invariant and slowly increasing on G/Γ (For the definitions of slowly increasing functions and rapidly decreasing functions on G/Γ, see R. Godement). Take a rapidly decreasing function Φ on G/Γ and put

$$\theta(f,\Phi;h) = \int_{G/\Gamma}\theta(g\cdot f,h)\Phi(g)\mathrm{d}g,$$

where $\mathrm{d}g$ is a Haar measure on G. Now assume that f satisfies (8.11) and set

$$\Theta(z, f, \Phi; h) = v^{-m/4}\theta(r(\sigma_z, Q)f, \Phi; h) \tag{8.14}$$

for $z = u + \mathrm{i}v \in \mathbb{H}$.

If no confusion is likely, we write

$$\Theta(z, h) = \Theta(z, f, \Phi; h).$$

Proposition 8.3 *Assume f satisfies* (8.11). *Then we have*

(1) *If* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *satisfies the condition* (8.13), *then*

$$\sqrt{\mathrm{i}}^{(p-q)\mathrm{sgn}(c)}\sqrt{J(\gamma, z)}^{-m}\Theta(\gamma(z), h) = \sum_{k\in L^*/L} c(h, k)_\gamma \Theta(z, k), \quad c \neq 0.$$

(2) *Assume that Φ satisfies the differential equation $L_Q\Phi = \lambda\Phi$ on G. Then*

$$\begin{aligned}&\left\{4v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) - 2\mathrm{i}mv\left(\frac{\partial}{\partial u} + \mathrm{i}\frac{\partial}{\partial v}\right)\right\}\Theta(z, h)\\ &= \left\{\lambda - m\left(\frac{m}{4} - 1\right) + n\left(\frac{n}{4} - 1\right)\right\}\Theta(z, h)\end{aligned} \tag{8.15}$$

for $z = u + \mathrm{i}v \in \mathbb{H}$.

Proof (1) It follows easily from (8.8) that

$$r(\gamma, Q)r(\sigma_z, Q) = r(\sigma_{\gamma(z)}, Q)r(k_\theta, Q),$$

where $\mathrm{e}^{-\mathrm{i}\theta} = J(\gamma, z)/|J(\gamma, z)|$ and $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Since f satisfies (8.11),

$$r(k_\theta, Q)f = \sqrt{\mathrm{i}}^{(p-q)\mathrm{sgn}(c)}\sqrt{J(\gamma, z)/|J(\gamma, z)|}^{-m} f$$

(see (8.9)). So, by Proposition 8.2, we have

$$\sqrt{\mathrm{i}}^{(p-q)\mathrm{sgn}(c)}\sqrt{J(\gamma, z)}^{-m}\Theta(\gamma(z), h) = \sum_{k\in L^*/L} c(h, k)_\gamma \Theta(z, k).$$

(2) By Lemma 8.3, we have

$$\begin{aligned}&\left\{4v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) - 2\mathrm{i}mv\left(\frac{\partial}{\partial u} + \mathrm{i}\frac{\partial}{\partial v}\right)\right\}\Theta(z, f, \Phi; h)\\ &= m\left(1 - \frac{m}{4}\right)\Theta(z, h) + \Theta(z, C_Q f, \Phi; h).\end{aligned}$$

By Lemma 8.2, Lemma 8.4 and integration by parts, we have (8.15). This completes the proof. ∎

Example 8.1 Let $n=1$, $Q=(2/N)$, $L=N\mathbb{Z}$ and $f(x)=\exp(-2\pi x^2/N)$. Then we have $p=1$, $q=0$, $L^*=\mathbb{Z}/2$, $r(k(\theta))f=(\cos\theta-\mathrm{i}\sin\theta)^{-1/2}f$ and $\theta(z,\ f,\ 0)=\theta(Nz)$, where $\theta(z,\ f,\ h)=v^{-1/4}\theta(r(\sigma_z,\ Q)f,\ h)$ and $\theta(z)$ is defined as in Chapter 1. From Proposition 8.3 we have for $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(4N)$ that

$$(\sqrt{\mathrm{i}})^{\mathrm{sgn}(c)}(cz+d)^{-1/2}\theta(N\sigma(z))=c(0,0)_\sigma\theta(Nz),$$
$$c(0,0)_\sigma=(\sqrt{\mathrm{i}})^{\mathrm{sgn}(c)}j(\sigma,z)(cz+d)^{-1/2}\left(\frac{N}{d}\right).$$

Of course these formulas are the same as the transformation formula for Theta-function in Chapter 1.

We note that $c(h,k)_\sigma$ in Proposition 8.2 does not depend on f. We can interpret the Weil representation by the so-called Fock representation. We define a map

$$I: L^2(\mathbb{R})\to H=L^2(\mathbb{C},\exp\{-\pi z\overline{z}\}\mathrm{d}z)$$

by the integral transformation

$$I(f)(z)=\int_{\mathbb{R}} k(x,z)f(x)\mathrm{d}x,$$

where $f\in L^2(\mathbb{R})$ and

$$k(x,z)=\exp\{-\pi mx^2\}e(x\sqrt{m}z)\exp\{\pi z^2/2\}.$$

Then I is bijective and maps the Hermite function $\exp(\pi mx^2)\dfrac{\mathrm{d}^s}{\mathrm{d}x^s}\Big|_{\sqrt{m}x}\exp(-2\pi x^2)$ in $L^2(\mathbb{R})$ to the polynomial z^s in H up to a constant multiple. Moreover, by a direct computation one can easily check that

$$I(r(k(\theta))f)=(\cos\theta-\mathrm{i}\sin\theta)^{-1/2}M(\mathrm{e}^{\mathrm{i}\theta})I(f),$$

where $f\in L^2(\mathbb{R})$, $Q=(m)$ and $M(\mathrm{e}^{\mathrm{i}\theta})$ is the map such that $M(\mathrm{e}^{\mathrm{i}\theta})g(z)=g(\mathrm{e}^{\mathrm{i}\theta}z)$ for $g(z)\in H$. In this way we can find a function $f_{1,s}\in L^2(\mathbb{R})$ satisfying

$$r(k(\theta))f_{1,s}=(\cos\theta-\mathrm{i}\sin\theta)^{-(2s+1)/2}f_{1,s}$$

for a positive integer s. Namely,

$$f_{1,s}(x)=H_s(2\sqrt{\pi m}x^2),$$

where

$$H_s(x)=(-1)^s\exp\{x^2/2\}\frac{\mathrm{d}^s}{\mathrm{d}x^s}\exp\{-x^2/2\}$$

is a Hermite polynomial.

Put again $m = 2/N$ and let L be as above. Then

$$\theta(z, f_{1,s}, 0) = \theta_{1,s}(z) = v^{-1/2} \sum_{x=-\infty}^{\infty} H(2\sqrt{2N\pi v}x) \exp\{2\pi i N z x^2\}$$

satisfies

$$\theta_{1,s}(\sigma(z)) = \left(\frac{N}{d}\right) j(\sigma, z)(cz + d)^s \theta_{1,s}(z)$$

according to the independence of $c(h, k)_\sigma$ to f. In the same way we can prove

$$\theta_{1,s}(-1/4Nz) = (2N)^{s/2}(\sqrt{-2iz})^{2s+1}\theta_s(z),$$

where

$$\theta_s(z) = (2v)^{-s/2} \sum_{x=-\infty}^{\infty} \exp\{2\pi i x^2 z\} H_s(2\sqrt{2\pi v}x).$$

□

Example 8.2 Now we consider the case $n = 2$, $Q = \begin{pmatrix} 0 & -4/N \\ -4/N & 0 \end{pmatrix}$, i.e.,

$$(x, y) = -\frac{4}{N}(x_1 y_2 + x_2 y_1)$$

and

$$L = (4N\mathbb{Z}) \oplus (N\mathbb{Z}/4).$$

Then $p = q = 1, r = r_0, L^* = (\mathbb{Z}) \oplus (\mathbb{Z}/16)$ and $4NL^* = L$ satisfies the assumption of Proposition 8.2. Put $L' = \mathbb{Z} \oplus (N\mathbb{Z}/4)$, $h \in L'$. Then for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$, $c(h, k)_\sigma = \delta_{k,ah}$ and $\theta(r_0(\sigma)f, h) = \theta(f, ah)$ are valid. If $f \in \phi(\mathbb{R}^2)$ satisfies $r(k(\theta))f = e^{is\theta} f$, and if we define $\theta_{2,s}(z, f)$ by

$$\theta_{2,s}(z, f) = \sum_{h \in L'/L^*} \overline{\chi_1}(h)\theta(z, f, h),$$

where $\chi_1 = \chi \left(\frac{-1}{*}\right)^\lambda$ with λ a positive integer and χ a character modulo $4N$. Then we have

$$\theta_{2,s}(\sigma(z), f) = \overline{\chi_1}(d)(cz + d)^s \theta_{2,s}(z, f).$$

We explain how to find f with this property. Put $Q = \begin{pmatrix} 0 & -2m \\ -2m & 0 \end{pmatrix}$, $m > 0$. We define a partial Fourier transformation F by

$$F(f)(x_1, x_2) = \sqrt{2m} \int_{-\infty}^{\infty} f(x_1, t) \exp\{4\pi i m t x_2\} dt,$$

$$F^{-1}(f)(x_1, x_2) = \sqrt{2m} \int_{-\infty}^{\infty} f(x_1, t) \exp\{-4\pi i m t x_2\} dt.$$

One can easily check that

$$r(\sigma)f = FR(\sigma)F^{-1}(f),$$

where

$$(R(\sigma)f)(x) = f((x_1, x_2)\sigma).$$

And so r is a representation of $SL_2(\mathbb{R})$ although Weil representation is not always a multiplicative representation. Put

$$f'(x_1, x_2) = (x_1 + \mathrm{i}x_2)^s \exp(-2m\pi(x_1^2 + x_2^2));$$
$$f_{2,s}(x) = F(f')(x) = \sqrt{2}(\sqrt{4\pi m})^{-s-1}H(\sqrt{4\pi m}(x_1 - x_2)) \exp(-2m\pi(x_1^2 + x_2^2)).$$

Then

$$R(k(\theta))f' = \mathrm{e}^{2\mathrm{i}s\theta}f',$$

and $f_{2,s}$ has the required property. Generally, the Weil representation commutes with the action of the orthogonal group of Q on $L^2(\mathbb{R}^n)$. In the present case, the elements of that group are diagonal matrices in $SL_2(\mathbb{R})$. Put $f_\eta(x_1, x_2) = f_{2,s}(\eta^{-1}x_1, \eta x_2)$, and $m = 2/N$. Put $\theta_{2,s}(z, \eta) = \theta_{2,s}(z, f_\eta)$. Then

$$\begin{aligned}\theta_{2,s}(z, \eta) =& v^{(1-s)/2} \sum_{x_1, x_2 \in \mathbb{Z}} \overline{\chi}_1(x_1) \exp\left\{ -2\pi \mathrm{i}ux_1x_2 - \frac{Nv}{4}\pi x_2^2\eta^2 - \frac{4v}{N}\pi x_1^2\eta^{-2} \right\} \\ &\times H_s\left(2\sqrt{\frac{2}{N}\pi v}\left(x_1\eta^{-1} - \frac{Nx_2}{4}\eta\right)\right).\end{aligned}$$

Observing that $f_{2,s} = F(f')$ and using the Poisson summation formula, we get a different expression for $\theta_{2,s}$:

$$\begin{aligned}\theta_{2,s}(z, \eta) =& \left(\sqrt{\frac{8\pi}{N}}\right)^{s+1} (\sqrt{2\pi})^{-1}\mathrm{i}^s\eta^{-s-1}v^{-s} \\ &\times \sum_{x_1, x_2 \in \mathbb{Z}} \overline{\chi_1}(x_1)(x_1\overline{z} + x_2)^s \exp\left\{ -\frac{4\pi}{N\eta^2 v}|x_1 z + x_2|^2 \right\}.\end{aligned}$$

□

Example 8.3 We denote by $r^{(i)}$ the Weil representation in the vector space V_i, $i = 1, 2, 3$, and by L_i, L_i^*, $r_0^{(i)}$, $h_i \in L_i^*$ and $c_i(h_i, k_i)_\sigma$ corresponding lattices, etc. If V_3 is the orthogonal sum of V_1 and V_2, then $r_0^{(3)} = r_0^{(1)} \otimes r_0^{(2)}, r^{(3)} = r^{(1)} \otimes r^{(2)}$, and $c_3(h_3, k_3)_\sigma = c_1(h_1, k_1)_\sigma c_2(h_2, k_2)_\sigma$ is obvious for $h_3 = (h_1, h_2), k_3 = (k_1, k_2)$. If $n = 3, Q = \frac{2}{N}\begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}$ and $L = 4N\mathbb{Z} \oplus N\mathbb{Z} \oplus (N\mathbb{Z}/4)$, then according to the preceding two examples, we have

$$c(h, k)_\sigma = \delta_{k,ah}(\sqrt{\mathrm{i}})^{\mathrm{sgn}(c)}j(\sigma, z)(cz + d)^{-1/2}\left(\frac{N}{d}\right)$$

for $f \in L^2(\mathbb{R}^3)$, $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$ and $h, k \in L'/L$ with $L' = \mathbb{Z} \oplus N\mathbb{Z} \oplus (N\mathbb{Z}/4)$. Therefore, if $r(k(\theta))(f) = (\cos\theta - \mathrm{i}\sin\theta)^{-\kappa/2} f$ is satisfied, then by Proposition 8.3 we have

$$\theta_\kappa(\sigma(z), f) = \overline{\chi}_1(d)\left(\frac{N}{d}\right) j(\sigma, z)(cz+d)^\lambda \theta_\kappa(z, f)$$

where $\kappa = 2\lambda + 1$, for $h = (h_1, h_2, h_3)$ we define, $\overline{\chi}_1(h) = \overline{\chi}_1(h_1)$ and

$$\theta_\kappa(z, f) = \sum_{h \in L'/L} \overline{\chi}_1(h)\theta(z, f, h).$$

One can take here $f_{1,s}(x_2) f_{2,\lambda-s}(x_1, x_3)$ ($s = 1, 2, \cdots, \lambda$), or their linear combinations for such $f(x)$. In view of

$$(x - \mathrm{i}y)^\lambda = \sum_{s=0}^{\lambda} \binom{\lambda}{s} H_{\lambda-s}(x) H_s(y)(-\mathrm{i})^s,$$

$f_3(x) = (x_1 - \mathrm{i}x_2 - x_3)^\lambda \exp\{-m\pi(2x_1^2 + x_2^2 + 2x_3^2)\}$ is available, too. On the other hand, the action of $SL_2(\mathbb{R})$ on $\mathbb{R}^3$ is defined as follows: $g \in SL_2(\mathbb{R})$ operates on $\mathbb{R}^3$ through the symmetric tensor representation, i.e., for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $gx = (x_1', x_2', x_3')$ is determined by

$$g \begin{pmatrix} x_1 & x_2/2 \\ x_2/2 & x_3 \end{pmatrix} g^{\mathrm{T}} = \begin{pmatrix} x_1' & x_2'/2 \\ x_2'/2 & x_3' \end{pmatrix},$$

and gives an isomorphism of $SL_2(\mathbb{R})$ with the orthogonal group of Q.

Let N be a positive integer, χ a character modulo $4N$ and $\chi_1 = \chi\left(\frac{-1}{*}\right)^\lambda$ with a positive integer λ. Define a function on $\mathbb{R}^3$ by

$$f(x) = (x_1 - \mathrm{i}x_2 - x_3)^\lambda \exp\{(-2\pi/N)(2x_1^2 + x_2^2 + 2x_3^2)\}.$$

For $\kappa = 2\lambda + 1$, $z = u + \mathrm{i}v \in \mathbb{H}$ and for the lattice $L' = \mathbb{Z} \oplus N\mathbb{Z} \oplus (N\mathbb{Z}/4) \in \mathbb{Q}^3$, we define a theta series $\theta(z, g)$ by

$$\theta(z, g) = \sum_{x \in L'} \overline{\chi_1}(x_1) v^{(3-\kappa)/4} (\exp\{2\pi\mathrm{i}(u/N)(x_2^2 - 4x_1x_3)\}) f(\sqrt{v} g^{-1} x),$$

where $\sqrt{v} \in \mathbb{R}$ is viewed as a scalar of the vector space $\mathbb{R}^3$, and $g \in SL_2(\mathbb{R})$ operates on $\mathbb{R}^3$ as above.

Let $gf \in L^2(\mathbb{R}^3)$ be defined by $(gf)(x) = f(g^{-1}x)$ and take $m = 2/N$ in $f_3(x)$. Then it is clear that $\theta(z, g) = \theta(z, gf_3)$. The action of $r_0(k(\theta))$ commutes with that of g in $L^2(\mathbb{R}^3)$, gf_3 has the same property as f_3, and the required transformation formula of $\theta(z, g)$ is

$$\theta(\sigma(z), g) = \overline{\chi}(d)\left(\frac{N}{d}\right) j(\sigma, z)^\kappa \theta(z, g).$$

We note that f_3 has the property $f_3(k(\alpha)x) = \mathrm{e}^{2\lambda \mathrm{i}\alpha} f_3(x)$, and so $\theta(z, gk(\alpha)) = \mathrm{e}^{-2\mathrm{i}\lambda\alpha}\theta(z, g)$. □

8.2 Shimura Lifting for Cusp Forms

Let $G(z) = \sum_{n=1}^{\infty} a(n)e(nz)$ be an element of $S(4N, k+1/2, \chi)$, t a square-free positive integer, put $\chi_t = \chi\left(\frac{-1}{*}\right)\left(\frac{t}{*}\right)$ and $\varPhi_t(w) = \sum_{n=1}^{\infty} A_t(n)e(nw)$ with $A_t(n)$ defined by the following equality

$$\sum_{n=1}^{\infty} A_t(n)e(nw) = \left(\sum_{m=1}^{\infty} \chi_t(m)m^{\lambda-1-s}\right)\left(\sum_{m=1}^{\infty} a(tm^2)m^{-s}\right).$$

Then $\varPhi_t(w)$ is called the Shimura $t-$lifting of $G(z)$. The main theorem of G. Shimura, 1973 asserted that $\varPhi_t$ belongs to $G(N_t,\ k-1,\ \chi^2)$, and in fact $\varPhi_t \in S(N_t,\ k-1,\ \chi^2)$ for $k \geqslant 5$ with a certain positive integer N_t. He proved this result through Weil theorem. He also conjectured the level N_t can be taken as $2N$, and for $k = 1$, $\varPhi(w)$ is a cusp form if and only if $G(z)$ is orthogonal to some theta series with respect to the Petersson inner product.

In this section we shall study these problems and prove these results. Our presentation is due to T. Shintani, S. Niwa, 1975, H. Kojima, 1980 and J. Sturm, 1982.

From now on, we always think of $\theta(z, g) = \theta(z, gf_3)$ as the function defined in Section 8.1. Now let $F(z)$ be in $S\left(4N,\ k/2,\ \overline{\chi}\left(\frac{N}{*}\right)\right)$ with $k = 2\lambda+1$ an odd positive integer. Since $F(z)$ is rapidly decreasing at each cusp of $\Gamma_0(4N)$, while $\theta(z, g)$ is at most slowly increasing there, so the following integral, which is the Petersson inner product of $F(z)$ and $\theta(z, g)$, is well-defined:

$$F(g) = \int_{D_0(4N)} F(z)\overline{\theta}(z, g)v^{k/2}\frac{\mathrm{d}u\mathrm{d}v}{v^2},$$

where $D_0(4N)$ is the fundamental domain of $\varGamma_0(4N)$. We have the following

Lemma 8.5 *The function $F(g)$ has the following properties:*

(1) *$F(g) \in C^{\infty}(SL_2(\mathbb{R}))$ is an eigenfunction of the* Casimir *operator D_g,* i.e., *$D_gF = \lambda(\lambda-1)F$, where*

$$D_g = \frac{1}{4}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^2 + 2\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + 2\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right);$$

(2) $F\left(g\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right) = \exp\{2\lambda\theta\mathrm{i}\}F(g);$

(3) $F(\gamma g) = \chi^2(d)F(g)$ *for every* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \Gamma_0(2N) \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$.

Proof The first conclusion is a direct consequence of the Proposition 8.3. In fact, by the proposition, we have

$$D_g\theta(z,g) = \left[4v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) - 2\mathrm{i}kv\left(\frac{\partial}{\partial u} + \mathrm{i}\frac{\partial}{\partial v}\right) + k\left(\frac{k}{4} - 1\right) + \frac{3}{4}\right],$$

where D_g is the Casimir operator on $SL_2(\mathbb{R})$. By Green's formula we have

$$D_g \int\limits_{D_0(4N)} F(z)\overline{\theta}(z,g)v^{k/2}\frac{\mathrm{d}u\mathrm{d}v}{v^2} = \lambda(\lambda-1)\int\limits_{D_0(4N)} F(z)\overline{\theta}(z,g)v^{k/2}\frac{\mathrm{d}u\mathrm{d}v}{v^2},$$

which is just (1).

Since $\theta(z, gk(\alpha)) = \mathrm{e}^{-2\mathrm{i}\lambda\alpha}\theta(z,g)$, so

$$\begin{aligned} F(gk(\alpha)) &= \int\limits_{D_0(4N)} F(z)\overline{\theta}(z,gk(\alpha))v^{k/2}\frac{\mathrm{d}u\mathrm{d}v}{v^2} \\ &= \int\limits_{D_0(4N)} F(z)\overline{\mathrm{e}^{-2i\lambda\alpha}\theta}(z,g)v^{k/2}\frac{\mathrm{d}u\mathrm{d}v}{v^2} \\ &= \exp\{2\mathrm{i}\lambda\alpha\}F(g), \end{aligned}$$

which is (2).

Now we prove that $\theta(z,\gamma g) = \chi^2(d)\theta(z,g)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \Gamma_0(2N)$ $\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$ from which (3) is deduced. Recalling the definition of $\theta(z,g)$:

$$\theta(z,g) = \sum_{x\in L'} \overline{\chi_1}(x_1)v^{(3-\kappa)/4}(\exp\{2\pi\mathrm{i}(u/N)(x_2^2 - 4x_1x_3)\})f(\sqrt{v}g^{-1}x),$$

where $L' = \mathbb{Z}\oplus N\mathbb{Z}\oplus(N\mathbb{Z}/4)$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \Gamma_0(2N) \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$, it is easy to verify that $a, d \in \mathbb{Z}, c \in N\mathbb{Z}/2, b \in 4\mathbb{Z}$. By the definition of the symmetric tensor representation, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\gamma x = (x_1', x_2', x_3')$ is determined by

$$\gamma\begin{pmatrix} x_1 & x_2/2 \\ x_2/2 & x_3 \end{pmatrix}\gamma^{\mathrm{T}} = \begin{pmatrix} x_1' & x_2'/2 \\ x_2'/2 & x_3' \end{pmatrix}.$$

That is,

$$\begin{aligned} x_1' &= a^2x_1 + abx_2 + b^2x_3, \\ x_2' &= 2cax_1 + (ad+bc)x_2 + 2bdx_3, \\ x_3' &= c^2x_1 + cdx_2 + d^2x_3. \end{aligned}$$

It is clear that both lattices $L = 4N\mathbb{Z}\oplus N\mathbb{Z}\oplus(N\mathbb{Z}/4)$ and L' are stable by γ and $x_1' \equiv a^2x_1 \pmod{4N}$ for $x = (x_1,\ x_2,\ x_3) \in L'$ which imply that $\theta(z,\ \gamma g) = (\overline{\chi}(a))^2\theta(z,\ g) = \chi^2(d)\theta(z,\ g)$ since $\overline{\chi}^2(a) = \chi^2(d)$. This completes the proof. □

We define two functions $\Psi(w)$ and $\Phi(w)$ $(w = \xi + \mathrm{i}\eta \in \mathbb{H})$ by

$$\Psi(w) = F\left(\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} \eta^{1/2} & \xi\eta^{-1/2} \\ 0 & \eta^{-1/2} \end{pmatrix}\right)(4\eta)^{-\lambda}$$

and

$$\Phi(w) = \Psi\left(-\frac{1}{2Nw}\right)(2N)^{\lambda}(-2Nw)^{-2\lambda}.$$

Let W be the isomorphism of $S\left(4N,\, k/2,\, \overline{\chi}\left(\frac{N}{*}\right)\right)$ onto $S(4N,\, k/2,\, \chi)$ defined by

$$G(z) = (F|[W(4N)])(z) = F(-1/4Nz)(4N)^{-k/4}(-\mathrm{i}z)^{-k/2}$$

for all $F(z) \in S\left(4N,\, k/2,\, \overline{\chi}\left(\frac{N}{*}\right)\right)$. Then $G(z)$ has the Fourier expansion

$$G(z) = \sum_{n=1}^{\infty} a(n)e(nz)$$

at ∞. Define a sequence $\{A(n)\}_{n=1}^{\infty}$ by the following relation

$$\sum_{n=1}^{\infty} A(n)n^{-s} = L(s-\lambda+1, \chi_1)\sum_{n=1}^{\infty} a(n^2)n^{-s},$$

where $\chi_1 = \chi\left(\frac{-1}{*}\right)^{\lambda}$. Then we define the Shimura lifting I_k $(k \geqslant 3)$ by

$$I_k(G(z)) = \sum_{n=1}^{\infty} A(n)e(nz) \quad \text{for } G(z) \in S(4N, k/2, \chi).$$

Now we can present the main result of this chapter as follows.

Theorem 8.1 *If $k \geqslant 3$, then $\Phi(w)$ belongs to $G(2N,\, k-1,\, \chi^2)$ and $\Phi(w) = cI_k(G(z))$ with*

$$c = \mathrm{i}^{k-1}N^{k/4}2^{(-9k+15)/4}\mathrm{Re}((2-\mathrm{i})^{(k-1)/2}).$$

Moreover, if $k \geqslant 5$, then $\Phi(w)$ belongs to $S(2N,\, k-1,\, \chi^2)$.

Proof By Lemma 8.5, we have

$$\theta(z, \gamma' g) = \chi^2(d')\theta(z, g)$$

for any

$$\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}\Gamma_0(2N)\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}.$$

And consequently, by the definition of $\Psi(w)$ we have

$$\Psi(\gamma(w)) = \overline{\chi}^2(d)(cw+d)^{2\lambda}\,\Psi(w)$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2N)$. This implies that

$$\Phi(\gamma(w)) = \chi^2(d)(cz+d)^{k-1}\,\Phi(w)$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2N)$. Therefore, if $\Phi(w)$ is holomorphic on $\mathbb{H}$, then we can conclude that $\Phi(w)$ is an integral modular form of weight $2\lambda = k-1$ for the congruence subgroup $\Gamma_0(2N)$. Now we prove that $\Phi(w)$ is holomorphic on $\mathbb{H}$. For the simplicity we assume $k=3$ though the method is applicable in all cases. By virtue of Lemma 8.5 and the invariance of the Casimir operator D_g, we have

$$\left(\eta^2\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right) - 2\mathrm{i}\eta\left(\frac{\partial}{\partial\xi}+\mathrm{i}\frac{\partial}{\partial\eta}\right)\right)\Phi(w) = 0.$$

Now $\Phi(w)$ has the Fourier expansion

$$\Phi(w) = \sum_{m=-\infty}^{\infty} a_m(\eta)\exp\{2\pi\mathrm{i}m\xi\}$$

at ∞. So $a_m(\eta)$ is a solution of the differential equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}+\frac{2}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}+(-4\pi^2m^2+4\pi m/\eta)\right)a_m(\eta) = 0.$$

Therefore we get

$$a_m(\eta) = \begin{cases} b_m\exp\{-2\pi m\eta\}+c_m u_m(\eta), & \text{if } m\neq 0,\\ b_0+c_0\eta^{-1}, & \text{if } m=0. \end{cases}$$

where

$$u_m(\eta) = \begin{cases} \exp\{-2\pi m\eta\}\displaystyle\int_1^{\eta}\eta^{-2}\exp\{4\pi m\eta\}\mathrm{d}\eta, & \text{if } m>0,\\ \exp\{-2\pi m\eta\}\displaystyle\int_{\eta}^{\infty}\eta^{-2}\exp\{4\pi m\eta\}\mathrm{d}\eta, & \text{if } m<0. \end{cases}$$

By integration by parts, we have the following asymptotic behavior of $u_m(\eta)$:

$$|u_m(\eta)| \geqslant (4\pi m-\pi)^{-1}\exp\{-2\pi m\eta\}|\exp\{(4\pi m-\pi)\eta\}-\exp\{4\pi m-\pi\}| \quad (8.16)$$

for $m>0$, and

$$u_m(\eta) = -\frac{\exp\{2\pi m\eta\}}{4\pi m\eta^2}+\alpha_m(\eta) \quad (8.17)$$

for $m < 0$, where

$$|\alpha_m(\eta)| \leqslant \exp\{2\pi m\eta\}(1/8\pi^2|m^2|\eta^3 + 15/32\pi^3|m^3|\eta^4).$$

Moreover we have

$$\eta\Phi(w) = O(\eta + \eta^{-1}) \quad \text{for } \eta \to 0 \text{ and } \eta \to \infty \;, \tag{8.18}$$

uniformly in ξ, which will be proved later. Since

$$\int_0^1 \eta^2|\Phi(w)|^2\mathrm{d}\xi = \sum_{m=-\infty}^{\infty} |a_m(\eta)|^2\eta^2,$$

we get from (8.18)

$$|a_m(\eta)| \leqslant M((\eta + \eta^{-1})\eta^{-1}), \tag{8.19}$$

where M is independent of m and η. Hence by (8.16) and (8.17), we have $c_m = 0$ for all $m > 0$ and $b_m = 0$ for all $m < 0$. Hence we see

$$\begin{aligned}\Phi(w) = &\sum_{m=1}^{\infty} b_m \exp\{-2\pi m\eta\} \exp\{2\pi i m\xi\} \\ &+ \sum_{m=1}^{\infty} c_{-m}u_{-m}(\eta) \exp\{-2\pi \mathrm{i} m\xi\} + a_0(\eta).\end{aligned} \tag{8.20}$$

By (8.19) we have $|a_m(1/|m|)| \leqslant M(1 + m^2)$. Hence we get $b_m = O(m^\nu)(m \to \infty)$ and $c_{-m} = O(m^\nu)(m \to \infty)$ for some $\nu > 0$. We shall prove that $\Phi(\mathrm{i}\eta)$ has the following asymptotic behavior later:

$$\Phi(\mathrm{i}\eta) = \begin{cases} O(\eta^{-\mu}), & \eta \to +\infty \quad \text{for all } \mu > 0, \\ O(\eta^{\mu}), & \eta \to 0 \quad\quad\; \text{for all } \mu > 0. \end{cases} \tag{8.21}$$

In particular, we see that $a_0(\eta) = 0$. Hence we have

$$\begin{aligned}\Phi(w) = &\sum_{m=1}^{\infty} b_m \exp\{-2\pi m\eta\} \exp\{2\pi \mathrm{i} m\xi\} \\ &+ \sum_{m=1}^{\infty} c_{-m}u_{-m}(\eta) \exp\{-2\pi \mathrm{i} m\xi\}.\end{aligned} \tag{8.22}$$

By virtue of (8.21), $\Phi(\mathrm{i}\eta)\eta^{l-1}$ belongs to $L_1(\mathbb{R}^+)$ for a sufficiently large $l > 0$. Let $\Omega(s)$ be the Mellin transformation of $\Phi(\mathrm{i}\eta)$, i.e.,

$$\Omega(s) = \int_0^\infty \Phi(\mathrm{i}\eta)\eta^{s-1}\mathrm{d}\eta.$$

Here we note that $\Phi(\mathrm{i}\eta)$ is a function with bounded variation on all compact subsets of $\mathbb{R}^+$ and $\Phi(\mathrm{i}\eta) = \frac{1}{2}(\Phi(\mathrm{i}(\eta+0)) + \Phi(\mathrm{i}(\eta-0)))$ for all $\eta > 0$. Hence the Mellin inversion formula gives

$$\Phi(\mathrm{i}\eta) = \frac{1}{2\pi\mathrm{i}}\int_{l-\mathrm{i}\infty}^{l+\mathrm{i}\infty} \Omega(s)\eta^{-s}\mathrm{d}s. \tag{8.23}$$

On the other hand, we shall compute that

$$\Omega(s) = c(2\pi)^{-s}\Gamma(s)L(s,\chi_1)\sum_{n=1}^{\infty} a(n^2)n^{-s} = (2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty} a'_n n^{-s},$$

where $G(z) = \sum_{n=1}^{\infty} a(n)e(nz)$. Consequently, we get

$$\Phi(\mathrm{i}\eta) = \sum_{n=1}^{\infty} a'_n \exp\{-2\pi n\eta\}. \tag{8.24}$$

Therefore, by (8.20), to prove that $\Phi(w)$ is holomorphic it is sufficient to show that $c_{-m} = 0$ for all $m \geqslant 1$. We assume that $c_{-m_0} \neq 0$ and $c_{-m} = 0$ for all $m < m_0$. Then by (8.20) and (8.24) we have

$$\begin{aligned}&\sum_{m>m_0} c_{-m}u_{-m}(\eta)/H_{m_0}(\eta) + c_{-m_0}u_{-m_0}(\eta)/H_{m_0}(\eta)\\ &=\sum_{n=1}^{\infty}(a'_n - b_n)\exp\{-2\pi n\eta\}/H_{m_0}(\eta),\end{aligned} \tag{8.25}$$

where $H_{m_0}(\eta) = \exp\{-2\pi m_0\eta\}/4\pi m_0\eta^2$.

We note that the series on both sides of (8.25) are uniformly convergent on $[1,\infty)$. Set $t = \exp\{-2\pi\eta\}$ for $\eta > 0$. The right hand side of (8.25) is equal to

$$\frac{m_0}{\pi}(\log t)^2\sum_{n=1}^{\infty}(a'_n - b_n)t^{n-m_0}.$$

By virtue of (8.17), we see that the left hand side of (8.25) converges to c_{-m_0} as $\eta \to +\infty$. Hence we get

$$\lim_{t\to 0, t>0}\left\{\frac{m_0}{\pi}(\log t)^2\sum_{n=1}^{\infty}(a'_n - b_n)t^{n-m_0}\right\} = c_{-m_0} \neq 0,$$

which is a contradiction and we proved that $\Phi(w)$ is holomorphic.

There still remains the investigation of the asymptotic behavior of $\Phi(\mathrm{i}\eta)$ as $\eta \to 0$ and ∞, and the computation of the Mellin transformation of $\Phi(\mathrm{i}\eta)$.

We first compute the Mellin transformation of $\Phi(\mathrm{i}\eta)$ for any $k \geqslant 3$. By the definition of Mellin transformation we have

$$\begin{aligned}\Omega(s) &= \int_0^\infty \Phi(\mathrm{i}\eta)\eta^{s-1}\mathrm{d}\eta = (-1)^\lambda(2N)^{\lambda-s}\int_0^\infty \Psi(\mathrm{i}\eta^{-1})\eta^{s-k}\mathrm{d}\eta \\ &= (-1)^\lambda(2N)^{\lambda-s}4^{-\lambda}\int_0^\infty \eta^{s-\lambda}\int_{D_0(4N)} v^{k/2}\overline{\theta}(z,\sigma_{4\mathrm{i}\eta-1})F(z)\mathrm{d}z\frac{\mathrm{d}\eta}{\eta},\end{aligned}$$

where $\mathrm{d}z = \dfrac{\mathrm{d}u\mathrm{d}v}{v^2}$ and $\sigma_w = \begin{pmatrix}\eta^{1/2} & \xi\eta^{-1/2}\\ 0 & \eta^{-1/2}\end{pmatrix}$ for $w = \xi + \mathrm{i}\eta \in \mathbb{H}$.

From the definition of $\theta(z,g)$ and the relation

$$(x-\mathrm{i}y)^\lambda = \sum_{\varepsilon=1}^{\lambda}\binom{\lambda}{\varepsilon}H_{\lambda-\varepsilon}(x)H_\varepsilon(y)(-\mathrm{i})^\varepsilon,$$

we have a simple expression

$$\theta(z,\sigma_{\mathrm{i}\eta}) = \left(2\sqrt{\frac{2\pi}{N}}\right)^{-\lambda}\sum_{\varepsilon=0}^{\lambda}\binom{\lambda}{\varepsilon}(-\mathrm{i})^\varepsilon\theta_{2,\lambda-\varepsilon}(z,\eta)\theta_{1,\varepsilon}(z),$$

where $\theta_{2,\lambda-\varepsilon}$, $\theta_{1,\varepsilon}$ are defined as in Example 8.1 and Example 8.2. Therefore by changing the order of integration whose justification can be deduced from the asymptotic behaviors (8.21) of $\Phi(\mathrm{i}\eta)$, we get

$$\Omega(s) = c_1(s)\sum_{\varepsilon=0}^{\lambda}\binom{\lambda}{\varepsilon}\mathrm{i}^\varepsilon\int_{D_0(4N)} v^{k/2}F(z)\overline{\theta}_{1,\varepsilon}(z)\left[\int_0^\infty \overline{\theta}_{2,\lambda-\varepsilon}(z,\eta^{-1})\eta^{s-\lambda}\frac{\mathrm{d}\eta}{\eta}\right]\mathrm{d}z$$

with $c_1(s) = (-1)^\lambda(2N)^{\lambda-s}4^{s-2\lambda}(2\sqrt{2\pi/N})^{-\lambda}$. Note that we can exchange the order of the summation and the integration as above. In terms of the different expressions of $\theta_{2,\varepsilon}$ given in Example 8.2, the integral in the bracket becomes an Eisenstein series

$$\begin{aligned}&\left(\sqrt{\frac{8\pi}{N}}\right)^{\lambda-\varepsilon+1}\sqrt{2\pi^{-1}}(-\mathrm{i})^{\lambda-\varepsilon}v^{-\lambda+\varepsilon}\left(\frac{1}{2}\right)\left(\frac{N}{4\pi}\right)^{(s-\varepsilon-1)/2}\\ &\times\Gamma\left(\frac{s-\varepsilon+1}{2}\right)\sum_{(x_1,x_2)\in\mathbb{Z}\times\mathbb{Z}}\chi_1(x_1)(x_1z+x_2)^{\lambda-\varepsilon}|x_1z+x_2|^{-s+\varepsilon-1}.\end{aligned}$$

Changing the variable z to $-1/4Nz$ and using $G(z) = F(-1/4Nz)(4N)^{-k/4}(-\mathrm{i}z)^{-k/2}$ and

$$\theta_{1,\varepsilon}(-1/4Nz) = (2N)^{\varepsilon/2}(\sqrt{-2\mathrm{i}z})^{2\varepsilon+1}\theta_\varepsilon(z),$$

we get

$$\Omega(s) = c_2(s)\sum_{\varepsilon=0}^{\lambda}\binom{\lambda}{\varepsilon}(\sqrt{2\pi})^{\lambda-\varepsilon+1}\mathrm{i}^{\varepsilon-\lambda}J_\varepsilon(s),$$

where $c_2(s)$ is like $c_1(s)$ above and $J_\varepsilon(s)$ is given by

$$J_\varepsilon(s) = \int_{D_0(4N)} G(z)\overline{\theta}_\varepsilon(z) v^{(s+\varepsilon+2)/2} \pi^{-(s-\varepsilon+1)/2} \Gamma\left(\frac{s-\varepsilon+1}{2}\right)$$
$$\times \sum_{x_1,x_2\in\mathbb{Z}} \chi_1(x_1)(4Nx_2z+x_1)^{\lambda-\varepsilon}|4Nx_2z+x_1|^{-s+\varepsilon-1}\mathrm{d}z$$
$$= \pi^{-(s-\varepsilon+1)/2}\Gamma\left(\frac{s-\varepsilon+1}{2}\right)L(s-\lambda+1,\chi_1)\int_0^\infty\int_0^1 G(z)\overline{\theta}_\varepsilon(z)v^{(s+\varepsilon+2)/2}\mathrm{d}z.$$

We note that $\theta_\varepsilon(z) = 0$ if ε is odd. The convolution appearing in $J_\varepsilon(s)$ is easily computed by Fourier expansion $\theta_\varepsilon(z) = \sum_{k=-\infty}^{\infty}(2v)^{-\varepsilon/2}H_\varepsilon(2\sqrt{2\pi v}k)\exp\{2\pi k^2 z\}$ and by partial integration, that is,

$$\int_0^\infty\int_0^1 G(z)\overline{\theta}_\varepsilon(z)\mathrm{d}uv^{(s+\varepsilon)/2}\frac{\mathrm{d}v}{v} = 2^{1-\varepsilon}(4\pi)^{-s/2}(s-1)(s-2)\cdots(s-\varepsilon)\Gamma\left(\frac{s-\varepsilon}{2}\right)D(s),$$

where $D(s) = \sum_{k=1}^{\infty} a(k^2)k^{-s}$ with $G(z) = \sum_{k=1}^{\infty} a(k)e(kz)$. Therefore we get

$$J_\varepsilon(s) = 2^{2-2s}\pi^{-s+\varepsilon/2}\Gamma(s)L(s-\lambda+1,\chi_1)D(s).$$

Hence we have

$$\Omega(s) = c(2\pi)^{-s}\Gamma(s)L(s-\lambda+1,\chi_1)D(s).$$

By the definition of the Shimura lifting I_k and the computation of the Mellin transformation of $\Phi(\mathrm{i}\eta)$, we see that $\Phi(w) = cI_k(G(z))$. For $k \geqslant 5$, the function $\Phi(w)$ belongs to $S(2N,\, k-1,\, \chi^2)$ by virtue of the magnitude of the growth of $A_1(n)$.

In order to complete the proof of the theorem we only need to give the proofs for (8.18) and (8.21). Now we first prove (8.18). It is easy to see that we only need to show it for $\Psi(w)$ by the relation between $\Phi(w)$ and $\Psi(w)$. In fact, we shall prove a more general result for any $k \geqslant 3$:

$$\eta^\lambda\,\Psi(w) = O(\eta+\eta^{-1}).$$

Recalling the definition of $\theta(z,g)$:

$$\theta(z,g) = \sum_{x\in L'}\overline{\chi_1}(x_1)v^{(3-k)/4}\exp\{2\pi\mathrm{i}(u/N)(x_2^2-4x_1x_3)\}f(\sqrt{v}g^{-1}x),$$

we get

$$|\theta(z,\sigma_{4w})| \leqslant v^{(3-k)/4}\sum_{x\in L'}|f(\sqrt{v}\sigma_{4w}^{-1}x)|.$$

Put $M = \mathbb{Z}/4\oplus\mathbb{Z}/4\oplus\mathbb{Z}/4$, then

$$\sum_{x\in L'}|f(\sqrt{v}\sigma_{4w}^{-1}x)| \leqslant \sum_{x\in M}|f(\sqrt{v}\sigma_{w}^{-1}x)| = \sum_{x\in M}|f(\sqrt{v}\sigma_{\gamma(w)}^{-1}x)| \quad \text{for } \gamma\in SL_2(\mathbb{Z}).$$

If $\eta > c_1 > 0$ and $|\xi| < c_2$, then there exist $0 < h_j(x) \in \phi(\mathbb{R})$, $j = 1, 2, 3$ such that

$$\left| \begin{pmatrix} 1 & \xi/\eta \\ 0 & 1 \end{pmatrix} f(x) \right| \leqslant h_1(x_1)h_2(x_2)h_3(x_3)$$

for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Thus

$$\begin{aligned} \sum_{x \in M} |f(\sqrt{v}\sigma_w^{-1} x)| &= \sum_{x \in M} \left| \begin{pmatrix} \sqrt{\eta} & 0 \\ 0 & \sqrt{\eta}^{-1} \end{pmatrix} \begin{pmatrix} 1 & \xi/\eta \\ 0 & 1 \end{pmatrix} f(\sqrt{v} x) \right| \\ &\leqslant \left(\sum_{x_1} h_1(\sqrt{v}\eta^{-1} x_1) \right) \left(\sum_{x_2} h_2(\sqrt{v} x_2) \right) \left(\sum_{x_3} h_3(\sqrt{v}\eta x_3) \right), \end{aligned}$$

where $x_j \in \mathbb{Z}/4$. Therefore

$$\sum_{x \in M} |f(\sqrt{v}\sigma_w^{-1} x)| = O((\sqrt{v}^{-1} + 1)^2 (\sqrt{v}^{-1}\eta + 1))$$

for $w = \xi + \mathrm{i}\eta$ with $|\xi| < c_2, \eta > c_1 > 0$. Put $U = \{w = \xi + \mathrm{i}\eta \,|\, |\xi| \leqslant 1/2, \eta > 0, |w| \geqslant 1\}$. Let $c_1 < \sqrt{3}/2, c_2 > 1/2$ and choose $\gamma \in SL_2(\mathbb{Z})$ for $w \in \mathbb{H}$ such that $\gamma(w) \in U$. Then

$$\begin{aligned} \sum_{x \in L'} |f(\sqrt{v}\sigma_{4w}^{-1} x)| &\leqslant \sum_{x \in M} |f(\sqrt{v}\sigma_{\gamma(w)}^{-1} x)| \\ &= O(\sqrt{v}^{-1} + 1)^3 (\mathrm{Im}(\gamma(w)) + 1) \\ &= O((v^{-3/2} + 1)(\eta + \eta^{-1})). \end{aligned}$$

Thus $|\theta(z, \sigma_{4w})| = O(v^{(3-k)/4}(v^{-3/2} + 1)(\eta + \eta^{-1}))$ for all $w \in \mathbb{H}, z \in \mathbb{H}$, and hence $\eta^\lambda \Psi(w) = O(\eta + \eta^{-1})$ for all $w \in \mathbb{H}$ by the definition of $\Psi(w)$.

Finally we prove (8.21). By the definition of $\theta_{2,\lambda-\varepsilon}(z, \eta)$, we know that it is majorized by $\eta^{-\lambda+\varepsilon-1} v^{-\lambda+\varepsilon} F_\varepsilon(z, \eta)$, where $F_\varepsilon(z, \eta)$ is defined by

$$F_\varepsilon(z, \eta) = \sum_{x_1, x_2} |x_1 z + x_2|^{\lambda-\varepsilon} \exp\left\{ -\frac{4\pi}{N\eta^2 v} |x_1 z + x_2|^2 \right\},$$

where $(0, 0) \neq (x_1, x_2) \in \mathbb{Z}^2$. Therefore, if β is the smallest integer $\geqslant (\lambda - \varepsilon)/2$, then

$$F_\varepsilon(z, \eta) \leqslant \begin{cases} l v^{\beta+1} \mathrm{e}^{-\pi h/v\eta^2}, & \text{if } \eta < 1, v > c > 0, c < \sqrt{3}/2, \\ l' \eta^{2(\lambda-\varepsilon+1)} v^{\beta+1} \mathrm{e}^{-\pi\eta^2 h/v}, & \text{if } \eta > 1, v > c > 0, c < \sqrt{3}/2, \end{cases}$$

where l, l' and h are positive constants depending only on ε and c. Put $U = \{z = u + \mathrm{i}v \in \mathbb{H} \,|\, |u| \leqslant 1/2, |z| \geqslant 1\}$, choose $\gamma_i \in SL_2(\mathbb{Z})$ such that $D_0(4N) \subset \bigcup_{i=1}^{t} \gamma_i(U)$ and put $T(z) = v^{k/2}\overline{\theta}_{1,\varepsilon}(z)F(z)$, then $T(\gamma_i(z)) = O(g_i(v))$ for $z \in U$ where the g_i's are some rapidly decreasing functions. Put $F'_\varepsilon(z, \eta) = \eta^{-\lambda+\varepsilon-1} v^{-\lambda+\varepsilon} F_\varepsilon(z, \eta)$, then

$$\int_{D_0(4N)} |T(z)\overline{\theta}_{2,\lambda-\varepsilon}(z,\eta^{-1})\mathrm{d}z \leqslant \sum_{i=1}^{t} c_i \int_U T(\gamma_i(z))F'_\varepsilon(\gamma_i(z),\eta^{-1})\mathrm{d}z$$
$$\leqslant \sum_{i=1}^{t} e_i \int_c^\infty v^{\nu_i}\eta^\alpha g_i(v)\exp\{-\pi\eta^2 hv^{-1}\}\mathrm{d}v$$

for all $\eta > 1$ with some constants c_i, e_i, ν_i, α. Since $\eta^{2\mu}v^{-\mu}\exp\{-\pi\eta^2 hv^{-1}\} < C_\mu$ for $\mu > 0$ with some constant C_μ and $\eta^{2\mu-\alpha}\int_c^\infty v^{\nu_i}\eta^\alpha g_i(v)\exp\{-\pi\eta^2 hv^{-1}\}\mathrm{d}v < C_\mu\int_c^\infty v^{\nu_i+\mu}g_i(v)\mathrm{d}v = C'_\mu$ with some constant C'_μ. Therefore

$$\int_{D_0(4N)} |T(z)\overline{\theta}_{2,\lambda-\varepsilon}(z,\eta^{-1})|\mathrm{d}z = O(\eta^{-\mu})$$

for any $\mu > 0$ if $\eta > 1$. In the same way, we get

$$\int_{D_0(4N)} |T(z)\overline{\theta}_{2,\lambda-\varepsilon}(z,\eta^{-1})|\mathrm{d}z = O(\eta^{\mu})$$

for any $\mu > 0$ if $\eta < 1$. Hence we get (8.21) by the above estimations, the definition of $\Phi(w)$ and

$$\theta(z,\sigma_{\mathrm{i}\eta}) = \left(2\sqrt{\frac{2\pi}{N}}\right)^{-\lambda}\sum_{\varepsilon=0}^{\lambda}\binom{\lambda}{\varepsilon}(-\mathrm{i})^\varepsilon\theta_{2,\lambda-\varepsilon}(z,\eta)\theta_{1,\varepsilon}(z).$$

This completes the proof. □

Let $G(z) = \sum_{n=1}^{\infty} a(n)e(nz)$ be an element of $S(4N,\, k/2,\, \chi)$, let t be a square-free positive integer, put $\chi_t = \chi\left(\frac{-1}{*}\right)\left(\frac{t}{*}\right)$ and $\Phi_t(w) = \sum_{n=1}^{\infty} A_t(n)e(nw)$ with $A_t(n)$ defined by

$$\sum_{n=1}^{\infty} A_t(n)n^{-s} = \left(\sum_{m=1}^{\infty}\chi_t(m)m^{\lambda-1-s}\right)\left(\sum_{m=1}^{\infty}a(tm^2)m^{-s}\right).$$

Then we have

Corollary 8.1 *$\Phi_t(w) \in G(2N,\, k-1,\, \chi^2)$ for all $k \geqslant 3$ and $\Phi_t(w) \in S(2N,\, k-1,\, \chi^2)$ if $k \geqslant 5$.*

Proof Since $G(tz) = \sum_{n=1}^{\infty} b(n)e(nz)$ belongs to $S\left(4tN,\, k/2,\, \chi\left(\frac{t}{*}\right)\right)$, Theorem 8.1

implies that $\widetilde{\Phi(w)} = \sum_{n=1}^{\infty} B_1(n)e(nw)$, defined by

$$\sum_{n=1}^{\infty} B_1(n)n^{-s} = \left(\sum_{m=1}^{\infty} \chi_t(m)m^{\lambda-1-s}\right)\left(\sum_{m=1}^{\infty} b(m^2)m^{-s}\right)$$

belongs to $G(2tN,\ k-1,\ \chi^2)$ for all $k \geqslant 3$ and $S(2tN,\ k-1,\ \chi^2)$ if $k \geqslant 5$. Since $b(m^2) = a(tj^2)$ or 0 according as $m = tj$ or t does not divide m, we know that

$$\sum_{n=1}^{\infty} B_1(n)n^{-s} = t^{-s}\sum_{m=1}^{\infty} A_t(m)m^{-s}$$

holds and so $B_1(n) = A_t(n/t)$ or 0 according as $t|n$ or $t \nmid n$. Hence we have

$$\widetilde{\Phi(w)} = \Phi_t(tw),$$

and so

$$\Phi_t(\sigma(w)) = (cw+d)^{2\lambda}\chi^2(d)\,\Phi_t(w)$$

for all $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^t(2N)$ with $\Gamma_0^t(2N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2N) | b \equiv 0 \pmod t\right\}$.

Put $\Gamma_\infty = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{Z}\right\}$. Since $\Gamma_0(2N)$ is generated by Γ_∞ and $\Gamma_0^t(2N)$, $\Phi_t(w)$ belongs to $G(2N,\ k-1,\ \chi^2)$ for all $k \geqslant 3$ and $S(2N,\ k-1,\ \chi^2)$ if $k \geqslant 5$. This completes the proof. □

Now we consider the Shimura lifting for cusp forms with weight 3/2. By Theorem 8.1 and Corollary 8.1 we know that, for any $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in S(4N,\ 3/2,\ \chi)$, t a square-free positive integer, the Shimura lifting $I_{3,t}(f)$ of f belongs to $G(2N,\ 2,\ \chi^2)$. It is clear that the Zeta function of $I_{3,t}(f)$ is

$$L(s, I_{3,t}(f)) = L\left(s, \chi\left(\frac{-t}{*}\right)\right)\sum_{m=1}^{\infty} a(tm^2)m^{-s}. \tag{8.26}$$

We shall prove that $I_{3,t}(f)$ is a cusp form if and only if $\langle f, h\rangle = 0$ for all $h \in T$, where T is the vector space spanned by all theta series of $S(4N,\ 3/2,\ \chi)$ associated with some Dirichlet characters.

Proposition 8.4 *Let ψ be a primitive character modulo r, put*

$$h(z,\psi) = \sum_{n=1}^{\infty} \psi(n)ne(n^2z), \quad \forall\ z \in \mathbb{H}.$$

Then $h \in S\left(4r^2,\ 3/2,\ \psi\left(\frac{-1}{*}\right)\right)$.

Proof This is one of the conclusions in Theorem 7.3. □

By (8.26) we get

$$L(s, I_{3,1}(h(z,\psi))) = L(s,\psi)L(s-1,\psi),$$

which shows that $I_{3,1}(h(z,\psi))$ is an Eisenstein series (not a cusp form).

Proposition 8.5 *Let α be a non-negative integer, A a positive integer, ϕ a primitive character modulo A. Define*

$$H_\alpha(s,z,\phi) = \pi^{-s}\Gamma(s)y^s \sum_{m,n}{}' \phi(n)(mAz+n)^\alpha |mAz+n|^{-2s},$$

where $z \in \mathbb{H}$, $(0,0) \neq (m,n) \in \mathbb{Z}^2$. Suppose that $\alpha > 0$ or $A > 1$, then the series above is absolutely convergent for $\mathrm{Re}(s) > 1 + \alpha/2$, $H_\alpha(s,z,\phi)$ can be continued to a holomorphic function on the whole s-plane and satisfies the following functional equation

$$H_\alpha(\alpha+1-s,z,\phi) = (-1)^\alpha g(\phi)A^{3s-\alpha-2}z^\alpha H_\alpha(s,-1/Az,\overline{\phi}),$$

where $g(\phi) = \sum_{k=1}^{A} \phi(k)e(k/A)$.

Proof We have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-\pi t|uz+v|^2/y\}e(ur+vs)\mathrm{d}u\mathrm{d}v \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-\pi t[(ux+v)^2+u^2y^2]/y\}e(ur+vs)\mathrm{d}u\mathrm{d}v \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-\pi t(v^2+u^2y^2)/y\}e(u(r-xs)+vs)\mathrm{d}u\mathrm{d}v \\
&= (ty)^{-1/2}\int_{-\infty}^{\infty} \exp\{-\pi u^2\}e(u(r-xs)/(ty)^{1/2})\mathrm{d}u(ty^{-1})^{-1/2} \\
&\quad \times \int_{-\infty}^{\infty} \exp\{-\pi v^2\}e(vsy^{1/2}/t^{1/2})\mathrm{d}v \\
&= t^{-1}\mathrm{e}^{-\pi[(r-xs)^2/(ty)+s^2y/t]} = t^{-1}\mathrm{e}^{-\pi|r-sz|^2/(ty)}. \qquad (8.27)
\end{aligned}$$

Since

$$\left(z\frac{\partial}{\partial r}+\frac{\partial}{\partial s}\right)e(ur+vs) = 2\pi\mathrm{i}(uz+v),$$

$$\left(z\frac{\partial}{\partial r}+\frac{\partial}{\partial s}\right)\exp\{-\pi|r-sz|^2/(ty)\} = -2\pi\mathrm{i}t^{-1}(r-sz)\exp\{-\pi|r-sz|^2/(ty)\},$$

applying α times the differential operator $\left(z\dfrac{\partial}{\partial r}+\dfrac{\partial}{\partial s}\right)$ on both sides of (8.27), we get

$$\begin{aligned}&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(uz+v)^\alpha\exp\{-\pi t|uz+v|^2/y\}e(ur+vs)\mathrm{d}u\mathrm{d}v\\ &=(-1)^\alpha t^{-\alpha-1}(r-sz)^\alpha\exp\{-\pi|r-sz|^2/(ty)\}.\end{aligned}\tag{8.28}$$

Put

$$\begin{aligned}\zeta(t,z,u,v)&=\sum_{m,n}((m+u)z+n+v)^\alpha\exp\{-\pi t|(m+u)z+n+v|^2/y\}\\ &=\sum_{m,n}c(m,n)e(mu+nv).\end{aligned}$$

By (8.28) we get

$$\begin{aligned}c(-m,-n)&=\int_0^1\int_0^1\sum_{m',n'}((m'+u)z+n'+v)^\alpha\\ &\quad\times\exp\{-\pi t|(m'+u)z+n'+v|^2/y\}e(mu+nv)\mathrm{d}u\mathrm{d}v\\ &=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(uz+v)^\alpha\exp\{-\pi t|uz+v|^2/y\}e(mu+nv)\mathrm{d}u\mathrm{d}v\\ &=(-1)^\alpha t^{-\alpha-1}(m-nz)^\alpha\exp\{-\pi|m-nz|^2/(ty)\}.\end{aligned}$$

Hence

$$\zeta(t,z,u,v)=(-1)^\alpha t^{-\alpha-1}\sum_{m,n}(mz+n)^\alpha\exp\{-\pi|mz+n|^2/(ty)\}e(mv-nu).\tag{8.29}$$

Suppose that p,q are integers, define

$$\xi(t,z,p,q)=\sum_{(m,n)\equiv(p,q)(\bmod A)}(mz+n)^\alpha\exp\{-\pi t|mz+n|^2/(A^2y)\}$$

and

$$\eta(t,z,p,q)=\sum_{k=1}^{A}\phi(k)\xi(t,z,kp,kq).\tag{8.30}$$

Suppose that $(p,q)\not\equiv(0,0)\pmod A$ if $A>1$. By (8.29) we have

$$\begin{aligned}\xi(t,z,p,q)&=A^\alpha\zeta(t,z,p/A,q/A)\\ &=(-A)^\alpha t^{-\alpha-1}\sum_{m,n}e((qm-pn)/A)(mz+n)^\alpha\exp\{-\pi|mz+n|^2/(ty)\}\\ &=(-A)^\alpha t^{-\alpha-1}\sum_{(a,b)\bmod A}e((qa-pb)/A)\xi(A^2t^{-1},z,a,b)\end{aligned}$$

and

$$\begin{aligned}\eta(t^{-1},z,p,q) &= \sum_{k=1}^{A}\phi(k)\xi(t^{-1},z,pk,qk)\\ &= (-A)^{\alpha}t^{\alpha+1}\sum_{k=1}^{A}\phi(k)\sum_{(a,b) \bmod A} e(k(qa-pb)/A)\xi(A^2t,z,a,b)\\ &= (-A)^{\alpha}t^{\alpha+1}g(\phi)\sum_{(a,b) \bmod A}\overline{\phi}(qa-pb)\xi(A^2t,z,p,q). \qquad (8.31)\end{aligned}$$

If $\alpha > 0$ or $A > 1$, the terms corresponding to $m = n = 0$ on both sides of (8.31) disappear. Hence by (8.30) and (8.31) we have

$$|\eta(t,z,p,q)| \leqslant \begin{cases} M\mathrm{e}^{-ct}, & \text{if } t > 1,\\ M't^{-\alpha-1}\mathrm{e}^{-c'/t}, & \text{if } t < 1,\end{cases} \qquad (8.32)$$

where M, M', c, c' are positive constants dependent only on z, p, q. We can integrate the following integral term by term

$$\begin{aligned}\int_0^{\infty}\eta(t,z,p,q)t^{s-1}\mathrm{d}t &= \sum_{k=1}^{A}\phi(k)\sum_{(m,n)\equiv k(p,q)(\bmod A)}(mz+n)^{\alpha}\\ &\quad\times\int_0^{\infty}\exp(-\pi t|mz+n|^2/(A^2y))t^{s-1}\mathrm{d}t\\ &= A^{2s}\pi^{-s}y^s\Gamma(s)\sum_{k=1}^{A}\phi(k)\sum_{(m,n)\equiv(p,q)(\bmod A)}(mz+n)^{\alpha}|mz+n|^{-2s}.\end{aligned} \qquad (8.33)$$

The series on the right hand side of (8.33) is absolutely convergent for $\mathrm{Re}(s) > 1+\alpha/2$. Divide the integral of the right hand side of (8.33) into two parts: $\int_0^1$ and $\int_1^{\infty}$. Using (8.32), we know that these two integrals are holomorphic functions on the s-plane which continues the series of the right hand side of (8.33) to a holomorphic function on the s-plane. And we have

$$A^{2s}H_{\alpha}(s,z,\phi) = \int_0^{\infty}\eta(t,z,0,1)t^{s-1}\mathrm{d}t. \qquad (8.34)$$

Therefore for $\alpha > 0$ or $A > 1$, $H_{\alpha}(s,z,\phi)$ can be continued to a holomorphic function on the s-plane. Substituting s by $\alpha+1-s$ in (8.34), we get

$$\begin{aligned}A^{2(\alpha+1-s)}H_{\alpha}(\alpha+1-s,z,\phi) &= \int_0^{\infty}\eta(t,z,0,1)t^{\alpha-s}\mathrm{d}t = \int_0^{\infty}\eta(t^{-1},z,0,1)t^{s-\alpha-2}\mathrm{d}t\\ &= (-A)^{\alpha}g(\phi)\sum_{(a,b) \bmod A}\overline{\phi}(a)\int_0^{\infty}\xi(A^2t,z,a,b)t^{s-1}\mathrm{d}t\end{aligned}$$

$$= (-A)^\alpha g(\phi) y^s \pi^{-s}\Gamma(s) \sum_{m,n}{}' \overline{\phi}(m)(mz+n)^\alpha |mz+n|^{-2s}$$
$$= (-1)^\alpha g(\phi) A^{\alpha+s} z^\alpha H_\alpha(s, -1/Az, \overline{\phi}),$$

which completes the proof. □

Proposition 8.6 *Let ω be a character modulo A, put*

$$G(s) = \Gamma(s) \sum_{m,n}{}' \omega(n)|mAz+n|^{-2s}.$$

Then $G(s)$ can be continued to a holomorphic function if ω is non-trivial; $G(s)$ can be continued to a meromorphic function with only two poles $s = 0,\ 1$ of order 1 if $A = 1$, and with the corresponding residues -1 and π/y respectively; and $G(s)$ can be continued to a meromorphic function with only one pole 1 of order 1 if $A > 1$ and ω is trivial and with the corresponding residue $\pi \prod_{p|A}(1-p^{-1})/(Ay)$.

Proof Let B be the conductor of ω and $A = BC$, let ϕ be the primitive character modulo B determined by ω. Then

$$G(s) = \Gamma(s) \sum_{m,n}{}' \phi(n) \sum_{d|(n,C)} \mu(d)|mAz+n|^{-2s}$$
$$= \Gamma(s) \sum_{d|C} \mu(d)\phi(d)d^{-2s} \sum_{m,n}{}' \phi(n) \left| m\frac{A}{d}z + n \right|^{-2s}. \qquad (8.35)$$

Hence, by Proposition 8.5, $G(s)$ can be continued to a holomorphic function if $B > 1$ (i.e. if ω is non-trivial).

Now suppose that $A = 1$, put

$$\eta(t, z) = \sum_{m,n} \exp\{-\pi t|mz+n|^2/y\}.$$

By (8.31) we get

$$\eta(t^{-1}, z) = t\eta(t, z).$$

We have, for all $\mathrm{Re}(s) > 1$, that

$$\pi^{-s} y^s G(s) = \int_0^\infty (\eta(t,z)-1)t^{s-1}\mathrm{d}t$$
$$= \int_1^\infty (\eta(t^{-1},z)-1)t^{-s-1}\mathrm{d}t + \int_1^\infty (\eta(t,z)-1)t^{s-1}\mathrm{d}t$$
$$= \int_1^\infty (t(\eta(t,z)-1)+t-1)t^{-s-1}\mathrm{d}t + \int_1^\infty (\eta(t,z)-1)t^{s-1}\mathrm{d}t$$
$$= \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty (\eta(t,z)-1)t^{-s}\mathrm{d}t + \int_1^\infty (\eta(t,z)-1)t^{s-1}\mathrm{d}t.$$

The two integrals on the right hand side of the above are holomorphic, so $G(s)$ can be continued to a meromorphic function with only two poles $s = 0, 1$ of order 1 and residues -1 and π/y respectively.

Now suppose that $B = 1, A > 1$. By (8.35) we get

$$G(s) = \sum_{d|A} \mu(d) d^{-2s} \Gamma(s) \sum_{m,n}{}' |mAz/d + n|^{-2s}.$$

Substituting $\frac{A}{d}z$ by z and using the above result for $A = 1$, we know that $G(s)$ can be continued to a meromorphic function with pole $s = 1$ and the residue

$$\sum_{d|A} \mu(d) d^{-2} \pi d/(Ay) = \pi \prod_{p|A} (1 - p^{-1})/(Ay).$$

This completes the proof. □

Now put

$$T = \{h(tz, \psi) | \psi \text{ is any odd primitive character, } t \text{ is any positive integer}\}$$

and $\widetilde{T}$ the vector space spanned by T. Also put

$$T_1 = \{h(tz, \psi) | \psi \text{ is any odd character, } t \text{ is any positive integer}\}$$

and

$$T_2 = \{\theta(tz, h, N) | t, h, N \in \mathbb{Z}, t > 0, N > 0\},$$

where

$$\theta(z, h, N) = \sum_{m \equiv h (\bmod N)} m e(m^2 z).$$

Denote by $\widetilde{T_i}$ the vector space spanned by T_i for $i = 1, 2$.

Lemma 8.6 *We have* $\widetilde{T} = \widetilde{T_1} = \widetilde{T_2}$.

Proof It is clear that $\widetilde{T} \subset \widetilde{T_1} \subset \widetilde{T_2}$. Let ψ be any odd character modulo N, $\widetilde{\psi}$ the primitive character determined by ψ. Then $\psi(d) = \widetilde{\psi}(d)$ for all $(d, N) = 1$, and

$$\begin{aligned}\sum_{m=1}^{\infty} \psi(m) m e(tm^2 z) &= \sum_{m=1}^{\infty} \sum_{d|(m,N)} \mu(d) \widetilde{\psi}(m) m e(tm^2 z) \\ &= \sum_{d|N} \mu(d) d \widetilde{\psi}(d) h(td^2 z, \widetilde{\psi}) \in \widetilde{T},\end{aligned}$$

which shows that $\widetilde{T} = \widetilde{T_1}$. Denote $d = (h, N)$. We have

$$\begin{aligned}\theta(tz,h,N) &= d\sum_{m\equiv hd^{-1}(\bmod Nd^{-1})} me(td^2m^2z)\\ &= d\phi(Nd^{-1})^{-1}\sum_{m=1}^{\infty}\sum_{\psi}\overline{\psi}(hd^{-1})\psi(m)me(td^2m^2z)\\ &= d\phi(Nd^{-1})\sum_{\psi}\overline{\psi}(hd^{-1})h(td^2z,\psi)\in\widetilde{T_1},\end{aligned}$$

where ψ runs over all characters modulo Nd^{-1}, ϕ is the Euler function. Therefore $\widetilde{T_1}=\widetilde{T_2}$, which completes the proof. □

If $f(z)=\sum_{n\in\mathbb{Q}} a(n)e(nz)$ is a formal series, define

$$\xi(f(z))=\sum_{n=0}^{\infty} a(n)e(nz).$$

Put

$$F=\{\theta(zA^{-1})|\theta(z)\in\widetilde{T},\ A \text{ is any positive integer}\}.$$

Lemma 8.7 *Let* $G(z)\in F$, $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$ *and* $H(z)=G(\gamma(z))(cz+d)^{-3/2}$. *Then* $H(z)\in F$, $\xi(G(z))\in\widetilde{T}$.

Proof Since $SL_2(\mathbb{Z})$ is generated by $\gamma_1=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\gamma_2=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, we only need prove that $H(z)\in F$ for γ_1, γ_2. Without loss of generality, we can assume that $G(z)=\theta(tA^{-1}z,\,h,\,N)$. It is easy to see

$$G(\gamma_1(z))=\sum_{\substack{g\equiv h(\bmod N),\\ g \bmod AN}} e(tg^2/A)\theta(tz/A,AN,g)\in F.$$

Using Lemma 7.5, we can prove that $H(z)\in F$ for γ_2. Now we prove $\xi(G(z))\in\widetilde{T}$. Assume again $G(z)=\theta(tz/A,h,N)$. Then

$$\xi(G(z))=\sum_{\substack{m\equiv h(\bmod N),\\ m^2\equiv 0(\bmod A)}} me(tm^2z/A).$$

Let $A=p_1^{e_1}\cdots p_j^{e_j}$ be the standard factorization of A. Take $B=p_1^{f_1}\cdots p_j^{f_j}$ such that f_i are the smallest positive integers with property $2f_i\geqslant e_i$ for all $1\leqslant i\leqslant j$. Then

$$\xi(G(z))=\sum_{\substack{m\equiv h(\bmod N),\\ m\equiv 0(\bmod B)}} me(tm^2z/A).$$

Denote $d = (B, N)$. If $d \nmid h$, then $\xi(G(z)) = 0$. If $d|h$, put $h' = h/d$, $N' = N/d$, $t' = tB^2/A$ and take B' such that $Bd^{-1}B' \equiv 1 \pmod{N'}$, then

$$\xi(G(z)) = \sum_{n \equiv h'B' (\bmod N')} nBe(tn^2B^2z/A) = \theta(t'z, h'B', N') \in \widetilde{T}.$$

This completes the proof. □

Theorem 8.2 *Let* $4|N$, $f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S(N,\ 3/2,\ \omega)$. *Then for any square-free positive integer* t, $I_{3,t}(f)$ *is a cusp form if and only if* $f(z)$ *is orthogonal to the subspace* $S(N,\ 3/2,\ \omega) \cap \widetilde{T}$.

Proof Let $I_{3,t}(f) = \sum_{n=0}^{\infty} b(n)e(nz) \in G(N/2,\ 2,\ \omega^2)$. By Theorem 7.13, $I_{3,t}(f)$ is a cusp form if and only if for all primitive character ψ and all positive integer r, $L(s, I_{3,t}(f), \psi, r)$ is holomorphic at $s = 2$. Substituting N by $[N, r]$ =l.c.m. of N, r, without loss of generality, we can assume that $r|N^{\infty}$. We have

$$\sum_{n=1}^{\infty} b(n)n^{-s} = L\left(s, \omega\left(\frac{-t}{*}\right)\right) \sum_{n=1}^{\infty} a(tn^2)n^{-s}.$$

Since ω is a character modulo N,

$$\begin{aligned} L(s, I_{3,t}(f), \psi, r) &= \sum_{n=1}^{\infty} \psi(n)b(rn)n^{-s} \\ &= L\left(s, \omega\psi\left(\frac{-t}{*}\right)\right) \sum_{n=1}^{\infty} \psi(n)a(tr^2n^2)n^{-s}. \end{aligned}$$

Put

$$h(z, \overline{\psi}) = \sum_{n=1}^{\infty} \overline{\psi}(n)n^{\nu}e(n^2z),$$

where $\psi(-1) = (-1)^{\nu}, \nu = 0, 1$. Taking a constant $\sigma > 0$, for $\mathrm{Re}(s) > \sigma$, we have

$$\begin{aligned} \int_0^{\infty}\int_0^1 f(z)\overline{h}(tr^2z, \overline{\psi})y^{s-1}\mathrm{d}x\mathrm{d}y &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a(n)\psi(m)m^{\nu}\int_0^{\infty} e(\mathrm{i}(n+tr^2m^2)y)y^{s-1}\mathrm{d}y \\ &\quad \times \int_0^1 e((n-tr^2m^2)x)\mathrm{d}x \\ &= (4\pi tr^2)^{-s}\Gamma(s)\sum_{m=1}^{\infty} \psi(m)a(tr^2m^2)m^{\nu-2s}. \end{aligned}$$

Denote by g the conductor of ψ. Then $h(tr^2z, \overline{\psi}) \in G\left(4tr^2g^2,\ (1+2\nu)/2,\ \overline{\psi}\left(\frac{(-1)^{\nu}tr^2}{*}\right)\right)$

by Theorem 7.3 and Theorem 5.16. Denote $\widetilde{N} = (4tr^2g^2, N)$, define $B(z,s) = f(z)\overline{h}(tr^2z, \overline{\psi})y^{s+1}$. Then for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = \Gamma_0(\widetilde{N})$, we have

$$B(\gamma(z), s) = \omega\psi(d)\left(\frac{-t}{d}\right)(cz+d)^{1-\nu}|cz+d|^{2\nu-1-2s}B(z,s).$$

Therefore

$$L(2s-\nu, I_{3,t}(f), \psi, r) = (4\pi tr^2)^s\Gamma(s)^{-1}\int\limits_{\Gamma\backslash\mathbb{H}} B(z,s)L\left(2s-\nu, \omega\psi\left(\frac{-t}{*}\right)\right)$$
$$\times \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma} \omega\psi(d)\left(\frac{-t}{d}\right)(cz+d)^{1-\nu}|cz+d|^{2\nu-1-2s}\frac{\mathrm{d}x\mathrm{d}y}{y^2}. \tag{8.36}$$

It is easy to see

$$L\left(2s-\nu, \omega\psi\left(\frac{-t}{*}\right)\right)\sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma} \omega\psi(d)\left(\frac{-t}{d}\right)(cz+d)^{1-\nu}|cz+d|^{2\nu-1-2s}$$
$$= \sum_{m,n}{}' \omega\psi(n)\left(\frac{-t}{n}\right)(m\widetilde{N}z+n)^{1-\nu}|m\widetilde{N}z+n|^{2\nu-1-2s}. \tag{8.37}$$

If $\nu = 0$, by Proposition 8.5, $L(s, I_{3,t}(f), \psi, r)$ is holomorphic at $s = 2$. If $\nu = 1$, by Proposition 8.6, we know that the series in (8.36) is holomorphic except the case $\omega = \overline{\psi}\left(\frac{-t}{*}\right)$. In that case, it has a pole $s = 3/2$ of order 1 with residue c/y and $c \neq 0$ a constant. Hence, by (8.36), only for $\omega = \overline{\psi}\left(\frac{-t}{*}\right)$, $L(s, I_{3,t}(f), \psi, r)$ has a possible pole $s = 2$ of order 1 with residue $c' < f, h(tr^2z, \overline{\psi}) >$ and $c' \neq 0$ a constant.

Now suppose that $I_{3,t}(f)$ is a cusp form. By the above argumentation we know that f is orthogonal to $h(tr^2z, \overline{\psi})$ if $\omega = \overline{\psi}\left(\frac{-t}{*}\right)$. If $\omega \neq \overline{\psi}\left(\frac{-t}{*}\right)$, put $\omega' = \overline{\psi}\left(\frac{-t}{*}\right)$. Then $f \in S(\widetilde{N}, 3/2, \omega)$, $h(tr^2z, \overline{\psi}) \in S(\widetilde{N}, 3/2, \omega')$. Therefore for any $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma_0(\widetilde{N})$ we have

$$\begin{aligned}\omega(d_\gamma)\overline{\omega'}(d_\gamma)\langle f, h(tr^2z,\overline{\psi})\rangle_{\Gamma_0(\widetilde{N})} &= \langle f|[\gamma], h(tr^2z,\overline{\psi})|[\gamma]\rangle_{\Gamma_0(\widetilde{N})} \\ &= \langle f, h(tr^2z,\overline{\psi})\rangle_{\Gamma_0(\widetilde{N})}.\end{aligned}$$

Since $\omega \neq \omega'$, we can find a $\gamma \in \Gamma_0(\widetilde{N})$ such that $\omega(d_\gamma) \neq \omega'(d_\gamma)$. Hence we get $\langle f, h(tr^2z,\overline{\psi})\rangle = 0$. But any positive integer u can be written as $u = tr^2$ with t square-free. So f is orthogonal to $\widetilde{T}$ and hence is orthogonal to $S(N, 3/2, \omega) \cap \widetilde{T}$.

Conversely, suppose f is orthogonal to $S(N, 3/2, \omega) \cap \widetilde{T}$. Take any $h(uz, \psi) \in T$. Then $h(uz, \psi) \in S\left(4ug^2, 3/2, \psi\left(\frac{-u}{*}\right)\right)$ where g is the conductor of ψ. Denote $\widetilde{N} = [4ug^2, N]$. Suppose $\omega = \psi\left(\frac{-u}{*}\right)$. Let $\Gamma_0(N) = \bigcup_{i=1}^{r} \Gamma(\widetilde{N})\gamma_i$ be the decomposition of $\Gamma_0(N)$ into right cosets with respect to $\Gamma(\widetilde{N})$. Let

$$\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}.$$

Then

$$g(z) = \sum_{i=1}^{r} \omega(a_i) h(uz, \psi)|[\gamma_i]$$

belongs to $S(N, 3/2, \omega)$. By Lemma 8.7 we know that $g(z) \in F$. Since $g(z+1) = g(z)$, $\xi(g(z)) = g(z)$. By Lemma 8.7 we know that $g(z) \in \widetilde{T}$, i.e., $g \in S(N, 3/2, \omega) \cap \widetilde{T}$. By hypothesis, we get

$$\begin{aligned} 0 &= \langle f(z), g(z) \rangle \\ &= \sum_{i=1}^{r} \overline{\omega}(a_i) \langle f(z), h(uz, \psi)|[\gamma_i] \rangle \\ &= \sum_{i=1}^{r} \overline{\omega}(a_i) \langle f|[\gamma_i^{-1}](z), h(uz, \psi) \rangle \\ &= r\langle f(z), h(uz, \psi) \rangle, \end{aligned}$$

which shows that f is orthogonal to $h(uz, \psi)$. Hence $L(s, I_{3,t}(f), \psi, r)$ is holomorphic at $s = 2$ (since whose residue at $s = 2$ is 0 or $c' \langle f, h(tr^2 z, \overline{\psi}) \rangle = 0$). This shows that $I_{3,t}(f)$ is a cusp form.

This completes the proof. □

8.3 Shimura Lifting of Eisenstein Spaces

In this section we deal with Shimura lifting of Eisenstein spaces.

Let χ be a Dirichlet character modulo N, and denote by $L(s, \chi)$ the associated L-series

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}.$$

For a positive integer k we have that $L(1-k, \chi) = -\frac{B_{k,\chi}}{k}$, where the numbers $B_{k,\chi}$ are defined by

$$\sum_{a=1}^{N} \frac{\chi(a) t \mathrm{e}^{at}}{\mathrm{e}^{Nt} - 1} = \sum_{k=0}^{\infty} B_{k,\chi} \frac{t^k}{k!}.$$

Fix an integer $k \geqslant 2$, we define rational numbers $H(k,n)$ by

$$H(k,n) := \begin{cases} \zeta(1-2k), & \text{if } n=0, \\ L(1-k,\chi_D)\sum_{d|f}\mu(d)\chi_D(d)d^{k-1}\sigma_{2k-1}(f/d), & \text{if } (-1)^k n = Df^2, \\ 0, & \text{otherwise,} \end{cases}$$

where ζ denotes the Riemann ζ-function, μ the Moebius function, D a fundamental discriminant, χ_D the quadratic character associated with $\mathbb{Q}(\sqrt{D})$ and the arithmetical function σ_r is defined by $\sigma_r(m) = \sum_{d|m} d^r$. H.Cohen introduced the rational numbers $H(k,n)$ and proved that

$$H_k(z) := \sum_{n=0}^{\infty} H(k,n)\exp\{2\pi i n z\} \tag{8.38}$$

is a modular form of half-integral weight $k+1/2$ for $\Gamma_0(4)$ in [C] which is now named Cohen-Eisenstein series. For $k=1$ and group $\Gamma_0(4p)$ with p a prime, Cohen-Eisenstein series are defined by

$$H_{1,p}(z) := \sum_{n=0}^{\infty} H(n)_p \exp\{2\pi i n z\}, \tag{8.39}$$

where $H(n)_p := H(p^2 n) - pH(n)$ with $H(n)$(for $n > 0$) the number of classes of positive definite binary quadratic forms of discriminant $-n$(where forms equivalent to a multiple of x^2+y^2 or x^2+xy+y^2 are counted with multiplicity $\frac{1}{2}$ or $\frac{1}{3}$ respectively) and with $H(0) = -\frac{1}{12}$. $H_{1,p}$ is a modular form of weight $3/2$ on $\Gamma_0(4p)$.

The problem of constructing Shimura lifting of non-cusp forms was first considered by W.Kohnen for the Cohen-Eisenstein series and later by A.G.Van Asch for the space of non-cusp forms of weight $k+1/2(k \geqslant 2)$ on $\Gamma_0(4)$ and $\Gamma_0(4p)$ with p an odd prime.

In this section we shall consider more general cases.

Let the rational numbers $H(k,\, l,\, N,\, N;\, n)$ and $H(k,\, l,\, m,\, N;\, n)$ be defined as in Section 7.4 with $N \neq m|N$.

Note that $H(k,\, 1,\, 1,\, 1;\, n) = H(k,\, n)$ is just the rational numbers defined by H.Cohen.

Theorem 8.3 *Let N be a square-free odd positive integer, l a divisor of N and D a fundamental discriminant with $\varepsilon(-1)^k D > 0$. Then*

(1) *If $k=1$ and $\left(\frac{D}{p}\right) \neq 1$ for all $p|N$, then the* Shimura *lifting defined by*

$$L_D\left(\sum_{n=0}^{\infty} a(n)q^n\right) := \frac{a(0)}{2} L_N\left(0, \left(\frac{D}{\cdot}\right)\right) + \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{D}{d}\right) a\left(|D|\frac{n^2}{d^2}\right)\right) q^n$$

gives an one-to-one correspondence from $E^{+}_{3/2}(4N, \mathrm{id.})$ *to* $\mathcal{E}(N, 2, \mathrm{id.})$.

(2) *If* $k \geqslant 2$, *then the* Shimura *lifting defined by*

$$L_D\left(\sum_{n=0}^{\infty} a(n)q^n\right) := \frac{a(0)}{2} L_N\left(1-k, \chi_l'\left(\frac{D}{\cdot}\right)\right) + \sum_{n=1}^{\infty}\left(\sum_{d|n} d^{k-1}\left(\frac{D}{d}\right)\chi_l'(d) a\left(|D|\frac{n^2}{d^2}\right)\right) q^n$$

gives a one-to-one correspondence from $E^{+}_{k+1/2}(4N, \chi_l)$ *to* $\mathcal{E}(N,\ 2k,\ \mathrm{id.})$.

Proof We denote by $U(m)(m|N^{\infty})$ the following operator defined by

$$U(m)\left(\sum_{n=0}^{\infty} a(n)q^n\right) = \sum_{n=0}^{\infty} a(mn)q^n$$

for any $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in G(4N,\ k+1/2,\ \chi_l)$ or $G(N,\ 2k,\ \mathrm{id.})$. Then $U(m)$ $(m|N^{\infty})$ map $G(N,\ 2k,\ \mathrm{id.})$ to $G(N,\ 2k,\ \mathrm{id.})$ and $U(m^2)$ $(m|N^{\infty})$ map $G(4N,\ k+1/2,\ \chi_l)$ to $G(4N,\ k+1/2,\ \chi_l)$. A direct calculation shows that $L_D \circ U(m^2) = U(m) \circ L_D$ for any $m|N^{\infty}$ and any fundamental discriminant D with $\varepsilon(-1)^k D > 0$.

(1) Since L_D is a linear map on the space consisting of all formal power series $\sum_{n=0}^{\infty} a(n)q^n$ with $a(n) \in \mathbb{C}$, we only need to prove that L_D maps a basis of $E^{+}_{3/2}(4N, \mathrm{id.})$ to a basis of $\mathcal{E}(N,\ 2,\ \mathrm{id.})$. We first consider the case that $N = p$ is a prime. Then the dimension of $E^{+}_{3/2}(4p,\ \mathrm{id.})$ equals to one and $H_1(\mathrm{id.},\ p,\ p) \in E^{+}_{3/2}(4p,\ \mathrm{id.})$. Denote that $H_1(\mathrm{id.},\ p,\ p) := \sum_{n=0}^{\infty} a(n)q^n$ and $L_D(H_1(\mathrm{id.},\ p,\ p)) := \sum_{n=0}^{\infty} b(n)q^n$. Then by the definition of L_D, we see that

$$\begin{aligned} b(n) &= \sum_{d|n}\left(\frac{D}{d}\right) a\left(|D|\frac{n^2}{d^2}\right) \\ &= \sum_{d|n}\left(\frac{D}{d}\right) L_p\left(0, \left(\frac{D'_{|D|\frac{n^2}{d^2}}}{\cdot}\right)\right) \sum_{d_1|f_{|D|\frac{n^2}{d^2}}} \mu(d_1)\left(\frac{D_{|D|\frac{n^2}{d^2}}}{d_1}\right) \sum_{\substack{e|f_{|D|\frac{n^2}{d^2}}/d_1 \\ (e,p)=1}} e \\ &= \sum_{d|n}\left(\frac{D}{d}\right) L_p\left(0, \left(\frac{D}{\cdot}\right)\right) \sum_{d_1|n/d} \mu(d_1)\left(\frac{D}{d_1}\right) \sum_{\substack{e|n/dd_1 \\ (e,p)=1}} e \\ &= L_p\left(0, \left(\frac{D}{\cdot}\right)\right) \sum_{s|n}\left(\frac{D}{s}\right) \sum_{e|n/s,(e,p)=1} e \sum_{d|s}\mu(d) = L_p\left(0, \left(\frac{D}{\cdot}\right)\right) \sum_{e|n,(e,p)=1} e, \end{aligned}$$

$$b(0) = \frac{1}{2}a(0)L_p\left(0, \left(\frac{D}{\cdot}\right)\right) = \frac{1}{2}L_p\left(0, \left(\frac{D}{\cdot}\right)\right)L_p(-1, \mathrm{id.}) = \frac{p-1}{24}L_p\left(0, \left(\frac{D}{\cdot}\right)\right).$$

Hence we obtain that

$$L_D(H_1(\mathrm{id.}, p, p)) = L_p\left(0, \left(\frac{D}{\cdot}\right)\right)E_2^{(p)}(z),$$

where

$$E_2^{(p)}(z) = \frac{p-1}{24} + \sum_{n=1}^{\infty}\left(\sum_{d|n, p|d} d\right)q^n \in \mathcal{E}(p, 2, \mathrm{id.})$$

is the normalized Eisenstein series of weight 2 on $\Gamma_0(p)$. By the hypothesis in Theorem 8.3 we see that

$$L_p\left(0, \left(\frac{D}{\cdot}\right)\right) = \left(1 - \left(\frac{D}{p}\right)\right)L\left(0, \left(\frac{D}{\cdot}\right)\right) = \left(1 - \left(\frac{D}{p}\right)\right)\frac{h(D)}{w_D} \neq 0,$$

where w_D is the half of the number of units in $\mathbb{Q}(\sqrt{D})$.

This shows that L_D is a bijection if $N = p$ is a prime. We now prove that this holds for any square-free positive integer $N > 1$. Suppose that $N = p_1p_2\cdots p_t$. For any prime divisor p_i of N, denote the following Eisenstein series by $E_2^{(p_i)}(z)$:

$$E_2^{(p_i)}(z) := \sum_{n=1}^{\infty} a_i(n)q^n := \frac{p_i - 1}{24} + \sum_{n=1}^{\infty}\left(\sum_{d|n, p_i|d} d\right)q^n,$$

which is the normalized Eisenstein series of weight 2 on $\Gamma_0(p_i)$.

Let

$$S_i = \{U(l)(E_2^{(p_i)}(z)) \mid l|N/(p_1\cdots p_i)\}$$

for $1 \leqslant i \leqslant t$. By the properties of $U(m)$ we know that $S_i \subset \mathcal{E}(N, 2, \mathrm{id.})$ for $1 \leqslant i \leqslant t$. We want to prove that $S := \bigcup_{i=1}^{t} S_i$ is a basis of $\mathcal{E}(N, 2, \mathrm{id.})$. Since $\dim(\mathcal{E}(N, 2, \mathrm{id.})) = 2^t - 1 =$ the number of elements in S, we only need to prove that the elements in S are linearly independent. We denote that $E_{2,l}^{(p_i)}(z) := U(l)(E_2^{(p_i)}(z))$. Suppose that there exist complex numbers $c_i(l)$ such that

$$\sum_{i=1}^{t}\sum_{l|N/(p_1\cdots p_i)} c_i(l)E_{2,l}^{(p_i)} = 0. \tag{8.40}$$

We must prove that $c_i(l) = 0$ for all $1 \leqslant i \leqslant t$ and $l|N/(p_1\cdots p_i)$. We prove this by induction on t. If $t = 1$, it is clear that $S = S_1 = \{E_2^{(p_1)}(z)\}$ is a basis of $\mathcal{E}(N, 2, \mathrm{id.}) = \mathcal{E}(p_1, 2, \mathrm{id.})$.

For any modular form $f(z) = \sum_{n=0}^{\infty} a(n)q^n$, let $L(s,f) := \sum_{n=1}^{\infty} a(n)n^{-s}$ be the corresponding Dirichlet series. Then by a direct calculation we see that

$$L(s, E_2^{(p_i)}) = \zeta(s)L(s-1, 1_{p_i}),$$

where 1_m denotes the trivial character modulo m for any positive integer m.

For $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in G(N, 2, \text{id.})$, $r|N$ and ψ any character modulo N, we define

$$L(s, f, \psi, r) := \sum_{n=1}^{\infty} \psi(n)a(rn)n^{-s}.$$

Then we have that $\psi(n) = 0$ if $(n, N) \neq 1$, and so

$$\begin{aligned} L(s, E_{2,l}^{(p_i)}(z), \psi, r) &= \sum_{n=1}^{\infty} \psi(n) \left(\sum_{d|nlr,(p_i,d)=1} d \right) n^{-s} \\ &= \sum_{(n,N)=1} \psi(n) \left(\sum_{d|nlr,(p_i,d)=1} d \right) n^{-s} \\ &= \left(\sum_{a|lr,(p_i,a)=1} a \right) \sum_{(n,N)=1} \psi(n) \left(\sum_{d|n} d \right) n^{-s} \\ &= \prod_{p_i \neq p|lr} (1 + p + p^2 + \cdots + p^{\nu_p(lr)}) L(s, \psi) L(s-1, \psi). \qquad (8.41) \end{aligned}$$

Hence from (8.40) and (8.41) we obtain that

$$\begin{aligned} 0 &= \sum_{i=1}^{t} \sum_{l|N/(p_1\cdots p_i)} c_i(l) L(s, E_{2,l}^{(p_i)}, \psi, r) \\ &= \sum_{i=1}^{t} \sum_{l|N/(p_1\cdots p_i)} c_i(l) \prod_{p_i \neq p|lr} (1 + p + p^2 + \cdots + p^{\nu_p(lr)}) L(s, \psi) L(s-1, \psi). \end{aligned}$$

This implies that

$$A_r := \sum_{i=1}^{t} \sum_{l|N/(p_1\cdots p_i)} c_i(l) \prod_{p_i \neq p|lr} (1 + p + p^2 + \cdots + p^{\nu_p(lr)}) = 0, \quad \forall\, r|N. \qquad (8.42)$$

That is, $c_i(l)$ must satisfy the above system of linear equations (8.42). Hence we only need to prove that the system of linear equations (8.42) has only the solution zero. It is clear that this holds for $t = 1$. Suppose that (8.42) has only the solution zero for $t-1$. Write that $N = p_1 N_1$ with $(p_1, N_1) = 1$. Let r_1 be a positive divisor of N_1. Then

$$\begin{aligned}
A_{p_1 r_1} - A_{r_1} =& \sum_{i=1}^{t} \sum_{l|N/(p_1\cdots p_i)} c_i(l) \prod_{p_i \neq p | l p_1 r_1} (1 + p + p^2 + \cdots + p^{\nu_p(l p_1 r_1)}) \\
& - \sum_{i=1}^{t} \sum_{l|N/(p_1\cdots p_i)} c_i(l) \prod_{p_i \neq p | l r_1} (1 + p + p^2 + \cdots + p^{\nu_p(l r_1)}) \\
=& \sum_{i=1}^{t} \sum_{l|N/(p_1\cdots p_i)} c_i(l) \Bigg(\prod_{p_i \neq p | l p_1 r_1} (1 + p + p^2 + \cdots + p^{\nu_p(l p_1 r_1)}) \\
& - \prod_{p_i \neq p | l r_1} (1 + p + p^2 + \cdots + p_p^{\nu}(l r_1)) \Bigg) \\
=& p_1 \sum_{i=2}^{t} \sum_{l|N/(p_1\cdots p_i)} c_i(l) \prod_{p_i \neq p | l r_1} (1 + p + p^2 + \cdots + p^{\nu_p(l r_1)}) \\
=& p_1 \sum_{i=2}^{t} \sum_{l|N_1/(p_2\cdots p_i)} c_i(l) \prod_{p_i \neq p | l r_1} (1+p+p^2 + \cdots + p^{\nu_p(l r_1)}) = 0, \quad \forall\, r_1 | N_1.
\end{aligned} \tag{8.43}$$

By the induction assumption, we know that (8.43) has only the solution zero. Therefore $c_i(l) = 0$ for all $2 \leqslant i \leqslant t$ and $l|N/(p_1 \cdots p_i)$. Then (8.42) becomes

$$\sum_{l|N/p_1} c_1(l) \prod_{p_1 \neq p | lr} (1+p+p^2+\cdots+p^{\nu_p(lr)}) = \sum_{l|N/p_1} c_1(l) \sum_{d|lr, p_1|d} d = 0, \quad \forall\, r|N. \tag{8.44}$$

This shows that $c_1(l)$ must satisfy the system of linear equations (8.44). So we only need to prove that (8.44) has only the solution zero. For any positive integer $N > 1$ and any prime p with $(p, N) = 1$, we define the following system of linear equations for $x(l)$ with $l|N$

$$B_{p,N}(r) := \sum_{l|N} x(l) \sum_{d|lr, p|d} d = 0, \quad \forall\, r|N. \tag{8.45}$$

It is clear that (8.44) has only the solution zero if we can prove that (8.45) has only the solution zero. We prove that (8.45) has only the solution zero by induction on the number of prime factors of N. For $t = 0$, it is obvious. Suppose that (8.45) has only the solution zero for $t-1$. We want to prove that our assertion holds also for $N = p_1 p_2 \cdots p_t$. Write $N = p_i N_i$ with $(p_i, N_i) = 1$ for all $1 \leqslant i \leqslant t$. Let $r_i | N_i$ be any positive divisor of N_i. Then

$$0 = B_{p,N}(r_i) - B_{p,N_i}(r_i) = \sum_{l|N} x(l) \sum_{d|lr_i, p|d} d - \sum_{l|N_i} x(l) \sum_{d|lr_i, p|d} d$$

$$= \left(\sum_{l|N_i} x(l) + \sum_{p_i|l|N} x(l)\right) \sum_{d|lr_i,p|d} d - \sum_{l|N_i} x(l) \sum_{d|lr_i,p|d} d$$
$$= \sum_{l_i|N_i} x(p_il_i) \sum_{d|p_il_ir_i,p|d} d$$
$$= (p_i+1) \sum_{l_i|N_i} x(p_il_i) \sum_{d|l_ir_i,p|d} d, \quad \forall\ r_i|N_i \text{ with } 1 \leqslant i \leqslant t,$$

where we used the fact that $(l_ir_i, p_i) = 1$ to deduce the last equality. Hence

$$\sum_{l_i|N_i} x(p_il_i) \sum_{d|l_ir_i,p|d} d = 0, \quad \forall\ r_i|N_i \text{ with } 1 \leqslant i \leqslant t.$$

By the induction hypothesis, we see that $x(p_il_i) = 0$ for all $l_i|N_i, 1 \leqslant i \leqslant t$. Therefore $x(l) = 0$ for all $l|N$ with $l \neq 1$. Substituting these into (8.45) we obtain that $x(1) = 0$. This shows that (8.45) and hence (8.44) has only the solution zero. We have proved that S is a basis of $\mathcal{E}(N, 2, \text{id.})$. Now let

$$S'_i = \{U(l^2)(H_1(\text{id.}, p_i, p_i)(z)) \mid l \mid N/(p_1 \cdots p_i)\}, \quad \text{for } 1 \leqslant i \leqslant t,$$
$$S' = \bigcup_{i=1}^{t} S'_i.$$

We know that $H_1(\text{id.}, p_i, p_i) \in E^+_{3/2}(4p_i, \text{id.}) \subset E^+_{3/2}(4N, \text{id.})$ and hence $U(l^2)(H_1(\text{id.}, p_i, p_i)(z)) \in E^+_{3/2}(4N, \text{id.})$ for all $l \mid N/(p_1 \cdots p_i)$. This shows that $S' \subset E^+_{3/2}(4N, \text{id.})$. Using the properties of $U(l^2)$ and L_D and the result proved above, we see that

$$\begin{aligned} & L_D \circ U(l^2)(H_1(\text{id.}, p_i, p_i)(z)) \\ =& U(l) \circ L_D(H_1(\text{id.}, p_i, p_i)(z)) \\ =& U(l) \left(L_{p_i}\left(0, \left(\frac{D}{\cdot}\right)\right) E_2^{(p_i)}(z)\right) \\ =& L_{p_i}\left(0, \left(\frac{D}{\cdot}\right)\right) U(l)(E_2^{(p_i)}(z)) \in \mathcal{E}(N, 2, \text{id.}), \quad \forall 1 \leqslant i \leqslant t \text{ and } l|N/(p_1 \cdots p_i), \end{aligned}$$

where we used the fact that $E_2^{(p_i)}(z) \in E_2(p_i, \text{id.}) \subset \mathcal{E}(N, 2, \text{id.})$ and $U(l)(E_2^{(p_i)}(z)) \in \mathcal{E}(N, 2, \text{id.})$ for all $l \mid N$. Since $\left(\frac{D}{p_i}\right) \neq 1$ for all $p_i|N$, then $L_{p_i}\left(0, \left(\frac{D}{\cdot}\right)\right) \neq 0$. Hence we have proved that L_D maps S' to a basis of $\mathcal{E}(N, 2, \text{id.})$. Because L_D is a linear operator, S' is a basis of $E^+_{3/2}(4N, \text{id.})$. This implies that L_D is a bijection from $E^+_{3/2}(4N, \text{id.})$ to $\mathcal{E}(N, 2, \text{id.})$.

(2) Since L_D is a linear operator, we only need to calculate the image of $H_k(\chi, m, N)$ under the Shimura lifting L_D. Denote that $H_k(\chi, N, N) := \sum_{n=0}^{\infty} a_N(n)q^n$ and $L_D(H_k(\chi,$

$N, N)) := \sum_{n=0}^{\infty} b_N(n)q^n$. Then by the definition of L_D, we see that

$$\begin{aligned}
b_N(n) &= \sum_{d|n} \chi_l'(d) \left(\frac{D}{d}\right) d^{k-1} a_N \left(|D|\frac{n^2}{d^2}\right) \\
&= \sum_{d|n} \chi_l'(d) \left(\frac{D}{d}\right) d^{k-1} \left(1-k, \left(\frac{D'_{|D|\frac{n^2}{d^2}}}{\cdot}\right)\right) \\
&\quad \times \sum_{d_1|f_{|D|\frac{n^2}{d^2}}} \mu(d_1)\chi_l'(d_1) \left(\frac{D_{|D|\frac{n^2}{d^2}}}{d_1}\right) d_1^{k-1} \sum_{\substack{e|f_{|D|\frac{n^2}{d^2}}/d_1 \\ (e,N)=1}} \mathrm{e}^{2k-1} \\
&= \sum_{d|n} \chi_l'(d) \left(\frac{D}{d}\right) d^{k-1} L_N \left(1-k, \chi_l' \left(\frac{D}{\cdot}\right)\right) \\
&\quad \times \sum_{d_1|n/d} \mu(d_1)\chi_l'(d_1) \left(\frac{D}{d_1}\right) d_1^{k-1} \sum_{\substack{e|n/dd_1 \\ (e,N)=1}} \mathrm{e}^{2k-1} \\
&= L_N \left(1-k, \chi_l' \left(\frac{D}{\cdot}\right)\right) \sum_{s|n} \chi_l'(s) \left(\frac{D}{s}\right) s^{k-1} \sum_{e|n/s,(e,N)=1} \mathrm{e}^{2k-1} \sum_{d|s} \mu(d) \\
&= L_N \left(1-k, \chi_l' \left(\frac{D}{\cdot}\right)\right) \sum_{e|n,(e,N)=1} \mathrm{e}^{2k-1}, \\
b(0) &= \frac{1}{2} a_N(0) L_N \left(1-k, \chi_l' \left(\frac{D}{\cdot}\right)\right) \\
&= \frac{1}{2} L_N \left(1-k, \chi_l' \left(\frac{D}{\cdot}\right)\right) L_N(1-2k, \mathrm{id.}).
\end{aligned}$$

Hence

$$L_D(H_k(\chi, N, N)) = L_N \left(1-k, \chi_l' \left(\frac{D}{\cdot}\right)\right) G_{2k,N}(z),$$

where

$$G_{2k,N}(z) := \frac{L_N(1-2k, \mathrm{id.})}{2} + \sum_{n=1}^{\infty} \left(\sum_{d|n,(d,N)=1} d^{2k-1}\right) q^n.$$

For $m|N$ with $m \neq N$ we can compute similarly and obtain that

$$\begin{aligned}
L_D(H_k(\chi, m, N)) =& L_m \left(1-k, \chi_l' \left(\frac{D}{\cdot}\right)\right) \left(\frac{(l, D)}{(l, D, m)}\right)^{2k-1} \\
& \times \prod_{p|N/m} \frac{1 - \chi_l'(p) \left(\frac{D}{p}\right) p^{-k}}{1 - p^{-2k}} G_{2k,m}(z),
\end{aligned}$$

where

$$G_{2k,m}(z) := \sum_{n=1}^{\infty} \left(\sum_{\substack{d|n,(d,m)=1,\\(n/d,N/m)=1}} d^{2k-1} \right) q^n, \quad \forall m|N \text{ with } m \neq N.$$

Hence we only need to prove that $\{G_{2k,m}(z) \mid m \mid N\}$ constitute a basis of $\mathcal{E}(N, 2k,$ id.) which is stated as the following

Lemma 8.8 *Let N be a square-free positive integer and $k \geqslant 4$ an even integer. Then*

$$G_{k,N}(z) := \frac{L_N(1-k, \text{id.})}{2} + \sum_{n=1}^{\infty} \left(\sum_{d|n,(d,N)=1} d^{k-1} \right) q^n,$$

$$G_{k,m}(z) := \sum_{n=1}^{\infty} \left(\sum_{\substack{d|n,(d,m)=1,\\(n/d,N/m)=1}} d^{k-1} \right) q^n, \forall m|N \textit{ with } m \neq N$$

constitute a basis of $\mathcal{E}(N, k, \text{id.})$.

Proof Let $E_k(z)$ be the Eisenstein series defined by

$$E_k(z) := \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{k-1} \right) q^n.$$

Then it is well known that $\{E_k(lz) \mid l \mid N\}$ constitute a basis of $\mathcal{E}(N, k, \text{id.})$. We define functions $q_{k,m}(z)$ as follows

$$q_{k,N}(z) := E_k(Nz),$$
$$q_{k,m}(z) := \sum_{l|N/m} \mu(l) E_k(mlz), \quad \forall\ m|N, m \neq N.$$

Then it is clear that $\{q_{k,m} \mid m \mid N\}$ constitute a basis of $\mathcal{E}(N, k, \text{id.})$. And

$$\begin{aligned} q_{k,m}(z) &= \sum_{n=1}^{\infty} \sum_{l|N/m} \mu(l) \sum_{l|n} \sum_{d|n/l} d^{k-1} q^{mn} \\ &= \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1} \sum_{l|(n/d,N/m)} \mu(l) q^{mn} \\ &= \sum_{n=1}^{\infty} \sum_{\substack{d|n\\(n/d,N/m)=1}} d^{k-1} q^{mn}. \end{aligned}$$

For $m|N$, denote by $G'_{k,m}(z)$ the following function

$$G'_{k,m}(z) := \sum_{s|m} \prod_{p|s} (1-p^{k-1}) q_{k,s}(z)$$
$$:= \sum_{n=1}^{\infty} a_m(n) q^n, \quad \forall\, m|N, m \neq N,$$

and

$$G'_{k,N}(z) := \sum_{s|N} \prod_{p|s} (1-p^{k-1}) q_{k,s}(z) := \sum_{n=1}^{\infty} a_N(n) q^n.$$

It is clear that these functions constitute a basis of $\mathcal{E}(N,k,\mathrm{id.})$. □

For any fixed n, let it be that $(n,m) = m_1, m = m_1 m_2, n = n' \prod_{p|M_1} p^{\nu_p(n)}$ with $(n',m) = 1$.

$$\begin{aligned} a_m(n) &= \sum_{s|m_1} \prod_{p|s} (1-p^{k-1}) \sum_{\substack{d|n/s \\ (n/sd, N/s)=1}} d^{k-1} \\ &= \sum_{s|m_1} \prod_{p|s} (1-p^{k-1}) \prod_{p|s} \left(\sum_{t=0}^{\nu_p(n)-1} p^{(k-1)t} \right) \prod_{p|m_1/s} p^{(k-1)\nu_p(n)} \sum_{\substack{d|n, (d,m)=1, \\ (n/d, N/m)=1}} d^{k-1} \\ &= \sum_{s|m_1} \prod_{p|s} (1-p^{(k-1)\nu_p(n)}) \prod_{p|m_1/s} p^{(k-1)\nu_p(n)} \sum_{\substack{d|n, (d,m)=1, \\ (n/d, N/m)=1}} d^{k-1} \\ &= \sum_{\substack{d|n, (d,m)=1, \\ (n/d, N/m)=1}} d^{k-1}. \end{aligned}$$

This shows that $G_{k,m}(z) = G'_{k,m}(z)$ for all $m|N$ with $m \neq N$. We can prove similarly that $G_{k,N}(z) = G'_{k,N}(z)$. Therefore $\{G_{k,m}(z) \mid m|N\}$ constitute a basis of $\mathcal{E}(N,k,\mathrm{id.})$.

This completes the proof of Theorem 8.3. □

As a Corollary of the above proof, we have

Corollary 8.2 *Let N be a square-free positive odd integer. Define*

$$G_{2,N}(z) : = -\frac{1}{24} \prod_{p|N} (1-p) + \sum_{n=1}^{\infty} \left(\sum_{d|n, (d,N)=1} d \right) q^n,$$
$$G_{2,m}(z) : = \sum_{n=1}^{\infty} \left(\sum_{\substack{d|n, (d,m)=1, \\ (n/d, N/m)=1}} d \right) q^n, \quad \forall\, m|N \text{ with } m \neq 1, N.$$

Then $\{G_{2,m} \mid m|N, m \neq 1\}$ *constitute a basis of* $\mathcal{E}(N, 2, \mathrm{id.})$.

Proof Completely similar to the proof of Theorem 8.3 (2), we can calculate the images of $H_1(\mathrm{id.}, m, N)$ under L_D for all $m|N$, $m \neq 1$. In particular, if we choose a negative fundamental discriminant D satisfying $D \equiv 0 \pmod N$, then

$$L_D(H_1(\mathrm{id.}, m, N)) = h(D) \prod_{p|N/m} \frac{1}{1-p^{-2}} G_{2,m}(z), \quad \forall\, m|N, m \neq 1.$$

We have shown in the proof of Theorem 8.3 (1) that L_D is a bijection from $E^+_{3/2}(4N, \mathrm{id.})$ to $\mathcal{E}(N,2, \mathrm{id.})$. Hence $G_{2,m}(z) = L_D(h(D)^{-1} \prod_{p|N/m} (1-p^{-2}) H_1(\mathrm{id.}, m, N)) \in \mathcal{E}(N,$ 2, id.) and constitute a basis of $\mathcal{E}(N, 2, \mathrm{id.})$. □

8.4 A Congruence Relation between Some Modular Forms

In this section we will give a congruence relation between some modular forms. A special case of our congruence (which was proved by Kohnen and J.A. Antoniadis, 1986) has important applications on the structure of the Selmer groups of some elliptic curves (Please compare J.A. Antoniadis, 1990).

Theorem 8.4 *Let* $N > 3$ *be a square-free positive odd integer with* $N \equiv 3 \pmod 4$, *and let* $l \geqslant 5$ *be a prime which divides the exact numerator of* $\frac{1}{12}\prod_{p|N}(p-1)$, *but does not divide the class number* $h(-N)$ *and* $\prod_{p|N/m}(p+1)$ *for any* $1 < m|N$. *We let*

$$G_{1,-N}(z) := \frac{1}{2}h(-N) + \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{-N}{d}\right)\right)q^n$$

be the Eisenstein series of weight 1 and Nebentypus $\left(\frac{-N}{\cdot}\right)$ *on* $\Gamma_0(N)$ *for the cusp* i∞. *Put*

$$\mathbb{C}_N := -\frac{1}{12}\prod_{p|N}(1-p)(G_{1,-N}(z))^2 - \frac{1}{2}h(-N)^2 \sum_{1<m|N}\left(\prod_{p|N/m}\frac{-p}{p+1}\right) G_{2,m}(z),$$

$$\mathbb{C}'_N := -\frac{1}{12}\prod_{p|N}(1-p)G_{1,-N}(4z)\theta(Nz) - \frac{1}{2}h(-N)\sum_{1<m|N}\left(\prod_{p|N/m}\frac{1-p}{p}\right)H_1(\mathrm{id.}, m, N)(z),$$

where

$$G_{2,N}(z) := -\frac{\prod_{p|N}(1-p)}{24} + \sum_{n=1}^{\infty}\left(\sum_{d|n,(d,N)=1} d\right)q^n,$$

$$G_{2,m}(z) := \sum_{n=1}^{\infty} \left(\sum_{\substack{d|n,(d,m)=1,\\(n/d,N/m)=1}} d \right) q^n, \quad \forall m|N \text{ with } m \neq 1, N.$$

Then

(1) *The function* $\mathbb{C}'_N(z) \in S^+_{3/2}(4N, \text{id.})$ *has* l*-integral* Fourier *coefficients, is non-zero modulo* l*, and the congruence*

$$\mathbb{C}'_N(z) \equiv -\frac{1}{2}h(-N) \sum_{1<m|N} \left(\prod_{p|N/m} \frac{1-p}{p} \right) H_1(\text{id.}, m, N)(z) \qquad (\text{mod } l)$$

holds.

(2) *The function* $\mathbb{C}_N(z) \in S(N,\, 2,\, \text{id.})$ *has* l*-integral* Fourier *coefficients, is non-zero modulo* l*, and the congruence*

$$\mathbb{C}_N(z) \equiv -\frac{1}{2}h(-N)^2 \sum_{1<m|N} \left(\prod_{p|N/m} \frac{-p}{p+1} \right) G_{2,m}(z) \qquad (\text{mod } l)$$

holds. And one has $L_{-N}(\mathbb{C}'_N(z)) = \mathbb{C}_N$.

(3) *Suppose that* $\mathbb{C}'_N(z)$ *belongs to a subspace* V *of* $S^+_{3/2}(4N, \text{id.})$ *which is isomorphic to a subspace of* $S(N,\, 2,\, \text{id.})$ *as modules over the* Hecke *algebra. And suppose that* V *has a basis* $\{f_i(z)\}_{i=1}^r$ *with* $f_i(z)$ *are all Hecke eigenforms and* $f_i(z) := \sum_{n\geqslant 1} c_i(n)q^n$ *corresponding to* $F_i \in S(N,\, 2,\, \text{id.})$*. Then one has*

$$\mathbb{C}'_N = -\frac{1}{12} \prod_{p|N} (1-p) \cdot \alpha' \cdot \sum_{i=1}^{r} \frac{L(F_i, 1)c_i(N)}{||f_i||^2} f_i,$$

where α' *is a non-zero constant not depending on* N*,* $L(F_i, s)$ *is the* L*-function associated with* F_i *and* $||f_i||^2 := \int_{\Gamma_0(4N)\backslash H} |f_i|^2 y^{-1/2} \mathrm{d}x\mathrm{d}y$ $(x = \text{Re}(z),\ y = \text{Im}(z))$ *the square of the* Petersson *norm of* f_i*.*

Proof (1) We first prove that $\mathbb{C}'_N(z)$ has l-integral Fourier coefficients. Since $\nu_l\left(\frac{1}{12}\prod_{p|N}(1-p)\right) > 0$ and $G_{1,-N}(4z)\theta(Nz)$ has rational Fourier coefficients, we only need to show that

$$\frac{1}{2}h(-N) \sum_{1<m|N} \left(\prod_{p|N/m} \frac{1-p}{p} \right) H_1(\text{id.}, m, N)(z)$$

has l-integral Fourier coefficients. By the definition of $H_1(\text{id.}, m, N)$ we see that the nth Fourier coefficient of

$$\left(\prod_{p|N/m} \frac{1-p}{p}\right) H_1(\text{id.}, m, N)(z)$$

equals to

$$\prod_{p|N/m} \frac{1-p}{p} H(1,1,m,N;n) = \prod_{p|N/m} \frac{1-p}{p} L_m(0,\chi_{D_n}) \prod_{p|N/m} \frac{1-p^{-1}\left(\frac{D_n}{p}\right)}{1-p^{-2}}$$
$$\times \sum_{d|f_n} \mu(d)\chi_{D_n}(d)\sigma_{m,N,1}(f_n/d)$$
$$= L_m(0,\chi_{D_n}) \prod_{p|N/m} \frac{\left(\frac{D_n}{p}\right) - p}{1+p}$$
$$\times \sum_{d|f_n} \mu(d)\chi_{D_n}(d)\sigma_{m,N,1}(f_n/d),$$

which is l-integral by hypothesis of Theorem 8.4, and hence $\mathbb{C}'_N(z)$ has l-integral Fourier coefficients.

Now we need only to prove that $\mathbb{C}'_N(z) \in S^+_{3/2}(4N, \text{id.})$, as the other assertions are obvious. We must show that $\mathbb{C}'_N(z) \in M^+_{3/2}(4N, \text{id.})$ and the values of $\mathbb{C}'_N(z)$ are zero at all cusp points. In order to do this we introduce the following Eisenstein series: For any positive integer k, and D_1, D_2 relatively prime fundamental discriminants with $(-1)^k D_1 D_2 > 0$ set

$$G_{k,D_1,D_2}(z) := \gamma_{k,D_1}^{-1} \times \frac{1}{2}{\sum_{m,n}}' \left(\frac{D_1}{n}\right)\left(\frac{D_2}{m}\right)(mD_1 z + n)^{-k},$$

where $\gamma_{k,D_1} := \left(\frac{D_1}{-1}\right)^{1/2} |D_1|^{-k+\frac{1}{2}} \frac{(-2\pi \mathrm{i})^k}{(k-1)!}$ and $\sum'$ means that (m,n) run over $\mathbb{Z}\times\mathbb{Z}$ except $(0,0)$. The function G_{k,D_1,D_2} is an Eisenstein series in $M_k\left(\Gamma_0(D), \left(\frac{D}{\cdot}\right)\right)$ $(D = D_1D_2)$ for the cusp $\frac{1}{D_1}$. The Fourier expansion of $G_{k,D_1,D_2}(z)$ is given by

$$G_{k,D_1,D_2}(z) = \sum_{n=0}^{\infty} \sigma_{k-1,D_1,D_2}(n) q^n,$$

where

$$\sigma_{k-1,D_1,D_2}(n) := \begin{cases} -L(1-k,\chi_{D_1})L(0,\chi_{D_2}), & \text{if } n = 0, \\ \displaystyle\sum_{\substack{d_1,d_2>0 \\ d_1d_2=n}} \left(\frac{D_1}{d_1}\right)\left(\frac{D_2}{d_2}\right) d_1^{k-1}, & \text{if } n > 0. \end{cases}$$

We must note that for $k = 1$ or 2 there is a slight problem of convergence. But we can define $G_{k,D_1,D_2}(z)$ as the holomorphic continuation to $s = 0$ of the corresponding non-holomorphic Eisenstein series of weight 1 or 2. Anyway the above formula for the Fourier expansion of G_{k,D_1,D_2} holds for $k = 1, 2$.

Hence we know that for any $k \geqslant 2$

$$\sigma_{k-1,D_1,D_2}(0) := \begin{cases} \frac{1}{2}L(1-k,\chi_D), & \text{if } D_2 = 1, \\ 0, & \text{if } D_2 \neq 1 \end{cases} \tag{8.46}$$

and

$$\sigma_{0,D_1,D_2}(0) := \begin{cases} \frac{1}{2}h(D), & \text{if } D_2 = 1 \text{ or } D_1 = 1, \\ 0, & \text{if } D_2 \neq 1 \text{ and } D_1 \neq 1. \end{cases} \tag{8.47}$$

Denote $G_{k,D}(z)$, $G_{k,4D}(z)$ the following Eisenstein series

$$G_{k,D}(z) := \frac{1}{2}L(1-k,\chi_D) + \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{D}{d}\right)d^{k-1}\right)q^n \in G\left(D, k, \left(\frac{D}{\cdot}\right)\right),$$

$$G_{k,4D}(z) := G_{k,D}(4z) - 2^{-k}\left(\frac{D}{2}\right)G_{k,D}(2z) \in G\left(4D, k, \left(\frac{D}{\cdot}\right)\right).$$

Now one can show that

$$(|D_1|z+1)^{-k}G_{k,D}\left(\frac{z}{|D_1|z+1}\right) = \left(\frac{D_2}{-1}\right)^{-1/2}\left(\frac{D_2}{|D_1|}\right)|D_2|^{-1/2}G_{k,D_1,D_2}\left(\frac{z+|D_1|^*}{|D_2|}\right), \tag{8.48}$$

where $|D_1|^*$ is an integer with $|D_1||D_1|^* \equiv 1(\bmod D_2)$.

And

$$(4|D_1|z+1)^{-k-\frac{1}{2}}G_{k,D}\left(\frac{4z}{4|D_1|z+1}\right)\theta\left(\frac{|D|z}{4|D_1|z+1}\right) = \left(\frac{D_2}{-|D_1|}\right)|D_2|^{-1}G_{k,D_1,D_2}\left(\frac{4z+|D_1|^*}{|D_2|}\right)\theta\left(\frac{|D_1|z+4^*}{|D_2|}\right), \tag{8.49}$$

where $a^* \in \mathbb{Z}$ with $aa^* \equiv 1 \pmod{D_2}$ (Please compare with W. Kohnen, 1981, 192-197).

From (8.47) we see immediately that

$$V\left(G_{k,D}(4z)\theta(|D|z), \frac{1}{4|D_1|}\right) = \lim_{z\to \mathrm{i}\infty}(4|D_1|z+1)^{-k-\frac{1}{2}}G_{k,D}\left(\frac{4z}{4|D_1|z+1}\right)\theta\left(\frac{|D|z}{4|D_1|z+1}\right)$$

$$= \lim_{z\to i\infty}\left(\frac{D_2}{-|D_1|}\right)|D_2|^{-1}G_{k,D_1,D_2}\left(\frac{4z+|D_1|^*}{|D_2|}\right)\theta\left(\frac{|D_1|z+4^*}{|D_2|}\right)$$
$$= \left(\frac{D_2}{-|D_1|}\right)|D_2|^{-1}V(G_{k,D_1,D_2}(z),i\infty)V(\theta(z),i\infty)$$
$$= \left(\frac{D_2}{-|D_1|}\right)|D_2|^{-1}\sigma_{k-1,D_1,D_2}(0).$$

Especially, from (8.47), we see that

$$V\left(G_{1,D}(4z)\theta(|D|z),\frac{1}{4|D_1|}\right)=\begin{cases}\frac{1}{2}\left(\frac{D_2}{-|D_1|}\right)|D_2|^{-1}h(D), & \text{if } D_2=1 \text{ or } D_1=1,\\ 0, & \text{if } D_2\neq 1 \text{ and } D_1\neq 1.\end{cases} \quad (8.50)$$

Since $4/|D_1|$ and $1/|D_1|$ are $\Gamma_0(4|D|)$-equivalent, we can also calculate the value of $G_{1,D}$ $(4z)\theta(|D|z)$ at the cusp point $1/|D_1|$ by Claim 1 of Theorem 10.9 and (8.48) as follows:

$$V\left(G_{1,D}(4z)\theta(|D|z),\frac{1}{|D_1|}\right) = V\left(G_{1,D}(4z),\frac{1}{|D_1|}\right)V\left(\theta(|D|z),\frac{1}{|D_1|}\right)$$
$$= \lim_{z\to i\infty}(-|D_1|z)G_{1,D}\left(4(z+\frac{1}{|D_1|})\right)$$
$$\times \lim_{z\to i\infty}(-|D_1|z)^{1/2}\theta\left(|D|(z+\frac{1}{|D_1|})\right)$$
$$= \frac{1}{4|D_2|^{1/2}}V\left(G_{1,D}(z),\frac{4}{|D_1|}\right)V\left(\theta(z),\frac{|D|}{|D_1|}\right)$$
$$= \frac{1}{4|D_2|^{1/2}}\left(\frac{D}{d}\right)V\left(G_{1,D}(z),\frac{1}{|D_1|}\right)V(\theta(z),|D_2|)$$
$$= \frac{1-i}{8|D_2|^{1/2}}\left(\frac{D}{d}\right)\left(\frac{D_2}{-1}\right)^{-1/2}\left(\frac{D_2}{|D_1|}\right)|D_2|^{-1/2}\sigma_{0,D_1,D_2}(0)$$
$$= -L(0,\chi_{D_1})L(0,\chi_{D_2})\frac{1-i}{8|D_2|}\left(\frac{D}{d}\right)\left(\frac{D_2}{-1}\right)^{-1/2}\left(\frac{D_2}{|D_1|}\right),$$

where d is an integer such that

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(|D|) \text{ and } \begin{pmatrix} a & b\\ c & d\end{pmatrix}(4/|D_1|)=1/|D_1|.$$

Therefore we get that

$$V\left(G_{1,D}(4z)\theta(|D|z),\frac{1}{|D_1|}\right)=\begin{cases}\frac{-1-i}{16|D|}h(D), & \text{if } D_1=1,\\ \frac{1-i}{16}h(D), & \text{if } D_1=D,\\ 0, & \text{if } D_1\neq 1 \text{ and } D_1\neq D.\end{cases} \quad (8.51)$$

Finally since $V(\theta(z), 1/2) = 0$, we see easily that

$$V\left((G_{1,D}(4z)\theta(|D|z), \frac{1}{2|D_1|})\right) = 0. \tag{8.52}$$

By Theorem 7.7 we can calculate the values of $H_1(\text{id.}, m, N)$ at cusp points as follows: For any positive divisor d of N,

$$\begin{aligned} V(H_1(\text{id.}, m, N), 1/d) &= V(L_m(-1, \text{id.})H_1^{'}(\text{id.}, m, N), 1/d) \\ &= -\frac{1}{12}\prod_{p|m}(1-p)(V(g_3(\text{id.}, 4m, 4N), 1/d) \\ &\quad -\frac{3}{2}V(g_3(\text{id.}, m, 4N), 1/d)) \\ &= -\frac{1}{12}\prod_{p|m}(1-p)\frac{1+\mathrm{i}}{8}\mu(m/d)dm^{-1}\varepsilon_d^{-1} \end{aligned} \tag{8.53}$$

and

$$V(H_1(\text{id.}, m, N), 1/2d) = 0 \tag{8.54}$$

and

$$\begin{aligned} V(H_1(\text{id.}, m, N), 1/4d) &= V(L_m(-1, \text{id.})H_1'(\text{id.}, m, N), 1/4d) \\ &= -\frac{1}{12}\prod_{p|m}(1-p)(V(g_1(\text{id.}, 4m, 4N), 1/4d) \\ &\quad -\frac{3}{2}V(g_1(\text{id.}, m, 4N), 1/4d)) \\ &= -\frac{1}{12}\prod_{p|m}(1-p)\mu(m/d)dm^{-1}. \end{aligned} \tag{8.55}$$

Using the above results we can compute the values of $\mathbb{C}_N'(z)$ at all cusp points. For example, we have for $D_1 \neq 1$ and $D_1 \neq -N$ by (8.50) and (8.55),

$$\begin{aligned} V(\mathbb{C}_N'(z), 1/4|D_1|) = V\bigg(&-\frac{1}{12}\prod_{p|N}(1-p)G_{1,-N}(4z)\theta(Nz)\bigg) \\ &-\frac{1}{2}h(-N)\sum_{1\neq m|N}\left(\prod_{p|N/m}\frac{1-p}{p}\right)H_1(\text{id.}, m, N)(z), 1/4|D_1|) \\ = &-\frac{1}{2}h(-N)\sum_{1\neq m|N}\left(\prod_{p|N/m}\frac{1-p}{p}\right)V(H_1(\text{id.}, m, N)(z), 1/4|D_1|) \\ = &-\frac{1}{2}h(-N)\sum_{1\neq m|N}\left(\prod_{p|N/m}\frac{1-p}{p}\right) \end{aligned}$$

$$\times \left(-\frac{1}{12} \prod_{p|m} (1-p) \right) \mu(m/|D_1|)|D_1|m^{-1}$$
$$= \frac{h(-N)}{24N} \prod_{p|N} (1-p)|D_1| \sum_{1<m|N} \mu\left(\frac{m}{|D_1|}\right) = 0.$$

For $D_1 = 1$, we have by (8.46) and (8.55)

$$\begin{aligned} V(\mathbb{C}'_N(z), 1/4) &= -\frac{1}{12} \prod_{p|N} (1-p) V(G_{1,-N}(4z)\theta(Nz), 1/4) \\ &\quad + \frac{h(-N)}{24N} \prod_{p|N} (1-p) \sum_{1<m|N} \mu(m) \\ &= \frac{h(-N)}{24N} \prod_{p|N} (1-p) - \frac{h(-N)}{24N} \prod_{p|N} (1-p) = 0. \end{aligned}$$

For $D_1 = -N$, we have by (8.46) and (8.55)

$$\begin{aligned} V(\mathbb{C}'_N(z), 1/4N) &= -\frac{1}{12} \prod_{p|N} (1-p) V(G_{1,-N}(4z)\theta(Nz), 1/4N) \\ &\quad + \frac{h(-N)}{24N} \prod_{p|N} (1-p) N \sum_{1<m|N} \mu(m/N) \\ &= -\frac{h(-N)}{24} \prod_{p|N} (1-p) + \frac{h(-N)}{24} \prod_{p|N} (1-p) = 0. \end{aligned}$$

This shows that for all positive divisors d of N, we have

$$V(\mathbb{C}'_N(z), 1/4d) = 0.$$

It is clear that $V(\mathbb{C}'_N(z), 1/2d) = 0$ for all positive divisors d of N from (8.52) and (8.54). We now compute the values of $\mathbb{C}'_N(z)$ at the cusp point $1/|D_1|$ by (8.51) and (8.53): for $D_1 \neq 1$ and $D_2 \neq 1$,

$$\begin{aligned} V(\mathbb{C}'_N(z), 1/|D_1|) &= 0 - \frac{1}{2} h(-N) \sum_{1<m|N} \prod_{p|N/m} \left(\frac{1-p}{p}\right) \\ &\quad \times L_m(-1, \mathrm{id.}) \frac{1+\mathrm{i}}{8} \mu\left(\frac{m}{|D_1|}\right) m^{-1} |D_1| \varepsilon_{|D_1|}^{-1} \\ &= \frac{(1+\mathrm{i})h(-N)}{192N} |D_1| \varepsilon_{|D_1|}^{-1} \prod_{p|N} (1-p) \sum_{1<m|N} \mu\left(\frac{m}{|D_1|}\right) = 0 \end{aligned}$$

and

$$\begin{aligned}
V(\mathbb{C}'_N(z),1) &= -\frac{1}{12}\prod_{p|N}(1-p)\left(-\frac{(1+\mathrm{i})h(-N)}{16N}\right) \\
&\quad -\frac{1}{2}h(-N)\sum_{1<m|N}\prod_{p|N/m}\left(\frac{1-p}{p}\right)L_m(-1,\mathrm{id.})\frac{1+i}{8}\mu(m)m^{-1}\varepsilon_1^{-1} \\
&= \frac{(1+\mathrm{i})h(-N)}{192N}\prod_{p|N}(1-p)+\frac{(1+\mathrm{i})h(-N)}{192N}\prod_{p|N}(1-p)\sum_{1<m|N}\mu(m) \\
&= \frac{(1+\mathrm{i})h(-N)}{192N}\prod_{p|N}(1-p)-\frac{(1+\mathrm{i})h(-N)}{192N}\prod_{p|N}(1-p)=0.
\end{aligned}$$

Since $N\equiv 3 \pmod 4$, we have $\varepsilon_N=\mathrm{i}$ and hence

$$\begin{aligned}
V(\mathbb{C}'_N(z),1/N) &= -\frac{1}{12}\prod_{p|N}(1-p)\left(-\frac{(1-\mathrm{i})h(-N)}{16}\right) \\
&\quad -\frac{1}{2}h(-N)\sum_{1<m|N}\prod_{p|N/m}\left(\frac{1-p}{p}\right)L_m(-1,\mathrm{id.})\frac{1+\mathrm{i}}{8}\mu\left(\frac{m}{N}\right)m^{-1}N\varepsilon_N^{-1} \\
&= -\frac{(1-\mathrm{i})h(-N)}{192}\prod_{p|N}(1-p)+\frac{(1+\mathrm{i})h(-N)}{192}\prod_{p|N}(1-p)\mathrm{i}^{-1}\sum_{1<m|N}\mu\left(\frac{m}{N}\right) \\
&= -\frac{(1-\mathrm{i})h(-N)}{192}\prod_{p|N}(1-p)+\frac{(1-\mathrm{i})h(-N)}{192}\prod_{p|N}(1-p)=0.
\end{aligned}$$

This shows that $V(\mathbb{C}'_N(z),1/d)=0$ for any positive divisor d of N. Hence $\mathbb{C}'_N(z)\in S(4N,3/2,\mathrm{id.})$ is a cusp form. On the other hand, we can prove that $G_{1,-N}(4z)\theta(Nz) = r\times pr(G_{1,-4N}(z)\theta(z))$ with r a constant by the method as exposed in W. Kohnen, 1981 where pr denotes the projection from the space $G(4N,\ 3/2,\ \mathrm{id.})$ to the space $M^+_{3/2}(4N,\mathrm{id.})$ (W. Kohnen, 1982). This shows that $\mathbb{C}'_N(z)\in M^+_{3/2}(4N,\mathrm{id.})$ and hence $\mathbb{C}'_N(z)\in S^+_{3/2}(4N,\mathrm{id.})$. This completes the proof of (1).

(2) It is clear that $\mathbb{C}_N(z)$ has l-integral Fourier coefficients by the hypothesis in Theorem 8.4. We only need to show that $L_{\text{-}N}(\mathbb{C}'_N(z))=\mathbb{C}_N$. The proof is similar to the arguments used in W. Kohnen, 1981. For the sake of completeness we give it as follows. Write $c(n)$ resp. $b(n)$ for the nth Fourier coefficient of $G_{1,-N}(4z)\theta(Nz)$ resp. $G_{1,-N}(z)$. Then

$$c(n)=\sum_{\substack{r\in\mathbb{Z},Nr^2\leqslant n\\ n\equiv Nr^2(\bmod 4)}} b\left(\frac{n-Nr^2}{4}\right).$$

Denote that $L_{-N}(G_{1,-N}(4z)\theta(Nz)):=\sum_{n=0}^{\infty}a(n)q^n$. Then for $n>0$ we have that

$$\begin{aligned} a(n) &= \sum_{d|n} \left(\frac{-N}{d}\right) c\left(N\frac{n^2}{d^2}\right) \\ &= \sum_{d|n} \left(\frac{-N}{d}\right) \sum_{\substack{r\in\mathbb{Z},|r|\leqslant\sqrt{n/d} \\ r\equiv n/d (\bmod 2)}} b\left(N\frac{n^2-r^2d^2}{4d^2}\right). \end{aligned}$$

Observing that $b(Nm) = b(m)$ for any $m \geqslant 0$ and writing $n_1 = \dfrac{n-rd}{2}, n_2 = \dfrac{n+rd}{2}$, we see that the coefficient

$$a(n) = \sum_{\substack{n_1,n_2\geqslant 0 \\ n_1+n_2=n}} \sum_{d|(n_1,n_2)} \left(\frac{-N}{d}\right) b\left(\frac{n_1n_2}{d^2}\right).$$

By the multiplicative properties of $b(n)$, the inner sum equals $b(n_1)b(n_2)$, hence

$$a(n) = \sum_{\substack{n_1,n_2\geqslant 0 \\ n_1+n_2=n}} b(n_1)b(n_2),$$

which is the nth Fourier coefficient of $G_{1,-N}(z)^2$. But

$$a(0) = \frac{c(0)}{2} L_N(0,\chi_{-N}) = \frac{1}{4}h(-N)^2,$$

which is the constant term of $G_{1,-N}(z)^2$. This shows that $L_{-N}(G_{1,-N}(4z)\theta(Nz)) = G_{1,-N}(z)^2$. But we know that from Corollary 8.2

$$L_{-N}(H_1(\text{id.}, m, N)) = h(-N) \prod_{p|N/m} \frac{1}{1-p^{-2}} G_{2,m}(z), \quad \forall\ m|N, m \neq 1,$$

which implies that $L_{-N}(\mathbb{C}'_N(z)) = \mathbb{C}_N(z)$ as desired.

(3) It can be proved by Rankin's trick, just as used in W. Kohnen, 1981 and J.A. Antoniadis, 1986. We omit the proof because of the complete similarity with the one in W. Kohnen, 1981 and J.A. Antoniadis, 1986. □

Proposition 8.7 *Let $p > 3$ be a prime with $p \equiv 3 \pmod 4$, and let $l \geqslant 5$ be a prime which divides the exact numerator of $\dfrac{p-1}{12}$, but does not divide the class number $h(-p)$. Then*

(1) *The function $\mathbb{C}'_p(z) \in S^+_{3/2}(4N, \text{id.})$ has l-integral* Fourier *coefficients, is non-zero modulo l, and the congruence*

$$\mathbb{C}'_p(z) \equiv -\frac{1}{2}h(-p)H_{1,p}(z) \pmod l$$

holds.

(2) *The function* $\mathbb{C}_p(z) \in S(N,\ 2,\ \text{id.})$ *has* l*-integral* Fourier *coefficients, is non-zero modulo* l*, and the congruence*

$$\mathbb{C}_p(z) \equiv -\frac{1}{2}h(-p)^2 G_{2,p}(z) \pmod{l}$$

holds. And one has

$$L_{-p}(\mathbb{C}'_p(z)) = \mathbb{C}_p.$$

(3) $\mathbb{C}'_N(z)$ *belongs to a subspace* V *of* $S^+_{3/2}(4N, \text{id.})$ *which is isomorphic to a subspace of* $S(N, 2, \text{id.})$ *as modules over the* Hecke *algebra. Suppose that* V *has a basis* $\{f_i(z)\}_{i=1}^r$ *with all* $f_i(z)$ *are* Hecke *eigenforms and* $f_i(z) := \sum_{n\geqslant 1} c_i(n)q^n$ *corresponding to* $F_i \in S(N,\ 2,\ \text{id.})$*. Then one has*

$$\mathbb{C}'_p = -\frac{1-p}{12}\cdot\alpha'\cdot\sum_{i=1}^{r}\frac{L(F_i,1)c_i(p)}{||f_i||^2}f_i,$$

where α' *is a non-zero constant not depending on* p*,* $L(F_i, s)$ *is the* L*-function associated with* F_i *and* $||f_i||^2 := \int_{\Gamma_0(4p)\backslash H}|f_i|^2 y^{-1/2}\mathrm{d}x\mathrm{d}y$ $(x = \mathrm{Re}(z), y = \mathrm{Im}(z))$ *the square of the* Petersson *norm of* f_i*.*

Proof This is a special case of Theorem 8.4. □

References

A. Weil. Sur certains groups dóperateurs unitaires. Acta Math. 1964. 111: 143–211.

G. Shimura. On modular forms of half-integral weight. Ann. Math. 1973. 97: 440–481.

H. Kojima. Cusp forms of weight 3/2. Nagoya Math. J. 1980, 79: 111–122.

I. Gelfand. M. I. Graev-Pyatetskii-Shapiro, Representation Theory and Automorphic Functions. Saunders Company.

J. Sturm. Theta series of weight 3/2, J. of Number Theory. 1982. 14: 353–361.

J. A. Antoniadis, W. Kohnen. Congruences between cusp forms and Eisenstein series of half-integral weight. Abh. Math. Sem. Univ. Hamburg, 1986. 57: 157–164.

J. A. Antoniadis, M. Bungert, G. Frey. Properties of twists of elliptic curves. J. reine angew. Math. 1990, 405: 1–28.

R. Godement. Introduction à la théorie de Langlands. Seminaire Bourbaki, Exposé 321, 1966.

S. Niwa, Modular forms of half-integral weight and the integral of certain theta functions. Nagoya Math. J., 1975. 56: 147–161.

T. Shintani. On construction of holomorphic cusp forms of half integral weight. Nagoya Math. J.. 1975. 58: 83–126.

T. Shintani. On construction of holomorphic cusp forms of half integral weight. Nagoya Math. J.. 1975. 58: 83–126.

W. Kohnen, D. Zagier. Values of L-series of modular forms at the center of the critical strip. Invent. Math. 1981. 64: 175–198.

W. Kohnen. Newforms of half-integral weight. J. reine. angew. Math. 1982. 333: 32–72.