# Chapter 7

# Construction of Eisenstein Series

## 7.1 Construction of Eisenstein Series with Weight $\geqslant 5/2$

In this section we study the following two problems: construct a basis of the Eisenstein space $\mathcal{E}(4N, k+1/2, \chi_l)$ which are eigenfunctions for all Hecke operators, and calculate their values at all cusp points.

Now we introduce some notations as in Chapter 2. For any odd positive integer $k$, let $\lambda = \dfrac{k-1}{2}$, and

$$\lambda_k(n, 4N) = L_{4N}(2\lambda, \text{id.})^{-1} L_{4N}(\lambda, \chi_{(-1)^\lambda n})\beta_k(n, \chi_N, 4N)$$

$$A_k(2,n) = \begin{cases} 2^{-k}(1+(-1)^\lambda \mathrm{i})\left(\dfrac{1-2^{(2-k)(\nu_2(n)-1)/2}}{1-2^{2-k}} - 2^{(2-k)(\nu_2(n)-1)/2}\right), \\ \qquad \text{if } 2 \nmid \nu_2(n), \\ 2^{-k}(1+(-1)^\lambda \mathrm{i})\left(\dfrac{1-2^{(2-k)\nu_2(n)/2}}{1-2^{2-k}} - 2^{(2-k)\nu_2(n)/2}\right), \\ \qquad \text{if } 2|\nu_2(n), (-1)^\lambda n/2^{\nu_2(n)} \equiv -1 \pmod 4, \\ 2^{-k}(1+(-1)^\lambda \mathrm{i})\left(\dfrac{1-2^{(2-k)\nu_2(n)/2}}{1-2^{2-k}} + 2^{(2-k)\nu_2(n)/2}\right. \\ \qquad \left.\left(1+2^{(3-k)/2}\left(\dfrac{(-1)^\lambda n/2^{\nu_2(n)}}{2}\right)\right)\right), \\ \qquad \text{if } 2|\nu_2(n), (-1)^\lambda n/2^{\nu_2(n)} \equiv 1 \pmod 4, \end{cases}$$

$$A_k(p,n) = \begin{cases} \dfrac{(p-1)(1-p^{(2-k)(\nu_p(n)-1)/2})}{p(p^{k-2}-1)} \\ \qquad - p^{(2-k)(\nu_p(n)+1)/2-1}, & \text{if } 2 \nmid \nu_p(n), \\ \dfrac{(p-1)(1-p^{(2-k)\nu_p(n)/2})}{p(p^{k-2}-1)} \\ \qquad + \left(\dfrac{(-1)^\lambda n/p^{\nu_p(n)}}{p}\right) p^{(2-k)(\nu_p(n)+1)/2-1/2}, & \text{if } 2|\nu_p(n), \end{cases}$$

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$$L_N(s,\chi) = \sum_{(n,N)=1}^{\infty} \chi(n)n^{-s} = \prod_{p\nmid N}(1-\chi(p)p^{-s})^{-1},$$

$$\beta_k(n,\chi_N,4N) = \sum_{\substack{(ab)^2|n,(ab,2N)=1\\ a,b \text{ positive integers}}} \mu(a)\left(\frac{(-1)^\lambda n}{a}\right) a^{-\lambda}b^{2-k},$$

$$\lambda_k'(n,4N) = \frac{(-2\pi \mathrm{i})^{k/2}}{\Gamma(k/2)}\lambda_k(n,4N).$$

We define functions $g_k(\chi_l,4m,4N)(z)$ $(m|N)$ and $g_k(\chi_l,m,4N)(z)$ $(m|N)$ as follows: For $k \geqslant 5$,

$$g_k(\chi_l,4N,4N)(z) = 1+\sum_{n=1}^{\infty}\lambda_k'(ln,4N)\prod_{p|2N}(A_k(p,ln)-\eta_p)(ln)^{k/2-1}q^n,$$

$$g_k(\chi_l,4m,4N)(z) = \sum_{n=1}^{\infty}\lambda_k'(ln,4N)\prod_{p|2m}(A_k(p,ln)-\eta_p)(ln)^{k/2-1}q^n, \quad \forall\ N \neq m|N,$$

$$g_k(\chi_l,m,4N)(z) = \sum_{n=1}^{\infty}\lambda_k'(ln,4N)\prod_{p|m}(A_k(p,ln)-\eta_p)(ln)^{k/2-1}q^n, \quad \forall\ m|N,$$

where $q = e(z) = \mathrm{e}^{2\pi \mathrm{i}z}$, $\eta_2 = \dfrac{1+(-1)^\lambda \mathrm{i}}{2^k-4}$ and $\eta_p = \dfrac{p-1}{p(p^{k-2}-1)}$ for $p \neq 2$.

**Lemma 7.1** *Let $k$ be a positive odd integer, $n$ a positive integer and $p$ a prime, $D$ a square free positive integer and $m|D$. Then*

(I) $\lambda_k(n,4m) = \lambda_k(n,4D)\prod_{p|D/m}(1+A_k(p,n))$,

(II) $A_k(p,p^2n)-\eta_p = p^{k-2}(A_k(p,n)-\eta_p)$.

**Proof** The second equality is clear from the definition of $A_k(p,n)$. The first equality can be proved from the definition of $\lambda_k(n,4D)$ and the properties of $\beta_k(n,\chi_D,4D)$. We omit the details. □

**Theorem 7.1** *Let $k \geqslant 5$ be an odd positive integer, $D$ a square-free positive odd integer and $l$ a divisor of $D$. Then the functions*

$$\{g_k(\chi_l,4m,4D), g_k(\chi_l,m,4D) \mid m|D\}$$

*constitute a basis of $\mathcal{E}(4D,k/2,\chi_l)$ and are eigenfunctions for all Hecke operators, and*

$$g_k(\chi_l,j,4D)(z)|T(p^2) = \begin{cases} g_k(\chi_l,j,4D)(z), & \text{if } p|j,\\ p^{k-2}g_k(\chi_l,j,4D)(z), & \text{if } p|8D/j,\\ (1+p^{k-2})g_k(\chi_l,j,4D)(z), & \text{if } p\nmid 2D,\end{cases}$$

*where $j = m$ or $4m, m|D$.*

**Proof** By the definition of a Hecke operator, we know that $g_k(\chi_l, j, 4D) = g_k(\text{id.}, j, 4D)|T(l)$. Hence we only need to prove Theorem 7.1 for $l = 1$. We first show that $g_k(\text{id.}, j, 4D)$ belongs to $\mathcal{E}(4D, k/2, \text{id.})$.

By Chapter 2, for square free odd positive integer $D$, the following functions belong to $\mathcal{E}(4D, k/2, \text{id.})$

$$E_k(\text{id.}, 4D)(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4D)} j(\gamma, z)^{-k}$$
$$= 1 + \sum_{n=1}^{\infty} \lambda'_k(n, 4D) \prod_{p|2D} A_k(p, n) n^{k/2-1} q^n,$$
$$E'_k(\chi_D, 4D)(z) = z^{-k/2} E_k(\chi_D, 4D)\left(-\frac{1}{4Dz}\right)$$
$$= \sum_{n=1}^{\infty} \lambda'_k(n, 4D) n^{k/2-1} q^n,$$

We introduce the following functions:

$$F_k(4D)(z) = E_k(\text{id.}, 4D)(z) = 1 + \sum_{n=1}^{\infty} \lambda'_k(n, 4D) \prod_{p|2D} A_k(p, n) n^{k/2-1} q^n,$$
$$F_k(4m)(z) = \sum_{n=1}^{\infty} \lambda'_k(n, 4D) \prod_{p|2m} A_k(p, n) n^{k/2-1} q^n, \tag{7.1}$$
$$F_k(m) = \sum_{n=1}^{\infty} \lambda'_k(n, 4D) \prod_{p|m} A_k(p, n) n^{k/2-1} q^n.$$

Since Lemma 7.1, we see that for any $m|D$,

$$E_k(\text{id.}, 4m)(z) = 1 + \sum_{n=1}^{\infty} \lambda'_k(n, 4m) \prod_{p|2m} A_k(p, n) n^{k/2-1} q^n$$
$$= 1 + \sum_{n=1}^{\infty} \lambda'_k(n, 4D) \prod_{p|2m} A_k(p, n) \prod_{p|D/m} (A_k(p, n) + 1) n^{k/2-1} q^n, \tag{7.2}$$
$$E'_k(\chi_m, 4m)(z) = \sum_{n=1}^{\infty} \lambda'_k(n, 4m) n^{k/2-1} q^n$$
$$= \sum_{n=1}^{\infty} \lambda'_k(n, 4D) \prod_{p|D/m} (A_k(p, n) + 1) n^{k/2-1} q^n.$$

Because

$$\prod_{p|2m} A_k(p, n) = \prod_{p|2m} A_k(p, n) \prod_{p|D/m} (1 + A_k(p, n) - A_k(p, n))$$
$$= \sum_{d|D/m} \mu(d) \prod_{p|2md} A_k(p, n) \prod_{p|D/(md)} (1 + A_k(p, n)),$$

$$\prod_{p|m} A_k(p,n) = \sum_{d|m} \mu(d) \prod_{p|m/d} (1 + A_k(p,n)). \tag{7.3}$$

By (7.1)–(7.3), we see that

$$F_k(4m) = \sum_{d|D/m} \mu(d) E_k(\text{id.}, 4md) \in E_{k/2}(4D, \text{id.}),$$
$$F_k(m) = \sum_{d|m} \mu(d) E'_k(\chi_{dD/m}, 4dD/m) \in E_{k/2}(4D, \text{id.}).$$

But

$$\begin{aligned} g_k(\text{id.}, 4m, 4D) &= \sum_{d|m} \mu(d) \prod_{p|d} \eta_p F_k(4m/d) - \sum_{d|m} \mu(d) \prod_{p|2d} \eta_{2p} F_k(m/d), \\ g_k(\text{id.}, m, 4D) &= \sum_{d|m} \mu(d) \prod_{p|d} \eta_p F_k(m/d), \end{aligned} \tag{7.4}$$

which implies that $g_k(\text{id.}, 4m, 4D)$ and $g_k(\text{id.}, m, 4D)$ belong to $\mathcal{E}(4D,\ k/2, \text{id.})$.

We now want to prove the equalities in Theorem 7.1. We recall the definition of Hecke operators: for any $f(z) = \sum_{n=0}^{\infty} a(n) e(nz) \in G(4D, k/2, \omega)$, we have that $f(z)|\mathrm{T}(p^2) = \sum_{n=0}^{\infty} b(n) e(nz)$ where

$$b(n) = a(p^2 n) + \omega(p) \left( \frac{(-1)^{\lambda} n}{p} \right) p^{\lambda - 1} a(n) + \omega(p^2) p^{k-2} a(n/p^2),$$

where $a(n/p^2) = 0$ if $p^2 \nmid n$.

In particular, if $p|4D$, then $b(n) = a(p^2 n)$. It is clear that $\beta_k(p^2 n, \chi_D, 4D) = \beta_k(n, \chi_D, 4D)$ for any $p|2D$. So the first two equalities in Theorem 7.1 can easily be deduced from Lemma 7.1 (II) and the obvious fact that $A_k(p, qn) = A_k(p, n)$ if $p \nmid q$. So we only need to prove the third equality. So suppose that $q$ is a prime with $q \nmid 2D$. We consider the action of $\mathrm{T}(q^2)$ on $f = g_k(\text{id.}, 4m, 4D)$. Denote

$$a(n) = \lambda'_k(n, 4D) \prod_{p|2m} (A_k(p,n) - \eta_p) n^{k/2-1}$$

and

$$f|\mathrm{T}(q^2) = \sum_{n=0}^{\infty} b(n) e(nz).$$

Since $q \nmid 2D$, then $A_k(p, q^2 n) = A_k(p, n)$ and

$$L_{4D}(\lambda, \chi_{(-1)^{\lambda} l q^2 n}) \prod_{p|2m} (A_k(p, lnq^2) - \eta_p) = L_{4D}(\lambda, \chi_{(-1)^{\lambda} n}) \prod_{p|2m} (A_k(p, ln) - \eta_p).$$

Now consider the term $\beta_k(ln, \chi_D, 4D)$. Denote $ln = \tau\sigma^2$ with $\tau$ a square free positive integer. Let $\nu_p(m)$ be the valuation of $m$ with respect to $p$. Then we have that

$$\beta_k(\tau\sigma^2, \chi_D, 4D) = \sum_{\substack{(ab)^2|\tau\sigma^2,(ab,2D)=1\\ a,b \text{ positive integers}}} \mu(a)\left(\frac{(-1)^\lambda ln}{a}\right)a^{-\lambda}b^{-k+2},$$
$$= \prod_{p\nmid 2D, p|\tau} \sum_{t=0}^{(\nu_p(\tau\sigma^2)-1)/2} p^{(-k+2)t}$$
$$\times \prod_{p\nmid 2D\tau, p|\sigma}\left(\sum_{t=0}^{\nu_p(\tau\sigma^2)/2} p^{(-k+2)t} - \chi_{(-1)^\lambda ln}(p)p^{-\lambda}\sum_{t=0}^{\nu_p(\tau\sigma^2)/2-1} p^{(-k+2)t}\right).$$

Therefore, if $\nu_q(ln) = 0$, i.e., $q \nmid ln$, then

$$\beta_k(\tau\sigma^2 q^2, \chi_D, 4D) = (1 + q^{-k+2} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda})\beta_k(\tau\sigma^2, \chi_D, 4D). \quad (7.5)$$

If $q|\tau$, then

$$\beta_k(\tau\sigma^2 q^2, \chi_D, 4D) = \left(\sum_{t=0}^{(\nu_q(\tau\sigma^2)+1)/2} q^{(-k+2)t}\right)\left(\sum_{t=0}^{(\nu_q(\tau\sigma^2)-1)/2} q^{(-k+2)t}\right)^{-1}\beta_k(\tau\sigma^2, \chi_D, 4D). \quad (7.6)$$

If $q \nmid \tau$, $q|\sigma$, then

$$\beta_k(\tau\sigma^2 q^2, \chi_D, 4D) = \left(\sum_{t=0}^{\nu_q(\tau\sigma^2)/2+1} q^{(-k+2)t} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda}\sum_{t=0}^{\nu_q(\tau\sigma^2)/2} q^{(-k+2)t}\right)$$
$$\times\left(\sum_{t=0}^{\nu_q(\tau\sigma^2)/2} q^{(-k+2)t} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda}\sum_{t=0}^{\nu_q(\tau\sigma^2)/2-1} q^{(-k+2)t}\right)^{-1}$$
$$\times\beta_k(\tau\sigma^2, \chi_D, 4D)$$

$$a(n) = \lambda'_k(n, 4D)\prod_{p|2m}(A_k(p,n) - \eta_p)n^{k/2-1}$$
$$= \frac{(-2\pi \mathrm{i})^{k/2}}{\Gamma(k/2)}\frac{L_{4D}(\lambda, \chi_{(-1)^\lambda ln})}{L_{4D}(2\lambda, \mathrm{id.})}\beta_k(ln, \chi_D, 4D)$$
$$\times\prod_{p|2m}(A_k(p, ln) - \eta_p)(ln)^{k/2-1}. \quad (7.7)$$

Hence we know that the coefficient $b(n)$ of $f|\mathrm{T}(q^2)$ is

(1) If $\nu_q(ln) = 0$, then by equality (7.5)

$$b(n) = a(q^2 n) + \chi_{(-1)^\lambda l}(q)\left(\frac{n}{q}\right)q^{\lambda-1}a(n) + q^{k-2}a(n/q^2)$$
$$= (1 + q^{-k+2} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda})q^{k-2}a(n) + \chi_{(-1)^\lambda ln}(q)q^{\lambda-1}a(n)$$
$$= (1 + q^{k-2})a(n).$$

(2) If $\nu_q(ln) = 1$, i.e., $q|\tau$, $q \nmid \sigma$, we see by equality (7.6) that

$$\begin{aligned} b(n) &= a(q^2n) + \chi_{(-1)^\lambda l\tau}(q)q^{\lambda-1}a(n) + q^{k-2}a(n/q^2) \\ &= a(q^2n) + \chi_{(-1)^\lambda l\tau}(q)q^{\lambda-1}a(n) \\ &= a(q^2n) = (1+q^{-k+2})q^{k-2}a(n) = (1+q^{k-2})a(n). \end{aligned}$$

(3) If $q|\tau$, $q|\sigma$, then $\nu_q(ln) \geqslant 3$, we have by equality (7.6),

$$\begin{aligned} b(n) &= a(q^2n) + \chi_{(-1)^\lambda l\tau}(q)q^{\lambda-1}a(n) + q^{k-2}a(n/q^2) = a(q^2n) + q^{k-2}a(n/q^2) \\ &= \left(\sum_{s=0}^{(\nu_q(ln)+1)/2} q^{(-k+2)s}\right)\left(\sum_{s=0}^{(\nu_q(ln)-1)/2} q^{(-k+2)s}\right)^{-1} q^{k-2}a(n) \\ &\quad + q^{k-2}\left(\sum_{s=0}^{(\nu_q(ln)-3)/2} q^{(-k+2)s}\right)\left(\sum_{s=0}^{(\nu_q(ln)-1)/2} q^{(-k+2)s}\right)^{-1} a(n)q^{-(k-2)} \\ &= \left(q^{k-2}\sum_{s=0}^{(\nu_q(ln)+1)/2} q^{(-k+2)s} + \sum_{s=0}^{(\nu_q(ln)-3)/2} q^{(-k+2)s}\right) \\ &\quad \cdot \left(\sum_{s=0}^{(\nu_q(ln)-1)/2} q^{(-k+2)s}\right)^{-1} a(n) \\ &= \left(q^{k-2}\sum_{s=0}^{(\nu_q(ln)-1)/2} q^{(-k+2)s} + q^{(-k+2)(\nu_q(ln)-1)/2}\right. \\ &\quad \left. + \sum_{s=0}^{(\nu_q(ln)-1)/2} q^{(-k+2)s} - q^{(-k+2)(\nu_q(ln)-1)/2}\right) \\ &\quad \cdot \left(\sum_{s=0}^{(\nu_q(ln)-1)/2} q^{(-k+2)s}\right)^{-1} a(n) \\ &= (1+q^{k-2})a(n). \end{aligned}$$

Finally, if $q \nmid \tau, q|\sigma$, then by equality (7.7), we have that

$$\begin{aligned} b(n) &= a(q^2n) + \chi_{(-1)^\lambda l\tau}(q)q^{\lambda-1}a(n) + q^{k-2}a(n/q^2) \\ &= q^{k-2}a(n)\left(\sum_{t=0}^{\nu_q(ln)/2+1} q^{(-k+2)t} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda}\sum_{t=0}^{\nu_q(ln)/2} q^{(-k+2)t}\right) \\ &\quad \cdot \left(\sum_{t=0}^{\nu_q(ln)/2} q^{(-k+2)t} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda}\sum_{t=0}^{\nu_q(ln)/2-1} q^{(-k+2)t}\right)^{-1} \end{aligned}$$

$$+ a(n)\left(\sum_{t=0}^{\nu_q(ln)/2-1} q^{(-k+2)t} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda}\sum_{t=0}^{\nu_q(ln)/2-2} q^{(-k+2)t}\right)$$
$$\cdot\left(\sum_{t=0}^{\nu_q(ln)/2} q^{(-k+2)t} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda}\sum_{t=0}^{\nu_q(ln)/2-1} q^{(-k+2)t}\right)^{-1}$$
$$= q^{k-2}a(n)\left(1 + (q^{(-k+2)(\nu_q(ln)/2+1)} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda+(-k+2)\nu_q(ln)/2}\right)$$
$$\cdot\left(\sum_{t=0}^{\nu_q(ln)/2} q^{(-k+2)t} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda}\right)\left(\sum_{t=0}^{\nu_q(ln)/2-1} q^{(-k+2)t}\right)^{-1}$$
$$+ a(n)\left(1 - (q^{(-k+2)\nu_q(ln)/2} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda+(-k+2)(\nu_q(ln)/2-1)}\right)$$
$$\cdot\left(\sum_{t=0}^{\nu_q(ln)/2} q^{(-k+2)t} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda}\sum_{t=0}^{\nu_q(ln)/2-1} q^{(-k+2)t}\right)^{-1}$$
$$= (1 + q^{k-2})a(n).$$

Hence we have proved that for any prime $q \nmid 2D$, $g(\chi_l, 4m, 4D)|\mathrm{T}(q^2) = (1+q^{k-2})g(\chi_l, 4m, 4D)$. Similarly, we can show that for any $q \nmid 2D$, $g(\chi_l, m, 4D)|\mathrm{T}(q^2) = (1 + q^{k-2})g(\chi_l, m, 4D)$.

Since the functions in Theorem 7.1 are eigenfunctions of Hecke operators with different eigenvalues, they are linearly independent. Thus they constitute a basis of $\mathcal{E}(4D,\ k/2,\ \chi_l)$ since the number of the functions is equal to the dimension of $\mathcal{E}(4D,\ k/2,\ \chi_l)$.

This completes the proof of Theorem 7.1. □

**Theorem 7.2** *Let $k \geqslant 5$ be an odd positive integer, $D$ a square-free positive odd integer, $m, l$ be divisors of $D$, $\alpha$ be a divisor of $m$, $\delta_k = 1$ or $-1$ according to $k \equiv 1$ or $-1 \pmod 4$ respectively. Then*

$$V(g_k(\chi_l, 4m, 4D), 1/\alpha) = -\frac{1+\mathrm{i}^{-\delta_k}}{2^k - 4}\mu(m/\alpha)\eta_{m/\alpha}l^{k/2-1}(l,\alpha)^{-k/2+1}\varepsilon^{\delta_k}_{\alpha/(l,\alpha)}\left(\frac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right).$$
$$V(g(\chi_l, 4m, 4D), 1/(4\alpha)) = \mu(m/\alpha)\eta_{m/\alpha}l^{k/2-1}(l,\alpha)^{-k/2+1}\varepsilon^{-\delta_k}_{l/(l,\alpha)}\left(\frac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right).$$
$$V(g(\chi_l, 4m, 4D), p) = 0,\ \text{if } p \neq 1/\alpha \text{ or } 1/4\alpha (\alpha|D), p \text{ a cusp point.}$$
$$V(g(\chi_l, m, 4D), 1/\alpha) = \mathrm{i}^{-\delta_k}\mu(m/\alpha)\eta_{m/\alpha}l^{k/2-1}(l,\alpha)^{-k/2+1}\varepsilon^{\delta_k}_{\alpha/(l,\alpha)}\left(\frac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right).$$
$$V(g(\chi_l, m, 4D), p) = 0, \quad \text{if } p \neq 1/\alpha\ (\alpha|D),$$

*where $p$ is a cusp point and $V(f,p)$ is the value of $f$ at the cusp point $p$, and* $\eta_\alpha = \prod_{p|\alpha}\eta_p$.

**Proof** In order to calculate the values of functions at cusp points, we first remember the definition of the value of a function at a cusp point. Let $f(z) \in G(N,\ k/2,\ \chi_l)$, and $s = d/c$ be a cusp point of $\Gamma_0(N)$. Let $\rho = \begin{pmatrix} a & b \\ -c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, then $\rho(s) = \mathrm{i}\infty$. We call the constant term of the Fourier expansion at $z = \mathrm{i}\infty$ of $f|\rho^{-1}$ the value of $f$ at the cusp point $s$. Denote it by $V(f, s)$. For $c \neq 0$, we have

$$\begin{aligned} V(f,s) &= \lim_{z\to i\infty} f\left(\frac{dz-b}{cz+a}\right)(cz+a)^{-k/2} \\ &= \lim_{z\to i\infty} f(-c^{-1}(cz+a)^{-1} + dc^{-1})(cz+a)^{-k/2} \\ &= \lim_{\tau\to 0} f(\tau + dc^{-1})(-c\tau)^{k/2}. \end{aligned} \tag{7.8}$$

In particular, for $s = 1/N$, we see that $V(f, 1/N) = V(f, \mathrm{i}\infty) = \lim_{z\to i\infty} f(z)$. □

An obvious, but useful fact is

**Lemma 7.2** *Let $f \in G(N, k/2, \omega)$. Suppose cusp point $s_1 = d_1/c_1$ and $s_2 = d_2/c_2$ are equivalent for the group $\Gamma_0(N)$,* i.e., *there exists $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ such that $\rho(s_1) = s_2$, then*

$$V(f, s_2) = \bar{\omega}\chi_c(d)\varepsilon_d^{-k} V(f, s_1).$$

A classical result for the values of Eisenstein series $E_k(\omega, N)(z)$, $E'_k(\omega, N)(z)$ is the following Lemma 7.3, which can be showed by the results in Chapter 2 and Lemma 7.2. Now we denote $S(N)$ a complete set of representatives of equivalence classes of cusp points for the group $\Gamma_0(N)$. In fact we can choose

$$S(N) = \{d/c \mid \ c|N, d \in (\mathbb{Z}/(c, N/c)\mathbb{Z})^* \text{ and } (d, c) = 1\}.$$

**Lemma 7.3** *Let $k \geqslant 5$ be an odd, $\omega$ a character modulo $N$. Then we have*

(1) $V(E'_k(\omega, N), 1) = \mathrm{i}^{-k}$, *and for any $d/c \in S(N)$ with $c \neq 1$, $V(E'_k(\omega, N), d/c) = 0$;*

(2) $V(E_k(\omega, N), i\infty) = 1$, *and for any $d/c \in S(N)$ with $c \neq N$, $V(E_k(\omega, N), d/c) =$ 0.*

We now return to our proof of Theorem 7.2. We need the following:

**Lemma 7.4** *Let $D$ be square free odd positive integer, $m, l$, and $\beta$ are divisors of $D$, $\alpha$ a divisor of $m$. And suppose that $f \in G(8D, k/2, \chi_l)$ satisfies*

$$f|\mathrm{T}(p^2) = f \quad \textit{for all prime } p|m,$$
$$f|\mathrm{T}(p^2) = p^{k-2} f \quad \textit{for all prime } p|Dm^{-1}.$$

*Then we have*

$$V(f, 1/\alpha) = \mu(\alpha)\eta_\alpha^{-1}(\alpha, l)^{-k/2+1}\varepsilon_{\alpha/(\alpha,l)}^{\delta_k}\left(\frac{l/(\alpha,l)}{\alpha/(\alpha,l)}\right)V(f, 1),$$

$$V(f,1/(4\alpha)) = \mu(\alpha)\eta_\alpha^{-1}(\alpha,l)^{-k/2+1}\varepsilon_{l/(\alpha,l)}^{\delta_k}\varepsilon_l^{-1}\left(\frac{\alpha/(\alpha,l)}{l/(\alpha,l)}\right)V(f,1/4),$$
$$V(f,1/(8\alpha)) = \mu(\alpha)\eta_\alpha^{-1}(\alpha,l)^{-k/2+1}\varepsilon_{l/(\alpha,l)}^{\delta_k}\varepsilon_l^{-1}\left(\frac{2}{(\alpha,l)}\right)\left(\frac{\alpha/(\alpha,l)}{l/(\alpha,l)}\right)V(f,1/8),$$

*where* $\eta_\alpha = \prod_{p|\alpha}\eta_p$, $\delta_k = 1$ *or* $-1$ *according to* $k \equiv 1$ *or* $-1 \pmod 4$ *respectively. And for* $(\beta, D/m) \neq 1, r = 0,1,2,3$, *we have that* $V(f,1/(2^r\beta)) = 0$.

**Proof** We only prove the Lemma 7.4 for the case $k \equiv 3 \pmod 4$. For the case $k \equiv 1 \pmod 4$ it can be proved by a similar method. We first prove the last result. Suppose $p$ prime, $p|(\beta, D/m)$. By our assumption in the lemma we have $f|\mathrm{T}(p^2) = p^{k-2}f$ and by the definition of Hecke operators, we see that

$$p^{k-2}f\left(z+\frac{1}{2^r\beta}\right) = p^{-2}\sum_{b=1}^{p^2} f\left(\frac{z}{p^2}+\frac{1+2^r\beta b}{2^r\beta p^2}\right).$$

Since $(1+2^r\beta b, 2^r\beta p^2) = 1$, the rational number $\frac{1+2^r\beta b}{2^r\beta p^2}$ is a cusp point. By equality (7.8), we know

$$p^{k-2}V\left(f,\frac{1}{2^r\beta}\right) = p^{-2}\sum_{b=1}^{p^2} V\left(f,\frac{1+2^r\beta b}{2^r\beta p^2}\right). \tag{7.9}$$

Since $(2^r\beta p^2, 8D) = 2^r\beta$ and $(2^r\beta, 8D/(2^r\beta)) = 1$ or $2$ according to $r = 0$, $3$ or $r = 1$, $2$, we know that the cusp point $\frac{1+2^r\beta b}{2^r\beta p^2}$ is equivalent to the cusp point $\frac{1}{2^r\beta}$ for the group $\Gamma_0(8D)$. Therefore there exists a matrix $\begin{pmatrix} a & e \\ c & d \end{pmatrix} \in \Gamma_0(8D)$ such that

$$\begin{pmatrix} a & e \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2^r\beta \end{pmatrix} = \begin{pmatrix} 1+2^r\beta b \\ 2^r\beta p^2 \end{pmatrix}.$$

Hence $a+2^r\beta e = 1+2^r\beta b$, $c+2^r\beta d = 2^r\beta p^2$. Noting $ad-ce=1$ and $8D|c$, we have that $a \equiv d \equiv 1 \pmod{2^r\beta}$, and $d \equiv p^2 \pmod{8D/(2^r\beta)}$. This shows that for $r=0$, 1, 2, 3, we have $\varepsilon_d = 1$ and

$$\left(\frac{c}{d}\right) = \left(\frac{2^r\beta p^2 - 2^r\beta d}{d}\right) = \left(\frac{2^r\beta}{d}\right) = 1.$$

By Lemma 7.2, we see

$$V\left(f,\frac{1+2^r\beta b}{2^r\beta p^2}\right) = V\left(f,\frac{1}{2^r\beta}\right).$$

By equality (7.9), we obtain

$$\begin{aligned}p^{k-2}V(f,1/(2^r\beta)) &= p^{-2}\sum_{b=1}^{p^2} V\left(f, \frac{1+2^r\beta b}{2^r\beta p^2}\right)\\ &= p^{-2}\sum_{b=1}^{p^2} V(f,1/(2^2\beta)) = V(f,1/(2^r\beta)),\end{aligned}$$

which implies that $V(f,1/(2^r\beta))=0$. Now we begin to prove the first equality in Lemma 7.4. It is clear that the equality holds for $\alpha=1$. We shall complete the proof by induction on the number of prime divisors of $\alpha$. We assume that the equality holds for $\alpha$ with $\alpha \neq m$. We must prove that the equality holds for $V(f,1/(\alpha p))$ with $p$ prime and satisfying $\alpha p|m$. Since $f|\mathrm{T}(p^2)=f$, we get

$$f(z+1/\alpha) = p^{-2}\sum_{b=1}^{p^2} f\left(\frac{z}{p^2}+\frac{1+b\alpha}{p^2\alpha}\right).$$

Because it is possible that $p|1+b\alpha$, in general the rational number $\dfrac{1+b\alpha}{p^2\alpha}$ is not reduced. We have to cancel the greatest common divisor in order to obtain a cusp point. Now there exists a unique integer $b_1$ such that $1\leqslant b_1\leqslant p$, $1+\alpha b_1=pt_1$. Similarly, there exists a unique integer $b_2$ such that $1\leqslant b_2\leqslant p^2$, $1+b_2\alpha=p^2t_2$, where $t_1$, $t_2$ are integers. Hence by the definition of values of a modular function at cusp points and equality (7.8), we obtain

$$\begin{aligned}V(f,1/\alpha) = p^{-2} &\sum_{\substack{1\leqslant b\leqslant p^2\\ p\nmid 1+b\alpha}} V\left(f,\frac{1+b\alpha}{p^2\alpha}\right)\\ &+p^{k/2-2}\sum_{\substack{1\leqslant b\leqslant p\\ p\nmid t_1+b\alpha}} V\left(f,\frac{t_1+b\alpha}{p\alpha}\right)+p^{k-2}V(f,t_2/\alpha). \qquad (7.10)\end{aligned}$$

The cusp points $\dfrac{1+b\alpha}{p^2\alpha}$ $(p\nmid 1+b\alpha)$, $\dfrac{t_1+b\alpha}{p\alpha}$ $(p\nmid t_1+b\alpha)$ and $t_2/\alpha$ are equivalent to $\dfrac{1}{p\alpha}$, $\dfrac{1}{p\alpha}$ and $1/\alpha$ under the group $\Gamma_0(8D)$ respectively. We now consider the case $p\nmid l$. Let $\begin{pmatrix} a & e\\ c & d\end{pmatrix}\in\Gamma_0(8D)$ such that

$$\begin{pmatrix} a & e\\ c & d\end{pmatrix}\begin{pmatrix} 1\\ p\alpha\end{pmatrix} = \begin{pmatrix} 1+b\alpha\\ p^2\alpha\end{pmatrix}, \qquad (7.11)$$

which deduces that $a+ep\alpha=1+b\alpha, c+dp\alpha=p^2\alpha$. But $ad-ce=1$. So we obtain that $d\equiv a\equiv 1 \pmod{\alpha}$, $d\equiv p\left(\bmod\ \dfrac{c}{p\alpha}\right)$. Since $8D|c, p\nmid l$, then $d\equiv p \pmod{4l/(l,\alpha)}$. By Lemma 7.2, we obtain

$$\begin{aligned}
V\left(f, \frac{1+b\alpha}{p^2\alpha}\right) &= \left(\frac{lc}{d}\right)\varepsilon_d V(f, 1/(p\alpha)) \\
&= \left(\frac{l/(l,\alpha)}{d}\right)\left(\frac{c/(l,\alpha)}{d}\right)\varepsilon_d V(f, 1/(p\alpha)) \\
&= \left(\frac{l/(l,\alpha)}{p}\right)\left(\frac{d}{\alpha/(l,\alpha)}\right)\varepsilon_{d\alpha/(l,\alpha)}\varepsilon^{-1}_{\alpha/(\alpha,l)}\varepsilon_d^{-1} V(f, 1/(p\alpha)) \\
&= \left(\frac{l/(l,\alpha)}{p}\right)\varepsilon_{p\alpha/(l,\alpha)}\varepsilon^{-1}_{\alpha/(\alpha,l)} V(f, 1/(p\alpha)). \qquad (7.12)
\end{aligned}$$

Similarly, we can deduce

$$\begin{cases} V\left(f, \dfrac{t_1+b\alpha}{p\alpha}\right) = \left(\dfrac{t_1+b\alpha}{p}\right)\left(\dfrac{p}{\alpha/(l,\alpha)}\right) V(f, 1/(p\alpha)), \\ V(f, t_2/\alpha) = V(f, 1/\alpha). \end{cases} \qquad (7.13)$$

Inserting equalities (7.12) and (7.13) into (7.10), we see that the second sum in equality (7.10) is zero, and hence

$$\begin{aligned}
V(f, 1/\alpha) &= p^{-2}\sum_{\substack{1\leqslant b\leqslant p^2 \\ p\nmid 1+b\alpha}} \varepsilon_{\alpha p/(l,\alpha)}\varepsilon^{-1}_{\alpha/(l,\alpha)}\left(\frac{l/(l,\alpha)}{p}\right) V(f, 1/(p\alpha)) + p^{k-2}V(f, 1/\alpha) \\
&= p^{-2}(p^2-p)\varepsilon_{\alpha p/(l,\alpha)}\varepsilon^{-1}_{\alpha/(l,\alpha)}\left(\frac{l/(l,\alpha)}{p}\right) V(f, 1/(p\alpha)) + p^{k-2}V(f, 1/\alpha),
\end{aligned}$$

which implies, by the induction assumption,

$$\begin{aligned}
V(f, 1/(p\alpha)) &= -\frac{(p^{k-2}-1)p}{p-1}\varepsilon^{-1}_{\alpha p/(l,\alpha)}\varepsilon_{\alpha/(l,\alpha)}\left(\frac{l/(l,\alpha)}{p}\right) V(f, 1/\alpha) \\
&= -\eta_p^{-1}\varepsilon^{-1}_{\alpha p/(l,\alpha)}\varepsilon_{\alpha/(l,\alpha)}\left(\frac{l/(l,\alpha)}{p}\right) V(f, 1/\alpha) \\
&= -\eta_p^{-1}\varepsilon^{-1}_{\alpha p/(l,\alpha)}\varepsilon_{\alpha/(l,\alpha)}\left(\frac{l/(l,\alpha)}{p}\right)\mu(\alpha)\eta_\alpha^{-1}(\alpha, l)^{-k/2+1}\varepsilon^{-1}_{\alpha/(l,\alpha)} \\
&\quad \cdot\left(\frac{l/(l,\alpha)}{\alpha/(\alpha,l)}\right) V(f, 1) \\
&= \mu(p\alpha)\eta_{p\alpha}^{-1}(p\alpha, l)^{-k/2+1}\varepsilon^{-1}_{\alpha p/(l,\alpha)}\left(\frac{l/(l,p\alpha)}{p\alpha/(p\alpha,l)}\right) V(f, 1),
\end{aligned}$$

where we assumed $p \nmid l$. Therefore for $p \nmid l$ we have proved the result. Now suppose $p|l$. In this case, from equality (7.11), we see

$$d \equiv a \equiv 1 \pmod{\alpha}, \quad d \equiv p \pmod{4l/(l,p\alpha)}, \quad (1+b\alpha)d \equiv 1 \pmod{p}.$$

Hence by Lemma 7.2,

$$V\left(f, \frac{1+b\alpha}{p^2\alpha}\right) = \left(\frac{lc}{d}\right)\varepsilon_d V(f, 1/(p\alpha))$$

$$= \left(\frac{l/(l,p\alpha)}{d}\right)\left(\frac{p\alpha/(l,\alpha)}{d}\right)\varepsilon_d V(f,1/(p\alpha))$$
$$= \left(\frac{l/(l,p\alpha)}{p}\right)\left(\frac{d}{p\alpha/(l,\alpha)}\right)\varepsilon_{\alpha/(l,\alpha)}\varepsilon^{-1}_{p\alpha/(l,\alpha)} V(f,1/(p\alpha))$$
$$= \varepsilon_{\alpha/(l,\alpha)}\varepsilon^{-1}_{p\alpha/(l,\alpha)}\left(\frac{(1+b\alpha)l/(l,p\alpha)}{p}\right) V(f,1/(p\alpha)).$$

Similarly we can show

$$V\left(f,\frac{t_1+b\alpha}{p\alpha}\right) = \left(\frac{p}{\alpha/(\alpha,l)}\right) V(f,1/(p\alpha)),$$
$$V(f,t_2/\alpha) = V(f,1/\alpha).$$

Inserting these results into the equality (7.10), we get that the first sum in the equality is zero, and hence

$$V(f,1/\alpha) = p^{k/2-2}\sum_{\substack{1\leqslant b\leqslant p\\ p\nmid t_1+b\alpha}}\left(\frac{p}{\alpha/(\alpha,l)}\right)V(f,1/(p\alpha)) + p^{k-2}V(f,1/\alpha)$$
$$= p^{k/2-2}(p-1)\left(\frac{p}{\alpha/(\alpha,l)}\right)V(f,1/(p\alpha)) + p^{k-2}V(f,1/\alpha),$$

which implies, by the induction assumption,

$$V(f,1/(p\alpha)) = -\frac{p^{k-2}-1}{p^{k/2-2}(p-1)}\left(\frac{p}{\alpha/(\alpha,l)}\right)V(f,1/\alpha)$$
$$= -\eta_p^{-1}p^{-k/2+1}\left(\frac{p}{\alpha/(\alpha,l)}\right)V(f,1/\alpha)$$
$$= -\eta_p^{-1}p^{-k/2+1}\left(\frac{p}{\alpha/(\alpha,l)}\right)\mu(\alpha)\eta_\alpha^{-1}(\alpha,l)^{-k/+1}\varepsilon^{-1}_{\alpha/(\alpha,l)}\left(\frac{l/(\alpha,l)}{\alpha/(\alpha,l)}\right)V(f,1)$$
$$= \mu(p\alpha)\eta_{p\alpha}^{-1}(p\alpha,l)^{-k/2+1}\varepsilon^{-1}_{p\alpha/(p\alpha,l)}\left(\frac{l/(p\alpha,l)}{p\alpha/(p\alpha,l)}\right)V(f,1),$$

where we assumed $p|l$. Hence for the case $p|l$ the first equality in the Lemma 7.4 holds. By induction, we know that this equality holds for any $\alpha|m$. The other two equalities in the Lemma 7.4 can be proved by a similar method which we omit. This completes the proof of Lemma 7.4.

Now we can prove Theorem 7.2 as follows.

Noting that $g_k(\text{id.},j,4D)|\mathrm{T}(l) = g_k(\chi_l,j,4D)$, we first consider the case $l=1$, i.e., $\chi_l = \text{id}$. For this case, by the equality (7.4), we have

$$g_k(\text{id.},4m,4D) = \sum_{d|m}\mu(d)\eta_d F_k(4m/d) - \sum_{d|m}\mu(d)\eta_{2d}F_k(m/d),$$

where

$$F_k(4D) = E_k(\text{id.},4D)(z),$$

$$F_k(4m) = \sum_{d|D/m} \mu(d)E_k(\text{id.}, 4md),$$
$$F_k(m) = \sum_{d|m} \mu(d)E'_k(\chi_{dD/m}, 4dD/m).$$

By Lemma 7.3, we have

$$V(F_k(4D), 1) = V(E_k(\text{id.}, 4D), 1) = 0,$$
$$V(F_k(4m), 1) = \sum_{d|D/m} \mu(d)V(E_k(\text{id.}, 4md), 1) = 0,$$
$$V(F_k(m), 1) = \sum_{d|m} \mu(d)V(E'_k(\chi_{dD/m}, 4dD/m), 1) = \sum_{d|m} \mu(d)\mathrm{i}^{-k}$$
$$= \mathrm{i}^{-k} \text{ or } 0 \text{ according to } m = 1 \text{ or } \neq 1.$$

Hence

$$V(g_k(\text{id.}, 4m, 4D), 1) = \sum_{d|m} \mu(d)\eta_d V(F_k(4m/d), 1) - \sum_{d|m} \mu(d)\eta_{2d} V(F_k(m/d), 1)$$
$$= -\mu(m)\eta_{2m}\mathrm{i}^{-k} = -\mathrm{i}^{-\delta_k}\mu(m)\eta_{2m}.$$

We now show that for any $\beta|D$, $V(g_k(\text{id.}, 4m, 4D), 1/(2\beta)) = 0$. In fact, since

$$g_k(\text{id.}, 4m, 4D)|\mathrm{T}(4) = g_k(\text{id.}, 4m, 4D),$$

we know

$$g_k(\text{id.}, 4m, 4D)(z + 1/(2\beta)) = 4^{-1}\sum_{b=1}^{4} g_k(\text{id.}, 4m, 4D)\left(z/4 + \frac{1+2\beta b}{8\beta}\right).$$

Because $(1+2\beta b, 8\beta) = 1$, $\dfrac{1+2\beta b}{8\beta}$ is a cusp point equivalent to the cusp point $1/(4\beta)$ for the group $\Gamma_0(4D)$. Therefore there exists a matrix $\begin{pmatrix} a_b & e_b \\ c_b & d_b \end{pmatrix} \in \Gamma_0(4D)$ such that

$$\begin{pmatrix} a_b & e_b \\ c_b & d_b \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1+2b\beta \\ 8\beta \end{pmatrix},$$

which implies that $a_b + 4\beta_b = 1 + 2\beta b$, $c_b + 4\beta d_b = 8\beta$, $d_b(1+2\beta b) \equiv 1 \pmod{4\beta}$. By equality (7.8) and Lemma 7.2, we obtain

$$V(g_k(\text{id.}, 4m, 4D), 1/2\beta) = 4^{-1}\sum_{b=1}^{4}\left(\frac{c_b}{d_b}\right)\varepsilon_{d_b}^{-k} V(g_k(\text{id.}, 4m, 4D), 1/4\beta)$$
$$= 4^{-1}\sum_{b=1}^{4}\left(\frac{8\beta - 4\beta d_b}{d_b}\right)\varepsilon_{1+2\beta b}^{-k} V(g_k(\text{id.}, 4m, 4D), 1/4\beta)$$
$$= 4^{-1}\sum_{b=1}^{4}\left(\frac{2\beta}{1+2\beta b}\right)\varepsilon_{1+2\beta b}^{-k} V(g_k(\text{id.}, 4m, 4D), 1/4\beta).$$

Since $\left(\dfrac{2\beta}{a+4\beta b}\right)=-\left(\dfrac{2\beta}{a}\right)$, it is clear that the above is equal to zero.

In order to compute the value of $g_k(\text{id.},4m,4D)$ at the cusp point $1/4$, we use the fact

$$g_k(\text{id.},4m,4D)|\mathrm{T}(4)=g_k(\text{id.},4m,4D),$$

Then

$$g_k(\text{id.},4m,4D)(z)=4^{-1}\sum_{b=1}^{4}g_k(\text{id.},4m,4D)(z/4+b/4).$$

Since $V(g_k(\text{id.},4m,4D),1/2)=0$, we see

$$\begin{aligned}&V(g_k(\text{id.},4m,4D),1)\\=&4^{-1}V(g_k(\text{id.},4m,4D),1/4)+4^{-1}V(g_k(\text{id.},4m,4D),3/4)\\&+2^{k-2}V(g_k(\text{id.},4m,4D),1)\end{aligned}\tag{7.14}$$

But the cusp point $3/4$ is equivalent to $1/4$ for the group $\Gamma_0(4D)$. Therefore there exists a matrix $\begin{pmatrix}a&e\\c&d\end{pmatrix}\in\Gamma_0(4D)$ such that

$$\begin{pmatrix}a&e\\c&d\end{pmatrix}\begin{pmatrix}1\\4\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix}.$$

Hence by Lemma 7.2, we have

$$\begin{aligned}V(g_k(\text{id.},4m,4D),3/4)&=\left(\frac{c}{d}\right)\varepsilon_d^{-\delta_k}V(g_k(\text{id.},4m,4D),1/4)\\&=\mathrm{i}^{-\delta_k}V(g_k(\text{id.},4m,4D),1/4).\end{aligned}$$

Combining with equality (7.14), we have

$$\begin{aligned}V(g_k(\text{id.},4m,4D),1/4)&=-\frac{2^k-4}{1+\mathrm{i}^{-\delta_k}}V(g_k(\text{id.},4m,4D),1)\\&=-\frac{2^k-4}{1+\mathrm{i}^{-\delta_k}}(-\mathrm{i}^{-\delta_k}\mu(m)\eta_{2m})\\&=\mu(m)\eta_m.\end{aligned}$$

By the above discussions, we know that

$$\begin{aligned}V(g_k(\text{id.},4m,4D),1)&=-\mathrm{i}^{-\delta_k}\mu(m)\eta_{2m}=-\frac{1+\mathrm{i}^{-\delta_k}}{2^k-4}\mu(m)\eta_m,\\V(g_k(\text{id.},4m,4D),1/4)&=\mu(m)\eta_m,\\V(g_k(\text{id.},4m,4D),1/2\beta)&=0,\ \text{for any }\beta|D.\end{aligned}$$

Hence by Theorem 7.1 and Lemma 7.4, we have proved that the first two equalities in Theorem 7.2 hold for $l=1$. Now we consider the function $g_k(\text{id.},m,4D)$. By Theorem 7.1, we have

$$g_k(\text{id.}, m, 4D)|\mathrm{T}(p^2) = g_k(\text{id.}, m, 4D) \quad \text{for all } p|m,$$
$$g_k(\text{id.}, m, 4D)|\mathrm{T}(p^2) = p^{k-2} g_k(\text{id.}, m, 4D) \quad \text{for all } p|2D/m.$$

In particular, we see

$$g_k(\text{id.}, m, 4D)|\mathrm{T}(4) = 2^{k-2} g_k(\text{id.}, m, 4D).$$

Noting that the cusp point $(1+4b\beta)/(16\beta)$ is equivalent to $1/(4\beta)$ for the group $\Gamma_0(4D)$, by equality (7.8) and Lemma 7.2, we see

$$\begin{aligned} 2^{k-2} V(g_k(\text{id.}, m, 4D), 1/4\beta) &= 4^{-1} \sum_{b=1}^{4} V\left( g_k(\text{id.}, m, 4D), \frac{1+4b\beta}{16\beta} \right) \\ &= V(g_k(\text{id.}, m, 4D), 1/4\beta), \end{aligned}$$

which implies that $V(g_k(\text{id.}, m, 4D), 1/4\beta) = 0$. In the same way, by equality (7.8), we have

$$2^{k-2} V(g_k(\text{id.}, m, 4D), 1/2\beta) = 4^{-1} \sum_{b=1}^{4} V\left( g_k(\text{id.}, m, 4D), \frac{1+2b\beta}{8\beta} \right).$$

Since the cusp point $(1+2b\beta)/(8\beta)$ is equivalent to $1/(4\beta)$ for the group $\Gamma_0(4D)$, the right hand side of the above equality is zero. So by Lemma 7.4, we only need to calculate the value of $g_k(\text{id.}, m, 4D)$ at the cusp point 1. But we know from the proof of Theorem 7.2,

$$g_k(\text{id.}, m, 4D) = \sum_{d|m} \mu(d) \eta_d F_k(m/d).$$

Noting that $V(F_k(m), 1) = \mathrm{i}^{-\delta_k}$ or 0 according to $m = 1$ or $m \neq 1$ respectively, we have

$$\begin{aligned} V(g_k(\text{id.}, m, 4D), 1) &= \sum_{d|m} \mu(d) \eta_d V(F_k(m/d), 1) \\ &= \mathrm{i}^{-\delta_k} \mu(m) \eta_m. \end{aligned}$$

Hence by Theorem 7.1 and Lemma 7.4, we have proved the claim for the values of $g_k(\text{id.}, m, 4D)$.

Now we consider the case $l \neq 1$. In this case we have

$$\begin{aligned} g_k(\chi_l, 4m, 4D)(z) &= g_k(\text{id.}, 4m, 4D)(z)|\mathrm{T}(l) \\ &= l^{-1} \sum_{b=1}^{l} g_k(\text{id.}, 4m, 4D) \left( \frac{z+b}{l} \right). \end{aligned}$$

Hence by the equality (7.8) and Lemma 7.2, we see

$$V(g_k(\chi_l, 4m, 4D), 1) = l^{-1} \sum_{d|l} d^{k/2} \sum_{\substack{b=1 \\ (b, l/d)=1}}^{l/d} V(g_k(\text{id.}, 4m, 4D), b/(ld^{-1}))$$

$$
\begin{aligned}
&= l^{-1}\sum_{d|l} d^{k/2}\sum_{b=1}^{l/d}\left(\frac{b}{ld^{-1}}\right)V(g_k(\mathrm{id.},4m,4D),1/(ld^{-1}))\\
&= l^{-1}\sum_{d|l} d^{k/2}V(g_k(\mathrm{id.},4m,4D),1/(ld^{-1}))\sum_{b=1}^{l/d}\left(\frac{b}{ld^{-1}}\right)\\
&= l^{-1}l^{k/2}V(g_k(\mathrm{id.},4m,4D),1)\\
&= l^{k/2-1}(-\mathrm{i}^{-\delta_k}\mu(m)\eta_{2m})\\
&= -\frac{1+\mathrm{i}^{-\delta_k}}{2^k-4}\mu(m)\eta_m l^{k/2-1}.
\end{aligned}
$$

Similar to the case $l=1$, we can prove $V(g_k(\chi_l,4m,4D),1/2\beta)=0$ for any $\beta|D$. Since $g_k(\chi_l,4m,4D)|\mathrm{T}(4)=g_k(\chi_l,4m,4D)$, we have

$$
\begin{aligned}
V(g_k(\chi_l,4m,4D),1)=&\,4^{-1}V(g_k(\chi_l,4m,4D),1/4)+4^{-1}V(g_k(\chi_l,4m,4D),3/4)\\
&+2^{k-2}V(g_k(\chi_l,4m,4D),1),
\end{aligned}
\tag{7.15}
$$

where we used the fact $V(g_k(\chi_l,4m,4D),1/2\beta)=0$ for any $\beta|D$. Because the cusp point $3/4$ is equivalent to $1/4$, so there exists a matrix $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(4D)$ such that

$$
\begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\4\end{pmatrix}=\begin{pmatrix}3\\4\end{pmatrix},
$$

which implies $d\equiv 3\pmod 4$, $d\equiv 1\pmod l$, $c\equiv 4\pmod d$. By Lemma 7.2, we have

$$
\begin{aligned}
V(g_k(\chi_l,4m,4D),3/4) &= \left(\frac{l}{d}\right)\mathrm{i}^{-\delta_k}V(g_k(\chi_l,4m,4D),1/4)\\
&= \mathrm{i}^{-\delta_k}\varepsilon_l^{d-1}\left(\frac{d}{l}\right)V(g_k(\chi_l,4m,4D),1/4)\\
&= \mathrm{i}^{-\delta_k}\varepsilon_l^{2}V(g_k(\chi_l,4m,4D),1/4).
\end{aligned}
$$

Inserting this into equality (7.15), we obtain

$$
\begin{aligned}
V(g_k(\chi_l,4m,4D),1/4) &= -\frac{2^k-4}{1+\mathrm{i}^{-\delta_k}\varepsilon_l^2}V(g_k(\chi_l,4m,4D),1)\\
&= -\frac{2^k-4}{1+\mathrm{i}^{-\delta_k}\varepsilon_l^2}\left(-\frac{1+\mathrm{i}^{-\delta_k}}{2^k-4}\mu(m)\eta_m l^{k/2-1}\right)\\
&= \mu(m)\eta_m l^{k/2-1}\varepsilon_l^{-\delta_k}.
\end{aligned}
$$

Similarly we can prove that $V(g_k(\chi_l,m,4D),1/2\beta)=V(g_k(\chi_l,m,4D),1/4\beta)=0$ for any $\beta|D$ and $V(g_k(\chi_l,4m,4D),1)=\mathrm{i}^{-\delta_k}\mu(m)\eta_m l^{k/2-1}$. Collecting all the above and Lemma 7.2 we proved our Theorem 7.2 for $l\neq 1$. This completes the whole proof for Theorem 7.2. □

## 7.2 Construction of Eisenstein Series with Weight 1/2

Let $\psi$ be a primitive character modulo $r$ with $\psi(-1) = (-1)^v$ ($v = 0$ or 1). Put

$$\theta_\psi(z) = \sum_{n=-\infty}^{\infty} \psi(n) n^v e(n^2 z), \quad z \in \mathbb{H}.$$

Then it is easy to see that

$$\theta_\psi(z) = \sum_{k=1}^{r} \psi(k)\theta(2rz; k, r),$$

where

$$\theta(z; k, r) = \sum_{m \equiv k \bmod r} m^v e(zm^2/(2r)), \quad z \in \mathbb{H}.$$

**Lemma 7.5** *We have the following transformation formula:*

$$\theta(-1/z; k, r) = (-1)^v r^{-1/2}(-\mathrm{i}z)^{(1+2v)/2} \sum_{j=1}^{r} e(jk/r)\theta(z; j, r).$$

**Proof** Set

$$g(x) = \sum_{m=-\infty}^{\infty} (x+m)^v e(\mathrm{i}rt(x+m)^2/2).$$

It is obvious that $g(x+1) = g(x)$. So by some computation we have a Fourier expansion:

$$g(x) = \sum_{m=-\infty}^{\infty} a(m)e(mx)$$

with

$$a(m) = (-\mathrm{i})^v (rt)^{-(1+2v)/2} \mathrm{e}^{-\pi m^2/(rt)},$$

so that

$$g(x) = (-\mathrm{i})^v (rt)^{-(1+2v)/2} \sum_{m=-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} mx - \pi m^2/(rt)}.$$

It is easy to see that

$$\theta(\mathrm{i}t; k, r) = r^v g(k/r) = (-\mathrm{i})^v r^{-1/2} t^{-(1+2v)/2} \sum_{j=1}^{r} e(jk/r)\theta(-1/(\mathrm{i}t); j, r),$$

which implies the lemma. This completes the proof. □

**Lemma 7.6** *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *with* $b$ *even and* $c \equiv 0 \pmod{2r}$. *Then*

$$\theta(\gamma(z); k, r) = e(abk^2/(2r))\varepsilon_d^{-1}\left(\frac{2cr}{d}\right)(cz+d)^{(1+2v)/2}\theta(z; ak, r).$$

**Proof** Assume that $c > 0$. By Lemma 7.5, we have

$$\begin{aligned}\theta(\gamma(z); k, r) &= \sum_{n \equiv k \bmod r} n^v e\left(n^2\left(a - \frac{1}{cz+d}\right)/(2cr)\right) \\ &= (-\mathrm{i})^v (cr)^{-1/2} (-\mathrm{i}(cz+d))^{(1+2v)/2} \sum_{t \bmod cr} \Phi(k,t) \\ &\qquad \sum_{n \equiv t \bmod cr} n^v e(n^2 z/(2r)),\end{aligned}$$

where

$$\Phi(k,t) = \sum_{\substack{g \bmod cr, \\ g \equiv k \bmod r}} e((\alpha g^2 + tg + \delta t^2)/(cr))$$

and $\alpha$, $\delta$ are integers such that $a \equiv 2\alpha \pmod{cr}$, $d \equiv 2\delta \pmod{cr}$. The remaining part of this proof is completely similar to the proof of Proposition 1.2. This completes the proof. □

**Theorem 7.3** *$\theta_\psi(z)$ is in $G(4r^2, 1/2, \psi)$ if $v = 0$ and $\theta_\psi(z)$ is in $S(4r^2, 3/2, \psi\chi_{-1})$ if $v = 1$.*

**Proof** Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4r^2)$. By Lemma 7.6, we see that

$$\begin{aligned}\theta_\psi(\gamma(z)) &= \sum_{k=1}^{r} \psi(k)\theta\left(\frac{2rza + 2rb}{2rz(c/(2r)) + d}; k, r\right) \\ &= \varepsilon_d^{-1}\left(\frac{c}{d}\right)(cz+d)^{(1+2v)/2} \sum_{k=1}^{r} \psi(k)\theta(2rz; ak, r) \\ &= \psi(d)\varepsilon_d^2 j(\gamma, z)^{1+2v}\theta_\psi(z).\end{aligned}$$

Consider the holomorphy of $\theta_\psi(z)$ at cusp points. Let $\rho = \begin{pmatrix} a & b \\ -c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with $c > 0$. Then we see that

$$|\theta_\psi(z)| \leqslant 1 - v + 2\sum_{n=1}^{\infty} n^v \mathrm{e}^{-2\pi n^2 y} < 1 - v + Cy^{-(1+v/2)}, \quad y \to \infty,$$

where $C$ is a constant. So that

$$\begin{aligned}|\theta_\psi(\rho^{-1}(z))(cz+a)^{-(1+2v)/2}| &\leqslant (1 - v + Cy^{-(1+v/2)}|cz+a|^{v+2})|cz+a|^{-(1+2v)/2} \\ &\leqslant (1 - v + C'y^{1+v/2})y^{-(1+2v)/2}, \quad y \to \infty,\end{aligned}$$

which implies that $\theta_\psi(z) \in G(4r^2, 1/2, \psi)$ or $S(4r^2, 3/2, \psi\chi_{-1})$ according to $v = 0$ or 1 respectively. This completes the proof. □

Let now $t$ be a positive integer, $\psi$ a primitive even character modulo $r$. Put

$$\theta_{\psi,t}(z) = \sum_{n=-\infty}^{\infty} \psi(n)e(tn^2z), \quad z \in \mathbb{H},$$

which is equal to $\theta_\psi|V(t)$, so $\theta_{\psi,t}(z)$ is in $G(4r^2t, 1/2, \psi\chi_t)$. Let $\omega$ be an even character modulo $N$, $\psi$ a primitive even character modulo $r(\psi)$, $t$ a positive integer. We denote by $\Omega(N,\omega)$ the set of pairings $(\psi, t)$ satisfying the following conditions:

(1) $4(r(\psi))^2t|N$;

(2) $\omega(n) = \psi(n)\chi_t(n)$ for any integer $n$ prime to $N$.

Let $\psi = \prod_{p|r(\psi)} \psi_p$ with $\psi_p$ the $p$-part of the character $\psi$. If every $\psi_p$ is an even character, then $\psi$ is called a totally even character. Denote by $\Omega_{\mathrm{e}}(N,\omega)$ the set of all parings $(\psi, t)$ in $\Omega(N,\omega)$ where $\psi$ is totally even. Set $\Omega_{\mathrm{c}}(N,\omega) = \Omega(N,\omega) - \Omega_{\mathrm{e}}(N,\omega)$. The following is our main result in this section.

**Theorem 7.4** (1) *The set* $\{\theta_{\psi,t}|(\psi,t) \in \Omega(N,\omega)\}$ *is a basis of* $G(N, 1/2, \omega)$;

(2) *The set* $\{\theta_{\psi,t}|(\psi,t) \in \Omega_{\mathrm{c}}(N,\omega)\}$ *is a basis of* $S(N, 1/2, \omega)$, *and the set* $\{\theta_{\psi,t}|(\psi,t) \in \Omega_{\mathrm{e}}(N,\omega)\}$ *is a basis of the orthogonal complement of* $S(N, 1/2, \omega)$ *in* $G(N, 1/2, \omega)$.

To show Theorem 7.4 we need some lemmas.

**Lemma 7.7** (1) *There exists a basis in* $G(N, k/2, \omega)$ *such that all Fourier coefficients of every function in the basis belong to some algebraic number field*;

(2) *let* $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N, k/2, \omega)$ *with* $a(n)$ *all algebraic numbers for* $n \geqslant 0$. *Then there exists an integer* $D$ *such that* $Da(n)$ *are all algebraic integers for all* $n \geqslant 0$.

**Proof** Put

$$f_0(z) = \theta(z)^{3k} = 1 + 6ke(z) + \cdots .$$

Define a map $\phi:\ f \mapsto ff_0$. Then $\phi$ maps $G(N, k/2, \omega)$ into $G(N, 2k, \omega)$. If $f$ has algebraic coefficients, so does $ff_0$. (2) holds for $ff_0$ (Please compare Theorem 3.52 of G. Shimura, 1971), so does (2) for $f$. Now show (1). $\theta(z)$ has no zeros in $\mathbb{H}$, and it is zero only at the cusp point $1/2 \in S(4) = \{1,\ 1/2,\ 1/4\}$. A function $g \in G(N,\ 2k,\ \omega)$ is an image of $\phi$ (i.e., $g/f_0 \in G(N,\ k/2,\ \omega)$) if and only if $g$ has high enough orders of zeros at all cusp points in $S(N)$ which are $\Gamma_0(N)$-equivalent to $1/2$. We know that the theorem we want to show holds for the spaces of modular forms integral weights. So there exists a basis $\{g_i\}$ in $G(N,\ 2k,\ \omega)$ such that the Fourier coefficients of $g_i$ at every cusp point are algebraic numbers. $g$ is a linear combination of $\{g_i\}$, and $g$ gets value zero with some orders at part of cusp points. This implies that the coefficients of the linear combination satisfy a system of some linear equations with algebraic numbers as the coefficients of these linear equations. Hence there exists a basis in $G(N, k/2, \omega)$ whose every element has algebraic coefficients. This completes the proof. □

**Lemma 7.8** *Let* $0 \neq f(z) = \sum_{n=0}^{\infty} a(n)e(nz)$ *be in* $G(N, 1/2, \omega), p \nmid N$ *a prime and* $f|\mathrm{T}(p^2) = c_p f$. *Assume that* $m$ *is a positive integer with* $p^2 \nmid m$. *Then*

(1) $a(mp^{2n}) = a(m)\omega(p)^n \left(\frac{m}{p}\right)^n$ *for any* $n \geqslant 0$;

(2) *if* $a(m) \neq 0$, *then* $p \nmid m$ *and* $c_p = \omega(p)\left(\frac{m}{p}\right)(1 + p^{-1})$.

**Proof** Since $\mathrm{T}(p^2)$ maps a modular form with algebraic coefficients to one of the same kind, by Lemma 7.7, we see that the eigenvalue $c_p$ of $\mathrm{T}(p^2)$ is an algebraic number and the corresponding eigenspace has a basis with algebraic coefficients. Without loss of generality, we may assume that the coefficients of $f$ are algebraic. Put

$$A(T) = \sum_{n=0}^{\infty} a(mp^{2n})T^n.$$

By Lemma 5.40 we have

$$A(T) = a(m)\frac{1 - \alpha T}{(1 - \beta T)(1 - \gamma T)},$$

where $\alpha = \omega(p)p^{-1}\left(\frac{m}{p}\right)$, $\beta + \gamma = c_p$, $\beta\gamma = \omega(p^2)p^{-1}$. Assume $a(m) \neq 0$. Then $A(T)$ is a non-zero rational function. We may think $A(T)$ as a $p$-adic $T$ function, i.e., think the coefficients of $A(T)$ as elements in some algebraic extension of the $p$-adic number field $\mathbb{Q}_p$. By Lemma 7.7 the $p$-adic absolute value of $a(mp^{2n})$ $(n \geqslant 0)$ are bounded. Therefore $A(T)$ is convergent for all $|T|_p < 1$. $A(T)$ has no poles in the unit disc $U = \{T \mid |T|_p < 1\}$. If $(1 - \beta T)(1 - \gamma T)$ is prime to $1 - \alpha T$, then $|\beta|_p < 1$, $|\gamma|_p < 1$. But $|\beta\gamma|_p = |\omega(p^2)p^{-1}|_p > 1$. So we see that one of $\beta$ and $\gamma$ must be $\alpha$. We may assume that $\beta = \alpha$ and hence $A(T) = a(m)/(1 - \gamma T)$, $a(mp^{2n}) = \gamma^n a(m)$. Since $\beta\gamma \neq 0$, we see that $\alpha \neq 0$, so $p \nmid m$ and

$$\gamma = \beta\gamma/\alpha = \frac{\omega(p^2)p^{-1}}{\omega(p)p^{-1}\left(\frac{m}{p}\right)} = \omega(p)\left(\frac{m}{p}\right).$$

This shows that $a(mp^{2n}) = a(m)\omega(p)^n\left(\frac{m}{p}\right)^n$ which is (1). And $c_p = \beta + \gamma = \alpha + \gamma = \omega(p)\left(\frac{m}{p}\right)(1 + p^{-1})$ which is (2). This completes the proof. □

**Lemma 7.9** *Let* $0 \neq f(z) = \sum_{n=0}^{\infty} a(n)e(nz)$ *be in* $G(N, 1/2, \omega)$, $N'$ *a multiple of* $N$.

*Assume that* $f|\mathrm{T}(p^2) = c_p f$ *for any* $p \nmid N'$. *Then there exists a unique square-free positive integer* $t$ *such that* $a(n) = 0$ *if* $n/t$ *is not a square and*

(1) $t|N'$;

(2) $c_p = \omega(p)\left(\dfrac{t}{p}\right)(1+p^{-1})$ *for any* $p \nmid N'$;

(3) $a(nu^2) = a(n)\omega(u)\left(\dfrac{t}{u}\right)$ *for any* $u \geqslant 1$ *with* $(u, N') = 1$.

**Proof** Let $m, m'$ be any positive integers with $a(m) \neq 0$ and $a(m') \neq 0$, $P$ the set of primes satisfying $p \nmid N'mm'$. For any $p \notin P$, by Lemma 7.8 we see that

$$\omega(p)\left(\frac{m}{p}\right)(1+p^{-1}) = \omega(p)\left(\frac{m'}{p}\right)(1+p^{-1}),$$

so that $\left(\dfrac{mm'}{p}\right) = 1$. This implies that $mm'$ must be a square. Therefore there exists a square-free positive integer $t$ with $m = tv^2, m' = t(v')^2$ which implies the first part of the lemma. Let now $p$ be any prime with $p \nmid N'$. Write $v = p^n u$, $p \nmid u$. Since $0 \neq a(m) = a(tp^{2n}u^2)$, we see that $a(tu^2) \neq 0$ by the part (1) of Lemma 7.8, so that $p \nmid t$ and $c_p = \omega(p)\left(\dfrac{t}{p}\right)(1+p^{-1})$ by the part (2) of Lemma 7.8. This showed (2) and (1) since $t$ is square-free. For the proof of the part (3), we only need to consider the case that $u = p, p \nmid N'$, then we can write $n = mp^{2a}$ with $p^2 \nmid m$. It is then clear that (3) can be deduced from the part (2) of Lemma 7.8. This completes the proof. □

**Corollary 7.1** *Let the assumptions be as in* Lemma 7.9. *And assume furthermore* $a(1) \neq 0$. *Then* $t = 1$ *and* $c_p = \omega(p)(1+p^{-1})$ *for any* $p \nmid N'$. *This implies that the character* $\omega$ *is determined uniquely by the set of eigenvalues* $c_p$.

**Corollary 7.2** *Under the assumptions of* Lemma 7.9 *we have that*

$$\sum_{n=1}^{\infty} a(n)n^{-s} = t^{-s}\left(\sum_{n|N'^{\infty}} a(tn^2)n^{-2s}\right)\prod_{p\nmid N'}\left(1-\omega(p)\left(\frac{t}{p}\right)p^{-2s}\right)^{-1}.$$

**Proof** This is a direct conclusion of the parts (1) and (3) of Lemma 7.9. □

From now on we always assume that

$$f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N, 1/2, \omega)$$

is a new form.

**Lemma 7.10** *Let* $f(z) = \sum_{n=0}^{\infty} a(n)e(nz)$ *be a new form in* $G(N, 1/2, \omega)$ *which is an eigenfunction of* $\mathrm{T}(p^2)$ *for almost all primes* $p$. *Then* $a(1) \neq 0$ *and* $t = 1$.

**Proof** If $a(1) = 0$, then $a(n) = 0$ for any $n$ with $(n, N') = 1$ by Lemma 7.9. By Corollary 6.3 we see that $f$ is in $G^{\text{old}}(N, 1/2, \omega)$ which is impossible, so that $a(1) \neq 0$ and hence $t = 1$ by Corollary 7.1. This completes the proof. □

From now on we always assume that $a(1) = 1$. In this case $f$ is called a normalized new form.

**Lemma 7.11** *Let $g \in G(N, 1/2, \omega)$ be an eigenfunction of* $\mathrm{T}(p^2)$ *for almost all primes $p$ and whose eigenvalues are equal to the ones of $f$. Then $g = cf$ with a constant $c$.*

**Proof** Let $c$ be the coefficient of $e(z)$ of the Fourier expansion of $g$. Then the coefficient of $e(z)$ of the Fourier expansion of $h = g - cf$ is zero. If $h \neq 0$, then $h$ is an eigenfunction of almost all Hecke operators. By Corollary 7.2 we can find $N'$ such that the coefficient of $e(nz)$ of the Fourier expansion of $h$ is zero for all $n$ with $(n, N') = 1$. By Corollary 6.3 we know that $h \in G^{\text{old}}(N, 1/2, \omega)$. Hence there exists a factor $N_1$ of $N$, a character $\psi$ modulo $N_1$ and a normalized new form $g_1$ in $G(N_1, 1/2, \psi)$ such that $g_1$, $f$ and $h$ have the same eigenvalues for almost all Hecke operators. But the character $\psi$ is determined uniquely by the set of all eigenvalues $c_p$ by Corollary 7.1. Hence $\psi = \omega$ and $g_1 \in G^{\text{old}}(N, 1/2, \omega)$. Similarly we have that $f - g_1 \in G^{\text{old}}(N, 1/2, \omega)$, so $f = g_1 + (f - g_1) \in G^{\text{old}}(N, 1/2, \omega)$ which contradicts that $f$ is a new form. This implies that $h = 0$, i.e., $g = cf$. This completes the proof. □

**Lemma 7.12** *Let $f$ be a new form in $G(N, 1/2, \omega)$ and be an eigenfunction of almost all* Hecke *operators. Then $f$ is an eigenfunction of all* Hecke *operators* $\mathrm{T}(p^2)$. *Assume that $f|\mathrm{T}(p^2) = c_p f$. Then*

$$\sum_{n=1}^{\infty} a(n)n^{-s} = \prod_{p|N}(1 - c_p p^{-2s})^{-1} \prod_{p\nmid N}(1 - \omega(p)p^{-2s})^{-1}$$

*and $c_p = 0$ if $4p|N$.*

**Proof** Let $p$ be any prime. Put $g = f|\mathrm{T}(p^2)$. By the assumptions of the lemma we know that $g$ and $f$ have the same eigenvalues with respect to the Hecke operators $\mathrm{T}(q^2)$ for almost all primes $q$. By Lemma 7.11 we have $g = cf$. This shows that $f$ is an eigenfunction of all Hecke operators. The Euler product can be deduced by Corollary 7.2. Assume that $4p|N$, then by Lemma 7.9 we see that $f|\mathrm{T}(p) \in G(N, 1/2, \omega\chi_p)$ and

$$f|\mathrm{T}(p) = \sum_{n=0}^{\infty} a(np)e(nz) = \sum_{m=0}^{\infty} a(m^2p^2)e(pm^2z) = (f|\mathrm{T}(p^2))|V(p) = c_p f|V(p).$$

If $c_p \neq 0$, applying Lemma 6.22 to $f|\mathrm{T}(p)$ we know that $\omega$ is well-defined modulo $N/p$ and there exists a $g \in G(N/p, 1/2, \omega)$ such that $f|\mathrm{T}(p) = g|V(p)$. Hence $g = c_p f$ which contradicts the fact that $f$ is a new form, so that $c_p = 0$. This completes the proof. □

**Lemma 7.13** *Let the assumptions be the same as in* Lemma 7.12. *Then $N$ is a square and $f|W(N) = cf|H$ with a constant $c$.*

**Proof** Let $p \nmid N$ be a prime. Then $f|\mathrm{T}(p^2) = c_p f$ and $c_p = \omega(p)(1+p^{-1})$. By Theorem 5.19 we see that

$$f|W(N)\mathrm{T}(p^2) = \overline{\omega}(p^2)c_p f|W(N) = \overline{c_p} f|W(N), \quad f|H\mathrm{T}(p^2) = (c_p f)|H = \overline{c_p} f|H.$$

Since $W(N), H$ send new forms to new forms, $f|W(N)$ is a new form in $G(N, 1/2, \overline{\omega}\chi_N)$ and $f|H$ a new form in $G(N, 1/2, \overline{\omega})$. Since they have the same eigenvalues with respect to $\mathrm{T}(p^2)$ for all $p \nmid N$, and the set of eigenvalues $\overline{c_p}$ determines uniquely the corresponding character, we know that $\overline{\omega}\chi_N = \overline{\omega}$. This shows that $N$ is a square. Lemma 7.11 implies that $f|W(N) = cf|H$ with a constant $c$. This completes the proof. □

**Theorem 7.5** *Let $f \in G(N, 1/2, \omega)$ be a normalized new form which is an eigenfunction of almost all* Hecke *operators. Denote by $r$ the conductor of $\omega$. Then $N = 4r^2$, $f = \dfrac{1}{2}\theta_\omega$.*

**Proof** Put

$$F(s) := \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_{p|N}(1 - c_p p^{-2s})^{-1} \prod_{p\nmid N}(1 - \omega(p)p^{-2s})^{-1},$$

$$\overline{F}(s) := \sum_{n=1}^{\infty} \overline{a(n)}n^{-s}.$$

By Theorem 5.22 we know that the above series is absolutely convergent for $\mathrm{Re}(s) > 3/2$ and we have the following functional equation:

$$(2\pi)^{-s}\Gamma(s)F(s) = c_1\left(\frac{2\pi}{N}\right)^{s-1/2}\Gamma(1/2 - s)\overline{F}(1/2 - s), \tag{7.16}$$

where we used the fact that $f|W(N) = cf|H$, $c_1$ and the following $c_2, c_3, c_4$ are all constants. Set

$$G(s) = L(2s, \omega) = \prod_{p\nmid r}(1 - \omega(p)p^{-2s})^{-1},$$

$$\overline{G}(s) = L(2s, \overline{\omega}).$$

Then we have

$$(2\pi)^{-s}\Gamma(s)G(s) = c_2\left(\frac{2\pi}{4r^2}\right)^{s-1/2}\Gamma(1/2 - s)\overline{G}(1/2 - s). \tag{7.17}$$

From (7.16) and (7.17) we see that

$$\prod_{p|m}\frac{1 - c_p p^{-2s}}{1 - \omega(p)p^{-2s}} = c_3\left(\frac{N}{4r^2}\right)^{s-1/2}\prod_{p|m}\frac{1 - \overline{c_p}p^{2s-1}}{1 - \overline{\omega}_p p^{2s-1}}, \tag{7.18}$$

where $m$ is the product of all prime divisors $p$ of $N$ with $c_p \neq \omega(p)$. If there exists a $p|m$ with $\omega(p) \neq 0$, then the function on the left (resp. right) hand side of (7.18) has infinite (resp. no) poles on the line $\mathrm{Re}(s) = 0$. Hence $\omega(p) = 0$ (i.e., $p|r$) for any $p|m$. In this case we have $c_p \neq 0$ since $c_p \neq \omega(p)$,

$$\prod_{p|m}(1 - c_p p^{-2s}) = c_4\left(\frac{Nm^2}{4r^2}\right)^s \prod_{p|m}(1 - c'_p p^{-2s}),$$

where $c'_p = p/\overline{c_p}$. Considering the zeros of the functions on both sides of the above equality we know that $c_p = c'_p$ for any $p|m$, so that $|c_p|^2 = p$ and hence $c_4 = 1$, $Nm^2 = 4r^2$. By Lemma 7.12 we know that $c_p = 0$ if $4p|N$. This implies that $m = 1$ or $m = 2$ by the definition of $m$. If $m = 1$, then $N = 4r^2$. If $m = 2$, then $c_2 \neq 0$, so $8 \nmid N$. But $\omega(2) = 0$, so $4|r$ which contradicts the fact that $4N = 4r^2$ and $8 \nmid N$. We have shown that $N = 4r^2$ and $F(s) = G(s)$. Thus for any $n \geqslant 1$ the coefficients of $e(nz)$ in the Fourier expansions of $f$ and $\frac{1}{2}\theta_\omega$ coincide with each other, i.e., $f - \frac{1}{2}\theta_\omega \in G(N, 1/2, \omega)$ is a constant, so that it must be zero. This completes the proof. □

**Lemma 7.14** *Let $\omega$ be an even character with conductor $r$. Then $\frac{1}{2}\theta_\omega \in G(4r^2, 1/2, \omega)$ is a normalized new form.*

**Proof** We know that $\theta_\omega$ is in $G(4r^2, 1/2, \omega)$. By Theorem 5.15 we see that

$$\theta_\omega|\mathrm{T}(p^2) = \omega(p)(1 + p^{-1})\theta_\omega, \quad \forall\, p \nmid 4r^2.$$

If $\theta_\omega$ is not a new form in $G(4r^2, 1/2, \omega)$, then there exists a proper divisor $N_1$ of $4r^2$, a character $\psi$ modulo $N_1$ and a new form $f$ in $G(N_1, 1/2, \psi)$ such that $f$ and $\theta_\omega$ have the same eigenvalues $\psi(p)(1 + p^{-1}) = \omega(p)(1 + p^{-1})$ for almost all Hecke operators $\mathrm{T}(p^2)$. Therefore $\omega = \psi$ and $N_1 = 4r^2$ by Theorem 7.5. This contradicts $N_1 < 4r^2$, hence $\theta_\omega \in G(4r^2, 1/2, \omega)$ is a new form. This completes the proof. □

Let

$$\varGamma_1(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N\right\}.$$

Suppose that $f(z) = \sum_{n=0}^{\infty} a(n)e(nz)$ is a modular form of weight $k/2$ for the group $\varGamma_1(N)$. Let $\varepsilon$ be a periodic function on $\mathbb{Z}$ with period $M$. Put

$$(f * \varepsilon)(z) = \sum_{n=0}^{\infty} a(n)\varepsilon(n)e(nz).$$

The Fourier transformation of $\varepsilon$ is

$$\hat{\varepsilon}(m) = M^{-1}\sum_{n=1}^{M}\varepsilon(n)e(-nm/M),$$

by the inverse Fourier transformation we have

$$\varepsilon(n) = \sum_{m=1}^{M}\hat{\varepsilon}(m)e(nm/M).$$

Hence we obtain that

$$(f*\varepsilon)(z) = \sum_{m=1}^{M}\hat{\varepsilon}(m)f(z+m/M),$$

It is clear that the function $f(z+m/M)$ is a modular form of weight $k/2$ for the group $\Gamma_1(NM^2)$.

**Lemma 7.15** *The following two assertions are equivalent*:

(1) *the values of $f$ at all cusp points $m/M$ $(m\in\mathbb{Z})$ are equal to zero* (*where $m$ and $M$ may not be co-prime to each other*);

(2) *for every periodic function $\varepsilon$ with period $M$, the function*

$$L(f*\varepsilon,s) = \sum_{n=1}^{\infty}a(n)\varepsilon(n)n^{-s}$$

*is holomorphic at $s=k/2$.*

The similar result holds also for modular forms of integral weights and the proof is completely similar to the following one.

**Proof** The assertion (1) is equivalent to the fact that for any periodic function $\varepsilon$ with period $M$ the function $f*\varepsilon$ takes value 0 at the cusp point $s=0$. By Theorem 5.22 the assertion (2) is equivalent to the fact that the function $f*\varepsilon|W(NM^2)$ takes value 0 at $i\infty$. But the value of $f*\varepsilon|W(NM^2)$ at $i\infty$ differs from the one of $f*\varepsilon$ at the cusp point $s=0$ by a constant multiple, so the lemma holds. This completes the proof. □

**Corollary 7.3** *$f$ is a cusp form if and only if $L(f*\varepsilon,s)$ is holomorphic at $s=k/2$ for any periodic function $\varepsilon$ on $\mathbb{Z}$.*

Since every cusp point is $\Gamma_0(N)$-equivalent to some cusp point $m/N$, ($m$ and $N$ may not be co-prime to each other), we only need to consider periodic functions with period $N$ for $f\in G(N,1/2,\omega)$.

**Lemma 7.16** *Let $\psi$ be an even character but not totally even. Then $\theta_\psi$ is a cusp form.*

**Proof** Let $\varepsilon$ be any periodic function on $\mathbb{Z}$ with period $N$. Without loss of generality, we may assume that $N$ is a multiple of the conductor $r(\psi)$ of $\psi$. By Corollary 7.3, we only need to show that

$$F_\varepsilon(s) = 2\sum_{n=1}^{\infty} \varepsilon(n^2)\psi(n)n^{-2s}$$

is holomorphic at $s = 1/2$. We have

$$F_\varepsilon(s) = 2\sum_{m=1}^{N} \varepsilon(m^2)\psi(m)F_{m,N}(2s),$$

where

$$F_{m,N}(s) = \sum_{\substack{n\equiv m \bmod N,\\ n\geqslant 1}} n^{-s}.$$

It is well known that $F_{m,N}(s)$ has a simple pole at $s = 1$ with residue $1/N$. Hence the residue of $F_\varepsilon(s)$ at $s = 1/2$ is equal to $R(\varepsilon,\psi)/N$ with $R(\varepsilon,\psi) = \sum_{m=1}^{N} \varepsilon(m^2)\psi(m)$. We now only need to show that $R(\varepsilon,\psi) = 0$. Since $\psi$ is not totally even, there exists a prime divisor $l$ of $r(\psi)$ such that the $l$-part $\psi_l$ of $\psi$ is odd. Write $N = l^a N'$ with $l \nmid N'$. Take an integer $l'$ such that $l' \equiv -1 \pmod{l^a}$, $l' \equiv 1 \pmod{N'}$. It is clear that $l'$ is invertible in $\mathbb{Z}/N\mathbb{Z}$ and $l'^2 \equiv 1(N), \psi(l') = -1$. Therefore

$$R(\varepsilon,\psi) = \sum_{m \bmod N} \varepsilon((l'm)^2)\psi(l'm) = -\sum_{m \bmod N} \varepsilon(m^2)\psi(m) = -R(\varepsilon,\psi),$$

i.e., $R(\varepsilon,\psi) = 0$. This completes the proof. □

**Lemma 7.17** *Let $\psi$ be a totally even character, $T$ a finite set of positive integers. If $f = \sum_{t\in T} c_t\theta_{\psi,t} (c_t \in \mathbb{C})$ is a cusp form, then $c_t = 0$ for all $t$.*

**Proof** Otherwise, let $t_0$ be the smallest number in $T$ such that $c_{t_0} \neq 0$. Take a positive integer $M$ such that $M$ is a common multiple of $2r(\psi)$ and all numbers of $T$. Since $\psi$ is totally even, there exists a character $\alpha$ modulo $M$ with $\alpha^2 = \psi$. Define a periodic function $\varepsilon$ on $\mathbb{Z}$ as follows:

$$\varepsilon(n) = \begin{cases} \overline{\alpha}(n/t_0), & \text{if } t_0|n \text{ and } (n/t_0, M) = 1,\\ 0, & \text{otherwise.}\end{cases}$$

We see that

$$\varepsilon(t_0 n^2) = \begin{cases} \overline{\psi}(n), & \text{if } (n, M) = 1,\\ 0, & \text{otherwise}\end{cases}$$

and

$$\varepsilon(tn^2) = 0, \quad \text{if } t \in T, t > t_0,$$

(since $(tn^2, M) \geqslant t > t_0$). Therefore

$$L(f * \varepsilon, s) = 2c_{t_0} \sum_{(n,M)=1, n\geqslant 1} \overline{\psi}(n)\psi(n)(t_0 n^2)^{-s} = 2c_{t_0} t_0^{-s} \sum_{(n,M)=1, n\geqslant 1} n^{-2s}$$

whose residue at $s = 1/2$ is

$$c_{t_0} t_0^{-1/2} \varphi(M)/M \neq 0.$$

By Corollary 7.3 we see that $f$ is not a cusp form which is impossible. This completes the proof. □

**Proof of Theorem 7.4** (1) We first prove that $\{\theta_{\psi,t} | (\psi, t) \in \Omega(N, \omega)\}$ are linearly independent. Since $\psi$ is determined uniquely by $\omega$ and $t$, $t$ appears only one time as the second entry of a paring $(\psi, t)$ in $\Omega(N, \omega)$. Assume

$$\sum_{i=1}^{m} \lambda_i \theta_{\psi_i, t_i} = 0,$$

where $t_1 < t_2 < \cdots < t_m$, $\lambda_i \neq 0$ $(1 \leqslant i \leqslant m)$. The coefficient of $e(t_1 z)$ of the Fourier expansion of $\theta_{\psi_1, t_1}$ is equal to 2, and the ones of $\theta_{\psi_i, t_i}$ $(i \geqslant 2)$ are equal to 0. This shows that $\lambda_1 = 0$ which contradicts $\lambda_1 \neq 0$.

We now show that $\{\theta_{\psi,t} | (\psi, t) \in \Omega(N, \omega)\}$ generate $G(N, 1/2, \omega)$. Let $f, g \in G(N, 1/2, \omega)$. For any $p \nmid N$, using Lemma 5.26 we have

$$\langle f | \mathrm{T}(p^2), g \rangle = \omega(p^2) \langle f, g | \mathrm{T}(p^2) \rangle,$$

which shows that $\overline{\omega}\mathrm{T}(p^2)$, $p \nmid N$ are Hermitian and commute each other. So there is a basis of $G(N, 1/2, \omega)$ whose every element is an eigenfunction of $\mathrm{T}(p^2)$, $p \nmid N$. Hence we only need to show that if $f$ is an eigenfunction of $\mathrm{T}(p^2)$ $(p \nmid N)$ then $f$ is a linear combination of $\{\theta_{\psi,t} | (\psi, t) \in \Omega(N, \omega)\}$. We apply induction on $N$. If $f$ is a new form, Theorem 7.5 gives what we want. If $f$ is an old form, then $f$ is either in $G(N/p, 1/2, \omega)$ and $\omega$ is well-defined for modulo $N/p$, or $f = g|V(p)$ with $g \in G(N/p, 1/2, \omega\chi_p)$ and $\omega\chi_p$ well-defined modulo $N/p$. In the first case, $f$ is a linear combination of $\{\theta_{\psi,t} | (\psi, t) \in \Omega(N/p, \omega)\}$ by the induction hypothesis. It is clear that $\Omega(N/p, \omega) \subset \Omega(N,\ \omega)$. For the second case, $g$ is a linear combination of $\{\theta_{\psi,t} | (\psi, t) \in \Omega(N/p, \omega\chi_p)\}$ due to the induction hypothesis, hence $f$ is a linear combination of $\{\theta_{\psi,t} | (\psi, t) \in \Omega(N, \omega)\}$. This completes the proof of the part (1).

(2) We only need to show the following three assertions: ① if $(\psi, t) \in \Omega_{\mathrm{c}}(N, \omega)$, then $\theta_{\psi,t}$ is a cusp form; ② any non-zero linear combination of $\{\theta_{\psi,t} | (\psi, t) \in \Omega_{\mathrm{e}}(N, \omega)\}$ is not a cusp form; ③ if $(\psi, t) \in \Omega_{\mathrm{c}}(N, \omega)$, $(\psi', t') \in \Omega_{\mathrm{e}}(N, \omega)$, then $\theta_{\psi,t}$ is orthogonal with $\theta_{\psi',t'}$ under the Petersson inner product.

The assertion ① is deduced from Lemma 7.16. Let now $V$ be the intersection of the set of linear combinations of $\{\theta_{\psi,t} | (\psi, t) \in \Omega_{\mathrm{e}}(N, \omega)\}$ and the space of cusp forms.

If $V \neq 0$, since $V$ is an invariant space for the Hecke operators $\mathrm{T}(p^2)(p \nmid N)$, there exists a $0 \neq f \in V$ which is an eigenfunction of all $\mathrm{T}(p^2)(p \nmid N)$. But $\psi(p)(1+p^{-1})$ is the eigenvalue of $\theta_{\psi,t}$ with respect to $\mathrm{T}(p^2)$. Hence $f$ is a linear combination of some $\theta_{\psi,t}$ with the same $\psi$. This contradicts Lemma 7.17 and hence $V = 0$ which shows the assertion ②. Finally we prove the assertion ③. Since $\overline{\psi'}\omega^2$ is a totally even character, we see that $\overline{\psi'}\omega^2 \neq \psi$. So there exists a prime $p$ with $\psi(p) \neq \overline{\psi'}\omega^2(p)$. Then $\psi(p)(1+p^{-1})$ and $\psi'(p)(1+p^{-1})$ are the eigenvalues of $\theta_{\psi,t}$ and $\theta_{\psi',t'}$ respectively with respect to $\mathrm{T}(p^2)$. By the properties of Petersson inner product we have

$$\langle \theta_{\psi,t}|\mathrm{T}(p^2), \theta_{\psi',t'}\rangle = \omega^2(p)\langle \theta_{\psi,t}, \theta_{\psi',t'}|\mathrm{T}(p^2)\rangle,$$

thus

$$\psi(p)\langle \theta_{\psi,t}, \theta_{\psi',t'}\rangle = \overline{\psi'}\omega^2(p)\langle \theta_{\psi,t}, \theta_{\psi',t'}\rangle,$$

i.e.,

$$\langle \theta_{\psi,t}, \theta_{\psi',t'}\rangle = 0,$$

which showed ③. This completes the proof of Theorem 7.4. □

## 7.3 Construction of Eisenstein Series with Weight 3/2

In this section we shall construct a basis of the Eisenstein space of weight 3/2 for a modular group $\Gamma_0(4N)$ with $N$ a square-free odd positive integer. The content of this section is due to D. Y. Pei, 1982. Considering the Eisenstein series in Chapter 2, we have

**Theorem 7.6** *For any $k > 3$ and $\omega$ not a real character, $E_k(\omega, N)$ and $E'_k(\overline{\omega}\chi_N, N)$ belong to $\mathcal{E}(N, k/2, \omega)$. The functions $f_2^*(\omega, N)$ and $f_2(\omega, N)$ belong to $\mathcal{E}(N, 3/2, \omega)$. If $D$ is a square-free odd positive integer, then the functions $f_1(\mathrm{id.}, 4D)$ and $f_1(\mathrm{id.}, 8D)$ belong to $\mathcal{E}(4D, 3/2, \mathrm{id.})$ and $\mathcal{E}(8D, 3/2, \mathrm{id.})$ respectively.*

**Proof** We only prove the theorem for $E_k(\omega, N)$ since the other assertion can be proved similarly. In Chapter 2 we proved that $E_k(\omega, N)$ is a holomorphic function on $\mathbb{H}$. We prove that it is also holomorphic at each cusp point. It is clear that $E_k(\omega, N)$ is holomorphic at i$\infty$. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with $c \neq 0$, we have

$$\begin{aligned}|E_k(\omega, N)(\gamma(z))(cz+d)^{-k/2}| &\leqslant \left(1 + \rho y^{-(k+5)/2}|cz+d|^{k+5}\right)|cz+d|^{-k/2} \\ &\leqslant \rho' y^{5/2} \quad (y \to \infty)\end{aligned}$$

by equality (2.31).

This shows that $E_k(\omega, N)$ is holomorphic at all cusp points which means that $E_k(\omega, N)$ belongs to $G(N, k/2, \omega)$. Now, we want to prove $E_k(\omega, N)$ is orthogonal to cusp forms. Let

$$f(z) = \sum_{n=1}^{\infty} c(n)e(nz) \in S(N, k/2, \omega)$$

and $\gamma \in \Gamma_0(N)$. Since $\int_0^1 \overline{f}(z)\mathrm{d}x = 0$ and

$$\overline{f}\big(\gamma(z)\big)\mathrm{Im}\big(\gamma(z)\big)^{(s+k)/2} = \overline{\omega}(d_\gamma)j(\gamma, z)^{-k}|j(\gamma, z)|^{-2s}\overline{f}(z)y^{(s+k)/2},$$

we have

$$\begin{aligned} 0 &= \int_0^\infty y^{(s+k)/2-2}\int_0^1 \overline{f}(z)\mathrm{d}x\mathrm{d}y = \int_{\Gamma_\infty\backslash\mathbb{H}} \overline{f}(x+\mathrm{i}y)y^{(s+k)/2-2}\mathrm{d}x\mathrm{d}y \\ &= \iint\limits_{\Gamma_0(N)\backslash\mathbb{H}} E_k(s, \overline{\omega}, N)(x+\mathrm{i}y)\overline{f}(x+\mathrm{i}y)y^{k/2-2}\mathrm{d}x\mathrm{d}y. \end{aligned}$$

To take $s=0$ gives the orthogonality. □

We can compute the values of $E_3'(\omega, N)$, $E_3(\omega, N)$, $f_1(\mathrm{id.}, 4D)$, $f_2^*(\mathrm{id.}, 4D)$, $f_2^*(\mathrm{id.}, 8D)$ and $f_2(\mathrm{id.}, 8D)$ at cusp points similarly as is done in Section 7.1.

**Lemma 7.18** (1) *Let* $\omega^2 \neq \mathrm{id.}$, *then* $V(E_3'(\omega, N), 1) = \mathrm{i}$. *For any* $d/c \in S(N)$ *and* $c \neq 1$, *we have* $V(E_3'(\omega, N), d/c) = 0$.

(2) *Let* $\omega^2 \neq \mathrm{id.}$, *then* $V(E_3(\omega, N), \mathrm{i}\infty) = 1$. *For any* $d/c \in S(N)$ *and* $c \neq N$, *we have* $V(E_3(\omega, N), d/c) = 0$.

**Proof** (1) By (2.7) we have

$$(-z)^{3/2}E_3'(\omega, N)(z) = \mathrm{i}E_3(\omega, N)(-1/(Nz)). \tag{7.19}$$

Hence, $V(E_3'(\omega, N), 1) = \mathrm{i}V(E_3(\omega, N), \mathrm{i}\infty) = \mathrm{i}$.

The other assertion can be proved by a method similar to the proof of Theorem 7.2.

(2) The first result is obvious and the second one is obvious from (7.19). □

**Lemma 7.19** *We have*

$$\begin{aligned} V(f_1(\mathrm{id.}, 4D), 1) &= -(1+\mathrm{i})(4D)^{-1}, \\ V(f_1(\mathrm{id.}, 8D), 1) &= -(1+\mathrm{i})(8D)^{-1}. \end{aligned}$$

**Proof** By the definition of $f_1(\mathrm{id.}, 4D)$, we have

$$f_1(\mathrm{id.}, 4D)(z) = E_3(0, \mathrm{id.}, 4D)(z) - (1-\mathrm{i})(4D)^{-1}z^{-3/2}E_3'(0, \chi_D, 4D)(-(4Dz)^{-1}).$$

Therefore,

$$\begin{aligned} z^{-3/2}f_1(\mathrm{id.}, 4D)(-(4Dz)^{-1}) &= E_3'(0, \mathrm{id.}, 4D)(z) - 2D^{1/2}(1+\mathrm{i})E_3(0, \chi_D, 4D)(z) \\ &= -2D^{1/2}(1+\mathrm{i})f_1(\chi_D, 4D)(z). \end{aligned}$$

By the definition of $V(f_1(\mathrm{id.}, 4D), 1)$ and (2.37), we have

$$\begin{aligned} V(f_1(\mathrm{id.}, 4D), 1) &= \lim_{z \to \mathrm{i}\infty} (4Dz)^{-3/2} f_1(\mathrm{id.}, 4D)(-(4Dz)^{-1}) \\ &= -(1+\mathrm{i})(4D)^{-1}. \end{aligned}$$

And the second result can be proved similarly. □

**Lemma 7.20** *We have*

$$\begin{aligned} V(f_2^*(\mathrm{id.}, 4D), 1/\beta) &= -4^{-1}(1+\mathrm{i})\mu(D/\beta)\beta/(D\varepsilon_\beta), \\ V(f_2^*(\mathrm{id.}, 4D), 1/(2\beta)) &= 0, \\ V(f_2^*(\mathrm{id.}, 4D), 1/(4\beta)) &= \mu(D/\beta)\beta/D. \end{aligned}$$

**Proof** We know that $f_2^*(\mathrm{id.}, 4D) \in G(4D, 3/2, \mathrm{id.})$ and for any prime factor $p|2D$, $f_2^*|\mathrm{T}(p^2) = f_2^*$ (This can be proved by (2.42)).

In particular, $f_2^*|\mathrm{T}(4) = f_2^*$. Hence

$$f_2^*(\mathrm{id.}, 4D)\left(z + \frac{1}{2\beta}\right) = 4^{-1} \sum_{k=1}^{4} f_2^*(\mathrm{id.}, 4D)\left(\frac{z}{4} + \frac{1+2k\beta}{8\beta}\right).$$

But $(1+2\beta k)/(8\beta)$ and $1/(4\beta)$ are $\Gamma_0(4D)$-equivalent. So we have

$$\begin{aligned} V(f_2^*(\mathrm{id.}, 4D), 1/(2\beta)) &= 4^{-1} \sum_{k=1}^{4} V\left(f_2^*(\mathrm{id.}, 4D), \frac{1+2\beta k}{8\beta}\right) \\ &= 4^{-1} \sum_{k=1}^{4} \left(\frac{2\beta}{1+2\beta k}\right) \varepsilon_{1+2k} V(f_2^*(\mathrm{id.}, 4D), 1/(4\beta)) = 0, \end{aligned}$$

where we used the fact $\left(\frac{2\beta}{a+4\beta}\right) = -\left(\frac{2\beta}{a}\right)$. Since $V(f_2^*(\mathrm{id.}, 4D), 1/(4D)) = 1$, by Lemma 7.1, we have $V(f_2^*(\mathrm{id.}, 4D), 1/4) = \mu(D)D^{-1}$. Hence we get the third equality by the second equality of Lemma 7.1. Using

$$f_2^*(\mathrm{id.}, 4D)(z) = 4^{-1} \sum_{k=1}^{4} f_2^*(\mathrm{id.}, 4D)\left(\frac{z}{4} + \frac{k}{4}\right)$$

and

$$V(f_2^*(\mathrm{id.}, 4D), 1/2) = 0,$$

we get

$$V(f_2^*(\mathrm{id.}, 4D), 1) = 4^{-1}(1+\mathrm{i})V(f_2^*(\mathrm{id.}, 4D), 1/4) + 2V(f_2^*(\mathrm{id.}, 4D), 1).$$

Since $3/4$ and $1/4$ are $\Gamma_0(4D)$-equivalent, we get

$$V(f_2^*(\mathrm{id.}, 4D), 1) = -4^{-1}(1+\mathrm{i})\mu(D)D^{-1}.$$

This proves the first assertion in Lemma 7.20 from Lemma 7.1. This completes the proof. □

**Lemma 7.21** *Let $m, \beta, l$ be factors of $D$. Let $f(z) \in G(8D, 3/2, \chi_{2l})$ satisfy*

$$\begin{aligned} f|\mathrm{T}(p^2) =& f, \quad \forall\ p|m, \\ f|\mathrm{T}(p^2) =& pf, \quad \forall\ p|Dm^{-1}. \end{aligned}$$

*Then*

$$\begin{aligned} V(f, 1/(2^r\alpha)) &= \mu(\alpha)\alpha(\alpha, l)^{-1/2}\varepsilon_{\alpha/(\alpha,l)}^{-1}\left(\frac{2^{1-r}l/(\alpha,l)}{\alpha/(\alpha,l)}\right)V(f, 1/2^r), \quad r = 0, 1, \\ V(f, 1/(8\alpha)) &= \mu(\alpha)\alpha(\alpha, l)^{-1/2}\varepsilon_{l/(\alpha,l)}\varepsilon_l^{-1}\left(\frac{\alpha/(\alpha,l)}{l/(\alpha,l)}\right)V(f, 1/8), \\ V(f, 1/(2^r\beta)) &= 0, \quad r = 0, 1, 3 \text{ and } (\beta, D/m) \neq 1. \end{aligned}$$

**Proof** This can be proved in a similar way as in the proof of Lemma 7.4. □

**Lemma 7.22** *Let $\beta$ be any factor of $D$. Then we have*

$$\begin{aligned} V(f_2^*(\chi_{2D}, 8D), 1/\beta) &= -2^{-3/2}(1 + \mathrm{i})\mu(D/\beta)\beta^{1/2}D^{-1/2}, \\ V(f_2^*(\chi_{2D}, 8D), 1/(2\beta)) &= 2^{-1}(1 + \mathrm{i})\mu(D/\beta)\beta^{1/2}D^{-1/2}, \\ V(f_2^*(\chi_{2D}, 8D), 1/(4\beta)) &= 0, \\ V(f_2^*(\chi_{2D}, 8D), 1/(8\beta)) &= \mu(D/\beta)\beta^{1/2}D^{-1/2}\varepsilon_{D/\beta}. \end{aligned}$$

**Proof** Put $h = f_2^*(\chi_{2D}, 8D)$. Then $h \in G(8D, 3/2, \chi_{2D})$ and $h|\mathrm{T}(p^2) = h$ for any prime factor $p|2D$. Using $h|\mathrm{T}(4) = h$ and $V(h, 1/(8D)) = 1$, we can prove $V(h, 1/(4\beta)) = 0$ for any $\beta|D$ and

$$\begin{aligned} V(h, 1) &= -2^{-3/2}(1 + \mathrm{i})\mu(D)D^{-1/2}, \\ V(h, 1/2) &= 2^{-1}(1 + \mathrm{i})\mu(D)D^{-1/2}, \\ V(h, 1/8) &= \mu(D)D^{-1/2}\varepsilon_D. \end{aligned}$$

Now taking $l = D$ in Lemma 7.21 gives Lemma 7.22. □

**Lemma 7.23** *Let $\beta$ be any factor of $D$. Then we have*

$$\begin{aligned} -2^{-1}(1 + \mathrm{i})\mu(D)V\big(f_2(\mathrm{id.}, 8D), 1/\beta\big) &= -16^{-1}(1 + \mathrm{i})\mu(D/\beta)\beta D^{-1}\varepsilon_\beta^{-1}, \\ -2^{-1}(1 + \mathrm{i})\mu(D)V\big(f_2(\mathrm{id.}, 8D), 1/(2\beta)\big) &= 0, \\ -2^{-1}(1 + \mathrm{i})\mu(D)V\big(f_2(\mathrm{id.}, 8D), 1/(4\beta)\big) &= -2^{-1}\mu(D/\beta)\beta D^{-1}, \\ -2^{-1}(1 + \mathrm{i})\mu(D)V\big(f_2(\mathrm{id.}, 8D), 1/(8\beta)\big) &= \mu(D/\beta)\beta D^{-1}. \end{aligned}$$

**Proof** By the definition of $f_2^*(\chi_{2D}, 8D)(z)$ and $f_2(\mathrm{id.}, 8D)(z)$, we have

$$f_2^*(\chi_{2D}, 8D)(-1/(8Dz))z^{-3/2} = 8\mathrm{i}Df_2(\mathrm{id.}, 8D)(z).$$

Let $c$ be a divisor of $8D$. Since

$$(-cz)^{3/2} f_2(\mathrm{id.}, 8D)(z+c^{-1}) = -\,\mathrm{i}(8D)^{-1} c^{3/2} f_2^*(\chi_{2D}, 8D) \times \left(\frac{cz}{8D(z+c^{-1})} - \frac{c}{8D}\right)\left(-\frac{z}{z+c^{-1}}\right)^{3/2}.$$

We have

$$V(f_2(\mathrm{id.}, 8D), 1/c) = -\mathrm{i}(8D)^{-1} c^{3/2} V(f_2^*(\chi_{2D}, 8D), -c/(8D)).$$

Since the cusp points $-c/(8D)$ and $c/(8D)$ are $\Gamma_0(8D)$-equivalent, we get the lemma by Lemma 7.22. □

**Lemma 7.24** *Let $f \in G(N, 3/2, \omega)$ be zero at all cusp points of $S(N)$ except $1/N$. Then $g = f|W(Q)$ is zero at all cusp points of $S(N)$ except $1/(NQ^{-1})$.*

**Proof** It is clear that the transformation $z \to \dfrac{Qz-1}{uNz+vQ}$ induces a permutation of the equivalent classes of cusp points of $\Gamma_0(N)$ and

$$\left.\frac{Qz-1}{uNz+vQ}\right|_{z=QN^{-1}} = \frac{Q-N/Q}{(u+v)N},$$

which is $\Gamma_0(N)$-equivalent to $1/N$. These two facts imply Lemma 7.24. □

Let $N = 2^r N'$, $r \geqslant 2$, $2 \nmid N'$ and $\omega$ be an even character modulo $N$ with conductor $r(\omega)$. Then by the dimension formula, we have

$$\dim\mathcal{E}(N, 3/2, \omega) = \begin{cases} 2 \displaystyle\sum_{\substack{c|N' \\ (c,N'/c)|N/r(\omega)}} \phi((c, N'/c)) - \dim\mathcal{E}(N, 1/2, \omega), & \text{if } r = 2, \\ 3 \displaystyle\sum_{\substack{c|N' \\ (c,N'/c)|N/r(\omega)}} \phi((c, N'/c)) - \dim\mathcal{E}(N, 1/2, \omega), & \text{if } r = 3, \\ \displaystyle\sum_{\substack{c|N \\ (c,N/c)|N/r(\omega)}} \phi((c, N/c)) - \dim\mathcal{E}(N, 1/2, \omega), & \text{if } r \geqslant 4. \end{cases}$$

By Theorem 7.4, we know that $\dim\mathcal{E}(N, 1/2, \omega)$ is the number of pairs $(\psi, t)$ of $\Omega_{\mathrm{e}}(N, \omega)$.

Now we always assume that $D$ is an odd square-free positive integer, $m, l$ and $\beta$ are divisors of $D$, $\alpha$ is a divisor of $m$ and $v$ is the number of prime divisors of $D$. Since $\Omega_{\mathrm{e}}(4D, \chi_l) = \{(\mathrm{id.}, l)\}$, $\Omega_{\mathrm{e}}(8D, \chi_l) = \{(\mathrm{id.}, l)\}$, $\Omega_{\mathrm{e}}(8D, \chi_{2l}) = \{(\mathrm{id.}, 2l)\}$, we have

$$\dim\mathcal{E}(4D, 3/2, \chi_l) = 2^{v+1} - 1,$$
$$\dim\mathcal{E}(8D, 3/2, \chi_l) = \dim\mathcal{E}(8D, 3/2, \chi_{2l}) = 3 \cdot 2^v - 1.$$

We shall construct a basic of $\mathcal{E}(4D, 3/2, \chi_l)$, $\mathcal{E}(8D, 3/2, \chi_l)$ and $\mathcal{E}(8D, 3/2, \chi_{2l})$ respectively. Since only Eisenstein series of weight 3/2 are considered, we shall omit all Subscripts 3. E.g., we define

$$\lambda(n, 4D) = \lambda_3(n, 4D) = L_{4D}(2, \mathrm{id.})^{-1} L_{4D}(1, \chi_{-n})\beta_3(n, 0, \chi_D, 4D)$$

and

$$A(p, n) = A_3(p, n), \quad \text{etc.}$$

Define functions

$$\begin{aligned} g(\chi_l, 4D, 4D)(z) =& 1 - 4\pi(1+\mathrm{i})l^{1/2} \sum_{n=1}^{\infty} \lambda(ln, 4D)(A(2, ln) - 4^{-1}(1-\mathrm{i})) \\ & \times \prod_{p|D} (A(p, ln) - p^{-1}) n^{1/2} e(nz), \\ g(\chi_l, 4m, 4D)(z) =& -4\pi(1+\mathrm{i})l^{1/2} \sum_{n=1}^{\infty} \lambda(ln, 4D)(A(2, ln) - 4^{-1}(1-\mathrm{i})) \\ & \times \prod_{p|m} (A(p, ln) - p^{-1}) n^{1/2} e(nz), \quad \forall m \neq D, \end{aligned}$$

$$g(\chi_l, m, 4D)(z) = 2\pi l^{1/2} \sum_{n=1}^{\infty} \lambda(ln, 4D) \prod_{p|m} (A(p, ln) - p^{-1}) n^{1/2} e(nz), \quad \forall\ m \neq 1.$$

**Theorem 7.7** (1) *The functions* $g(\chi_l, 4m, 4D)$, $(\forall m|D)$ *and* $g(\chi_l, m, 4D)$ $(\forall\ 1 \neq m|D)$ *constitute a basis of* $\mathcal{E}(4D, 3/2, \chi_l)$.

(2) *For any* $p \in S(4D)$, *we have*

$$V(g(\chi_l, 4m, 4D), p) = \begin{cases} -4^{-1}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1} l^{1/2} (l, \alpha)^{-1/2} \varepsilon_{\alpha/(l,\alpha)}^{-1} \left(\dfrac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right), \\ \qquad \text{if } p = 1/\alpha,\ \alpha|m, \\ \mu(m/\alpha)\alpha m^{-1} l^{1/2} (l, \alpha)^{-1/2} \varepsilon_{l/(l,\alpha)} \left(\dfrac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right), \\ \qquad \text{if } p = 1/(4\alpha),\ \alpha|m, \\ 0, \qquad \text{otherwise.} \end{cases}$$

(3) *For any* $p \in S(4D)$, *we have*

$$V(g(\chi_l, m, 4D), p) = \begin{cases} -4^{-1}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1} l^{1/2} (l, \alpha)^{-1/2} \varepsilon_{\alpha/(l,\alpha)}^{-1} \left(\dfrac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right), \\ \qquad \text{if } p = 1/\alpha,\ \alpha|m, \\ 0, \qquad \text{otherwise.} \end{cases}$$

**Proof** We first prove (2) for $l = 1$. By equality (2.45), we have $g(\mathrm{id.}, 4D, 4D) = f_2^*(\mathrm{id.}, 4D)$. Hence the theorem holds for $g(\mathrm{id.}, 4D, 4D)$ by Theorem 7.6 and Lemma 7.20. Now suppose $m \neq D$. We have

$$g(\mathrm{id.}, 4m, 4D) = -4\pi(1+\mathrm{i}) \prod_{p|D/m} p(1+p)^{-1} \sum_{n=1}^{\infty} \lambda(n, 4D)(A(2,n) - 4^{-1}(1-\mathrm{i}))$$
$$\times \prod_{p|m}(A(p,n)-p^{-1}) \prod_{p|D/m} \left\{1+A(p,n)-(A(p,n)-p^{-1})\right\} n^{1/2}e(nz)$$
$$= \prod_{p|D/m} p(1+p)^{-1} \sum_{d|D/m} \mu(d) f_2^*(\mathrm{id.}, 4md).$$

Therefore $g(\mathrm{id.}, 4m, 4D) \in \mathcal{E}(4D, 3/2, \mathrm{id.})$. But

$$\begin{aligned} A(2,4n) - 4^{-1}(1-\mathrm{i}) &= 2^{-1}(A(2,n) - 4^{-1}(1-\mathrm{i})), \\ A(p,p^2n) - p^{-1} &= p^{-1}(A(p,n) - p^{-1}), \quad p \neq 2 \end{aligned} \tag{7.20}$$

implies that

$$\begin{aligned} g(\mathrm{id.}, 4m, 4D)|\mathrm{T}(p^2) &= g(\mathrm{id.}, 4m, 4D), \quad p|2m \\ g(\mathrm{id.}, 4m, 4D)|\mathrm{T}(p^2) &= pg(\mathrm{id.}, 4m, 4D), \quad p|D/m. \end{aligned} \tag{7.21}$$

By Lemma 7.20, we have

$$V(g(\mathrm{id.}, 4m, 4D), 1) = \prod_{p|D/m} p(1+p)^{-1} \sum_{d|D/m} \mu(d) V(f_2^*(\mathrm{id.}, 4md), 1)$$
$$= -4^{-1}(1+\mathrm{i}) \prod_{p|D/m} p(1+p)^{-1} \sum_{d|D/m} \mu(d)\mu(md)(md)^{-1}$$
$$= -4^{-1}(1+\mathrm{i})\mu(m)m^{-1}.$$

Using $g(\mathrm{id.}, 4m, 4D)|\mathrm{T}(4) = g(\mathrm{id.}, 4m, 4D)$ and the method for showing Lemma 7.20, we can prove that

$$V(g(\mathrm{id.}, 4m, 4D), 1/(2p)) = 0$$

and

$$V(g(\mathrm{id.}, 4m, 4D), 1/4) = -4(1+\mathrm{i})^{-1} V(g(\mathrm{id.}, 4m, 4D), 1) = \mu(m)m^{-1}.$$

By Lemma 7.4 we get part (2) of the theorem for $l = 1$.

For $l \neq 1$, we have

$$g(\chi_l, 4m, 4D)(z) = g(\mathrm{id.}, 4m, 4D)(z)|\mathrm{T}(l) = l^{-1} \sum_{k=1}^{l} g(\mathrm{id.}, 4m, 4D)\left(\frac{z+k}{l}\right).$$

Hence $g(\chi_l, 4m, 4D) \in \mathcal{E}(4D, 3/2, \chi_l)$ and we have

$$V(g(\chi_l, 4m, 4D), 1) = l^{-1} \sum_{d|l} d^{3/2} \sum_{\substack{k=1 \\ (k,l/d)=1}}^{l/d} V(g(\mathrm{id.}, 4m, 4D), k/(ld^{-1}))$$

$$= l^{-1} \sum_{d|l} l^{3/2} \sum_{k=1}^{l/d} \left( \frac{k}{ld^{-1}} \right) V(g(\text{id.}, 4m, 4D), 1/(ld^{-1}))$$
$$= -4^{-1}(1+\mathrm{i})\mu(m)m^{-1}l^{1/2}$$

by Lemma 7.2. Since (7.21) holds also for $g(\chi_l, 4m, 4D)$, we can prove that the part (2) of the theorem holds also for $g(\chi_l, 4m, 4D)$. This completes the proof of the part (2).

Now we prove part (3) of the theorem. Similar to the above, we only need to consider the case $l = 1$. Suppose $g(\text{id.}, m, 4D) \in \mathcal{E}(4D, 3/2, \text{id.})$, then by (7.20) we have

$$\begin{aligned} & g(\text{id.}, m, 4D)|\mathrm{T}(p^2) = g(\text{id.}, m, 4D), \quad \forall\ p|m, \\ & g(\text{id.}, m, 4D)|\mathrm{T}(p^2) = pg(\text{id.}, m, 4D), \quad \forall\ p|2D/m. \end{aligned} \tag{7.22}$$

Using (7.22) for $p = 2$, we have

$$\begin{aligned} 2V(g(\text{id.}, m, 4D), 1/(4\beta)) &= 4^{-1} \sum_{k=1}^{4} V\left( g(\text{id.}, m, 4D), \frac{1+4\beta k}{4\beta} \right) \\ &= V(g(\text{id.}, m, 4D), 1/(4\beta)), \end{aligned}$$

which implies $V(g(\text{id.}, m, 4D), 1/(4\beta)) = 0$.

Using again (7.22) for $p = 2$, we have also

$$2V(g(\text{id.}, m, 4D), 1/(2\beta)) = 4^{-1} \sum_{k=1}^{4} V\left( g(\text{id.}, m, 4D), \frac{1+2\beta k}{8\beta} \right) = 0.$$

So if $V(g(\text{id.}, m, 4D), 1)$ is known, then the values of $g(\text{id.}, m, 4D)$ at all cusp points can be computed by Lemma 7.4. Put

$$f_3(\text{id.}, 4D)(z) = 2\pi \sum_{n=1}^{\infty} \lambda(n, 4D) \left( \prod_{p|D} A(p, n) - D^{-1} \right) n^{1/2} e(nz).$$

Then

$$\begin{aligned} f_1(\text{id.}, 4D) &= -f_3(\text{id.}, 4D) + 1 - 4\pi(1+\mathrm{i}) \sum_{n=1}^{\infty} \lambda(n, 4D) \big( A(2, n) - 4^{-1}(1-\mathrm{i}) \big) \\ &\quad \times \prod_{p|D} A(p, n) n^{1/2} e(nz) \\ &= D^{-1} \sum_{m|D} mg(\text{id.}, 4m, 4D) - f_3(\text{id.}, 4D), \end{aligned}$$

which implies that $f_3(\text{id.}, 4D) \in \mathcal{E}(4D, 3/2, \text{id.})$ and

$$V(f_3(\text{id.}, 4D), 1) = D^{-1} \sum_{m|D} mV(g(\text{id.}, 4m, 4D), 1) - V(f_1(\text{id.}, 4D), 1)$$

$$= -4^{-1}(1+\mathrm{i})D^{-1}\sum_{m|D}\mu(m) + (1+\mathrm{i})(4D)^{-1}$$
$$= (1+\mathrm{i})(4D)^{-1}. \tag{7.23}$$

We shall prove that $g(\mathrm{id.}, m, 4D) \in \mathcal{E}(4D, 3/2, \mathrm{id.})$ and calculate $V(g(\mathrm{id.}, m, 4D), 1)$ by induction, and hence will complete the proof of part (3).

If $D = p$ is a prime, then $g(\mathrm{id.}, p, 4p) = f_3(\mathrm{id.}, 4p) \in \mathcal{E}(4p, 3/2, \mathrm{id.})$ and then (7.23) implies the part (3). Now we use induction on the number of prime divisors of $D$. Since

$$\prod_{p|\beta}(1+p)^{-1}\prod_{p|D}(A(p,n)-p^{-1})$$
$$= \prod_{p|D/\beta}(A(p,n)-p^{-1})\prod_{p|\beta}\left\{(1+A(p,n))(1+p)^{-1}-p^{-1}\right\}$$
$$= \sum_{d|\beta}\mu(\beta|d)d\beta^{-1}\prod_{p|D/\beta}(A(p,n)-p^{-1})\prod_{p|d}(1+A(p,n))(1+p)^{-1},$$

we get

$$\sum_{D\neq\beta|D}\mu(\beta)\prod_{p|\beta}(1+p)^{-1}\prod_{p|D}(A(p,n)-p^{-1})$$
$$= \prod_{p|D}A(p,n) - D^{-1} + \sum_{D\neq\beta|D}\sum_{1\neq d|\beta}\mu(d)d\beta^{-1}$$
$$\prod_{p|D/\beta}(A(p,n)-p^{-1})\prod_{p|d}(1+A(p,n))(1+p)^{-1}.$$

But

$$\lambda_k(n,4m) = \lambda_k(n,4D)\prod_{p|D/m}(1+A_k(p,n)),$$

we get

$$\sum_{D\neq\beta|D}\mu(\beta)\prod_{p|\beta}(1+p)^{-1}g(\mathrm{id.}, D, 4D)$$
$$= f_3(\mathrm{id.}, 4D) + \sum_{D\neq\beta|D}\sum_{1\neq d|\beta}\mu(d)d\beta^{-1}g(\mathrm{id.}, D/\beta, 4D/d).$$

By induction hypothesis, we get $g(\mathrm{id.}, D, 4D) \in \mathcal{E}(4D, 3/2, \mathrm{id.})$ and

$$\sum_{D\neq\beta|D}\mu(\beta)\prod_{p|\beta}(1+p)^{-1}V(g(\mathrm{id.}, D, 4D), 1)$$
$$= (1+\mathrm{i})(4D)^{-1} + \sum_{D\neq\beta|D}\sum_{1\neq d|\beta}\mu(d)d\beta^{-1}\prod_{p|d}(1+p)^{-1}\left(-4^{-1}(1+\mathrm{i})\mu(D/\beta)\beta D^{-1}\right)$$
$$= -(4D)^{-1}(1+\mathrm{i})\mu(D)\sum_{D\neq\beta|D}\mu(\beta)\prod_{p|\beta}(1+p)^{-1}.$$

Therefore, $V(g(\mathrm{id.}, D, 4D), 1) = -(4D)^{-1}(1 + \mathrm{i})\mu(D)$, which completes the proof of part (3) for $m = D$.

For $m|D$, by the method used in the proof of the part (2), we get

$$g(\mathrm{id.}, m, 4D) = \prod_{p|D/m} p(1+p)^{-1} \sum_{d|D/m} \mu(d) g(\mathrm{id.}, md, 4md).$$

Using the induction hypothesis and the above result, $g(\mathrm{id.}, md, 4md) \in \mathcal{E}(4D, 3/2, \mathrm{id.})$, and hence $g(\mathrm{id.}, m, 4D) \in \mathcal{E}(4D, 3/2, \mathrm{id.})$ and as well as

$$\begin{aligned} V(g(\mathrm{id.}, m, 4D), 1) &= \prod_{p|D/m} p(1+p)^{-1} \sum_{d|D/m} \mu(d) V(g(\mathrm{id.}, md, 4md), 1) \\ &= -4^{-1}(1+\mathrm{i}) \prod_{p|D/m} p(1+p)^{-1} \sum_{d|D/m} \mu(d)\mu(md)(md)^{-1} \\ &= -(4m)^{-1}(1+\mathrm{i})\mu(m), \end{aligned}$$

we complete the proof of part (3).

Finally we prove part (1). For each prime divisor $p$ of $D$, we define

$$\begin{aligned} G(\chi_l, p, 4D) &= 2(\mathrm{i}-1) l^{-l/2} (l,p)^{1/2} \varepsilon_{p/(l,p)} \left(\frac{l/(l,p)}{p/(l,p)}\right) g(\chi_l, p, 4D), \\ G(\chi_l, 4, 4D) &= l^{-1/2} \varepsilon_l^{-1} g(\chi_l, 4, 4D). \end{aligned}$$

We define the following function by induction on the number of prime factors of $m$:

$$\begin{aligned} G(\chi_l, 4m, 4D) = {} & l^{-l/2}(l,m)^{1/2} \varepsilon_{l/(l,m)}^{-1} \left(\frac{m/(l,m)}{l/(l,m)}\right) \Big\{ g(\chi_l, 4m, 4D) - g(\chi_l, m, 4D) \\ & - \mu(m) m^{-1} l^{1/2} \sum_{m \neq \alpha | m} \mu(\alpha) \alpha (l,\alpha)^{-1/2} \varepsilon_{l/(l,\alpha)} \\ & \times \left(\frac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right) G(\chi_l, 4\alpha, 4D) \Big\} \end{aligned}$$

and

$$\begin{aligned} G(\chi_l, m, 4D) = {} & 2(\mathrm{i}-1) l^{-1/2} (l,m)^{1/2} \varepsilon_{m/(l,m)} \left(\frac{l/(l,m)}{m/(l,m)}\right) \\ & \times \Big\{ g(\chi_l, m, 4D) + (1+\mathrm{i})(4m)^{-1} \\ & \times \mu(m) \sum_{1, m \neq \alpha | m} \mu(\alpha) \alpha l^{1/2} (l,\alpha)^{-1/2} \varepsilon_{\alpha/(l,\alpha)}^{-1} \\ & \times \left(\frac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right) G(\chi_l, \alpha, 4D) \Big\}. \end{aligned}$$

We can prove that for $r = 0$ or 2, $V(G(\chi_l, 2^r m, 4D), p) = 0$ for all $p \in S(4D)$ except for $p = 1$ and $1/(2^r m)$ and

$$V(G(\chi_l,4m,4D),1/(4m))=V(G(\chi_l,m,4D),1/m)=1,$$
$$V(G(\chi_l,4m,4D),1)=-(4m)^{-1}(1+\mathrm{i})(l,m)^{1/2}\varepsilon_{l/(l,m)}^{-1}\left(\frac{m/(l,m)}{l/(l,m)}\right),$$
$$V(G(\chi_l,m,4D),1)=-m^{-1}(l,m)^{1/2}\varepsilon_{m/(l,m)}\left(\frac{l/(l,m)}{m/(l,m)}\right).$$

These equalities imply that $G(\chi_l,4m,4D)\,(\forall m|D)$ and $G(\chi_l,m,4D)\,(1\neq m|D)$ are linearly independent. But the number of these functions is equal to the dimension of $\mathcal{E}(4D,3/2,\chi_l)$. So they constitute a basis of $\mathcal{E}(4D,3/2,\chi_l)$, so do $g(\chi_l,4m,4D)$ and $g(\chi_l,m,4D)$. This completes the proof of the theorem. □

We shall construct a basis of $\mathcal{E}(8D,3/2,\chi_l)$ and $\mathcal{E}(8D,3/2,\chi_{2l})$ respectively. Put

$$R=\{n\in\mathbb{Z}|n\geqslant 1,n\equiv 1 \text{ or } 2 \pmod 4\}.$$

Define

$$f_4(\mathrm{id.},4D)=2\pi\sum_{n\in R}\lambda(n,4D)\prod_{p|D}(A(p,n)-p^{-1})n^{1/2}e(nz).$$

Then

$$f_2^*(\mathrm{id.},4D)+2^{-1}(1+\mathrm{i})\mu(D)f_2(\mathrm{id.},8D)=\frac{3}{2}f_4(\mathrm{id.},8D),$$

where we used the fact $A(2,n)-4^{-1}(1-\mathrm{i})=\dfrac{3}{8}(\mathrm{i}-1)$ for $n\in R$. It follows that $f_4(\mathrm{id.},8D)\in\mathcal{E}(8D,3/2,\mathrm{id.})$. By Lemma 7.21 and Lemma 7.23, we get

$$\begin{aligned}V(f_4(\mathrm{id.},8D),1/(8\beta))&=V(f_4(\mathrm{id.},8D),1/(2\beta))=0,\\V(f_4(\mathrm{id.},8D),1/\beta)&=-8^{-1}(1+\mathrm{i})\mu(D/\beta)\beta D^{-1}\varepsilon_\beta^{-1},\\V(f_4(\mathrm{id.},8D),1/(4\beta))&=\mu(D/\beta)\beta D^{-1}.\end{aligned}\tag{7.24}$$

For any $m|D$, define

$$g(\chi_l,4m,8D)=2\pi l^{1/2}\sum_{ln\in R}\lambda(ln,4D)\prod_{p|m}(A(p,ln)-p^{-1})n^{1/2}e(nz).$$

**Theorem 7.8** (1) *The functions* $g(\chi_l,4m,8D)$ $(\forall\, m|D), g(\chi_l,4m,4D)$ $(\forall\, m|D)$ $g(\chi_l,m,4D)$ $(\forall\, 1\neq m|D))$ *constitute a basis of* $\mathcal{E}(8D,3/2,\chi_l)$.

(2)

$$V(g(\chi_l,4m,8D),p)=\begin{cases}-8^{-1}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{\alpha/(l,\alpha)}^{-1}\left(\dfrac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right),\\ \qquad\text{if } p=1/\alpha,\ \alpha|m,\\ \mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{l/(l,\alpha)}\left(\dfrac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right),\\ \qquad\text{if } p=1/(4\alpha),\ \alpha|m,\\ 0, \qquad\text{otherwise.}\end{cases}$$

**Proof** We first prove (2). Since $g(\chi_l, 4m, 8D) = g(\text{id.}, 4m, 8D)|\mathrm{T}(l)$. So we only need to prove (2) for $l = 1$. We can get

$$g(\text{id.}, 4m, 8D) = \prod_{p|D/m} p(1+p)^{-1} \sum_{d|D/m} \mu(d) f_4(\text{id.}, 8md) \in \mathcal{E}(8D, 3/2, \text{id.})$$

by a similar method used in the proof of theorem 7.7. By (7.24) we have

$$V(g(\text{id.}, 4m, 8D), 1/(8\beta)) = V(g(\text{id.}, 4m, 8D), 1/(2\beta)) = 0,$$
$$V(g(\text{id.}, 4m, 8D), 1) = -8^{-1}(1+\mathrm{i})\mu(m)m^{-1},$$
$$V(g(\text{id.}, 4m, 8D), 1/4) = \mu(m)m^{-1}.$$

But

$$g(\text{id.}, 4m, 8D)|\mathrm{T}(p^2) = \begin{cases} g(\text{id.}, 4m, 8D), & \forall p|m, \\ pg(\text{id.}, 4m, 8D), & \forall p|D/m \end{cases}$$

implies (2) by Lemma 7.4.

Now we prove (1) by a method similar to the proof of Theorem 7.7. Since $\dfrac{1}{8\alpha}$ and $\dfrac{1}{4\alpha}$ are $\Gamma_0(4D)$-equivalent, we have

$$V(g(\chi_l, 4m, 4D), 1/(8\alpha)) = \mu(m/\alpha)\alpha m^{-1} l^{1/2} (l, \alpha)^{-1/2} \varepsilon_{l/(l,\alpha)} \left(\frac{2\alpha(l,\alpha)}{l/(l,\alpha)}\right).$$

Define

$$G(\chi_l, 4, 8D) = l^{-1/2}\varepsilon_l^{-1} g(\chi_l, 4, 8D),$$
$$G(\chi_l, 8, 8D) = l^{-1/2}\varepsilon_l^{-1}\left(\frac{2}{l}\right)\{g(\chi_l, 4, 4D) - g(\chi_l, 4, 8D)\}.$$

Then we define by induction

$$\begin{aligned} G(\chi_l, 8m, 8D) = {} & l^{-1/2}(l,m)^{1/2}\varepsilon_{l/(l,m)}^{-1}\left(\frac{2m/(l,m)}{l/(l,m)}\right)\Big\{g(\chi_l, 4m, 4D) \\ & - g(\chi_l, 4m, 8D) - 2^{-1}g(\chi_l, m, 4D) \\ & - \mu(m)m^{-1}l^{1/2} \sum_{m \neq \alpha | m} \mu(\alpha)\alpha(l,\alpha)^{-1/2} \\ & \times \varepsilon_{l/(l,\alpha)}\left(\frac{2\alpha/(l,\alpha)}{l/(l,\alpha)}\right) G(\chi_l, 8\alpha, 8D)\Big\} \end{aligned}$$

and

$$\begin{aligned} G(\chi_l, 4m, 8D) = {} & l^{-1/2}(l,m)^{1/2}\varepsilon_{l/(l,m)}^{-1}\left(\frac{m/(l,m)}{l/(l,m)}\right)\Big\{g(\chi_l, 4m, 8D) \\ & - 2^{-1}g(\chi_l, m, 4D) - \mu(m)m^{-1}l^{1/2} \sum_{m \neq \alpha | m} \mu(\alpha)\alpha(l,\alpha)^{-1/2} \\ & \times \varepsilon_{l/(l,\alpha)}\left(\frac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right) G(\chi_l, 4\alpha, 4D)\Big\}. \end{aligned}$$

We define also $G(\chi_l, m, 8D) = G(\chi_l, m, 4D)$ for $m \neq 1$. We can prove that for $r = 0, 2, 3, V(G(\chi_l, 2^r m, 8D), p) = 0$ for all $p \in S(8D)$ except $p = 1$ and $1/(2^r m)$ by induction, and

$$V(G(\chi_l, m, 8D), 1/m) = 1, \quad m \neq 1,$$
$$V(G(\chi_l, 4m, 8D), 1/(4m)) = V(G(\chi_l, 8m, 8D), \quad 1/(8m)) = 1,$$
$$V(G(\chi_l, m, 8D), 1) = -m^{-1}(l, m)^{1/2}\varepsilon_{m/(l,m)}\left(\frac{l/(l,m)}{m/(l,m)}\right),$$
$$V(G(\chi_l, 4m, 8D), 1) = -8^{-1}(1+\mathrm{i})m^{-1}(l, m)^{1/2}\varepsilon^{-1}_{l/(l,m)}\left(\frac{m/(l,m)}{l/(l,m)}\right),$$
$$V(G(\chi_l, 8m, 8D), 1) = -8^{-1}(1+\mathrm{i})m^{-1}(l, m)^{1/2}\varepsilon^{-1}_{l/(l,m)}\left(\frac{2m/(l,m)}{l/(l,m)}\right).$$

Gathering the values of $G(\chi_l, m, 4D)$ at $1/m$ and 1 computed in the proof of Theorem 7.7, we know that $G(\chi_l, 8m, 8D)$ ($\forall\ m|D$), $G(\chi_l, 4m, 4D)$ ($\forall\ m|D$) and $G(\chi_l, m, 8D)$ ($\forall\ 1 \neq m|D$) constitute a basis of $\mathcal{E}(8D, 3/2, \chi_l)$. This completes the proof. □

Finally we consider $\mathcal{E}(8D, 3/2, \chi_{2l})$. Define

$$g(\chi_{2l}, m, 8D) = g(\chi_l, m, 4D)|\mathrm{T}(2), \quad \forall\ 1 \neq m|D,$$
$$g(\chi_{2l}, 2m, 8D) = g(\chi_l, 4m, 8D)|\mathrm{T}(2), \quad \forall\ m|D,$$
$$g(\chi_{2l}, 8m, 8D) = g(\chi_l, 4m, 4D)|\mathrm{T}(2), \quad \forall\ m|D.$$

Then we have

**Theorem 7.9** (1) *The functions* $g(\chi_{2l}, m, 8D)$ ($\forall\ 1 \neq m|D$), $g(\chi_{2l}, 2m, 8D)$ ($\forall\ m|D$) *and* $g(\chi_{2l}, 8m, 8D)$ ($\forall\ m|D$) *constitute a basis of* $\mathcal{E}(8D, 3/2, \chi_{2l})$.

(2) *For* $p \in S(8D)$, *we have*

$$V(g(\chi_{2l},\ m, 8D), p) = \begin{cases} -2^{-3/2}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon^{-1}_{\alpha/(l,\alpha)}\left(\dfrac{2l/(l,\alpha)}{\alpha/(l,\alpha)}\right), \\ \qquad\qquad \text{if } p = 1/\alpha,\ \alpha|m, \\ 0, \qquad\qquad \text{otherwise,} \end{cases}$$

$$V(g(\chi_{2l}, 2m, 8D), p) = \begin{cases} -2^{-5/2}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon^{-1}_{\alpha/(l,\alpha)}\left(\dfrac{2l/(l,\alpha)}{\alpha/(l,\alpha)}\right), \\ \qquad\qquad \text{if } p = 1/\alpha,\ \alpha|m, \\ -2^{-1}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon^{-1}_{\alpha/(l,\alpha)}\varepsilon_l^{-1}\left(\dfrac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right), \\ \qquad\qquad \text{if } p = 1/(2\alpha),\ \alpha|m, \\ 0, \qquad\qquad \text{otherwise,} \end{cases}$$

$$V(g(\chi_{2l}, 8m, 8D), p) = \begin{cases} -2^{-3/2}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1} l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{\alpha/(l,\alpha)}^{-1}\left(\dfrac{2l/(l,\alpha)}{\alpha/(l,\alpha)}\right), \\ \qquad\qquad \text{if } p = 1/\alpha,\ \alpha | m, \\ -2^{-1}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1} l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{\alpha/(l,\alpha)}^{-1}\varepsilon_l^{-1}\left(\dfrac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right), \\ \qquad\qquad \text{if } p = 1/(2\alpha),\ \alpha | m, \\ \mu(m/\alpha)\alpha m^{-1} l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{l/(l,\alpha)}\left(\dfrac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right), \\ \qquad\qquad \text{if } p = 1/(8\alpha),\ \alpha | m, \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

**Proof** Since $\dim\mathcal{E}(8D, 3/2, \chi_{2l}) = \dim\mathcal{E}(8D, 3/2, \chi_l)$ and T(2) is a linear operator from $\mathcal{E}(8D, 3/2, \chi_l)$ to $\mathcal{E}(8D, 3/2, \chi_{2l})$, we get the part (1) by Theorem 7.8. The part (2) can be proved by Theorem 7.7, Theorem 7.8 and the definitions of $g(\chi_{2l}, 2^r m, 8D)$ $(r = 0, 1, 3)$. □

Several applications of the basis given in Theorems 7.1–7.9 will be described in the rest part of the book:

(1) Construct certain generalization of Cohen-Eisenstein (Section 7.4);

(2) Prove Siegel theorem for positive definite ternary quadratic forms (Section 10.1);

(3) Determine the eligible numbers of certain positive definite ternary quadratic forms (Section 10.3).

It is worth mentioning one more application briefly, which is due to G. Shimura, [S5] here. Let

$$f(z) = \sum_{n=1}^{\infty} a(n)\exp\{2\pi \mathrm{i} n z\}, \quad g(z) = \sum_{n=0}^{\infty} b(n)\exp\{2\pi \mathrm{i} n z\}$$

be a cusp form with the weight $k/2$ and a modular form with the weight $l/2$ respectively, where $k$ and $l (l < k)$ are positive odd numbers and the Fourier coefficients $a(n)$ and $b(n)$ are algebric numbers. Define Zeta function

$$D(s, f, g) = \sum_{n=1}^{\infty} a(n)b(n)n^{-s}.$$

Shimura proved that the number $D(t/2, f, g)$, where $1 \leqslant t \leqslant k-2$, multiplied by the number $\pi^{-r}u_{-}(F)$ is a algebric number, where the integer $r$ is determined by $t, l, k$ and $u_{-}(F)$ is the period of a modular form $F$ determined by $f$ with the weight $k-1$. In the Shimura's proof of the above result the basis constructed in Theorems 7.7–7.9 were used when $k = 3$.

## 7.4 Construction of Cohen-Eisenstein Series

Let $\chi$ be a Dirichlet character modulo $N$, and denote by $L(s,\chi)$ the associated L-series

$$L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s}.$$

For a positive integer $k$ we have that $L(1-k,\chi)=-\dfrac{B_{k,\chi}}{k}$, where the numbers $B_{k,\chi}$ are defined by

$$\sum_{a=1}^{N}\frac{\chi(a)t\mathrm{e}^{at}}{\mathrm{e}^{Nt}-1}=\sum_{k=0}^{\infty}B_{k,\chi}\frac{t^k}{k!}.$$

Fix an integer $k\geqslant 2$ and define rational numbers $H(k,n)$ by

$$H(k,n):=\begin{cases}\zeta(1-2k), & \text{if } n=0,\\ L(1-k,\chi_D)\sum_{d|f}\mu(d)\chi_D(d)d^{k-1}\sigma_{2k-1}(f/d), & \text{if } (-1)^k n=Df^2,\\ 0, & \text{otherwise,}\end{cases}$$

where $\zeta$ denotes the Riemann $\zeta$-function, $\mu$ the Moebius function, $D$ a fundamental discriminant, $\chi_D$ the quadratic character associated with $\mathbb{Q}(\sqrt{D})$ and the arithmetical function $\sigma_r$ is defined by $\sigma_r(m)=\sum_{d|m}d^r$. H.Cohen introduced the rational numbers $H(k,n)$ and proved that

$$H_k(z):=\sum_{n=0}^{\infty}H(k,n)\exp(2\pi \mathrm{i}nz) \tag{7.25}$$

is a modular form of half-integral weight $k+1/2$ for $\Gamma_0(4)$ in [C] which is now named Cohen-Eisenstein series. For $k=1$ and group $\Gamma_0(4p)$ with $p$ a prime, Cohen-Eisenstein series is defined by

$$H_{1,p}(z):=\sum_{n=0}^{\infty}H(n)_p\exp(2\pi \mathrm{i}nz), \tag{7.26}$$

where $H(n)_p:=H(p^2n)-pH(n)$ with $H(n)$ (for $n>0$) the number of classes of positive definite binary quadratic forms of discriminant $-n$ (where forms equivalent to a multiple of $x^2+y^2$ or $x^2+xy+y^2$ are counted with multiplicity $1/2$ or $1/3$ respectively) and $H(0)=-1/12$. $H_{1,p}$ is a modular form of weight $3/2$ on $\Gamma_0(4p)$.

We shall construct some explicit modular forms in the space $E^+_{k+1/2}(4N,\chi_l)$ with $k\geqslant 1$ which can be viewed as a generalization of Cohen-Eisenstein series and constitute a basis of $E^+_{k+1/2}(4N,\chi_l)$.

Let $B_{k,\chi}$ be the generalized Bernoulli number defined by

$$\sum_{a=1}^{N}\frac{\chi(a)t\mathrm{e}^{at}}{\mathrm{e}^{Nt}-1}=\sum_{k=0}^{\infty}B_{k,\chi}\frac{t^k}{k!},$$

where $N$ is a square free odd positive integer and $\chi$ is a Dirichlet character modulo $N$. And let $M^+_{k+1/2}(4N,\chi_l)$ be Kohnen's "+ space" defined by

$$M^+_{k+1/2}(4N,\chi_l) := \Big\{ f(z) = \sum_{n=0}^{\infty} a(n)q^n | f \in G(4N, k+1/2, \chi_l) \\ \text{with } a(n) = 0 \text{ whenever } \varepsilon(-1)^k n \equiv 2,3 \pmod 4 \Big\},$$

$S^+_{k+1/2}(4N,\chi_l)$ the Kohnen's "space" defined by

$$S^+_{k+1/2}(4N,\chi_l) := \Big\{ f(z) = \sum_{n=0}^{\infty} a(n)q^n | f \in S(4N, k+1/2, \chi_l) \\ \text{with } a(n) = 0 \text{ whenever } \varepsilon(-1)^k n \equiv 2,3 \pmod 4 \Big\},$$

$E^+_{k+1/2}(4N,\chi_l)$ the Kohnen's "space" defined by

$$E^+_{k+1/2}(4N,\chi_l) := \Big\{ f(z) = \sum_{n=0}^{\infty} a(n)q^n | f \in \mathcal{E}(4N, k+1/2, \chi_l) \\ \text{with } a(n) = 0 \text{ whenever } \varepsilon(-1)^k n \equiv 2,3 \pmod 4 \Big\}.$$

We define the following rational numbers $H(k,l,N,N;n)$ and $H(k,l,m,N;n)$ with $N \neq m|N$:

$$H(k,l,N,N;n) := \begin{cases} L_N(1-2k, \text{id.}), & \text{if } n = 0, \\ L_N(1-k, \chi_{D'_n}) \sum_{d|f_n} \mu(d)\chi'_l(d)\chi_{D_n}(d)d^{k-1}\sigma_{N,2k-1}(f_n/d), & \\ & \text{if } \varepsilon(-1)^k n = D_n f_n^2 \text{ and } (-1)^k ln = D'_n(f'_n)^2, \\ 0, & \text{otherwise,} \end{cases}$$

where $\sigma_{N,2k-1}$ is the arithmetical function defined by $\sigma_{N,2k-1}(t) := \sum_{d|t,(d,N)=1} d^{2k-1}$, and

$$H(k,l,m,N;n) := \begin{cases} 0, & \text{if } n = 0, \\ L_m(1-k, \chi_{D'_n}) \prod_{p|N/m} \dfrac{1 - p^{-k}\left(\frac{D'_n}{p}\right)}{1-p^{-2k}} \left(\dfrac{(l,D_n)}{(l,D_n,m)}\right)^{2k-1} & \\ \quad \times \sum_{d|f_n} \mu(d)\chi'_l(d)\chi_{D_n}(d)d^{k-1}\sigma_{m,N,2k-1}(f_n/d), & \\ & \text{if } \varepsilon(-1)^k n = D_n f_n^2 \text{ and } (-1)^k ln = D'_n(f'_n)^2, \\ 0, & \text{otherwise,} \end{cases}$$

where $\sigma_{m,N,2k-1}$ is the arithmetical function defined by

$$\sigma_{m,N,2k-1}(t) := \sum_{\substack{d|t,(d,m)=1,\\ (t/d,N/m)=1}} d^{2k-1}.$$

Note that $H(k,1,1,1;n) = H(k,n)$ are just the rational numbers defined by H.Cohen.

**Theorem 7.10** *Let $N$ be a square-free odd positive integer and $l$ a divisor of $N$. Then*

(1) *If $k=1$ and $N>1$, then the functions defined by*

$$H_1(\chi_l,N,N)(z) := \sum_{n=0}^{\infty} H(1,l,N,N;n)q^n,$$

$$H_1(\chi_l,m,N)(z) := \sum_{n=0}^{\infty} H(1,l,m,N;n)q^n \quad \text{for all } m \text{ with } 1,N \neq m|N$$

*belong to $E_{3/2}^{+}(4N,\chi_l)$ and constitute a basis of the space $E_{3/2}^{+}(4N,\chi_l)$.*

(2) *If $k \geqslant 2$, then the functions defined by*

$$H_k(\chi_l,N,N)(z) := \sum_{n=0}^{\infty} H(k,l,N,N;n)q^n,$$

$$H_k(\chi_l,m,N)(z) := \sum_{n=0}^{\infty} H(k,l,m,N;n)q^n \quad \text{for all } m \text{ with } N \neq m|N$$

*belong to $E_{k+1/2}^{+}(4N,\chi_l)$ and constitute a basis of the space $E_{k+1/2}^{+}(4N,\chi_l)$.*

**Remark 7.1** $H_k(\text{id.},1,1)(z)$ is just the Cohen-Eisenstein series $H_k(z)$. Since

$$L_N(-1,\text{id.}) = -\frac{1}{12}\prod_{p|N}(1-p)$$

and

$$H(n) = \frac{h(D)}{w(D)}\sum_{d|f}\mu(d)\left(\frac{D}{d}\right)\sigma_1(f/d),$$

where $-n = Df^2$ with $D$ a negative fundamental discriminant, $w(D)$ half the number of units in $\mathbb{Q}(\sqrt{D})$, we see that $H_1(\text{id.},p,p)$ is just the Cohen-Eisenstein series $H_{1,p}(z)$ by class number formula.

We need the following:

**Lemma 7.25** *Let $n$ be a positive integer with $(-1)^k n = D(2^r f)^2$ where $D$ is a fundamental discriminant, $f$ is a positive odd integer and $r \geqslant -1$ is an integer. Then*

$$(A_k(2,n)-\eta_2)2^{k-2}(1-(-1)^{\lambda}\mathrm{i})(1-2^{k-2})\left(1-2^{-\lambda}\left(\frac{D}{2}\right)\right)(1-2^{1-k})^{-1}$$

$$= 2^{-r(k-2)}\left(1 - 2^{\lambda-1}\left(\frac{D}{2}\right)\right),$$

$$(A_k(p,n) - \eta_p)p^{\nu_p(f)(k-2)}(1-p^{k-2})\left(1 - p^{-\lambda}\left(\frac{D}{p}\right)\right)(1-p^{1-k})^{-1}$$

$$= 1 - p^{\lambda-1}\left(\frac{D}{p}\right), \quad p \text{ is an odd prime },$$

*where* $\lambda = (k-1)/2$ *for an odd integer* $k$.

**Proof** The lemma can be proved by the definitions and some direct calculations. □

**Proof of Thorem 7.10** (1) We know that the dimension of $E^+_{3/2}(4N, \chi)$ is $2^{t(N)} - 1$. So we only need to prove that $H_1(\chi_l, m, N)(z)$ $(1 \neq m|N)$ belong to $E^+_{3/2}(4N, \chi)$ and are linearly independent.

By the results in Section 7.3 we know that the following functions

$$H_1'(\chi_l, m, N) := g(\chi_l, 4m, 4N) - \frac{3}{2}g(\chi_l, m, 4N), \quad \forall\ 1 \neq m|N \tag{7.27}$$

belong to $\mathcal{E}(4N, 3/2, \chi_l)$ and are linearly independent. We now prove that $H_1'(\chi_l, m, N)$ belongs to $E^+_{3/2}(4N, \chi_l)$ and is a non-zero multiple of $H_1(\chi_l, m, N)$ with $1 \neq m|N$. By the definition, we see that

$$H_1'(\chi_l, m, N) := \sum_{n=1}^{\infty} a_m(n)q^n = -4\pi(1+\mathrm{i})\sum_{n=1}^{\infty}\lambda_3(ln, 4N)(A_3(2, ln) + 2^{-3}(1-\mathrm{i}))$$
$$\times \prod_{p|m}(A_3(p, ln) - p^{-1})(ln)^{1/2}q^n, \quad \forall\ m|N, m \neq 1, N, \tag{7.28}$$
$$H_1'(\chi_l, N, N) := \sum_{n=1}^{\infty} a_N(n)q^n = 1 - 4\pi(1+\mathrm{i})\sum_{n=1}^{\infty}\lambda_3(ln, 4N)(A_3(2, ln) + 2^{-3}(1-\mathrm{i}))$$
$$\times \prod_{p|N}(A_3(p, ln) - p^{-1})(ln)^{1/2}q^n.$$

Denote

$$I(l, n) := A_3(2, ln) + 2^{-3}(1-\mathrm{i}). \tag{7.29}$$

By the definition of $A(2, ln)$, we see easily that $I(l, n) = 0$ if $ln \equiv 1, 2 \pmod 4$ and hence $a_m(n) = 0$, $a_N(n) = 0$ if $ln \equiv 1, 2 \pmod 4$. This implies that $H_1'(\chi_l, m, N) \in E^+_{3/2}(4N, \chi_l)$. When $ln \equiv 0, 3 \pmod 4$, $\varepsilon = (-1)^{\frac{l-1}{2}} \equiv l \pmod 4$ which implies that $\varepsilon n \equiv 0, 3 \pmod 4$. Hence we can suppose that $-\varepsilon n = D_n f_n^2$ and $-ln = D_n'(f_n')^2$ with $D_n$ and $D_n'$ fundamental discriminants, $f_n$ and $f_n'$ positive integers. It is clear that $D_n' = \varepsilon l D_n/(l, D_n)^2$, $f_n' = (l, D_n)f_n$. From these we see that if $p \nmid N$ then $p|D_n$ if and only if $p|D_n'$ and $\nu_p(f_n) = \nu_p(f_n')$. By the definition of $A_3(p, ln)$ and some calculations we have that

$$I(l,n) = \begin{cases} 4^{-1}(1-\mathrm{i})\left(1+\frac{1}{2}\left(\frac{D_n'}{2}\right)\right), & \text{if } ln \equiv 3 \pmod 4, \\ \frac{3}{16}(1-\mathrm{i})\sum_{t=0}^{\nu_2(f_n')} 2^{-t}, & \text{if } ln \equiv 0 \pmod 4 \text{ and } 2 \nmid \nu_2(ln), \\ 4^{-1}(1-\mathrm{i})\left(1+\frac{1}{2}\left(\frac{D_n'}{2}\right)\right)\left(\sum_{t=0}^{\nu_2(f_n')} 2^{-t} - \frac{1}{2}\left(\frac{D_n'}{2}\right)\sum_{t=0}^{\nu_2(f_n')-1} 2^{-t}\right), & \\ & \text{if } ln \equiv 0 \pmod 4 \text{ and } 2|\nu_2(ln), 2 \nmid D_n', \\ \frac{3}{16}(1-\mathrm{i})\sum_{t=0}^{\nu_2(f_n')} 2^{-t}, & \text{if } ln \equiv 0 \pmod 4 \text{ and } 2|\nu_2(ln), 2|D_n'. \end{cases} \tag{7.30}$$

By Lemma 7.25 we obtain that for $ln \equiv 0, 3 \pmod 4$

$$\begin{aligned} \prod_{p|m}(A_3(p,ln)-p^{-1})(ln)^{1/2} &= | D_n' |^{1/2} \prod_{p|m}\left(1-\left(\frac{D_n'}{p}\right)\right)(1-p)^{-1} \\ &\quad \times \left(1-p^{-1}\left(\frac{D_n'}{p}\right)\right)^{-1}(1-p^{-2})\prod_{p\nmid m} p^{\nu_p(f_n')} \\ &= | D_n' |^{1/2}\, \frac{(l,D_n)}{(l,D_n,m)} \prod_{p|m}\left(1-\left(\frac{D_n'}{p}\right)\right)(1-p)^{-1} \\ &\quad \times \left(1-p^{-1}\left(\frac{D_n'}{p}\right)\right)^{-1}(1-p^{-2})\prod_{p\nmid m} p^{\nu_p(f_n)}. \end{aligned} \tag{7.31}$$

We also have that

$$\begin{aligned} &\beta_3(ln, \chi_N, 4N) \\ &= \sum_{\substack{(ab)^2|ln,(ab,2N)=1 \\ a,b \text{ positive integers}}} \mu(a)\left(\frac{-ln}{a}\right)(ab)^{-1} \\ &= \prod_{p|D_n', p\nmid 2N} \sum_{t=0}^{\nu_p(f_n')} p^{-t} \prod_{p\nmid 2ND_n'}\left(\sum_{t=0}^{\nu_p(f_n')} p^{-t} - p^{-1}\left(\frac{D_n'}{p}\right)\sum_{t=0}^{\nu_p(f_n')-1} p^{-t}\right) \\ &= \prod_{p|D_n, p\nmid 2N} \sum_{t=0}^{\nu_p(f_n)} p^{-t} \prod_{p\nmid 2ND_n}\left(\sum_{t=0}^{\nu_p(f_n)} p^{-t} - p^{-1}\chi_l'(p)\left(\frac{D_n}{p}\right)\sum_{t=0}^{\nu_p(f_n)-1} p^{-t}\right), \end{aligned} \tag{7.32}$$

where we have used the fact that $p|D_n$ if and only if $p|D_n'$ and $\nu_p(f_n) = \nu_p(f_n')$ for $p \nmid N$. By the functional equation of L-functions we see that

$$\frac{-4\pi(1+\mathrm{i})L_{4N}\left(1,\left(\frac{D'_n}{\cdot}\right)\right)}{L_{4N}(2,\mathrm{id.})}$$
$$=2(1+\mathrm{i})\mid D'_n\mid^{-1/2}\frac{L\left(0,\left(\frac{D'_n}{\cdot}\right)\right)}{\zeta(-1)}\prod_{p|2N}\frac{\left(1-p^{-1}\left(\frac{D'_n}{p}\right)\right)}{1-p^{-2}}. \tag{7.33}$$

Using these equalities (7.28)–(7.33), we finally find that for $1, N \neq m|N$ and $n \geqslant 1$

$$a_m(n)=\frac{L_m\left(0,\left(\frac{D'_n}{\cdot}\right)\right)}{L_m(-1,\mathrm{id.})}\frac{(l,D_n)}{(l,D_n,m)}\prod_{p|N/m}\frac{\left(1-p^{-1}\left(\frac{D'_n}{p}\right)\right)}{1-p^{-2}}$$
$$\times\sum_{d|f_n}\mu(d)\chi'_l(d)\left(\frac{D_n}{d}\right)\sum_{\substack{e|f_n/d,(e,m)=1\\(f_n/de,N/m)=1}}e,$$

where we used the fact that

$$\prod_{p\nmid m}p^{\nu_p(f_n)}\prod_{p|D_n,p\nmid N}\sum_{t=0}^{\nu_p(f_n)}p^{-t}\prod_{p\nmid ND_n}\left(\sum_{t=0}^{\nu_p(f_n)}p^{-t}-p^{-1}\chi'_l(p)\left(\frac{D_n}{p}\right)\sum_{t=0}^{\nu_p(f_n)-1}p^{-t}\right)$$
$$=\prod_{p|N/m}p^{\nu_p(f_n)}\prod_{p|D_n,p\nmid N}\sum_{t=0}^{\nu_p(f_n)}p^{t}\prod_{p\nmid ND_n}\left(\sum_{t=0}^{\nu_p(f_n)}p^{t}-p^{-1}\chi'_l(p)\left(\frac{D_n}{p}\right)\sum_{t=0}^{\nu_p(f_n)-1}p^{t}\right)$$
$$=\sum_{d|f_n}\mu(d)\chi'_l(d)\left(\frac{D_n}{d}\right)\sum_{\substack{e|f_n/d,(e,m)=1,\\(f_n/de,N/m)=1}}e.$$

Similarly we have that

$$a_N(n)=\frac{L_N\left(0,\left(\frac{D'_n}{\cdot}\right)\right)}{L_N(-1,\mathrm{id.})}\sum_{d|f_n}\mu(d)\chi'_l(d)\left(\frac{D_n}{d}\right)\sum_{\substack{e|f_n/d\\(e,N)=1}}e.$$

These show that

$$H'_1(\chi_l,N,N)=1+\sum_{\substack{n>0,\\ln\equiv 0,3(\mathrm{mod}\ 4)}}\left\{\frac{L_N\left(0,\left(\frac{D'_n}{\cdot}\right)\right)}{L_N(-1,\mathrm{id.})}\sum_{d|f_n}\mu(d)\chi'_l(d)\left(\frac{D_n}{d}\right)\sum_{\substack{e|f_n/d\\(e,N)=1}}e\right\}q^n,$$

$$H_1'(\chi_l, m, N) = \sum_{\substack{n>0,\\ ln\equiv 0,3 (\mathrm{mod}\ 4)}} \left\{ \frac{L_m\left(0, \left(\frac{D_n'}{\cdot}\right)\right)}{L_m(-1, \mathrm{id.})} \frac{(l, D_n)}{(l, D_n, m)} \right.$$
$$\left. \times \prod_{p|N/m} \frac{\left(1 - p^{-1}\left(\frac{D_n'}{p}\right)\right)}{1-p^{-2}} \sum_{d|f_n} \mu(d)\chi_l'(d)\left(\frac{D_n}{d}\right) \sum_{\substack{e|f_n/d, (e,m)=1\\ (f_n/de, N/m)=1}} e \right\} q^n.$$

Comparing the coefficients of $H_1(\chi_l, m, N)$ and $H_1'(\chi_l, m, N)$, we find that

$$\begin{aligned} H_1(\chi_l, m, N) &= L_m(-1, \mathrm{id.})H_1'(\chi_l, m, N) \\ &= -\frac{1}{12}\prod_{p|m}(1-p)H_1'(\chi_l, m, N) \end{aligned}$$

for all $1 \neq m|N$. This completes the proof of (1).

(2) We define the following functions

$$H_k'(\chi_l, m, N) := g_{2k+1}(\chi_l, 4m, 4N) + (2^{-2k-1}(1+(-1)^k i) + \eta_2) g_{2k+1}(\chi_l, m, 4N).$$

Similar to the proof of (1), we want to prove that $H_k'(\chi_l, m, N)$ with $m|N$ constitute a basis of $E_{k+1/2}^+(4N, \chi_l)$ and is a non-zero multiple of $H_k(\chi_l, m, N)$. Since the dimension of $E_{k+1/2}^+(4N, \chi_l)$ is equal to the number of positive divisors of $N$, by Theorem 7.1 we only need to show that $H_k'(\chi_l, m, N) \in E_{k+1/2}^+(4N, \chi_l)$ and is a non-zero multiple of $H_k(\chi_l, m, N)$. By results in Section 7.1 we see that

$$\begin{aligned} H_k'(\chi_l, m, N) &:= \sum_{n=1}^{\infty} a_m(n) q^n \\ &= \sum_{n=1}^{\infty} \lambda_{2k+1}'(ln, 4N)(A_{2k+1}(2, ln) + 2^{-2k-1}(1+(-1)^k \mathrm{i})) \\ &\quad \times \prod_{p|m}(A_{2k+1}(p, ln) - \eta_p)(ln)^{k-1/2} q^n, \quad \forall m|N, m \neq N, \\ H_k'(\chi_l, N, N) &:= \sum_{n=1}^{\infty} a_N(n) q^n \\ &= 1 + \sum_{n=1}^{\infty} \lambda_{2k+1}'(ln, 4N)(A_{2k+1}(2, ln) \\ &\quad + 2^{-2k-1}(1+(-1)^k \mathrm{i})) \prod_{p|N}(A_{2k+1}(p, ln) - \eta_p)(ln)^{k-1/2} q^n. \end{aligned} \tag{7.34}$$

Let

$$I_k(l, n) := A_{2k+1}(2, ln) + 2^{-2k-1}(1+(-1)^k \mathrm{i}).$$

By the definition of $A_k(2,\ ln)$, we see that $I_k(l,n) = 0$ if $(-1)^k ln \equiv 2,3 \pmod 4$. This shows that $a_m(n) = 0$ and $a_N(n) = 0$ whenever $(-1)^k ln \equiv 2,3 \pmod 4$ and hence $H_k'(\chi_l, m, N) \in E_{k+1/2}^+(4N, \chi_l)$. Now we must compute the coefficients $a_m(n)$ of $H_k'(\chi_l, m, N)$ for all $m|N$. When $(-1)^k ln \equiv 0,1 \pmod 4$, we denote that $\varepsilon = (-1)^{\frac{l-1}{2}} \equiv l \pmod 4, (-1)^k \varepsilon n = D_n f_n^2$ and $l(-1)^k ln = D_n'(f_n')^2$ with $D_n, D_n'$ fundamental discriminants, $f_n, f_n'$ positive integers. It is clear that $D_n' = \varepsilon l D_n/(l, D_n)^2$ and $f_n' = (l, D_n) f_n$.

By the definition of $A_k(p, ln)$ and some calculations we have that

$$I_k(l,n) = \begin{cases} 2^{-2k}(1+(-1)^k\mathrm{i})\left(1+2^{-k}\left(\dfrac{D_n'}{2}\right)\right), & \text{if } (-1)^k ln \equiv 1 (\mathrm{mod}\ 4), \\ 2^{-2k}(1+(-1)^k\mathrm{i})(1-2^{-2k})\displaystyle\sum_{t=0}^{\nu_2(f_n')} 2^{(1-2k)t}, & \\ \qquad \text{if } (-1)^k ln \equiv 0(\mathrm{mod}\ 4) \text{ and } 2 \nmid \nu_2(ln), & \\ 2^{-2k}(1+(-1)^k\mathrm{i})\left(1+2^{-k}\left(\dfrac{D_n'}{2}\right)\right) & \\ \quad \times\left(\displaystyle\sum_{t=0}^{\nu_2(f_n')} 2^{(1-2k)t} - 2^{-k}\left(\dfrac{D_n'}{2}\right)\sum_{t=0}^{\nu_2(f_n')-1} 2^{(1-2k)t}\right), & \\ \qquad \text{if } (-1)^k ln \equiv 0(\mathrm{mod}\ 4), 2|\nu_2(ln) \text{ and } 2 \nmid D_n', & \\ 2^{-2k}(1+(-1)^k\mathrm{i})(1-2^{-2k})\displaystyle\sum_{t=0}^{\nu_2(f_n')} 2^{(1-2k)t}, & \\ \qquad \text{if } ln \equiv 0(\mathrm{mod}\ 4), 2|\nu_2(ln) \text{ and } 2|D_n'. & \end{cases} \tag{7.35}$$

By Lemma 7.25 we obtain that for $(-1)^k ln \equiv 0,1 \pmod 4$

$$\begin{aligned} &\prod_{p|m}(A_{2k+1}(p, ln) - \eta_p)(ln)^{k-1/2} \\ &= | D_n' |^{k-1/2} \prod_{p|m}\left(1 - p^{k-1}\left(\frac{D_n'}{p}\right)\right)(1-p^{2k-1})^{-1} \\ &\quad \times \left(1 - p^{-k}\left(\frac{D_n'}{p}\right)\right)^{-1}(1-p^{-2k})\prod_{p\nmid m} p^{(2k-1)\nu_p(f_n')} \\ &= | D_n' |^{k-1/2}\left(\frac{(l, D_n)}{(l, D_n, m)}\right)^{2k-1}\prod_{p|m}\left(1 - p^{k-1}\left(\frac{D_n'}{p}\right)\right)(1-p^{2k-1})^{-1} \\ &\quad \times \left(1 - p^{-k}\left(\frac{D_n'}{p}\right)\right)^{-1}(1-p^{-2k})\prod_{p\nmid m} p^{(2k-1)\nu_p(f_n)}. \end{aligned} \tag{7.36}$$

We also have that

$$
\begin{aligned}
&\beta_{2k+1}(ln, \chi_N, 4N) \\
&= \sum_{\substack{(ab)^2 | ln, (ab, 2N)=1 \\ a,b \text{ positive integers}}} \mu(a) \left(\frac{(-1)^k ln}{a}\right) a^{-k} b^{1-2k} \\
&= \prod_{p | D_n', p \nmid 2N} \sum_{t=0}^{\nu_p(f_n')} p^{(1-2k)t} \prod_{p \nmid 2ND_n'} \left( \sum_{t=0}^{\nu_p(f_n')} p^{(1-2k)t} - p^{-k} \left(\frac{D_n'}{p}\right) \sum_{t=0}^{\nu_p(f_n')-1} p^{(1-2k)t} \right) \\
&= \prod_{p | D_n, p \nmid 2N} \sum_{t=0}^{\nu_p(f_n)} p^{(1-2k)t} \prod_{p \nmid 2ND_n} \left( \sum_{t=0}^{\nu_p(f_n)} p^{(1-2k)t} - p^{-k} \chi_l'(p) \left(\frac{D_n}{p}\right) \sum_{t=0}^{\nu_p(f_n)-1} p^{(1-2k)t} \right),
\end{aligned}
\tag{7.37}
$$

where we have used the fact that $p|D_n$ if and only if $p|D_n'$ and $\nu_p(f_n) = \nu_p(f_n')$ for $p \nmid N$. By the functional equation of L-function we see that

$$
\begin{aligned}
\lambda_k'(ln, 4N) = 2^{2k-1}(1-(-1)^k \mathrm{i}) \mid D_n' \mid^{1/2-k} \frac{L\left(1-k, \left(\frac{D_n'}{\cdot}\right)\right)}{\zeta(1-2k)} \\
\times \prod_{p|2N} \frac{\left(1 - p^{-k}\left(\frac{D_n'}{p}\right)\right)}{1-p^{-2k}}.
\end{aligned}
\tag{7.38}
$$

Using these equalities (7.33)–(7.37), we finally find that for $N \neq m|N$ and $n \geqslant 1$

$$
\begin{aligned}
a_m(n) = &\frac{L_m\left(1-k, \left(\frac{D_n'}{\cdot}\right)\right)}{L_m(1-2k, \mathrm{id.})} \left(\frac{(l, D_n)}{(l, D_n, m)}\right)^{2k-1} \prod_{p|N/m} \frac{\left(1-p^{-k}\left(\frac{D_n'}{p}\right)\right)}{1-p^{-2k}} \\
&\times \sum_{d|f_n} \mu(d) \chi_l'(d) \left(\frac{D_n}{d}\right) d^{k-1} \sum_{\substack{e|f_n/d, (e,m)=1 \\ (f_n/de, N/m)=1}} \mathrm{e}^{2k-1}
\end{aligned}
$$

where we used the fact that

$$
\begin{aligned}
&\prod_{p \nmid m} p^{(2k-1)\nu_p(f_n)} \prod_{p|D_n, p \nmid N} \sum_{t=0}^{\nu_p(f_n)} p^{(1-2k)t} \\
&\times \prod_{p \nmid ND_n} \left( \sum_{t=0}^{\nu_p(f_n)} p^{(1-2k)t} - p^{-k} \chi_l'(p) \left(\frac{D_n}{p}\right) \sum_{t=0}^{\nu_p(f_n)-1} p^{(1-2k)t} \right) \\
= &\prod_{p|N/m} p^{(2k-1)\nu_p(f_n)} \prod_{p|D_n, p \nmid N} \sum_{t=0}^{\nu_p(f_n)} p^{(2k-1)t}
\end{aligned}
$$

$$\times \prod_{p\nmid ND_n} \left( \sum_{t=0}^{\nu_p(f_n)} p^{(2k-1)t} - p^{k-1}\chi_l'(p)\left(\frac{D_n}{p}\right) \sum_{t=0}^{\nu_p(f_n)-1} p^{(2k-1)t} \right)$$
$$= \sum_{d|f_n} \mu(d)\chi_l'(d)\left(\frac{D_n}{d}\right) d^{k-1} \sum_{\substack{e|f_n/d,(e,m)=1 \\ (f_n/de,N/m)=1}} \mathrm{e}^{2k-1}.$$

Similarly we have that

$$a_N(n) = \frac{L_N\left(1-k,\left(\frac{D_n'}{\cdot}\right)\right)}{L_N(1-2k,\mathrm{id.})} \sum_{d|f_n} \mu(d)\chi_l'(d)\left(\frac{D_n}{d}\right) d^{k-1} \sum_{e|f_n/d,(e,N)=1} \mathrm{e}^{2k-1}.$$

These show that

$$H_k'(\chi_l, N, N) = 1 + \sum_{\substack{n>0, \\ (-1)^k ln \equiv 0,1 \pmod 4}} \left\{ \frac{L_N\left(1-k,\left(\frac{D_n'}{\cdot}\right)\right)}{L_N(1-2k,\mathrm{id.})} \right.$$
$$\left. \times \sum_{d|f_n} \mu(d)\chi_l'(d)\left(\frac{D_n}{d}\right) d^{k-1} \sum_{\substack{e|f_n/d \\ (e,N)=1}} \mathrm{e}^{2k-1} \right\} q^n;$$

$$H_k'(\chi_l, m, N) = \sum_{\substack{n>0, \\ (-1)^k ln \equiv 0,1 \pmod 4}} \left\{ \frac{L_m\left(1-k,\left(\frac{D_n'}{\cdot}\right)\right)}{L_m(1-2k,\mathrm{id.})} \left(\frac{(l,D_n)}{(l,D_n,m)}\right)^{2k-1} \right.$$
$$\times \prod_{p|N/m} \frac{\left(1-p^{-k}\left(\frac{D_n'}{p}\right)\right)}{1-p^{-2k}} \sum_{d|f_n} \mu(d)\chi_l'(d)\left(\frac{D_n}{d}\right) d^{k-1}$$
$$\left. \times \sum_{\substack{e|f_n/d,(e,m)=1 \\ (f_n/de,N/m)=1}} \mathrm{e}^{2k-1} \right\} q^n$$

Comparing the coefficients of $H_k(\chi_l, m, N)$ and $H_k'(\chi_l, m, N)$ show that $H_k(\chi_l, m, N) = L_m(1-2k,\mathrm{id.})H_k'(\chi_l, m, N) = -\frac{B_{2k}}{2k} H_k'(\chi_l, m, N)$ for all $m|N$ where $B_r := B_{r,\mathrm{id.}}$ is the $r$-th Bernoulli number. This completes the proof of (2). □

## 7.5 Construction of Eisenstein Series with Integral Weight

Let $N$ and $k$ be positive integers, $\omega$ a character modulo $N$ with $\omega(-1) = (-1)^k$. Take a positive integer $Q$ such that $Q|N$ and $(Q, N/Q) = 1$. Define a matrix

$$W(Q) = \begin{pmatrix} Qs & t \\ Nu & Qv \end{pmatrix} \in GL_2^+(\mathbb{Z}), \quad \det(W(Q)) = Q.$$

We see that $W(Q)\Gamma_0(N)W(Q)^{-1} = \Gamma_0(N)$.

**Lemma 7.26** *Let* $W(Q)$ *be as above,* $\omega = \omega_1\omega_2$*, where* $\omega_1$ *and* $\omega_2$ *are characters modulo* $Q$ *and* $N/Q$ *respectively. If* $f \in G(N, k, \omega)$ (resp. $\mathcal{E}(N, k, \omega)$)*, then* $g = f|[W(Q)]_k \in G(N, k, \overline{\omega_1}\omega_2)$ (resp.$\mathcal{E}(N, k, \overline{\omega_1}\omega_2)$).

**Proof** Take any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, set $W(Q)\gamma W(Q)^{-1} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$. It is easy to check that $c_0 \equiv 0 \pmod N$, $d_0 \equiv a \pmod Q$, $d_0 \equiv d \pmod{(N/Q)}$. Hence we see that

$$g|[\gamma] = f|[W(Q)\gamma W(Q)^{-1}W(Q)] = \omega(d_0)f|[W(Q)] = \omega(d_0)g,$$

i.e., $g \in G(N, k, \overline{\omega_1}\omega_2)$. Similar to Lemma 5.35, we have for $N|M$ that

$$\mathcal{E}(N, k, \omega) = G(N, k, \omega) \bigcap \mathcal{E}(\Gamma(M), k),$$

from which the last conclusion of the lemma can be deduced. This completes the proof. □

Let now $E_k(z, \omega_1, \omega_2)$ be as in Section 2.2. By the computation in Section 2.2 we see that $E_k(z, \omega_1, \omega_2)$ is a common eigenfunction of all Hecke operators and

$$E_k(z, \omega_1, \omega_2)|\mathrm{T}(p) = (\omega_1(p) + p^{k-1}\omega_2(p))E_k(z, \omega_1, \omega_2).$$

Similar to Theorem 5.18 we have the following:

**Lemma 7.27** *Let* $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N, k, \omega)$*. Assume that* $t$ *is the conductor of* $\omega$ *and* $\psi$ *is a primitive character modulo* $r$*. Put*

$$h(z) = \sum_{u=1}^{r} \overline{\psi}(u)f(z + u/r) = \sum_{u=1}^{r} \overline{\psi}(u)e(u/r) \sum_{n=1}^{\infty} \psi(n)a(n)e(nz),$$

*then* $h(z) \in G(M, k, \omega\psi^2)$ *with* $M = [N, rt, r^2]$*. If* $f(z) \in S(N, k, \omega)$ (resp. $\mathcal{E}(N, k, \omega)$)*, then* $h(z) \in S(M, k, \omega\psi^2)$ (resp. $\mathcal{E}(M, k, \omega\psi^2)$)*.*

Let $E_k(z, \omega, N)$ be as in Section 2.2. From the transformation formula of $E_k(z, \omega, N)$ and a standard method invented by Petersson we know that $E_k(z, \omega, N) \in \mathcal{E}(N, k, \omega)$ for $k \neq 2$ or $k = 2$, $\omega \neq \mathrm{id}$. Hence we know that $E_k(z, \omega, N)|[W(Q)] \in \mathcal{E}(N, k, \overline{\omega_1}\omega_2)$ from Lemma 7.26. Let now $\omega = \omega_1\omega_2$. Assume that $r_1$ and $r_2$ are the conductors of $\omega_1$ and $\omega_2$ respectively. Write

$$r_1 = \prod_{i=1}^{m} p_i^{\alpha_i}, r_2 = \prod_{i=1}^{m} p_i^{\beta_i}, \quad \omega_1 = \prod_{i=1}^{m} \omega_{1,i}, \omega_2 = \prod_{i=1}^{m} \omega_{2,i},$$

where $\omega_{1,i}$ and $\omega_{2,i}$ have conductors $p_i^{\alpha_i}$ and $p_i^{\beta_i}$ respectively. Without loss of generality, we may assume that there is a positive integer $m_1$ such that $\alpha_i \geqslant \beta_i$ for $1 \leqslant i \leqslant m_1 \leqslant m$ and $\alpha_i < \beta_i$ for $m_1 < i \leqslant m$. In terms of Lemma 7.26, we know that there is a $\widetilde{E}_k(z)$ such that

$$\begin{aligned}\widetilde{E}_k(z) &= E_k\left(z, \prod_{i=1}^{m_1} \omega_{1,i}\overline{\omega}_{2,i}, \prod_{i=m_1+1}^{m} \overline{\omega}_{1,i}\omega_{2,i}\right) \\ &\in \mathcal{E}\left(\prod_{i=1}^{m_1} p_i^{\alpha_i} \prod_{i=m_1+1}^{m} p_i^{\beta_i}, k, \prod_{i=1}^{m_1} \omega_{1,i}\overline{\omega}_{2,i} \prod_{i=m_1+1}^{m} \overline{\omega}_{1,i}\omega_{2,i}\right).\end{aligned}$$

Put $\psi = \prod_{i=1}^{m_1} \omega_{2,i} \prod_{i=m_1+1}^{m} \omega_{1,i}$, then the conductor of $\psi$ is $r = \prod_{i=1}^{m_1} p_i^{\beta_i} \prod_{i=m_1+1}^{m} p_i^{\alpha_i}$. Set

$$E_k(z, \omega_1, \omega_2) = \left(\sum_{u=1}^{r} \overline{\psi}(u) e(u/r)\right)^{-1} \sum_{u=1}^{r} \overline{\psi}(u) \widetilde{E}_k(z + u/r), \tag{7.39}$$

then $E_k(z, \omega_1, \omega_2) \in \mathcal{E}(r_1 r_2, k, \omega)$ by Lemma 7.27. And we have also that

$$\begin{aligned}L(s, E_k(z, \omega_1, \omega_2)) &= L\left(s, \psi \prod_{i=1}^{m_1} \omega_{1,i}\overline{\omega}_{2,i}\right) L\left(s - k + 1, \psi \prod_{i=m_1+1}^{m} \overline{\omega}_{1,i}\omega_{2,i}\right) \\ &= L(s, \omega_1) L(s - k + 1, \omega_2).\end{aligned}$$

Let $l$ be a positive integer, $\omega$ a character modulo $N$ with conductor $r$, $\omega_1$ and $\omega_2$ two primitive characters modulo $r_1$ and $r_2$ respectively. Denote by $A(N, r)$ the number of $(l, \omega_1, \omega_2)$ satisfying

$$\omega = \omega_1 \omega_2, l r_1 r_2 | N. \tag{7.40}$$

For any such $(l, \omega_1, \omega_2)$ there is a function

$$E_k(lz, \omega_1, \omega_2) \in \mathcal{E}(l r_1 r_2, k, \omega) \subset \mathcal{E}(N, k, \omega)$$

such that

$$L(s, E_k(lz, \omega_1, \omega_2)) = l^{-s} L(s, \omega_1) L(s - k + 1, \omega_2).$$

**Lemma 7.28** *We have that*

$$A(N, r) = \sum_{c|N, (c, N/c)|N/r} \varphi((c, N/c)).$$

**Proof** Let $B(N, r)$ be the right hand side of the above equality. If $N = N_1 N_2$, $r = r_1 r_2$ with $(N_1, N_2) = 1$, $r_1 | N_1$, $r_2 | N_2$, then we see that $A(N, r) = A(N_1, r_1) A(N_2, r_2)$, $B(N, r) = B(N_1, r_1) B(N_2, r_2)$. Hence we only need to show the lemma for the case

$N = p^a$, $r = p^b$ with $b \leqslant a$. If $(p^i, \omega_1, \omega_2)$ satisfies (7.40), then one of the $r_1$ and $r_2$ must be a multiple of $r$, so $0 \leqslant i \leqslant a-b$. If one of the $r_1$ and $r_2$ is larger than $r$, then $r_1 = r_2$. Since $\omega_2 = \omega\overline{\omega_1}$, we see that $\omega_2$ is determined by $\omega_1$.

We assume first that $2b \leqslant a$. If $0 \leqslant i \leqslant a-2b$, the maximal possible value of $r_1$ is $p^{[(a-i)/2]}$. We see that $[(a-i)/2] \geqslant b$ and $\omega_1$ can be any character modulo $p^{[(a-i)/2]}$. If $a-2b+1 \leqslant i \leqslant a-b$, then $b \geqslant 1$, $2b+i > a$ and it is impossible that $p^b|r_1$ and $p^b|r_2$. But one of $r_1$ and $r_2$ must be $p^b$, so $\omega_1$ can be $\chi$ or $\omega\chi$ where $\chi$ is any character modulo $p^{a-b-i}$. Hence we see that

$$A(p^a, p^b) = 2\sum_{i=0}^{b-1} \varphi(p^i) + \sum_{i=0}^{a-2b} \varphi(p^{[(a-i)/2]})$$
$$= \begin{cases} 2\displaystyle\sum_{i=0}^{a/2-1} \varphi(p^i) + \varphi(p^{a/2}) = B(p^a, p^b), & \text{if } 2|a, \\ 2\displaystyle\sum_{i=0}^{(a-1)/2} \varphi(p^i) = B(p^a, p^b), & \text{if } 2 \nmid a. \end{cases}$$

Assume now $a < 2b$. Then one of $r_1$ and $r_2$ must be $p^b$ and $\omega_1$ can be $\chi$ or $\omega\chi$ with $\chi$ any character modulo $p^{a-b-i}$. Therefore

$$A(p^a, p^b) = 2\sum_{i=0}^{a-b} \varphi(p^i) = B(p^a, p^b).$$

This completes the proof. □

By Theorem 5.9 we see that $-L(0,\omega_1)L(1-k,\omega_2)$ is the constant term of the Fourier expansion at $\infty$ of $E_k(lz, \omega_1, \omega_2)$. And if $\omega$ is a primitive character modulo $r \neq 1$ with $\omega(-1) = (-1)^\nu$ ($\nu = 0$ or $1$), then the function

$$R(s,\omega) := (r/\pi)^{(s+\nu)/2}\Gamma\left(\frac{s+\nu}{2}\right)L(s,\omega)$$

is holomorphic on the whole $s$-plane. It is well known that the function

$$\pi^{-s/2}s(s-1)\Gamma(s/2)\zeta(s)$$

is holomorphic on the whole $s$-plane. Since $s = 0$ and negative integers are poles of $\Gamma(s)$ with order 1, we know that $L(0,\omega) = 0$ (resp.$L(1-k,\omega) = 0$) if $\omega$ is a non-trivial even character(resp. if $k > 1$ is odd and $\omega$ is even or $k$ is even and $\omega$ is odd.). Hence

$$-L(0,\omega_1)L(1-k,\omega_2) = \begin{cases} 0, & \text{if } k \neq 1 \text{ and } \omega_1 \text{ is nontrivial,} \\ & \text{or both } \omega_1 \text{ and } \omega_2 \text{ are non-trivial,} \\ \dfrac{L(1-k,\omega)}{2}, & \text{otherwise,} \end{cases}$$

where we used the fact that $\zeta(0) = -1/2$.

Let $N = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ be a positive integer. We introduce an order in the set of all factors of $N$ as follows: if $l = p^{\beta_1} \cdots p_n^{\beta_n}$ and $l' = p_1^{\gamma_1} \cdots p_n^{\gamma_n}$ are two divisors of $N$, then we define $l \succ l'$ if there exist $i$ with $0 \leqslant i \leqslant n$ such that $\beta_j = \gamma_j$ for $1 \leqslant j \leqslant i$ and $\beta_{i+1} > \gamma_{i+1}$.

**Theorem 7.11** *Let $\omega$, $\omega_1$, $\omega_2$, $r_1$, $r_2$ be as above. Then*

(1) *For $k \geqslant 3$ or $k = 2, \omega \neq$ id., the functions*

$$E_k(lz, \omega_1, \omega_2) = -L(0, \omega_1)L(1-k, \omega_2) + \sum_{n=1}^{\infty} \left( \sum_{d|n} \omega_1(n/d)\omega_2(d)d^{k-1} \right) e(lnz),$$

*constitute a basis of $\mathcal{E}(N, k, \omega)$ where $(l, \omega_1, \omega_2)$ runs over all triples satisfying* (7.40).

(2) *The functions*

$$E_1(lz, \omega_1, \omega_2) = -L(0, \omega_1)L(0, \omega_2) + \sum_{n=1}^{\infty} \left( \sum_{d|n} \omega_1(n/d)\omega_2(d) \right) e(lnz)$$

*constitute a basis of $\mathcal{E}(N, 1, \omega)$ where $(l, \omega_1, \omega_2)$ runs over all triples satisfying* (7.40) *but only one of $(l, \omega_1, \omega_2)$ and $(l, \omega_2, \omega_1)$ can be taken.*

**Proof** (1) It is clear that $E_k(lz, \omega_1, \omega_2) \in \mathcal{E}(N, k, \omega)$. By dimension formula and Lemma 7.28 we have that $\dim(\mathcal{E}(N, k, \omega)) = A(N, r)$. Hence it is sufficient to show that the functions are linearly independent. Assume

$$0 = \sum_{n=0}^{\infty} b(n)e(nz) = \sum_{(l,\omega_1,\omega_2)} c(l, \omega_1, \omega_2)E_k(lz, \omega_1, \omega_2),$$

where $(l, \omega_1, \omega_2)$ runs over the set of triples satisfying (7.40). Let $1_N$ be the trivial character modulo $N$. For any given $(1, \omega_1, \omega_2)$ satisfying (7.40), we see that

$$\begin{aligned} 0 =& \sum_{n=1}^{\infty} 1_N \overline{\omega_2}(n) b(n) n^{-s} \\ =& c(1, \omega_1, \omega_2) L(s, \omega_1 \overline{\omega_2} 1_N) L(s-k+1, 1_N) \\ &+ \sum_{\omega_2' \neq \omega_2} c(1, \omega_1', \omega_2') L(s, \omega_1' \overline{\omega_2} 1_N) L(s-k+1, \omega_2' \omega_2 1_N), \end{aligned} \tag{7.41}$$

where the last summation is taken for triples $(l, \omega_1, \omega_2')$ satisfying (7.40) but $\omega_2 \neq \omega_2'$. The first term on the right hand side of (7.41) has a pole at $s = k$ with order 1 and the others have no poles at $s = k$. Hence $c(1, \omega_1, \omega_2) = 0$ for any $(1, \omega_1, \omega_2)$. Assume that $c(l', \omega_1, \omega_2) = 0$ for all $l' \prec l$ and that $(l, \omega_1, \omega_2)$ satisfies (7.40), we see that

$$\begin{aligned} 0 =& \sum_{n=1}^{\infty} 1_N \overline{\omega_2}(n) b(ln) n^{-s} \\ =& c(l, \omega_1, \omega_2) L(s, \omega_1 \overline{\omega_2} 1_N) L(s-k+1, 1_N) \\ &+ \sum_{\omega_2' \neq \omega_2} c(l, \omega_1', \omega_2') L(s, \omega_1' \overline{\omega_2} 1_N) L(s-k+1, \omega_2' \overline{\omega_2} 1_N), \end{aligned}$$

so $c(l, \omega_1, \omega_2) = 0$ by a similar argumentation. By induction we see that $c(l, \omega_1, \omega_2) = 0$ for any $(l, \omega_1, \omega_2)$.

(2) It is clear that $E_1(lz, \omega_1, \omega_2) \in \mathcal{E}(N, 1, \omega)$. By the dimension formula we see that $\dim(\mathcal{E}(N, 1, \omega)) = \frac{1}{2}A(N, r)$. Therefore we only need to show that the functions are linearly independent. But this can be done similarly as we did in the proof of (1). This completes the proof. □

Recall the definition of the function $g_t^*(z)$ in Section 2.2:

$$g_t^*(z) = -\frac{1}{24}\prod_{p|t}(1-p) + \sum_{n=1}^{\infty}\left(\sum_{d|n,(d,t)=1} d\right)e(nz).$$

It is easy to show that $g_t^* \in \mathcal{E}(t, 2, \text{id.})$. For any positive integer $l$, put $t(l) = \prod_{p|l} p$. For $l \neq 1$ we define

$$E_2(lz, \text{id.}, \text{id.}) = g_{t(l)}^*(lz/t(l)) \in \mathcal{E}(l, 2, \text{id.}).$$

It is easy to see that

$$L(s, E_2(lz, \text{id.}, \text{id.})) = (l/t(l))^{-s}\zeta(s)L(s-1, 1_{t(l)}).$$

It should be noticed that the symbol $E_2(z, \text{id.}, \text{id.})$ is not defined. If $\omega_1$ is non-trivial but $\omega_1^2 = \text{id.}$, we define

$$E_2(z, \omega_1, \omega_2) = \left(\sum_{u=1}^{r_1}\omega_1(u)e(u/r_1)\right)^{-1}\sum_{u=1}^{r_1}\omega_1(u)g_{t(r_1)}^*(z+u/r_1),$$

then $E_2(z, \omega_1, \omega_2) \in \mathcal{E}(r_1^2, 2, \text{id.})$ by Lemma 7.27, and

$$L(s, E_2(z, \omega_1, \omega_2)) = L(s, \omega_1)L(s-1, \omega_2).$$

If $\omega_1^2 \neq \text{id.}$, we define

$$E_2(z, \omega_1, \omega_2) = \left(\sum_{u=1}^{r_1}\overline{\omega_1}(u)e(u/r_1)\right)^{-1}\sum_{u=1}^{r_1}\overline{\omega_1}(u)E_2(z+u/r_1, \text{id.}, \overline{\omega_1}^2),$$

where $E_2(z, \text{id.}, \overline{\omega_1}^2)$ is well defined as in (7.39) since $\omega_1^2 \neq \text{id.}$. It is not difficult to show that $E_2(z, \omega_1, \omega_2) \in \mathcal{E}(r_1^2, 2, \text{id.})$ and

$$L(s, E_2(z, \omega_1, \omega_2)) = L(s, \omega_1)L(s-1, \omega_2).$$

So we have a function $E_2(lz, \omega_1, \omega_2) \in \mathcal{E}(N, 2, \text{id.})$ for every triple $(l, \omega_1, \omega_2)$ satisfying

$$\omega_1\omega_2 = \text{id.}, \quad lr_1r_2|N \quad \text{and} \quad l \neq 1 \quad \text{if} \quad r_1 = r_2 = 1. \tag{7.42}$$

Let $a_0(l,\omega_1,\omega_2)$ be the constant term of the Fourier expansion of $E_2(lz,\omega_1,\omega_2)$. It is easy to see that

$$a_0(l,\omega_1,\omega_2) = \begin{cases} 0, & \text{if } \omega_1 \text{ is non-trivial,} \\ -\dfrac{1}{24}\displaystyle\prod_{p|l}(1-p), & \text{if } \omega_1 \text{ is trivial,} \end{cases}$$

**Theorem 7.12** *The functions*

$$E_2(lz,\omega_1,\omega_2) = a_0(l,\omega_1,\omega_2) + \sum_{n=1}^{\infty}\left(\sum_{d|n}\omega_1(n/d)\omega_2(d)d\right)e(lnz)$$

*constitute a basis of* $\mathcal{E}(N,2,\text{id.})$, *where* $(l,\omega_1,\omega_2)$ *runs over the set of triples* $(l,\omega_1,\omega_2)$ *satisfying* (7.42).

**Proof** We only need to show that the functions are linearly independent. Assume

$$\sum c(l,\omega_1,\omega_2)E_2(lz,\omega_1,\omega_2) = 0, \tag{7.43}$$

where the summation was taken over all triples $(l,\omega_1,\omega_2)$ satisfying (7.42).

Let $f(z) = \displaystyle\sum_{n=0}^{\infty} a(n)e(nz) \in G(N,k,\omega)$, $r|N$ and $\psi$ any character modulo $N$. Define

$$L(s,f,\psi,r) = \sum_{n=1}^{\infty}\psi(n)a(rn)n^{-s}.$$

We have that $L(s,E_2(lz,\text{id.},\text{id.}),\psi,r) = 0$ if $l/t(l)\nmid r$. If $l/t(l)|r$, then

$$\begin{aligned} L(s,E_2(lz,\text{id.},\text{id.}),\psi,r) &= \sum_{n=1}^{\infty}\psi(n)\left(\sum_{\substack{d|nrt(l)/l,\\ (d,l)=1}} d\right)n^{-s} \\ &= \prod_{p|r,p\nmid l}(1+p+\cdots+p^{\nu_p(r)})L(s,\psi)L(s-1,\psi), \end{aligned}$$

where $\nu_p(r)$ is the $p$-adic valuation of $r$. If $\psi$ is non-trivial, then $L(s,E_2(lz,\text{id.},\text{id.}),\psi,r)$ is holomorphic at $s=2$, by the same argumentation as in the proof of Theorem 7.11 and (7.43) we know that $c(l,\omega_1,\omega_2)=0$ if $\omega_2$ is a non-trivial character.

Denote by $f$ the left hand side of (7.43). It is clear that $L(s,f,1_N,r)$ has no pole at $s=2$. Hence

$$A_r = \sum_{\substack{l|N, l\neq 1,\\ l/t(l)|r}}\ \prod_{\substack{p|r,\\ p\nmid l}}(1+p+\cdots+p^{\nu_p(r)})c(l) = 0, \quad N\neq r|N, \tag{7.44}$$

where $c(l) = c(l,\text{ id., id.})$. The equality (7.44) is a system of linear equations with respect to $\{c(l)|1\neq l|N\}$. We shall prove the system has only zero as solution which implies the theorem. If $N = p^n$ with $p$ a prime, it is then clear that $A_1 = 0$, $A_p = 0$, $\cdots$, $A_{p^{n-1}} = 0$, so $c(p) = 0$, $c(p^2) = 0$, $\cdots$, $c(p^n) = 0$. We apply induction to the number of prime factors of $N$: let $N = p_1^n N_1$ with $(p_1, N_1) = 1$, suppose that (7.44) has only zero as solution if $N = N_1$. Now suppose that $r_1 | N_1$, then

$$A_{p_1^n r_1} - A_{p_1^{n-1} r_1} = p_1^n \sum_{\substack{1 \neq l | N, \\ l/t(l) | r_1}} \prod_{\substack{p | r_1, \\ p \nmid l}} (1 + p + \cdots + p^{\nu_p(r_1)}) c(l) = 0, \quad N_1 \neq r_1 | N_1.$$

By induction hypothesis we see that $c(l) = 0$ if $p_1 \nmid l$. But $p_1$ can be any prime factor of $N$, we see that $c(l) = 0$ if there exists some prime factor $p$ of $N$ such that $p \nmid l$. Hence

$$A_{r_1} = \sum_{\substack{1 \neq l | N_1, \\ l/t(l) | r_1}} \prod_{\substack{p | r_1, \\ p \nmid l}} (1 + p + \cdots + p^{\nu_p(r_1)}) c(p_1 l) = 0, \quad N_1 \neq r_1 | N_1.$$

By induction hypothesis again we see that $c(p_1 l) = 0$ for $l | N_1$. Similarly using the fact that $A_{p_1 r_1} = 0, A_{p_1^2 r_1} = 0, \cdots, A_{p_1^{n-1} r_1} = 0$ for $N_1 \neq r_1 | N_1$, we obtain that $c(p_1^2 l) = 0, \cdots, c(p_1^n l) = 0$ for $l | N_1$. This shows that the system (7.44) has only zero solution. This completes the proof. □

**Theorem 7.13** *Let* $f(z) = \sum_{n=0}^{\infty} a(n) e(nz) \in G(N, k, \omega)$. *Then* $f(z)$ *is a cusp form if and only if the function* $L(s, f, \psi, r)$ *is holomorphic at* $s = k$ *for any proper divisor* $r$ *of* $N$ *and any character* $\psi$ *modulo* $N$.

**Proof** The necessity can be deduced from Lemma 7.15. We now assume that the function $L(s, f, \psi, r)$ is holomorphic at $s = k$. Since $G(N, k, \omega) = \mathcal{E}(N, k, \omega) \oplus S(N, k, \omega)$, we have

$$f(z) = \sum c(l, \omega_1, \omega_2) E_k(lz, \omega_1, \omega_2) + g(z),$$

where the summation was taken over the set of triples satisfying the conditions in Theorem 7.11 or Theorem 7.12 according to $k \neq 2, k = 2, \omega \neq \text{id.}$ or $k = 2, \omega = \text{id.}$ respectively, and $g(z) \in S(N, k, \omega)$. By the holomorphy of $L(s, f, \psi, r)$ at $s = k$ and applying the similar argumentation used in the proofs of Theorem 7.11 and Theorem 7.12, we can prove that $c(l, \omega_1, \omega_2) = 0$. Hence $f(z) \in S(N, k, \omega)$. This completes the proof. □

**Remark 7.2** The hypothesis in Theorem 7.13 can be represented as follows: $L(s, f, \psi, r)$ is holomorphic at $s = k$ for any proper divisor $r$ of $N$ and any primitive character $\psi$ induced from any character modulo $N$. The necessity can be deduced from Lemma 7.15. We now assume the above condition is satisfied. Let $\chi$ be any character modulo $N$ and $\psi$ the primitive character induced by $\chi$. Then

$$L(s,f,\chi,r) = \sum_{n=1}^{\infty} \chi(n)a(rn)n^{-s} = \sum_{n=1}^{\infty} \psi(n) \sum_{d|(n,N)} \mu(d)a(rn)n^{-s}$$
$$= \sum_{d|N} \psi(d)d^{-s} L(s,f,\psi,rd),$$

which implies the holomorphy of $L(s,f,\chi,r)$ at $s = k$. Hence $f$ is a cusp form by Theorem 7.13. Also the condition can be represented as follows: $L(s,f,\psi,r)$ is holomorphic at $s = k$ for any positive integer $r|N$ and any primitive character $\psi$.

## References

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