Chapter 6

New Forms and Old Forms

6.1 New Forms with Integral Weight

Let N, k positive integers, χ a character modulo N. We know that the Hecke operators $\mathrm{T}(n), (n, N) = 1$ can be diagonalized simultaneously in the space $S(N, k, \chi)$. On the other hand, if f is an eigenfunction of all Hecke operators $\mathrm{T}(n)$, then $L(s, f)$ has an Euler product. So we want to ask the following question: Can all Hecke operators $\mathrm{T}(n)$ be diagonalized simultaneously in the space $S(N, k, \chi)$. The following example gives a counterexample to the question:

Example 6.1 Consider the space $V = S(2, 12, \mathrm{id.})$ which has dimension 2. Then

$$f_1(z) = \Delta(z) := \frac{64\pi^{12}}{27}((E_4(z))^3 - (E_6(z))^2) \in V,$$
$$f_2(z) = \Delta(2z) \in V.$$

For any odd prime p, they have the same eigenvalue for $\mathrm{T}(p)$. If there exists a basis $\{g_1, g_2\}$ of V such that g_1, g_2 are eigenfunctions of all Hecke operators $\mathrm{T}(p)$ for any prime p, then by the properties of f_1, f_2, we see that $(g_1 - g_2)|\mathrm{T}(p) = 0$ for any odd prime p. Hence $(g_1 - g_2)|\mathrm{T}(n) = 0$ if n has an odd divisor. That is, the n-th Fourier coefficient $c(n)$ of $g_1 - g_2$ is equal to 0 if n has an odd divisor. This implies that $g_1 - g_2 = 0$ by the following Lemma 6.1. This contradicts the assertion. □

Lemma 6.1 (1) *Let* $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$ *with* $(a, b, c, d) = 1, \det(\alpha) = n > 1$, $(n, N) = 1$. *Assume that* $f \in G_k(\Gamma(N))$ *and* $f|[\alpha]_k \in G_k(\Gamma(N))$, *then* $f = 0$.

(2) *Let* $p \nmid N$ *be a prime and* $f(z) = \sum_{n=0}^{\infty} c(n)\mathrm{e}^{2\pi i n z/N} \in G_k(\Gamma(N))$ *satisfy*

$$c(n) = 0, \quad \text{for all } n \not\equiv 0 \pmod{p}.$$

Then $f = 0$.

(3) *Let* p *and* f *be as above. If*

$$c(n) = 0, \quad \text{for all } n \equiv 0 \pmod{p},$$

then $f = 0$.

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Proof Since $\Gamma(N)$ is a normal subgroup of $\Gamma(1)$, we may assume that $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$. Put $\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $f|[\alpha]_k[\tau^N]_k = f|[\alpha]_k$, i.e., $f|[\alpha\tau^N\alpha^{-1}]_k = f$. But $\alpha\tau^N\alpha^{-1} = n^{-1}\begin{pmatrix} n & N \\ 0 & n \end{pmatrix}$, so that

$$f\left|\left[\begin{pmatrix} n & N \\ 0 & n \end{pmatrix}\right]_k = f.\right. \tag{6.1}$$

Take $\gamma \in \Gamma(1)$ such that $\gamma \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \pmod{n}, \gamma \equiv I \pmod{N}$. Then $\gamma \in \Gamma(N)$. Put

$$\beta = \gamma\begin{pmatrix} n & N \\ 0 & n \end{pmatrix} \equiv N\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \pmod{n}.$$

Then $\beta^l \equiv N^l\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \pmod{n}$ and $\det(\beta^l) = n^{2l}$ for any positive integer l. This implies that β^l is primitive (i.e., the entries of β^l are co-prime.). By (6.1), we have $f|[\beta]_k = f$ and hence

$$f|[\beta^l]_k = f$$

for any positive integer l. Take a positive integer l such that $n^l \equiv 1 \pmod{N}$, then

$$\beta^l \equiv \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}^l \equiv I \pmod{N}.$$

Since β^l is primitive, its elementary divisors are $\{1, n^{2l}\}$. Therefore there exist $\delta, \epsilon \in \Gamma(1)$ such that $\beta^l = \delta\begin{pmatrix} 1 & 0 \\ 0 & n^{2l} \end{pmatrix}\epsilon = \delta\alpha^{2l}\epsilon$. By the choice of l, we see that $\delta\epsilon \equiv \epsilon\delta \equiv I \pmod{N}$, i.e., $\delta\epsilon$, $\epsilon\delta \in \Gamma(N)$, so that

$$f|[\delta]_k[\alpha^{2l}]_k = f|[\delta]_k. \tag{6.2}$$

Put $g = f|[\delta]_k$, then $g \in G_k(\Gamma(N))$. Let

$$g(z) = \sum_{s=0}^{\infty} a(s)\mathrm{e}^{2\pi \mathrm{i} sz/N}$$

be the Fourier expansion of g at $\mathrm{i}\infty$. Then by (6.2) we see that $g\left(\frac{z}{r}\right) = r^{m/2}g(z)$ with $r = n^{2l}$, so that

$$a(s) = 0, \quad \forall r \nmid s, \quad a(sr) = r^{m/2}a(s).$$

This implies that $a(s) = 0$ for all $s \geqslant 1$, so that $g = 0$ and $f = 0$. This shows (1).

By the assumption of (2), we see that $f(z + N/p) = f(z)$, so that $f \in G_k(\Gamma(N))$

and $f|[\alpha]_k = f \in G_k(\Gamma(N))$ with $\alpha = \begin{pmatrix} p & N \\ 0 & p \end{pmatrix}$. Since α is primitive, we obtain (2) by (1).

By Lemma 5.18, we have

$$p^{1-k/2} f|\mathrm{T}(p) = f\left|\left[\sigma_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]_k + \sum_{b=0}^{p-1} f\right|\left[\begin{pmatrix} 1 & bt \\ 0 & p \end{pmatrix}\right]_k,$$

where $t|N$. By the assumption of (3), we see that

$$\begin{aligned} \sum_{b=0}^{p-1} f\left|\left[\begin{pmatrix} 1 & bt \\ 0 & p \end{pmatrix}\right]_k\right. &= p^{-k/2} \sum_{b=0}^{p-1} \left(\frac{z+bt}{p}\right) \\ &= p^{-k/2} \sum_{n=0, p\nmid n}^{\infty} c(n) \mathrm{e}^{2\pi i n z/p} \sum_{b=0}^{p-1} \mathrm{e}^{2\pi i n t b/p} = 0, \end{aligned}$$

where we used the fact $\sum_{b=0}^{p-1} \mathrm{e}^{2\pi i n t b/p} = 0$ (since $p \nmid nt$). Therefore

$$f\left|\left[\sigma_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]_k = p^{1-k/2} f|\mathrm{T}(p) \in G_k(\Gamma(N)).\right.$$

Since $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ is primitive, we see that $f|[\sigma]_k = 0$ by (1), so that $f = 0$. This completes the proof. □

Let k, l be positive integers, put $\delta_l = \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}$. It is clear that, for any function f on $\mathbb{H}$, we have

$$f(lz) = l^{-k/2}(f|[\delta_l]_k)(z).$$

For any element $\gamma = \begin{pmatrix} a & b \\ clN & d \end{pmatrix} \in \Gamma_0(lN)$, we have

$$\delta_l \gamma \delta_l^{-1} = \begin{pmatrix} a & bl \\ cN & d \end{pmatrix} \in \Gamma_0(N).$$

For any $f \in G(N, k, \chi)$, put $g = f|[\delta_l]_k$. Then

$$g|[\gamma]_k = (f|[\delta_l \gamma \delta_l^{-1}]_k)|[\delta_l]_k = \chi(d) f|[\delta_l]_k = \chi(d) g,$$

so that we have the following:

Lemma 6.2 *Let $f \in G(N, k, \chi)$. Then, for any positive integer l, we have*

$$f(lz) = l^{-k/2}(f|[\delta_l]_k)(z) \in G(Nl, k, \chi).$$

Furthermore, $f(lz)$ is a cusp form if f is a cusp form.

Remark 6.1 We denote by $V(l)$ the operator in Lemma 6.2 and call it translation operator. It is clear that it is an analog of the translation operator for modular forms with half integral weight (see Theorem 5.16). Similar to Theorem 5.19, we can prove the following:

Lemma 6.3 *Let $f \in G(N, k, \chi)$, l a positive integer. Then we have*

$$(f|V(l))|\mathrm{T}(n) = (f|\mathrm{T}(n))|V(l), (n, l) = 1.$$

Let χ be a primitive character modulo m with $m|N$. Then $S(N, k, \chi)$ contains the following set

$$\left\{ f(z), f(lz) \middle| f(z) \in S(L, k, \chi), m|L, L|N, l \middle| \frac{N}{L} \right\}. \tag{6.3}$$

The functions f_1, f_2 are in the corresponding set (6.3) of $S(2, 12, \mathrm{id.})$. We shall show that all Hecke operators can be diagonalized in the orthogonal complement of the space spanned by (6.3) in $S(N, k, \chi)$ with respect to Petersson inner product.

Put

$$\Delta_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \middle| c \equiv 0 \pmod N, (a, N) = 1, ad - bc > 0 \right\},$$
$$\Delta_0^*(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \middle| c \equiv 0 \pmod N, (d, N) = 1, ad - bc > 0 \right\}.$$

Lemma 6.4 *Let $\alpha \in \Delta_0(N)$ or $\in \Delta_0^*(N)$ respectively. Then there exist positive integers l, m satisfying $l|m$, $(l, N) = 1$ such that*

$$\Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N) \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \Gamma_0(N)$$

or

$$\Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N) \begin{pmatrix} m & 0 \\ 0 & l \end{pmatrix}$$

respectively.

Proof Let $\alpha = \begin{pmatrix} a & b \\ cN & d \end{pmatrix}$, $a' = (a, c)$. Then $(a, cN) = a'$. Let u, v be integers such that $(u, v) = 1$, $au + cNv = a'$. Then $\begin{pmatrix} u & v \\ -cN/a' & a/a' \end{pmatrix} \in \Gamma_0(N)$ and

$$\begin{pmatrix} u & v \\ -cN/a' & a/a' \end{pmatrix} \begin{pmatrix} a & b \\ cN & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \in \Delta_0(N).$$

It is clear that $0 < a' \leqslant |a|$, and $0 < a' < |a|$ if $a \nmid c$. Put $a_1 = (a', b')$, then $0 < a_1 \leqslant a'$, and $0 < a_1 < a'$ if $a' \nmid b'$. It is easy to see that $(a', b'N) = a_1$. Let u_1, v_1

be integers such that $(u_1, v_1) = 1, a'u_1 + b'Nv_1 = a_1$, then $\begin{pmatrix} u_1 & -b'/a_1 \\ v_1 N & a'/a_1 \end{pmatrix} \in \Gamma_0(N)$ and

$$\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \begin{pmatrix} u_1 & -b'/a_1 \\ v_1 N & a'/a_1 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ c_1 N & d_1 \end{pmatrix} \in \Delta_0(N).$$

The above process shows that, if $a \nmid b$ or c, then there exist $\gamma_1, \gamma_2 \in \Gamma_0(N)$ such that $\gamma_1 \alpha \gamma_2 \in \Delta_0(N)$ and the upper left entry a_1 of $\gamma_1 \alpha \gamma_2$ satisfies $1 \leqslant |a_1| < |a|$. Repeating the above process, we may assume that $\alpha \in \Delta_0(N)$ satisfies $a|(b, c)$. Then $\begin{pmatrix} 1 & 0 \\ -cN/a & 1 \end{pmatrix} \in \Gamma_0(N), \begin{pmatrix} 1 & -b/a \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$ and

$$\begin{pmatrix} 1 & 0 \\ -cN/a & 1 \end{pmatrix} \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \begin{pmatrix} 1 & -b/a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d_1 \end{pmatrix} \in \Delta_0(N).$$

Put $l = (a,\ d_1)$, then $l = (a,\ d_1 N)$. Take integers $a_2,\ c_2$ such that $(a_2,\ c_2) = 1$, $a_2 a - c_2 N d_1 = l$, then $\begin{pmatrix} 1 & -1 \\ -d_1 c_2 N/l & a a_2/l \end{pmatrix} \in \Gamma_0(N)$, $\begin{pmatrix} a_2 & d_1/l \\ c_2 N & a/l \end{pmatrix} \in \Gamma_0(N)$ and

$$\begin{pmatrix} 1 & -1 \\ -d_1 c_2 N/l & a a_2/l \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & d_1/l \\ c_2 N & a/l \end{pmatrix} = \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \in \Delta_0(N).$$

Taking determinants, we obtain that $ad_1 = lm = \det(\alpha)$, so that $m > 0$, $l|m$ since $l = (a, d_1)$. This shows the assertion for $\Delta_0(N)$. We can prove the assertion for $\Delta_0^*(N)$ similarly. This completes the proof. □

Lemma 6.5 *Let $f \in G(N, k, \chi)$. Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_0(N)$ satisfy*

(1) $\det(\alpha) > 1$;
(2) $(\det(\alpha), N) = 1$;
(3) $(a, b, c, d) = 1$.

If $f|[\alpha^{-1}]_k \in G(N, k, \chi)$, then $f = 0$.

Proof By (2), we see that $\alpha \in \Delta_0^*(N)$, by Lemma 6.4, there exist $\gamma_1,\ \gamma_2 \in \Gamma_0(N)$ such that $\gamma_1 \alpha \gamma_2 = \begin{pmatrix} m & 0 \\ 0 & l \end{pmatrix}$ with $l|m,\ l,\ m > 0$. By (3), $(l, m) = 1$, so that $l = 1$. By (1), $m > 1$ and

$$\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ N/m & 1 \end{pmatrix} \notin \Gamma_0(N),$$

hence $\alpha \Gamma_0(N) \alpha^{-1} \not\subset \Gamma_0(N)$. Take $\gamma \in \Gamma_0(N)$ such that $\alpha \gamma \alpha^{-1} \notin \Gamma_0(N)$. Since $\det(\alpha)\alpha^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \Delta_0(N)$, $\det(\alpha)\alpha\gamma\alpha^{-1} \in \Delta_0(N)$, by Lemma 6.4, there exist $\gamma_3,\ \gamma_4 \in \Gamma_0(N)$ such that

$$\det(\alpha)\gamma_3 \alpha \gamma \alpha^{-1} \gamma_4 = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \quad u|v, u, v > 0. \tag{6.4}$$

Taking the determinants, we have $(\det(\alpha))^2 = uv$. If $u = v$, then $\alpha\gamma\alpha^{-1} = \gamma_3^{-1}\gamma_4^{-1} \in \Gamma_0(N)$ which is impossible. Therefore, $h = v/u > 1$. Considering the action of both sides of (6.4) on $g = f|[\alpha^{-1}]_k$, we obtain that

$$g(z/h) = (\det(\alpha))^{-k} v^k \chi(\gamma_3)\chi(\gamma)\chi(\gamma_4) g(z) := cg(z).$$

Let $g(z) = \sum_{n=0}^{\infty} a(n)e(nz)$ be the Fourier expansion of g. Then, for any positive integer s, we have

$$a(n) = c^{-1}a(n/h) = c^{-s}a(n/h^s),$$

so that $a(n) = 0$ for any $n \geqslant 0$ since $k > 0$ and $|c| = h^{k/2} > 1$. Therefore $g = 0$ and hence $f = 0$. This completes the proof. □

Theorem 6.1 *Let l be a positive integer, f a function on $\mathbb{H}$ satisfying:*

(i) $f(z+1) = f(z)$;

(ii) $f(lz) \in G(N, k, \chi)$.

Then the following two assertions hold:

(1) $f(z) \in G(N/l, k, \chi)$ *if* $lm_\chi | N$;

(2) $f(z) = 0$ *if* $lm_\chi \nmid N$,

where m_χ is the conductor of χ. Furthermore, $f(z) \in S(N/l, k, \chi)$ if $f(lz) \in S(N, k, \chi)$.

Proof We need only to show the theorem for l a prime since we can apply induction on the number of prime factors of l. So we assume now that l is a prime. Because of the assumptions in the theorem, we have

$$\begin{aligned} G(N,k,\chi) \ni f(lz)|\mathrm{T}(l) &= l^{k/2-1}\left(f(lz)\left|\left[\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}\right]_k + \sum_{m=0}^{l-1} f(lz)\right|\left[\begin{pmatrix} l & m \\ 0 & l \end{pmatrix}\right]_k \right) \\ &= l^{k-1} f(l^2 z) + \frac{1}{l}\sum_{m=0}^{l-1} f(l(z+m/l)) \\ &= l^{k-1} f(l^2 z) + f(lz). \end{aligned}$$

Hence $f(l^2 z) \in G(N, k, \chi)$ since $f(lz) \in G(N, k, \chi)$. If $l \nmid N$, taking $\alpha = \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}$ in Lemma 6.5, we see that $f(l^2 z) = 0$, so that $f(z) = 0$. Therefore we assume now $l|N$.

We consider first the case $lm_\chi \nmid N$. For any element $\gamma_1 = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$, owing to the assumptions in the theorem, we see that

$$f\left|\left[\begin{pmatrix} a & bl \\ N/l & d \end{pmatrix}\right]_k = f|[\delta_l \gamma_1 \delta_l^{-1}] = \chi(d) f.\right. \tag{6.5}$$

For any given positive integers m, n, put

$$\gamma = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ N/l & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+mN/l & m+n(1+mN/l) \\ N/l & 1+nN/l \end{pmatrix} \in \Gamma_0(N/l). \tag{6.6}$$

In particular, if n, m are chosen such that $nN/l + 1 \not\equiv 0 \pmod{l}$ and

$$n(1+mN/l)+m = n+(nN/l+1)m \equiv 0 \pmod{l}, \tag{6.7}$$

then, by (6.6) and (6.7), we have

$$\begin{pmatrix} 1+mN/l & l^{-1}(m+n(1+mN/l)) \\ N & 1+nN/l \end{pmatrix} \in \Gamma_0(N).$$

Then we obtain

$$f|[\gamma]_k = \chi(1+nN/l)f$$

by (6.5). But $\delta_l \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \delta_l^{-1} = \begin{pmatrix} 1 & 0 \\ N/l & 1 \end{pmatrix}$, so by assumptions (i) and (ii), we see that

$$f|[\gamma]_k = f\left|\left[\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ N/l & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right]_k\right. = f.$$

This shows that $\chi(1+nN/l) = 1$ for any $(1+nN/l, l) = 1$ if $f \neq 0$. This implies that the conductor m_χ of χ satisfies $m_\chi | N/l$. This contradicts $lm_\chi \nmid N$. Hence we have $f = 0$ if $lm_\chi \nmid N$.

We now assume that $lm_\chi | N$. For any $\gamma = \begin{pmatrix} a & b \\ cN/l & d \end{pmatrix} \in \Gamma_0(N/l)$, we can find an m satisfying $l \nmid (a + mcN/l)$ since $(a, cN/l) = 1$, then take an n such that $l | (a + mcN/l)n + b + md$, so that

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ cN/l & d \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a' & b'l \\ c'N/l & d' \end{pmatrix}$$

with a', b', c', d' integers. Hence $\begin{pmatrix} a' & b' \\ c'N & d' \end{pmatrix} \in \Gamma_0(N)$ and $d' \equiv d \pmod{N/l}$. Put $z = lw$, $g(w) = f(lw)$, by (i), (ii) and $m_\chi | N/l$, we have

$$\begin{aligned}
(f|[\gamma]_k)(z+n) &= \left(f\left|\left[\begin{pmatrix} a' & b'l \\ c'N/l & d' \end{pmatrix}\right]_k\right.\right)(z) \\
&= (c'Nz/l + d')^{-k} f\left(\frac{a'z + b'l}{c'Nz/l + d'}\right) \\
&= (c'Nw + d')^{-k} f\left(\frac{l(a'w + b')}{c'Nw + d'}\right) \\
&= \left(g\left|\left[\begin{pmatrix} a' & b' \\ c'N & d' \end{pmatrix}\right]_k\right.\right)(w) \\
&= \chi(d')g(w) = \chi(d)f(z).
\end{aligned}$$

This shows that $f| \in G(N/l, k, \chi)$. It is clear that $f(z) \in S(N/l, k, \chi)$ if $f(lz) \in S(N, k, \chi)$. This completes the proof. □

Lemma 6.6 *Let* $f = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N,k,\chi)$ *and* L *a positive integer. Put* $g(z) = \sum_{(n,L)=1} a(n)e(nz)$. *Then* $g(z) \in G(M,k,\chi)$ *with* $M = N \prod_{p|L, p|N} p \prod_{q|L, q\nmid N} q^2$, *where* p, q *are primes. Furthermore,* $g(z)$ *is a cusp form if* $f(z)$ *is a cusp form.*

Proof We only need to show the lemma for L a prime since we can apply induction on the number of prime factors of L. So we assume now that L is a prime. Put

$$N' = \begin{cases} N, & \text{if } p|N, \\ pN, & \text{if } p \nmid N. \end{cases}$$

Then $p|N'$. By Lemma 5.17, we have

$$\Gamma_0(N') \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(N') = \bigcup_{m=0}^{p-1} \Gamma_0(N') \begin{pmatrix} 1 & m \\ 0 & p \end{pmatrix}. \tag{6.8}$$

Since $G(N,k,\chi) \subset G(N',k,\chi)$, we see that

$$f|\mathrm{T}(p) \in G(N',k,\chi)$$

holds in $G(N',k,\chi)$. By (6.8), we have

$$(f|\mathrm{T}(p))(z) = p^{-1} \sum_{n=0}^{\infty} a(n) \sum_{m=0}^{p-1} \mathrm{e}^{2\pi i n(z+m)/p} = \sum_{n=0}^{\infty} a(np)e(nz).$$

By Lemma 6.2, we see that

$$(f|\mathrm{T}(p))(pz) = \sum_{n=0}^{\infty} a(np)e(npz) \in G(N'p,k,\chi).$$

Put $M = N'p$, then

$$g(z) = f(z) - (f|\mathrm{T}(p))(pz) \in G(M,k,\chi).$$

This completes the proof. □

Lemma 6.7 *Let* N *be a positive integer,* p *a prime. Then*

$$\Gamma_0(pN) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(N)$$
$$= \begin{cases} \bigcup_{m=0}^{p-1} \Gamma_0(pN) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, & \text{if } p|N, \\ \Gamma_0(pN) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \sigma_p \bigcup_{m=0}^{p-1} \Gamma_0(pN) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, & \text{if } p \nmid N, \end{cases}$$

where σ is a matrix satisfying

$$\sigma_p \in \Gamma_0(N), \quad \sigma_p \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N, \quad \sigma_p \equiv \begin{pmatrix} 0 & -l \\ l' & 0 \end{pmatrix} \pmod p$$

with l any fixed integer such that $p \nmid l$ and l' an integer such that $ll' \equiv 1 \pmod p$.

Proof Assume first that $p|N$. Let $\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$. Then $(a, cN) = 1$ and hence $p \nmid a$. Take $0 \leqslant v \leqslant p-1$ with $av \equiv b \pmod p$. Put $b_1 = (b - av)/p$, $d_1 = d - vcN$. Then $\gamma_1 = \begin{pmatrix} a & b_1 \\ cpN & d_1 \end{pmatrix} \in \Gamma_0(pN)$ and

$$\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \begin{pmatrix} a & b_1 \\ cpN & d_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} = \gamma.$$

This shows the first case in the lemma.

Now assume that $p \nmid N$. For any $\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$, if $p \nmid a$, then similar to the first case, there exists $\gamma_1 \in \Gamma_0(pN)$, $0 \leqslant v \leqslant p-1$ such that

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}.$$

If $p|a$, since $p \nmid N$, there exists a_1 such that $a_1 p \equiv 1 \pmod N$. Take c_1 such that $c_1 N \equiv l' \pmod p$ and $(c_1, a_1 p) = 1$ (since $p \nmid c_1$, if necessary, take an integer t such that $pt + c_1$ is a prime larger than a_1, then $(pt + c_1, a_1 p) = 1$). Then $(a_1 p^2, c_1 N^2) = 1$. Take b_1, $d_1 \in \mathbb{Z}$ such that $d_1 a_1 p^2 - b_1 c_1 N^2 = 1$, then $\sigma_p = \begin{pmatrix} a_1 p & b_1 N \\ c_1 N & d_1 p \end{pmatrix}$ satisfies the conditions in the lemma. And

$$\gamma \sigma_p^{-1} = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \begin{pmatrix} d_1 p & -b_1 N \\ -c_1 N & a_1 p \end{pmatrix} = \begin{pmatrix} a_2 & b_2 p \\ c_2 N & d_2 \end{pmatrix} \in \Gamma_0(N)$$

and a_2, b_2, c_2, $d_2 \in \mathbb{Z}$. Therefore $\begin{pmatrix} a_2 & b_2 \\ c_2 pN & d_2 \end{pmatrix} \in \Gamma_0(pN)$, and

$$\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 pN & d_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} a_2 & b_2 p \\ c_2 N & d_2 \end{pmatrix} = \gamma \sigma_p^{-1}.$$

This shows the second case in the lemma. This completes the proof. □

Lemma 6.8 *Let χ be a character modulo N, l a positive integer, $p \nmid l$ a prime. Put $M = lN$, then we have the following two commutative diagrams*:

(1)

$$\begin{array}{ccc}
G(pN,k,\chi) & \xrightarrow{\Gamma_0(pN)\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_0(N)} & G(N,k,\chi) \\
\Big\downarrow \text{Embedding} & & \Big\downarrow \text{Embedding} \\
G(pM,k,\chi) & \xrightarrow[\Gamma_0(pM)\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_0(M)]{} & G(M,k,\chi)
\end{array}$$

(2)

$$\begin{array}{ccc}
G(pN,k,\chi) & \xrightarrow{\Gamma_0(pN)\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_0(N)} & G(N,k,\chi) \\
{\scriptstyle[\delta_l]_k}\Big\downarrow & & \Big\downarrow{\scriptstyle[\delta_l]_k} \\
G(pM,k,\chi) & \xrightarrow[\Gamma_0(pM)\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_0(M)]{} & G(M,k,\chi).
\end{array}$$

And similar results hold for cusp forms.

Proof The diagram (1) is an immediate conclusion of Lemma 6.7. We show now the second diagram. Let $f(z) \in G(pN,k,\chi)$. Put $g(z) = f|[\delta_l]_k$. By Lemma 6.7, we have

$$\begin{aligned}
& g|\Gamma_0(pM)\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_0(M) \\
&= \sum_{v=0}^{p-1} g\left|\left[\begin{pmatrix}1&0\\0&p\end{pmatrix}\begin{pmatrix}1&v\\0&1\end{pmatrix}\right]_k + g\right|\left[\begin{pmatrix}1&0\\0&p\end{pmatrix}\sigma_p\right]_k \\
&\qquad \text{(where the last term disappears if } p|M). \\
&= \sum_{v=0}^{p-1} f\left|\left[\delta_l\begin{pmatrix}1&0\\0&p\end{pmatrix}\begin{pmatrix}1&v\\0&1\end{pmatrix}\right]_k + f\right|\left[\delta_l\begin{pmatrix}1&0\\0&p\end{pmatrix}\sigma_p\right]_k \\
&= \sum_{v=0}^{p-1} f\left|\left[\begin{pmatrix}1&0\\0&p\end{pmatrix}\delta_l\begin{pmatrix}1&v\\0&1\end{pmatrix}\delta_l^{-1}\delta_l\right]_k + f\right|\left[\begin{pmatrix}1&0\\0&p\end{pmatrix}\delta_l\sigma_p\delta_l^{-1}\delta_l\right]_k \\
&= \sum_{v=0}^{p-1} f\left|\left[\begin{pmatrix}1&0\\0&p\end{pmatrix}\begin{pmatrix}1&vl\\0&1\end{pmatrix}\delta_l\right]_k + f\right|\left[\begin{pmatrix}1&0\\0&p\end{pmatrix}\widehat{\sigma_p}\delta_l\right]_k,
\end{aligned}$$

where $\widehat{\sigma_p} \in \Gamma_0(N)$ satisfies $\widehat{\sigma_p} \equiv \begin{pmatrix}1&0\\0&1\end{pmatrix} \pmod N$, and furthermore $\widehat{\sigma_p} \equiv \begin{pmatrix}0&-ml\\(ml)'&0\end{pmatrix}$ (mod p) if $\sigma_p \equiv \begin{pmatrix}0&-m\\m'&0\end{pmatrix} \pmod p$. Hence, by Lemma 6.7, we see that

$$(f|[\delta_l]_k)|\Gamma_0(pM)\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_0(M) = \left(f|\Gamma_0(pN)\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_0(N)\right)|[\delta_l]_k.$$

This completes the proof. □

Lemma 6.9 *Let l be a square free positive integer, $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N,k,\chi)$ such that $a(n) = 0$ if $(n,l) = 1$. Then*

$$f(z) = \sum_{p|l} g_p(pz),$$

where $g_p(z) \in G(Nl^2,k,\chi)$ and moreover $g_p(z) \in G(Nl,k,\chi)$ if $l|N$. Furthermore, all g_p are cusp forms if $f(z)$ is a cusp form.

Proof We assume first that l is a prime. Put $g(z) = f(z/l)$. By Theorem 6.1, we see that $g(z) \in G(N/l,k,\chi)$ or $g(z) = 0$ if $lm_\chi|N$ or $lm_\chi \nmid N$ respectively. Anyway, $g(z) \in G(Nl,k,\chi)$ and $f(z) = g(lz)$, the lemma holds. Now assume that l is a composite and the lemma holds for any proper factor of l. Let p be a prime factor of l. Put $l' = l/p$ and $h(z) = \sum_{p\nmid n} a(n)e(nz)$. By Lemma 6.6, we see that $h(z) \in G(Np^2,k,\chi)$. Put $f(z) - h(z) = \sum_{n=0}^{\infty} b(n)e(nz)$. It is clear that $b(n) = 0$ if $p \nmid n$. Set $g_p(z) = f(z/p) - h(z/p)$, by Theorem 6.1, we have that $g_p(z) \in G(Np,k,\chi)$ and

$$f(z) = g_p(pz) + h(z).$$

Since $h(z), Np^2, l'$ satisfy the conditions in the lemma, by induction hypothesis, we have

$$h(z) = \sum_{q|l'} g_q(qz), g_q(z) \in G(Nl'^2,k,\chi) \subset G(Nl^2,k,\chi),$$

with q primes. It is clear that, by Lemma 6.6 and the above proof, $g_p \in G(Nl,k,\chi)$ if $l|N$. This completes the proof. □

Theorem 6.2 *Let $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N,k,\chi)$, l a positive integer. Assume that $a(n) = 0$ if $(l,n) = 1$. Then*

(1) $f(z) = 0$ if $(l, N/m_\chi) = 1$;

(2) if $(l, N/m_\chi) \neq 1$, then for any prime factor p of $(l, N/m_\chi)$ there exists $f_p(z) \in G(N/p, k, \chi)$ such that

$$f(z) = \sum_{p|(l,N/m_\chi)} f_p(pz),$$

where m_χ is the conductor of χ. Furthermore, all f_p are cusp forms if f is a cusp form.

Proof Without loss of generality, we may assume that l is square free. It is clear that, by Theorem 6.1, the theorem holds for l a prime. Now assume that l is a composite and the theorem holds for any proper factor of l. Let p be a prime factor of l and $l' = l/p$. Set

$$\begin{aligned} h(z) &= \sum_{(n,l')\neq 1} a(n)e(nz), \\ g(z) &= f(z) - h(z) = \sum_{(n,l')=1} a(n)e(nz). \end{aligned} \tag{6.9}$$

By Lemma 6.6, $g(z) \in G(Nl'^2, k, \chi)$ and so $h(z) \in G(Nl'^2, k, \chi)$. It is clear that the Fourier coefficient $a(n)$ of $g(z)$ must be zero if $p \nmid n$, so that $g_p(z+1) = g_p(z)$ where $g_p(z) = g(z/p)$. If $pm_\chi \nmid N$, then $pm_\chi \nmid Nl'^2$, and $g(z) = 0$ by Theorem 6.1. Therefore $f(z) = h(z) = \sum_{(n,l')\neq 1} a(n)e(nz)$. This shows that the theorem holds by the induction hypothesis. Now assume that $pm_\chi | N$. By Theorem 6.1, we see that $g_p(z) \in G(Nl'^2/p, k, \chi)$. Lemma 6.7 gives

$$\Gamma_0(Nl'^2)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_0(Nl'^2/p) = \Gamma_0(Nl'^2)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\sigma_p \bigcup_{v=0}^{p-1}\Gamma_0(Nl'^2)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix},$$

where the first term disappears if $p^2|N$, so that,

$$\begin{aligned} &\left(g|\Gamma_0(Nl'^2)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_0(Nl'^2/p)\right)(z) \\ &= p^{k/2-1}\sum_{v=0}^{p-1}\left(g\left|\left[\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}\right]_k\right.\right)(z) + p^{k/2-1}\left(g\left|\left[\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\sigma_p\right]_k\right.\right)(z) \\ &= p^{-1}\sum_{v=0}^{p-1}\left(g_p\left|\left[\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}\right]_k\right.\right)(z) + p^{-1}(g_p|[\sigma_p]_k)(z) \\ &= \frac{d}{p}g_p(z), \end{aligned}$$

where $d = \begin{cases} p, & \text{if } p^2|N, \\ p+1, & \text{if } p^2 \nmid N \end{cases}$. Therefore

$$g(z) = g_p(pz) = \frac{d}{p}\left(g\left|\Gamma_0(Nl'^2)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_0(Nl'^2/p)\right.\right)(pz). \tag{6.10}$$

Since

$$f_p(z) = \frac{d}{p}\left(f\left|\Gamma_0(N)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_0(N/p)\right.\right)(z) \in G(N/p, k, \chi),$$

we have that, by Lemma 6.8,

$$f_p(z) = \frac{d}{p}\left(f \Big| \Gamma_0(Nl'^2) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(Nl'^2/p)\right)(z). \tag{6.11}$$

We want to show that $f(z) - f_p(pz)$ satisfies the conditions in the theorem for l', and hence we can complete the proof by induction. It is clear that $f(z) - f_p(pz) \in G(N, k, \chi)$. By (6.9)–(6.11), we see that

$$\begin{aligned} f(z) - f_p(pz) &= f(z) - f_p(pz) - g(z) + g_p(pz) \\ &= h(z) - \frac{d}{p}\left(h \Big| \Gamma_0(Nl'^2) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(Nl'^2/p)\right)(pz). \end{aligned} \tag{6.12}$$

Applying the induction hypothesis for $h(z)$, Nl'^2 and l', we have

$$h(z) = \sum_{q|l'} h_q(qz), \quad h_q(z) \in G(Nl'^2, k, \chi) \tag{6.13}$$

with q primes. By Lemma 6.8, for any prime factor q of l', we have

$$h\big|\Gamma_0(Nl'^2) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(Nl'^2/p) = h\big|\Gamma_0(Nl'^3q) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(Nl'^3q/p) \tag{6.14}$$

and this holds also if h is substituted by h_q. By (6.13), (6.14) and (2) of Lemma 6.8, we have

$$\begin{aligned} &\left(h \Big| \Gamma_0(Nl'^2) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(Nl'^2/p)\right)(z) \\ &= \left(\sum_{q|l'} (q^{-k/2} h_q|[\delta_q]_k) \Big| \Gamma_0(Nl'^3q) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(Nl'^3q/p)\right)(z) \\ &= \sum_{q|l'} \left(h_q \Big| \Gamma_0(Nl'^3) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(Nl'^3/p)\right)(qz). \end{aligned}$$

This implies that the Fourier coefficient $b(n)$ of $\left(h \Big| \Gamma_0(Nl'^2) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(Nl'^2/p)\right)(z)$ must be zero if $(n, l') = 1$, and hence, by (6.12) and (6.13), so is the Fourier coefficient $c(n)$ of $f(z) - f_p(pz)$. This shows that $f(z) - f_p(pz)$ satisfies the conditions in the theorem for l'. Hence

$$f(z) - f_p(pz) = \sum_{q|l'} f_q(qz), \quad f_q(z) \in G(N/q, k, \chi),$$

where q runs over all prime factors of $(l', N/m_\chi)$. This completes the proof. □

Definition 6.1 *Denote by $S^{\mathrm{old}}(N, k, \chi)$ the subspace of $S(N, k, \chi)$ generated by*

$$\bigcup_{\substack{m_\chi|M|N, \\ M \neq N}} \bigcup_{l|N/M} \{f(lz) | f(z) \in S(M, k, \chi)\}.$$

And any modular form in $S^{\mathrm{old}}(N, k, \chi)$ is called an old form.

Definition 6.2 *Denote by* $S^{\text{new}}(N,k,\chi)$ *the orthogonal complement subspace of* $S^{\text{old}}(N,k,\chi)$ *in* $S(N,k,\chi)$ *with respect to the* Petersson *inner product. And any modular form in* $S^{\text{new}}(N,k,\chi)$ *is called a new form.*

By the definitions, we have

Lemma 6.10 (1) $S(N,k,\chi)=S^{\text{new}}(N,k,\chi)$ *if* χ *is a primitive character modulo* N;

(2) $S(M,k,\chi)\subset S^{\text{old}}(N,k,\chi)$ *if* $m_\chi|M|N$ *and* $M\neq N$;

(3) $S(N,k,\chi)$ *is generated by* $\displaystyle\bigcup_{m_\chi|M|N}\ \bigcup_{l|N/M}\{f(lz)|f(z)\in S^{\text{new}}(M,k,\chi)\}$.

Lemma 6.11 *Let* n *be a positive integer with* $(n,N)=1$. *Then* $\mathrm{T}(n)$ *sends* $S^{\text{old}}(N,k,\chi)$ (*and* $S^{\text{new}}(N,k,\chi)$ resp.) *into* $S^{\text{old}}(N,k,\chi)$ (*and* $S^{\text{new}}(N,k,\chi)$ *resp.*).

Proof Let $f(z)\in S^{\text{old}}(N,k,\chi)$. By the definition of old forms, we have

$$f(z)=\sum_v f_v(l_vz),\quad f_v\in S(M_v,k,\chi), l_vM_v|N, M_v\neq N.$$

Put $g_v(z)=f_v(l_vz)$. Since $\mathrm{T}(n)$ commutes with $[\delta_l]_k$ for any $(n,l)=1$, we see that

$$(f|\mathrm{T}(n))(z)=\sum_v(g_v|\mathrm{T}(n))(z)=\sum_v(f_v|\mathrm{T}(n))(l_vz).$$

Since $f_v\in S(M_v,k,\chi)$, we have that $f_v|\mathrm{T}(n)\in S(M_v,k,\chi)$, so that $f|\mathrm{T}(n)\in S^{\text{old}}(N,k,\chi)$. This shows that $\mathrm{T}(n)$ sends $S^{\text{old}}(N,k,\chi)$ into itself. The next lemma will show that $\overline{\chi}(n)\mathrm{T}(n)$ is the conjugate operator of $\mathrm{T}(n)$ on the space $S(N,k,\chi)$ with respect to the Petersson inner product, so that $\mathrm{T}(n)$ sends $S^{\text{new}}(N,k,\chi)$ into itself. This completes the proof. □

Lemma 6.12 *Let* $f(z)=\displaystyle\sum_{m=1}^{\infty}a(m)e(mz)\in S(N,k,\chi)$ *and* $f(z)|\mathrm{T}(n)=\displaystyle\sum_{m=1}^{\infty}b(m)e(mz)\in S(N,k,\chi)$. *Then*

(1) $b(m)=\displaystyle\sum_{1\leqslant d|(m,n)}\chi(d)d^{k-1}a(mn/d^2)$;

(2) *the conjugate operator* $\mathrm{T}(n)^*$ *of* $\mathrm{T}(n)$ (*with respect to the* Petersson *inner product*) *is equal to* $\overline{\chi}(n)\mathrm{T}(n)$ *for any* $(n,N)=1$.

Proof (1) is a direct conclusion of (5.14).

(2) is a direct conclusion of Lemma 5.18 and Lemma 5.26. □

By Lemma 6.11, there is a basis in $S^{\text{new}}(N,k,\chi)$ (and in $S^{\text{old}}(N,k,\chi)$ resp.) whose elements are eigenfunctions of all Hecke operators $\mathrm{T}(n)$ with $(n,N)=1$.

Lemma 6.13 *Let L be a positive integer, and*

$$0 \neq f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in S^{\text{new}}(N, k, \chi)$$

an eigenfunction of all Hecke *operators* $\mathrm{T}(n)$ *with* $(n, L) = 1$. *Then* $a_1 \neq 0$.

Proof Assume that $a_1 = 0$. If $a(n) = 0$ for any $(n, L) = 1$, then, by Theorem 6.2, $f(z) \in S^{\text{old}}(N, k, \chi)$ which is impossible. Hence

$$m = \min\{n|(n, L) = 1, a(n) \neq 0\} > 1.$$

Let p be a prime factor of m. Then $f|\mathrm{T}(p) = c_p f$ with c_p a constant. By Lemma 6.12, we see that $c_p a(m/p) = a(m) + \chi(p)p^{k-1}a(m/p^2)$. By the definition of m, we have $a(m/p) = a(m/p^2) = 0$, so that $a(m) = 0$, which is impossible. This completes the proof. □

Theorem 6.3 *Let L be a positive integer, f and $g \in S(N, k, \chi)$ such that $f|\mathrm{T}(n) = \lambda_n f$, $g|\mathrm{T}(n) = \lambda_n g$ for all $(n, L) = 1$ with λ_n constants. Then $f = cg$ for a constant c if $0 \neq f \in S^{\text{new}}(N, k, \chi)$.*

Proof Let $f(z) = \sum_{n=1}^{\infty} a(n)e(nz)$. Without loss of generality, we can assume that $a(1) = 1$ by Lemma 6.13. We may assume also that $N|L$. Set

$$g(z) = g^{(0)}(z) + g^{(1)}(z), \quad g^{(0)}(z) \in S^{\text{new}}(N, k, \chi), \quad g^{(1)}(z) \in S^{\text{old}}(N, k, \chi).$$

By Lemma 6.11, we see that

$$g^{(0)}|\mathrm{T}(n) = \lambda_n g^{(0)}, \quad g^{(1)}|\mathrm{T}(n) = \lambda_n g^{(1)}, \quad (n, L) = 1.$$

Hence, by Lemma 6.13, $b(1) \neq 0$ if $g^{(0)}(z) = \sum_{n=1}^{\infty} b(n)e(nz) \neq 0$. By Lemma 6.12, we have

$$f|\mathrm{T}(n) = a(n)f, \quad g^{(0)}|\mathrm{T}(n) = \frac{b(n)}{b(1)}g^{(0)}, \quad (n, L) = 1.$$

This shows that $a(n)b(1) = b(n)$ for all $(n, L) = 1$. Put

$$g^{(0)} - b(1)f = \sum_{n=1}^{\infty} c(n)e(nz),$$

then $c(n) = 0$ for all $(n, L) = 1$, so that $g^{(0)} - b(1)f \in S^{\text{old}}(N, k, \chi)$ by Theorem 6.2. This implies that $g^{(0)} - b(1)f = 0$. We shall now prove that $g^{(1)} = 0$. If $m_\chi = N$, then $S^{\text{old}}(N, k, \chi) = 0$. So we may assume that $m_\chi \neq N$. Suppose that $g^{(1)} \neq 0$, then

$$g^{(1)}(z) = \sum_v h_v(l_v z), \quad h_v \in S^{\text{new}}(M_v, k, \chi), \quad l_v M_v|N, \quad M_v \neq N. \tag{6.15}$$

Since there is a basis in $S^{\mathrm{new}}(M_v,k,\chi)$ whose elements are eigenfunctions for all $\mathrm{T}(n)$ $((n,M_v)=1)$, we may assume that $h_v(z)$ is an eigenfunction of all $\mathrm{T}(n)$ $((n,M_v)=1)$, so that, by Lemma 6.3, $h_v(l_vz)$ is an eigenfunction of all $\mathrm{T}(n)$ $((n,L)=1)$. Since eigenfunctions corresponding to different eigenvalues are linearly independent, the sum of $h_v(l_vz)$ with eigenvalue different from $a(n)$ with respect to $\mathrm{T}(n)$ must be zero. Therefore every $h_v(z)$ on the right hand side of (6.15) must satisfy

$$h_v|\mathrm{T}(n)=a(n)h_v, \quad (n,L)=1.$$

Denote by h any fixed one of these h_v. Let d be the first coefficient of the Fourier expansion of h, then $d\neq 0$ by Lemma 6.13. Put

$$h(z)-df(z)=\sum_{n=1}^{\infty}d(n)e(nz),$$

then $d(n)=0$ for all $(n,L)=1$, so that $h(z)-df(z)\in S^{\mathrm{old}}(N,k,\chi)$ by Theorem 6.2. Therefore

$$f(z)=-\frac{1}{d}\left(h(z)-df(z)\right)+\frac{1}{d}h(z)\in S^{\mathrm{old}}(N,k,\chi),$$

which implies that $f(z)=0$ since $f(z)\in S^{\mathrm{new}}(N,k,\chi)$. This contradicts the hypothesis $f\neq 0$. This completes the proof. □

Theorem 6.4 *Let* $R_0(N)$ *and* $R_0^*(N)$ *be the* Hecke *algebras* $R(\Gamma_0(N),\Delta_0(N))$ *and* $R(\Gamma_0(N),\Delta_0^*(N))$ *respectively. Then there is a basis in* $S^{\mathrm{new}}(N,k,\chi)$ *whose elements are common eigenfunctions of* $R_0(N)$ *and* $R_0^*(N)$.

Proof By Theorem 5.5, $R_0(N)$ and $R_0^*(N)$ are commutative and $\mathrm{T}(n)\in R_0(N)$ for any $(n,N)=1$. Let $\{f_1,f_2,\cdots,f_r\}$ be a basis of $S^{\mathrm{new}}(N,k,\chi)$ such that every f_i is a common eigenfunction of $\mathrm{T}(n)$ for all $(n,N)=1$. Put $f_i|\mathrm{T}(n)=a(n,i)f_i$, $(n,N)=1$ with $a(n,i)$ a constant. For any $T\in R_0(N)$, since $\mathrm{T}(n)$ $((n,N)=1)$ commutes with T, we see that

$$(f_i|T)|\mathrm{T}(n)=(f_i|\mathrm{T}(n))|T=a(n,i)f_i|T, \quad (n,N)=1.$$

That is, $f_i|T$ is a common eigenfunction of all $\mathrm{T}(n)$ with eigenvalue $a(n,i)$. By Theorem 6.3, we have that $f_i|T=cf_i$ with a constant c. This shows that f_i is a common eigenfunction of $R_0(N)$. This shows the first part of the theorem. Since $\mathrm{T}(n)^*\in R_0^*(N)$ $((n,N)=1)$ commutes with any $T\in R_0^*(N)$, and $\mathrm{T}(n)^*=\overline{\chi}(n)\mathrm{T}(n)$, $(n,N)=1$, we see that $\mathrm{T}(n)$ commutes with $T\in R_0^*(N)$. Similar to the above process, $f_i|T=c'f_i$ with a constant c' for any $T\in R_0^*(N)$, so that, f_i is also a common eigenfunction of $R_0^*(N)$. Therefore f_i $(1\leqslant i\leqslant r)$ are common eigenfunctions of $R_0(N)$ and $R_0^*(N)$. This completes the proof. □

Definition 6.3 $f(z)=\sum_{n=1}^{\infty}a(n)e(nz)\in S(N,k,\chi)$ *is called a primitive cusp form* *if it satisfies the following two conditions*:

(1) $f \in S^{\text{new}}(N,k,\chi)$ *and it is a common eigenfunction of* $R_0(N)$;

(2) $a(1) = 1$.

By Theorem 6.4, a primitive cusp form is also a common eigenfunction of $R_0^*(N)$, and there exists a basis in $S^{\text{new}}(N,k,\chi)$ whose elements are primitive cusp forms.

Lemma 6.14 *Let* $f \in S(N,k,\chi)$ *be a common eigenfunction of all* $\mathrm{T}(n)$ *with* $(n,N)=1$, *and* $f|\mathrm{T}(n) = a(n)f$, $(n,N)=1$. *Then there exists a factor* M *of* N *and a primitive cusp form* g *of* $S^{\text{new}}(M,k,\chi)$ *such that*

$$g|\mathrm{T}(n) = a(n)g, \quad (n,N)=1.$$

Furthermore, we can take $M \neq N$ *if* $f \notin S^{\text{new}}(N,k,\chi)$.

Proof If $f \in S^{\text{new}}(N,k,\chi)$, the lemma is obvious. So assume $f \notin S^{\text{new}}(N,k,\chi)$. By the proof of Theorem 6.3, there exists $N \neq M|N$ and $h \in S^{\text{new}}(M,k,\chi)$ such that

$$h|\mathrm{T}(N) = a(n)h, (n,N)=1.$$

Take $g = \frac{1}{d}h$ with d the first Fourier coefficient of h. This completes the proof. □

Lemma 6.15 *Let* $f \in G(N,k,\chi)$. *Then*

$$(f|\mathrm{T}(l,m))|[W(N)]_k = (f|[W(N)]_k)|\mathrm{T}(m,l)^*,$$
$$(f|\mathrm{T}(n))|[W(N)]_k = (f|[W(N)]_k)|\mathrm{T}(n)^*.$$

Proof It is clear that we only need to show the first equality in the lemma. It is clear that the map: $\alpha \mapsto W(N)^{-1}\alpha W(N)$ is an isomorphism from $\Delta_0(N)$ to $\Delta_0^*(N)$, and $W(N)^{-1}\Gamma_0(N)\ W(N) = \Gamma_0(N)$. For any $\alpha \in \Delta_0(N)$, we have

$$\chi(W(N)^{-1}\alpha W(N)) = \chi(\alpha)^{-1}.$$

Let $\Gamma_0(N)\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}\Gamma_0(N) = \bigcup_v \Gamma_0(N)\alpha_v$ be a disjoint union, then

$$\Gamma_0(N)\begin{pmatrix} m & 0 \\ 0 & l \end{pmatrix}\Gamma_0(N) = \bigcup_v \Gamma_0(N)(W(N)^{-1}\alpha_v W(N)).$$

Hence, for any $g \in G(N,k,\overline{\chi})$, we have

$$\begin{aligned} & g|[W(N)^{-1}]_k\mathrm{T}(l,m)[W(N)]_k \\ =&(lm)^{k/2-1}\sum_v \chi(\alpha_v)^{-1} g|[W(N)^{-1}\alpha_v W(N)]_k \\ =&(lm)^{k/2-1}\sum_v \overline{\chi}(W(N)^{-1}\alpha_v W(N))^{-1} g|[W(N)^{-1}\alpha_v W(N)]_k \\ =&g|\mathrm{T}(m,l)^*. \end{aligned}$$

Since $W(N)$ is an isomorphism from $G(N,k,\chi)$ to $G(N,k,\overline{\chi})$, we see that the first equality holds in the lemma. This completes the proof. □

Theorem 6.5 (1) *The map: $f \mapsto f|[W(N)]_k$ induces the following isomorphisms.*

$$S^{\mathrm{new}}(N,k,\chi) \simeq S^{\mathrm{new}}(N,k,\overline{\chi}),$$
$$S^{\mathrm{old}}(N,k,\chi) \simeq S^{\mathrm{old}}(N,k,\overline{\chi});$$

(2) *Let*

$$f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S(N,k,\chi)$$

be a primitive cusp form, then

$$g(z) := \sum_{n=1}^{\infty} \overline{a(n)}e(nz)$$

is a primitive cusp form of $S(N,k,\overline{\chi})$, and $f|[W(N)]_k = cg$ with a constant c.

Proof (1) We show first that $[W(N)]_k$ sends $S^{\mathrm{old}}(N,k,\chi)$ into $S^{\mathrm{old}}(N,k,\overline{\chi})$. This is equivalent to show the following assertion: let $N \neq M|N$, $m_\chi|M$, $l|N/M$, and let $h \in S(M,k,\chi)$ such that $f(z) = h|[\delta_l]_k$, then $f|[W(N)]_k \in S^{\mathrm{old}}(N,k,\overline{\chi})$. We show now the assertion. Put $l' = N/(lM)$. Then $\delta_l W(N)\delta_{l'}^{-1} = lW(M)$, so that

$$f|[W(N)]_k = h|[\delta_l W(N)\delta_{l'}^{-1}\delta_{l'}]_k = (h|[W(M)]_k)|[\delta_{l'}]_k.$$

Since $h|[W(M)]_k \in S(M,k,\overline{\chi})$, $f|[W(N)]_k \in S^{\mathrm{old}}(N,k,\overline{\chi})$. Now suppose $f \in S^{\mathrm{new}}(N,k,\chi)$. Then, for any $f_1 \in S^{\mathrm{old}}(N,k,\overline{\chi})$, we have

$$\langle f|[W(N)]_k, f_1\rangle = \langle f, f_1|[W(N)^\tau]_k = (-1)^k\langle f, f_1|[W(N)]_k\rangle = 0,$$

since $f_1|[W(N)]_k \in S^{\mathrm{old}}(N,k,\chi)$. Therefore $f|[W(N)]_k \in S^{\mathrm{new}}(N,k,\overline{\chi})$. This shows (1).

(2) By (1), we have $f|[W(N)]_k \in S^{\mathrm{new}}(N,k,\overline{\chi})$. By Lemma 6.15, we have

$$(f|[W(N)]_k)|\mathrm{T}(n) = (f|\mathrm{T}(n)^*)|[W(N)]_k = \overline{a(n)}f|[W(N)]_k$$

for any positive integer n. Hence $f|[W(N)]_k$ must be a constant multiple of some primitive cusp form g. Let $b(n)$ be the n-th Fourier coefficient of $f|[W(N)]_k$, then $b(n) = \overline{a(n)}b(1)$, so that

$$(f|[W(N)]_k)(z) = b(1)\sum_{n=1}^{\infty} \overline{a(n)}e(nz).$$

Since $a(1) = 1$ and the first Fourier coefficient of g is also equal to 1, we see that

$$g(z) = \sum_{n=1}^{\infty} \overline{a(n)}e(nz), \quad f|[W(N)]_k = b(1)g.$$

This completes the proof. □

Let $f(z)=\sum_{n=1}^{\infty}a(n)e(nz)\in S(N,k,\chi)$ be a primitive cusp form. Then

$$L(s,f)=\sum_{n=1}^{\infty}a(n)n^{-s}=\prod_{p}\left(1-a(p)p^{-s}+\chi(p)p^{k-1-2s}\right)^{-1}$$
$$=\prod_{p\nmid N}\left(1-a(p)p^{-s}+\chi(p)p^{k-1-2s}\right)^{-1}\prod_{p|N}\left(1-a(p)p^{-s}\right)^{-1}.$$

For any $p\nmid N$, by the Ramanujan-Petersson Conjecture (proved by Deligne), we have $|a(p)|\leqslant 2p^{(k-1)/2}$. We discuss now $a(p)$ for $p|N$. For any $p|N$, set $N=N_pN_p'$ with $p\nmid N_p'$, and χ_p the character modulo N_p induced from χ. Fix a prime factor q of N, put $\chi'=\prod_{p\neq q}\chi_p$. Let γ_q, $\gamma_q'\in SL_2(\mathbb{Z})$ satisfy

$$\gamma_q\equiv\begin{cases}\begin{pmatrix}0&-1\\1&0\end{pmatrix} & (\text{mod } N_q^2),\\ \begin{pmatrix}1&0\\0&1\end{pmatrix} & (\text{mod } (N/N_q)^2),\end{cases}\qquad \gamma_q'\equiv\begin{cases}\begin{pmatrix}1&0\\0&1\end{pmatrix} & (\text{mod } N_q^2),\\ \begin{pmatrix}0&-1\\1&0\end{pmatrix} & (\text{mod } (N/N_q)^2).\end{cases}$$

Set

$$\eta_q=\gamma_q\begin{pmatrix}N_q&0\\0&1\end{pmatrix},\quad \eta_q'=\gamma_q'\begin{pmatrix}N/N_q&0\\0&1\end{pmatrix},$$

then

$$\eta_q\Gamma_0(N)\eta_q^{-1}=\Gamma_0(N),\quad \eta_q'\Gamma_0(N)\eta_q'^{-1}=\Gamma_0(N)$$

and for any $\gamma\in\Gamma_0(N)$, we have

$$\chi(\eta_q\gamma\eta_q^{-1})=(\chi'\overline{\chi_q})(\gamma),\quad \chi(\eta_q'\gamma\eta_q'^{-1})=(\overline{\chi'}\chi_q)(\gamma).$$

Hence we have the following two isomorphisms:

$$S(N,k,\chi)\xrightarrow{[\eta_q]_k}S(N,k,\chi'\overline{\chi_q}),$$
$$S(N,k,\chi)\xrightarrow{[\eta_q']_k}S(N,k,\overline{\chi'}\chi_q).$$

And the following two diagrams are commutative:

$$\begin{array}{ccc}S(N,k,\chi) & \xrightarrow{\overline{\chi_q}(n)\mathrm{T}(n)} & S(N,k,\chi)\\ {\scriptstyle[\eta_q]_k}\downarrow & & \downarrow{\scriptstyle[\eta_q]_k}\\ S(N,k,\chi'\overline{\chi_q}) & \xrightarrow{\mathrm{T}(n)} & S(N,k,\chi'\overline{\chi_q})\end{array},\quad (n,N_q)=1;$$

$$\begin{array}{ccc}S(N,k,\chi) & \xrightarrow{\overline{\chi_q'}(n)\mathrm{T}(n)} & S(N,k,\chi)\\ {\scriptstyle[\eta_q']_k}\downarrow & & \downarrow{\scriptstyle[\eta_q']_k}\\ S(N,k,\overline{\chi'}\chi_q) & \xrightarrow{\mathrm{T}(n)} & S(N,k,\overline{\chi'}\chi_q)\end{array},\quad (n,N/N_q)=1.$$

These can be proved along similar lines as in the proof of Lemma 6.15. In particular, we see that $f|[\eta_q]_k \in S(N,k,\chi'\overline{\chi_q})$ and $f|[\eta_q']_k \in S(N,k,\overline{\chi'}\chi_q)$ are common eigenfunctions of all $\mathrm{T}(n)$ $((n,N)=1)$ if $f \in S(N,k,\chi)$ is a common eigenfunction of all $\mathrm{T}(n)$ $((n,N)=1)$. Therefore we see that the assertion (1) of the following theorem holds:

Theorem 6.6 (1) *We have the following isomorphisms*:

$$\begin{aligned}
[\eta_q]_k &: S^{\mathrm{new}}(N,k,\chi) \simeq S^{\mathrm{new}}(N,k,\chi'\overline{\chi_q}),\\
[\eta_q]_k &: S^{\mathrm{old}}(N,k,\chi) \simeq S^{\mathrm{old}}(N,k,\chi'\overline{\chi_q}),\\
[\eta_q']_k &: S^{\mathrm{new}}(N,k,\chi) \simeq S^{\mathrm{new}}(N,k,\overline{\chi'}\chi_q),\\
[\eta_q']_k &: S^{\mathrm{old}}(N,k,\chi) \simeq S^{\mathrm{old}}(N,k,\overline{\chi'}\chi_q).
\end{aligned}$$

(2) *For any* $f \in S(N,k,\chi)$, *we have*

$$\begin{aligned}
f|[\eta_q^2]_k &= \chi_q(-1)\overline{\chi'}(N_q)f,\\
f|[{\eta'}_q^2]_k &= \chi'(-1)\overline{\chi_q}(N/N_q)f,\\
f|[\eta_q\eta_q']_k &= \overline{\chi'}(N_q)f|[W(N)]_k.
\end{aligned}$$

(3) *If* $f = \sum_{n=1}^{\infty} a(n)e(nz) \in S^{\mathrm{new}}(N,k,\chi)$ *is a primitive cusp form, set*

$$f|[\eta_q]_k = c\sum_{n=1}^{\infty} b(n)e(nz), b(1)=1, \quad g_q(z) = \sum_{n=1}^{\infty} b(n)e(nz),$$

then $g_q(z)$ *is a primitive cusp form of* $S(N,k,\chi'\overline{\chi_q})$ *and*

$$b(p) = \begin{cases} \overline{\chi_q}(p)a(p), & \text{if } p \neq q,\\ \chi'(q)\overline{a(q)}, & \text{if } p = q. \end{cases}$$

Proof (2) Put $\eta_q^2 = N_q\gamma$, then $\gamma \in \varGamma(1)$ and

$$\gamma \equiv \begin{cases} \begin{pmatrix} -1 & 0\\ 0 & -1 \end{pmatrix} & (\mathrm{mod}\ N_q),\\ \begin{pmatrix} N_q & 0\\ 0 & N_q^{-1} \end{pmatrix} & (\mathrm{mod}\ (N/N_q)). \end{cases}$$

So that, $\gamma \in \varGamma_0(N)$, and hence $f|[\eta_q^2]_k = \chi_q(-1)\overline{\chi'}(N_q)f$. Similarly set ${\eta'}_q^2 = \dfrac{N}{N_q}\gamma_1$, then $\gamma_1 \in \varGamma_0(N)$ and

$$\gamma_1 \equiv \begin{cases} \begin{pmatrix} N/N_q & 0\\ 0 & (N/N_q)^{-1} \end{pmatrix} & (\mathrm{mod}\ N/N_q),\\ \begin{pmatrix} -1 & 0\\ 0 & -1 \end{pmatrix} & (\mathrm{mod}\ N_q). \end{cases}$$

Hence

$$f|[\eta'^2_q]_k = \chi'(-1)\overline{\chi_q}(N/N_q)f.$$

Set $\gamma_2 = \eta_q\eta'_q W(N)^{-1}$, then $\gamma_2 \in \Gamma_0(N)$ and

$$\gamma_2 \equiv \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & (\text{mod } N_q), \\ \begin{pmatrix} N_q & 0 \\ 0 & N_q^{-1} \end{pmatrix} & (\text{mod } (N/N_q)). \end{cases}$$

Hence

$$f|[\eta_q\eta'_q]_k = \overline{\chi'}(N_q)f|[W(N)]_k.$$

(3) If $(n,q)=1$, then

$$(f|[\eta_q]_k)|\mathrm{T}(n) = \overline{\chi_q}(n)(f|\mathrm{T}(n))|[\eta_q]_k = \overline{\chi_q}(n)a(n)f|[\eta_q]_k. \tag{6.16}$$

If $(n, N/N_q)=1$, then

$$(f|[\eta'_q]_k)|\mathrm{T}(n) = \overline{\chi'}(n)a(n)f|[\eta'_q]_k. \tag{6.17}$$

Since $f|[\eta_q]_k \in S^{\text{new}}(N,k,\chi'\overline{\chi_q})$ by (1), $f|[\eta_q]_k$ is a constant multiple of a primitive cusp form by Lemma 6.14, and by (6.16) we have

$$b(p) = \overline{\chi_q}(p)a(p), \qquad \text{if } p \neq q.$$

By (2), we see that $f|[\eta_q]_k = cf|[W(N)\eta'_q]_k$ with $c = \overline{\chi'}(-N_q)\overline{\chi_q}(N/N_q)$, so that

$$(f|[\eta_q]_k)|\mathrm{T}(n) = c((f|[W(N)]_k)|[\eta'_q]_k)|\mathrm{T}(n).$$

Since $f|[W(N)]_k \in S(N,k,\overline{\chi})$, we see that, by (6.17) and Lemma 6.15,

$$\begin{aligned}(f|[\eta_q]_k)|\mathrm{T}(n) &= c\chi'(q)((f|[W(N)]_k)|\mathrm{T}(n))|[\eta'_q]_k \\ &= c\chi'(q)\overline{a(n)}f|[W(N)\eta'_q]_k \\ &= \chi'(q)\overline{a(n)}f|[\eta_q]_k.\end{aligned}$$

Therefore $b(q) = \chi'(q)\overline{a(q)}$. This completes the proof. □

Theorem 6.7 *Let $f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S(N,k,\chi)$ be a primitive cusp form, m the conductor of χ. For any prime $q|N$, put $N = N_qN'_q$, $m = m_qm'_q$ with $q \nmid N'_q$ and $q \nmid m'_q$. Then*

(1) $|a_q| = q^{(k-1)/2}$, *if* $N_q = m_q$;

(2) $a_q^2 = \overline{\chi'}(q)q^{k-2}$, *if* $N_q = q$ *and* $m_q = 1$;

(3) $a_q = 0$, *if* $q^2|N$ *and* $N_q \neq m_q$.

Proof (1) Let γ_q, η_q be as above, a a positive integer prime to q. Take a positive integer b such that $ab+1 \equiv 0 \pmod{N_q}$ and $a \equiv b \pmod{N/N_q}$. Let γ be a matrix satisfying

$$\begin{pmatrix} 1 & a \\ 0 & q^e \end{pmatrix} \gamma_q = \gamma \begin{pmatrix} 1 & b \\ 0 & q^e \end{pmatrix}, \quad N_q = q^e,$$

then $\gamma \in SL_2(\mathbb{Z})$ and

$$\gamma \equiv \begin{pmatrix} a & * \\ 0 & -b \end{pmatrix} \pmod{N_q}, \quad \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N/N_q},$$

so that $\gamma \in \mathit{\Gamma}_0(N)$ and $\chi(\gamma) = \chi_q(-b)$. Therefore we obtain

$$f \left| \left[\begin{pmatrix} 1 & a \\ 0 & q^e \end{pmatrix} \gamma_q \right]_k \right. = \chi_q(-b) f \left| \left[\begin{pmatrix} 1 & b \\ 0 & q^e \end{pmatrix} \right]_k \right. .$$

Let a run over a reduced residue system modulo N_q, then we get

$$\begin{aligned} & q^{e(k/2-1)} \sum_{(a,N_q)=1} \left(f \left| \left[\begin{pmatrix} 1 & a \\ 0 & q^e \end{pmatrix} \right]_k \right. \right) \Big| [\eta_q]_k \\ &= q^{e(k/2-1)} \left(\sum_{(b,N_q)=1} \chi_q(-b) f \left| \left[\begin{pmatrix} 1 & b \\ 0 & q^e \end{pmatrix} \right]_k \right. \right) \left| \left[\begin{pmatrix} q^e & 0 \\ 0 & 1 \end{pmatrix} \right]_k \right. \\ &= q^{e(k/2-1)} \chi_q(-1) \left(\sum_{n=1}^{\infty} \sum_{(b,N_q)=1} \chi_q(b) \mathrm{e}^{2\pi i n b/q^e} \right) a(n) e(nz) \\ &= q^{e(k/2-1)} W(\chi_q) \sum_{n=1}^{\infty} \overline{\chi_q}(-n) a(n) e(nz), \end{aligned} \tag{6.18}$$

where $W(\chi_q)$ is the Gauss sum of χ_q. Since

$$f|\mathrm{T}(n) = n^{k/2-1} \sum_{\substack{ad=n, a>0, \\ (a,N)=1}} \sum_{b \bmod d} f \left| \left[\sigma_a \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right]_k \right. ,$$

we see that

$$\begin{aligned} q^{e(k/2-1)} \sum_{(a,N_q)=1} f \left| \left[\begin{pmatrix} 1 & a \\ 0 & q^e \end{pmatrix} \right]_k \right. &= f|\mathrm{T}(q^e) - q^{k/2-1} (f|\mathrm{T}(q^{e-1})) \left| \left[\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \right]_k \right. \\ &= a(q^e) f - q^{k/2-1} a(q^{e-1}) f \left| \left[\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \right]_k \right. . \end{aligned}$$

Hence we obtain

$$\begin{aligned} & q^{e(k/2-1)} \sum_{(a,N_q)=1} \left(f \left| \left[\begin{pmatrix} 1 & a \\ 0 & q^e \end{pmatrix} \right]_k \right. \right) \Big| [\eta_q]_k \\ &= a(q^e) f|[\eta_q]_k - \chi'(q) q^{k/2-1} a(q^{e-1}) (f|[\eta_q]_k \left| \left[\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \right]_k \right. , \end{aligned} \tag{6.19}$$

where we used the facts: $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \eta_q \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}^{-1} \eta_q^{-1} \in SL_2(\mathbb{Z})$ and $\chi(\gamma) = \chi'(q)$. Let $g(z) = \sum_{n=1}^{\infty} b(n)e(nz)$ be as in (3) of Theorem 6.6, then $f|[\eta_q]_k = cg$ with a constant c. Comparing the coefficients of $e(z)$ and $e(qz)$ of (6.18), (6.19), we obtain

$$ca(q^e) = q^{e(k/2-1)}W(\chi_q), \quad ca(q^e)b(q) - c\chi'(q)q^{k-1}a(q^{e-1}) = 0.$$

Hence we have, by Theorem 6.6,

$$|a(q)|^2 = q^{k-1}, \quad c = W(\chi_q)q^{e(k/2-1)}a(q^e)^{-1}.$$

(2) By Lemma 5.17 and Lemma 6.8, since $N_q = q$, we see that

$$\Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_0(N/q) = \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_0(N) \bigcup \Gamma_0(N)\eta_q,$$

since we can take $\sigma_q = \gamma_q$ and $\gamma = \gamma_q \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \sigma_q^{-1} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}^{-1} \in \Gamma_0(N)$. Therefore

$$f\Big|\Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_0(N/q) = f|\mathrm{T}(q) + q^{k/2-1}f|[\eta_q]_k.$$

If $(n, N) = 1$, then $\mathrm{T}(n)$ commutes with $\mathrm{T}(q)$ and $[\eta_q]_k$, so that

$$g := f\Bigg|\Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_0(N/q) \in S(N/q, k, \chi)$$

is a common eigenfunction of all $\mathrm{T}(n)$, $(n, N) = 1$ and the eigenvalues are the same as the ones of f. By Theorem 6.3, g is a constant multiple of f. This implies that $g = 0$ since $g \in S(N/q, k, \chi)$ and f is a new form. So that, we get

$$q^{k/2-1}f|[\eta_q]_k = -a(q)f,$$

and hence, by (2) of Theorem 6.6, we have

$$q^{k/2-1}\chi_q(-1)\overline{\chi'}(q)f = q^{k/2-1}f|[\eta_q^2]_k = -a(q)f|[\eta_q]_k = q^{1-k/2}a(q)^2 f.$$

That is, $a(q)^2 = \chi_q(-1)\overline{\chi'}(q)q^{k-2}$. Since $m_q = 1$, $\chi_q(-1) = 1$, $a(q)^2 = \overline{\chi'}(q)q^{k-2}$.

(3) Similar to the proof of (2), we have

$$\Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_0(N/q) = \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_0(N).$$

Hence we get, along similar arguments for the assertion (2),

$$f|\mathrm{T}(q) = f\Bigg|\left[\Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_0(N/q)\right]_k = 0.$$

This implies that $a(q) = 0$, which completes the proof. □

During the proof of Theorem 6.7, we have also shown the following:

Corollary 6.1 (1) *If* $N_q = m_q$, *then*

$$f|[\eta_q]_k = a(q^e)^{-1}q^{e(k/2-1)}W(\chi_q)g$$

with g *a primitive cusp form of* $S(N, k, \chi'\overline{\chi_q})$.

(2) *if* $N_q = q$, $m_q = 1$, *then*

$$f|[\eta_q]_k = -a(q)q^{1-k/2}f, \quad \overline{a(q)} = \chi'(q)a(q).$$

Theorem 6.8 *Let* $f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S(N, k, \chi)$ *be a common eigenfunction of* $R_0(N)$ *and* $R_0^*(N)$, $a(1) = 1$ *and* $g = \sum_{n=1}^{\infty} b(n)e(nz) \in S(M, k, \omega)$ *a primitive cusp form. Assume that there exists a positive integer* L *such that* $a(n) = b(n)$ *for all* $(n, L) = 1$. *Then* $N = M$, $\chi = \omega$ *and* $f = g$.

Proof Without loss of generality, we may assume that L is a common multiple of M and N. If $p \nmid L$, by Lemma 6.12, we have

$$p^{k-1}\chi(p) = a(p)^2 - a(p^2), \quad p^{k-1}\omega(p) = b(p)^2 - b(p^2).$$

But $b(p) = a(p)$ and $a(p^2) = b(p^2)$ for any $p \nmid L$, so that $\chi(p) = \omega(p)$ for any $p \nmid L$. Hence we obtain

$$\chi(n) = \omega(n), \quad \text{if } (n, L) = 1.$$

By the functional equation in Theorem 5.9, we see that

$$\frac{R_N(s, f)}{R_M(s, g)} = \frac{R_N(k - s, f|[W(N)]_k)}{R_M(k - s, g|[W(M)]_k)}. \tag{6.20}$$

Since $L_N(s, f)$ and $L_M(s, g)$ have Euler products for $\mathrm{Re}(s) > 1 + k/2$ respectively, we see that for $\mathrm{Re}(s) > 1 + k/2$

$$\frac{R_N(s, f)}{R_M(s, g)} = \left(\frac{\sqrt{N}}{\sqrt{M}}\right)^s \prod_{p|L} \frac{1 - b(p)p^{-s} + \omega(p)p^{k-1-2s}}{1 - a(p)p^{-s} + \chi(p)p^{k-1-2s}}. \tag{6.21}$$

By the analytic continuation principle, we know that (6.21) holds for all s. Similarly, by (2) of Theorem 6.5 and Lemma 6.15, we have

$$\frac{R_N(k - s, f|[W(N)]_k)}{R_M(k - s, g|[W(M)]_k)} = c\left(\frac{\sqrt{N}}{\sqrt{M}}\right)^{k-s} \prod_{p|L} \frac{1 - \overline{b(p)}p^{s-k} + \overline{\omega}(p)p^{2s-k-1}}{1 - \overline{a(p)}p^{s-k} + \overline{\chi}(p)p^{2s-k-1}} \tag{6.22}$$

with a constant c. Comparing (6.20)–(6.22), we obtain

$$\left(\frac{N}{M}\right)^s \prod_{p|L} \frac{1-b(p)p^{-s}+\omega(p)p^{k-1-2s}}{1-a(p)p^{-s}+\chi(p)p^{k-1-2s}} = c\left(\frac{\sqrt{N}}{\sqrt{M}}\right)^k \prod_{p|L} \frac{1-\overline{b(p)}p^{s-k}+\overline{\omega}(p)p^{2s-k-1}}{1-\overline{a(p)}p^{s-k}+\overline{\chi}(p)p^{2s-k-1}}. \tag{6.23}$$

Let M_p and N_p be the p-parts (i.e., $M_p = p^{\nu_p(M)}$ and $N_p = p^{\nu_p(N)}$, where $\nu_p(*)$ is the p-valuation.) of M and N respectively. By (6.23) and the uniqueness of Dirichlet series, for $p|L$ we have that

$$\left(\frac{N_p}{M_p}\right)^s \frac{1-b(p)p^{-s}+\omega(p)p^{k-1-2s}}{1-a(p)p^{-s}+\chi(p)p^{k-1-2s}} = c_p \frac{1-\overline{b(p)}p^{s-k}+\overline{\omega}(p)p^{2s-k-1}}{1-\overline{a(p)}p^{s-k}+\overline{\chi}(p)p^{2s-k-1}}$$

with c_p a constant. Set $x = p^{-s}$, then

$$1-a(p)p^{-s}+\chi(p)p^{k-1-2s} = 1-a(p)x+\chi(p)p^{k-1}x^2,$$
$$1-b(p)p^{-s}+\omega(p)p^{k-1-2s} = 1-b(p)x+\omega(p)p^{k-1}x^2.$$

Denote by u, v the degrees of the above polynomials with respect to x. It is clear that $0 \leqslant u, v \leqslant 2$.

(1) If $u = v = 0$, we see that $M_p = N_p$.

(2) If $u = 0$, $v = 1$, set $N_p/M_p = p^e$, then we see that

$$1-b(p)x = c_p x^e(1-\overline{b(p)}p^{-k}x^{-1}), b(p) \neq 0.$$

Therefore $|b(p)|^2 = p^k$ which contradicts Theorem 6.7, so that it is impossible that $u = 0$ and $v = 1$.

(3) If $u = 1$, $v = 0$, similar to (2), it is easy to see that $M_p = pN_p$.

(4) If $u = 0$, $v = 2$, set $N_p/M_p = p^e$, then

$$1-b(p)x+\omega(p)p^{k-1}x^2 = c_p x^e(1-\overline{b(p)}p^{-k}x^{-1}+\overline{\omega}(p)p^{-k-1}x^{-2}).$$

This implies that $e = 2$ and hence $|\omega(p)| = p$ which is impossible, so that it is impossible that $u = 0$, $v = 2$.

(5) If $u = 2$, $v = 0$, similar to (4), it is easy to see that $M_p = p^2 N_p$.

(6) If $u = 1$, $v = 2$, set $N_p/M_p = p^e$, then

$$\frac{1-b(p)x+\omega(p)p^{k-1}x^2}{1-a(p)x} = c_p x^e \frac{1-\overline{b(p)}p^{-k}x^{-1}+\overline{\omega}(p)p^{-k-1}x^{-2}}{1-\overline{a(p)}p^{-k}x^{-1}}.$$

This implies that $e = 1$, so that

$$\begin{aligned}&(1-b(p)x+\omega(p)p^{k-1}x^2)(x-\overline{a(p)}p^{-k})\\ &= c_p(1-a(p)x) \times (x^2-\overline{b(p)}p^{-k}x+\overline{\omega}(p)p^{-k-1}).\end{aligned} \tag{6.24}$$

By comparing the coefficients on both sides of (6.24), we obtain

$$|a(p)| = p^{k/2-1}, \quad |c_p| = p^{k/2}. \tag{6.25}$$

By (6.24) and (6.25), we see that $a(p)^{-1} = p^{-k+2}\overline{a(p)}$ should be a root of $1 - b(p)x + \omega(p)p^{k-1}x^2 = 0$, i.e.,

$$1 - b(p)p^{-k+2}\overline{a(p)} + \omega(p)p^{3-k}\overline{a(p)}^2 = 0,$$

so that,

$$b(p) = a(p) + \omega(p)p\overline{a(p)} = a(p) - c(p). \tag{6.26}$$

By (6.25) and (6.26), we have

$$\left|1 - |b(p)|p^{-k/2}\right| < p^{-1},$$

which contradicts Theorem 6.7, and it is impossible that $u = 1$, $v = 2$.

(7) If $u = 2$, $v = 1$, similar to (6), it is easy to see that $M_p = pN_p$.

(8) If $u = v = 2$, it is easy to see that $M_p = N_p$.

Anyway, we proved that $N|M$ and $\chi(n) = \omega(n)$ if $(n, M) = 1$. This implies that $S(N, k, \chi) \subset S(M, k, \omega)$. By Theorem 6.3, we have $f = g$, and hence $M = N$ in terms of Lemma 6.14. This completes the proof. □

By Lemma 6.14 and Theorem 6.8, it is easy to show the following:

Corollary 6.2 (1) *Let* $0 \neq f(z) \in S(N, k, \chi)$*, and*

$$f|\mathrm{T}(n) = a(n)f, \quad (n, N) = 1.$$

Then there exists a unique factor M *of* N *and a unique primitive cusp form* $g(z)$ *of* $S(M, k, \chi)$ *such that*

$$g|\mathrm{T}(n) = a(n)g, \quad (n, N) = 1.$$

(2) *Let* $f(z) \in S(N, k, \chi)$ *be a common eigenfunction of* $R_0(N)$ *and* $R_0^*(N)$*. Then* $f(z)$ *is a constant multiple of some primitive cusp form of* $S^{\mathrm{new}}(N, k, \chi)$*.*

6.2 New Forms with Half Integral Weight

In this section we discuss the Kohnen's theory of new forms with half integral weight. Here and after, we always assume that N is an odd square free positive integer, χ a quadratic character modulo N with conductor t. Put $\varepsilon = \chi(-1)$ and $\chi_1 = \left(\frac{4\varepsilon}{\cdot}\right)\chi$.

We define $S_{k+1/2}(N, \chi)$ as the space of cusp forms of weight $k + 1/2$ and character χ_1 on $\Gamma_0(4N)$ which have a Fourier expansion $\sum_{n=1}^{\infty} a(n)e(nz)$ with $a(n) = 0$ for $\varepsilon(-1)^k n \equiv 2, 3 \pmod 4$. We write $S_{k+1/2}(N)$ for $S_{k+1/2}(N, \mathrm{id.})$ and we call this space Kohnen's "+" space. It is clear that $S_{k+1/2}(N, \chi) \subset S(4N, k + 1/2, \chi_1)$.

Put

$$\xi := \xi_{k,\varepsilon} = \left\{ \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \varepsilon^{1/2} e^{\pi i/4} \right\},$$
$$Q := Q_{k,N,\chi_1} = [\Delta_0(4N,\chi_1)\xi_{k,\varepsilon}\Delta_0(4N,\chi_1)],$$

where $\Delta_0(M,\omega) := \left\{ (A,\phi) | A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M), \phi(z) = \omega(d)\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-1/2}(cz+d)^{1/2} \right\}$. We usually omit the subscripts $k+1/2, 4N, \chi_1$ and write just ξ, Q.

Lemma 6.16 *The operator Q satisfies the quadratic equation $(Q-\alpha)(Q-\beta)=0$ where $\alpha = (-1)^{[(k+1)/2]}\varepsilon 2\sqrt{2}$ and $\beta = -\frac{\alpha}{2}$. It is Hermitian, and its α eigenspace is just $S_{k+1/2}(N,\chi)$.*

Proof It is easy to check that

$$\xi^{\mp}\Delta_0(4N,\chi_1)\xi^{\pm} \bigcap \Delta_0(4N,\chi_1) = \Delta_0(16N,\chi_1).$$

Therefore

$$\Delta_0(4N,\chi_1)\xi^{\pm}\Delta_0(4N,\chi_1) = \bigcup \Delta_0(16N,\chi_1)\xi^{\pm}\xi_u \tag{6.27}$$

is a disjoint union, where $\{\xi_u\}$ is a set of representatives for $\Delta_0(4N,\chi_1)/\Delta_0(16N,\chi_1)$. For any $v \in \mathbb{Z}$, put $A_v = \begin{pmatrix} 1 & 0 \\ 4Nv & 1 \end{pmatrix}$. Then $\{A_v^* | v \bmod 4\}$ is a set of representatives for $\Delta_0(4N,\chi_1)/\Delta_0$ $(16N,\chi_1)$, by (6.27), we see that

$$f|Q = \sum_{v \bmod 4} f|[\xi A_v^*],$$
$$f|Q^2 = \sum_{v \bmod 4}\sum_{u \bmod 4} f|[\xi A_v^* \xi A_u^*].$$

Now

$$\begin{aligned} \xi A_0 \xi A_u &= \left\{ 8\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \varepsilon i^{1/2} \right\} A_u^* \\ &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \chi_1(-2Nu+1)\left(\frac{4Nu}{-2Nu+1}\right)\left(\frac{-4}{-2Nu+1}\right)^{1/2} \right\} \\ &\quad \times \begin{pmatrix} 1+2Nu & -Nu \\ 4Nu & 1-2Nu \end{pmatrix}^* \left\{ 8\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \varepsilon i^{1/2} \right\}. \end{aligned}$$

By the invariance of f under the operation of elements in $\Delta_0(4N,\chi_1)$ and the fact that

$$\sum_{u \bmod 4} \chi_1(-2Nu+1)\left(\frac{4Nu}{-2Nu+1}\right)\left(\frac{-4}{-2Nu+1}\right)^{-k-1/2} = 0,$$

we obtain that

$$\sum_{u \bmod 4} f|[\xi A_0^* \xi A_u^*] = 0.$$

Next we observe that

$$\xi A_{\pm 1}^* \xi = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \chi_1(1 \pm N + N^2) \left(\frac{-4}{1 \pm N + N^2} \right)^{-1/2} \mathrm{e}^{\pi \mathrm{i}/4} \right\} \times \begin{pmatrix} 1 \mp N + N^2 & \left(\dfrac{N \pm 1}{2}\right)^2 \\ -4N^2 & 1 \pm N + N^2 \end{pmatrix}^* \xi A_{\pm 1}^*,$$

hence

$$\sum_{u \bmod 4} f|[\xi A_{\pm 1}^* \xi A_u^*] = \chi_1(1 \pm N + N^2) \left(\frac{-4}{1 \pm N + N^2} \right)^{k-1/2} \varepsilon^{-k-1/2} \mathrm{e}^{-(2k+1)\pi \mathrm{i}/4} f|Q.$$

Since

$$\chi_1(1 + N + N^2) \left(\frac{-4}{1 + N + N^2} \right)^{k-1/2} + \chi_1(1 - N + N^2) \left(\frac{-4}{1 - N + N^2} \right)^{k-1/2} = 1 + \varepsilon(-1)^k \mathrm{i},$$

we obtain

$$\sum_{u \bmod 4} (f|[\xi A_1^* \xi A_u^*] + f|[\xi A_{-1}^* \xi A_u^*]) = (1 + \varepsilon(-1)^k \mathrm{i}) \varepsilon^{-k-1/2} \mathrm{e}^{-(2k+1)\pi \mathrm{i}/4} f|Q.$$

Finally

$$\xi A_2^* \xi = \left\{ \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}, 1 \right\} \begin{pmatrix} 1 + 2N & \dfrac{1+N}{2} \\ 8N & 1 + 2N \end{pmatrix}^*$$

and so

$$\sum_{u \bmod 4} f|[\xi A_2^* \xi A_u^*] = 4f.$$

Summarizing the facts above we showed that

$$Q^2 = (1 + \varepsilon(-1)^k \mathrm{i}) \varepsilon^{-k-1/2} \mathrm{e}^{-(2k+1)\pi \mathrm{i}/4} Q + 4,$$

that is,

$$(Q - \alpha)(Q - \beta) = 0.$$

The adjoint operator of Q is given by

$$f|\widetilde{Q} = \sum_{\xi} f|[\xi],$$

where ξ runs through a set of representatives of the right cosets of $\Delta_0(4N, \chi_1)$ in $\Delta_0(4N, \chi_1)\ \xi' \Delta_0(4N, \chi_1)$ with $\xi' = \left\{ \begin{pmatrix} 4 & -1 \\ 0 & 4 \end{pmatrix}, \varepsilon^{-k-1/2} \mathrm{e}^{-(2k+1)\pi \mathrm{i}/4} \right\}$, but

$$\xi' = \begin{pmatrix} 1-2N & \dfrac{N-1}{2} \\ 8N & 1-2N \end{pmatrix}^* \xi \begin{pmatrix} 1 & 0 \\ -8N & 1 \end{pmatrix}^*,$$

so that Q is Hermitian.

Let $f = \sum_{n=1}^{\infty} a(n)e(nz)$ be an element of $S(4N, k+1/2, \chi_1)$. Then

$$\begin{aligned} f|[\xi + \xi'] &= \varepsilon^{-k-1/2}\mathrm{e}^{-(2k+1)\pi\mathrm{i}/4} f(z+1/4) + \varepsilon^{k+1/2}\mathrm{e}^{(2k+1)\pi\mathrm{i}/4} f(z-1/4) \\ &= \varepsilon^k \sum_{n=1}^{\infty} (\varepsilon^{-1/2}\mathrm{i}^{-k}\mathrm{e}^{-\pi\mathrm{i}/4}\mathrm{e}^{\pi\mathrm{i}n/2} + \varepsilon^{1/2}\mathrm{i}^{k}\mathrm{e}^{\pi\mathrm{i}/4}\mathrm{e}^{-\pi\mathrm{i}n/2}) a(n)e(nz) \end{aligned}$$

and hence

$$f|[\xi+\xi'] = (-1)^{[(k+1)/2]}\varepsilon\sqrt{2}\left(\sum_{\varepsilon(-1)^k n \equiv 0,1 \bmod 4} a(n)e(nz) - \sum_{\varepsilon(-1)^k n \equiv 2,3 \bmod 4} a(n)e(nz) \right). \tag{6.28}$$

This shows that f is in $S_{k+1/2}(N, \chi)$ if and only if $f|[\xi + \xi'] = \dfrac{\alpha}{2} f$. Now by the definition of the trace operator in Section 5.4, we see that, by (6.27),

$$f|Q = (f|[\xi])|\mathrm{Tr}, \quad f|\widetilde{Q} = (f|[\xi'])|\mathrm{Tr}, \tag{6.29}$$

where Tr is the trace operator from $S(16N, k+1/2, \chi_1)$ to $S(4N, k+1/2, \chi_1)$. Thus, if $f \in S_{k+1/2}(N, \chi)$, we see that

$$f|Q = \frac{1}{2} f|[Q + \widetilde{Q}] = \frac{1}{2}((f|[\xi])|\mathrm{Tr} + (f|[\xi'])|\mathrm{Tr}) = \frac{\alpha}{4} f|\mathrm{Tr} = \alpha f.$$

Conversely, suppose that $f|Q = \alpha f$. Then

$$(f|[\xi - \alpha/4])|\mathrm{Tr} = (f|[\xi' - \alpha/4])|\mathrm{Tr} = 0$$

and so

$$(f|[\xi + \xi' - \alpha/2])|\mathrm{Tr} = 0. \tag{6.30}$$

By the definition of Tr, the equation (6.30) implies that the function $f' := f|[\xi + \xi' - \alpha/2]$ is in the orthogonal complement of $S(4N, k+1/2, \chi_1)$ in $S(16N, k+1/2, \chi_1)$. In particular, we have

$$\langle f', f \rangle = 0.$$

Since $(f|[\xi + \xi'])|[\xi + \xi'] = 2f$, we see that

$$\langle f', f|[\xi+\xi'] \rangle = \langle f'|[\xi+\xi'], f \rangle = \left\langle 2f - \frac{\alpha}{2} f|[\xi+\xi'], f \right\rangle = -\frac{\alpha}{2}\langle f', f \rangle = 0.$$

Together with $\langle f', f \rangle = 0$, this implies that $\langle f', f' \rangle = 0$, i.e. $f|[\xi+\xi'] = \dfrac{\alpha}{2} f$. Therefore f is in $S_{k+1/2}(N, \chi)$. This completes the proof. □

For each prime divisor p of N, we defined an operator $W(p)$ in Section 5.4 by

$$W(p) = \left\{ \begin{pmatrix} p & a \\ 4N & pb \end{pmatrix},\ \varepsilon_p^{-1} p^{1/4} (4Nz + pb)^{1/2} \right\},$$

where a, b are integers with $p^2b - 4Na = p$. Then $W(p)$ maps $S(4N, k+1/2, \chi_1)$ to $S\left(4N, k+1/2, \chi_1\left(\frac{4p}{\cdot}\right)\right)$ and $\left(\frac{-4}{p}\right)^{-(2k+1)/4} W(p)$ acts as an unitary involution on the sum of these spaces (see Section 5.4).

Lemma 6.17 *$W(p)$ maps the space $S_{k+1/2}(N, \chi)$ isomorphically onto the space $S_{k+1/2}\left(N, \chi\left(\frac{\cdot}{p}\right)\right)$.*

Proof We must show that $S_{k+1/2}(N, \chi)|W(p) \subset S_{k+1/2}(N, \chi(\frac{\cdot}{p}))$. In view of Lemma 6.16 we only need to show that

$$(f|W(p))|Q_{k,N,\left(\frac{4p}{\cdot}\right)\chi_1} = \left(\frac{-4}{p}\right)(f|Q_{k,N,\chi_1})|W(p) \tag{6.31}$$

holds for $f \in S(4N, k+1/2, \chi_1)$. It is easy to verify that for every $v \in \mathbb{Z}$ there is some $\gamma_v \in \Gamma_0(4N)$ such that

$$W(p)\xi_{k,\left(\frac{-4}{p}\right)\varepsilon} A_v^* = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \left(\frac{-4}{p}\right) \right\} \gamma_v^* \xi_{k,\varepsilon} A_u^* W(p),$$

where u is determined mod 4 by $Nu \equiv -1 - b(1 + Nv) + N/p \pmod 4$. This implies (6.31) since $f|Q = \sum_{v \bmod 4} (f|[\xi_{k,\left(\frac{-4}{p}\right)\varepsilon}])|A_v^*$. This completes the proof. □

Let $m|N^\infty$ and $U(m)$ be the operator defined as in Lemma 5.38. For any prime divisor p of N, put

$$w := w_{p,k+1/2,N} := p^{-(2k-1)/4} U(p) W(p)$$

and define $S_{k+1/2}^{\pm p}(N)$ as the subspace of $S_{k+1/2}(N)$ consisting of forms whose n-th Fourier coefficients vanish for $\left(\frac{(-1)^k n}{p}\right) = \mp 1$. Then we set

$$w_{p,\chi} := w_{p,k+1/2,N,\chi} := U(t)^{-1} w_{p,k+1/2,N} U(t),$$
$$S_{k+1/2}^{\pm p}(N, \chi) = S_{k+1/2}^{\pm p}(N)|U(t),$$

where we used the fact that $U(t)$ is an isomorphism from $S_{k+1/2}(N)$ to $S_{k+1/2}(N, \chi)$ which will be proved in (1) of the following lemma.

Lemma 6.18 (1) *The operator $U(t)$ maps isomorphically $S_{k+1/2}(N)$ onto $S_{k+1/2}(N,\chi)$ where t is the conductor of χ.*

(2) *The operator $w_{p,k+1/2,N,\chi}$ is a Hermitian involution on $S_{k+1/2}(N,\chi)$ whose (± 1)-eigen -space is $S^{\pm p}_{k+1/2}(N,\chi)$. In particular, for any $p|N$, we have an orthogonal decomposition*

$$S_{k+1/2}(N,\chi) = S^{+p}_{k+1/2}(N,\chi) \oplus S^{-p}_{k+1/2}(N,\chi).$$

If $p \nmid t$, then $w_{p,\chi}$ coincides with the restriction of $\left(\frac{t}{p}\right) p^{-(2k-1)/4}U(p)W(p)$ to $S_{k+1/2}(N,\chi)$, and $S^{\pm p}_{k+1/2}(N,\chi)$ coincides with the subspace of $S_{k+1/2}(N,\chi)$ consisting of forms whose $n-th$ Fourier coefficients vanish for $\left(\frac{(-1)^k tn}{p}\right) = \mp 1$.

Proof We prove first the following assertion: suppose $p \nmid t$, then $p^{-(2k-1)/4}U(p)W(p)$ defines a Hermitian involution on $S_{k+1/2}(N,\chi)$ whose (± 1)-eigenspace consists of those functions f which have a Fourier expansion $f = \sum_{n=1}^{\infty} a(n)e(nz)$ with $a(n) = 0$ for $\left(\frac{(-1)^k n}{p}\right) = \mp 1$.

In fact, by the definition of $U(p)$, we see that

$$f|U(p) = p^{(2k-3)/4} \sum_{v \bmod p} f\left|\left[\left\{ \begin{pmatrix} 1 & v \\ 0 & p \end{pmatrix}, p^{1/4} \right\}\right]\right.$$

and so

$$\begin{aligned} & f|p^{-(2k-1)/4}U(p)W(p) \\ = & p^{-1/2} \sum_{v \bmod p} f\left|\left[\left\{ \begin{pmatrix} p+4Nv & a+pbv \\ 4Np & p^2 b \end{pmatrix}, \left(\frac{-4}{p}\right)^{-1/2} (4Nz+pb)^{1/2} \right\}\right]\right.. \end{aligned}$$

If $1+4Nv/p \not\equiv 0 \pmod p$, then $4N$ and $1+4Nv/p$ are co-prime, and so we can find integers α, β such that $\alpha(-1-4Nv/p) - 4N\beta = 1$. Thus $\begin{pmatrix} \alpha & \beta \\ 4N & -1-4Nv/p \end{pmatrix} \in \Gamma_0(4N)$, by $f \in S_{k+1/2}(N,\chi)$ and $p \nmid t$, we see that

$$\begin{aligned} & f\left|\left[\left\{ \begin{pmatrix} p+4Nv & a+pbv \\ 4Np & p^2 b \end{pmatrix}, (4Nz+pb)^{1/2} \right\}\right]\right. \\ = & \left(\frac{N/p}{p}\right) f\left|\left[\left\{ \begin{pmatrix} p & -a\alpha \\ 0 & p \end{pmatrix}, \left(\frac{a\alpha}{p}\right) \right\}\right]\right.. \end{aligned}$$

Hence we have

$$f|p^{-(2k-1)/4}U(p)W(p) = \left(\frac{N/p}{p}\right)\left(\frac{-4}{p}\right)^{k+1/2} p^{-1/2} \sum_{\substack{\alpha \bmod p, \\ (\alpha,p)=1}} f\left|\left[\left\{ \begin{pmatrix} p & \alpha \\ 0 & p \end{pmatrix}, \left(\frac{-\alpha}{p}\right) \right\}\right]\right.$$

$$+p^{-1/2}\left(f\bigg|\left[\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}\right]\right)\bigg|W(p), \tag{6.32}$$

where v_0 is an integer with $1+4Nv_0/p \equiv 0 \pmod p$. Since

$$\sum_{\substack{\alpha \bmod p,\\ (\alpha,p)=1}} \left(f\bigg|\left[\left\{\begin{pmatrix}p & \alpha\\ 0 & p\end{pmatrix}, \left(\frac{-\alpha}{p}\right)\right\}\right]\right)\bigg|U(p)=0,$$

we see from (6.32) that

$$\begin{aligned}&f|(p^{-(2k-1)/4}U(p)W(p))^2\\ =&p^{-(2k+1)/4}f\bigg|\left[\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}\right]\bigg|W(p)\bigg|U(p)W(p)\\ =&\frac{1}{p}\sum_{u \bmod p}\left(f\bigg|\left[\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}\right]\right)\bigg|W(p)\bigg|\left[\left\{\begin{pmatrix}1 & u\\ 0 & p\end{pmatrix}, p^{1/4}\right\}\right]\bigg|W(p).\end{aligned}$$

Since $p\nmid t$, it is easy to check that

$$\left\{\begin{pmatrix}p^{-2} & 0\\ 0 & p^{-2}\end{pmatrix}, 1\right\}\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}W(p)\left\{\begin{pmatrix}1 & u\\ 0 & p\end{pmatrix}, p^{1/4}\right\}W(p)\in \varDelta_0(4N,\chi),$$

so that, we have

$$f|(p^{-(2k-1)/4}U(p)W(p))^2=f.$$

Since the adjoint of $\left\{\begin{pmatrix}p & \alpha\\ 0 & p\end{pmatrix}, \left(\frac{-\alpha}{p}\right)\left(\frac{-4}{p}\right)^{-1/2}\right\}$ is $\left\{\begin{pmatrix}p & -\alpha\\ 0 & p\end{pmatrix}, \left(\frac{\alpha}{p}\right)\left(\frac{-4}{p}\right)^{-1/2}\right\}$, and the adjoint of $\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}W(p)$ can be written as $C^*\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}W(p)$ with $C\in \varGamma_0(4N)$, it follows that $p^{-(2k-1)/4}U(p)W(p)$ is Hermitian.

Finally, by Gauss sum and (6.32), we have

$$\begin{aligned}f|p^{-(2k-1)/4}U(p)W(p)=&\left(\frac{N/p}{p}\right)\sum_{n=1}^{\infty}\left(\frac{(-1)^k n}{p}\right)a(n)e(nz)\\ &+p^{-1/2}f\bigg|\left[\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}\right]W(p).\end{aligned} \tag{6.33}$$

Therefore to complete the proof of our assertion we only need to show that

$$f|U(p)=\pm\left(\frac{-4}{p}\right)^{-k-1/2}p^{(2k-1)/4}f|W(p)$$

is equivalent to the identity

$$p^{-1/2}\left(f\bigg|\left[\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}\right]W(p)\right)(z)=\pm\left(\frac{N/p}{p}\right)(f|U(p))(pz),$$

which can be derived from the following fact

$$\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}W(p) = \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \left(\frac{-4}{p}\right)^{1/2}\left(\frac{N/p}{p}\right)\right\}$$
$$C^* W(p)\left\{\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}, p^{-1/4}\right\} \tag{6.34}$$

with $C \in \Gamma_0(4N)$, and hence the assertion is proved. Since we have the following commutation rule

$$f|U(t)W(p) = \left(\frac{t}{p}\right) f|W(p)U(t), \quad p \nmid t,$$

the assertions in (2) of the lemma will be clear once (1) will have been proved. By Lemma 6.17, we have that $\dim(S_{k+1/2}(N)) = \dim(S_{k+1/2}(N,\chi))$. So we only need to show that $U(t)$ is injective on $S_{k+1/2}(N)$. But we have shown above that $U(p)W(p)$ is injective on $S_{k+1/2}(N,\chi)$ for $p \nmid t$, so $U(p)$ is injective on $S_{k+1/2}(N,\chi)$ for $p \nmid t$, and hence we conclude by induction that $U(t)$ is injective on $S_{k+1/2}(N)$. This completes the proof. □

We introduce now the Hecke operators on $S_{k+1/2}(N,\chi)$. Let

$$\mathrm{pr} := \mathrm{pr}_{k,N,\chi_1} := \frac{1}{\alpha-\beta}(Q_{k,N,\chi_1} - \beta)$$

be the orthogonal projection onto $S_{k+1/2}(N,\chi)$. For a prime $p \nmid N$, we define $\mathrm{T}(p) := \mathrm{T}_{N,k,\chi}(p)$ as the restriction of

$$\nu_p p^{k-3/2}\left[\Delta_0(4N,\chi_1)\left\{\begin{pmatrix}1 & 0\\ 0 & p^2\end{pmatrix}, p^{1/2}\right\}\Delta_0(4N,\chi_1)\right]\mathrm{pr}$$

to $S_{k+1/2}(N,\chi)$, where $\nu_p = 1$ or $3/2$ according to $p \neq 2$ or $p = 2$. It is clear that for an odd p, $\mathrm{T}_{N,k,\chi}(p)$ is the restriction of the Hecke operator $\mathrm{T}_{N,k,\chi_1}(p^2)$. We write $\mathrm{T}_{N,k}(p)$ for $\mathrm{T}_{N,k,\mathrm{id.}}(p)$.

Lemma 6.19 *Let* $f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S_{k+1/2}(N,\chi)$. *Put* $f|\mathrm{T}_{N,k,\chi}(p) = \sum_{n=1}^{\infty} b(n)e(nz)$. *Then*

$$b(n) = \begin{cases} a(p^2 n) + \chi(p)\left(\dfrac{\varepsilon(-1)^k n}{p}\right)p^{k-1}a(n) + a(n/p^2), & \text{if } \varepsilon(-1)^k n \equiv 0,1 \pmod 4,\\ 0, & \text{if } \varepsilon(-1)^k n \equiv 2,3 \pmod 4. \end{cases} \tag{6.35}$$

The operators $\mathrm{T}(p)$ *generate a commutative* $\mathbb{C}$*-algebra of Hermitian operators.*

Proof Since $\mathrm{T}(p)$ is just the Hecke operator $\mathrm{T}(p^2)$ for $p \neq 2$, so (6.35) is clear for p odd by Theorem 5.15. Let us now prove (6.35) for $p = 2$. We use the same notations as in the proof of Lemma 6.16. By the definition of $U(m)$, we see that

$$U(4) = 2^{k-3/2}\left[\Delta_0(4N,\chi_1)\left\{\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix}, 2^{1/2}\right\}\Delta_0(4N,\chi_1)\right].$$

By the definition of T(2) and (6.29), we have

$$f|\mathrm{T}(2) = \frac{1}{\alpha}\left((f|U(4))|[\xi]\right)|\mathrm{Tr} + \frac{1}{2}f|U(4) = f_1 + f_2 + f_3$$

with

$$\begin{aligned} f_1 &= \frac{1}{\alpha}((f|U(4))|[\xi])|[A_0^* + A_2^*] + \frac{1}{2}f|U(4),\\ f_2 &= \frac{1}{\alpha}((f|U(4))|[\xi])|[A_N^*],\\ f_3 &= \frac{1}{\alpha}((f|U(4))|[\xi])|[A_{-N^3}^*]. \end{aligned}$$

Since

$$A_0^* = \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, 1\right\}, \quad \xi A_2^* = \begin{pmatrix}1+2N & \dfrac{N+1}{2}\\ 8N & 1+2N\end{pmatrix}^* \xi'$$

and $f \in S_{k+1/2}(N,\chi)$, we see that

$$f_1 = \frac{1}{\alpha}(f|U(4))|[\xi + \xi'] + \frac{1}{2}f|U(4).$$

By (6.28) and Lemma 5.38, we have

$$f_1 = \sum_{\varepsilon(-1)^k n \equiv 0,1 \bmod 4} a(4n)e(nz).$$

But we have also

$$f|U(4) = 2^{k-3/2}\sum_{v \bmod 4} f\left|\left[\left\{\begin{pmatrix}1 & v\\ 0 & 4\end{pmatrix}, 2^{1/2}\right\}\right]\right.,$$

so that

$$\begin{aligned} f_2 &= \frac{2^{k-3/2}}{\alpha}\sum_{v \bmod 4} f\left|\left[\left\{\begin{pmatrix}1 & v\\ 0 & 4\end{pmatrix}, 2^{1/2}\right\}\right]\right|A_N^*\\ &= 2^{k-3/2}\sum_{v \bmod 4} f\left|\left[\left\{\begin{pmatrix}4+4N^2(4v+1) & 4v+1\\ 64N^2 & 16\end{pmatrix}, \varepsilon^{1/2}e^{\pi i/4}(8N^2z+2)^{1/2}\right\}\right]\right.. \end{aligned}$$

For $v \in \mathbb{Z}$ we can find an integer a such that

$$-a(1+N^2(4v+1)) + 2(4v+1) \equiv 0 \pmod{16},$$

so that

$$\left\{ \begin{pmatrix} 4+4N^2(4v+1) & 4v+1 \\ 64N^2 & 16 \end{pmatrix}, \varepsilon^{1/2}\mathrm{e}^{\pi \mathrm{i}/4}(8N^2z+2)^{1/2} \right\}$$
$$= \begin{pmatrix} \dfrac{1+N^2(4v+1)}{2} & \dfrac{-a(1+N^2(4v+1))+2(4v+1)}{16} \\ 8N^2 & -aN^2+2 \end{pmatrix}^*$$
$$\times \left\{ \begin{pmatrix} 8 & a \\ 0 & 8 \end{pmatrix}, \chi(2)\left(\frac{4\varepsilon}{a}\right)\left(\frac{8}{a}\right)\left(\frac{-4}{a}\right)^{-1/2} \right\}.$$

Moreover, if v runs through integers mod 4, a runs through a reduced residue system mod 8. Thus

$$f_2 = \chi(2)\frac{2^{k-3/2}}{\alpha} \sum_{\substack{a \bmod 8, \\ a \text{ odd}}} f \bigg| \left[\left\{ \begin{pmatrix} 8 & a \\ 0 & 8 \end{pmatrix}, \varepsilon^{1/2}\mathrm{e}^{\pi \mathrm{i}/4}\left(\frac{4\varepsilon}{a}\right)\left(\frac{8}{a}\right)\left(\frac{-4}{a}\right)^{-1/2} \right\} \right].$$

From this equality, it is easy to verify that

$$f_2 = \chi(2)\sum_{n=1}^{\infty}\left(\frac{\varepsilon(-1)^k n}{2}\right)a(n)e(nz).$$

We want now to compute f_3. By the proof of Lemma 6.16, we know that

$$f|[\xi+\xi'] = \frac{\alpha}{2}f. \tag{6.36}$$

Since

$$\left\{ \begin{pmatrix} 1 & \pm 1 \\ 0 & 4 \end{pmatrix}, 2^{1/2} \right\} \left\{ \begin{pmatrix} -N^4+1 & 1 \\ -4N^4 & 4 \end{pmatrix}, \varepsilon^{1/2}\mathrm{e}^{\pi \mathrm{i}/4}2^{1/2}(-N^4z+1)^{1/2} \right\}$$
$$= \begin{pmatrix} \mp N^4 + \dfrac{1-N^4}{4} & \pm 1 + \dfrac{(4\pm 1)(1-N^4)}{16} \\ -4N^4 & \mp N^4+4 \end{pmatrix}^* \xi^{\mp 1},$$

so (6.36) implies

$$\frac{\alpha}{2}f = \sum_{v=1,-1} f \bigg| \left[\left\{ \begin{pmatrix} 1 & v \\ 0 & 4 \end{pmatrix}, 2^{1/2} \right\} \left\{ \begin{pmatrix} -N^4+1 & 1 \\ -4N^4 & 4 \end{pmatrix}, \varepsilon^{1/2}\mathrm{e}^{\pi \mathrm{i}/4}2^{1/2}(-N^4z+1)^{1/2} \right\} \right],$$

and hence

$$\sum_{v=1,-1} f \bigg| \left[\left\{ \begin{pmatrix} 1 & v \\ 0 & 4 \end{pmatrix}, 2^{1/2} \right\} \right]$$
$$= \frac{\alpha}{2} f \bigg| \left[\left\{ \begin{pmatrix} 4 & -1 \\ 4N^4 & -N^4+1 \end{pmatrix}, \varepsilon^{-1/2}\mathrm{e}^{-\pi \mathrm{i}/4}2^{-1/2}(4N^4z-N^4+1)^{1/2} \right\} \right]. \tag{6.37}$$

Since $a(n)=0$ for $n \equiv 2 \pmod 4$, we have

$$\sum_{v=1,-1} f \bigg| \left[\left\{ \begin{pmatrix} 1 & v \\ 0 & 4 \end{pmatrix}, 2^{1/2} \right\} \right] = 2^{1/2-k} f|U(4).$$

From (6.37) we obtain

$$f|U(4) = 2^{k-3/2}\alpha f\left|\left[\left\{\begin{pmatrix}4 & -1\\ 4N^4 & -N^4+1\end{pmatrix}, \varepsilon^{-1/2}\mathrm{e}^{-\pi i/4}2^{-1/2}(4N^4z-N^4+1)^{1/2}\right\}\right]\right.$$
$$= 2^{k-3/2}\alpha\left(f\left|\left[\left\{\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}, 2^{-1/2}\right\}\right]\right.\right)\Big|[A^*_{N^3}]\Big|[\xi^{-1}].$$

Hence

$$f_3 = \frac{1}{\alpha}((f|U(4))|[\xi])|[A^*_{-N^3}] = 2^{k-3/2}f\left|\left[\left\{\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}, 2^{-1/2}\right\}\right]\right.$$
$$= 2^{2k-1}\sum_{n=1}^{\infty} a(n/4)e(nz). \tag{6.38}$$

Putting together all expansions for f_1, f_2 and f_3, we get (6.35) for $p=2$. It is clear that the operators $\mathrm{T}_{N,k,\chi}(p)$ commute each other from (6.35). $\mathrm{T}_{N,k,\chi}(p)$ ($p \nmid 2N$) is Hermitian since the operator $\left[\left\{\Delta_0(4N,\chi_1)\begin{pmatrix}1 & 0\\ 0 & p^2\end{pmatrix}, p^{1/2}\right\}\Delta_0(4N,\chi_1)\right]$ is Hermitian for $p \nmid N$. So we only need to show that $\mathrm{T}(2)$ is Hermitian. Let f, g be in $S_{k+1/2}(N,\chi)$. Then

$$\frac{2}{3}\langle f|\mathrm{T}(2), g\rangle = \langle f|U(4)\mathrm{pr}, g\rangle = \langle f|U(4), g|\mathrm{pr}\rangle$$
$$= \langle f|U(4), g\rangle = 2^{k-3/2}\sum_{v \bmod 4}\left\langle f\left|\left[\left\{\begin{pmatrix}1 & v\\ 0 & 4\end{pmatrix}, 2^{1/2}\right\}\right], g\right.\right\rangle$$
$$= 2^{k-3/2}\sum_{v \bmod 4}\left\langle f, g\left|\left[\left\{\begin{pmatrix}4 & -v\\ 0 & 1\end{pmatrix}, 2^{-1/2}\right\}\right]\right.\right\rangle$$
$$= 2^{k+1/2}\left\langle f, g\left|\left[\left\{\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}, 2^{-1/2}\right\}\right]\right.\right\rangle.$$

Now we have

$$\frac{1}{\alpha}((g|U(4))|[\xi])|[A^*_{-N^3}] = 2^{k-3/2}g\left|\left[\left\{\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}, 2^{-1/2}\right\}\right]\right.$$
$$= \frac{1}{\alpha}((g|U(4))|[\xi^{-1}])|[A^*_{N^3}], \tag{6.39}$$

and the first equality is derived from (6.38), and the second can be proved similarly. By (6.39), we see easily that

$$2^{k+1/2}g\left|\left[\left\{\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix}, 2^{-1/2}\right\}\right]\right. = \frac{2}{\alpha}(g|U(4))|[\xi A^*_{-N^3} + \xi^{-1}A^*_{N^3}].$$

Thus

$$\frac{2}{3}\langle f|\mathrm{T}(2), g\rangle = \frac{2}{\alpha}\langle f, g|U(4))|[\xi A^*_{-N^3} + \xi^{-1}A^*_{N^3}]\rangle$$
$$= \frac{2}{\alpha}\langle f|[A^*_{N^3}\xi^{-1} + A^*_{-N^3}\xi], g|U(4)\rangle$$

$$= \frac{2}{\alpha}\langle f|[\xi + \xi^{-1}], g|U(4)\rangle$$
$$= \langle f, g|U(4)\rangle = \frac{2}{3}\langle f, g|\mathrm{T}(2)\rangle.$$

This completes the proof. □

For a positive divisor d of N we set $S_{k+1/2}(d,\chi) = S_{k+1/2}(d)|U(t)$. Put

$$S^{\mathrm{old}}_{k+1/2}(N,\chi) = \sum_{N\neq d|N} \left(S_{k+1/2}(d,\chi) + S_{k+1/2}(d,\chi)|U(N^2/d^2)\right),$$

which is called the space of old forms in $S_{k+1/2}(N,\chi)$. And we define the space of new forms, denoted by $S^{\mathrm{new}}_{k+1/2}(N,\chi)$, to be the orthogonal complement of the space of old forms in $S_{k+1/2}(N,\chi)$ with respect to the Petersson inner product. We write

$$S^{\mathrm{new}}_{k+1/2}(N) = S^{\mathrm{new}}_{k+1/2}(N, \mathrm{id.}).$$

Lemma 6.20 *We have*

$$S^{\mathrm{new}}_{k+1/2}(N,\chi) = S^{\mathrm{new}}_{k+1/2}(N)|U(t).$$

Proof By Lemma 6.18 it suffices to show the inclusion

$$S^{\mathrm{new}}_{k+1/2}(N)|U(t) \subset S^{\mathrm{new}}_{k+1/2}(N,\chi).$$

Let $f \in S^{\mathrm{new}}_{k+1/2}(N)$. We must show that

$$\langle g|U(t), f|U(t)\rangle = 0$$

for all old forms g in $S_{k+1/2}(N)$. Let $t = p_1 \cdots p_r$ be the standard factorization of t. Then we have

$$\langle g|U(t), f|U(t)\rangle = p_r^{k+1/2}\langle g|U(t/p_r), f|U(t/p_r)\rangle,$$

since $W(p_r)$ is unitary and $p_r^{-(2k+1)/4}U(p_r)W(p_r)$ is a Hermitian involution on $S_{k+1/2}(N)|U\left(\frac{t}{p_r}\right)$ (by the proof of Lemma 6.18). By induction, we see that

$$\langle g|U(t), f|U(t)\rangle = t^{k+1/2}\langle g, f\rangle = 0.$$

This completes the proof. □

We shall carry over the basic facts about the space of new forms $S^{\mathrm{new}}(N, 2k)$ to $S^{\mathrm{new}}_{k+1/2}(N, \chi)$. Recall that for every prime divisor p of N the operator $U(p)$ preserves $S^{\mathrm{new}}(N,2k) \subset S(N,2k)$ and that $U(p) = -p^{k-1}W_{p,2k,N}$ on $S^{\mathrm{new}}(N,2k)$, where $W_{p,2k,N}$ is the Atkin-Lehner involution on $S(N,2k)$ defined by

$$(f|W_{p,2k,N})(z) = p^k(4Nz+pb)^{-2k}f\left(\frac{pz+a}{4Nz+pb}\right), \quad a,b\in\mathbb{Z}, p^2b - 4Na = p.$$

We shall now prove an analogous result for new forms of half integral weight.

Theorem 6.9 *For every prime $p|N$, the operators $U(p^2)$ and $w_{p,\chi} := w_{p,k,N,\chi}$ preserve the space of new forms. And we have $U(p^2) = -p^{k-1}w_{p,\chi}$ on $S^{\text{new}}_{k+1/2}(N,\chi)$.*

Proof We first show that $w_{p,\chi} := w_{p,k,N,\chi}$ maps new forms to new forms. Since $w_{p,\chi}$ is Hermitian it is sufficient to show that $w_{p,\chi}$ maps old forms to old forms. By the definitions we only need to show this for $\chi = \text{id}$. Now set $w_p := w_{p,k,N}$. We only need to show that w_p maps $S_{k+1/2}(N/l)$ and $S_{k+1/2}(N/l)|U(l^2)$ to old forms for every prime divisor l of N.

Let $f \in S_{k+1/2}(N/l)$. If $p \neq l$, by (2) of Lemma 6.18, $f|w_p$ is in $S_{k+1/2}(N/l)$ and so an old form. The same is true for $f|U(l^2)|w_p = f|w_p|U(l^2)$. Thus we assume that $p = l$. Let $f(z) = \sum_{n=1}^{\infty} a(n)e(nz)$. Then, by (6.34) and (6.35) in the proof of Lemma 6.18, we see that

$$f|w_p = \left(\frac{N/p}{p}\right)\sum_{n=1}^{\infty}\left(\frac{(-1)^k n}{p}\right)a(n)e(nz) + \left(\frac{-4}{p}\right)^{-k-1/2}\left(\frac{N/p}{p}\right)p^{-1/2}(f|W(p))\Bigg|\left[\left\{\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}, p^{-1/4}\right\}\right].$$

Since $f \in S_{k+1/2}(N/p)$, we have

$$\begin{aligned} f|W(p) &= \left(f\Bigg|\begin{pmatrix} -1 & 0\\ 4N/p & -1\end{pmatrix}^*\right)\Bigg|W(p)\\ &= f\Bigg|\left[\left\{\begin{pmatrix} -p & -a\\ 0 & -1\end{pmatrix}, \left(\frac{-4}{p}\right)^{-1/2}p^{-1/4}\right\}\right]\\ &= f\Bigg|\left[\left\{\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}, \left(\frac{-4}{p}\right)^{-1/2}p^{-1/4}\right\}\right]. \end{aligned}$$

Thus we obtain that

$$f|w_p = \left(\frac{N/p}{p}\right)\sum_{n=1}^{\infty}\left(\left(\frac{(-1)^k n}{p}\right)a(n) + p^k a(n/p^2)\right)e(nz),$$

i.e.

$$f|w_p = \left(\frac{N/p}{p}\right)p^{-k+1}(-f|U(p^2) + f|\mathrm{T}_{N/p,k}(p^2)). \tag{6.40}$$

This shows that $f|w_p$ is an old form. Moreover, applying w_p on both sides of (6.40) and noting $w_p^2 = \text{id}$. we see that $(f|U(p^2))|w_p$ is an old form. This shows that w_p maps old forms to old forms, and so that, new forms to new forms.

Finally, we must now prove that on $S^{\text{new}}_{k+1/2}(N,\chi)$

$$U(p^2) = -p^{k-1}w_{p,k,N,\chi}, \quad p \text{ prime}, p|N. \tag{6.41}$$

But Lemma 6.20 and the injectivity of $U(t)$ on $S_{k+1/2}(N)$ (see Lemma 6.18) allows us to assume $\chi = \text{id.}$ for the proof of (6.41). Denote by $\mathrm{Tr} := \mathrm{Tr}^N_{N/p} : S(N, k+1/2) \to S(N/p,\, k+1/2)$ the trace operator. It is easy to verify that $\mathrm{Tr}^N_{N/p}$ maps $S_{k+1/2}(N)$ to $S_{k+1/2}(N/p)$ by Lemma 6.16. Let $f \in S^{\text{new}}_{k+1/2}(N)$. Since f is orthogonal to $S_{k+1/2}(N/p)$, it follows that $f|\mathrm{Tr} = 0$. On the other hand, $\begin{pmatrix} 1 & 0 \\ 4N/p & 1 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ (u mod p) together with $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ form a complete set of representatives for $\Gamma_0(4N)/\Gamma_0(4N/p)$. Thus we have

$$f|\mathrm{Tr} = f + \sum_{u \bmod p} f \left| \left[\begin{pmatrix} 1 & 0 \\ 4N/p & 1 \end{pmatrix}^* \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}^* \right] \right. .$$

But

$$\begin{pmatrix} 1 & 0 \\ 4N/p & 1 \end{pmatrix}^* \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}^* = \left\{ \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, \left(\frac{-4}{p}\right)^{1/2} \right\} W(p) \left\{ \begin{pmatrix} 1 & u-a \\ 0 & 1 \end{pmatrix}, p^{1/4} \right\},$$

so that

$$f|\mathrm{Tr} = f + \left(\frac{-4}{p}\right)^{-k-1/2} p^{-k/2+3/4} f|W(p)U(p).$$

Since $f|\mathrm{Tr} = 0$, we obtain that

$$f|W(p)U(p) = -\left(\frac{-4}{p}\right)^{k+1/2} p^{(2k-3)/4} f.$$

By (2) of Lemma 6.18 and the fact that $w_{p,k,N,\chi}$ preserves the space of new forms, we see that $U(p)W(p)$ is an isomorphism of $S^{\text{new}}_{k+1/2}(N)$. Thus replacing f with $f|U(p)W(p)$ in the above equality, we see that

$$\begin{aligned} \left(\frac{-4}{p}\right)^{k+1/2} f|U(p^2) &= f|U(p)W(p)W(p)U(p) \\ &= -\left(\frac{-4}{p}\right)^{k+1/2} p^{(2k-3)/4} f|U(p)W(p), \end{aligned}$$

i.e.

$$f|U(p^2) = -p^{k-1} f|w_p.$$

This completes the proof. □

Lemma 6.21 *Let* $f = \sum_{n=1}^{\infty} a(n)e(nz) \in S(4N, k+1/2, \chi_1)$ *satisfy that* $a(n) = 0$ *for* $n \equiv 2 \pmod 4$. *Then* f *is in* $S_{k+1/2}(N, \chi)$.

Proof The hypothesis $a(n) = 0$ for $n \equiv 2 \pmod 4$ is equivalent to

$$f\left|\left[\left\{\begin{pmatrix}4 & 1\\ 0 & 4\end{pmatrix},1\right\}\right]+f\right|\left[\left\{\begin{pmatrix}4 & -1\\ 0 & 4\end{pmatrix},1\right\}\right]$$
$$=2^{-k+1/2}(f|U(4))\left|\left[\left\{\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix},2^{-1/2}\right\}\right].\right.$$

Now apply the trace operator $\mathrm{Tr} := \mathrm{Tr}_{4N}^{16N}$ from $S(16N, k+1/2, \chi_1)$ to $S(4N, k+1/2, \chi)$ on both sides of the above equation. Because of the identity (6.29) and the fact that Q is Hermitian, we obtain that

$$\varepsilon(-1)^{[(k+1)/2]}\sqrt{2}f|Q = 2^{-k+1/2}\left((f|U(4))\left|\left[\left\{\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix},2^{-1/2}\right\}\right]\right)\right|\mathrm{Tr}. \tag{6.42}$$

Since $U(4)$ and $\left\{\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix},2^{-1/2}\right\}\Big|\mathrm{Tr}$ equal $2^{k-3/2}\left[\Delta_0(4N,\chi_1)\left\{\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix},2^{1/2}\right\}\Delta_0(4N,\chi_1)\right]$ and $\left[\Delta_0(4N,\chi_1)\left\{\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix},2^{-1/2}\right\}\Delta_0(4N,\chi_1)\right]$ respectively, and also since

$$\begin{aligned}
&\Delta_0(4N,\chi_1)\left\{\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix},2^{1/2}\right\}\Delta_0(4N,\chi_1)\\
&\cdot\Delta_0(4N,\chi_1)\left\{\begin{pmatrix}4 & 0\\ 0 & 1\end{pmatrix},2^{-1/2}\right\}\Delta_0(4N,\chi_1)\\
&=4\Delta_0(4N,\chi_1)\left\{\begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix},1\right\}\Delta_0(4N,\chi_1)\\
&+\Delta_0(4N,\chi_1)\left\{\begin{pmatrix}4 & 1\\ 0 & 4\end{pmatrix},1\right\}\Delta_0(4N,\chi_1)\\
&+\Delta_0(4N,\chi_1)\left\{\begin{pmatrix}4 & -1\\ 0 & 4\end{pmatrix},1\right\}\Delta_0(4N,\chi_1)\\
&+\Delta_0(4N,\chi_1)\left\{\begin{pmatrix}4 & 2\\ 0 & 4\end{pmatrix},1\right\}\Delta_0(4N,\chi_1),\\
&\Delta_0(4N,\chi_1)\left\{\begin{pmatrix}4 & 2\\ 0 & 4\end{pmatrix},1\right\}\Delta_0(4N,\chi_1)=0,
\end{aligned}$$

the right hand side of (6.42) equals

$$\frac{1}{2}\left(4f+\varepsilon(-1)^{[(k+1)/2]}\sqrt{2}f|Q\right),$$

so that

$$f|Q = \varepsilon(-1)^{[(k+1)/2]}2\sqrt{2}f$$

and hence f is in $S_{k+1/2}(N,\chi)$ by Lemma 6.16. This completes the proof. □

Lemma 6.22 *Let p be a prime and $0 \neq f = \sum_{n=1}^{\infty} a(n)e(nz) \in G(N, k/2, \omega)$. Assume* *that* $a(n) = 0$ *for all* n *with* $p \nmid n$*. Then* $p|N/4$, $\omega\chi_p$ *is well-defined modulo* N/p *and* $f = g|V(p)$ *with* $g \in G(N/p, k/2, \omega\chi_p)$ *where* $\chi_p = \left(\frac{p}{*}\right)$.

Proof Put

$$g(z) = f(z/p) = \sum_{n=0}^{\infty} a(np)e(nz) = p^{k/4} f \bigg| \left[\left\{ \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, p^{1/4} \right\} \right]. \tag{6.43}$$

Set

$$N' = \begin{cases} N/p, & \text{if } p|N/4, \\ N, & \text{if } p \nmid N/4. \end{cases} \qquad \Gamma_0(N', p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N') \big| p|b \right\}.$$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N', p)$, then $A_1 = \begin{pmatrix} a & b/p \\ cp & d \end{pmatrix} \in \Gamma_0(N)$ and we see that

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, p^{1/4} \right\} A^* = \{1, \chi_p(d)\} A_1^* \left\{ \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, p^{1/4} \right\}.$$

Hence

$$g|[A^*] = \omega(d)\chi_p(d)g. \tag{6.44}$$

By (6.43) we have

$$g \bigg| \left[\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right\} \right] = g.$$

Since $\Gamma_0(N')$ can be generated by $\Gamma_0(N', p)$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we see that (6.44) holds for any $A \in \Gamma_0(N')$. We declare that $\omega\chi_p$ must be well-defined modulo N'. Otherwise, there exist integers a and d such that $ad \equiv 1 \pmod{N'}$ and $\omega\chi_p(a) \cdot \omega\chi_p(d) \neq 1$. Take

$$B = \begin{pmatrix} a & b \\ N' & d \end{pmatrix} \in \Gamma_0(N'),$$

we have that $g = g|[B^*(B^{-1})^*] = \omega\chi_p(a)\omega\chi_p(d)g$, which is impossible since $g \neq 0$. Therefore $\omega\chi_p$ must be well-defined modulo N', so that $p|N/4$ and $N' = N/p$. It is therefore clear that g is in $G(N/p, k/2, \omega\chi_p)$ and $f = g|V(p)$. This completes the proof. □

Lemma 6.23 *Let* m *be a positive integer, and*

$$f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N, k/2, \omega).$$

Suppose that $a(n) = 0$ *for any* n *with* $(n, m) = 1$*. Then*

$$f = \sum f_p|V(p), \quad f_p \in G(N/p, k/2, \omega\chi_p),$$

where the prime p *runs over the set of common factors of* m *and* $N/4$*. And* $\omega\chi_p$ *is well-defined modulo* N/p*.* f_p *can be chosen as cusp forms if* f *is a cusp form.* f_p *are eigenfunctions for almost all* Hecke *operators* $\mathrm{T}(p^2)$ *if* f *is an eigenfunction for almost all* Hecke *operators* $\mathrm{T}(p^2)$.

Proof We can assume that m is square-free. Let r be the number of different prime factors of m. If $r = 0$, then $f = 0$ and the lemma holds. If $r = 1$, this is the Lemma 6.22. We now prove the lemma by induction on r. Let $m = p_0 m_0$. Take a prime p and put $K(p) = 1 - \mathrm{T}(p,\ Np)V(p)$ where $\mathrm{T}(p,\ Np)$ is the Hecke operator $\mathrm{T}_{Np,k,\omega}(p)$ on the space $G(pN,\ k/2,\ \omega)$. By the properties of Hecke operators, we have

$$f|K(p) = \sum_{(n,p)=1} a(n)e(nz) \in G(p^2N, k/2, \omega).$$

So

$$h := \sum_{(n,m_0)=1} a(n)e(nz) = f\Big|\prod_{p|m_0} K(p) \in G(m_0^2N, k/2, \omega).$$

If $h = 0$, replacing m by m_0, we see that the lemma holds by induction hypothesis. Now suppose that $h \neq 0$. If $(n,\ m_0) = 1$ and $a(n) \neq 0$, then $p_0|n$. By Lemma 6.22, there is $g_{p_0} \in G(m_0^2N/p,\ k/2,\ \omega\chi_{p_0})$ such that $h = g_{p_0}|V(p_0)$, and $\omega\chi_{p_0}$ is well-defined modulo m_0^2N/p_0. Hence $p_0|N/4$ and $\omega\chi_{p_0}$ is well-defined modulo N/p_0. We have

$$f - h = f - g_{p_0}|V(p_0) = \sum_{n=0}^{\infty} b(n)e(nz).$$

Noting that $b(n) = 0$ if $(n,\ m_0) = 1$ and applying induction hypothesis, we have

$$f - g_{p_0}|V(p_0) = \sum_{p} g_p|V(p),$$

where p runs over the set of prime factors of m_0, and $\omega\chi_p$ is well-defined modulo m_0^2N/p. Therefore by Theorem 5.21, we see that

$$f|S(\omega) - g_{p_0} = \sum_{p} (g_p|S(\omega\chi_p, m_0^2N/p, p_0))|V(p).$$

Put $f_{p_0} = f|S(\omega)$. Then $f_{p_0} \in G(N/p_0, k/2, \omega\chi_{p_0})$. If we write

$$f_{p_0}|V(p_0) = \sum_{n=0}^{\infty} c(n)e(nz),$$

then the nth Fourier coefficient of $f_{p_0}|V(p_0) - g_{p_0}|V(p_0)$ is not zero only for $(n, m_0) \neq 1$. So we get $c(n) = a(n)$ for $(n, m_0) = 1$, and hence the nth Fourier coefficient of $f - f_{p_0}|V(p_0)$ is zero for $(n, m_0) = 1$. By the induction hypothesis we get the decomposition of f as stated in the lemma. The other results can be proved also by induction. This completes the proof. □

Corollary 6.3 *Let f be as in* Lemma 6.23. *If f is an eigenfunction of almost all Hecke operators, then $f \in G^{\mathrm{old}}(N, k/2, \omega)$.*

Theorem 6.10 *We have the following decomposition:*

$$S_{k+1/2}(N, \chi) = \bigoplus_{r,d \geqslant 1, rd|N} S_{k+1/2}^{\mathrm{new}}(d, \chi)|U(r^2).$$

Proof We now prove the decomposition for the case $N = q$ with q an odd prime. We can prove the general case by induction. First assume $\chi = 1$. Suppose that $f \in S_{k+1/2}(1)$ and $f|U(q^2) \in S_{k+1/2}(1)$. We may assume that f is an eigenfunction of all Hecke operators $\mathrm{T}(p) := \mathrm{T}_{1,k,1}(p)$. To prove the decomposition we must show that $f = 0$. If otherwise, since f and $f|U(q^2)$ have the same eigenvalues for all $\mathrm{T}(p)$ with $p \neq q$, we conclude that $f|U(q^2) = cf$ with some constant $c \in \mathbb{C}$ (in fact, by Theorem 6.3, a non-zero Hecke eigenform in $S(1, 2k, \text{id.})$ is completely determined up to a constant factor by prescribing all up to finitely many of its eigenvalues, so is also a non zero Hecke eigenform in $S_{k+1/2}(1)$ by Theorem 9.7).

Now let λ_q be the eigenvalue of f with respect to $\mathrm{T}(q)$ and write $f = \sum_{n=1}^{\infty} a(n)e(nz)$. Then, by the definition of $\mathrm{T}(q)$ and the fact that $f|U(q^2) = cf$, we have

$$\left(\lambda_q - c - \left(\frac{(-1)^k n}{q}\right) q^{k-1}\right) a(n) = q^{2k-1} a(n/q^2), \quad \forall n \in \mathbb{N}. \tag{6.45}$$

By Lemma 6.22 we can choose n' such that $q \nmid n'$ and $a(n') \neq 0$. We see then that

$$\lambda_q = c + \left(\frac{(-1)^k n'}{q}\right) q^{k-1}. \tag{6.46}$$

Substituting (6.46) into (6.45) we have

$$\left(\left(\frac{(-1)^k n'}{q}\right) - \left(\frac{(-1)^k n}{q}\right)\right) a(n) = q^k a(n/q^2), \quad \forall n \in \mathbb{N},$$

so that

$$f|U(q^2) = \left(\frac{(-1)^k n'}{q}\right) q^k f, \quad \forall n \equiv 0 \pmod{q^2},$$

i.e.,

$$c = \left(\frac{(-1)^k n'}{q}\right) q^k.$$

Thus by (6.46) we see that

$$|\lambda_q| = q^k + q^{k-1},$$

which is impossible by Ramanujan-Petersson-Deligne's Theorem. Thus we proved that

$$S_{k+1/2}(1) \bigcap S_{k+1/2}(1)|U(q^2) = \{0\}.$$

Hence by the definitions of new forms and old forms, we have

$$\begin{aligned} S_{k+1/2}(q) &= S^{\text{new}}_{k+1/2}(q) \oplus \left(S_{k+1/2}(1) + S_{k+1/2}(1)|U(q^2)\right) \\ &= S^{\text{new}}_{k+1/2}(q) \oplus S_{k+1/2}(1) \oplus S_{k+1/2}(1)|U(q^2) \\ &= S^{\text{new}}_{k+1/2}(q) \oplus S^{\text{new}}_{k+1/2}(1) \oplus S^{\text{new}}_{k+1/2}(1)|U(q^2). \end{aligned}$$

Thus the theorem is proved for $\chi = 1$. If χ is primitive modulo q, the theorem follows from the following facts (see Lemma 6.18 and Lemma 6.20) :

$$S_{k+1/2}(q)|U(q) = S_{k+1/2}(q,\chi), \quad S^{\text{new}}_{k+1/2}(q)|U(q) = S^{\text{new}}_{k+1/2}(q,\chi).$$

This completes the proof. □

Theorem 6.11 (1) *The space* $S^{\text{new}}_{k+1/2}(N,\chi)$ *has an orthogonal basis of common eigenfunctions for all operators* $\mathrm{T}(p) := \mathrm{T}_{N,k,\chi}(p)$ *(*p *prime,* $p \nmid N$*) and* $U(p^2)$ *(*p *prime,* $p|N$*), uniquely determined up to multiplication with non-zero complex numbers, the eigenvalues corresponding to* $U(p^2)$ *with* $p|N$ *are* $\pm p^{k-1}$*. If* f *is such an eigenfunction and* λ_p *the eigenvalue corresponding to* $\mathrm{T}(p)$ *resp.* $U(p^2)$*, then there is an eigenfunction* $F \in S^{\text{new}}_{2k}(N)$*, uniquely determined up to multiplication with a non-zero complex number, which satisfies* $F|\mathrm{T}_{N,2k}(p) = \lambda_p F$ *resp.* $F|U(p^2) = \lambda_p F$ *for all primes* p *with* $p \nmid N$ *resp.* $p|N$*. Let* $f = \sum_{n=1}^{\infty} a(n)e(nz)$ *and* $F = \sum_{n=1}^{\infty} A(n)e(nz)$*, and* D *a fundamental discriminant with* $\varepsilon(-1)^k D > 0$*. Then we have*

$$L(s-k+1,\chi\chi_D)\sum_{n=1}^{\infty} a(|D|n^2)n^{-s} = a(|D|)\sum_{n=1}^{\infty} A(n)n^{-s}.$$

(2) *Let the map* $L_{D,N,k,\chi}$ *be defined by*

$$\sum_{n=1}^{\infty} b(n)e(nz) \mapsto \sum_{n=1}^{\infty}\Big(\sum_{d|n}\chi(d)\chi_D(d)d^{k-1}b(n^2|D|/d^2)\Big)e(nz).$$

Then $L_{D,N,k,\chi}$ *maps* $S_{k+1/2}(N,\chi)$ *to* $S(N,2k,\text{id.}), S^{\text{new}}_{k+1/2}(N,\chi)$ *to* $S^{\text{new}}(N,2k,\text{id.})$ *and* $S^{\pm p}_{k+1/2}(N,\chi)\bigcap S^{\text{new}}_{k+1/2}(N,\chi)$ *to* $S^{\pm p}(N,2k,\text{id.})\bigcap S^{\text{new}}(N,2k,\text{id.})$ *with* p *any prime divisor of* N *where* $S^{\pm p}(N,2k,\text{id.}) = \{f \in S(N,2k,\text{id.}) \,|\, f|W_{p,2k,N} = \pm f\}$*. It satisfies*

$$\mathrm{T}_{N,k,\chi}(p)L_{D,N,k,\chi} = L_{D,N,k,\chi}\mathrm{T}_{N,2k,1}(p), \quad \forall p \nmid N,$$
$$U(p^2)L_{D,N,k,\chi} = L_{D,N,k,\chi}U(p), \quad \forall p|N.$$

There exists a linear combination of the $L_{D,N,k,\chi}$ *which maps* $S^{\text{new}}_{k+1/2}(N,\chi)$ *resp.*

$$S^{\pm p}_{k+1/2}(N,\chi)\bigcap S^{\text{new}}_{k+1/2}(N,\chi)$$

isomorphically onto $S^{\text{new}}(N,2k,\text{id.})$ *resp.* $S^{\pm p}(N,2k,\text{id.})\bigcap S^{\text{new}}(N,2k,\text{id.})$*.*

Proof Since $\mathrm{T}(p)$ commutes with $U(d^2)$ for $d|N$, and since for $f \in S_{k+1/2}(N)$ we have

$$f|U(t)|\mathrm{T}(p) = f|\mathrm{T}(p)|U(t),$$

it follows that the Hecke operator $\mathrm{T}(p)$ preserves the space of old forms and so preserves also $S_{k+1/2}^{\mathrm{new}}(N,\chi)$. We now have that

$$\mathrm{Tr}(\mathrm{T}_{N,k,\chi}(n), S_{k+1/2}^{\mathrm{new}}(N,\chi)) = \mathrm{Tr}(\mathrm{T}_{N,2k}(n), S^{\mathrm{new}}(N,2k)) \tag{6.47}$$

for all $n \in \mathbb{N}$ with $(n,2N)=1$. In fact, this follows by induction from the decompositions:

$$S_{k+1/2}(N,\chi) = \bigoplus_{r,d\geqslant 1, rd|N} S_{k+1/2}^{\mathrm{new}}(d,\chi)|U(r^2),$$
$$S^{\mathrm{new}}(N) = \bigoplus_{r,d\geqslant 1, rd|N} S^{\mathrm{new}}(d,2k)|U(r)$$

and from the Theorem 9.7.

By (6.47) and the corresponding statement for $S^{\mathrm{new}}(N,2k)$ (see Section 6.1), we deduce that $S_{k+1/2}^{\mathrm{new}}(N,\chi)$ has an orthogonal basis of common eigenfunctions for all operators $\mathrm{T}_{N,k,\chi}(p)$ $(p \nmid 2N)$, uniquely determined up to multiplication with non-zero complex numbers. Since $\mathrm{T}_{N,k,\chi}(p)$ $(p \nmid 2N)$, $U(p^2)(p|N)$ and $\mathrm{T}_{N,k,\chi}(2)$ commute, so these functions are also eigenfunctions of $U(p^2)(p|N)$ and $\mathrm{T}_{N,k,\chi}(2)$. Furthermore, by Theorem 6.9 and in particular the fact that $w_{N,p,k+1/2,\chi}$ is an involution shows that the eigenvalues with respect to $U(p^2)(p|N)$ are $\pm p^{k-1}$. Now let $f = \sum_{n=1}^{\infty} a(n)e(nz)$ be an eigenfunction and assume that $f|\mathrm{T}(p) = \lambda_p f$ resp. $f|U(p^2) = \lambda_p f$ for $p \nmid N$ resp. $p|N$. Then a formal computation as in Lemma 5.40 and Theorem 5.23 shows that

$$L(s-k+1,\chi\chi_D)\sum_{n=1}^{\infty} a(|D|n^2) = a(|D|)\prod_p \left(1-\lambda_p p^{-s} + \left(\frac{N}{p}\right)^2 p^{2k-1-2s}\right)^{-1}$$

for every fundamental discriminant D with $\varepsilon(-1)^k D > 0$.

Let us show the assertions about the maps $L_D := L_{D,N,k,\chi}$. Note that the Hecke operators $\mathrm{T}_{N,k,\chi}(p)$ and $\mathrm{T}_{N,2k,\mathrm{id.}}(p)$ act in a natural way on the formal power series in $q = e(z)$. It is clear that for a formal power series $f = \sum_{\varepsilon(-1)^k n \equiv 0,1 \bmod 4} a(n)q^n$, we have

$$f|\mathrm{T}_{N,k,\chi}(p)|L_D = f|L_D|\mathrm{T}_{N,2k,\mathrm{id.}}(p), \quad \forall p \nmid N,$$
$$f|U(p^2)|L_D = f|L_D|U(p), \quad \forall p|N,$$

by a formal computation.

The other assertions will be shown first under the assumption that $D \equiv 0 \pmod 4$. Write $D = 4t$ with t square free and $t \equiv 2,3 \pmod 4$. For

$$f = \sum_{n=1}^{\infty} a(n)e(nz) \in S_{k+1/2}(N,\chi),$$

put

$$f|L_{t,4N,k,\chi_1} = \sum_{n=1}^{\infty}\Bigg(\sum_{d|n}\Big(\frac{4\varepsilon t}{d}\Big)\chi(d)d^{k-1}a(n^2|t|/d^2)\Big)e(nz).$$

Then $f|L_{t,4N,k,\chi_1}$ is a cusp form of weight $2k$ on $\Gamma_0(2N)$ by the results of Chapter 8. Since $f \in S_{k+1/2}(N,\chi)$, the nth Fourier coefficients of $f|L_{t,4N,k,\chi_1}$ are zero for any odd n. Hence the function $(f|L_{t,4N,k,\chi_1})|U(2) = f|L_{D,N,k,\chi}$ is in $S(N, 2k, \mathrm{id.})$.

If $f \in S^{\mathrm{new}}_{k+1/2}(N,\chi)$ is a Hecke eigenfunction, then from Theorem 6.9 we see that

$$f|U(p^2) = \pm p^{k-1}f, \quad \forall\ p|N.$$

Therefore $F = f|L_D$ is a Hecke eigenform in $S(N, 2k, \mathrm{id.})$ with $F|U(p) = \pm p^{k-1}F$ for all $p|N$, and this implies that F must be in $S^{\mathrm{new}}(N, 2k, \mathrm{id.})$ by the results in Section 6.1.

That L_D maps $S^{\pm p}_{k+1/2}(N,\chi) \bigcap S^{\mathrm{new}}_{k+1/2}(N,\chi)$ to $S^{\pm p}(N, 2k, \mathrm{id.}) \bigcap S^{\mathrm{new}}(N, 2k, \mathrm{id.})$ follows from Theorem 6.9, the identity $U(p^2)L_D = L_D U(p)$ and the fact that $U(p) = -p^{k-1}W_{p,N,2k}$ on $S^{\mathrm{new}}(N, 2k, \mathrm{id.})$.

We shall now prove that there is a linear combination of L_D with $D \equiv 0 \pmod 4$ which gives an isomorphism of $S^{\mathrm{new}}_{k+1/2}(N,\ \chi)$ onto $S^{\mathrm{new}}(N,\ 2k,\ \mathrm{id.})$. Now suppose that $f \in S^{\mathrm{new}}_{k+1/2}(N,\ \chi)$ is a non-zero Hecke eigenfunction. We declare that there is a fundamental discriminant $D \equiv 0 \pmod 4$ with $\varepsilon(-1)^k D > 0$ such that the Fourier coefficient of f at $e(|D|z)$ is non-zero. Otherwise, then the n-th Fourier coefficients of $g = f|U(4)$ are zero for all $n \equiv 2 \pmod 4$, and so that g is in $S_{k+1/2}(N,\ \chi)$ by Lemma 6.21. It follows that $g = cf$ for some constant c. In fact, by Theorem 9.7 and identity (6.47), we see that there exists an isomorphism $\psi : S_{k+1/2}(N,\ \chi) \rightarrow S(N,\ 2k,$ id.) which maps new forms onto new forms and $\psi \mathrm{T}_{N,k+1/2,\chi}(p) = \mathrm{T}_{N,2k}(p)\psi$ for all primes $p \nmid 2N$. So $f|\psi$ is a new form with the same eigenvalues as $g|\psi$ for all Hecke operators $\mathrm{T}_{N,2k}(p)$ with $p \nmid 2N$, and so that $g|\psi \in \mathbb{C}f|\psi$ by the results in Section 6.1. This shows that $g = cf$ for some constant c. Now note that f is an eigenfunction of $\mathrm{T}_{N,k,\chi}(2)$. Denote by λ_2 the corresponding eigenvalue, then similar to the proof of Theorem 6.10, we have

$$|\lambda_2| = 2^k + 2^{k-1},$$

which contradicts the Ramanujan-Petersson-Deligne Theorem. Thus we proved the above claim.

Let $f_1, f_2, \cdots, f_r \in S_{k+1/2}(N,\chi)$ be an orthogonal basis of common eigenfunctions of the operators $\mathrm{T}_{N,k,\chi}(p)(p \nmid N)$ resp. $U(p^2)(p|N)$, and write $f_i = \sum_{n=1}^{\infty} a_i(n)e(nz)$. For every i find a fundamental discriminant $D_i \equiv 0 \pmod 4$ with $\varepsilon(-1)^k D_i > 0$ and $a_i(|D_i|) \neq 0$. Then the polynomial

$$P(x_1, x_2, \cdots, x_r) = \prod_{1 \leqslant i \leqslant r} (a_i(|D_1|)x_1 + \cdots + a_i(|D_r|)x_r)$$

is non-zero. Choose $c_1, \cdots, c_r \in \mathbb{C}$ such that $P(c_1, \cdots, c_r) \neq 0$ and put

$$L_{N,k,\chi} = \sum_i c_i L_{D_i,N,k,\chi}.$$

Then it is immediate that $L_{N,k,\chi}$ is an isomorphism of $S^{\text{new}}_{k+1/2}(N,\chi)$ onto $S^{\text{new}}(N, 2k, \text{id.})$. By Lemma 6.18 and the fact that $S^{\pm p}_{k+1/2}(N,\chi)$ is the (± 1)-eigenspace of the involution $w_{p,k+1/2,N,\chi}$, we see that $L_{N,k,\chi}$ maps $S^{\pm p}_{k+1/2}(N,\chi) \bigcap S^{\text{new}}_{k+1/2}(N,\chi)$ onto $S^{\pm p}(N, 2k, \text{id.}) \bigcap S^{\text{new}}(N, 2k, \text{id.})$.

Finally we must prove the assertions about $L_{D,N,k,\chi}$ for $D \equiv 1 \pmod 4$. It is enough to show that $L_{D,N,k,\chi}$ maps $S^{\text{new}}_{k+1/2}(N,\chi)$ to $S^{\text{new}}(N, 2k, \text{id.})$. In fact, for any prime divisor $l|N$, it is easy to verify that

$$L_{D,N/l,k,\chi} = L_{D,N,k,\chi}\left(1 - \left(\frac{D}{l}\right)l^{k-1}V(l)\right),$$
$$U(t)L_{D,N,k,\chi} = L_{D_0,N,k,\text{id.}}U((D,t)^2),$$

where $V(l)$ is the translation operator defined by $(f|V(l))(z) = f(lz)$ and $\left(\frac{D_0}{*}\right)$ is the primitive character induced by $\left(\frac{D}{*}\right)\chi$. It then follows inductively that $S_{k+1/2}(N,\chi)$ is mapped to $S(N, 2k, \text{id.})$. And the same argument as in the case $D \equiv 0 \pmod 4$ shows that $S^{\pm p}_{k+1/2}(N,\chi) \bigcap S^{\text{new}}_{k+1/2}(N,\chi)$ is mapped to $S^{\pm p}(N, 2k, \text{id.}) \bigcap S^{\text{new}}(N, 2k, \text{id.})$.

Now let F be a normalized eigenform in $S^{\text{new}}(N, 2k, \text{id.})$ with $F|\mathrm{T}_{N,2k}(p) = \lambda_p F$ resp. $F|U(p) = \lambda_p F$ for all primes $p \nmid N$ resp. $p|N$. Then $F = \sum_{n=1}^{\infty} \lambda_n e(nz)$ and λ_n is determined by

$$\sum_{n=1}^{\infty} \lambda_n n^{-s} = \prod_p (1 - \lambda_p p^{-s} + \chi_N(p)^2 p^{2k-1-2s})^{-1}.$$

Write $\phi_{N,k,\chi}$ for the inverse of $L_{N,k,\chi}$ and put $G = F|\phi_{N,k,\chi}L_{D,N,k,\chi}$. Then G is a power series in $q = e(z)$ which converges on $\mathbb{H}$ and satisfies $G|\mathrm{T}_{N,2k}(p) = \lambda_p G$ resp. $G|U(p) = \lambda_p G$ for all primes $p \nmid N$ resp. $p|N$. Hence it follows that the coefficient of G at $e(nz)$ equals $c\lambda_n$ with c the first Fourier coefficient of G. Thus we have that $(F|\phi_{N,k,\chi})|L_{D,N,k,\chi} = cF$. This shows that $L_{D,N,k,\chi}$ maps $S^{\text{new}}_{k+1/2}(N,\chi)$ to $S^{\text{new}}(N, 2k, \text{id.})$. This completes the proof. □

Corollary 6.4 *Let N_1 and N_2 be two square free positive integers, f_1 and f_2 two new forms in $S^{\text{new}}_{k+1/2}(N_1,\omega_1)$ and $S^{\text{new}}_{k+1/2}(N_2,\omega_2)$ respectively such that f_1 and f_2 have the same eigenvalues with respect to infinitely many operators $\mathrm{T}(p)$ for $(p, N_1N_2) = 1$. Then $N_1 = N_2$ and $f_1 = cf_2$ with some constant c.*

Proof This is a direct conclusion of Theorem 6.11 and Theorem 6.8. □

6.3 Dimension Formulae for the Spaces of New Forms

In this section we shall give some dimension formulae of the spaces of new forms. Recall first the following result:

Theorem 6.12 *Let k be any even positive integer and N a positive integer. Then we have*

$$d_0(N,k) = \frac{k-1}{12} N s_0(N) - \frac{1}{2}\nu_\infty(N) + c_2(k)\nu_2(N) + c_3(N)\nu_3(N) + \delta_{1,k/2},$$

where $d_0(N,k)$ is the dimension of the space of cusp forms with weight k on the group $\Gamma_0(N)$, $\delta_{x,y}$ is zero or 1 according to $x = y$ or $x \neq y$ respectively, and the functions s_0, ν_∞, ν_2, ν_3, c_2 and c_3 are defined as follows:

s_0 : *the multiplicative function defined by* $s_0(p^t) = 1 + \dfrac{1}{p}$ *for all* $t \geqslant 1$;

ν_∞ : *the multiplicative function defined by*

$$\nu_\infty(p^t) = \begin{cases} 2p^{(t-1)/2}, & \textit{if } t \textit{ is odd}, \\ p^{t/2} + p^{t/2-1}, & \textit{if } t \textit{ is even}. \end{cases}$$

ν_2 : *the multiplicative function defined by*

$$\nu_2(p^t) = \begin{cases} 1, & \textit{if } p = 2, t = 1, \\ 0, & \textit{if } p = 2, t \geqslant 2, \\ 2, & \textit{if } p \equiv 1(4), t \geqslant 1, \\ 0, & \textit{if } p \equiv 3(4), t \geqslant 1 \end{cases}$$

ν_3 : *the multiplicative function defined by*

$$\nu_3(p^t) = \begin{cases} 1, & \textit{if } p = 3, t = 1, \\ 0, & \textit{if } p = 3, t \geqslant 2, \\ 2, & \textit{if } p \equiv 1(3), t \geqslant 1, \\ 0, & \textit{if } p \equiv 2(3), t \geqslant 1. \end{cases}$$

c_2 : *the function defined by* $c_2(k) = \dfrac{1}{4} + \left\lfloor \dfrac{k}{4} \right\rfloor$;

c_3 : *the function defined by* $c_3(k) = \dfrac{1}{3} + \left\lfloor \dfrac{k}{3} \right\rfloor$.

Proof This is a direct conclusion of the dimension formula of the space of cusp forms with integral weight in Section 4.1. □

We now denote by $d_0^{\text{new}}(N,k)$ the dimension of the space of new forms with weight k on the group $\Gamma_0(N)$.

Then we have the following

Theorem 6.13 *Let k be any even positive integer and N a positive integer. Then*

$$d_0^{\text{new}}(N,k) = \frac{k-1}{12} N s_0^{\text{new}}(N) - \frac{1}{2}\nu_\infty^{\text{new}}(N) + c_2(k)\nu_2^{\text{new}}(N) + c_3(k)\nu_3^{\text{new}}(N) + \delta_{1,k/2}\mu(N),$$

where the function $c_2, c_3, \delta_{1,k/2}$ are as in Theorem 6.12, μ is the Moebius function and s_0^{new}, ν_∞^{new}, ν_2^{new}, ν_3^{new} are defined as follows:

s_0^{new} : *the multiplicative function defined by*

$$s_0^{\text{new}}(p^t) = \begin{cases} 1 - \dfrac{1}{p}, & \text{if } t = 1, \\ 1 - \dfrac{1}{p} - \dfrac{1}{p^2}, & \text{if } t = 2, \\ \left(1 - \dfrac{1}{p}\right)\left(1 - \dfrac{1}{p^2}\right), & \text{if } t \geqslant 3. \end{cases}$$

ν_∞^{new} : *the multiplicative function defined by*

$$\nu_\infty^{\text{new}} = \begin{cases} 0, & \text{if } t \text{ is odd}, \\ p-2, & \text{if } t = 2, \\ p^{t/2-2}(p-1)^2, & \text{if } t \geqslant 4 \text{ even}. \end{cases}$$

ν_2^{new} : *the multiplicative function defined by*

$$\nu_2^{\text{new}}(p^t) = \begin{cases} -1, & \text{if } p = 2, t = 1 \text{ or } 2, \\ 1, & \text{if } p = 2, t = 3, \\ 0, & \text{if } p = 2, t \geqslant 4, \\ 0, & \text{if } p \equiv 1(4), t = 1 \text{ or } t \geqslant 3, \\ -1, & \text{if } p \equiv 1(4), t = 2, \\ -2, & \text{if } p \equiv 3(4), t = 1, \\ 1, & \text{if } p \equiv 3(4), t = 2, \\ 0, & \text{if } p \equiv 3(4), t \geqslant 3. \end{cases}$$

ν_3^{new} : *the multiplicative function defined by*

$$\nu_3^{\text{new}}(p^t) = \begin{cases} -1, & \text{if } p = 3, t = 1 \text{ or } 2, \\ 1, & \text{if } p = 3, t = 3, \\ 0, & \text{if } p = 3, t \geqslant 4, \\ 0, & \text{if } p \equiv 1(3), t = 1 \text{ or } t \geqslant 3, \\ -1, & \text{if } p \equiv 1(3), t = 2, \\ -2, & \text{if } p \equiv 2(3), t = 1, \\ 1, & \text{if } p \equiv 2(3), t = 2, \\ 0, & \text{if } p \equiv 2(3), t \geqslant 3. \end{cases}$$

Proof We recall first the following facts about arithmetic functions: the set of arithmetic functions $f : \mathbb{N} \to \mathbb{C}$ forms a ring under the usual addition of functions and the Dirichlet convolution as the multiplication operation:

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d) \tag{6.48}$$

for any two arithmetic functions f and g. And the function $\delta(n) := \delta_{1,n}$ is the multiplicative identity of the ring. And the set of all multiplicative functions f with $f(1) \neq 0$ forms a multiplicative subgroup under the Dirichlet convolution. In fact, if $f(1) \neq 0$, then the function g defined as follows:

$$g(n) = \begin{cases} \dfrac{1}{f(1)}, & \text{if } n = 1, \\ -\dfrac{1}{f(1)} \displaystyle\sum_{d|n, d\neq n} f(n/d)g(d), & \text{if } n > 1 \end{cases} \tag{6.49}$$

is the inverse of f. By Moebius inversion formula we see that the Moebius function μ is the inverse of the function $1(n)$ which takes the value 1 at all positive integers:

$$(\mu * 1)(n) = \sum_{d|N} \mu(d) = \delta(n).$$

And in general we use the following Moebius inversion formula: for any two arithmetic functions f and g, we have

$$f(n) = \sum_{d|n} g(d), \quad \forall\ n \in \mathbb{N}$$

if and only if

$$g(n) = \sum_{d|n} \mu(n/d) f(d), \quad \forall\ n \in \mathbb{N}.$$

In fact, we have

$$f(n) = \sum_{d|n} g(d) = (1 * g)(n)$$

if and only if

$$\begin{aligned} g(n) &= ((\mu * 1) * g)(n) \\ &= (\mu * (1 * g))(n) \\ &= (\mu * f)(n) \\ &= \sum_{d|n} \mu(n/d) f(d). \end{aligned}$$

From the results in Section 6.1 we have

$$S(N, k) = \bigoplus_{l|N} \bigoplus_{m|N/l} S^{\text{new}}(l, k)|V(m),$$

where $V(m)$ is the translation operator defined by $f|V(m) = f(mz)$ which is an injection from $S(l,k)$ to $S(N,k)$. Therefore we have

$$d_0(N,k) = \sum_{l|N} \sum_{m|N/l} d_0^{\text{new}}(l,k) = \sum_{l|N} d_0^{\text{new}}(l,k)\tau(N/l), \tag{6.50}$$

where $\tau(n) = \sum_{d|n} 1$ is the number of positive divisors of n. In terms of Dirichlet convolution, we see that from (6.50)

$$d_0 = d_0^{\text{new}} * \tau$$

holds for any fixed k. Let λ be the inverse of τ. Since $\tau = 1 * 1$, we see that

$$\lambda = \tau^{-1} = (1 * 1)^{-1} = 1^{-1} * 1^{-1} = \mu * \mu.$$

Hence, from (6.48), λ is the multiplicative function defined by

$$\lambda(p^t) = \begin{cases} -2, & \text{if } t = 1, \\ 1, & \text{if } t = 2, \\ 0, & \text{if } t \geqslant 3 \end{cases}$$

Therefore we see that $d_0^{\text{new}} = d_0 * \lambda$, and so that

$$\begin{aligned} d_0^{\text{new}}(N,k) = & \frac{k-1}{12}((i_0 s_0) * \lambda)(N) - \frac{1}{2}(\nu_\infty * \lambda)(N) \\ & + c_2(k)(\nu_2 * \lambda)(N) + c_3(k)(\nu_3 * \lambda)(N) + \delta_{1,k/2}(1 * \lambda)(N) \end{aligned}$$

from Theorem 6.12 and the fact that the set of arithmetic functions forms a ring under the usual addition and the Dirichlet convolution, where $i_0(n) = n$ is the identity function on $\mathbb{N}$. But we see that $1 * \lambda = 1 * (\mu * \mu) = (1 * \mu) * \mu = \delta * \mu = \mu$, and $\nu_\infty * \lambda$, $\nu_2 * \lambda$, $\nu_3 * \lambda$ are multiplicative functions which equal ν_∞^{new}, ν_2^{new}, ν_3^{new} respectively by (6.48) and the definitions of ν_∞^{new}, ν_2^{new}, ν_3^{new}. Finally we see that

$$i_0(p^t)s_0(p^t) * \lambda(p^t) = \sum_{m=0}^{t} p^m s_0(p^m)\lambda(p^{t-m}) = p^t s_0^{\text{new}}(p^t),$$

i.e. the multiplicative function $((i_0 s_0) * \lambda)(N) = N s_0^{\text{new}}(N)$. This completes the proof. □

By Theorem 6.11, there exists a linear combination of the Shimura lifting $L_{D,N,k,\chi}$ which maps $S_{k+1/2}^{\text{new}}(N,\chi)$ isomorphically onto $S^{\text{new}}(N,2k)$, so that

$$\dim(S_{k+1/2}^{\text{new}}(N,\chi)) = \dim\left(S^{\text{new}}(N,2k)\right).$$

Hence by Theorem 6.13 we have the following:

Corollary 6.5 *Let k be a positive integer, N a square free positive integer and χ a quadratic character modulo N. Then*

$$\begin{aligned} d_0^{\text{new}}(N, k+1/2) = &\frac{2k-1}{12} N s_0^{\text{new}}(N) - \frac{1}{2}\nu_\infty^{\text{new}}(N) \\ &+ c_2(2k)\nu_2^{\text{new}}(N) + c_3(2k)\nu_3^{\text{new}}(N) + \delta_{1,k}\mu(N), \end{aligned}$$

where $d_0^{\text{new}}(N, k+1/2) := \dim(S_{k+1/2}^{\text{new}}(N, \chi))$.