Chapter 10

Integers Represented by Positive Definite Quadratic Forms

10.1 Theta Function of a Positive Definite Quadratic Form and Its Values at Cusp Points

In the first chapter we introduced the theta function of a positive definite quadratic form and discussed its transformation formula under the action of the modular group. We want now to show that the theta function is a modular form.

Let $f(x_1, \cdots, x_k)$ be a positive definite quadratic form with integral coefficients. Define the matrix A of $f(x_1, \cdots, x_k)$ as follows:

$$A = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right).$$

It is clear that A is a symmetric matrix with even diagonal entries. Put

$$\theta_f(z) = \sum_{m \in \mathbb{Z}^k} e(zmAm^{\mathrm{T}}/2), \quad z \in \mathbb{H}.$$

It is clear that $\theta_f(z)$ is a holomorphic function on $\mathbb{H}$. Let N be the level of $f(x_1, \cdots, x_k)$, i.e., the minimal positive integer N such that NA^{-1} is an integral matrix with even diagonal entries. Set

$$\chi = \begin{cases} \left(\dfrac{2 \det A}{\cdot} \right), & \text{if } k \text{ is odd}, \\ \left(\dfrac{(-1)^{k/2} \det A}{\cdot} \right), & \text{if } k \text{ is even}. \end{cases}$$

Theorem 10.1 *$\theta_f(z)$ is in $G(N, k/2, \chi)$.*

Proof By the results in Chapter 1 we need only to consider the behavior of $\theta_f(z)$ at the cusp points of $\Gamma_0(N)$. It is clear that

$$\lim_{z \to i\infty} \theta_f(z) = 1,$$

i.e., $\theta_f(z)$ is holomorphic at i∞. Let a/c be any cusp point with $c > 0$. Take

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$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, then $\rho(\infty) = a/c$. We have that

$$\theta_f(z)\left(\frac{az+b}{cz+d}\right) = \sum_{x \bmod c} e(axAx^{\mathrm{T}}/2c) \sum_{m\in\mathbb{Z}^k} e(-(m+x/c)A(m+x/c)^{\mathrm{T}}/2(z+d/c)), \tag{10.1}$$

where $x \in \mathbb{Z}^k$. By the proof of Proposition 1.2 we see that

$$\sum_{m\in\mathbb{Z}^k} e(-(x+m)A(x+m)^{\mathrm{T}}/2z) = (-\mathrm{i}z)^{k/2}(\det A)^{-1/2} \sum_{m\in\mathbb{Z}^k} e(zmA^{-1}m^{\mathrm{T}}/2 + x\cdot m^{\mathrm{T}}),$$

where $x \in \mathbb{R}^k$. Replacing x by x/c in the above equality we get

$$\begin{aligned}\theta_f\left(\frac{az+b}{cz+d}\right) = (-\mathrm{i}(z+d/c))^{k/2}(\det A)^{-1/2} \sum_{m\in\mathbb{Z}^k} e(zmAm^{\mathrm{T}}/2) \\ \times \sum_{x \bmod c} e(axAx^{\mathrm{T}}/2c + x\cdot m^{\mathrm{T}}/c + dmA^{-1}m^t/2c),\end{aligned}$$

hence

$$\lim_{z\to i\infty}(z+d/c)^{-k/2}\theta_f\left(\frac{az+b}{cz+d}\right) = (-\mathrm{i})^{k/2}(\det A)^{-1/2} \sum_{x \bmod c} e(axAx^{\mathrm{T}}/2c), \tag{10.2}$$

i.e., $\theta_f(z)$ is holomorphic at the cusp point a/c. This completes the proof. □

Let $f_1 = f_1(x_1, \cdots, x_k)$ and $f_2 = f_2(x_1, \cdots, x_k)$ be two positive definite quadratic forms with integral coefficients, A_1 and A_2 the corresponding matrices of f_1 and f_2 respectively. f_1 and f_2 are called equivalent if there exists an integral matrix S with determinant ± 1 such that $SA_1S^{\mathrm{T}} = A_2$. f_1 and f_2 are called equivalent over the real field $\mathbb{R}$ if there exists a real invertible matrix S_r such that $S_rA_1S_r^{\mathrm{T}} = A_2$. Let p be a prime and take A_1, A_2 as matrices over the finite field $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. f_1 and f_2 are called equivalent over $\mathbb{F}_p$ if there exists an invertible matrix S_p on $\mathbb{F}_p$ such that $S_pA_1S_p^{\mathrm{T}} = A_2$. f_1 and f_2 are called in the same genus if f_1 and f_2 are equivalent over $\mathbb{R}$ and over $\mathbb{F}_p$ for any prime p. It is clear that f_1 and f_2 are in the same genus if they are equivalent. It can be proved that there are only finite equivalence classes in a genus.

Let $f = f(x_1, x_2, \cdots, x_k)$ be a positive definite quadratic form, and $f_1, f_2, \cdots, f_h$ be a full system of representations of all different classes in the genus of f. Let n be an arbitrary non-negative integer, and $r(f_i, n)$ denote the number of integral solutions of the equation $f_i(x) = n$. It is difficult to find an analytical expression for the number $r(f_i, n)$ in general cases.

Denote by $M_k(\mathbb{Z})$ the set of all $k \times k$ integral matrices. Put $O(f) = \#\{S \in M_k(\mathbb{Z}) | SAS^{\mathrm{T}} = A\}$, define the theta function θ of the genus of f:

$$\theta(\mathrm{gen}.f, z) = \left(\sum_{i=1}^{h} \frac{1}{O(f_i)} \right)^{-1} \sum_{i=1}^{h} \frac{\theta_{f_i}(z)}{O(f_i)},$$

Then

$$\begin{aligned} \theta(\mathrm{gen}.f, z) &= \sum_{n=0}^{\infty} r(\mathrm{gen}.f, n) \exp\{2\pi i n z\} \\ &= \sum_{i=1}^{h} \left(\frac{1}{O(f_i)} \right)^{-1} \sum_{i=1}^{h} \sum_{n=0}^{\infty} \frac{r(f_i, n)}{O(f_i)} \exp\{2\pi i n z\}, \end{aligned}$$

it follows that

$$r(\mathrm{gen}.f, n) = \sum_{i=1}^{h} \left(\frac{1}{O(f_i)} \right)^{-1} \sum_{i=1}^{h} \frac{r(f_i, n)}{O(f_i)},$$

i.e., the number $r(\mathrm{gen}.f, n)$ is a mean of the numbers $r(f_i, n)$, $(n \geqslant 0)$ when $k \geqslant 5$. This result is called Siegel theorem C.L.Siegel, 1966, which is equivalent to the fact that the function is an Eisenstein series of the weight $k/2$. A.N. Andrianov, 1980 obtained the same conclusion of Siegel theorem in the case of $k = 4$. Finally R. Schulze, 1984 reduced the same result of Siegel theorem in the case of $k = 3$. He proved that the function $\theta(\mathrm{gen}.f, z)$ is an Eisenstein series of the weight $3/2$ when $k = 3$. Under certain conditions, if the function $\theta(\mathrm{gen}.f, z)$ belongs to the space $\mathcal{E}(4D, 3/2, \chi_l)$ or $\mathcal{E}(8D, 3/2, \chi_l)$ then it can be represented as a linear combination of the basis functions for these spaces given in the Theorem 7.7 and Theorem 7.8 respectively. The coefficients of the linear combination can be determined using the values of the function $\theta(\mathrm{gen}.f, z)$, thus an analytic expression for the number $r(\mathrm{gen}.f, n)$ can be reduced in this way.

The Scholze-Pillot's Proof for Siegel theorem will be described below.

Let f_1 and f_2 be in the same genus. Then the corresponding matrices of f_1 and f_2 have the same determinant. If a/c is a cusp point with $c > 0$, then there exists an integral matrix S such that $(\det S, 2c) = 1$ and $SA_1S^{\mathrm{T}} \equiv A_2 \pmod{2c}$ by the above definitions and the Chinese remainder theorem. This shows that $\theta_{f_1}(z)$ and $\theta_{f_2}(z)$ have the same value at the cusp point a/c by (10.2). Hence $\theta_{f_1}(z) - \theta_{f_2}(z)$ is a cusp from.

Theorem 10.2 *Let p be a prime, $p \nmid N$. Set*

$$\lambda_p = \begin{cases} p^{k-2} + 1, & \text{if } 2 \nmid k, \\ p^{k-2} + 2p^{k/2-1} \left(\dfrac{(-1)^{k/2} \det A}{p} \right) + 1, & \text{if } 2 | k. \end{cases}$$

Then

$$\theta(\text{gen}.f, z)|\mathrm{T}(p^2) = \lambda_p \theta(\text{gen}.f, z),$$

where $\mathrm{T}(p^2)$ *is the* Hecke *operator on the space* $G(N, k/2, \chi)$.

Proof Please see R. Schulze, 1984 and P. Ponomarev, 1981. □

Theorem 10.3 *The function* $\theta(\text{gen}.f, z)$ *is in the space* $\mathcal{E}(N, k/2, \chi)$.

Proof We assume first that $k \geqslant 4$ is an even. Since

$$G(N, k/2, \chi) = \mathcal{E}(N, k/2, \chi) \oplus S(N, k/2, \chi),$$

there exist two functions $g_1(z)$ and $g_2(z)$ such that

$$\theta(\text{gen}.f, z) = g_1(z) + g_2(z), \quad g_1(z) \in S(N, k/2, \chi), g_2(z) \in \mathcal{E}(N, k/2, \chi).$$

Let $g_1(z) = \sum_{n=n_0}^{\infty} c(n)e(nz), c(n_0) \neq 0$. For any $p \nmid N$, by Theorem 10.2, we see that $g_1(z)|\mathrm{T}(p^2) = \lambda_p g_1(z)$, and hence

$$\lambda_p c(n_0) = c(n_0 p^2) + \chi(p)\left(\frac{-n_0}{p}\right)a(n_0).$$

By Lemma 7.24 we have that $c(n) = O(n^{k/4})$, so $\lambda_p = O(p^{k/2})$. If $k \geqslant 6$, we see that $\lambda_p \sim p^{k-2}$ $(p \to \infty)$ which contradicts $\lambda_p = O(p^{k/2})$. Hence we have $g_1(z) = 0$, which shows the theorem. If $k = 4$, we can prove the theorem similarly in terms of a more precise estimation $c(n) = O(n^{k/4-1/5})$ proved by R.A. Rankin, 1939. This shows the theorem for $k \geqslant 4$ even.

Now assume that k is an odd. For $k \geqslant 5$ we can prove the theorem by a similar method as for the case $k \geqslant 6$ an even. Now let $k = 3$ and $V := S(N, 3/2, \chi) \bigcap \widetilde{T}$ be as in Theorem 8.2. Denote by $V^\perp$ the orthogonal complement of V in $S(N, 3/2, \chi)$. Then we have

$$\theta(\text{gen}.f, z) = g_1 + g_2 + g_3, \quad g_1 \in V, \quad g_2 \in V^\perp, \quad g_3 \in \mathcal{E}(N, 3/2, \chi).$$

By Theorem 10.2 we see that $g_i|\mathrm{T}(p^2) = (p+1)g_i$ for any $p \nmid N$ and $i = 1, 2, 3$. But by the definition of $\widetilde{T}$ we know that g_1 is a finite linear combination of functions $h(tz; \psi)$ with $\chi = \psi\left(\frac{-t}{\cdot}\right)$. Hence we have

$$h(tz; \psi)|\mathrm{T}(p^2) = \chi(p)\left(\frac{-t}{p}\right)(p+1)h(tz; \psi).$$

There must be a prime p such that

$$h(tz; \psi)|\mathrm{T}(p^2) = -(p+1)h(tz; \psi)$$

holds for all finite functions $h(tz;\psi)$, so that $g_1(z)|\mathrm{T}(p^2)=-(p+1)g_1$, from which we get $g_1=0$ since we have also $g_1(z)|\mathrm{T}(p^2)=(p+1)g_1$. $g_2(z)$ is mapped in $S(N/2,2,\mathrm{id.})$ under the Shimura lifting S and the image $S(g_2)$ of g_2 is also an eigenfunction of $\mathrm{T}(p)$ with eigenvalue $p+1$. In terms of Rankin's estimation $c(n)=O(n^{4/5})$ we can show that $g_2=0$. Therefore $\theta(\mathrm{gen}.f,z)\in\mathcal{E}(N,3/2,\chi)$. This completes the proof. □

Let $f(x_1,x_2,\cdots,x_k)$ be a positive definite quadratic form with integral coefficients. Put

$$\theta_f(z)=\sum_{m\in\mathbb{Z}^k} e(zmAm^{\mathrm{T}}/2),\quad z\in\mathbb{H},$$
$$O(f)=\#\{S\in M_k(\mathbb{Z})|SAS^{\mathrm{T}}=A\},$$
$$\theta(\mathrm{gen}.f,z)=\left(\sum_{f_i}\frac{1}{O(f_i)}\right)^{-1}\sum_{f_i}\frac{\theta_{f_i}(z)}{O(f_i)},$$

where the f_i run over a complete set of representatives of the equivalence classes in the genus of f.

Suppose that N is the level of f, i.e.,

$$N=\min\{N|NA^{-1}\text{ is integral and the diagonal entries are even, }N\text{ positive integer}\}.$$

Let now $S(N)$ denote a complete set of representatives of equivalence classes of cusp points for the group $\Gamma_0(N)$. In fact we can choose $S(N)=\{d/c\,|\,c|N,d\in(\mathbb{Z}/(c,N/c)\mathbb{Z})^*$ and $(d,c)=1\}$.

We want to compute the values of $\theta_f(z)$ at cusp points for $\Gamma_0(N)$. It is clear that

$$\lim_{z\to i\infty}\theta_f(z)=1.$$

Now suppose that a/c is a cusp point, where $(a,c)=1, c|N, a\in(\mathbb{Z}/(c,N/c)\mathbb{Z})^*$. Choose a matrix $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$, then $\gamma(\mathrm{i}\infty)=a/c$. So in terms of the equality (10.2) we obtain

$$\begin{aligned}V(\theta_f,a/c)&=\lim_{z\to\mathrm{i}\infty}(cz+d)^{-k/2}\theta_f\left(\frac{az+b}{cz+d}\right)\\&=(-\mathrm{i})^{k/2}(\det A)^{-1/2}c^{-k/2}\sum_{x\bmod c}e(axAx^{\mathrm{T}}/2c)\end{aligned}$$

This shows that in order to get the values of $\theta_f(z)$ at cusp points we only need to evaluate the Gauss sum

$$\sum_{x\bmod c}e(axAx^{\mathrm{T}}/2c)$$

where c,a are co-prime positive integers.

Now we will calculate the Gauss sum

$$G(a,c) := \sum_{x \bmod c} e(axAx^{\mathrm{T}}/2c), \quad (c,a)=1.$$

Lemma 10.1 *If* $(c,c')=1$, *then*

$$G(a,cc') = G(ac,c')G(ac',c).$$

Proof Let $x = cy + c'z$, then

$$\begin{aligned} G(a,cc') &= \sum_{x \bmod cc'} e(axAx^{\mathrm{T}}/2cc') \\ &= \sum_{y \bmod c'} \sum_{z \bmod c} e(a(cy+c'z)A(cy+c'z)^{\mathrm{T}}/2cc') \\ &= \sum_{y \bmod c'} e(acyAy^{\mathrm{T}}/2c') \sum_{z \bmod c} e(ac'zAz^{\mathrm{T}}/2c) \\ &= G(ac,c')G(ac',c). \end{aligned}$$

This completes the proof. □

By Lemma 10.1, we only need to evaluate the Gauss sum $G(a,p^m)$ where $p \nmid a$ a prime and m is a positive integer.

We first assume that p is an odd prime. Then there exists an invertible matrix S over the ring $\mathbb{Z}_p$ of p-adic integers such that

$$SAS^{\mathrm{T}} = \mathrm{diag}\{\alpha_1 p^{\beta_1}, \alpha_2 p^{\beta_2}, \cdots, \alpha_k p^{\beta_k}\},$$

where $\alpha_i, \det S \in \mathbb{Z}_p^*$, $0 \leqslant \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_k$ are rational integers. Let $l_m = \#\{\beta_i | \beta_i \geqslant m\}$.

Hence

$$\begin{aligned} G(a,p^m) &= \sum_{x \bmod p^m} e(axAx^{\mathrm{T}}/2p^m) \\ &= \sum_{x \bmod p^m} e\left(ax\left(\bigoplus_{i=1}^{k} \alpha_i p^{\beta_i}\right)x^{\mathrm{T}}/2p^m\right) \\ &= \sum_{x=(x_1,\cdots,x_k) \bmod p^m} \prod_{i=1}^{k} e(a\alpha_i p^{\beta_i} x_i^2/2p^m) \\ &= p^{ml_m} \prod_{\beta_i<m} \left(\sum_{x \bmod p^m} e(a\alpha_i' x^2/p^{m-\beta_i})\right) \quad (\text{where } \alpha_i' \equiv 2^{-1}\alpha_i \bmod p^{m-\beta_i}) \\ &= p^{ml_m} \prod_{\beta_i<m} \left(\sum_{z \bmod p^{\beta_i}} \sum_{y \bmod p^{m-\beta_i}} e(a\alpha_i'(y+p^{m-\beta_i}z)^2/p^{m-\beta_i})\right) \end{aligned}$$

$$= p^{ml_m} \prod_{\beta_i < m} \left(\sum_{z \bmod p^{\beta_i}} \sum_{y \bmod p^{m-\beta_i}} e(a\alpha_i' y^2 / p^{m-\beta_i}) \right)$$
$$= p^{ml_m} \prod_{\beta_i < m} p^{\beta_i} S(a\alpha_i', p^{m-\beta_i})$$
$$= p^{ml_m} \prod_{\beta_i < m} p^{\beta_i} \left(\frac{a\alpha_i'}{p^{m-\beta_i}} \right) \varepsilon_{p^{m-\beta_i}} p^{\frac{m-\beta_i}{2}}$$
$$= p^{ml_m} \prod_{\beta_i < m} \left(\frac{a\alpha_i'}{p^{m-\beta_i}} \right) \varepsilon_{p^{m-\beta_i}} p^{\frac{m+\beta_i}{2}},$$

where $S(\alpha, p^\beta) = \sum_{x \bmod p^\beta} e(\alpha x^2/p^\beta)$ is the classical Gauss sum, and $\varepsilon_d = 1$ or i according to $d \equiv 1$ or $3 \pmod 4$ respectively.

Now consider the case $p = 2$. In this case, there exists an invertible matrix S over the ring $\mathbb{Z}_2$ of 2-adic integers such that

$$SAS^{\mathrm{T}} = \bigoplus_{i=1}^{l} \alpha_i 2^{s_i} \bigoplus_{j=1}^{l_1} \beta_j 2^{t_j} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bigoplus_{s=1}^{l_2} \gamma_s 2^{u_s} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

where α_i, β_j, $\gamma_s \in \mathbb{Z}_2^*$, $s_i \geqslant 1$, t_j, $u_s \geqslant 0$ are rational integers.

Hence we have

$$G(a, 2^m) = \sum_{x \bmod 2^m} e(axAx^{\mathrm{T}}/2^{k+1})$$
$$= \sum_{x \bmod 2^m} e\Bigg(ax \Bigg(\bigoplus_{i=1}^{l} \alpha_i 2^{s_i} \bigoplus_{j=1}^{l_1} \beta_j 2^{t_j} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bigoplus_{s=1}^{l_2} \gamma_s 2^{u_s} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \Bigg) x^{\mathrm{T}}/2^{k+1} \Bigg),$$

which implies that we only need to evaluate the following kinds of Gauss sums:

$$G_{1,t}(a\alpha_i, 2^m) := \sum_{x \bmod 2^m} e(a\alpha_i x^2/2^t),$$
$$G_{2,t}(a\beta_j, 2^m) := \sum_{(x,y) \bmod 2^m} e(a\beta_j xy/2^t),$$
$$G_{3,t}(a\gamma_s, 2^m) := \sum_{(x,y) \bmod 2^m} e(a\gamma_s (x^2 + xy + y^2)/2^t),$$

where t is a positive integer and $t \leqslant m$.

Now we compute the above Gauss sums:

$$
\begin{aligned}
G_{1,t}(a\alpha_i, 2^m) &= \sum_{x \bmod 2^m} e(a\alpha_i x^2/2^t) \\
&= \sum_{y \bmod 2^t} \sum_{z \bmod 2^{m-t}} e(a\alpha_i (y + 2^t z)^2/2^t) \\
&= \sum_{z \bmod 2^{m-t}} \sum_{y \bmod 2^t} e(a\alpha_i y^2/2^t) = 2^{k-t} S(a\alpha_i, 2^t) \\
&= \begin{cases} 0, & \text{if } t = 1, \\ (1 + i^{a\alpha_i}) 2^{m-\frac{t}{2}}, & \text{if } t \text{ is even}, \\ 2^{m-\frac{t-1}{2}} e^{\frac{\pi i a \alpha_i}{4}}, & \text{if } t > 1 \text{ and odd.} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
G_{2,t}(a\beta_j, 2^m) &= \sum_{(x,y) \bmod 2^m} e(a\beta_j xy/2^t) = \sum_{x \bmod 2^m} \sum_{y \bmod 2^m} e(a\beta_j xy/2^t) \\
&= \sum_{x \bmod 2^m} 2^{m-t} \sum_{y \bmod 2^t} e(a\beta_j xy/2^t) = 2^{m-t} \sum_{\substack{x \bmod 2^m, \\ 2^t | x}} 2^t = 2^{2m-t},
\end{aligned}
$$

$$
\begin{aligned}
G_{3,t}(a\gamma_s, 2^m) &= \sum_{(x,y) \bmod 2^m} e(a\gamma_s (x^2 + xy + y^2)/2^t) \\
&= \sum_{x \bmod 2^m} \sum_{y \bmod 2^m} e(a\gamma_s (x^2 + xy + y^2)/2^t) \\
&= \sum_{x \bmod 2^m} e(a\gamma_s x^2/2^t) \sum_{y \bmod 2^m} e(a\gamma_s (xy + y^2)/2^t) \\
&= \sum_{x \bmod 2^m} e(a\gamma_s x^2/2^t) \sum_{z \bmod 2^{m-t}} \sum_{y \bmod 2^t} \\
&\quad e(a\gamma_s (x(y + 2^t z) + (y + 2^t z)^2)/2^t) \\
&= \sum_{x \bmod 2^m} e(a\gamma_s x^2/2^t) \sum_{z \bmod 2^{m-t}} \sum_{y \bmod 2^t} e(a\gamma_s (xy + y^2)/2^t) \\
&= 2^{m-t} \sum_{x \bmod 2^m} e(a\gamma_s x^2/2^t) \sum_{y \bmod 2^t} e(a\gamma_s (xy + y^2)/2^t) \\
&= 2^{2(m-t)} \sum_{x \bmod 2^t} e(a\gamma_s x^2/2^t) \sum_{y \bmod 2^t} e(a\gamma_s (xy + y^2)/2^t).
\end{aligned}
$$

Now let $w = \left[\frac{t+1}{2}\right]$, then

$$
\begin{aligned}
&\sum_{y \bmod 2^t} e(a\gamma_s (xy + y^2)/2^t) \\
&= \sum_{u \bmod 2^w} \sum_{v \bmod 2^{t-w}} e(a\gamma_s (x(u + 2^w v) + (u + 2^w v)^2)/2^t) \\
&= \sum_{u \bmod 2^w} e(a\gamma_s (xu + u^2)/2^t) \sum_{v \bmod 2^{t-w}} e(a\gamma_s (x + 2u) v/2^{t-w})
\end{aligned}
$$

$$= \sum_{\substack{u \bmod 2^w,\\ 2^{t-w}|(x+2u)}} 2^{t-w} e(a\gamma_s(xu+u^2)/2^t).$$

Therefore, we obtain

$$\begin{aligned}
&G_{3,t}(a\gamma_s, 2^m)\\
&= 2^{2(m-t)} \sum_{x \bmod 2^t} e(a\gamma_s x^2/2^t) \sum_{\substack{u \bmod 2^w,\\ 2^{t-w}|(x+2u)}} 2^{t-w} e(a\gamma_s(xu+u^2)/2^t)\\
&= 2^{2m-t-w} \sum_{u \bmod 2^w} e(a\gamma_s u^2/2^t) \sum_{\substack{x \bmod 2^t,\\ x+2u\equiv 0(2^{t-w})}} e(a\gamma_s(xu+x^2)/2^t)\\
&= 2^{2m-t-w} \sum_{u \bmod 2^w} e(a\gamma_s u^2/2^t) \sum_{y \bmod 2^w}\\
&\quad e(a\gamma_s((-2u+2^{t-w}y)u+(-2u+2^{t-w}y)^2)/2^t)\\
&= 2^{2m-t-w} \sum_{u \bmod 2^w} e(3a\gamma_s u^2/2^t) \sum_{y \bmod 2^w} e(-3a\gamma_s yu/2^w) e(a\gamma_s 2^{2(t-w)} y^2/2^t).
\end{aligned}$$

Now, if $t = 2g$ is even, then $w = \left[\dfrac{t+1}{2}\right] = g$, and $t - w = g$, $2^{2(t-w)}y^2/2^t = y^2$. Therefore we get

$$\begin{aligned}
G_{3,t}(a\gamma_s, 2^m) &= 2^{2m-t-w} \sum_{u \bmod 2^w} e(3a\gamma_s u^2/2^t) \sum_{y \bmod 2^w} e(-3a\gamma_s yu/2^w)\\
&= 2^{2m-t-w} \sum_{\substack{u \bmod 2^w,\\ 2^w|u}} 2^w e(3a\gamma_s u^2/2^t)\\
&= 2^{2m-t}.
\end{aligned}$$

If $t = 2g+1$ is odd, then $w = \left[\dfrac{t+1}{2}\right] = g+1$, and $t-w = g$, $2^{2(t-w)}y^2/2^t = y^2/2$. Therefore we get

$$\begin{aligned}
&G_{3,t}(a\gamma_s, 2^m)\\
&= 2^{2m-t-w} \sum_{u \bmod 2^w} e(3a\gamma_s u^2/2^t) \sum_{y \bmod 2^w} e(-3a\gamma_s yu/2^w) e(a\gamma_s y^2/2)\\
&= 2^{2m-t-w} \sum_{u \bmod 2^w} e(3a\gamma_s u^2/2^t) \Bigg(- \sum_{\substack{y \bmod 2^w,\\ y \text{ is odd}}} e(-3a\gamma_s yu/2^w)\\
&\quad + \sum_{\substack{y \bmod 2^w,\\ y \text{ is even}}} e(-3a\gamma_s yu/2^w) \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= 2^{2m-t-w} \sum_{u \bmod 2^w} e(3a\gamma_s u^2/2^t)\Bigg(- \sum_{y \bmod 2^w} e(-3a\gamma_s yu/2^w) \\
&\quad + 2 \sum_{\substack{y \bmod 2^w,\\ y \text{ is even}}} e(-3a\gamma_s yu/2^w)\Bigg) \\
&= -2^{2m-t-w} \sum_{u \bmod 2^w} e(3a\gamma_s u^2/2^t) \sum_{y \bmod 2^w} e(-3a\gamma_s yu/2^w) \\
&\quad + 2^{2m-t-w+1} \sum_{u \bmod 2^w} e(3a\gamma_s u^2/2^t) \sum_{\substack{y \bmod 2^w,\\ y \text{ is even}}} e(-3a\gamma_s yu/2^w) \\
&= -2^{2m-t-w} \sum_{u \bmod 2^w, 2^w | u} 2^w e(3a\gamma_s u^2/2^t) \\
&\quad + 2^{2m-t-w+1} \sum_{u \bmod 2^w} e(3a\gamma_s u^2/2^t) \sum_{y \bmod 2^{w-1}} e(-3a\gamma_s yu/2^{w-1}) \\
&= -2^{2m-t} + 2^{2m-t-w+1} \sum_{\substack{u \bmod 2^w\\ 2^{w-1}|u}} 2^{w-1} e(3a\gamma_s u^2/2^t) \\
&= -2^{2m-t} + 2^{2m-t}(1 + e(3a\gamma_s(2^{w-1})^2/2^t)) \\
&= -2^{2m-t} + 2^{2m-t}(1 + e(3a\gamma_s/2)) \\
&= -2^{2m-t},
\end{aligned}
$$

where $e(3a\gamma_s/2) = -1$ since $3a\gamma_s \equiv 1 \pmod 2$.

Therefore we have proved

$$G_{3,t}(a\gamma_s, 2^m) = (-1)^t 2^{2m-t}.$$

Now let $l_m = \#\{s_i | s_i \geqslant m+1\} + 2\#\{t_j | t_j \geqslant m\} + 2\#\{u_s | u_s \geqslant m\}$. Finally we have

$$
\begin{aligned}
&G(a, 2^m) \\
&= 2^{ml_m} \prod_{s_i < m+1} G_{1,m+1-s_i}(a\alpha_i, 2^m) \prod_{t_j < m} G_{2,m-t_j}(a\beta_j, 2^m) \prod_{u_s < m} G_{3,m-u_s}(a\gamma_s, 2^m) \\
&= 2^{ml_m} \prod_{s_i < m+1} G_{1,m+1-s_i}(a\alpha_i, 2^m) \prod_{t_j < m} 2^{2m-(m-t_j)} \prod_{u_s < m} (-1)^{m-u_s} 2^{2m-(m-u_s)} \\
&= 2^{ml_m} \prod_{s_i < m+1} G_{1,m+1-s_i}(a\alpha_i, 2^m) \prod_{t_j < m} 2^{m+t_j} \prod_{u_s < m} (-1)^{m-u_s} 2^{m+u_s}.
\end{aligned}
$$

So we can compute the values of $\theta_f(z)$ at each cusp point.

Example 10.1 Let $f(x, y) = ax^2 + bxy + cy^2$ be an integral primitive, positive definite, binary quadratic form with fundamental discriminant D. We want to evaluate $\theta_f(z)$ at cusp point $1/\alpha$ where $\alpha | D$. Since D is a fundamental discriminant, the odd part of D is square free. If $p|D$ is an odd prime, then $p \nmid a$ or $p \nmid c$ since f is primitive. Hence we have

(1) If $p \nmid a$, then

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \sim \begin{pmatrix} 2a & 0 \\ 0 & (2a)^{-1}\det A \end{pmatrix}$$

over $\mathbb{Z}_p$.

(2) If $p \nmid c$, then

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \sim \begin{pmatrix} 2c & 0 \\ 0 & (2c)^{-1}\det A \end{pmatrix}$$

over $\mathbb{Z}_p$.

Therefore

$$G(n,p) = \begin{cases} pS(an,p) = \left(\dfrac{an}{p}\right)\varepsilon_p p^{3/2}, & \text{if } p \nmid a, \\ pS(cn,p) = \left(\frac{cn}{p}\right)\varepsilon_p p^{3/2}, & \text{if } p \nmid c. \end{cases}$$

So for $\alpha = p_1 p_2 \cdots p_s | D$, p_i odd, we have

$$G(1,\alpha) = \prod_{i=1}^{s} G(\alpha/p_i, p_i) = \prod_{i=1}^{s}\left(\frac{\delta_i\alpha/p_i}{p_i}\right)\varepsilon_{p_i} p_i^{3/2} = \alpha^{3/2}\prod_{i=1}^{s}\left(\frac{\delta_i\alpha/p_i}{p_i}\right)\varepsilon_{p_i},$$

where $\delta_i = a$ or c according to $p_i \nmid a$ or $p_i \nmid c$. Hence

$$\begin{aligned} V(\theta_f, 1/\alpha) &= -i(\det A)^{-1/2}\alpha^{-1}G(1,\alpha) \\ &= -i\left(\frac{\alpha}{\det A}\right)^{1/2}\prod_{i=1}^{s}\left(\frac{\delta_i\alpha/p_i}{p_i}\right)\varepsilon_{p_i} = -\left(\frac{\alpha}{D}\right)^{1/2}\prod_{i=1}^{s}\left(\frac{\delta_i\alpha/p_i}{p_i}\right)\varepsilon_{p_i}. \end{aligned}$$

We now compute the Gauss sum for $p = 2$.

(3) If $D = b^2 - 4ac \equiv 1 \pmod 4$, and $a \equiv c \equiv 1 \pmod 2$, then

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

over $\mathbb{Z}_2$. Therefore

$$G(n, 2^m) = (-1)^m 2^m \quad \text{for any odd positive integer } n.$$

(4) If $D \equiv 1 \pmod 4$, $ac \equiv 0 \pmod 2$, then

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

over $\mathbb{Z}_2$. Therefore

$$G(n, 2^m) = 2^m, \quad \text{for any odd positive integer } n.$$

(5) If $D \equiv 0 \pmod 4$, then $2|b$. Denote $b = 2b'$. It is clear that $2 \nmid a$ or $2 \nmid c$ since $(a, b, c) = 1$. We assume that $2 \nmid a$. Hence

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} = 2\begin{pmatrix} a & b' \\ b' & c \end{pmatrix} \sim 2\begin{pmatrix} a & 0 \\ 0 & a^{-1}\frac{D}{4} \end{pmatrix}$$

over $\mathbb{Z}_2$. Therefore we have

$$G(n, 2^m) = G_{1,m}(na, 2^m)G_{1,m-t}(n\beta, 2^m),$$

where $t = \nu_2(c - a^{-1}b'^2) = \nu_2(D/4)$, $\beta = (c - a^{-1}b'^2)2^{-t} = a^{-1}\dfrac{D}{2^{2+t}}$, and we think

$$G_{1,m-t}(n\beta, 2^m) = 2^m$$

for any $m \leqslant t$. In particular, we know that

$$G(n, 2) = G(n, 2^{t+1}) = 0.$$

Since D is a fundamental discriminant, $t = \nu_2(D/4) = 0$ or 1 according to $D \equiv 12$ or $8 \pmod{16}$ respectively.

So for $\alpha = 2^m|D$, we have

$$\begin{aligned} V(\theta_f, 1/2^m) &= -\mathrm{i}(\det A)^{-1/2}2^{-m}G(1, 2^m) \\ &= -(D)^{-1/2}2^{-m}G_{1,m}(a, 2^m)G_{1,m-t}(\beta, 2^m). \end{aligned}$$

In particular

$$V(\theta_f, 1/\alpha) = 0$$

for any $\alpha = 2^m\alpha_1|D$ where $m = 1$ or $t + 1$, $2 \nmid \alpha_1$. For $\alpha = 2^m\alpha_1 = 2^m\prod_{i=1}^{s} p_i|D$ with $m \neq 1, t + 1$, we have

$$\begin{aligned} V(\theta_f, 1/\alpha) &= -\mathrm{i}(\det A)^{-1/2}\alpha^{-1}G(1, \alpha) \\ &= -(D)^{-1/2}\alpha^{-1}G(2^m, \alpha_1)G(\alpha_1, 2^m) \\ &= -(D)^{-1/2}\alpha^{-1}\alpha_1^{3/2}G_{1,m}(a\alpha_1, 2^m)G_{1,m-t}(\alpha_1\beta, 2^m)\prod_{i=1}^{s}\left(\frac{\delta_i\alpha/p_i}{p_i}\right)\varepsilon_{p_i} \\ &= -(\alpha/D)^{1/2}2^{-3m/2}G_{1,m}(a\alpha_1, 2^m)G_{1,m-t}(\alpha_1\beta, 2^m)\prod_{i=1}^{s}\left(\frac{\delta_i\alpha/p_i}{p_i}\right)\varepsilon_{p_i}. \end{aligned}$$

□

Remark 10.1 If D is an odd fundamental discriminant, our result is just Lemma IV(2.3) in B.H.Gross, D.B.Zagier 1986. If D is even, our result is just Proposition 2 in I. Kiming, 1995.

Example 10.2 Let $f = f(x_1, \cdots, x_k)$ be a positive definite quadratic form with k odd. Suppose that the level of f is $4D$ with D square free odd integer. Let

$D = p_1 p_2 \cdots p_t$. Since D is square free, there exists an invertible matrix S_i over $\mathbb{Z}_{p_i}$ such that

$$S_i A S_i^{\mathrm{T}} = \mathrm{diag}\{\alpha_{i,1}, \alpha_{i,2}, \cdots, \alpha_{i,s_i}, \alpha_{i,s_i+1} p_i, \cdots, \alpha_{i,k} p_i\}$$

with $\alpha_{i,j} \in \mathbb{Z}_{p_i}^*$. Hence

$$G(n, p_i) = p_i^{k-s_i} \prod_{g=1}^{s_i} \left(\frac{n\alpha'_{i,g}}{p_i} \right) \varepsilon_{p_i} p_i^{1/2} = p_i^{k-\frac{s_i}{2}} \varepsilon_{p_i}^{s_i} \left(\frac{n^{s_i} A_i}{p_i} \right),$$

where $A_i = \prod_{g=1}^{s_i} \alpha'_{i,g}$ and $\alpha'_{i,g} \equiv 2^{-1}\alpha_{i,g} \pmod{p_i}$. Therefore for any $\alpha = \prod_{i=1}^{t} p_i^{\delta_i} | D, \delta_i = 0$ or 1, we can evaluate

$$G(1, \alpha) = \prod_{i=1}^{t} G(\alpha/p_i^{\delta_i}, p_i^{\delta_i}) = \prod_{i=1}^{t} \left(p_i^{k-\frac{s_i}{2}} \varepsilon_{p_i}^{s_i} \left(\frac{n^{s_i} A_i}{p_i} \right) \right)^{\delta_i}.$$

Since $4D$ is the level of f and D square free, there exists an invertible matrix S over $\mathbb{Z}_2$ such that

$$SAS^{\mathrm{T}} = \bigoplus_{i=1}^{l} \alpha_i 2^{a_i} \bigoplus_{j=1}^{l_1} \beta_j 2^{t_j} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bigoplus_{s=1}^{l_2} \gamma_s 2^{u_s} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Since k is odd, α_i appears at least one time and $s_i = 1, t_j, u_s \leqslant 2$. Hence we have

$$\begin{aligned} G(n, 2) &= 0, \\ G(n, 4) &= 2^{4a} \prod_{i=1}^{l} G_{1,2}(n\alpha_i, 4) \prod_{t_j<2} 2^{2+t_j} \prod_{u_s<2} (-1)^{u_s} 2^{2+u_s} \\ &= (-1)^e 2^{2a+l+2b+2c+d+e} \prod_{i=1}^{l} (1 + i^{n\alpha_i}), \end{aligned}$$

where $a = \#\{t_j, u_s | t_j = u_s = 2\}, b = \#\{t_j | t_j < 2\}$, $c = \#\{u_s | u_s < 2\}$, $d = \sum_{t_j<2} t_j$, $e = \sum_{u_s<2} u_s$. From the above calculation we obtain the value

$$V(\theta_f, 1/\alpha) = (-\mathrm{i})^{k/2} (\det A)^{-1/2} \alpha^{-k/2} G(1, \alpha)$$

for any $\alpha | 4D$. In particular we know that $V(\theta_f, 1/2\beta) = 0$ for any $\beta | D$ since $G(n, 2) = 0$ for any odd integer n. □

10.2 The Minimal Integer Represented by a Positive Definite Quadratic Form

We consider the following problem: for a given positive definite quadratic form f, find an upper bound on the size for the minimal positive integer represented by f.

We first consider the case that the level of f is equal to 1. Let

$$E_k(z) = \frac{1}{2}\sum_{l,m}{}' \frac{1}{(lz+m)^k}, \quad k = 4, 6, 8, \cdots, \tag{10.3}$$

where (l, m) run over all pairs of integers except $(0, 0)$. By Section 7.5 we know that

$$E_k(z) = \zeta(k) + \frac{(2\pi \mathrm{i})^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n, \tag{10.4}$$

where

$$\sigma_g(n) = \sum_{d|n} d^g.$$

In view of

$$\zeta(k) = -\frac{(2\pi \mathrm{i})^k B_k}{2(k)!}, \tag{10.5}$$

$E_k(z)$ can be expressed by the formulae

$$E_k(z) = \zeta(k)G_k(z), G_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n, \quad k = 4, 6, 8, \cdots. \tag{10.6}$$

In particular, we have the Bernoulli numbers:

$$B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{14} = \frac{7}{6},$$

and hence

$$\begin{cases} G_4(z) = 1 + 240\sum \sigma_3(n)q^n, \\ G_6(z) = 1 - 504\sum \sigma_5(n)q^n, \\ G_8(n) = 1 + 480\sum \sigma_7(n)q^n, \\ G_{10}(z) = 1 - 264\sum \sigma_9(n)q^n, \\ G_{14}(z) = 1 - 24\sum \sigma_{13}(n)q^n \end{cases} \tag{10.7}$$

with integral coefficients and constant 1. By the dimension formula we see that the dimension r_h of the linear space of modular forms of weight h is equal to $\left[\frac{h}{12}\right] + 1$ or $\left[\frac{h}{12}\right]$ according to $h \not\equiv 2 \pmod{12}$ or $h \equiv 2 \pmod{12}$ respectively. In particular we have

$$G_4^2 = G_8, \quad G_4G_6 = G_{10}, \quad G_4^2G_6 = G_{14}, \quad G_lG_{14-l} = G_{14}, \\ l = h - 12r_h + 12 = 0, 4, 6, 8, 10, 14 \tag{10.8}$$

and for the modular form

$$\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

of weight 12,

$$1728\Delta = G_4^3 - G_6^2.$$

Let

$$j(z) = G_4^3/\Delta(z) = q^{-1} + \cdots \tag{10.9}$$

be the absolute invariant, then

$$\begin{aligned}\Delta^2\frac{\mathrm{d}j}{\mathrm{d}z} &= 3G_4^2\frac{\mathrm{d}G_4}{\mathrm{d}z}\Delta - G_4^3\frac{\mathrm{d}\Delta}{\mathrm{d}z}\\ &= \frac{1}{1728}G_4^2G_6\left(2G_4\frac{\mathrm{d}G_6}{\mathrm{d}z} - 3G_6\frac{\mathrm{d}G_4}{\mathrm{d}z}\right)\end{aligned}$$

and the expression in the brackets is a modular form of weight 12 and indeed a cusp form which can therefore differ from Δ at most by a constant factor. Comparing the coefficients of q in the Fourier expansions, we get

$$\frac{\mathrm{d}j}{\mathrm{d}\log q} = -G_{14}\Delta^{-1}. \tag{10.10}$$

Let hereafter, $h > 2$, and hence $r_h > 0$. The power-products $G_4^aG_6^b$, where the exponents a, b run over all non-negative rational integer solutions of

$$4a + 6b = h$$

form a basis of the space $G(1, h, \mathrm{id.}) := G(h)$. It follows from this that, for every function $M \in G(h)$, $MG_{h-12r+12}^{-1}$ always belongs to $G(12r-12)$. Since Δ^{r-1} is a modular form of weight $12r-12$, not vanishing anywhere in the interior of the upper half-plane,

$$MG_{h-12r+12}^{-1}\Delta^{1-r} := w(f) := w, \tag{10.11}$$

is an entire modular function and hence a polynomial in j with constant coefficients.

Let

$$T_h = G_{12r-h+2}\Delta^{-r} \tag{10.12}$$

with Fourier expansion

$$T_h = c_{hr}q^{-r} + \cdots + c_{h1}q^{-1} + c_{h0} + \cdots \tag{10.13}$$

and first coefficient $c_{hr} = 1$. Since

$$\Delta^{-1} = q^{-1} \prod_{n=1}^{\infty} (1 + q^n + q^{2n} + \cdots)^{24}, \tag{10.14}$$

all the Fourier coefficients of T_h turn out to be rational integers.

Theorem 10.4 *Let*

$$M = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots \tag{10.15}$$

be the Fourier *series of a modular form M of weight h. Then*

$$c_{h0}a_0 + c_{h1}a_1 + \cdots + c_{hr}a_r = 0.$$

Proof For $l = 0, 1, 2, \cdots$, we have

$$j^l \frac{\mathrm{d}j}{\mathrm{d}z} = \frac{1}{l+1} \frac{\mathrm{d}j^{l+1}}{\mathrm{d}z},$$

and hence, by (10.9), it has a Fourier series without constant term. Since the function w defined by (10.11) is a polynomial in j, the product $w\frac{\mathrm{d}j}{\mathrm{d}z}$ has also a Fourier series without constant term. Because of (10.8) and (10.10), we have

$$-\frac{1}{2\pi i} w \frac{\mathrm{d}j}{\mathrm{d}z} = MG^{-1}_{h-12r+12}\Delta^{1-r}G_{14}\Delta^{-1} = MG_{12r-h+2}\Delta^{-r} = MT_h$$

from which the theorem follows on substituting the series (10.13) and (10.15) for T_h and M respectively. □

Put $c_{h0} := c_h$ for brevity. We have the following:

Theorem 10.5 *We have $c_h \neq 0$.*

Proof First, consider the case $h \equiv 2 \pmod 4$. So that $h \equiv 2t \pmod{12}$ with $t = 1, 3, 5$. Then correspondingly $12r = h - 2$, $h + 6$, $h + 2$, hence $12r - h + 2 = 0$, 8, 4 and

$$G_{12r-h+2} = G_0, G_4^2, G_4.$$

Since by (10.7), G_4 has all its Fourier coefficients positive and the same holds for Δ^{-r} as a consequence of (10.14). We conclude from (10.12) that all the coefficients in the expansion (10.13) are positive. Therefore the integers $c_{h0}, c_{h1}, \cdots, c_{hr}$ are all positive and in particular, $c_h = c_{h0} > 0$, i.e., $c_h \neq 0$.

Let now $h \equiv 0 \pmod 4$, so that $h \equiv 4t \pmod{12}$ with $t = 0, 1, 2$ whence $12r = h - 4t + 12$, $h - 12r + 12 = 4t$ and

$$g_{h-12r+12} = G_{4t} = G_4^t.$$

Furthermore we have now

$$\begin{aligned} T_h &= -G_{12r-h+2}\Delta^{1-r}G_{14}^{-1}\frac{\mathrm{d}j}{\mathrm{d}\log q} \\ &= -G_4^{-t}\Delta^{1-r}\frac{\mathrm{d}j}{\mathrm{d}\log q} = \frac{3}{t-3}\Delta^{1-r-t/3}\frac{\mathrm{d}j^{1-t/3}}{\mathrm{d}\log q} \\ &= \frac{3}{t-3}\frac{\mathrm{d}(G_4^{3-t}\Delta^{-r})}{\mathrm{d}\log q} + \frac{3r+t-3}{(3-t)r}G_4^{3-t}\frac{\mathrm{d}\Delta^{-r}}{\mathrm{d}\log q}; \end{aligned}$$

hence c_{h0} is also the constant term in the Fourier expansion of the function

$$V_h = \frac{3r+t-3}{(3-t)r}G_4^{3-t}\frac{\mathrm{d}\Delta^{-r}}{\mathrm{d}\log q}.$$

Because of the assumption $h > 2$, we see that $3r+t-3 > 0$. The series for G_4^{3-t} begins with 1 and has again all its coefficients positive. Furthermore, by (10.14), the coefficients of the negative powers $q^{-1}, \cdots, q^{-r}$ of q in the derivative of Δ^{-r} with respect to $\log q$ are all negative while the constant term is absent. Hence the constant term in V_h is negative and $c_h = c_{h0} < 0$, i.e., $c_h \neq 0$. This completes the proof. □

A most important consequence of Theorem 10.4 and Theorem 10.5 is the fact that, for every modular form M of weight h and level 1, the constant term a_0 in its Fourier expansion is determined by the r Fourier coefficients $a_1, \cdots, a_r$, which comes out of the formula

$$a_0 = c_h^{-1}(c_{h1}a_1 + \cdots + c_{hr}a_r). \tag{10.16}$$

If, in particular, $a_0 \neq 0$, then there must be some i $(1 \leqslant i \leqslant r)$ such that $a_i \neq 0$. In particular, if taking the theta function of a positive definite even unimodular quadratic form Q in $2h$ variables as our M, we have that $a_0 = 1 \neq 0$, and hence conclude that Q represents a positive integer $n \leqslant r_h$(Please compare [?]).

We now want to extend Siegel's results above to the case with level 2.

Let $G(2,h)$ be the vector space of holomorphic modular forms of weight h for $\Gamma_0(2)$, $r = r(2,h) := \dim(G(2,h))$. Then by the dimension formula we see that $r(2,h) = 1 + \left[\frac{h}{4}\right]$ for any even nonnegative number h.

We introduce some analogues of the above function T_h. In order to do this, we need some more Eisenstein series.

Put

$$\sigma_k^{\mathrm{odd}}(n) := \sum_{\substack{0<d|n \\ 2\nmid d}} d^k, \quad \sigma_k^{\mathrm{alt}}(n) := \sum_{0<d|n} (-1)^d d^k, \quad \sigma_{N,k}^*(n) := \sum_{\substack{0<d|n \\ N\nmid(n/d)}} d^k.$$

Since $r(2,2) = \left[\frac{2}{4}\right] + 1 = 1$, let $E_{\infty,2}$ be the unique normalized modular form (in fact, the Eisenstein series) in $G(2,2)$ defined by

$$E_{\infty,2}(z) := 1 + 24\sum_{n=1}^{\infty}\sigma_1^{\text{odd}}(n)q^n.$$

Since $r(2,4) = \left[\frac{4}{4}\right] + 1 = 2$, the vector space $G(2,4)$ is spanned by two Eisenstein series $E_{0,4}(z)$ and $E_{\infty,4}(z)$ with respect to the cusp points 0 and ∞ respectively. They have Fourier expansions:

$$E_{0,4} = 1 + 16\sum_{n=1}^{\infty}\sigma_3^{\text{alt}}(n)q^n, \quad E_{\infty,4} = \sum_{n=1}^{\infty}\sigma_{2,3}^{*}(n)q^n.$$

In fact, in terms of the results in Section 7.5, we can easily see that all the functions $E_{\infty,2}(z)$ and $E_{0,4}(z), E_{\infty,4}(z)$ are in $\mathcal{E}(2,2,\text{id.})$ and $\mathcal{E}(2,4,\text{id.})$ respectively.

We also denote by $j_2 = j_2(z)$ the following modular function for $\Gamma_0(2)$:

$$j_2(z) := E_{\infty,2}^2 E_{\infty,4}^{-1},$$

which is a level two analogue of $j(z)$ for $\Gamma_0(1)$. Finally, we introduce analogues of the T_h:

$$T_{2,h} := E_{\infty,2}E_{0,4}E_{\infty,4}^{-r} \text{ if } r = r(2,h) \equiv 0 (\text{mod } 4),$$
$$T_{2,h} := E_{\infty,2}^2 E_{0,4}E_{\infty,4}^{-1-r} \text{ if } r = r(2,h) \equiv 2 (\text{mod } 4).$$

We need the following:

Lemma 10.2 *The function j_2 is a modular function for $\Gamma_0(2)$. It is holomorphic on $\mathbb{H}$ with a simple pole at infinity and defines a bijection of $\mathbb{H}/\Gamma_0(2)$ onto $\mathbb{C}$ by passage to the quotient.*

Proof The first two conclusions are clear. Let $S : z \to -1/z$ and $T : z \to z+1$ be two linear fractional transformations. Let

$$F = \{z \in \mathbb{H} \mid |z| > 1, |\text{Re}(z)| < 1/2\}$$

be the fundamental domain of $\Gamma_0(1)$. Denote by V the closure of $F \bigcup S(F) \bigcup ST(F)$, and put $F_2 = V \bigcup \{\text{i}\infty\}$. Then F_2 is a fundamental domain for $\Gamma_0(2)$ which has two $\Gamma_0(2)$-inequivalent cusp points: zero and $\text{i}\infty$. The only non-cusp in F_2 fixed by a map in $\Gamma_0(2)$ is $\gamma = -\frac{1}{2} + \frac{1}{2}\text{i}$. The number of zeros in a fundamental domain of a non zero function in $G(2,h)$ is $h/4$. Now let $f_\lambda = E_{\infty,2}^2 - \lambda E_{\infty,4}$ for any $\lambda \in \mathbb{C}$. Then $f_\lambda \in G(2,4)$. The sum of its zero orders in a fundamental domain is 1. If f_λ has multiple zeros in a fundamental domain, there must be exactly two of them in the equivalence class of γ, or exactly three in the one of $\rho = \text{e}^{2\pi\text{i}/3}$. This completes the proof. □

Lemma 10.3 *Let f be a meromorphic function on $\mathbb{H}^*$. Then the following statements are equivalent*:

(1) *f is a modular function for $\Gamma_0(2)$*;

(2) *f is a quotient of two modular forms for $\Gamma_0(2)$ of equal weight*;

(3) *f is a rational function of j_2.*

Proof It is clear that (3) $\Rightarrow$ (2) $\Rightarrow$ (1). for $z \in \mathbb{H}^*$, denote by $[z]$ the equivalence class of z in $\mathbb{H}/\Gamma_0(2)$. By an abuse of the notation we may take f as in (1) as a function from $\mathbb{H}^*/\Gamma_0(2)$ to $\widehat{\mathbb{C}}$. The function j_2, also regarded in this fashion, is invertible. Let $\widetilde{f} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ satisfy $\widetilde{f} = f \circ j_2^{-1}$. Then $\widetilde{f}$ is meromorphic on $\widehat{\mathbb{C}}$, so that it is rational. If $z \in \widehat{\mathbb{C}}$, let $u = j_2^{-1}(z) \in \mathbb{H}^*/\Gamma_0(2)$. Then $f(u) = f(j_2^{-1}(z)) = \widetilde{f}(z) = \widetilde{f}(j_2(z))$. Thus f is a rational function in j_2. □

Lemma 10.4 *For $z \in \mathbb{H}$, we have that*

$$\frac{\mathrm{d}}{\mathrm{d}z} j_2(z) = -2\pi \mathrm{i} E_{\infty,2}(z) E_{0,4}(z) E_{\infty,4}^{-1}(z).$$

Proof It is clear from the definition of a modular function that the derivative of a modular function has weight two. Therefore both sides of the equality in the lemma are meromorphic modular forms of weight 2 for $\Gamma_0(2)$. The only poles of either functions lie at infinity. On both sides, the principal parts of the Fourier expansions at infinity consist only of the term $-2\pi \mathrm{i} q^{-1}$. Hence the modular form

$$\alpha := \frac{\mathrm{d}}{\mathrm{d}z} j_2(z) + 2\pi \mathrm{i} E_{\infty,2}(z) E_{0,4}(z) E_{\infty,4}^{-1}(z)$$

is holomorphic with weight two. For a non zero modular form in $G(2, h)$, the number of zeros in a fundamental domain is $h/4$, we can easily check that the exponent of the first nonzero Fourier coefficient in the expansion of α exceeds $h/4 = 1/2$. This exponent counts the number of zeros at $\mathrm{i}\infty$. Hence $\alpha = 0$ and the lemma holds. □

We now introduce an analogue of the map w in (10.11).

For $h \equiv 0 \pmod 4$ and $f \in G(2, h)$, let

$$W_2(f) = f E_{\infty,4}^{-h/4}.$$

For $h \equiv 2 \pmod 4$ and $f \in G(2, h)$, let

$$W_2(f) = f E_{\infty,2} E_{\infty,4}^{-(h+2)/4}.$$

Lemma 10.5 *Let h be an even positive integer. Then*

(1) *the restriction of W_2 to $G(2, h)$ is an isomorphism from the vector space $G(2, h)$ to the vector space of polynomials in j_2 of degree less than $r = r(2, h)$ or of degree between 1 and r inclusive according to $r \equiv 0 \pmod 4$ or $h \equiv 2 \pmod 4$ respectively.*

(2) *for any $f \in G(2, h)$, the constant term in the Fourier expansion at infinity of $f T_{2,h}$ is zero.*

Proof (1) Suppose $h \equiv 0 \pmod 4$ and $f \in G(2,h)$, then

$$W_2(f) = fE_{\infty,4}^{-h/4} = fE_{\infty,4}^{1-r}.$$

For $d = 0, 1, 2, \cdots, r-1$, the products $j_2^d E_{\infty,4}^{r-1}$ belong to $G(2,h)$. We have $W_2(j_2^d E_{\infty,4}^{r-1}) = j_2^d$. Let V be the subspace of $G(2,h)$ generated by the modular forms $j_2^d E_{\infty,4}^{r-1}$ for $d = 0, 1, 2, \cdots, r-1$. And denote by V_1 the space of polynomials in j_2 of degree at most $r-1$. W_2 carries V isomorphically onto V_1. Hence $\dim(V) = r$ which implies that $V = G(2,h)$. Now let $h \equiv 2 \pmod 4$. Then

$$W_2(f) = fE_{\infty,2}E_{\infty,4}^{-r}.$$

For $d = 0, 1, 2, \cdots, r-1$, the products $j_2^d E_{\infty,2}E_{\infty,4}^{r-1}$ belong to $G(2,h)$ and

$$W_2(j_2^d E_{\infty,2}E_{\infty,4}^{r-1}) = j_2^{d+1}.$$

W_2 carries $E_{\infty,2}V$ isomorphically onto j_2V_1. Therefore $\dim(E_{\infty,2}V) = r$. Hence $E_{\infty,2}V = G(2,h)$.

(2) Suppose $h \equiv 0 \pmod 4$. Then

$$W_2(f)\frac{\mathrm{d}j_2}{\mathrm{d}z} = -fE_{\infty,4}^{1-r}2\pi\mathrm{i}E_{\infty,2}E_{0,4}E_{\infty,4}^{-1} = -2\pi\mathrm{i}fT_{2,h}.$$

We can obtain the same result for $h \equiv 2 \pmod 4$ by a similar computation. Thus $fT_{2,h}$ is the derivative of a polynomial in j_2, so it can be expressed in a neighborhood of infinity as the derivative with respect to z of a power series in the variable $q = \mathrm{e}^{2\pi\mathrm{i}z}$. This derivative is a power series in q with vanishing constant term. This completes the proof. □

Lemma 10.6 (1)

$$E_{\infty,4}(z) = q\prod_{0<n\in 2\mathbb{Z}}(1-q^n)^8\prod_{0<n\in\mathbb{Z}\backslash 2\mathbb{Z}}(1-q^n)^{-8};$$

(2) *For a given set A and a given arithmetical function f, the number $p_{A,f}(n)$ defined by the equation*

$$\prod_{n\in A}(1-x^n)^{-f(n)/n} = 1 + \sum_{n=1}^{\infty}p_{A,f}(n)x^n$$

satisfies the recursion formula

$$np_{A,f}(n) = \sum_{k=1}^{n}f_A(k)p_{A,f}(n-k),$$

where $p_{A,f}(0) = 1$ and $f_A(k) = \sum_{d|k,d\in A}f(d)$.

Proof (1) This is equivalent to show that

$$E_{\infty,4}(z) = \eta(2z)^{16}\eta(z)^{-8}.$$

Denote by $f(z)$ the right hand side of the above. The function f is holomorphic on $\mathbb{H}$ because η is non-vanishing on $\mathbb{H}$. We see that f has the product expansion

$$f(z) = q \prod_{0<n\in 2\mathbb{Z}} (1-q^n)^8 \prod_{0<n\in \mathbb{Z}\setminus 2\mathbb{Z}} (1-q^n)^{-8}$$

from the product expansion of η. It follows that f has a simple zero at infinity. The number of zeros in a $\Gamma_0(2)$ fundamental domain for a modular form in $G(2,4)$ is one. But from the transformation formula of the η function we know easily that f is in $G(2,4)$. This shows that f and $E_{\infty,4}$ are monic modular forms with the same weight, level and divisor (both equal to $1\cdot \mathrm{i}\infty$), hence identical.

(2) By induction. □

Theorem 10.6 *For any even positive integer h, the constant term in the Fourier expansion at infinity of $T_{2,h}$ is non zero.*

Proof Let $h \equiv 0 \pmod 4$. Put $u = 2\pi \mathrm{i} z = \log q$. Write D for the operator $\frac{\mathrm{d}}{\mathrm{d}u}$. It is clear that $D(q^n) = nq^n$. Put $m_2 = j_2 - 64$. It is easy to see that $E_{\infty,2}^2 = E_{0,4} + 64E_{\infty,4}$. So that $m_2 = E_{0,4}E_{\infty,4}^{-1}$. Thus

$$\frac{\mathrm{d}m_2}{\mathrm{d}z} = \frac{\mathrm{d}j_2}{\mathrm{d}z} = -2\pi\mathrm{i}E_{\infty,2}E_{0,4}E_{\infty,4}^{-1}$$

and $D(m_2) = -E_{\infty,2}E_{0,4}E_{\infty,4}^{-1}$. It follows that

$$T_{2,h} = -E_{\infty,4}^{1-r}D(m_2).$$

Therefore

$$\begin{aligned}
E_{\infty,4}^{1-r}D(m_2) &= D(E_{\infty,4}^{1-r}m_2) - m_2D(E_{\infty,4}^{1-r}) \\
&= D(E_{\infty,4}^{1-r}m_2) - m_2(1-r)E_{\infty,4}^{-r}D(E_{\infty,4}) \\
&= D(E_{\infty,4}^{1-r}m_2) + (r-1)m_2E_{\infty,4}^{-r}\left(-\frac{1}{r}E_{\infty,4}^{1+r}D(E_{\infty,4}^{-r})\right) \\
&= D(E_{\infty,4}^{1-r}m_2) + \frac{1-r}{r}m_2E_{\infty,4}D(E_{\infty,4}^{-r}) \\
&= D(E_{\infty,4}^{1-r}m_2) + \frac{1-r}{r}E_{0,4}D(E_{\infty,4}^{-r}).
\end{aligned}$$

The term $D(E_{\infty,4}^{1-r}m_2)$ makes no contribution to the constant term. Hence the constant term of $T_{2,h}$ is equal to that of $\frac{r-1}{r}E_{0,4}D(E_{\infty,4}^{-r})$. We now compute the principal part of $D(E_{\infty,4}^{-r})$.

By Lemma 10.6, for fixed s, if we write

$$E_{\infty,4}^{-s} = q^{-s}\sum_{n=0}^{\infty} R(n)q^n,$$

then $R(0) = 1$ and

$$R(n) = \frac{8s}{n}\sum_{a=1}^{n} \sigma_1^{\mathrm{alt}}(a)R(n-a), \quad \forall\, n > 0. \tag{10.17}$$

Because $\sigma_1^{\mathrm{alt}}(a)$ alternates sign, the alternation of the sign of $R(n)$ follows by an induction from (10.16). So we can write $R(n) = U_n(-1)^n$ with some $U_n > 0$. Therefore we have

$$E_{\infty,4}^{-r} = U_0(-1)^0q^{-r} + U_1(-1)^1q^{-r+1} + \cdots + U_{r-1}(-1)^{r-1}q^{-1} + 0 + \cdots ,$$

hence

$$\begin{aligned} D(E_{\infty,4}^{-r}) &= -rU_0(-1)^0q^{-r} + (1-r)U_1(-1)^1q^{1-r} \\ &\quad + \cdots + (-1)U_{r-1}(-1)^{r-1}q^{-1} + 0 + \cdots \\ &= V_r(-1)^1q^{-r} + V_{r-1}(-1)^2q^{1-r} + \cdots + V_1(-1)^rq^{-1} + 0 + \cdots , \end{aligned}$$

where $V_i = iU_{r-i} > 0$ for $1 \leqslant i \leqslant r$. On the other hand, the Fourier coefficient of $q^n (n \geqslant 0)$ in the expansion of $E_{0,4}$ is $W_n(-1)^n$ for positive W_n, by the definition of $E_{0,4}$. Therefore the constant term of $E_{0,4}D(E_{\infty,4}^{-r})$ is equal to

$$\sum_{n=1}^{r} V_n(-1)^{r+1-n}W_n(-1)^n = (-1)^{r+1}\sum_{n=1}^{r} V_nW_n,$$

so that the constant term of $T_{2,h}$ is equal to

$$\frac{r-1}{r}(-1)^{r+1}\sum_{n=1}^{r} V_nW_n \neq 0$$

for $h \geqslant 4, h \equiv 0(4)$ (since $r > 1$ in this case).

Now we assume that $h \equiv 2 \pmod 4$. We have proved the following equality above

$$\frac{\mathrm{d}}{\mathrm{d}z}m_2(z) = \frac{\mathrm{d}}{\mathrm{d}z}j_2(z) = -2\pi\mathrm{i}E_{\infty,2}(z)E_{0,4}(z)E_{\infty,4}^{-1}(z).$$

So $D(m_2(z)) = -E_{\infty,2}(z)E_{0,4}(z)E_{\infty,4}^{-1}(z)$. This implies that

$$T_{2,h} = E_{\infty,2}^2E_{0,4}E_{\infty,4}^{-1-r} = -E_{\infty,2}E_{\infty,4}^{-r}D(m_2).$$

Therefore

$$
\begin{aligned}
&E_{\infty,2}E_{\infty,4}^{-r}D(m_2)\\
&= D(E_{\infty,2}E_{\infty,4}^{-r}m_2) - E_{\infty,2}m_2D(E_{\infty,4}^{-r}) - E_{\infty,4}^{-r}m_2D(E_{\infty,2})\\
&= D(E_{\infty,2}E_{\infty,4}^{-r}m_2) - E_{\infty,2}m_2(-r)E_{\infty,4}^{-r-1}D(E_{\infty,4}) - E_{0,4}E_{\infty,4}^{-r-1}D(E_{\infty,2})\\
&= D(E_{\infty,2}E_{\infty,4}^{-r}m_2) - E_{\infty,2}m_2(-r)E_{\infty,4}^{-r-1}\left(\frac{1}{-r-1}\right)E_{\infty,4}^{r+2}D(E_{\infty,4}^{-r-1})\\
&\quad -E_{0,4}E_{\infty,4}^{-r-1}D(E_{\infty,2})\\
&= D(E_{\infty,2}E_{\infty,4}^{-r}m_2) - \frac{r}{r+1}E_{\infty,2}m_2E_{\infty,4}^{-r-1}E_{\infty,4}^{r+2}D(E_{\infty,4}^{-r-1}) - E_{0,4}E_{\infty,4}^{-r-1}D(E_{\infty,2})\\
&= D(E_{\infty,2}E_{\infty,4}^{-r}m_2) - \frac{r}{r+1}E_{0,4}E_{\infty,2}D(E_{\infty,4}^{-r-1}) - E_{0,4}E_{\infty,4}^{-r-1}D(E_{\infty,2})\\
&= D(E_{\infty,2}E_{\infty,4}^{-r}m_2) - \frac{r}{r+1}E_{0,4}\left(D(E_{\infty,2}E_{\infty,4}^{-r-1}) - E_{\infty,4}^{-r-1}D(E_{\infty,2})\right)\\
&\quad -E_{0,4}E_{\infty,4}^{-r-1}D(E_{\infty,2})\\
&= D(E_{\infty,2}E_{\infty,4}^{-r}m_2) - \frac{r}{r+1}E_{0,4}D(E_{\infty,2}E_{\infty,4}^{-r-1}) - \frac{1}{r+1}E_{0,4}E_{\infty,4}^{-r-1}D(E_{\infty,2}).
\end{aligned}
$$

The term $D(E_{\infty,2}E_{\infty,4}^{-r}m_2)$ makes no contribution to the constant term of $T_{2,h}$ because for any formal series $\sum_{n=0}^{\infty} b_nq^n$ we have that $D\left(\sum_{n=0}^{\infty} b_nq^n\right) = \sum_{n=0}^{\infty} nb_nq^n$ which has no constant term. Hence we only need to compute the constant terms of $\frac{r}{r+1}E_{0,4}$ $D(E_{\infty,2}E_{\infty,4}^{-r-1})$ and $\frac{1}{r+1}E_{0,4}E_{\infty,4}^{-r-1}D(E_{\infty,2})$.

For any positive integer s, we write

$$E_{\infty,4}^{-s} := q^{-s}\sum_{n=0}^{\infty} R_s(n)q^n.$$

Then by Lemma 10.6 and by an easy induction we can prove that $R_s(n) = (-1)^nU_s(n)$ with $U_s(n) > 0$.

But we know

$$E_{\infty,2}(z) := 1 + 24\sum_{n=1}^{\infty} \sigma_1^{\mathrm{odd}}(n)q^n.$$

Hence we have

$$
\begin{aligned}
E_{\infty,2}E_{\infty,4}^{-r-1} &:= q^{-r-1}\sum_{i=0}^{\infty} a_iq^i\\
&= \left(1 + 24\sum_{n=1}^{\infty}\sigma_1^{\mathrm{odd}}(n)q^n\right)\left(\sum_{n=0}^{\infty}(-1)^nU_{r+1}(n)q^n\right),
\end{aligned}
$$

where

$$a_i = 24\sum_{j=0}^{i} \sigma_1^{\text{odd}}(j)U_{r+1}(i-j)(-1)^{i-j}, \quad \sigma_1^{\text{odd}}(0) := \frac{1}{24}. \tag{10.18}$$

Hence

$$D(E_{\infty,2}E_{\infty,4}^{-r-1}) = q^{-r-1}\sum_{i=0}^{\infty}(i-r-1)a_i q^i.$$

Noting that the nth Fourier coefficient of

$$E_{0,4} = 1 + 16\sum_{n=1}^{\infty}\sigma_3^{\text{alt}}(n)q^n$$

has the form $(-1)^n W_n$ with $W_n = (-1)^n 16\sigma_3^{\text{alt}}(n)$ a positive integer, we see that

$$\begin{aligned} E_{0,4}D(E_{\infty,2}E_{\infty,4}^{-r-1}) &:= \sum_{n=-r-1}^{\infty} a'_n q^n \\ &= \left(\sum_{n=0}^{\infty}(-1)^n W_n q^n\right)\left(\sum_{i=0}^{\infty}(i-r-1)a_i q^{i-r-1}\right). \end{aligned}$$

In particular, we have

$$a'_0 = \sum_{i=0}^{r}(i-r-1)a_i(-1)^{r+1-i}W_{r+1-i}. \tag{10.19}$$

On the other hand, we have

$$E_{0,4}E_{\infty,4}^{-r-1} := \sum_{i=0}^{\infty} b_i q^{i-r-1} = \left(\sum_{n=0}^{\infty}(-1)^n W_n q^n\right)\left(\sum_{n=0}^{\infty}(-1)^n U_{r+1}(n)q^n\right),$$

where

$$b_i := \sum_{j=0}^{i}(-1)^i U_{r+1}(i-j)W_j \tag{10.20}$$

and

$$D(E_{\infty,2}) = 24\sum_{n=1}^{\infty} n\sigma_1^{\text{odd}}(n)q^n$$

Hence

$$E_{0,4}E_{\infty,4}^{-r-1}D(E_{\infty,2}) := \sum_{n=-r}^{\infty} b'_n q^n = \left(\sum_{i=0}^{\infty} b_i q^{i-r-1}\right)\left(24\sum_{n=1}^{\infty} n\sigma_1^{\text{odd}}(n)q^n\right).$$

In particular, we have

$$b'_0 = 24\sum_{i=0}^{r} b_i(r+1-i)\sigma_1^{\text{odd}}(r+1-i) \tag{10.21}$$

From (10.17)–(10.20) we see

$$\begin{aligned} a_0' &= \sum_{i=0}^{r}\sum_{j=0}^{i} 24\sigma_1^{\text{odd}}(j)U_{r+1}(i-j)(-1)^{r+1-j}(i-r-1)W_{r+1-i} \\ &= 24\sum_{i=0}^{r}(i-r-1)W_{r+1-i}\sum_{j=0}^{i}(-1)^{r+1-j}U_{r+1}(i-j)\sigma_1^{\text{odd}}(j), \\ b_0' &= 24\sum_{i=0}^{r}\sum_{j=0}^{i}U_{r+1}(i-j)W_j(-1)^i(r+1-i)\sigma_1^{\text{odd}}(r+1-i) \\ &= 24\sum_{i=0}^{r}(-1)^i(r+1-i)\sigma_1^{\text{odd}}(r+1-i)\sum_{j=0}^{i}U_{r+1}(i-j)W_j. \end{aligned}$$

Therefore the constant term of $T_{2,h}$ is equal to

$$\begin{aligned} &-\frac{r}{r+1}a_0' - \frac{1}{r+1}b_0' \\ &= -\frac{24r}{r+1}\sum_{i=0}^{r}(i-r-1)W_{r+1-i}\sum_{j=0}^{i}(-1)^{r+1-j}U_{r+1}(i-j)\sigma_1^{\text{odd}}(j) \\ &\quad -\frac{24}{r+1}\sum_{i=0}^{r}(-1)^i(r+1-i)\sigma_1^{\text{odd}}(r+1-i)\sum_{j=0}^{i}U_{r+1}(i-j)W_j \\ &= -\frac{24}{r+1}\left((-1)^r r\sum_{i=0}^{r}(r+1-i)W_{r+1-i}\sum_{j=0}^{i}(-1)^jU_{r+1}(i-j)\sigma_1^{\text{odd}}(j)\right. \\ &\quad \left.+\sum_{i=0}^{r}(-1)^i(r+1-i)\sigma_1^{\text{odd}}(r+1-i)\sum_{j=0}^{i}U_{r+1}(i-j)W_j\right) \\ &= -\frac{24}{r+1}\sum_{i=0}^{r}(r+1-i)((-1)^r rW_{r+1-i} + (-1)^i\sigma_1^{\text{odd}}(r+1-i)) \\ &\quad \times\sum_{j=0}^{i}((-1)^j\sigma_1^{\text{odd}}(j)+W_j). \end{aligned}$$

For any nonnegative even integer n, it is clear that $(-1)^n\sigma_1^{\text{odd}}(n)+W_n > 0$ because $\sigma_1^{\text{odd}}(n) > 0$ and $W_n > 0$ for any nonnegative integer n. For any odd integer n we have

$$\begin{aligned} (-1)^n\sigma_1^{\text{odd}}(n) + W_n &= -\sum_{\substack{0<d\mid n\\ 2\nmid d}} d - 16\sum_{0<d\mid n}(-1)^d d^3 \\ &= \sum_{\substack{0<d\mid n\\ 2\nmid d}}(16d^3-d) - \sum_{\substack{0<d\mid n\\ 2\mid d}}(16d^3) = \sum_{\substack{0<d\mid n\\ 2\nmid d}}(16d^3-d) > 0. \end{aligned}$$

And it is clear that $rW_{r+1-i}+(-1)^{r-i}\sigma_1^{\text{odd}}(r+1-i)>0$ if $r-i$ is even. If $r-i$ is odd, then $r+1-i$ is even, so we see that

$$\begin{aligned}
&rW_{r+1-i}+(-1)^{r-i}\sigma_1^{\text{odd}}(r+1-i)\\
&=16r\sum_{0<d\mid r+1-i}(-1)^d d^3-\sum_{\substack{0<d\mid n\\ 2\nmid d}} d\\
&=16r\sum_{\substack{0<d\mid r+1-i\\ 2\mid d}} d^3-\sum_{\substack{0<d\mid n\\ 2\nmid d}}(16d^3+d)\geqslant r\sum_{i=1}^{t}16(2d_i)^3-\sum_{i=1}^{t}(16d_i^3+d_i)\\
&=\sum_{i=1}^{t}(128rd_i^2-16d_i^2-1)d_i>0\quad\text{for all } r\geqslant 1,
\end{aligned}$$

where d_i with $1\leqslant i\leqslant t$ are all distinct odd divisors of $r+1-i$.

This shows that

$$\begin{aligned}
-\frac{r}{r+1}a_0'-\frac{1}{r+1}b_0'&=-\frac{24}{r+1}\sum_{i=0}^{r}(r+1-i)((-1)^r rW_{r+1-i}+(-1)^i\sigma_1^{\text{odd}}(r+1-i))\\
&\quad\times\sum_{j=0}^{i}U_{r+1}(i-j)((-1)^j\sigma_1^{\text{odd}}(j)+W_j)\\
&=(-1)^{r+1}\frac{24}{r+1}\sum_{i=0}^{r}X_i\sum_{j=0}^{i}Y_j,
\end{aligned}$$

where

$$X_i:=(r+1-i)(rW_{r+1-i}+(-1)^{r-i}\sigma_1^{\text{odd}}(r+1-i))>0$$

and

$$Y_j:=U_{r+1}(i-j)((-1)^j\sigma_1^{\text{odd}}(j)+W_j)>0,$$

This proves that the constant term of $T_{2,h}$ is

$$\frac{r}{r+1}a_0'+\frac{1}{r+1}b_0'\neq 0$$

for any positive integer r. This completes the proof. □

Theorem 10.7 *Suppose $f\in G(2,h)$ with* Fourier *expansion at infinity*

$$f(z)=\sum_{n=0}^{\infty}A_nq^n\quad\textit{with } A_0\neq 0.$$

If $h\equiv 0 \pmod 4$, then there is some $A_n\neq 0$ for $1\leqslant n\leqslant r(2,h)$. If $h\equiv 2 \pmod 4$, then there is some $A_n\neq 0$ for $1\leqslant n\leqslant 1+r(2,h)$.

Proof First suppose that $h \equiv 0 \pmod 4$. We denote the coefficient of q^n in the Fourier coefficient of any modular form g at infinity as $c_n(g)$. The meromorphic modular form $T_{2,h}$ has a Fourier expansion

$$T_{2,h} = \sum_{n=-r}^{\infty} c_n(T_{2,h}) q^n$$

with $c_{-r}(T_{2,h}) = 1$. By the part (2) of Lemma 10.5, we see that

$$0 = c_0(T_{2,h} f) = \sum_{i=0}^{r} c_{-i}(T_{2,h}) A_i.$$

By hypothesis, $A_0 \neq 0$. By Theorem 10.6, $c_0(T_{2,h}) \neq 0$, so

$$A_0 = -(c_0(T_{2,h}))^{-1} \sum_{i=1}^{r} c_{-i}(T_{2,h}) A_i,$$

which implies that there exists an n with $1 \leqslant n \leqslant r$ such that $A_n \neq 0$.

If $h \equiv 2 \pmod 4$, then

$$T_{2,h} = \sum_{n=-r-1}^{\infty} c_n(T_{2,h}) q^n$$

with $c_{-r-1}(T_{2,h}) = 1$. By the part (2) of Lemma 10.5, we see that

$$0 = c_0(T_{2,h} f) = \sum_{i=0}^{r+1} c_{-i}(T_{2,h}) A_i.$$

By hypothesis, $A_0 \neq 0$. By Theorem 10.6, $c_0(T_{2,h}) \neq 0$, so that

$$A_0 = -(c_0(T_{2,h}))^{-1} \sum_{i=1}^{r+1} c_{-i}(T_{2,h}) A_i,$$

which implies that there exists an n with $1 \leqslant n \leqslant r+1$ such that $A_n \neq 0$. This completes the proof. □

Theorem 10.8 *Let Q be an even positive definite quadratic form of level two in v variables. Then Q represents a positive integer $2n \leqslant 2 + v/4$ or a positive integer $2n \leqslant 3 + v/4$ according to $v \equiv 0 \pmod 8$ or $v \equiv 4 \pmod 8$ respectively.*

Proof Suppose that Q is an even positive definite quadratic form of level two in v variables with $v \equiv 4 \pmod 8$. Put $v = 8k + 4$. Then by the well-known facts on θ-functions we know that the function defined by

$$\theta_Q(z) := \sum_{n=0}^{\infty} \# Q^{-1}(2n) q^n \in G(2, v/2)$$

is a holomorphic modular form where

$$\#Q^{-1}(2n) := \#\{(x_1, x_2, \cdots, x_v) \in \mathbb{Z}^v | Q(x_1, x_2, \cdots, x_v) = 2n\}.$$

It is clear that $\#Q^{-1}(0) = 1$. Hence by Theorem 10.7 we know that there exists an n_0 with $1 \leqslant n_0 \leqslant 1 + r(2, v/2)$ such that $\#Q^{-1}(2n_0) > 0$. That means Q represents the integer $2n_0$ with $n_0 \leqslant 1 + r(2, v/2) = 1 + r(2, 4k+2) = 2 + \left[\dfrac{4k+2}{4}\right] = 2 + k$. Hence Q represents the integer $2n_0 \leqslant 2(2+k) = 4 + 2k = 3 + v/4$. We can prove the case $h \equiv 0 \pmod 8$ similarly. This completes the proof. □

10.3 The Eligible Numbers of a Positive Definite Ternary Quadratic Form

In this section we study the problem of how to find the integers represented by a positive definite ternary quadratic form. It is a classical result that, taken together, the forms of a genus represent all numbers not ruled out by some corresponding congruences B.W. Jones, 1931; B.W. Jones, 1950. Following Kaplansky, we call these the eligible numbers of the genus I. Kaplansky, 1995. But it is very difficult to determine which of these eligible numbers can be represented by a form in the genus. In general we have the following results:

(R1) A positive definite ternary quadratic form f represents all of sufficiently large numbers which are represented by the spinor genus of f. (cf. W. Duke, 1990.)

(R2) Let n_0 be a square-free positive integer represented primitively by the genus of a positive definite ternary quadratic form f with discriminant d, then f primitively represents all of sufficiently large integers n_0t^2 if $(t, 2d) = 1$ and n_0t^2 are primitively represented by the spinor genus of f. (cf. J. Hsia, 1997.)

But there are no effective algorithm to determine all exceptions because (R1) and (R2) are dependent on Siegel's ineffective lower bound for the class numbers and the Iwaniec's estimation for the coefficients of cusp forms (cf. Remark 10.3). Even for the simplest cases, we can not do this. For example, let $f_1 = x^2 + y^2 + 7z^2$, $f_2 = x^2 + 7y^2 + 7z^2$. Then f_1 and $g_1 = x^2 + 2y^2 + 4z^2 + 2yz$ belong to the same genus, f_2 and $g_2 = 2x^2 + 4y^2 + 7z^2 - 2xy$ belong to another genus. The eligible numbers of f_1 and g_1 (f_2 and g_2 respectively)are numbers which are not the product of an odd (even respectively) power of 7 and a number congruent to 3, 5 or 6 mod 7 (see Example 10.1 and Example 10.2). We also can not determine which of them are represented by f_1 and f_2 respectively.

In I. Kaplansky, 1995 Kaplansky proved the following result and pointed out the following tables:

Theorem The form f_1 represents all eligible numbers which are multiples of 9;

it also represents all eligible numbers congruent to 2 mod 3 which are not of the form $14t^2$.

List I: Up to 100, 000 there are 27 eligible numbers prime to 7 not represented by f_1: 3, 6, 19, 22, 31, 51, 55, 66, 94, 139, 142, 159, 166, 214, 235, 283, 439, 534, 559, 670, 874, 946, 1726, 2131, 2419, 3559, 4759.

List II: Up to $100,000$ there are 26 eligible numbers congruent to $1, 2$ or 4 mod 7 which are not represented by f_2: 2, 22, 46, 58, 85, 93, 102, 205, 298, 310, 330, 358, 466, 478, 697, 862, 949, 1222, 1402, 1513, 1957, 1978, 2962, 3502, 7165, 10558.

List III: Up to $100,000$ there are 3 eligible numbers prime to 7 not represented by $f_3 = x^2 + 2y^2 + 7y^2$: 5, 20, 158.

List IV: Up to $100,000$ there are 3 eligible numbers congruent to $1, 2$ or $4 \mod 7$ which are not represented by $f_4 = x^2 + 7y^2 + 14z^2$: 2, 74, 506.

It is clear that $14 \cdot 7^{2k} \equiv 2 \pmod 3$ and f_1 does not represent $14 \cdot 7^{2k}$ for any non-negative integer k by a simple induction. We call the numbers of $14 \cdot 7^{2k}$ to be of trivial type. Hence there are indeed eligible numbers of the form $14t^2$ which are missed by f_1. But as Kaplansky pointed out, **List II** shows, that up to $700,000$ there are no further eligible numbers of form $14t^2$ that are missed by f_1 and which are not of trivial type. This motivated Kaplansky to make the following:

Conjecture f_1 represents all eligible numbers congruent to 2 mod 3 which are not of trivial type.

Kaplansky also conjectured that these four lists describe all exceptions, and so our knowledge of the integers represented by f_1 and g_1 would be complete.

In this section we want to show some general results about the eligible numbers of positive definite ternary forms by using modular forms of weight $3/2$, and give an algorithm for the number of representations of a positive integer n by a genus of positive definite ternary quadratic forms which is of an independent interest because it is a generalization of the classical theorem of Gauss concerning the number of representations of a natural number as a sum of three squares. By this algorithm, we can more precisely deal with eligible numbers and prove that the above **Conjecture** holds. We will also show how to use the algorithm to compute the number of representations and eligible numbers of a positive integer n by a genus of a positive definite ternary quadratic forms. We will study the relationships between the numbers of representations of ternary positive definite quadratic forms and elliptic curves.

Now let α, β, γ be square-free positive odd integers with $(\alpha, \beta, \gamma) = 1$, $D = [\alpha, \beta, \gamma]$, and λ_{4m} $(m|D)$ and λ_m $(1 \neq m|D)$ be the unique solution of the following system of linear equations:

$$(\star)\begin{cases} \sum_{m|D}(C_{4m}\cdot\mu(m/d)m^{-1}) + \sum_{1\neq m|D}(C_m\cdot\mu(m/d)m^{-1} \\ \quad = \dfrac{1}{D}\left(\dfrac{-1}{d}\right)\left(\dfrac{\alpha\beta/(\alpha,\beta)^2}{(d,\alpha,\beta)(d,l,\gamma)}\right)\left(\dfrac{\beta\gamma/(\beta,\gamma)^2}{d,\beta,\gamma)(d,l,\alpha)}\right)\left(\dfrac{\gamma\alpha/(\gamma,\alpha)^2}{(d,\gamma,\alpha)(d,l,\beta)}\right), \\ \sum_{m|D} C_{4m}\cdot\mu(m/d)m^{-1} = \dfrac{1}{D}\dfrac{-1}{(D/d)}\left(\dfrac{\alpha\beta/(\alpha,\beta)^2}{\gamma(\alpha,\beta)(\alpha,\beta,d)^{-1}(\gamma,\alpha\beta d)^{-1}}\right) \\ \quad\times\left(\dfrac{\beta\gamma/(\beta,\gamma)^2}{\alpha(\beta,\gamma)(\alpha,\beta\gamma d)^{-1}(\beta,\gamma,d)^{-1}}\right)\left(\dfrac{\gamma\alpha/(\gamma,\alpha)^2}{\beta(\gamma,\alpha)(\beta,\alpha\gamma d)^{-1}(\gamma,\alpha,d)^{-1}}\right) \quad d|D, \end{cases}$$

which will be proved to have a unique solution later (cf. The proof of Theorem 10.9). It is clear that λ_{4m} $(m|D)$ and λ_m $(1\neq m|D)$ are only dependent on α,β,γ.

For positive integers n, D, l we define:

$$\alpha(n) = \begin{cases} 3\times 2^{-(1+\nu_2(n))/2}, & \text{if } 2\nmid \nu_2(n), \\ 3\times 2^{-(1+\nu_2(n)/2)}, & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)}\equiv 1(\text{mod } 4), \\ 2^{-\nu_2(n)/2}, & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)}\equiv 3(\text{mod } 8), \\ 0, & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)}\equiv 7(\text{mod } 8) \end{cases}$$

and

$$\beta_{l,p}(n) = \begin{cases} (1+p)p^{(1-\nu_p(ln))/2}, & \text{if } 2\nmid \nu_p(ln), \\ 2p^{1-\nu_p(ln)/2}, & \text{if } 2|\nu_p(ln), \left(\dfrac{-ln/p^{\nu_p(ln)}}{p}\right) = -1, \\ 0, & \text{if } 2|\nu_p(ln), \left(\dfrac{-ln/p^{\nu_p(ln)}}{p}\right) = 1. \end{cases}$$

and

$$\beta_3(n,\chi_D,4D) = \sum_{\substack{(ab)^2|n,(ab,2D)=1\\ a,b \text{ positive integers}}} \mu(a)\left(\frac{-n}{a}\right)(ab)^{-1}.$$

Note that $\beta_3(n,\chi_D,4D)=1$ if n is square-free.

Let f be a positive definite ternary quadratic form, $\{f_1=f,\ f_2,\ \cdots,\ f_t\}$ a set of representatives of equivalence class in the genus of f. Denote by $r_i(n)=r(f_i,n)$ the number of representations of n by f_i. Put $G(n)=\sum_{i=1}^{t}\dfrac{r_i(n)}{O(f_i)}$. With these notations we get the following

Theorem 10.9 *Let α, β, γ be square-free odd positive integers such that $(\alpha,\ \beta,\ \gamma)=1$, $f=\alpha x^2+\beta y^2+\gamma z^2$. Let $\mathbb{A}=\{f_1=f,\ f_2,\ \cdots,\ f_t\}$ be a set of representatives for the equivalence classes in the genus of f. Then for any positive integer n we have that*

$$G(n)=r(\alpha,\beta,\gamma;n)\cdot h(-ln),$$

where $l = \alpha\beta\gamma/((\alpha,\beta)^2(\alpha,\gamma)^2(\beta,\gamma)^2)$ *and* $r(\alpha,\beta,\gamma;n)$ *is given by the following formula:*

$$\begin{aligned}
&r(\alpha,\beta,\gamma;n)\\
&= \frac{32}{\omega_{ln}}\alpha(ln)(1-2^{-1}\chi_{-ln}(2))\left(\frac{ln}{\delta_{ln}}\right)^{\frac{1}{2}}\beta_3(ln,\chi_D,4D)\left(\sum_{i=1}^{t}\frac{1}{0(f_i)}\right)\\
&\quad\times\Bigg(\sum_{m|D}(-1)^{t(m)}\lambda_{4m}\prod_{p|D/m}\frac{(1-\chi_{-ln}(p)p^{-1})p^2}{p^2-1}\prod_{p|m}\frac{(1-\chi_{-ln}(p)p^{-1})}{p^2-1}\beta_{l,p}(n)\\
&\quad+\sum_{1\neq m|D}(-1)^{t(m)}\lambda_m\prod_{p|D/m}\frac{(1-\chi_{-ln}(p)p^{-1})p^2}{p^2-1}\prod_{p|m}\frac{(1-\chi_{-ln}(p)p^{-1})}{p^2-1}\beta_{l,p}(n)\Bigg).
\end{aligned}$$

Proof We recall the following notations introduced in Section 7.3

$$\lambda_3(n,4D) = L_{4D}(2,\mathrm{id.})^{-1}L_{4D}(1,\chi_{-n})\beta_3(n,\chi_D,4D)$$

$$A_3(2,n) = \begin{cases} 4^{-1}(1-\mathrm{i})(1-3\cdot 2^{-(1+\nu_2(n))/2}), & \text{if } 2\nmid\nu_2(n),\\ 4^{-1}(1-\mathrm{i})(1-3\cdot 2^{-(1+\nu_2(n)/2)}), & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)}\equiv 1 \pmod 4,\\ 4^{-1}(1-\mathrm{i})(1-2^{-\nu_2(n)/2}), & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)}\equiv 3 \pmod 8,\\ 4^{-1}(1-\mathrm{i}), & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)}\equiv 7 \pmod 8.\end{cases}$$

$$A_3(p,n) = \begin{cases} p^{-1}-(1+p)p^{-(3+\nu_p(n))/2}, & \text{if } 2\nmid\nu_p(n),\\ p^{-1}-2p^{-1-\nu_p(n)/2}, & \text{if } 2|\nu_p(n), \left(\frac{-n/p^{\nu_p(n)}}{p}\right)=-1,\\ p^{-1}, & \text{if } 2|\nu_p(n), \left(\frac{-n/p^{\nu_p(n)}}{p}\right)=1,\end{cases}$$

$$L_N(s,\chi) = \sum_{(n,N)=1}^{\infty}\chi(n)n^{-s} = \prod_{p\nmid N}(1-\chi(p)p^{-S})^{-1},$$

$$\beta_3(n,\chi_D,4D) = \sum_{\substack{(ab)^2|n,(ab,2D)=1\\ a,b \text{ positive integers}}}\mu(a)\left(\frac{-n}{a}\right)(ab)^{-1},$$

where $\nu_2(n)$ is the maximal non–negative integer such $p^{\nu_2(n)}|n$.

We define functions $g(\chi_l,4m,4D)(z)$ $(m|D)$ and $g(\chi_l,m,4D)(z)$ $(m\neq 1, m|D)$, where D is a square–free odd positive integer and $l|D$ as follows:

$$\begin{aligned}
g(\chi_l,4D,4D)(z) &= 1-4\pi(1+\mathrm{i})l^{\frac{1}{2}}\sum_{n=1}^{\infty}\lambda_3(ln,4D)(A(2,ln)-4^{-1}(1-\mathrm{i}))\\
&\quad\times\prod_{p|D}(A(p,ln)-p^{-1})n^{\frac{1}{2}}\exp\{2\pi\mathrm{i}nz\},
\end{aligned}$$

$$g(\chi_l,4m,4D)(z) = -4\pi(1+\mathrm{i})l^{\frac{1}{2}}\sum_{n=1}^{\infty}\lambda_3(ln,4D)(A(2,ln)-4^{-1}(1-\mathrm{i}))$$

$$\times \prod_{p|m}(A(p,ln)-p^{-1})n^{\frac{1}{2}}\exp\{2\pi i n z\}, \forall D \neq m|D,$$

$$g(\chi_l, m, 4D)(z) = 2\pi l^{\frac{1}{2}}\sum_{n=1}^{\infty}\lambda(ln, 4D)\prod_{p|m}(A(p,ln)-p^{-1})n^{\frac{1}{2}}\exp\{2\pi i n z\}.$$

By the results of Section 7.3, the set of functions

$$g(\chi_l, 4m, 4D)(m|D), \quad g(\chi_l, m, 4D), \quad 1 \neq m|D$$

is a basis of $\mathcal{E}(4D,\ 3/2,\ \chi_l)$, and we have

$$\begin{aligned}
&V(g(\chi_l,4m,4D),1/\alpha) = -4^{-1}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1}l^{\frac{1}{2}}(l,\alpha)^{-\frac{1}{2}}\varepsilon^{-1}_{\alpha/(l,\alpha)}\left(\frac{l/(l,\alpha)}{d/(l,\alpha)}\right),\\
&V(g(\chi_l,4m,4D),1/(4\alpha)) = \mu(m/\alpha)\alpha m^{-1}l^{\frac{1}{2}}(l,\alpha)^{-\frac{1}{2}}\varepsilon_{l/(l,\alpha)}\left(\frac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right),\\
&V(g(\chi_l,4m,4D),1/(2\alpha)) = 0,\\
&V(g(\chi_l,m,4D),1/\alpha) = -4^{-1}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1}l^{\frac{1}{2}}(l,\alpha)^{-\frac{1}{2}}\varepsilon^{-1}_{\alpha/(l,\alpha)}\left(\frac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right),\\
&V(g(\chi_l,m,4D),1/(2\alpha)) = 0,\\
&V(g(\chi_l,m,4D),1/(4\alpha)) = 0,
\end{aligned}$$

where α is any positive divisor odd D and $V(f,p)$ represents the value of f at the cusp point p.

For $f = \alpha x^2 + \beta y^2 + \gamma z^2$, we see that $\theta_f(z) \in G(4D, 3/2, \chi_l)$ and $\theta(\text{gen}.f, z) \in \mathcal{E}(4D, 3/2, \chi_l)$ by the results in Section 10.1, where $D = [\alpha, \beta, \gamma]$ and $l = \alpha\beta\gamma/((\alpha,\beta)^2 \cdot (\alpha,\gamma)^2(\beta,\gamma)^2)$. Therefore there exist complex numbers $c_{4m}(m|D)$ and $c_m(m|D, m \neq 1)$ such that

$$\theta(\text{gen}.f, z) = \sum_{m|D} c_{4m}g(\chi_l, 4m, 4D) + \sum_{1\neq m|D} c_m g(\chi_l, m, 4D).$$

If we can compute explicitly these complex numbers, then we can obtain the explicit expression of $G(n) := \sum_{i=1}^{t}\frac{r_i(n)}{O(f_i)}$ by comparing the Fourier coefficients of the two sides of the above equality. In order to do this, we only need to calculate the values of $\theta(\text{gen}.f, z)$ at cusp points.

Claim 1 Let d/c be a cusp point $(c > 0, (c,d) = 1)$. Then

$$V(\theta, d/c) = \begin{cases} \varepsilon_d^{-1}\left(\dfrac{d}{c}\right), & \text{if } 4|c,\\ \dfrac{1-i}{2}\varepsilon_c\left(\dfrac{d}{c}\right), & \text{if } 2 \nmid c,\\ 0, & \text{if } 2 \parallel c,\end{cases}$$

where $\theta(z) = \sum_{m=-\infty}^{\infty} \exp\{m^2 z\}$.

Claim 2 Let d be a square-free odd positive integer, then

$$\varepsilon_d = \prod_{p|d} \varepsilon_p \left(\frac{dp^{-1}}{p}\right).$$

The proofs of these two claims are just simple calculations, and hence they are omitted.

It is easy to see that for square-free positive odd D, $S(4D) := \{1/d,\ 1/2d,\ 1/4d \mid d|D\}$ is a representative system of all equivalent classes of cusp points of $\varGamma_0(4D)$.

Claim 3 Let be $f = \alpha x^2 + \beta y^2 + \gamma z^2$, where α, β, γ are square–free positive odd integers such that $(\alpha, \beta, \gamma) = 1$. Then

$$\begin{aligned}
V(\theta_f, 1/d) &= -\frac{(1+\mathrm{i})dl^{1/2}}{4D(l,d)^{1/2}} \varepsilon^{-1}_{d/(d,l)} \left(\frac{-1}{d}\right)\left(\frac{l/(l,d)}{d/(l,d)}\right) \\
&\quad \cdot \left(\frac{\alpha\beta/(\alpha,\beta)^2}{(d,\alpha,\beta)(d,l,\gamma)}\right)\left(\frac{\beta\gamma/(\beta,\gamma)^2}{(d,\beta,\gamma)(d,l,\alpha)}\right) \times \left(\frac{\gamma\alpha/(\gamma,\alpha)^2}{(d,\gamma,\alpha)(d,l,\beta)}\right), \\
V(\theta_f, 1/4d) &= dD^{-1} l^{1/2}(l,d)^{-1/2} \varepsilon_{l/(l,d)} \left(\frac{-1}{D/d}\right)\left(\frac{d/(l,d)}{l/(l,d)}\right) \\
&\quad \times \left(\frac{\alpha\beta/(\alpha,\beta)^2}{\gamma(\alpha,\beta)(\alpha,\beta,d)^{-1}(\gamma,\alpha\beta d)^{-1}}\right) \\
&\quad \times \left(\frac{\beta\gamma/(\beta,\gamma)^2}{\alpha(\beta,\gamma)(\alpha,\beta\gamma d)^{-1}(\beta,\gamma,d)^{-1}}\right)\left(\frac{\gamma\alpha/(\gamma,\alpha)^2}{\beta(\gamma,\alpha)(\beta,\alpha\gamma d)^{-1}(\gamma,\alpha,d)^{-1}}\right), \\
V(\theta_f, 1/2d) &= 0,
\end{aligned}$$

where $d|D$.

This is a special case of our general result in Section 10.1. But now we can give a new proof for this fact. We have that

$$\begin{aligned}
V(\theta_f, 1/d) &= \lim_{z\to 0}(-dz)^{3/2}\theta_f\left(z + \frac{1}{d}\right) \\
&= \lim_{z\to 0}(-dz)^{3/2}\theta(\alpha(z+1/d))\theta(\beta(z+1/d))\theta(\gamma(z+1/d)) \\
&= \lim_{z\to 0}(-dz)^{3/2}\theta\left(\alpha z + \frac{\alpha/(\alpha,d)}{d/(\alpha,d)}\right)\theta\left(\beta z + \frac{\beta/(\beta,d)}{\alpha/(\beta,d)}\right)\theta\left(\gamma z + \frac{\gamma/(\gamma,d)}{d/(\gamma,d)}\right) \\
&= \left(\frac{(\alpha,d)(\beta,d)(\gamma,d)}{\alpha\beta\gamma}\right)^{\frac{1}{2}} V\left(\theta, \frac{\alpha/(\alpha,d)}{d/(\alpha,d)}\right) \\
&\quad \cdot V\left(\theta, \frac{\beta/(\beta,d)}{d/(\beta,d)}\right)\cdot V\left(\theta, \frac{\gamma/(\gamma,d)}{d/(\gamma,d)}\right).
\end{aligned}$$

We express d as $d = (d,l) \times \dfrac{d}{(d,l)}$. Suppose that p is a prime factor of d. Then $p|(d,l)$

if and only if only one of α, β, γ is divisible by p, $p|d/(d,l)$ if and only if only two of α, β, γ are divisible by p. This shows that $\alpha\beta\gamma = D^2/l$, $(\alpha,d)(\beta,d)(\gamma,d) = d^2/(d,l)$. Hence by the above claims we obtain that

$$V(\theta_f, 1/d) = -4^{-1}(1+\mathrm{i})dD^{-1}l^{1/2}(d,l)^{-1/2}V_1,$$

where

$$\begin{aligned}
V_1 =& \varepsilon_{d/(\alpha,d)}\varepsilon_{d/(\beta,d)}\varepsilon_{d/(\gamma,d)} \left(\frac{\alpha/(\alpha,d)}{d/(\alpha,d)}\right)\left(\frac{\beta/(\beta,d)}{d/(\beta,d)}\right)\left(\frac{\gamma/(\gamma,d)}{d/(\gamma,d)}\right) \\
=& \prod_{p|d}\varepsilon_p^2 \prod_{p|d/(d,l)}\varepsilon_p^{-1} \prod_{p|d/(\alpha,d)}\left(\frac{\alpha d/p}{p}\right) \\
& \prod_{p|d/(\beta,d)}\left(\frac{\beta d/p}{p}\right) \prod_{p|d/(\gamma,d)}\left(\frac{\gamma d/p}{p}\right) \\
=& \left(\frac{-1}{d}\right)\varepsilon_{d/(d,l)}^{-1} \prod_{p|d/(d,l)}\left(\frac{d(p(d,l))^{-1}}{p}\right) \\
& \prod_{p|d/(\alpha,d)}\left(\frac{\alpha d/p}{p}\right) \prod_{p|d/(\beta,d)}\left(\frac{\beta d/p}{p}\right) \prod_{p|d/(\gamma,d)}\left(\frac{\gamma d/p}{p}\right) \\
=& \left(\frac{-1}{d}\right)\varepsilon_{d/(d,l)}^{-1}\left(\frac{\alpha(d,l)}{(d,\beta,\gamma)}\right)\left(\frac{\beta(d,l)}{(d,\gamma,\alpha)}\right) \\
& \left(\frac{\gamma(d,l)}{(d,\alpha,\beta)}\right)\left(\frac{\alpha\beta}{(d,l,\gamma)}\right)\left(\frac{\beta\gamma}{(d,l,\alpha)}\right)\left(\frac{\gamma\alpha}{(d,l,\beta)}\right) \\
=& \left(\frac{-1}{d}\right)\varepsilon_{d/(d,l)}^{-1}\left(\frac{l/(d,l)}{d/(d,l)}\right)\left(\frac{\alpha\beta/(\alpha,\beta)^2}{(d,l,\gamma)(d,\alpha,\beta)}\right) \\
& \left(\frac{\beta\gamma/(\beta,\gamma)^2}{(d,l,\alpha)(d,\beta,\gamma)}\right)\left(\frac{\gamma\alpha/(\gamma,\alpha)^2}{(d,l,\beta)(d,\gamma,\alpha)}\right),
\end{aligned}$$

which implies the expression of $V(\theta_f, 1/d)$.

Similarly we have that

$$\begin{aligned}
V(\theta_f, 1/4d) =& \lim_{z\to 0}(-4dz)^{\frac{3}{2}}\theta_f(z+1/4d) \\
=& \lim_{z\to 0}(-4dz)^{\frac{3}{2}}\theta(\alpha(z+1/4d))\theta(\beta(z+1/4d))\theta(\gamma(z+1/4d)) \\
=& \left(\frac{(\alpha,d)(\beta,d)(\gamma,d)}{\alpha\beta\gamma}\right)^{\frac{1}{2}} V\left(\theta, \frac{\alpha/(\alpha,d)}{4d/(\alpha,d)}\right)
\end{aligned}$$

$$\begin{aligned}&\times V\left(\theta, \frac{\beta/(\beta,d)}{4d/(\beta,d)}\right) V\left(\theta, \frac{\gamma/(\gamma,d)}{4d/(\gamma,d)}\right)\\ &= dD^{-\frac{1}{2}} l^{\frac{1}{2}} (l,d)^{-\frac{1}{2}} V_2,\end{aligned}$$

where

$$\begin{aligned}V_2 &= \varepsilon_{\alpha/(\alpha,d)}^{-1} \varepsilon_{\beta/(\beta,d)}^{-1} \varepsilon_{\gamma/(\gamma,d)}^{-1} \left(\frac{d/(\alpha,d)}{\alpha/(\alpha,d)}\right) \left(\frac{d/(\beta,d)}{\beta/(\beta,d)}\right) \left(\frac{d/(\gamma,d)}{\gamma/(\gamma,d)}\right)\\ &= \prod_{p|D/p} \varepsilon_p^{-2} \prod_{p|l/(l,\alpha)} \varepsilon_p \prod_{p|\alpha/(\alpha,d)} \left(\frac{\alpha d/p}{p}\right) \prod_{p|\beta/(\beta,d)} \left(\frac{\beta d/p}{p}\right) \prod_{p|\gamma/(\gamma,d)} \left(\frac{\gamma d/p}{p}\right)\\ &= \varepsilon_{l/(l,d)} \left(\frac{-1}{D/d}\right) \prod_{p|l/(l,d)} \left(\frac{l(p(l,d))^{-1}}{p}\right) \prod_{p|\alpha/(\alpha,d)} \left(\frac{\alpha d/p}{p}\right)\\ &\quad\times \prod_{p|\beta/(\beta,d)} \left(\frac{\beta d/p}{p}\right) \prod_{p|\gamma/(\gamma,d)} \left(\frac{\gamma d/p}{p}\right),\end{aligned}$$

since

$$l/(l,d) = \alpha/(\alpha,\beta\gamma d) \times \beta/(\beta,\gamma\alpha d) \times \gamma/(\gamma,\alpha\beta d).$$

Hence,

$$\begin{aligned}V_2 &= \varepsilon_{l/(l,d)} \left(\frac{-1}{D/d}\right) \left(\frac{\alpha\beta/(\alpha,\beta)^2}{(\alpha,\beta)/(\alpha,\beta,d)}\right) \left(\frac{\beta\gamma/(\beta,\gamma)^2}{(\beta,\gamma)/(\beta,\gamma,d)}\right) \left(\frac{\gamma,\alpha/(\gamma,\alpha)^2}{(\gamma,\alpha)/(\gamma,\alpha,d)}\right)\\ &\quad\times \left(\frac{\alpha dl(d,l)^{-1}(\alpha,l)^{-2}}{\alpha/(\alpha,\beta\gamma d)}\right) \left(\frac{\beta dl(d,l)^{-1}(\beta,l)^{-2}}{\beta/(\beta,\gamma\alpha d)}\right) \left(\frac{\gamma dl(d,l)^{-1}(\gamma,l)^{-2}}{\gamma/(\gamma,\alpha\beta,d)}\right)\\ &= \varepsilon_{l/(l,d)} \left(\frac{-1}{D/d}\right) \left(\frac{d/(d,l)}{l/(d,l)}\right) \left(\frac{\alpha\beta/(\alpha,\beta)^2}{(\alpha,\beta)/(\alpha,\beta,d) \times \gamma/(\gamma,\alpha\beta d)}\right)\\ &\quad\times \left(\frac{\beta\gamma/(\beta,\gamma)^2}{(\beta,\gamma)/(\beta,\gamma,d) \times \alpha/(\alpha,\beta\gamma d)}\right) \left(\frac{\gamma\alpha/(\gamma,\alpha)^2}{(\gamma,\alpha)/(\gamma,\alpha,d) \times \beta/(\beta,\gamma\alpha d)}\right),\end{aligned}$$

which implies the expressions for $V(\theta_f, 1/4d)$. Finally we can show that $V(\theta_f, 1/2d) = 0$ by the fact that $V(\theta, 1/2) = 0$. This completes the proof of Claim 3.

Since $\theta_f(z)$ and $\theta(\text{gen}.f, z)$ have the same values at each cusp point, we see that

$$\begin{aligned}&V(\theta(\text{gen}.f, z), p) = V(\theta_f(z), p)\\ &= \sum_{m|D} C_{4m} V(g(\chi_l, 4m, 4D), p) + \sum_{1\neq m|D} C_m V(g(\chi_l, m, 4D), p)\end{aligned}$$

for each cusp point p. Hence we obtain a system of equations:

$$\begin{cases} \displaystyle\sum_{m|D} C_{4m} V(g(\chi_l, 4m, 4D), 1/\alpha) + \sum_{1\neq m|D} C_m V(g(\chi_l, m, 4D), 1/\alpha) \\ = V(\theta_f, 1/\alpha), (\alpha|D), \\ \displaystyle\sum_{m|D} C_{4m} V(g(\chi_l, 4m, 4D), 1/(2\alpha)) + \sum_{1\neq m|D} C_m V(g(\chi_l, m, 4D), 1/(2\alpha)) \\ = V(\theta_f, 1/(2\alpha)) = 0, (\alpha|D), \\ \displaystyle\sum_{m|D} C_{4m} V(g(\chi_l, 4m, 4D), 1/(4\alpha)) + \sum_{1\neq m|D} C_m V(g(\chi_l, m, 4D), 1/(4\alpha)) \\ = V(\theta_f, 1/(4\alpha)), (\alpha|D). \end{cases} \tag{10.22}$$

Inserting the values of the functions at cusp points into equality (10.22), we have that

$$\begin{cases} \displaystyle\sum_{m|D} (C_{4m} \cdot \mu(m/d) m^{-1}) + \sum_{1\neq m|D} (C_m \cdot \mu(m/d) m^{-1} \\ \displaystyle = \frac{1}{D}\left(\frac{-1}{d}\right)\left(\frac{\alpha\beta/(\alpha,\beta)^2}{(d,\alpha,\beta)(d,l,\gamma)}\right)\left(\frac{\beta\gamma/(\beta,\gamma)^2}{d,\beta,\gamma)(d,l,\alpha)}\right)\left(\frac{\gamma\alpha/(\gamma,\alpha)^2}{(d,\gamma,\alpha)(d,l,\beta)}\right), \\ \displaystyle\sum_{m|D} C_{4m} \cdot \mu(m/d) m^{-1} = \frac{1}{D}\frac{-1}{(D/d)}\left(\frac{\alpha\beta/(\alpha,\beta)^2}{\gamma(\alpha,\beta)(\alpha,\beta,d)^{-1}(\gamma,\alpha\beta d)^{-1}}\right) \\ \displaystyle\qquad \times\left(\frac{\beta\gamma/(\beta,\gamma)^2}{\alpha(\beta,\gamma)(\alpha,\beta\gamma d)^{-1}}(\beta,\gamma,d)^{-1}\right) \\ \displaystyle\qquad \times\left(\frac{\gamma\alpha/(\gamma,\alpha)^2}{\beta(\gamma,\alpha)(\beta,\alpha\gamma d)^{-1}(\gamma,\alpha,d)^{-1}}\right), \quad (d|D). \end{cases} \tag{10.23}$$

We must prove that the system (10.23) has a unique solution for C_{4m} $(m|D)$ and C_m $(1 \neq m|D)$. This is equivalent to proving that the corresponding homogeneous system has only zero as a solution. Otherwise, suppose that $C_{4m} = \lambda_{4m}$ $(m|D)$ and $C_m = \lambda_m$ $(1 \neq m|D)$ is a non-zero solution of (10.23), i.e.,

$$\begin{cases} \displaystyle\sum_{m|D} (\lambda_{4m} \cdot \mu(m/d) m^{-1}) + \sum_{1\neq m|D} (\lambda_m \cdot \mu(m/d) m^{-1} = 0, \\ \displaystyle\sum_{m|D} \lambda_{4m} \cdot \mu(m/d) m^{-1} = 0, \quad d|D. \end{cases} \tag{10.24}$$

Consider the following function:

$$h(z) = \sum_{m|D} \lambda_{4m} g(\chi_l, 4m, 4D) + \sum_{1\neq m|D} \lambda_m g(\chi_l, m, 4D),$$

which belongs to the space $\mathcal{E}(4D, 3/2, \chi_l)$. We now compute the values of $h(z)$ at all cusp points. For any $d|D$, we see that:

$$\begin{aligned}
V(h(z),1/d) &= \sum_{m|D} \lambda_{4m} V(g(\chi_l,4m,4D),1/d) + \sum_{1\neq m|D} \lambda_m V(g(\chi_l,m,4D),1/d) \\
&= -4^{-1}(1+i)dl^{\frac{1}{2}}(l,d)^{-\frac{1}{2}}\varepsilon_{d/(l,d)}^{-1} \\
&\quad \times \left(\frac{l/(l,d)}{d/(l,d)}\right)\left(\sum_{m|D}\lambda_{4m}\mu(m/d)m^{-1} + \sum_{1\neq m|D}\lambda_m\mu(m/d)m^{-1}\right) \\
&= 0, \\
V(h(z),1/(2d)) &= \sum_{m|D} \lambda_{4m} V(g(\chi_l,4m,4D),1/(2d)) + \sum_{1\neq m|D} \lambda_m V(g(\chi_l,m,4D),1/(2d)) \\
&= \sum_{m|D} \lambda_{4m}\cdot 0 + \sum_{1\neq m|D} \lambda_m \cdot 0 = 0, \\
V(h(z),1/(4d)) &= \sum_{m|D} \lambda_{4m} V(g(\chi_l,4m,4D),1/(4d)) + \sum_{1\neq m|D} \lambda_m V(g(\chi_l,m,4D),1/(4d)) \\
&= dl^{\frac{1}{2}}(l,d)^{-\frac{1}{2}}\varepsilon_{l/(l,d)}\left(\frac{d/(l,d)}{l/(l,d)}\right)\left(\sum_{m|D}\lambda_{4m}\mu(m/d)m^{-1}\right) = 0.
\end{aligned}$$

These imply that the values of modular form $h(z)$ are equal to zero at all cusp points of $\Gamma_0(4D)$. Hence $h(z) \in S(4D,3/2,\chi_l)$ which shows that $h(z) \in S(4D, 3/2,\chi_l) \bigcap \mathcal{E}(4D,3/2,\chi_l) = \{0\}$, i.e.,

$$\sum_{m|D} \lambda_{4m} g(\chi_l,4m,4D) + \sum_{1\neq m|D} \lambda_m g(\chi_l,m,4D) = 0.$$

But $g(\chi_l,4m,4D)$ $(m|D)$ and $g(\chi_l,m,4D)$ $(1 \neq m|D)$ are linearly independent. Therefore $\lambda_{4m} = 0$ $(m|D)$ and $\lambda_m = 0$ $(1 \neq m|D)$ which contradicts the assumption for λ_{4m} and λ_m and hence show that the system (10.23) has only zero as a solution.

From (10.23) we can easily calculate explicitly all the $C_m(1 \neq m|D)$ and $C_{4m}(m|D)$, it is clear that all these are rational numbers and only dependent on α,β,γ.

That is, we obtain explicitly rational numbers C_m and C_{4m} such that

$$\theta(\text{gen.}f,z) = \sum_{m|D} C_{4m} g(\chi_l,4m,4D) + \sum_{1\neq m|D} C_m g(\chi_l,m,4D). \tag{10.25}$$

On the other hand, let

$$\begin{aligned}
\alpha(n) &= 2(1+i)(4^{-1}(1-i) - A_3(2,n)) \\
&= \begin{cases} 3\times 2^{-(1+\nu_2(n))/2}, & \text{if } 2\nmid \nu_2(n), \\ 3\times 2^{-(1+\nu_2(n)/2)}, & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)} \equiv 1 (\text{mod } 4), \\ 2^{-\nu_2(n)/2}, & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)} \equiv 3 (\text{mod } 8), \\ 0, & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)} \equiv 7 (\text{mod } 8) \end{cases}
\end{aligned} \tag{10.26}$$

and

$$\beta_{l,p}(n) = p^2(p^{-1} - A_3(p, ln)) \\ = \begin{cases} (1+p)p^{(1-\nu_p(ln))/2}, & \text{if } 2 \nmid \nu_p(ln), \\ 2p^{1-\nu_p(ln)/2}, & \text{if } 2|\nu_p(ln), \left(\dfrac{-ln/p^{\nu_p(ln)}}{p}\right) = -1, \\ 0, & \text{if } 2|\nu_p(ln), \left(\dfrac{-ln/p^{\nu_p(ln)}}{p}\right) = 1. \end{cases} \tag{10.27}$$

Let δ_{ln} be the conductor of the character χ_{-ln} and $h(-ln)$ be the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-ln})$. Then the class number formula shows that

$$h(-ln) = (2\pi)^{-1}\delta_{ln}^{\frac{1}{2}}\omega_{ln}L(1, \chi_{-ln}),$$

where

$$\omega_{ln} = \begin{cases} 6, & \text{if } \delta_{ln} = 3, \\ 4, & \text{if } \delta_{ln} = 4, \\ 2, & \text{if otherwise.} \end{cases}$$

Hence

$$\begin{aligned} \lambda_3(ln, 4D) &= L_{4D}(2, \mathrm{id})^{-1}L_{4D}(1, \chi_{-ln})\beta_3(ln, \chi_D, 4D) \\ &= L(2, \mathrm{id})^{-1}\prod_{p|4D}(1-p^{-2})^{-1}L(1, \chi_{-ln}) \\ &\quad \cdot \prod_{p|4D}(1-\chi_{-ln}(p)p^{-1}) \cdot \beta_3(ln, \chi_D, 4D) \\ &= \frac{6}{\pi^2} \cdot \prod_{p|4D}(1-p^{-2})^{-1}(1-\chi_{-ln}(p)p^{-1}) \\ &\quad \cdot h(-ln) \cdot 2\pi \cdot \omega_{ln}^{-1}\delta_{ln}^{-\frac{1}{2}}\beta_3(ln, \chi_D, 4D) \\ &= \frac{12}{\pi}\prod_{p|4D}\frac{(1-\chi_{-ln}(p)p^{-1})p^2}{p^2-1} \cdot \frac{h(-ln)}{\omega_{ln}\sqrt{\delta_{ln}}} \cdot \beta_3(ln, \chi_D, 4D). \end{aligned}$$

This implies that

$$\begin{aligned} g(\chi_l, 4D, 4D) &= 1 + (-1)^{t(D)}32\sum_{n=1}^{\infty} h(-ln)\omega_{ln}^{-1}\alpha(ln)(1-2^{-1}\chi_{-ln}(2)) \\ &\quad \times \prod_{p|D}\left[\frac{(1-\chi_{ln}(p)p^{-1})}{p^2-1}\beta_{l,p}(n)\right] \\ &\quad \cdot \left(\frac{ln}{\delta_{ln}}\right)^{1/2}\beta_3(ln, \chi_D, 4D)\exp\{2\pi i n z\}, \\ g(\chi_l, 4m, 4D) &= (-1)^{t(m)}32\sum_{n=1}^{\infty} h(-ln)\omega_{ln}^{-1}\alpha(ln)(1-2^{-1}\chi_{-ln}(2)) \end{aligned}$$

$$\times \prod_{p|D/m} \frac{(1-\chi_{-ln}(p)p^{-1})p^2}{p^2-1} \prod_{p|m} \frac{(1-\chi_{-ln}(p)p^{-1})}{p^2-1} \beta_{l,p}(n)$$

$$\times \left(\frac{ln}{\delta_{ln}}\right)^{\frac{1}{2}} \beta_3(ln, \chi_D, 4D) \exp\{2\pi i n z\},$$

$$g(\chi_l, m, 4D) = (-1)^{t(m)} 32 \sum_{n=1}^{\infty} h(-ln)\omega_{ln}^{-1}(1-2^{-1}\chi_{ln}(2)) \tag{10.28}$$

$$\times \prod_{p|D/m} \frac{(1-\chi_{-ln}(p)p^{-1})p^2}{p^2-1} \prod_{p|m} \frac{(1-\chi_{-ln}(p)p^{-1})}{p^2-1} \beta_{l,p}(n)$$

$$\times \left(\frac{ln}{\delta_{ln}}\right)^{\frac{1}{2}} \beta_3(ln, \chi_D, 4D) \exp\{2\pi i n z\},$$

where $t(m)$ is the number of distinct prime factors of m. Let be $ln = ds^2$ with d square-free, then $\delta_{ln} = d$ or $4d$ according to $d \equiv 1 \pmod 4$ or $d \equiv 2, 3 \pmod 4$ which implies that $\left(\frac{ln}{\delta_{ln}}\right)^{1/2} = \left(\frac{ds^2}{d}\right)^{1/2} = s$ or $\left(\frac{ln}{\delta_{ln}}\right)^{1/2} = \left(\frac{ds^2}{4d}\right)^{1/2} = \frac{s}{2}$ according to $d \equiv 1 \pmod 4$ or $d \equiv 2, 3 \pmod 4$. Anyway, $\left(\frac{ln}{\delta_{ln}}\right)^{1/2}$ is an explicitly determined rational number. Now we compare the Fourier coefficients of the two sides of (10.24), and use (10.27) to obtain that

$$G(n) = r(\alpha, \beta, \gamma; n)h(-ln),$$

where $r(\alpha, \beta, \gamma; n)$ is defined as in 10.9. This completes the proof of the theorem. □

By Theorem 10.9 we obtain the following:

An Algorithm for $G(n)$ and eligible numbers of f:

Input: A positive definite ternary quadratic form f;

Output: $G(n)$ and the set $\mathbb{E}$ of eligible numbers of f;

Step 1: Solve the system $(\star)$;

Step 2: Use Theorem 10.9 to compute $G(n)$;

Step 3: Put $\mathbb{E} = \{n \in \mathbb{N} | r(\alpha, \beta, \gamma; n) = 0\}$.

We will compute some examples with this algorithm.

It is clear that Theorem 10.9 holds indeed for any positive definite ternary quadratic form f with level $4D$ (D a square-free odd positive integer). Hence by Theorem 10.9 we can always give the precise major part for the number $r(f, n)$ of representations for n by f. Especially if the space $S(N, 3/2, \chi_l)$ is the null space, we can obtain the precise formula for $r(f, n)$ by Theorem 10.9. For example, by the dimension formulae for the space of modular forms, we can find that the following spaces are all null spaces:

$$\begin{array}{lll} S(4,3/2,\chi_1), & S(8,3/2,\chi_1), & S(8,3/2,\chi_2), \\ S(12,3/2,\chi_1), & S(12,3/2,\chi_3), & S(16,3/2,\chi_1), \\ S(20,3/2,\chi_1), & S(20,3/2,\chi_5), & S(24,3/2,\chi_1), \\ S(24,3/2,\chi_2), & S(24,3/2,\chi_3), & S(24,3/2,\chi_6), \\ S(32,3/2,\chi_1), & S(32,3/2,\chi_2), & S(64,3/2,\chi_2). \end{array}$$

Hence we can obtain the following formulae: Let be $N(a,b,c;n) = r(ax^2+by^2+cz^2,n)$, $\delta(x) = 1$ or 0 according to x is an integer or not, then

$$\begin{aligned} N(1,1,1;n) &= 2\pi n^{\frac{1}{2}}\lambda(n,4)\alpha(n), \quad \text{(Gauss formula)} \\ N(1,2,2;n) &= 2\pi n^{\frac{1}{2}}\lambda(n,4)\left(\alpha(n) - \delta\left(\frac{n-1}{4}\right) - \delta\left(\frac{n-2}{4}\right)\right), \\ N(1,3,3;n) &= 2\pi n^{\frac{1}{2}}\lambda(n,12)(1/3 - A(3,n))(2-\alpha(n)), \\ N(1,5,5;n) &= 2\pi n^{\frac{1}{2}}\lambda(n,20)\alpha(n)(A(5,n)+1/5), \\ N(2,3,6;n) &= 2\pi n^{\frac{1}{2}}\lambda(n,12)(1/3 + A(3,n))\left(\alpha(n) - \delta\left(\frac{n-1}{4}\right) - \delta\left(\frac{n-2}{4}\right)\right), \text{etc.} \end{aligned}$$

From this point of view we see that Theorem 10.9 is a generalization of the classical result of Gauss concerning the number of representations of a natural number as a sum of three squares.

Corollary 10.1 *Let $f = x^2 + y^2 + pz^2$, p an odd prime, then*

$$G(n) := \sum_{i=1}^{t} \frac{r_i(n)}{O(f_i)} = \begin{cases} \dfrac{32}{\omega_{pn}(p^2-1)} h(-pn)\alpha(pn)(2p - \beta_{p,p}(n))\gamma_p(n) \cdot \left(\displaystyle\sum_{i=1}^{t} \frac{1}{O(f_i)}\right), \\ \qquad\qquad \text{if} p \equiv 1 (\text{mod } 4), \\ \dfrac{32}{\omega_{pn}(p^2-1)} h(-pn)(2-\alpha(pn))\beta_{p,p}(n)\gamma_p(n) \cdot \left(\displaystyle\sum_{i=1}^{t} \frac{1}{O(f_i)}\right), \\ \qquad\qquad \text{if} p \equiv 3 (\text{mod } 4), \end{cases}$$

where $\gamma_p(n) = (1 - 2^{-1}\chi_{-pn}(2))(pn/\delta_{pn})^{1/2} \displaystyle\sum_{\substack{(ab)^2 \mid n \\ (ab,2p)=1}} \mu(a)\chi_{-pn}(a)(ab)^{-1}.$

Proof Just as in the proof of Theorem 10.9, we have that $D = l = p$. So by (10.28) we see that $\mathcal{E}(4p,\, 3/2,\, \chi_p)$ has a basis as follows:

$$g(\chi_p, 4p, 4p) = 1 - \frac{32}{p^2-1}\sum_{n=1}^{\infty} h(-pn)\omega_{pn}^{-1}\alpha(pn)\beta_{p,p}(n)\gamma_p(n)\exp\{2\pi i n z\},$$
$$g(\chi_p, 4, 4p) = \frac{32p^2}{p^2-1}\sum_{n=1}^{\infty} h(-pn)\omega_{pn}^{-1}\alpha(pn)\gamma_p(n)\exp\{2\pi i n z\},$$
$$g(\chi_p, p, 4p) = -\frac{32}{p^2-1}\sum_{n=1}^{\infty} h(-pn)\omega_{pn}^{-1}\beta_{p,p}(n)\gamma_p(n)\exp\{2\pi i n z\}.$$

We can easily calculate the solution of the system of equations (10.23):

$$\begin{pmatrix} c_4 \\ c_{4p} \\ c_p \end{pmatrix} = \begin{pmatrix} \dfrac{2}{p} \\ 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

according to $p \equiv 1$ or $3 \pmod 4$. Hence we see that

$$\theta(\text{gen}.f, z) = \begin{cases} g(\chi_p, 4p, 4p) + 2p^{-1}g(\chi_p, 4, 4p), & \text{if } p \equiv 1 (\text{mod } 4), \\ g(\chi_p, 4p, 4p) - 2g(\chi_p, p, 4p), & \text{if } p \equiv 3 (\text{mod } 4). \end{cases}$$

Hence we see that

$$G(n) = \begin{cases} \dfrac{32}{\omega_{pn}(p^2-1)} h(-pn)\alpha(pn)(2p - \beta_{p,p}(n))\gamma_p(n) \cdot \left(\displaystyle\sum_{i=1}^{t} \frac{1}{O(f_i)}\right), \\ \qquad \text{if } p \equiv 1 \pmod 4 \\ \dfrac{32}{\omega_{pn}(p^2-1)} h(-pn)(2 - \alpha(pn))\beta_{p,p}(n)\gamma_p(n) \cdot \left(\displaystyle\sum_{i=1}^{t} \frac{1}{O(f_i)}\right), \\ \qquad \text{if } p \equiv 3 \pmod 4 \end{cases}$$

as stated in the corollary. □

Example 10.3 Let $p = 7$, then $f = f_1 = x^2 + y^2 + 7z^2$ and $g_1 = x^2 + 2y^2 + 4z^2 + 2yz$ belong to the same genus, $O(f_1) = 8$, $O(g_1) = 4$. Hence

$$G_1(n) = \frac{r_1(n)}{8} + \frac{r_1'(n)}{4} = \frac{1}{4}\omega_{7n}^{-1} \cdot (2 - \alpha(7n))\beta_{7,7}(n)\gamma_7(n)h(-7n).$$

□

Corollary 10.2 *Let $f = x^2 + py^2 + pz^2$, p an odd prime, then*

$$G(n):=\sum_{i=1}^{t}\frac{r_i(n)}{O(f_i)}$$
$$=\begin{cases}\dfrac{32}{\omega_n(p^2-1)}h(-n)\alpha(n)(2p-\beta_{1,p}(n))\gamma_p'(n)\left(\displaystyle\sum_{i=1}^{t}\frac{1}{O(f_i)}\right), & \text{if } p\equiv 1(\text{mod } 4),\\ \dfrac{32}{\omega_n(p^2-1)}h(-n)(2-\alpha(n))\beta_{1,p}(n)\gamma_p'(n)\left(\displaystyle\sum_{i=1}^{t}\frac{1}{O(f_i)}\right), & \text{if } p\equiv 3(\text{mod } 4),\end{cases}$$

where $\gamma_p'(n)=(1-2^{-1}\chi_{-n}(2))(1-\chi_{-n}(p)\cdot p^{-1})(n/\delta_n)^{1/2}\sum\mu(a)\chi_{-n}(a)(ab)^{-1}$.

Proof Just as in the proof of Theorem 10.9, we have that $D=p$, $l=1$. So by (10.28) we see that $\mathcal{E}(4P,3/2,\chi_1)$ has a basis as follows:

$$g(\chi_1,4p,4p)=1-\frac{32}{p^2-1}\sum_{n=1}^{\infty}h(-n)\omega_n^{-1}\alpha(n)\beta_{1,p}(n)\gamma_p'(n)\exp\{nz\},$$
$$g(\chi_1,4,4p)=\frac{32p^2}{p^2-1}\sum_{n=1}^{\infty}h(-n)\omega_n^{-1}\alpha(n)\gamma_p'(n)\exp\{nz\},$$
$$g(\chi_1,p,4p)=-\frac{32}{p^2-1}\sum_{n=1}^{\infty}h(-n)\omega_n^{-1}\beta_{1,p}(n)\gamma_p'(n)\exp\{nz\}.$$

We can also calculate the solution of the system of equations (10.23):

$$\begin{pmatrix} c_4\\ c_{4p}\\ c_p\end{pmatrix}=\begin{pmatrix}\frac{2}{p}\\ 1\\ 0\end{pmatrix}\text{ or }\begin{pmatrix}0\\ 1\\ -2\end{pmatrix}$$

according to $p\equiv 1$ or $3 \pmod 4$. Hence we see that

$$\theta(\text{gen}.f,z)=\begin{cases} g(\chi_1,4p,4p)+2p^{-1}g(\chi_1,4,4p), & \text{if } p\equiv 1\,(\text{mod } 4),\\ g(\chi_1,4p,4p)-2g(\chi_1,p,4p), & \text{if } p\equiv 3\,(\text{mod } 4).\end{cases}$$

Therefore we see that

$$G(n)=\begin{cases}\dfrac{32}{\omega_n(p^2-1)}h(-n)\alpha(n)(2p-\beta_{1,p}(n))\gamma_p'(n)\cdot\left(\displaystyle\sum_{i=1}^{t}\frac{1}{O(f_i)}\right),\\ \qquad\qquad\text{if } p\equiv 1 \pmod 4,\\ \dfrac{32}{\omega_n(p^2-1)}h(-pn)(2-\alpha(n))\beta_{1,p}(n)\gamma_p'(n)\cdot\left(\displaystyle\sum_{i=1}^{t}\frac{1}{O(f_i)}\right),\\ \qquad\qquad\text{if } p\equiv 3 \pmod 4.\end{cases}$$

This completes the proof. □

Example 10.4 Let $f = f_2 = x^2+7y^2+7z^2$, then f_2 and $g_2 = 2x^2+4y^2+7z^2-2xy$ belong to the same genus, $O(f_2) = 8$, $O(g_2) = 4$. Hence

$$G_2(n) := \frac{r_2(n)}{8} + \frac{r_2'(n)}{4} = \frac{1}{4}\omega_n^{-1} \cdot (2-\alpha(n))\beta_{1,7}(n)\gamma_7'(n)h(-n).$$

□

By Corollary 10.1 and Corollary 10.2, we can prove the following

Corollary 10.3 *Let $f_{(p)} = x^2+y^2+pz^2$, p an odd prime, then*

(1) *if $p \equiv 3 \pmod 4$, the eligible numbers of the genus of $f_{(p)}$ are numbers which are not the product of an odd power of p and a number n satisfying $\left(\frac{-n}{p}\right) = 1$;*

(2) *if $p \equiv 1 \pmod 8$, the eligible numbers of the genus of $f_{(p)}$ are numbers which are not the product of an even power of 2 and a number congruent to 7 mod 8;*

(3) *if $p \equiv 5 \pmod 8$, the eligible numbers of the genus of $f_{(p)}$ are numbers which are not the product of an even power of 2 and a number congruent to 3 mod 8.*

Corollary 10.4 *Let $g_{(p)} = x^2+py^2+pz^2$, p an odd prime, then*

(1) *if $p \equiv 3 \pmod 4$, the eligible numbers of the genus of $g_{(p)}$ are numbers which are not the product of an even power of p and a number n satisfying $\left(\frac{-n}{p}\right) = 1$;*

(2) *if $p \equiv 1 \pmod 4$, the eligible numbers of the genus of $g_{(p)}$ are numbers which are not the numbers n satisfying $\left(\frac{n}{p}\right) = -1$ or the product of an even power of 2 and a number congruent to 7 mod 8.*

Proof By definition, a positive integer n is eligible if and only if $G(n) > 0$, i.e., n is not an eligible integer if and only if $G(n) = 0$. If $p \equiv 3 \pmod 4$, then

$$G(n) = \frac{32}{\omega_{pn}} h(-pn)(2-\alpha(pn))\beta_{p,p}(n)\gamma_p(n) \cdot \left(\sum_{i=1}^{t} \frac{1}{O(f_i)}\right),$$

which implies that $G(n) = 0$ if and only if one of the factors at the right end of the above equality equals zero. But it is clear that $\frac{32}{\omega_{pn}} h(-pn)\left(\sum_{i=1}^{t} \frac{1}{O(f_i)}\right) > 0$. So we only need to consider the other three factors. By (10.26) we see that $2-\alpha(pn) \geqslant 2-3/2 = 1/2$. So the only possibilities are that $\beta_{p,p}(n) = 0$ or $\gamma_p(n) = 0$. By (10.27) we know that $\beta_{p,p}(n) = 0$ if and only if $\nu_p(n) \equiv 1 \pmod 2$ and $\left(\frac{-n/p^{\nu_p(n)}}{p}\right) = 1$. Hence if we can prove that $\gamma_p(n) \neq 0$, then this completes the proof of (1). In fact, we can prove the following claim which completes the proof of (1). The proofs of (2) and Corollary 10.4 are similar.

Claim Let D be a square-free positive integer, then

$$\beta_3(n,\chi_D,4D) = \sum_{\substack{(ab)^2|n,(ab,2D)=1\\ a,b \text{ positive integers}}} \mu(a)\left(\frac{-n}{a}\right)(ab)^{-1} \neq 0$$

for any positive integer n.

In fact, by definition, we see that

$$\begin{aligned}\beta_3(n,\chi_D,4D) &= \sum_{\substack{(ab)^2|n,(ab,2D)=1\\ a,b \text{ positive integers}}} \mu(a)\left(\frac{-n}{a}\right)(ab)^{-1}\\ &= \prod_{p\nmid 2D,p|D_n} \sum_{t=0}^{h(p,f_n)} p^{-t}\cdot \prod_{p\nmid 2DD_n}\left(\sum_{t=0}^{\nu_p(f_n)} p^{-t} - p^{-1}\left(\frac{D_n}{p}\right)\sum_{t=0}^{\nu_p(f_n)-1} p^{-t}\right),\end{aligned}$$

where $-n = D_n f_n^2$ such that D_n is a fundamental discriminant and f_n is a positive integer. The above equality implies that $\beta_3(n,\chi_D,4D) \neq 0$. This completes the proofs. □

Example 10.5 The eligible numbers of $f_1 = f_{(7)} = x^2 + y^2 + 7z^2$ are numbers which are not the product of an odd power of 7 and a number congruent to $3,5$ or 6 mod 7 since $\left(\frac{-n}{7}\right) = 1$ if and only if n congruent to $3,5$ or 6 mod 7. □

Example 10.6 The eligible numbers of $f_2 = g_{(7)} = x^2 + 7y^2 + 7z^2$ are numbers which are not the product of an even power of 7 and a number congruent to $3,5$ or 6 mod 7 since $\left(\frac{-n}{7}\right) = 1$ if and only if n is congruent to 3, 5 or 6 mod 7. □

Theorem 10.10 *Let f be a positive definite quadratic form with matrix A. Then there are only finitely many square-free eligible integers which are prime to $2|A|$ and not represented by f.*

Proof The proof of this theorem is similar to the one in W. Duke, 1990. For the sake of completeness we include it here. In order to prove the theorem, we need some of the results in B.W. Jones, 1950, esp. Theorem 86 in B.W. Jones, 1950 which can be described as the following claim:

Claim: Let f be a positive definite ternary quadratic form with matrix A, $d = |A|$, Ω the g.c.d. of the 2-rowed minor determinants of A and $\Delta = qd/\Omega^2$ with q prime to $2d$, then for any eligible number q of the genus of f with $(q,2d) = 1$ we have that

$$G(A,q) = 2^{-t(d/\Omega^2)}H(\Delta)\rho_\Delta$$

where $t(w)$ is the number of odd prime factors of w, $H(\Delta)$ is the number of properly primitive classes of positive binary forms $ax^2+2bxy+cy^2$ of determinant $\Delta = ac-b^2$,

ρ_Δ is a rational number equal to 1/8, 1/6, 1/4, 1/3, 1/2, 2/3, 1, 2, 4 according to the different cases of the values of Δ, and $G(A,q)$ is the number of essentially distinct primitive representations of q by the genus of f. Please compare Theorem 86 in B.W. Jones, 1950 for details.

Now let $\mathbb{G} = \{f = f_1, f_2, \cdots, f_t\}$ be a set of representatives of the genus of f. Define

$$\theta_f(z) = \sum_{m\in\mathbb{Z}^3} e(zmAm^{\mathrm{T}}/2), \quad z \in \mathbb{H},$$
$$O(f) = \#\{S \in M_3(\mathbb{Z}) | SAS^{\mathrm{T}} = A\},$$
$$\theta(\text{gen}.f, z) = \left(\sum_{f_i} \frac{1}{O(f_i)}\right)^{-1} \sum_{f_i} \frac{\theta_{f_i}(z)}{O(f_i)},$$

then we have that

$$\theta_f(z) - \theta(\text{gen}.f, z) \in S(N, 3/2, \chi)$$

by the results in Section 10.1. Now let $r_i(n)$ be the number of representations of n by f_i, then

$$\begin{aligned}\theta_f(z) - \theta(\text{gen}.f, z) &:= \sum_{n=1}^{\infty} a(n)q^n \\ &= \sum_{n=1}^{\infty} r_1(n)q^n - \left(\sum_{f_i} \frac{1}{O(f_i)}\right)^{-1} \sum_{n=1}^{\infty} \left(\sum_{f_i} \frac{r_i(n)}{O(f_i)}\right) q^n.\end{aligned}$$

Now suppose that n_0 is a square-free eligible number of $\mathbb{G}$ which can not be represented by $f = f_1$, i.e., $r_1(n_0) = 0$. Then by Iwaniec's H. Iwaniec, 1987 and Duke's W. Duke, 1988 we have that

$$|a(n_0)| = \left(\sum_{f_i} \frac{1}{O(f_i)}\right)^{-1} \left(\sum_{f_i} \frac{r_i(n)}{O(f_i)}\right) \ll \tau(n_0) n_0^{\frac{3}{7}} (\log 2n_0)^2.$$

On the other hand, let $G_i(n)$ be the essentially distinct primitive representations of n by f_i, it is clear that $2G_i(n) \leqslant r_i(n)$ because every positive definite ternary quadratic form has at least two automorphs. So we see that

$$\begin{aligned}G(A, n) &= \sum_{f_i} G_i(n) \leqslant \frac{1}{2} \sum_{f_i} r_i(n) \\ &\leqslant \frac{O(\mathbb{G})}{2} \sum_{f_i} \frac{r_i(n)}{O(f_i)} = \frac{O(\mathbb{G}) \left(\sum_{f_i} \frac{1}{O(f_i)}\right)}{2} |a(n)|,\end{aligned}$$

where $O(\mathbb{G}) = \max\{O(f_i)\}$. So by the above **Claim** and Siegel's lower bounds for the class numbers we see that

$$|a(n_0)| \gg G(A, n_0) \gg H(\Delta) = H(n_0 d/\Omega^2) \gg n_0^{1/2-\epsilon}.$$

Comparing these two estimations we see that there are only finitely many square-free eligible integers prime to $2|A|$ which can not be represented by f. This completes the proof. □

Remark 10.2 Notice that there are some similarities between our Theorem 10.9 and Theorem 86 in B.W. Jones, 1950, but they differ from one another in the following aspects:

(1) In general $G(n) \neq G(A, n)$ and there is no simple equality between them. Of course we have the inequality $G(n) \leqslant G(A, n) \leqslant \dfrac{O(\mathbb{G})}{2} G(n)$ just as we saw in the proof of Theorem 10.10;

(2) In Jones' Theorem 86, it is assumed that $(n, N) = 1$ where N is the level of the quadratic form f. But we need not this assumption in our Theorem 10.9;

(3) Jones' Theorem 86 can not tell us which are the eligible numbers for the genus but our Theorem 10.9 can do this (cf. Example 10.5 and Example 10.6). Anyway neither does our Theorem 10.9 contain Jones' Theorem 86, nor is the converse the case.

Since we employed Theorem 86 (i.e., our **Claim**) in B.W. Jones, 1950 in our proof of Theorem 10.10, we have to limit ourselves to the case with n_0 prime to $2d$. For the case with n_0 not prime to $2|A|$, we may employ our Theorem 10.9. For a concrete positive definite ternary quadratic form f, we can always investigate any square-free natural number n (prime or not prime to $2|A|$) by Theorem 10.9. For example we take the forms in Corollary 10.1 and Corollary 10.2. Suppose that $p \equiv 3 \pmod 4$, N a square-free eligible number not represented by $f_p = x^2 + x^2 + pz^2$ or $f_p = x^2 + py^2 + pz^2$, by (10.26), (10.26):

$$\begin{aligned}
\alpha(pN) &= \alpha(N) = \frac{3}{2},\ 1 \text{ or } 0,\\
\beta_{p,p}(N) &= p + 1 \text{ or } 2,\\
\gamma_p(N) &= \left(1 - \frac{\chi_{-pN}(2)}{2}\right)\left(\frac{pN}{\delta_{pN}}\right)^{\frac{1}{2}} \geqslant \frac{1}{4},\\
\beta_{1,p}(N) &= p + 1 \text{ or } 2p,\\
\gamma'_p(N) &= \left(1 - \frac{\chi_{-N}(2)}{2}\right)\left(1 - \frac{\chi_{-N}(p)}{p}\right)\left(\frac{N}{\delta_N}\right)^{\frac{1}{2}} \geqslant \frac{p-1}{4p}.
\end{aligned}$$

Then Corollary 10.1 and Corollary 10.2 imply that

$$|a(N)| = \left(\sum_{f_i} \frac{1}{O(f_i)}\right)^{-1} \left(\sum_{f_i} \frac{r_i(n)}{O(f_i)}\right) \gg h(-pN) \gg N^{1/2-\epsilon},$$

$$|a(N)| = \left(\sum_{f_i} \frac{1}{O(f_i)}\right)^{-1} \left(\sum_{f_i} \frac{r_i(n)}{O(f_i)}\right) \gg h(-N) \gg N^{1/2-\epsilon}$$

because of Siegel's lower bounds for class numbers. Together with the estimations in H. Iwaniec, 1987 and W. Duke, 1988 as above, we obtain that there exist at most finitely many square-free eligible integers which are not represented by $f_p = x^2 + y^2 + pz^2$ or $f_p = x^2 + py^2 + pz^2$ for $p \equiv 3 \pmod 4$. We can similarly discuss this phenomenon for $p \equiv 1 \pmod 4$.

Remark 10.3 Even though there exist only finitely many square-free eligible numbers prime to $2|A|$ which can not be represented by a positive definite ternary quadratic forms, it is not implementable to find all of these eligible numbers through computation for two reasons: ① Siegel's lower bounds for class numbers are not effective; ② it is impossible to obtain a contradiction through computation even if we assume that the lower bounds are effective since we have to compute all of n with $n^{1/2} \leqslant \tau(n) n^{3/7} (\log(2n))^2$ which requires that n is about 10^{75}. Even if we replace Iwaniec's bound by a sharper bound, cf. V.A. Bykovskii, 1998, we also can not implement the algorithm to find all of these exceptional eligible integers by calculation.

Theorem 10.11 *Let $\mathbb{A} = \{f_1, f_2, \cdots, f_t\}$ be a set of representatives of the genus of a positive definite ternary quadratic form of level N. Assume that there are the following linear combinations of Theta- functions:*

$$\widetilde{f_i}(z) := \sum_{n=1}^{\infty} b_i(n) q^n = \sum_{j=1}^{i+1} \alpha_{i,j} \theta(f_j)$$

with $\alpha_{i,1}\alpha_{i,i+1} \neq 0$ for $1 \leqslant i \leqslant t-1$, such that $\widetilde{f_i}(z)$ is an eigenfunction for all Hecke *operators whose Shimura lifting is a cusp form corresponding to an elliptic curve E_i. Then we can find an effectively determinable finite set $P_{\mathbb{A}} = \{p_0, p_1, \cdots, p_s\}$ of primes such that for every square-free eligible number n_0 of $\mathbb{A}$ with $(n_0, N) = 1$* (i.e., *$(n_0, N) = 1$ and n_0 can be represented by one of the forms in $\mathbb{A}$) and for every prime p not in $P_{\mathbb{A}}$, we have that $p^2 n_0$ can be represented by f_1.*

Proof We only consider the case $t = 3$ because the general case is similar. Let N be the level of f_1, P_N the set of all distinct prime factors of N, and $F_i(z) := \sum_{n=1}^{\infty} B_i(n) q^n$ the Shimura lifting of $\widetilde{f_i}(z)$. Since $\widetilde{f_i}(z)$ is an eigenfunction for all Hecke operators, there exist complex numbers α_{ip} such that $T_{p^2}(\widetilde{f_i}(z)) = \alpha_{ip} \widetilde{f_i}(z)$. But Hecke operators commute with Shimura liftings. Therefore

$$T_p(F_i(z)) = T_p(S(\widetilde{f}_i(z))) = S(T_{p^2}(\widetilde{f}_i(z))) = S(\alpha_{ip}\widetilde{f}_i(z)) = \alpha_{ip}F_i(z).$$

But because $F_i(z)$ is a new form corresponding to the elliptic curve E_i, it shows that $T_p(F_i(z)) = B_i(p)F_i(z)$. Hence we see that $\alpha_{ip} = B_i(p)$ for any $p \notin P_N$. This implies that

$$B_i(p)b_i(n) = b_i(p^2n) + \chi(p)\left(\frac{-n}{p}\right)b_i(n) + pb_i(n/p^2)$$

for any prime p with $(p, N) = 1$ and any positive integer n. Especially for any square-free positive integer n we have that

$$B_i(p)b_i(n) = b_i(p^2n) + \chi(p)\left(\frac{-n}{p}\right)b_i(n).$$

Hence we see that

$$\alpha_{11}r_1(p^2n) + \alpha_{12}r_2(p^2n) = (\alpha_{11}r_1(n) + \alpha_{12}r_2(n))\left(B_1(p) - \chi(p)\left(\frac{-n}{p}\right)\right) \quad (10.29)$$

$$\begin{aligned}&\alpha_{21}r_1(p^2n) + \alpha_{22}r_2(p^2n) + \alpha_{23}r_3(p^2n)\\ =&(\alpha_{21}r_1(n) + \alpha_{22}r_2(n) + \alpha_{23}r_3(n))(B_2(p) - \chi(p)\left(\frac{-n}{p}\right),\end{aligned} \quad (10.30)$$

where $r_i(n)$ is the number of representations of n by f_i. We want to prove that for any square-free eligible number n_0 of $\mathbb{A}$ which is prime to N and not represented by f_1, p^2n_0 can be represented by f_1 where $p \notin P_{\mathbb{A}}$ and $P_{\mathbb{A}}$ containing P_N is an effectively determinable finite set of primes. Otherwise, suppose that $p \notin P_N$ is a prime such that p^2n_0 can not be represented by f_1. Let be $n = n_0$ in (10.29) and (10.30), then

$$r_2(p^2n_0) = r_2(n_0)\left(B_1(p) - \chi(p)\left(\frac{-n_0}{p}\right)\right), \quad (10.31)$$

$$\alpha_{22}r_2(p^2n_0) + \alpha_{23}r_3(p^2n_0) = \alpha_{22}r_2(n_0) + \alpha_{23}r_3(n_0))\left(B_2(p) - \chi(p)\left(\frac{-n_0}{p}\right)\right), \quad (10.32)$$

since $r_1(n_0) = r_1(p^2n_0) = 0$. By (10.31) and (10.32) it is clear that

$$\begin{aligned}\alpha_{23}r_3(p^2n_0) :=& \alpha r_2(n_0) + \beta r_3(n_0)\\ =& \alpha_{22}(B_2(p) - B_1(p))r_2(n_0) + \alpha_{23}\left(B_2(p) - \chi(p)\left(\frac{-n_0}{p}\right)\right)r_3(n_0).\end{aligned}$$

Now let $G(n)$ and $G_i(n)$ be the essentially distinct primitive representations of n by $\mathbb{A}$ and f_i respectively. Then we have that $2G_i(n) \leqslant r_i(n)$ and $G_i(n) \geqslant \dfrac{r_i(n)}{O(f_i)}$. So

$$G(n) = \sum_{i=1}^{t} G_i(n) \leqslant \frac{1}{2}\sum_{i=1}^{t} r_i(n) \leqslant \frac{O(\mathbb{A})}{2}\sum_{i=1}^{t}\frac{r_i(n)}{O(f_i)},$$

$$G(n) = \sum_{i=1}^{t} G_i(n) \geqslant \sum_{i=1}^{t}\frac{r_i(n)}{O(f_i)},$$

where $O(\mathbb{A}) = \max\{O(f_i)\}$. From these and the **Claim** in the proof of Theorem 10.10 we see that

$$\frac{H(p^2\Delta)\rho_{p^2\Delta}}{H(\Delta)\rho_\Delta} = \frac{G(p^2n_0)}{G(n_0)} \leqslant \frac{O(\mathbb{A})}{2}\frac{\sum_{i=1}^{t}\frac{r_i(p^2n_0)}{O(f_i)}}{\sum_{i=1}^{t}\frac{r_i(n_0)}{O(f_i)}}$$

$$= \frac{O(\mathbb{A})}{2}\frac{\delta_2 r_2(p^2n_0)+\delta_3 r_3(p^2n_0)}{\delta_2 r_2(n_0)+\delta_3 r_3(n_0)}$$

$$= \frac{O(\mathbb{A})}{2}\frac{\delta_2 r_2(p^2n_0)+\delta_3\alpha_{23}^{-1}\alpha r_2(n_0)+\delta_2\alpha_{23}^{-1}\beta r_3(n_0)}{\delta_2 r_2(n_0)+\delta_3 r_3(n_0)}, \quad (10.33)$$

where $\delta_i = \dfrac{1}{O(f_i)}$ and $\Delta = n_0 d/\Omega^2$ as in the proof of Theorem 10.10. Now consider two cases:

Case (1) Suppose that $r_3(n_0) \leqslant r_2(n_0)$, then (10.31)–(10.33) show that

$$\frac{1}{3}(p-1) \leqslant \frac{O(\mathbb{A})}{2}\frac{\left|\delta_2\frac{r_2(p^2n_0)}{r_2(n_0)}+\delta_3\alpha\alpha_{23}^{-1}+\delta_3\beta\alpha_{23}^{-1}\frac{r_3(n_0)}{r_2(n_0)}\right|}{\delta_2+\delta_3\frac{r_3(n_0)}{r_2(n_0)}}$$

$$\leqslant \frac{O(\mathbb{A})}{2}\frac{\delta_2\left|B_1(p)-\chi(p)\left(\frac{-n_0}{p}\right)\right| + |\ \delta_3\alpha\alpha_{23}^{-1}\ | + |\ \delta_3\beta\alpha_{23}^{-1}\ |}{\delta_2}.$$

Case (2) Suppose that $r_2(n_0) \leqslant r_3(n_0)$, a similar computation shows that

$$\frac{1}{3}(p-1) \leqslant \frac{O(\mathbb{A})}{2}\frac{\left|\delta_2\frac{r_2(p^2n_0)}{r_3(n_0)}+\delta_3\beta\alpha_{23}^{-1}+\delta_3\alpha\alpha_{23}^{-1}\frac{r_2(n_0)}{r_3(n_0)}\right|}{\delta_3+\delta_2\frac{r_2(n_0)}{r_3(n_0)}}$$

$$\leqslant \frac{O(\mathbb{A})}{2}\frac{\delta_2\left|\frac{r_2(p^2n_0)}{r_2(n_0)}\right| + |\ \delta_3\beta\alpha_{23}^{-1}\ | + |\ \delta_3\alpha\alpha_{23}^{-1}\ |}{\delta_3}$$

$$\leqslant \frac{O(\mathbb{A})}{2}\frac{\delta_2\left|B_1(p)-\chi(p)\left(\frac{-n_0}{p}\right)\right| + |\ \delta_3\alpha\alpha_{23}^{-1}\ | + |\ \delta_3\beta\alpha_{23}^{-1}\ |}{\delta_3},$$

where we used the facts that $H(p^2\Delta)/H(\Delta) = p - \left(\dfrac{\Delta}{p}\right)$ and $\rho_{p^2\Delta}/\rho_\Delta \geqslant 1/3$ (cf. Theorem 86 in B.W. Jones, 1950). Anyway we have obtained the following inequality:

$$p-1 \leqslant C_1 \mid B_1(p) \mid + C_2 \mid B_2(p) \mid + C_3,$$

where C_1, C_2, C_3 are positive constants only dependent on α_{ij} and $O(f_i)$. On the other hand we have that $| B_i(p) | \leqslant 2p^{1/2}$ which implies that

$$p - 1 \leqslant 2(C_1 + C_2)\sqrt{p} + C_3.$$

It is clear that this inequality only holds for finitely many primes. Denote it by P. Then for any $p \notin P_{\mathbb{A}} = P \bigcup P_N$ we have that $p^2 n_0$ can be represented by f_1 which completes the proof. □

The argumentation in the above proof implies the following

Corollary 10.5 *Let $\mathbb{A} = \{f, g\}$ be a genus consisting of two equivalence classes such that $\widetilde{f}(z) = \alpha\theta(f) + \beta\theta(g)$ is an eigenfunction for all* Hecke *operators and its* Shimura *lifting is a cusp form corresponding to an elliptic curve E. Then for any eligible integer n_0 which is prime to $2|A|$ and not represented by f and any prime $p \notin P_{\mathbb{A}}$, $p^2 n_0$ can be represented by f where $P_{\mathbb{A}} = \left\{p \text{ prime} \mid p|N \text{ or } \frac{1}{3}(p-1) \leqslant \frac{O(\mathbb{A})}{2}(2\sqrt{p}+1)\right\}$ and N is the level of f.*

Remark 10.4 Just as pointed out in Remark 10.2, to investigate the case n not prime to the level or to obtain more precise result about the set $P_{\mathbb{A}}$, we may employ our Theorem 10.9. The following proof of Theorem 10.12 is an example together with the ideas in Theorem 10.11 and Theorem 10.9.

Theorem 10.12 *Let be $f_2 = x^2 + 7y^2 + 7z^2$. If n is a positive integer with $\left(\frac{n}{7}\right) = 1$* (i.e., *$n$ is an eligible integer prime to* 7) *which can not be represented by f_2, then n is square-free.*

Proof By Example 10.4 and the fact that n is an eligible integer, we know that

$$0 < G(n) := \frac{r_2(n)}{8} + \frac{r_2'(n)}{4} = \frac{1}{4}\omega_n^{-1}(2 - \alpha(n))\beta_{1,7}(n)\gamma_7'(n)h(-n), \tag{10.34}$$

where $r_2(n)$ and $r_2'(n)$ denote the numbers of representations of n by f_2 and $g_2 = 2x^2 + 4y^2 + 7z^2 - 2xy$ respectively. We also easily know that

$$\widetilde{f_2}(z) := \sum_{n=1}^{\infty} b(n)\mathrm{e}^{2\pi \mathrm{i} nz} = \frac{1}{2}\sum_{n=1}^{\infty}(r_2(n) - r_2'(n))\exp\{2\pi \mathrm{i} nz\},$$

is an eigenfunction of all Hecke operators T_{n^2} in the space $S(28, 3/2, \chi_1)$ by a direct computation. And the Shimura lifting $F_2(z) = S(\widetilde{f_2}(z))$ of $\widetilde{f_2}(z)$ is a new form with weight 2, character χ_1 and level 14, i.e., $F_2(z) \in S^{\text{new}}(14, 3/2, \chi_1)$. So there exist complex numbers α_n such that $T_{n^2}(\widetilde{f_2}(z)) = \alpha_n \widetilde{f_2}(z)$. But Hecke operators commute with the Shimura lifting. So we see that

$$T_n(F_2(z)) = T_n(S(\widetilde{f_2}(z))) = S(T_{n^2}(\widetilde{f_2}(z))) = S(\alpha_n \widetilde{f_2}(z)) = \alpha_n S(\widetilde{f_2}(z)) = \alpha_n F_2(z)$$

which implies that α_n are also the eigenvalues of T_n for $F_2(z)$. But because $F_2(z)$ is a new form with weight 2 shows that for any positive integer m with $(m, 14) = 1$, $\alpha_m = B(m)$ where $F_2(z) = \sum_{n=1}^{\infty} B(n)\mathrm{e}^{2\pi \mathrm{i} nz}$ is the Fourier expansion of $F_2(z)$. These facts show that

$$B(p)b(n) = \alpha_p b(n) = b(p^2 n) + \left(\frac{-n}{p}\right) b(n) + pb(n/p^2) \tag{10.35}$$

for any prime p with $(p, 14) = 1$ and any positive integer n. We obtain by $\frac{r_2(n) - r_2'(n)}{2}$ instead of $b(n)$ that

$$r_2(p^2 n) - r_2'(p^2 n) = \left(B(p) - \left(\frac{-n}{p}\right)\right)(r_2(n) - r_2'(n)) + p(r_2(n/p^2) - r_2'(n/p^2)).$$

In particular, if n is a square-free positive integer, then for any prime p with $(p, 14) = 1$, we see that

$$r_2(p^2 n) - r_2'(p^2 n) = \left(B(p) - \left(\frac{-n}{p}\right)\right)(r_2(n) - r_2'(n)). \tag{10.36}$$

For a prime p such that $p|14$, by the definition of Hecke operators, we see that $T_{p^2}(\widetilde{f}_2(z)) = \sum_{n=1}^{\infty} b(p^2 n)\mathrm{e}^{2\pi \mathrm{i} nz}$ which implies that

$$\alpha_p b(n) = b(p^2 n),$$

i.e.

$$r_2(p^2 n) - r_2'(p^2 n) = \alpha_p(r_2(n) - r_2'(n)). \tag{10.37}$$

An easy calculation shows that $\alpha_2 = -1$ and $\alpha_7 = 1$. We now want to prove that if n_0 is square-free eligible number such that $r_2(n_0) = 0$ (i.e., n_0 is not represented by f_2) then $r_2(p^2 n_0) \neq 0$ (i.e., $p^2 n_0$ can be represented by f_2) for any prime p with $(p, 7) = 1$. Otherwise, we have by (10.36), (10.37) that

$$\begin{aligned} \frac{r_2'(p^2 n_0)}{r_2'(n_0)} &= B(p) - \left(\frac{-n_0}{p}\right) \leqslant B(p) + 1, \\ \frac{r_2'(2^2 n_0)}{r_2'(n_0)} &= \alpha_2 = -1. \end{aligned} \tag{10.38}$$

On the other hand, we have that by (10.34)

$$\begin{aligned} \frac{r_2'(p^2 n_0)}{r_2'(n_0)} &= \frac{G_2(p^2 n_0)}{G_2(n_0)} \\ &= \frac{\omega_{p^2 n_0}^{-1}(2 - \alpha(p^2 n_0))\beta_{1,7}(p^2 n_0)\gamma_7'(p^2 n_0)h(-p^2 n_0)}{\omega_{n_0}^{-1}(2 - \alpha(n_0))\beta_{1,7}(n_0)\gamma_7'(n_0)h(-n_0)} \\ &= \frac{(2 - \alpha(p^2 n_0))\beta_{1,7}(p^2 n_0)\gamma_7'(p^2 n_0)}{(2 - \alpha(n_0))\beta_{1,7}(n_0)\gamma_7'(n_0)}. \end{aligned} \tag{10.39}$$

We now suppose that $p \neq 7$ and 2, then by the definitions of $\alpha(n), \beta_{1,7}(n), \gamma'_7(n)$ and n_0 a square-free integer, we easily obtain that

$$\begin{aligned}
\alpha(p^2 n_0) &= \alpha(n_0), \\
\beta_{1,7}(p^2 n_0) &= \beta_{1,7}(n_0), \\
\gamma'_7(p^2 n_0) &= (p+1)\gamma'_7(n_0), \\
\alpha(2^2 n_0) &= \frac{1}{2}\alpha(n_0), \\
\beta_{1,7}(2^2 n_0) &= \beta_{1,7}(n_0), \\
\gamma'_7(2^2 n_0) &= \frac{3}{1 - 2^{-1}\chi_{-n_0}(2)}\gamma'_7(n_0), \\
\alpha(7^2 n_0) &= \alpha(n_0), \\
\beta_{1,7}(7^2 n_0) &= \frac{1}{7}\beta_{1,7}(n_0), \\
\gamma'_7(7^2 n_0) &= \frac{8}{1 - \chi_{-n_0}(7)7^{-1}}\gamma'_7(n_0).
\end{aligned}$$

Hence we see that

$$\frac{r'_2(p^2 n_0)}{r'_2(n_0)} = \begin{cases} p+1, & \text{if } p \neq 2, 7, \\ 5, & \text{if } p = 2, \nu_2(n_0) = 1, \\ 15, & \text{if } p = 2, n_0 \equiv 1 \pmod 4, \\ 9, & \text{if } p = 2, n \equiv 3 \pmod 8, \\ 6, & \text{if } p = 2, n \equiv 7 \pmod 8, \\ \dfrac{8}{7 - \chi_{-n_0}(7)}, & \text{if } p = 7. \end{cases} \tag{10.40}$$

For any prime $p \neq 2, 7$, by equalities (10.38) and (10.40) we have that

$$B(p) \geqslant p$$

and

$$0 < \frac{r'_2(2^2 n_0)}{r'_2(n_0)} = -1 < 0,$$

which is impossible, since n_0 is an eligible integer. On the other hand, it is well known that $B(p) \leqslant 2p^{\frac{1}{2}}$ by Deligne's estimation for coefficients of modular forms. This implies that $2p^{\frac{1}{2}} \geqslant p$ for any prime $p \neq 2$ and 7 which is impossible.

What we have proved is that if n is any square-free eligible number of the genus of f_2 which is not represented by f_2, then $p^2 n$ can be represented by f_2 for any prime p with $p \neq 7$. This, of course, is equivalent to saying that if an eligible number n prime to 7 can not be represented by f_2 then n is square-free. This completes the proof. □

As a conclusion of Theorem 10.12 we have that

Theorem 10.13 *The form $f_1 = x^2 + y^2 + 7z^2$ represents all eligible numbers which are multiples of 9; it also represents all eligible numbers congruent to 2 mod 3 except those of the trivial type. In other words, the* Kaplansky's Conjecture *holds.*

Proof We first show the following fact: $f_1 = x^2 + y^2 + 7z^2$ does not represent $7A$ if and only if $f_2 = x^2 + 7y^2 + 7z^2$ does not represent A.

In fact, it is obvious that if f_2 represents A, i.e., there are integers a, b, c such that $a^2 + 7b^2 + 7c^2 = A$, then $7A = (7b)^2 + (7c)^2 + 7a^2$. Conversely, if $7A = x^2 + y^2 + 7z^2$, then $x^2 + y^2 \equiv 0 \pmod 7$ which implies that $x \equiv 0 \pmod 7$ and $y \equiv 0 \pmod 7$. Let be $x = 7x'$, $y = 7y'$, we see that $A = z^2 + 7(x')^2 + 7(y')^2$ which shows that f_2 represents A.

By Example 10.6, we know that the eligible numbers of f_2 are precisely all integers which are not the product of an even power of 7 and a number congruent to 3, 5, 6 mod 7. Hence, to prove Theorem 10.13 we only need to show that f_2 represents all eligible numbers which are congruent to 1, 2, 4 mod 7 and of form $2t^2$ with $t \neq 1$ and $7 \nmid t$. If $2|t$, it is clear that f_2 represents $2t^2$ because f_2 represents 8. Hence we can assume that t is an odd integer. This shows that Theorem 10.12 implies Theorem 10.13. □

Remark 10.5 If n is not prime to 7, the result in Theorem 10.12 does not hold. For example $n = 98 = 2 \cdot 7^2$ can not be represented by f_2. In fact, for $p = 7$, the above proof is not suitable because we can not obtain a contradiction as above for $p \neq 7$. For if we assume that n_0 is an eligible number such that $r_2(n_0) = r_2(7^2 n_0) = 0$, then the calculations above show that $\dfrac{8}{7 - \chi_{-n_0}(7)} = \dfrac{r_2'(7^2 n_0)}{r_2'(n_0)} = \alpha_7 = 1$, which possibly holds, e.g., $n_0 = 2$ makes it hold. In this proof we need not introduce the concept of essentially distinct primitive representations. And for the formula giving the number of representations for a genus of positive definite ternary quadratic forms, we also need not assume that our discussion is limited to the integers prime to the level of the quadratic form because we do not employ Theorem 86 in B.W. Jones, 1950. In fact, the argumentation of the above proof can also be applied to other genera consisting of two equivalent classes. For example, we can prove the following result:

Corollary 10.6 *Let $f_{(p)} = x^2 + py^2 + pz^2$ with an odd prime p and assume that the genus of $f_{(p)}$ consists of two equivalence classes which we denote by $f_{(p)}$ and $g_{(p)}$. Denote*

$$\widetilde{f}_{(p)}(z) := \sum_{n=1}^{\infty} b(n) e^{2\pi i n z} = \frac{1}{2} \sum_{n=1}^{\infty} (r(n) - r'(n)) e^{2\pi i n z},$$

where $r(n)$ and $r'(n)$ are the numbers of representations of n by $f_{(p)}$ and $g_{(p)}$ respectively. And assume that the Shimura *lifting $F_{(p)}(z) = S(\widetilde{f}_{(p)}(z))$ of $\widetilde{f}_{(p)}(z)$ is a new form of weight 2 corresponding to a modular elliptic curve, then every eligible number prime to $2p$ of the genus of $f_{(p)}$ not represented by $f_{(p)}$ is square-free.*

Proof It is completely similar to the proof of Theorem 10.12. □

Example 10.7 Every eligible integer prime to 34 not represented by $f_{(17)} = x^2 + 17y^2 + 17z^2$ is square free. This is because that the genus for $f_{(17)}$ consists of $f_{(17)}$ and $g_{(17)} = 2x^2 + 9y^2 + 17z^2 + 2xy$ and the Shimura lifting of $\widetilde{f}_{(17)}(z)$ is the new form corresponding to the modular elliptic curve (34A). □

Combining Theorem 10.10, Theroem 10.12, Remark 10.2 and the result of Corollary 10.6 we indeed obtain:

Corollary 10.7 *Let $f_{(p)} = x^2 + py^2 + pz^2$ be as in* Corollary 10.6*. Then there are only finitely many eligible numbers which are prime to* $2p$ *and not represented by the quadratic form $f_{(p)}$. In particular, there are only finitely many eligible numbers prime to* 7 *and* 34 *not represented by the forms $f_{(7)}$ and $f_{(17)}$ respectively.*

We now consider the following problem: Let n be a square free positive integer, f and g be two ternary positive definite quadratic forms in the same genus, then when do we have that $r(f,n) \neq r(g,n)$ where $r(f,n)$ and $r(g,n)$ are the numbers of representation of n by f and g respectively. For example, if $f_{(7)} = x^2 + 7y^2 + 7z^2$, $g_{(7)} = 2x^2 + 4y^2 + 7z^2 - 2xy$, then $f_{(7)}$ and $g_{(7)}$ are in the same genus, and we want to know when do we have that $r(f_{(7)}, n) \neq r(g_{(7)}, n)$ for a positive integer. It is clear that we only need to consider eligible numbers n because $r(f,n) = r(g,n) = 0$ if n is not eligible.

We now assume always that f and g are in the same genus and $r(f,1) \neq r(g,1)$. Let

$$\widetilde{f}(z) =: \sum_{n=1}^{\infty} b_n \mathrm{e}^{2\pi i n z} = \frac{1}{r} \sum_{n=1}^{\infty} (r(f,n) - r(g,n)) \exp\{2\pi i n z\},$$

where $r = r(f,1) - r(g,1) \neq 0$. Then $\widetilde{f}(z) \in S(N,\ 3/2,\ \chi_l)$. For example, we have that

$$\begin{aligned}\widetilde{f}_{(7)}(z) &= \frac{1}{2} \sum_{n=1}^{\infty} (r(f_{(7)}, n) - r(g_{(7)}, n)) \exp\{2\pi i n z\} \\ &= q + \cdots \in S(28, 3/2, \chi_1), \quad q = \exp\{2\pi i z\}.\end{aligned}$$

We assume further that the Shimura lifting $F(z)$ of $\widetilde{f}(z)$ is a new form corresponding to a modular elliptic curve $E/\mathbb{Q}$. For example, we see that $F_{(7)}(z) = S(\widetilde{f}_{(7)}(z))$ is the new form corresponding to the modular elliptic curve (14C):

$$(14\mathrm{C}): \qquad y^2 = x^3 + x^2 + 72x - 368.$$

$F_{(11)}(z) = S(\widetilde{f}_{(11)}(z))$ is the new form corresponding to the modular elliptic curve (11B) where $f_{(11)} = x^2 + 11y^2 + 11z^2$:

$$(11\mathrm{B}): \qquad y^2 + y = x^3 - x^2 - 10x - 20.$$

By the definition of $\widetilde{f}(z)$, what we want to know is that when are the coefficients of $\widetilde{f}(z)$ not equal to zero. In order to do this we need the following result of Waldspurger:

Lemma 10.7 *Assume that $E/\mathbb{Q}$ is a modular elliptic curve with corresponding cusp form f_E, and that*

$$F \in S(N, 3/2, \chi_t)\bigcap S_0(N, \chi_t)^{\perp}$$

with

$$S(F) = f_E, \quad F = \sum_{n=1}^{\infty} a_n \mathrm{e}^{2\pi i n z},$$

where $S_0(N, \psi)$ is the subspace of $S(N, 3/2, \psi)$ generated by the form F of the following type: There is a $t \in \mathbb{N}$ and a quadratic character χ with conductor r such that $F = \sum_{m=1}^{\infty} \chi(m) m q^{tm^2}$ and $N = 4r^2 t$, $\psi = \chi \cdot \chi_t \cdot \chi_{-1}$. Assume that d and d_0 are natural square free numbers with

$$d \equiv d_0 \mod \left(\prod_{p|N} \mathbb{Q}_p^{*2}\right), \qquad \textit{and } (dd_0, N) = 1.$$

Then

$$L_{E_{-td}}(1)\sqrt{d} a_{d_0}^2 = L_{E_{-td_0}}(1)\sqrt{d_0} a_d^2.$$

So especially: if

$$L_{E_{-td_0}} a_{d_0} \neq 0,$$

then

$$L_{E_{-td}}(1) = 0 \textit{ if and only if } a_d = 0,$$

where $L_{E_D}(s)$ is the Hasse-Weil Zeta function of the D-th twist of elliptic curves E.

Now denote the set of representatives of all inequivalent integers mod $\prod_{p|N} \mathbb{Q}_p^{*2}$ which are eligible numbers for the genus of f and prime to N by D_N, then D_N is finite. Let be $D_N = \{d_1, d_2, \cdots, d_l\}$.

We have that for any square free eligible natural integer d such $(d, N) = 1$, there exist unique $d_i \in D_N$ such that

$$\frac{L_{E_{-td}}(1)\sqrt{d}}{a_d^2} = \frac{L_{E_{-td_i}}(1)\sqrt{d_i}}{a_{d_i}^2}.$$

Using this equality, we can deduce when the coefficients a_d are different from zero.

Example 10.8 Let $f = f_{(7)}$, $g = g_{(7)}$, $E = (14\mathrm{C})$, then

$$\widetilde{f}_{(7)}(z) = \frac{1}{2}\left(\theta(f_{(7)}) - \theta(g_{(7)})\right) \in S_{3/2}(28, \chi_1)$$

and

$$F_{(7)}(z) = S(\widetilde{f}_{(7)}(z)) \in S_2^{\text{new}}(14)$$

corresponding to the modular elliptic curve (14C). And we can calculate that

$$\begin{aligned}
D_{28} &= \{1, 11, 15, 29\}, \\
L_{E_{-d_i}} &\neq 0, \text{for all} d_i \in D_{28}, \\
b_1 &= \frac{1}{2}(r(f_{(7)}, 1) - r(g_{(7)}, 1)) = 1, \\
b_{11} &= \frac{1}{2}(r(f_{(7)}, 11) - r(g_{(7)}, 11)) = \frac{1}{2}(8-8) = 0, \\
b_{15} &= \frac{1}{2}(r(f_{(7)}, 15) - r(g_{(7)}, 15)) = \frac{1}{2}(8-8) = 0, \\
b_{29} &= \frac{1}{2}(r(f_{(7)}, 29) - r(g_{(7)}, 29)) = \frac{1}{2}(8-4) = 2.
\end{aligned}$$

These calculations and Waldspurger's Theorem show that for square free eligible numbers d such that $(d, 14) = 1$:

$$r(f_{(7)}, d) = r(g_{(7)}, d), \quad \text{if } d \equiv 11, 15 \mod \left(\prod_{p|28} \mathbb{Q}_p^{*2}\right),$$

$$r(f_{(7)}, d) \neq r(g_{(7)}, d) \text{ if and only if } L_{E_{-d}}(1) \neq 0 \text{ for } d \equiv 1, 29 \mod \left(\prod_{p|28} \mathbb{Q}_p^{*2}\right).$$

□

Hence we have the following:

Theorem 10.14 *Let be* $f_{(7)} = x^2 + 7y^2 + 7z^2, g_{(7)} = 2x^2 + 4y^2 + 7z^2 - 2xy$, *$E$ the corresponding modular elliptic curve of the cusp form* $\frac{1}{2}(\theta(f_{(7)}) - \theta(g_{(7)}))$ *and E_{-d} the $-d$-twist of E. Then for any square free eligible numbers d such that $(d, 14) = 1$, we have that*

(1) $r(f_{(7)}, d) = r(g_{(7)}, d)$, *if* $d \equiv 11, 15 \mod \left(\prod_{p|28} \mathbb{Q}_p^{*2}\right)$;

(2) $r(f_{(7)}, d) \neq r(g_{(7)}, d)$ *if and only if* $L_{E_{-d}}(1) \neq 0$ *for* $d \equiv 1, 29 \mod \left(\prod_{p|28} \mathbb{Q}_p^{*2}\right)$,

where $L_{E_{-d}}(s)$ is the Hasse-Weil *L-function of the elliptic curve E_{-d}. Especially, if n is a square free natural number such that*

$$n \equiv 3 \pmod 8 \text{ and } \left(\frac{n}{7}\right) = 1$$

or

$$n \equiv 7 \,(\mathrm{mod}\ 8) \ \textit{and}\ \left(\frac{n}{7}\right) = 1,$$

then

$$r(f_{(7)}, n) = r(g_{(7)}, n).$$

Proof Above all proved except for the last assertion. But

$$n \equiv 3 \,(\mathrm{mod}\ 8) \quad \text{and} \quad \left(\frac{n}{7}\right) = 1$$

implies that

$$n \equiv 11 \,\mathrm{mod} \left(\prod_{p|28} \mathbb{Q}_p^{*2}\right).$$

And

$$n \equiv 7 \,(\mathrm{mod}\ 8) \quad \text{and} \quad \left(\frac{n}{7}\right) = 1$$

implies that

$$n \equiv 15 \,\mathrm{mod} \left(\prod_{p|28} \mathbb{Q}_p^{*2}\right),$$

which shows this theorem. □

From this theorem, we see that for the cases of $d \equiv 11, 15 \mod \left(\prod_{p|28} \mathbb{Q}_p^{*2}\right)$, the result (1) is completely pleasant. And for the cases of $d \equiv 1, 29 \mod \left(\prod_{p|28} \mathbb{Q}_p^{*2}\right)$, the result (2) is not so pleasant because it is not an easy task to determine if $L_{E_{-d}}(1) = 0$. But we have the following:

Theorem 10.15 *Let* $p \equiv 1 \,\mathrm{mod} \left(\prod_{p|28} \mathbb{Q}_p^{*2}\right)$ *be a prime not dividing* 14*, then* $r(f_{(7)}, p) \neq r(g_{(7)}, p)$ *if* p *is represented by* $2X^2 + 7Y^2$*.*

Proof As in J.A. Antoniadis, 1990, we denote

$$F_0 = \left(\theta(X^2 + 14Y^2) - \theta(2X^2 + 7Y^2)\right) \cdot \theta_{\mathrm{id},14} := \sum_{n=1}^{\infty} a_n \mathrm{e}^{2\pi \mathrm{i} n z} \in S_{3/2}(56, \chi_1),$$

where

$$\theta_{\mathrm{id},14} := \sum_{n=-\infty}^{\infty} q^{14n^2} \in M_{1/2}(56, \chi_{14}), \quad \theta(X^2 + 14Y^2) - \theta(2X^2 + 7Y^2) \in S_1(56, \chi_{-14}).$$

Then by the results in J.A. Antoniadis, 1990, we know that F_0 is mapped to the cusp form corresponding to the modular elliptic curve (14C) under Shimura lifting and $a_p \neq 0$ if p is a prime not dividing 14 and represented by $2X^2 + 7Y^2$. Since $p \equiv 1 \mod \left(\prod_{p|28} \mathbb{Q}_p^{*2}\right)$, by Waldspurger's Theorem, we see that

$$L_{E_{-p}}(1)\sqrt{p}a_1^2 = L_{E_{-1}}(1)a_p^2.$$

A direct computation shows that $a_1 \cdot L_{E_{-1}}(1) \neq 0$ which implies that

$$L_{E_{-p}}(1) = 0 \text{ if and only if } a_p = 0.$$

Therefore by Lemma 10.7, we have proved that

$$r(f_{(7)}, p) \neq r(g_{(7)}, p) \text{ if and only if } a_p \neq 0,$$

which completes the proof since $a_p \neq 0$ if $(p, 14) = 1$ and represented by $2X^2 + 7Y^2$. □

Our method can be used for other ternary positive definite quadratic forms. For example, we can similarly study the forms $f_{(11)}, g_{(11)}$. In this case, we calculate:

$$\begin{aligned} &D_{44} = \{1, 3, 5, 15\}, \\ &L_{E_{-d_i}} \neq 0, \quad \text{for all } d_i \in D_{44}, \\ &b_1 = 1, \quad b_3 = -1m \quad b_5 = -1, \quad b_{15} = 1. \end{aligned}$$

Hence we conclude that

Theorem 10.16 *Let be $f_{(11)} = x^2 + 11y^2 + 11z^2$, $g_{(11)} = 3x^2 + 4y^2 + 11z^2 + 2xy$, E the corresponding modular elliptic curve of the cusp form $\frac{1}{2}(\theta(f_{(11)}) - \theta(g_{(11)}))$ and E_{-d} the $-d$-twist of E. Then for square free eligible numbers d such that $(d, 22) = 1$, we have that*

$$r(f_{(11)}, d) \neq r(g_{(11)}, d) \quad \text{if and only if } L_{E_{-d}}(1) \neq 0,$$

where $L_{E_{-d}}(s)$ is the Hasse-Weil *L-function of the elliptic curve E_{-d}. Especially, we have that $r(f_{(11)}, d) \neq r(g_{(11)}, d)$ if d satisfies one of the following conditions*:

(1) $d = p$ is a prime not splitting in $\mathbb{Q}(\sqrt{-11})_{(2)}/\mathbb{Q}(\sqrt{-11})$, where $\mathbb{Q}(\sqrt{-11})_{(2)}$ is the class field of $\mathbb{Q}(\sqrt{-11})$ with conductor 2;

(2) $d = p$ is a prime with $(p, 22) = 1$ such that p is represented by $3X^2+2XY+4Y^2$;

(3) $5 \nmid h(-d)$.

Proof Since $L_{E_{-d_i}}(1) \cdot b_{d_i} \neq 0$ for all $d_i \in D_{44}$, we know that $r(f_{(11)}, d) \neq r(g_{(11)}, d)$ if and only if $L_{E_{-d}}(1) \neq 0$ by Waldspurger's Theorem. All other assertions are immediate conclusions of Proposition 4.2 and Proposition 4.8 in J.A. Antoniadis, 1990. □

Remark 10.6 Our method in this section can be used to any other positive definite quadratic forms satisfying our assumptions in the paragraph before Lemma 10.7. For example, we can study similarly the forms $f_{(17)}$ and $g_{(17)}$, etc.

Finally we consider the following problem: for a given positive definite quadratic form with integral coefficients, find an exact formula for the number of representations of integers by this form. In general it is a difficult classical problem. Even for the simplest cases, i.e., binary forms and ternary forms, the problem is still open. For the general case, what we know is that the sum of the numbers of representations of an integer by all classes in a fixed genus is in relation to the coefficients of some modular forms in an Eisenstein subspace. But even for the sum, it is non-trivial to give an exact formula for a form given generally. In any case, the number of representations of an integer by one form in the genus has never been formulated if the class number of the genus is larger than one.

We shall consider some ternary quadratic forms with class number two of their genus, and give exact formulae for the numbers of representations of an integer by these forms. The main idea is as follows. For a positive definite ternary form f, let f and g be the representatives of classes in the genus of f. On the one hand, some linear combination of the numbers of representations of an integer by f and g can be related to the class number of a certain quadratic field; on the other hand, sometimes, we can find another linear combination of these numbers which is related to the L-function of an elliptic curve. By these two linear combinations, in terms of class number of a quadratic field and the special value of the L-function of an elliptic curve, we can get exact formulae for the number of representations of an integer by f and g respectively. This also shows the difficulty of the classical problem mentioned above because of the mysterious properties of the special values of L-functions and class numbers.

Theorem 10.17 *Let $f = \alpha x^2 + \beta y^2 + \gamma z^2$ be a positive definite ternary quadratic form with level N. Suppose the genus of f consists of two classes, f and g are the representatives of the classes. We assume further that $\mu O(f) - \nu O(g) \neq 0$, and*

$$\mu\theta_f + \nu\theta_g = \sum_{n=1}^{\infty} a_n \mathrm{e}^{2\pi i n z} \in S(N, 3/2, \chi_t) \bigcap S_0(N, \chi_t)^{\perp}$$

and the Shimura *lifting $F(z)$ of $\mu\theta_f + \nu\theta_g$ is a new form corresponding to an elliptic curve $E/\mathbb{Q}$. Let n with $(n, N) = 1$ be any square-free eligible number of the genus* (i.e., *d can be represented by the genus of f) with $n \equiv d_i \bmod \prod_{p|N} \mathbb{Q}_p^{*2}$ and $L_{E_{-ld_i}}(1) \neq 0$, then*

$$r(f,n) = \frac{O(f)a_{d_i}\sqrt{\dfrac{L_{E_{-ln}}(1)}{L_{E_{-ld_i}}(1)}} - \nu O(f)O(g)r(a,b,c;n)h(-ln)}{\mu O(f) - \nu O(g)},$$

$$r(g,n) = \frac{\mu O(f)O(g)r(a,b,c;n)h(-ln) - O(g)a_{d_i}\sqrt{\dfrac{L_{E_{-ln}}(1)}{L_{E_{-ld_i}}(1)}}}{\mu O(f) - \nu O(g)},$$

where $d_i \in D_N = \{d_1, d_2, \cdots, d_l\}$, $L_{E_D}(s)$ *is the* Hasse-Weil Zeta *function of the D-th twist of the elliptic curve E.*

Proof In Lemma 10.7, we take $F(z) = \mu\theta_f(z) + \nu\theta_g(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$. Then by Theorem 10.9 we obtain the following system of equations:

$$\begin{cases} \mu r(f,n) + \nu r(g,n) = a_n, \\ \dfrac{r(f,n)}{O(f)} + \dfrac{r(g,n)}{O(g)} = r(a,b,c;n)h(-ln). \end{cases} \tag{10.41}$$

For the positive integer n, there is a unique $d_i \in D_N$ with $n \equiv d_i \mod \left(\prod_{p|N} \mathbb{Q}_p^{*2}\right)$. By the above Lemma 10.7, under the assumptions of the theorem, we have that

$$a_n = a_{d_i}\sqrt{\frac{L_{E_{-ln}}(1)}{L_{E_{-ld_i}}(1)}},$$

solving the system (10.41) for $r(f,n), r(g,n)$, and inserting above the expression for a_n, we get the results desired, which completes the proof. □

Remark 10.7 Because the set $D_N = \{d_1, d_2, \cdots, d_l\}$ is finite, we see that $r(f,n)$ and $r(g,n)$ can be represented explicitly in terms of the classnumber $h(-ln)$ and the special value $L_{E_{-ln}}(1)$ of L-function of the twist of the elliptic curve E.

Example 10.9 Let be $f_1 = x^2 + 7y^2 + 7z^2$, $g_1 = 2x^2 + 4y^2 + 7z^2 - 2xy$.Then $O(f_1) = 8$, $O(g_1) = 4$, and

$$\begin{aligned} \widetilde{f}(z) &= \sum_{n=1}^{\infty} a_n \exp\{2\pi i n z\} \\ &:= \frac{1}{2}\theta_{f_1}(z) - \frac{1}{2}\theta_{g_1}(z) = \frac{1}{2}\sum_{n=1}^{\infty}(r(f_1,n) - r(g_1,n))\exp\{2\pi i n z\} \\ &= q + \cdots \in S(28, 3/2, \chi_1), \quad q = \exp\{2\pi i z\} \end{aligned}$$

and $F(z) = S(\widetilde{f}(z))$ is the new form corresponding to the elliptic curve (14C):

$$(14\text{C}): \qquad y^2 = x^3 + x^2 + 72x - 368.$$

We can easily calculate that

$$\begin{aligned}
D_{28} &= \{1, 11, 15, 29\},\\
L_{E_{-d_i}} &\neq 0 \quad \text{for all } d_i \in D_{28},\\
a_1 &= \frac{1}{2}(r(f,1) - r(g,1)) = 1,\\
a_{11} &= \frac{1}{2}(r(f,11) - r(g,11)) = \frac{1}{2}(8-8) = 0,\\
a_{15} &= \frac{1}{2}(r(f,15) - r(g,15)) = \frac{1}{2}(8-8) = 0,\\
a_{29} &= \frac{1}{2}(r(f,29) - r(g,29)) = \frac{1}{2}(8-4) = 2.
\end{aligned}$$

Hence by Theorem 10.17, for any square-free eligible integer n , we have that

$$\begin{aligned}
r(f_1,n) &= \frac{4}{3}\sqrt{\frac{L_{E_{-n}}(1)}{L_{E_{-1}}(1)}} + \frac{8}{3}r(1,7,7;n)h(-n), \quad \text{if } n \equiv 1 \bmod \prod_{p|28} \mathbb{Q}_p^{*2},\\
r(g_1,n) &= \frac{8}{3}r(1,7,7;n)h(-n) + \frac{1}{3}\sqrt{\frac{L_{E_{-n}}(1)}{L_{E_{-1}}(1)}}, \quad \text{if } n \equiv 1 \bmod \prod_{p|28} \mathbb{Q}_p^{*2},\\
r(f_1,n) &= r(g_1,n) = \frac{8}{3}r(1,7,7;n)h(-n), \quad \text{if } n \equiv 11 \bmod \prod_{p|28} \mathbb{Q}_p^{*2},\\
r(f_1,n) &= r(g_1,n) = \frac{8}{3}r(1,7,7;n)h(-n), \quad \text{if } n \equiv 15 \bmod \prod_{p|28} \mathbb{Q}_p^{*2},\\
r(f_1,n) &= \frac{8}{3}\sqrt{\frac{L_{E_{-n}}(1)}{L_{E_{-29}}(1)}} + \frac{8}{3}r(1,7,7;n)h(-n), \quad \text{if } n \equiv 29 \bmod \prod_{p|28} \mathbb{Q}_p^{*2},\\
r(g_1,n) &= \frac{8}{3}r(1,7,7;n)h(-n) + \frac{2}{3}\sqrt{\frac{L_{E_{-n}}(1)}{L_{E_{-29}}(1)}}, \quad \text{if } n \equiv 29 \bmod \prod_{p|28} \mathbb{Q}_p^{*2},
\end{aligned}$$

where

$$\begin{aligned}
r(1,7,7;n) &= \frac{1}{4}\omega_n^{-1} \cdot (2 - \alpha(n))\beta_{1,7}(n)\gamma_7'(n);\\
\gamma_p'(n) &= (1 - 2^{-1}\chi_{-n}(2))(1 - \chi_{-n}(p)\cdot p^{-1})(n/\delta_n)^{\frac{1}{2}}\\
&\quad \times \sum_{\substack{(ab)^2|n\\(ab,2p)=1}} \mu(a)\chi_{-n}(a)(ab)^{-1}
\end{aligned}$$

for any prime p; In particular we know that $\gamma_p'(n) = (1 - 2^{-1}\chi_{-n}(2))(1 - \chi_{-n}(p)\cdot p^{-1})(n/\delta_n)^{1/2}$ for any square-free positive integer n.

From these results, we can get very explicit formulae for the number of representations of the square-free positive eligible number n with $(n,28)=1$ by f and g. E.g., for any square-free positive integer $n>3$ with $n \equiv 3 \mod 8$ and $\left(\frac{n}{7}\right)=1$, then $n \equiv 11 \mod \left(\prod_{p|28} \mathbb{Q}_p^{*2}\right)$. By the definitions of $\alpha(n), \beta_{1,7}(n)$ and $\gamma_7'(n)$, we have that

$$\alpha(n)=1, \quad \beta_{1,7}(n)=14, \quad \gamma_7'(n)=\frac{12}{7}.$$

So

$$r(f_1,n)=r(g_1,n)=8h(-n).$$

Of course, we can discuss also other square-free positive integers n in a similar way. □

Example 10.10 Let be $f_2=x^2+11y^2+11z^2$, $g_2=3x^2+4y^2+11z^2+2xy$. Then we have that

$$\begin{aligned} &D_{44}=\{1,3,5,15\}, \quad L_{E_{-d_i}} \neq 0, \quad \text{for all } d_i \in D_{44}, \\ &a_1=1, \quad a_3=-1, \quad a_5=-1, \quad a_{15}=1. \end{aligned}$$

And $O(f_2)=8, O(g_2)=4$,

$$\begin{aligned} \widetilde{f}(z) &= \sum_{n=1}^{\infty} a_n \exp(2\pi i n z) := \frac{1}{2}\theta_{f_2}(z)-\frac{1}{2}\theta_{g_2}(z) \\ &= \frac{1}{2}\sum_{n=1}^{\infty}(r(f_2,n)-r(g_2,n))\exp\{2\pi i n z\} \\ &= q+\cdots \in S(28,3/2,\chi_1), \quad q=\exp\{2\pi i z\} \end{aligned}$$

and $F(z)=S(\widetilde{f}(z))$ is the new form corresponding to the elliptic curve (11B):

$$(11\text{B}): \qquad y^2+y=x^3-x^2-10x-20,$$

So by Theorem 10.17, we can get the exact formulae for the number of representations of any square-free eligible integer n with $(n,22)=1$ by f and g in terms of $h(-n)$ and $L_{E_{-n}}(1)$. We omit the calculations. □

Theorem 10.18 *Suppose that n is an odd square-free positive integer congruent to 1 or 3 modulo 8. $f_3=x^2+2y^2+32z^2, g_3=2x^2+4y^2+9z^2-4yz$. Then*

$$r(f_3,n)=c(n)h(-n)+2\sqrt{\frac{L_{E_{n^2}}(1)}{\omega\sqrt{n}}},$$

$$r(g_3,n)=c(n)h(-n)-2\sqrt{\frac{L_{E_{n^2}}(1)}{\omega\sqrt{n}}},$$

where $c(n) = 2$ *or* 6 *according to* $n \equiv 1$ *or* $3 \pmod 8$, ω *is the real period of the elliptic curve* $E : y^2 = 4x^3 - 4x$ *and* $L_{E_{n^2}}(s)$ *is the L-function of the congruent elliptic curve defined by* $y^2 = x^3 - n^2 x$.

Proof Let $f_3 = x^2 + 2y^2 + 32z^2$, $g_3 = 2x^2 + 4y^2 + 9z^2 - 4yz$. We want to give the formula for the number of representations of n by f_3 and g_3. It is clear that $r(f_3, n) = r(g_3, n) = 0$ for any $n \equiv 5$ or $7 \pmod 8$. So we only need to consider positive integers congruent to 1 or 3 modulo 8. Now let $f_3' = 2x^2 + y^2 + 32z^2$, $g_3' = 2x^2 + y^2 + 8z^2$, then by Tunnell's paper J.B. Tunnell, 1983, for any odd positive integer n, we have

$$\frac{L_{E_{n^2}}(1)}{\omega\sqrt{n}} = \frac{1}{4}a(n)^2,$$

where E_{n^2} is the congruent elliptic curve defined by $y^2 = x^3 - n^2 x$, ω is the real period of the elliptic curve $y^2 = 4x^3 - 4x$ and $a(n) = r(f_3', n) - \frac{1}{2}r(g_3', n)$. It is not difficult to see that $a(n) = \frac{1}{2}(r(f_3, n) - r(g_3, n))$ for any odd n. So we have

$$\frac{L_{E_{n^2}}(1)}{\omega\sqrt{n}} = \frac{1}{4}a(n)^2, \tag{10.42}$$

where $a(n) = \frac{1}{2}(r(f_3, n) - r(g_3, n))$.

In order to get the formulae for the number of representations of n by f_3 and g_3, we only need to find the number $r(f_3, n) + r(g_3, n)$ by (10.42). But by the definitions of $r(f_3, n)$ and $r(g_3, n)$, we see that $r(f_3, n) + r(g_3, n) = r(x^2 + 2y^2 + 8z^2, n)$. So we only need to calculate the number $r(x^2 + 2y^2 + 8z^2, n)$. We shall prove that for $n > 3$ square-free,

$$r(x^2 + 2y^2 + 8z^2, n) = \begin{cases} 4h(-n) \text{ if } n \equiv 1 \pmod 8, \\ 12h(-n) \text{ if } n \equiv 3 \pmod 8. \end{cases}$$

In fact, if $n \equiv 1 \pmod 8$, then for any triple $(x, y, z) \in \mathbb{Z}^3$ such that $x^2 + 2y^2 + 2z^2 = n$, the x must be odd and y, z are both even. So we have a one-to-one correspondence:

$$\{(x, y, z) \in \mathbb{Z}^3 | x^2 + 2y^2 + 2z^2 = n\} \leftrightarrow \{(x, y, z) \in \mathbb{Z}^3 | x^2 + 2y^2 + 8z^2 = n\},$$
$$(x, y, z) \leftrightarrow (x, y, z/2).$$

If $n \equiv 3 \pmod 8$, then for any triple $(x, y, z) \in \mathbb{Z}^3$ such that $x^2 + 2y^2 + 2z^2 = n$, the x must be odd and there is exactly one of y, z that is odd. We let z be the even one. Then we have a two-to-one correspondence:

$$\{(x, y, z) \in \mathbb{Z}^3 | x^2 + 2y^2 + 2z^2 = n\} \leftrightarrow \{(x, y, z) \in \mathbb{Z}^3 | x^2 + 2y^2 + 8z^2 = n\}$$
$$\begin{cases} (x, y, z) \\ (x, z, y) \end{cases} \leftrightarrow (x, y, z/2).$$

So we have

$$r(x^2+2y^2+8z^2,n)=\begin{cases} r(x^2+2y^2+2z^2,n) \text{ if } n\equiv 1\,(\text{mod } 8),\\ \dfrac{1}{2}r(x^2+2y^2+2z^2,n) \text{ if } n\equiv 3\,(\text{mod } 8).\end{cases}$$

Now we can compute the number $r(x^2+2y^2+2z^2,n)$ in terms of our Theorem 10.9. By Theorem 10.9 it can be proved that for any positive integer n

$$\begin{aligned} r(x^2+2y^2+2z^2,n)=&\frac{32h(-n)\sqrt{n}}{\omega_n\sqrt{\delta_n}}\left(1-\frac{1}{2}\chi_{-n}(2)\right)\\ &\times\left(\alpha(n)-\delta\left(\frac{n-1}{4}\right)-\left(\frac{n-2}{n}\right)\right)\\ &\sum_{\substack{(ab)^2|n,(ab,2)=1\\ a,b \text{ positive integers}}}\mu(a)\left(\frac{-n}{a}\right)(ab)^{-1},\end{aligned}$$

where $\delta(x)=1$ or 0 according to x an integer or not.

In particular, for any square-free odd positive integer n, the sum is equal to 1, and since the conductor δ_n of χ_{-n} is equal to $4n$ or n according to $n\equiv 1$ or 3 (mod 4), we have

$$r(x^2+2y^2+2z^2,n)=\begin{cases} 2, & \text{if } n=1,\\ 8, & \text{if } n=3,\\ 4h(-n), & \text{if } n\equiv 1\,(\text{mod } 8), n\neq 1,\\ 24h(-n), & \text{if } n\equiv 3\,(\text{mod } 8), n\neq 3.\end{cases}$$

Therefore we have for any square-free odd positive integer $n>3$

$$r(f_3,n)+r(g_3,n)=r(x^2+2y^2+8z^2,n)=\begin{cases} 4h(-n), & \text{if } n\equiv 1\,(\text{mod } 8),\\ 12h(-n), & \text{if } n\equiv 3\,(\text{mod } 8),\end{cases} \tag{10.43}$$

By the above (10.40) and (10.42) we have proved the theorem. □

Let $N=p_1p_2\cdots p_m$ with $p_1,p_2,\cdots,p_m$ distinct odd primes, at most two of them congruent to 3 modulo 8 and others congruent to 1 modulo 8. If there is at most one of p_i congruent to 3 modulo 8, then we define a simple graph $G_N=(V(G_N),E(G_N))$ with vertices $V(G_N)=\{p_1,p_2,\cdots,p_m\}$ and edges $E(G_N)=\left\{(p_i,p_j)\,\middle|\,\left(\frac{p_j}{p_i}\right)=-1\right\}$ where $(-)$ is the Legendre symbol as usual. Otherwise, without loss of generality, we may assume $p_1\equiv p_2\equiv 3$ (mod8) and $p_i\equiv 1$ (mod8) for $i\geqslant 3$. We define a simple graph $G_N=(V(G_N),E(G_N))$ with vertices $V(G_N)=\{p_1,p_2,\cdots,p_m\}$ and edges $E(G_N)=\left\{(p_1,p_2)\bigcup(p_i,p_j)\,\middle|\,\left(\frac{p_j}{p_i}\right)=-1,\{i,j\}\neq\{1,2\}\right\}$. By the quadratic reciprocity law, the graph G_N is a non-directed graph. We denote the number of spanning trees of G_N by $\tau(G_N)$ if N has at most one prime factor congruent to 3 modulo 8, otherwise τ is the number of spanning trees containing the special edge (p_1, p_2)(a subgraph of a non-directed simple graph is called a spanning tree if it is a tree and its vertices coincide with that of the original graph). Let $\nu_2(n)$ be the 2-adic additive valuation normalized by $\nu_2(2) = 1$.

Theorem 10.19 *Let $N = p_1p_2\cdots p_m > 3$ congruent to 1 or 3 modulo 8, with $p_1, p_2, \cdots, p_m$ distinct odd primes, at most two of them congruent to 3 modulo 8 and all others congruent to 1 modulo 8. Let f_3, g_3 be as in* Theorem 10.18. *Then*

(1) $\nu_2(r(f_3, N)) \geqslant m, \nu_2(r(g_3, N)) \geqslant m$;

(2) *if all $p_i (i = 1, 2, \cdots, m)$ are congruent to 1 modulo 8, then the equality in (1) holds if and only if $\nu_2(h(-N)) = m - 1$;*

(3) *if there is only one or two p_i $(i = 1, 2, \cdots, m)$ congruent to 3 modulo 8, then the equality in (1) holds if and only if one of the following conditions is satisfied*: i) $\nu_2(h(-N)) = m - 1$ *and $\tau(G_N)$ is even*; ii) $\nu_2(h(-N)) > m - 1$ *and $\tau(G_N)$ is odd.*

Proof In order to prove the theorem, we need the following facts(for the proofs of these facts please see C. Zhao, 1991, C. Zhao, 2001, C. Zhao, 2003):

Claim Let the notations be as in the theorem. Then

(1) $\nu_2\left(\dfrac{L_{E_{N^2}}(1)}{\omega\sqrt{N}}\right) \geqslant 2m$ if all p_i $(i = 1, 2, \cdots, m)$ are congruent to 1 modulo 8;

(2) $\nu_2\left(\dfrac{L_{E_{N^2}}(1)}{\omega\sqrt{N}}\right) \geqslant 2m - 2$ if one or two of $p_i (i = 1, 2, \cdots, m)$ are congruent to 3 modulo 8 and others are congruent to 1 modulo 8. Moreover, the equality holds if and only if $\tau(G_N)$ is odd.

We consider the 2-adic valuation of the terms on the right side of the conclusion of Theorem 10.18. It is clear that $\nu_2(c(N)) = 1$. From the Gauss genus theory we know that

$$\nu_2(h(-N)) \geqslant m - 1, \tag{10.44}$$

where m is the number of prime factors of N. By the claim we see that $\nu_2 \left(4\dfrac{L_{E_{N^2}}(1)}{\omega\sqrt{N}}\right) \geqslant 2m$. So the first conclusion (1) of the theorem is valid.

Now suppose that $N = p_1p_2\cdots p_m$ with all $p_i \equiv 1 \pmod 8$. Then we have that

$$\nu_2\left(4\frac{L_{E_{N^2}}(1)}{\omega\sqrt{N}}\right) \geqslant 2m + 2.$$

Therefore, by Theorem 10.18, (10.43) and (10.44), we see that $\nu_2(r(f_3, N)) = \nu_2(r(g_3, N)) = m$ if and only if $\nu_2(c(N)h(-N)) = m$, which is equivalent to $\nu_2(h(-N)) = m - 1$. This is the second assertion (2) of the theorem.

Finally, suppose that $N = p_1 p_2 \cdots p_m$ as in (3) of the theorem. By the claim we have

$$\nu_2 \left(4 \frac{L_{E_{N^2}}(1)}{\omega \sqrt{N}} \right) \geqslant 2m. \tag{10.45}$$

And the equality holds if and only if $\tau(G_N)$ is odd. By (10.43), we have

$$\nu_2(c(N)h(-N)) \geqslant m \tag{10.46}$$

and the equality holds if and only if $\nu_2(h(-N)) = m - 1$. Therefore by Theorem 10.18, $\nu_2(r(f_3, N)) = \nu_2(r(g_3, N)) = m$ if and only if one of the inequalities in (10.45) and (10.46) holds while the other one does not hold. This is the assertion (3) of the theorem which completes the proof. □

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