Xueli Wang
Dingyi Pei

Modular Forms with Integral and Half-Integral Weights

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With 2 figures

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Preface

The theory of modular forms is an important subject of number theory. Also it has very important applications to other areas of number theory such as elliptic curves, quadratic forms, etc. Its contents is vast. So any book on it must necessarily make a rather limited selection from the fascinating array of possible topics. Our focus is on topics which deal with the fundamental theory of modular forms of one variable with integral and half-integral weight. Even for such a selection we have to make further limitations on the themes discussed in this book. The leading theme of the book is the development of the theory of Eisenstein series.

A fundamental problem is the construction of a basis of the space of modular forms. It is well known that, for any weight $\geqslant 2$ and the weight 1, the orthogonal complement of the space of cusp forms is spanned by Eisenstein series. Does this conclusion hold for the half-integral weight < 2? The problem for weight $1/2$ was solved by J.P.Serre and H.M.Stark. Then one of the authors of this book, Dingyi Pei, proved that the conclusion holds for weight $3/2$ by constructing explicitly a basis of the orthogonal complement of the space of cusp forms. To introduce this result and some of its applications is our motivation for writing this book, which is a large extension version of the book "Modular forms and ternary quadratic forms" (in Chinese) written by Dingyi Pei.

Chapter 1 can be viewed as an introduction to the themes discussed in the book. Starting from the problem of representing integers by quadratic forms we introduce the concept of modular forms. In Chapter 2, we discuss the analytic continuation of Eisenstein series with integral and half-integral weight, which prepares the construction of Eisenstein series in Chapter 7.

In Chapters 3-5, some fundamental concepts, notations and results about modular forms are introduced which are necessary for understanding later chapters. More specifically, we introduce in Chapter 3 the modular group and its congruence subgroups and the Riemannian surface associated with a discrete subgroup of $SL_2(\mathbb{R})$. Furthermore, the concept of cusp points for a congruence subgroup is presented. In Chapter 4, we define modular forms with integral and half-integral weight, calculate the dimension of the space of modular forms using the theorem of Riemann-Roch. Chapter 5 is dedicated to define Hecke rings and discuss some of their fundamental properties. Also in this chapter the Zeta function of a modular form with integral or half-integral weight is described. In particular, we deduce the functional equation of the Zeta function of a modular form, and discuss Weil's Theorem.

In Chapter 6, the definitions of new forms and old forms with integral and half-integral weight are given. In particular the Atkin-Lehner's theory and the Kohnen's theory, with respect to new forms for integral and half-integral weight, are discussed at length respectively.

In Chapter 7, we construct Eisenstein series. The first objective is to construct Eisenstein series with half-integral weight $\geqslant 5/2$. The second objective is the construction of Eisenstein series with weight $1/2$ according to Serre and Stark. Then the method of the construction for Eisenstein series of weight $3/2$ is introduced, followed by the construction of Cohen-Eisenstein series. For completeness, the construction of Eisenstein series with integral weight, which is due to Hecke, is also given in the last section of the chapter.

The Shimura lifting is the main objective of Chapter 8 where we follow the way depicted by Shintani. Weil representation is introduced first and some elementary properties of Weil representation are discussed. Then the Shimura lifting from cusp forms with half-integral weight to ones with integral weight is constructed. Also the Shimura lifting for Eisenstein spaces is deduced in this chapter.

In Chapter 9, we discuss the Eichler-Selberg trace formula for the space of modular forms with integral and half-integral weight. The simplest case of the Eichler-Selberg trace formula on $SL_2(\mathbb{Z})$ is deduced in terms of Zagier's method. Then the trace formula on a Fuchsian group is obtained by Selberg's method. Finally the Niwa's and Koknen's trace formulae are obtained for the space of modular forms with half-integral weight and the group $\Gamma_0(N)$.

In Chapter 10, some applications of modular forms and Eisenstein series to the arithmetic of quadratic forms are described. We first present the Schulze-Pillot's proof of Siegel theorem. Then some results of representation of integers by ternary quadratic forms are explained. We also give an upper bound of the minimal positive integer represented by a positive definite even quadratic form with level 1 or 2.

Although many modern results on modular forms with half-integral weight are contained in this book, it is written as elementarily as possible and it's content is self-contained. We hope it can be used as a reference book for researchers and as a textbook for graduate students.

The authors would like to thank Ms. Yuzhuo Chen for her many helps. Also many thanks should be given to Dr. Junwu Dong for his helpful suggestions and carefully typesetting the draft of this book. We especially wish to thank Dr. Wolfgang Happle Happle for carefully reading the draft of this book and correcting some errors in the draft. The author Xueli Wang wishes to thank Prof. Dr. Gerhard Frey for stimulating discussions and providing the environment of I.E.M in Essen University, where part

of the draft has been done. Xueli Wang hope to give deepest gratitude for his lovely and beautiful wife, Dr. Dongping Xu, who assumed all of the housework over the years. Finally, the author Xueli Wang would like to dedicate this book to the 80th birthday of his father.

Xueli Wang Dingyi Pei

Guangzhou

September, 2011

Contents

Chapter 1

Theta Functions and Their Transformation Formulae

In this chapter, we introduce theta functions of positive definite quadratic forms and study their transformation properties under the action of the modular group.

Let a, b, c and n be positive integers with $(a, b, c) = 1$. Denote by $N(a, b, c; n)$ the number of integral solutions $(x, y, z) \in \mathbb{Z}^3$ of the following equation:

$$ax^2 + by^2 + cz^2 = n.$$

Define the theta function by

$$\theta(z) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{2\pi\mathrm{i}n^2 z}, \quad z \in \mathbb{H},$$

where $\mathbb{H}$ is the upper half of the complex plane, i.e., $\mathbb{H} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$. It is clear that $\theta(z)$ is holomorphic on $\mathbb{H}$. Put

$$f(z) = \theta(az)\theta(bz)\theta(cz),$$

then

$$f(z) = 1 + \sum_{n=1}^{\infty} N(a, b, c; n)\mathrm{e}^{2\pi\mathrm{i}nz}.$$

Hence the number $N(a, b, c; n)$ is the n-th Fourier coefficient of the function. This shows that we know the number $N(a, b, c; n)$ if the Fourier coefficients of f can be computed explicitly. It is clear that there is a close relationship between $f(z)$ and the θ function. We shall see later that $f(z)$ is a modular form of weight $3/2$ from the transformation properties of θ under the action of linear fractional transformations. After having studied some properties of modular forms, we shall resume this topic later. Firstly, we shall consider some more general problems.

Now let t be a positive real number, put

$$\varphi(x) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi t(n+x)^2}.$$

The series satisfies $\varphi(x+1)=\varphi(x)$. Hence it has the following Fourier expansion:

$$\varphi(x)=\sum_{m=-\infty}^{\infty} c_m \mathrm{e}^{2\pi \mathrm{i} m x},$$

where

$$c_m=\int_0^1 \varphi(x)\mathrm{e}^{-2\pi\mathrm{i}mx}\mathrm{d}x=\int_{-\infty}^{\infty}\mathrm{e}^{-\pi t x^2-2\pi\mathrm{i}mx}\mathrm{d}x=t^{-1/2}\mathrm{e}^{-\pi m^2/t}.$$

Hence

$$\varphi(x)=t^{-1/2}\sum_{m=-\infty}^{\infty}\mathrm{e}^{-\pi m^2+2\pi\mathrm{i}mx}. \tag{1.1}$$

Taking $x=0$ in equation (1.1) we get

$$\tilde{\theta}(\mathrm{i}t)=t^{-1/2}\tilde{\theta}(-1/(\mathrm{i}t)),$$

where $\tilde{\theta}(z)=\theta(z/2)$. Because $\tilde{\theta}(z)$ is a holomorphic function on the upper half plane, we have that

$$\tilde{\theta}(-1/z)=(-\mathrm{i}z)^{1/2}\tilde{\theta}(z),\quad \forall z\in\mathbb{H}. \tag{1.2}$$

For the multi-valued function $z^{1/2}$, we choose $\arg(z^{1/2})$ such that $-\pi/2<\arg(z^{1/2})\leqslant \pi/2$. In general, we have that $(z_1z_2)^{1/2}=\pm z_1^{1/2}z_2^{1/2}$ where we take "$-$" if one of the following conditions is satisfied:

(1) $\mathrm{Im}(z_1)<0$, $\mathrm{Im}(z_2)<0$, $\mathrm{Im}(z_1z_2)>0$;

(2) $\mathrm{Im}(z_1)<0$, $\mathrm{Im}(z_2)>0$, $\mathrm{Im}(z_1z_2)<0$;

(3) z_1 and z_2 are both negative, or one of them is negative and the imaginary of the other one is positive.

Otherwise we take "+".

Let $f(x_1,\cdots,x_k)$ be an integral positive definite quadratic form in k variables. Define the matrix

$$A=\left(\frac{\partial^2 f}{\partial x_i\partial x_j}\right).$$

Then A is a positive definite symmetric integral matrix with even entries on the diagonal. It is clear that

$$f(x_1,\cdots,x_k)=\frac{1}{2}xAx^{\mathrm{T}},$$

where $x=(x_1,\cdots,x_k)\in\mathbb{Z}^k$ is a row vector, x^{T} is the transposal of x. We now define the θ function of f as

$$\theta_f(z)=\sum_{x\in\mathbb{Z}^k}\mathrm{e}^{2\pi\mathrm{i}f(x)z}\qquad\text{for all } z\in\mathbb{H}.$$

It is clear that

$$\theta_f(z) = \sum_{x\in\mathbb{Z}^k} \mathrm{e}^{\pi \mathrm{i} x A x^{\mathrm{T}} z} = \sum_{n=0}^{\infty} r(f,n)\mathrm{e}^{2\pi \mathrm{i} n z},$$

where $r(f,n)$ is the number of the solutions of $f(x) = n$ with $x \in \mathbb{Z}^k$. $\theta_f(z)$ is absolutely and uniformly convergent in any bounded domain of $\mathbb{H}$, so it is holomorphic on the whole of $\mathbb{H}$.

Let N be the least positive integer such that all the entries of the matrix NA^{-1} are integers and the entries on the diagonal are even. This implies that $\det A$ is a divisor of N^k. Hence the prime divisors of $\det A$ are also prime divisors of N. But it is clear that $N|2\det A$. So all the odd prime divisors of N are certainly prime divisors of $\det A$.

If we consider A as a matrix on the ring $\mathbb{Z}_2$ of 2-adic integers, it can be proved that there exists an inverse matrix S on $\mathbb{Z}_2$ such that

$$SAS^{\mathrm{T}} = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & & \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_r \end{pmatrix},$$

where A_i is either an integer of $2\mathbb{Z}_2$ or a symmetric matrix $\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ with $a, b, c \in \mathbb{Z}_2$.

It is clear that there is at least one A_i which is a 1×1 matrix if k is odd. So we get the following

Lemma 1.1 *If k is odd, then $2|\det A$ and $4|N$; if k is even, then $N|\det A$. If $4|k$, then $\det A \equiv 0$ or $1 \mod 4$; if $k \equiv 2 \mod 4$, then $\det A \equiv 0$ or $3 \mod 4$. Hence $(-1)^{k/2}\det A$ is always 1 or 0 modulo by 4 if k is even.*

Let h be a vector in $\mathbb{Z}^k$ such that $hA \in N\mathbb{Z}^k$ and define a function on $\mathbb{H}$ as follows

$$\theta(z; h, A, N) = \sum_{m\equiv h(N)} e\Big(\frac{zmAm^{\mathrm{T}}}{2N^2}\Big),$$

where $e(z) = \mathrm{e}^{2\pi \mathrm{i} z}$.

Proposition 1.1 *We have the following transformation formula*

$$\theta(-1/z; h, A, N) = (\det A)^{-1/2}(-\mathrm{i}z)^{k/2} \sum_{k \bmod N,\ kA\equiv 0(N)} e(hAk^{\mathrm{T}}/N^2)\theta(z; k, A, N).$$

Proof Let v be a positive real number, $x = (x_1, \cdots, x_k) \in \mathbb{R}^k$, and

$$g(x) = \sum_{m\in\mathbb{Z}^k} e(\mathrm{i}v(x+m)A(x+m)^{\mathrm{T}}/2).$$

Then $g(x)$ has Fourier expansion

$$g(x) = \sum_{m \in \mathbb{Z}^k} a_m e(x \cdot m^{\mathrm{T}}), \tag{1.3}$$

where

$$a_m = \int \cdots \int_{0 \leqslant x_j < 1} g(x) e(-x \cdot m^{\mathrm{T}}) \mathrm{d}x = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e(\mathrm{i} v x A x^{\mathrm{T}}/2 - x \cdot m^{\mathrm{T}}) \mathrm{d}x.$$

There exists a real orthogonal matrix S such that SAS^{T} is a diagonal matrix $\mathrm{diag}\{\alpha_1, \cdots, \alpha_k\}$ with $\alpha_i > 0\,(1 \leqslant i \leqslant k)$. We make a variable change $x = yS$ in the above integral and denote $Sm^{\mathrm{T}} = (u_1, \cdots, u_k)^{\mathrm{T}}$. Then

$$\begin{aligned} a_m &= \prod_{j=1}^{k} \int_{-\infty}^{\infty} \mathrm{e}^{-\pi v \alpha_j y^2 - 2\pi \mathrm{i} u_j y} \mathrm{d}y \\ &= \prod_{j=1}^{k} \int_{-\infty}^{\infty} \mathrm{e}^{-\pi v \alpha_j \left(y + \frac{\mathrm{i} u_j}{v \alpha_j}\right)^2 - \frac{\pi u_j^2}{v \alpha_j}} \mathrm{d}y \\ &= v^{-k/2} \prod_{j=1}^{k} \alpha_j^{-1/2} \mathrm{e}^{-\frac{\pi u_j^2}{v \alpha_j}} \\ &= v^{-k/2} (\det A)^{-1/2} \mathrm{e}^{-\pi m A^{-1} m^{\mathrm{T}}/v}. \end{aligned} \tag{1.4}$$

For any $m \in \mathbb{Z}^k$, let $k \equiv mNA^{-1} \pmod N$. Then $kA \equiv 0 \pmod N$ and m can be written as $(Nu + k)A/N (u \in \mathbb{Z}^k)$. Inserting (1.4) into (1.3), we get

$$\begin{aligned} g(x) = v^{-k/2} (\det A)^{-1/2} &\sum_{\substack{k \bmod N, \\ kA \equiv 0 (N)}} e(xAk^{\mathrm{T}}/N) \\ &\cdot \sum_{u} e(xAu^{\mathrm{T}} + \mathrm{i}(Nu + k)A(Nu + k)^{\mathrm{T}}/(2vN^2)). \end{aligned}$$

Since $\theta(\mathrm{i}v; h, A, N) = g(h/N)$, we get by the above equality

$$\theta(\mathrm{i}v; h, A, N) = v^{-k/2} (\det A)^{-1/2} \sum_{\substack{k \bmod N, \\ kA \equiv 0 (N)}} e(hAk^{\mathrm{T}}/N^2) \theta\left(-\frac{1}{\mathrm{i}v}; k, A, N\right),$$

which shows that Proposition 1.1 holds for $z = -1/\mathrm{i}v$. This implies that the proposition holds because $\theta(z; h, A, N)$ is holomorphic on the whole of $\mathbb{H}$. □

Now we define the full modular group of order 2 as follows

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

We want to find the transformation formula of $\theta(z;\, h,\, A,\, N)$ under the transformation $z \mapsto \gamma(z) = (az+b)/(cz+d)$. We first assume that $c > 0$, then we get by Proposition 1.1 that

$$\begin{aligned}
\theta(\gamma(z); h, A, N) &= \sum_{m \equiv h(N)} e\left(mAm^{\mathrm{T}}\left(a - \frac{1}{cz+d}\right)/(2cN^2)\right) \\
&= \sum_{\substack{g \bmod (cN), \\ g \equiv h(N)}} e(agAg^{\mathrm{T}}/(2cN^2)) \\
&\quad \cdot \sum_{m \equiv g \bmod (cN)} e(-cmAm^{\mathrm{T}}/[2(cz+d)(cN)^2]) \\
&= (\det A)^{-1/2} c^{-k/2} (-\mathrm{i}(cz+d))^{k/2} \\
&\quad \cdot \sum_{\substack{k \bmod (cN), \\ kA \equiv 0(N)}} \varPhi(h,k)\theta(cz; k, cA, cN),
\end{aligned} \tag{1.5}$$

where

$$\varPhi(h,k) = \sum_{\substack{g \bmod (cN), \\ g \equiv h(N)}} e([agAg^{\mathrm{T}} + 2kAg^{\mathrm{T}} + dkAk^{\mathrm{T}}]/(2cN^2))$$

and we also used the fact that mAm^{T} is even for any $m \in \mathbb{Z}^k$. Since $ad = bc+1$, it follows

$$\begin{aligned}
\varPhi(h,k) &= \sum_{\substack{g \bmod (cN), \\ g \equiv h(N)}} e\big(a(g+dk)A(g+dk)^{\mathrm{T}}/(2cN^2)\big) e\big(-b[2gAk^{\mathrm{T}} + dkAk^{\mathrm{T}}]/(2N^2)\big) \\
&= e\big(-b[2hAk^{\mathrm{T}} + dkAk^{\mathrm{T}}]/(2N^2)\big)\, \varPhi(h+dk, 0),
\end{aligned}$$

which implies that $\varPhi(h,k)$ is only dependent on $k \bmod N$. By equality (1.5) we get

$$\begin{aligned}
&\theta(\gamma(z); h, A, N)(\det A)^{1/2} c^{k/2} (-\mathrm{i}(cz+d))^{-k/2} \\
&= \sum_{\substack{k \bmod (N), \\ kA \equiv 0(N)}} \varPhi(h,k) \sum_{\substack{g \bmod (cN), \\ g \equiv k(N)}} \theta(cz; g, cA, cN) \\
&= \sum_{\substack{k \bmod (N), \\ kA \equiv 0(N)}} \varPhi(h,k)\theta(z; k, A, N).
\end{aligned}$$

Substituting z by $-1/z$, we get by Proposition 1.1

$$\theta\left(\frac{bz-a}{dz-c}; h, A, N\right) \det A c^{k/2}\big(-\mathrm{i}(d-c/z)\big)^{-k/2}(-\mathrm{i}z)^{-k/2}$$
$$= \sum_{\substack{l \mod N,\\ lA\equiv 0(N)}} \left\{ \sum_{\substack{k \mod N,\\ kA\equiv 0(N)}} e(lAk^{\mathrm{T}}/N^2)\Phi(h,k) \right\} \theta(z; l, A, N). \tag{1.6}$$

Now suppose that $d \equiv 0(N)$. Since NA^{-1} is an integral matrix with even entries on the diagonal,

$$kAk^{\mathrm{T}}/(2N) = (N^{-1}kA \cdot NA^{-1} \cdot N^{-1}Ak^{\mathrm{T}})/2$$

is an integer. Hence

$$\Phi(h,k) = e(-bhAk^{\mathrm{T}}/N^2)\Phi(h,0)$$

and the right hand of (1.6) becomes

$$\Phi(h,0) \sum_{\substack{l \bmod N,\\ lA\equiv 0(N)}} \left\{ \sum_{\substack{k \bmod N,\\ kA\equiv 0(N)}} e((l-bh)Ak^{\mathrm{T}}/N^2) \right\} \theta(z; l, A, N).$$

We now compute the inner summation of the formula above. There exist modular matrices P, Q, such that $PAQ = \mathrm{diag}\{\alpha_1, \cdots, \alpha_k\}$. Since NA^{-1} is an integral matrix, then $\alpha_i | N (1 \leqslant i \leqslant k)$. Since

$$kA \equiv (l-bh)A \equiv 0(N),$$

a direct computation shows that

$$\sum_{\substack{k \mathrm{mod}(N),\\ kA\equiv 0(N)}} e((l-bh)Ak^{\mathrm{T}}/N^2) = \begin{cases} 0, & \text{if } 1 \not\equiv bh(N), \\ \det A, & \text{if } 1 \equiv bh(N). \end{cases}$$

Now substituting $\begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we assume that $c \equiv 0(N), d < 0$. Then we have that

$$\theta((az+b)/(cz+d); h, A, N) = (-\mathrm{i}(c+d/z))^{k/2}(-\mathrm{i}z)^{k/2} W \theta(z; ah, A, N), \tag{1.7}$$

where

$$W = |d|^{-k/2} \sum_{\substack{g \mathrm{mod}(|d|N),\\ g\equiv h(N)}} e\big(-bgAg^{\mathrm{T}}/(2|d|N^2)\big).$$

Since $\mathrm{Im}(-\mathrm{i}) < 0$, $\mathrm{Im}(c + d/z) > 0$, then $(-\mathrm{i}(c + d/z))^{k/2} = (-\mathrm{i})^{k/2}(c + d/z)^{k/2}$. Similarly, since $\mathrm{Im}(-\mathrm{i}) < 0$, $\mathrm{Im}(z) > 0$, we get $(-\mathrm{i}z)^{k/2} = (-\mathrm{i})^{k/2}z^{k/2}$. Again since $\mathrm{Im}(cz + d) = c\mathrm{Im}(z)$, it follows

$$z^{k/2}(c + d/z)^{k/2} = \mathrm{sgn}(c)^k(cz + d)^{k/2},$$

where

$$\mathrm{sgn}(c) = \begin{cases} 1, & \text{if } c \geqslant 0, \\ -1, & \text{if } c < 0. \end{cases}$$

Therefore

$$(-\mathrm{i}(c + d/z))^{k/2}(-\mathrm{i}z)^{k/2} = (-\mathrm{i}\mathrm{sgn}(c))^k(cz + d)^{k/2}. \tag{1.8}$$

Since $ad \equiv 1(N)$, we can express g in W as $adh + Nu$ with $u \in (\mathbb{Z}/|d|\mathbb{Z})^k$. Then

$$W = e(abhAh^{\mathrm{T}}/(2N^2))w(b, |d|), \tag{1.9}$$

where

$$w(b, |d|) = |d|^{-k/2} \sum_{u \bmod |d|} e(-buAu^{\mathrm{T}}/(2|d|)).$$

If $c = 0$ or $b = 0$, then $d = -1$ and hence $w(b, |d|) = 1$. Now suppose that $bc \neq 0$ and d is an odd. We substitute z by $z + 8m(m \in \mathbb{Z})$ in (1.7) such that $d + 8mc < 0$. By (1.8) and (1.9) we know that

$$w(b, |d|) = w(b + 8ma, |d + 8mc|).$$

Because d and $8c$ are co-prime, we can find an integer m such that $-d - 8mc$ is an odd prime which will be denoted by p. Let $\beta = -(b + 8ma)$. Then

$$w(b, |d|) = w(-\beta, p) = p^{-k/2} \sum_{u \bmod p} e(\beta uAu^{\mathrm{T}}/(2p)).$$

Suppose that $\beta \equiv 2\beta'(p)$. Since $c \equiv 0(N)$, d and c are co-prime, then p and N are co-prime, and hence p and $\det A$ are co-prime. There exists an integral matrix S such that $\det S$ is prime to p and SAS^t is congruent to $\mathrm{diag}\{q_1, \cdots, q_k\}$ modulo p. By Gauss sum, we have that

$$w(b, |d|) = p^{-k/2} \prod_{i=1}^{k} \left(\sum_{x=1}^{k} e(\beta' q_i x^2/p) \right) = \varepsilon_p^k \left(\frac{(\beta')^k \det A}{p} \right),$$

where $\left(\dfrac{q}{p}\right)$ is the Legendre symbol

$$\left(\frac{q}{p}\right) = \begin{cases} 1, & \text{if } q \text{ is a quadratic residue modulo } p, \\ -1, & \text{otherwise.} \end{cases}$$

The symbol ε_n is defined for all odd integers:

$$\varepsilon_n = \begin{cases} 1, & \text{if } n \equiv 1(4), \\ \mathrm{i}, & \text{if } n \equiv 3(4). \end{cases}$$

It is clear that $\varepsilon_p = \varepsilon_{-d} = \mathrm{i}\varepsilon_d^{-1}$. Since all prime divisors of $\det A$ are divisors of N, $p \equiv -d(8N)$,

$$\left(\frac{\det A}{p}\right) = \left(\frac{\det A}{-d}\right).$$

Since $\begin{pmatrix} a & -\beta \\ c & -p \end{pmatrix} \in SL_2(\mathbb{Z})$, i.e., $\beta c - ap = 1$, we get $2\beta' c \equiv 1(p)$. Hence

$$\left(\frac{\beta'}{p}\right) = \left(\frac{2c}{p}\right) = \left(\frac{2c}{-d}\right).$$

Let a be an integer, $b \neq 0$ be an odd. We define a new quadratic residue symbol $\left(\frac{a}{b}\right)$ satisfying the following properties:

(1) $\left(\frac{a}{b}\right) = 0$ if $(a, b) \neq 1$;

(2) $\left(\frac{0}{\pm 1}\right) = 1$;

(3) If $b > 0$, then $\left(\frac{a}{b}\right)$ is the Jacobi symbol, i.e., if $b = \prod p^r$, then $\left(\frac{a}{b}\right) = \prod \left(\frac{a}{p}\right)^r$;

(4) If $b < 0$, then $\left(\frac{a}{b}\right) = \operatorname{sgn}(a) \left(\frac{a}{|b|}\right)$.

Hereafter, the symbol $\left(\frac{a}{b}\right)$ will be defined as above. Then we have

$$w(b, |d|) = \varepsilon_d^{-k} (\operatorname{sgn}(c)\mathrm{i})^k \left(\frac{2c \det A}{d}\right) \tag{1.10}$$

and (1.10) holds for $c = 0$ or $c \neq 0$.

Define a subgroup of the full modular group as follows

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, c \equiv 0(N) \right\}.$$

Proposition 1.2 *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. *If k is odd, then we have*

$$\theta(\gamma(z); h, A, N) = e\big(abhAh^{\mathrm{T}}/(2N^2)\big) \left(\frac{\det A}{d}\right) \left(\frac{2c}{d}\right)^k \varepsilon_d^{-k} (cz + d)^{k/2} \theta(z; ah, A, N), \tag{1.11}$$

If k is even, then we have

$$\theta\big(\gamma(z);h,A,N\big)=e\big(abhAh^{\mathrm{T}}/(2N^2)\big)\Big(\frac{(-1)^{k/2}\det A}{d}\Big)(cz+d)^{k/2}\theta(z;ah,A,N), \tag{1.12}$$

Proof First assuming that k is odd. By Lemma 1.1, $N\equiv 0(4)$. Hence d is odd. For $d<0$, inserting (1.8), (1.9) and (1.10) into (1.7), we can get (1.11) immediately. For $d>0$, substituting γ by $-\gamma$ and noting that $(-\gamma)(z)=\gamma(z)$, we have

$$\begin{aligned}\theta(\gamma(z);h,A,N)=e(abhAh^{\mathrm{T}}/(2N^2))&\left(\frac{\det A}{d}\right)\left(\frac{-2c}{-d}\right)^k\\ &\times\varepsilon_{-d}^{-k}(-cz-d)^{k/2}\theta(z;-ah,A,N).\end{aligned}$$

It is clear that $\theta(z;-ah,A,N)=\theta(z;ah,A,N)$. If $c=0$, then $d=1$ and

$$\left(\frac{-2c}{-d}\right)^k\varepsilon_{-d}^{-k}(-cz-d)^{k/2}=\mathrm{i}^{-k}(-1)^{k/2}=1.$$

If $c\neq 0$, we have

$$\begin{aligned}\left(\frac{-2c}{-d}\right)^k\varepsilon_{-d}^{-k}(-cz-d)^{k/2}&=(-\mathrm{sgn}(c))^k\left(\frac{-2c}{d}\right)^k\mathrm{i}^{-k}\varepsilon_d^{-k}(-\mathrm{i}\,\mathrm{sgn}(c))^k(cz+d)^{k/2}\\ &=\varepsilon_d^{-k}\left(\frac{2c}{d}\right)^k(cz+d)^{k/2}.\end{aligned}$$

This shows that (1.12) holds also for $d>0$. Now assuming that k is even. If d is odd, we can get (1.12) by proceeding similarly as above. If d is even, then c is odd, and N is also odd. By the result for the case d odd, we have

$$\begin{aligned}&\theta\left(\frac{az+aN+b}{cz+cN+d};h,A,N\right)\\ &=e\Big(\frac{abhAh^{\mathrm{T}}}{2N^2}\Big)\left(\frac{(-1)^{k/2}\det A}{cN+d}\right)(cz+cN+d)^{k/2}\theta(z;ah,A,N),\end{aligned} \tag{1.13}$$

where we used the fact that $hAh^{\mathrm{T}}/(2N)$ is an integer. By Lemma 1.1 and Lemma 1.2 which will be proved later, we have

$$\left(\frac{(-1)^{k/2}\det A}{cN+d}\right)=\left(\frac{(-1)^{k/2}\det A}{d}\right),$$

where d is even. So the right hand side of above is equal to $\left(\frac{(-1)^{k/2}\det A}{\det A+d}\right)$. Substituting z by $z-N$ in (1.13) we get (1.12). □

It is clear that $\theta_f(z) = \theta(z; 0, A, N)$. Thus we obtain the main theorem of this chapter:

Theorem 1.1 *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. *If* k *is odd, then*

$$\theta_f(\gamma(z)) = \left(\frac{2 \det A}{d}\right) \varepsilon_d^{-k} \left(\frac{c}{d}\right)^k (cz+d)^{k/2} \theta_f(z).$$

If k *is even, then*

$$\theta_f(\gamma(z)) = \left(\frac{(-1)^{k/2} \det A}{d}\right) (cz+d)^{k/2} \theta_f(z).$$

In particular, taking $k = 1, A = 2$, then $N = 4$. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, by Theorem 1.1, we have

$$\theta(\gamma(z)) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) (cz+d)^{1/2} \theta(z).$$

We define the symbol

$$j(\gamma, z) = \varepsilon_d^{-1} \left(\frac{c}{d}\right) (cz+d)^{1/2}, \quad \gamma \in \Gamma_0(4).$$

If $\gamma_1, \gamma_2 \in \Gamma_0(4)$, by the above result, we have

$$\theta(\gamma_1\gamma_2(z)) = j(\gamma_1\gamma_2, z)\theta(z)$$

and

$$\theta(\gamma_1\gamma_2(z)) = j(\gamma_1, \gamma_2(z))\theta(\gamma_2(z)) = j(\gamma_1, \gamma_2(z)) j(\gamma_2, z)\theta(z).$$

Therefore

$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2(z)) j(\gamma_2, z). \tag{1.14}$$

Lemma 1.2 *Let* $a = ds^2 \neq 0$ *be an integer,* d *square-free. Let*

$$D = \begin{cases} |d|, & \text{if } d \equiv 1(4), \\ 4|d|, & \text{if } d \equiv 2, 3(4). \end{cases}$$

Then the map $b \mapsto \left(\frac{a}{b}\right)$ *(*b *is odd) defines a character modulo* $4a$ *with conductor* D.

Proof If a, b are co-prime, it is clear that

$$\left(\frac{a}{b}\right) = \left(\frac{d}{b}\right).$$

(1) Suppose $d > 0$ and d odd. If $b > 0$, then

$$\left(\frac{d}{b}\right) = \begin{cases} \left(\frac{b}{d}\right), & \text{if } d \equiv 1(4), \\ \left(\frac{-1}{b}\right)\left(\frac{b}{d}\right), & \text{if } d \equiv 3(4). \end{cases}$$

If $b < 0$, $d \equiv 1(4)$, then

$$\left(\frac{d}{b}\right) = \left(\frac{d}{|b|}\right) = \left(\frac{|b|}{d}\right) = \left(\frac{b}{d}\right).$$

If $b < 0$, $d \equiv 3(4)$, then

$$\left(\frac{d}{b}\right) = \left(\frac{d}{|b|}\right) = \left(\frac{-1}{|b|}\right)\left(\frac{|b|}{d}\right) = \left(\frac{-1}{b}\right)\left(\frac{b}{d}\right).$$

These conclusions show that the lemma holds in this case.

(2) Suppose $d < 0$, d is odd. If $b > 0$, then

$$\left(\frac{d}{b}\right) = \left(\frac{-1}{b}\right)\left(\frac{|d|}{b}\right) = \begin{cases} \left(\frac{b}{|d|}\right), & \text{if } d \equiv 1(4), \\ \left(\frac{-1}{b}\right)\left(\frac{b}{|d|}\right), & \text{if } d \equiv 3(4). \end{cases}$$

If $b < 0$, $d \equiv 1(4)$, then

$$\left(\frac{d}{b}\right) = -\left(\frac{d}{|b|}\right) = -\left(\frac{|b|}{|d|}\right) = \left(\frac{b}{|d|}\right).$$

If $b < 0$, $d \equiv 3(4)$, then

$$\left(\frac{d}{b}\right) = -\left(\frac{d}{|b|}\right) = -\left(\frac{-1}{|b|}\right)\left(\frac{|b|}{|d|}\right) = \left(\frac{-1}{b}\right)\left(\frac{b}{|d|}\right).$$

These conclusions show that the lemma holds in this case.

(3) Suppose $d = 2d'$, then

$$\left(\frac{d}{b}\right) = \left(\frac{2}{b}\right)\left(\frac{d'}{b}\right).$$

$\left(\frac{2}{b}\right)$ is a character modulo 8, gathering the results in (1) and (2), we proved the lemma. □

Remark 1.1 If $a \equiv 1(4)$, $b \mapsto \left(\frac{a}{b}\right)$ is a character modulo a. In this case, b can be an even integer.

Chapter 2

Eisenstein Series

2.1 Eisenstein Series with Half Integral Weight

In this section we always assume that k is an odd integer, N is a positive integer such that $4|N$, ω is an even character modulo N, i.e., $\omega(-1) = 1$. We shall construct a class of holomorphic functions which are named as Eisenstein series with the following property

$$f(\gamma(z)) = \omega(d_\gamma) j(\gamma, z)^k f(z), \quad \gamma = \begin{pmatrix} * & * \\ * & d_\gamma \end{pmatrix} \in \Gamma_0(N).$$

Lemma 2.1 *Let $k > 2$ be a positive integer, $z \in \mathbb{H}$. Put*

$$L = \{mz + n | m, n \in \mathbb{Z}\}.$$

Then the series

$$E_k(z) = \sum_{w \in L \setminus \{0\}} w^{-k} = \sum_{m,n}{}'(mz + n)^{-k}$$

is a holomorphic function on the upper half plane $\mathbb{H}$ where $\sum'$ indicates the summation over all $(m, n) \neq (0, 0)$.

Proof Let P_m be the parallelogram with vertices $\pm mz \pm m$. Denote

$$r = \min\{|w|, w \in P_1\},$$

for any $w \in P_m$, we have that $|w| \geqslant mr$. Since there are $8m$ points in $L \bigcap P_m$, then

$$\sum_{w \in L \setminus \{0\}} |w|^{-k} = \sum_{m=1}^{\infty} \sum_{w \in P_m} |w|^{-k} \leqslant 8 \sum_{m=1}^{\infty} m(mr)^{-k}.$$

It is clear that the right hand side of the above is convergent for $k > 2$. So $E_k(z)$ is absolutely and uniformly convergent in any bounded domain of $\mathbb{H}$. This shows that $E_k(z)$ is holomorphic on the whole of $\mathbb{H}$. □

Let

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\},$$

which is clearly a subgroup of $\Gamma_0(N)$. Suppose $k \geqslant 5$ and define

$$E_k(\omega, N)(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \omega(d_\gamma) j(\gamma, z)^{-k}, \tag{2.1}$$

where γ runs over a complete set of representatives of right cosets of $\mathit{\Gamma}_\infty$ in $\mathit{\Gamma}_0(N)$. For $\gamma' \in \mathit{\Gamma}_\infty$, by (1.14), we have that

$$\omega(d_{\gamma'\gamma}) j(\gamma'\gamma, z)^{-k} = \omega(d_\gamma) j(\gamma, z)^{-k},$$

which implies that $E_k(\omega, N)(z)$ is well defined. By Lemma 2.1 it is a holomorphic function on $\mathbb{H}$. For any $\gamma' \in \mathit{\Gamma}_0(N)$, it is easy to verify

$$E_k(\omega, N)(\gamma'(z)) = \overline{\omega}(d_{\gamma'}) j(\gamma', z)^k E_k(\omega, N)(z).$$

For $1 \leqslant k < 5$, the series defined in (2.1) is not absolutely convergent. We now introduce the following function

$$E_k(s, \omega, N)(z) = y^{s/2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \omega(d_\gamma) j(\gamma, z)^{-k} |j(\gamma, z)|^{-2s}, \tag{2.2}$$

where $y = \mathrm{Im}(z) > 0$, s is a complex variable and we will therefore call $|j(\gamma, z)|^{-2s}$ Hecke convergence factor because it was first introduced by Hecke. It is clear that for $\mathrm{Re}(s) > 2 - k/2$ the series (2.2) is absolutely convergent and has the following transformation property

$$E_k(s, \omega, N)(\gamma(z)) = \overline{\omega}(d_\gamma) j(\gamma, z)^k E_k(s, \omega, N)(z), \quad \gamma \in \mathit{\Gamma}_0(N). \tag{2.3}$$

We shall study the meromorphic continuation of $E_k(s, \omega, N)$ to the whole s-plane. Then we get a holomorphic function on $\mathbb{H}$ for $s = 0$. By (2.3)

$$E_k(s, \omega, N)(z + 1) = E_k(s, \omega, N)(z),$$

i.e., $E_k(s, \omega, N)(z)$ has period 1. We shall first compute the Fourier expansion of $E_k(s, \omega, N)(z)$ with respect to $\mathrm{e}^{2\pi \mathrm{i} z}$. Then we can get the analytic continuation with respect to s. Now we assume that $k \geqslant 1$. We need some lemmas.

Lemma 2.2 *Let $\lambda, y \in \mathbb{R}, \beta \in \mathbb{C}$, and $y > 0, \mathrm{Re}(\beta) > 0$. Then*

$$\int_{y - \mathrm{i}\infty}^{y + \mathrm{i}\infty} v^{-\beta} \mathrm{e}^{\lambda v} \mathrm{d}v = \begin{cases} 2\pi \mathrm{i} \lambda^{\beta - 1} \Gamma(\beta)^{-1}, & \text{if } \lambda > 0, \\ 0, & \text{if } \lambda \leqslant 0. \end{cases}$$

Proof We only need to prove the lemma for $0 < \mathrm{Re}(\beta) < 1$. Let

$$\beta = a + \mathrm{i}b, \quad v = |v| \mathrm{e}^{\mathrm{i}\varphi} = s + \mathrm{i}t, \quad s, t \in \mathbb{R}.$$

For $\lambda \leqslant 0$, we integrate along a path shown in Figure 2.1. Since

$$|v^{-\beta}\mathrm{e}^{\lambda v}| = \mathrm{e}^{-a\lg|v|+b\varphi+\lambda s} \to 0, \quad |v| \to \infty, s \geqslant y,$$

by the Cauchy Theorem for path integrals, we know that the lemma holds. For $\lambda > 0$,

$$\int_{y-\mathrm{i}\infty}^{y+\mathrm{i}\infty} v^{-\beta}\mathrm{e}^{\lambda v}\mathrm{d}v = \lambda^{\beta-1}\int_{\lambda y-\mathrm{i}\infty}^{\lambda y+\mathrm{i}\infty} v^{-\beta}\mathrm{e}^{v}\mathrm{d}v,$$

we integrate along the path as in Figure 2.2. When v runs over the small circle with radius r, we get

$$r|v^{-\beta}\mathrm{e}^{v}| = r^{1-a}|\mathrm{e}^{v}| \to 0, \quad r \to 0,$$

since $0 < a < 1$. On the other hand,

$$|v^{-\beta}\mathrm{e}^{v}| = \mathrm{e}^{-a\lg|v|+b\varphi+s} \to 0, \quad |v| \to \infty, s \leqslant \lambda y.$$

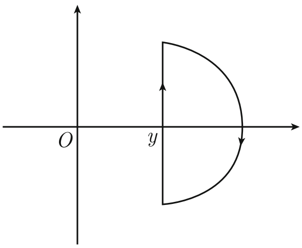

Figure 2.1

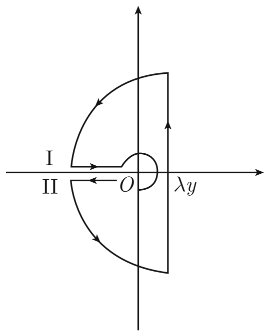

Figure 2.2

Hence by the Cauchy Theorem we have

$$\int_{\lambda y-\mathrm{i}\infty}^{\lambda y+\mathrm{i}\infty} v^{-\beta}\mathrm{e}^{v}\mathrm{d}v = -\int_{-\infty}^{0} v^{-\beta}\mathrm{e}^{v}\mathrm{d}v - \int_{0}^{-\infty} v^{-\beta}\mathrm{e}^{v}\mathrm{d}v,$$

where the variable v in the first integral runs above the negative real axis and the variable v in the second integral runs underneath the negative real axis. Therefore

$$\int_{-\infty}^{0} v^{-\beta}\mathrm{e}^{v}\mathrm{d}v = \mathrm{e}^{-\mathrm{i}\pi\beta}\int_{0}^{\infty} x^{-\beta}\mathrm{e}^{-x}\mathrm{d}x = \mathrm{e}^{-\mathrm{i}\pi\beta}\Gamma(1-\beta),$$
$$\int_{0}^{-\infty} v^{-\beta}\mathrm{e}^{v}\mathrm{d}v = -\mathrm{e}^{\mathrm{i}\pi\beta}\int_{0}^{\infty} x^{-\beta}\mathrm{e}^{-x}\mathrm{d}x = -\mathrm{e}^{\mathrm{i}\pi\beta}\Gamma(1-\beta).$$

But

$$(\mathrm{e}^{\mathrm{i}\pi\beta} - \mathrm{e}^{-\mathrm{i}\pi\beta})\Gamma(1-\beta) = 2\mathrm{i}\Gamma(1-\beta)\sin\pi\beta = 2\pi\mathrm{i}\Gamma(\beta)^{-1},$$

which completes the proof. □

Let $y > 0,\ \alpha,\ \beta \in \mathbb{C}$ and define

$$W(y,\alpha,\beta) = \Gamma(\beta)^{-1}\int_0^\infty (1+u)^{\alpha-1}u^{\beta-1}\mathrm{e}^{-yu}\mathrm{d}u,$$

which is called the Whittaker function. It is clear that the integral is convergent for $\mathrm{Re}(\beta) > 0$. Applying integration by parts we get

$$W(y,\alpha,\beta) = yW(y,\alpha,\beta+1) + (1-\alpha)W(y,\alpha-1,\beta+1). \tag{2.4}$$

Due to the above equality $W(y,\alpha,\beta)$ can be continued analytically to $\mathbb{C}^2$ for (α,β). We will also denote the continued function by $W(y,\alpha,\beta)$.

Lemma 2.3 $W(y,\alpha,0) = 1, W(y,\alpha,-1/2) = y^{1/2}$.

Proof Taking $\beta = 0$ in equality (2.4), we have

$$\begin{aligned}
W(y,\alpha,0) &= yW(y,\alpha,1) + (1-\alpha)W(y,\alpha-1,1)\\
&= y\int_0^\infty (1+u)^{\alpha-1}\mathrm{e}^{-yu}\mathrm{d}u + (1-\alpha)\int_0^\infty (1+u)^{\alpha-2}\mathrm{e}^{-yu}\mathrm{d}u\\
&= y\int_0^\infty (1+u)^{\alpha-1}\mathrm{e}^{-yu}\mathrm{d}u - \int_0^\infty \mathrm{e}^{-yu}\mathrm{d}(1+u)^{\alpha-1}\\
&= -\mathrm{e}^{-yu}(1+u)^{\alpha-1}\ |_0^\infty = 1.
\end{aligned}$$

Similarly taking $\beta = -1/2$ in (2.4), we have

$$W(y,1,-1/2) = yW(y,1,1/2) = y\Gamma(1/2)^{-1}\int_0^\infty u^{-1/2}\mathrm{e}^{-yu}\mathrm{d}u = y^{1/2},$$

which completes the proof. □

Lemma 2.4 *Let* $y > 0, \alpha, \beta \in \mathbb{C}$. *Then*

$$y^\beta W(y,\alpha,\beta) = y^{1-\alpha}W(y,1-\beta,1-\alpha).$$

Proof Taking the Mellin transformation of $\Gamma(\beta)W(y,\alpha,\beta)$ (assume $\mathrm{Re}(s) > 0$), we see

$$\begin{aligned}
\Gamma(\beta)\int_0^\infty W(y,\alpha,\beta)y^{s-1}\mathrm{d}y &= \int_0^\infty (u+1)^{\alpha-1}u^{\beta-1}\int_0^\infty y^{s-1}\mathrm{e}^{-yu}\mathrm{d}y\mathrm{d}u\\
&= \Gamma(s)\int_0^\infty (u+1)^{\alpha-1}u^{\beta-s-1}\mathrm{d}u.
\end{aligned}$$

Suppose $\mathrm{Re}(1-\alpha) > 0$ and insert the following equality into the formula above

$$(u+1)^{\alpha-1} = \Gamma(1-\alpha)^{-1}\int_0^\infty \mathrm{e}^{-x(u+1)}x^{-\alpha}\mathrm{d}x,$$

we get

$$\Gamma(1-\alpha)\Gamma(\beta)\int_0^{\infty} W(y,\alpha,\beta)y^{s-1}\mathrm{d}y = \Gamma(s)\Gamma(\beta-s)\Gamma(1-\alpha-\beta+s).$$

By the inverse Mellin transformation, we see

$$W(y,\alpha,\beta) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\Gamma(s)\Gamma(\beta-s)\Gamma(1-\alpha-\beta+s)}{\Gamma(1-\alpha)\Gamma(\beta)} y^{-s}\mathrm{d}s,$$

where c satisfies the inequalities $c > 0$, $\mathrm{Re}(\beta) > c > \mathrm{Re}(\alpha+\beta-1)$. There exists such a c if $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(1-\alpha) > 0$. Let $S = s - \beta$, we have

$$y^{\beta}W(y,\alpha,\beta) = \frac{1}{2\pi\mathrm{i}}\int_{-p-\mathrm{i}\infty}^{-p+\mathrm{i}\infty} \frac{\Gamma(-S)\Gamma(\beta+S)\Gamma(1-\alpha+S)}{\Gamma(1-\alpha)\Gamma(\beta)} y^{-S}\mathrm{d}S,$$

where p satisfies $0 < p < \min\{\mathrm{Re}(1-\alpha), \mathrm{Re}(\beta)\}$. The right hand side of the above equality is stable under the transformation $\alpha \to 1-\beta, \beta \to 1-\alpha$. This shows that the lemma holds for $\mathrm{Re}(1-\alpha) > 0, \mathrm{Re}(\beta) > 0$. But $W(y,\alpha,\beta)$ is analytic on $\mathbb{C}^2$. So the lemma holds for any $(\alpha,\beta) \in \mathbb{C}^2$, which completes the proof. □

Lemma 2.5 *Suppose that* $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\alpha+\beta) > 1, z = x + \mathrm{i}y \in \mathbb{H}$, *then*

$$\sum_{m=-\infty}^{+\infty} (z+m)^{-\alpha}(\overline{z}+m)^{-\beta} = \sum_{n=-\infty}^{+\infty} t_n(y,\alpha,\beta)\mathrm{e}^{2\pi\mathrm{i}nx},$$

where

$$\mathrm{i}^{\alpha-\beta}(2\pi)^{-\alpha-\beta}t_n(y,\alpha,\beta) = \begin{cases} n^{\alpha+\beta-1}\mathrm{e}^{-2\pi ny}\Gamma(\alpha)^{-1}W(4\pi ny,\alpha,\beta), & \text{if } n > 0, \\ |n|^{\alpha+\beta-1}\mathrm{e}^{-2\pi|n|y}\Gamma(\beta)^{-1}W(4\pi|n|y,\beta,\alpha), & \text{if } n < 0, \\ \Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}\Gamma(\alpha+\beta-1)(4\pi y)^{1-\alpha-\beta}, & \text{if } n = 0. \end{cases}$$

Proof Let

$$f(x) = \sum_{m=-\infty}^{+\infty} (x+\mathrm{i}y+m)^{-\alpha}(x-\mathrm{i}y+m)^{-\beta}.$$

This series is absolutely convergent for $\mathrm{Re}(\alpha+\beta) > 1$. Since $f(x+1) = f(x)$, we have

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n\mathrm{e}^{2\pi\mathrm{i}nx},$$

where

$$\begin{aligned}
c_n &= \int_0^1 f(x)\mathrm{e}^{-2\pi \mathrm{i} nx}\mathrm{d}x \\
&= \int_{-\infty}^{+\infty} (x+\mathrm{i}y)^{-\alpha}(x-\mathrm{i}y)^{-\beta}\mathrm{e}^{-2\pi \mathrm{i} nx}\mathrm{d}x \\
&= \mathrm{i}^{\beta-\alpha}\int_{-\infty}^{+\infty} (y-\mathrm{i}x)^{-\alpha}(y+\mathrm{i}x)^{-\beta}\mathrm{e}^{-2\pi \mathrm{i} nx}\mathrm{d}x \\
&= \mathrm{i}^{\beta-\alpha-1}\mathrm{e}^{2\pi ny}\int_{y-\mathrm{i}\infty}^{y+\mathrm{i}\infty} v^{-\beta}(2y-v)^{-\alpha}\mathrm{e}^{-2\pi nv}\mathrm{d}v \\
&= \mathrm{i}^{\beta-\alpha-1}\mathrm{e}^{2\pi ny}\Gamma(\alpha)^{-1}\int_{y-\mathrm{i}\infty}^{y+\mathrm{i}\infty} v^{-\beta}\mathrm{e}^{-2\pi nv}\int_0^{\infty}\mathrm{e}^{-\xi(2y-v)}\xi^{\alpha-1}\mathrm{d}\xi\mathrm{d}v \\
&= \mathrm{i}^{\beta-\alpha-1}\mathrm{e}^{2\pi ny}\Gamma(\alpha)^{-1}\int_0^{\infty}\xi^{\alpha-1}\mathrm{e}^{-2y\xi}\left\{\int_{y-\mathrm{i}\infty}^{y+\mathrm{i}\infty} v^{-\beta}\mathrm{e}^{(\xi-2\pi n)v}\mathrm{d}v\right\}\mathrm{d}\xi,
\end{aligned}$$

where we used the fact that

$$(2y-v)^{-\alpha} = \Gamma(\alpha)^{-1}\int_0^{\infty}\mathrm{e}^{-\xi(2y-v)}\xi^{\alpha-1}\mathrm{d}\xi$$

for $\mathrm{Re}(\alpha) > 0$.

Now let $\xi = 2\pi p, u = \max\{0, n\}$. Since $\mathrm{Re}(\beta) > 0$, by Lemma2.2 we have

$$\begin{aligned}
c_n &= 2\pi\mathrm{i}^{\beta-\alpha}\mathrm{e}^{2\pi ny}\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}\int_{2\pi u}^{\infty}\xi^{\alpha-1}(\xi-2\pi n)^{\beta-1}\mathrm{e}^{-2y\xi}\mathrm{d}\xi \\
&= (2\pi)^{\alpha+\beta}i^{\beta-\alpha}\mathrm{e}^{2\pi ny}\Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}\int_u^{\infty} p^{\alpha-1}(p-n)^{\beta-1}\mathrm{e}^{-4\pi py}\mathrm{d}p.
\end{aligned}$$

If $n > 0$, then $u = n$, let $p - n = nq$. If $n < 0$, then $u = 0$, let $p = -nq$. Hence we have

$$\begin{aligned}
&\int_u^{\infty} p^{\alpha-1}(p-n)^{\beta-1}\mathrm{e}^{-4\pi py}\mathrm{d}p \\
&= \begin{cases}
n^{\alpha+\beta-1}\displaystyle\int_0^{\infty}(q+1)^{\alpha-1}q^{\beta-1}\mathrm{e}^{-4\pi n(1+q)y}\mathrm{d}q, & \text{if } n > 0, \\
|n|^{\alpha+\beta-1}\displaystyle\int_0^{\infty}(q+1)^{\beta-1}q^{\alpha-1}\mathrm{e}^{-4\pi|n|qy}\mathrm{d}q, & \text{if } n < 0, \\
\displaystyle\int_0^{\infty} p^{\alpha+\beta-2}\mathrm{e}^{-4\pi py}\mathrm{d}p, & \text{if } n = 0
\end{cases} \\
&= \begin{cases}
n^{\alpha+\beta-1}\mathrm{e}^{-4\pi ny}W(4\pi ny, \alpha, \beta), & \text{if } n > 0, \\
|n|^{\alpha+\beta-1}W(4\pi|n|y, \beta, \alpha), & \text{if } n < 0, \\
(4\pi y)^{1-\alpha-\beta}\Gamma(\alpha+\beta-1), & \text{if } n = 0,
\end{cases}
\end{aligned}$$

which completes the proof. □

Now we can compute the Fourier expansion of $E_k(s,\omega,N)(z)$. Let

$$W=\{(c,d)|c,d\in\mathbb{Z},\gcd(c,d)=1,N|c,c\geqslant 0,d=1 \text{ if } c=0\}.$$

Then we can prove that there exists a one-to-one correspondence between W and the set of representatives of right cosets of Γ_∞ in $\Gamma_0(N)$. Suppose $\mathrm{Re}(s)>2-k/2$, by Lemma 2.5 we have (substituting c by cN)

$$\begin{aligned}E_k(s,\omega,N)(z)&=y^{s/2}\left\{1+\sum_{d=-\infty}^{+\infty}\sum_{c=1}^{+\infty}\omega(d)\varepsilon_d^k\left(\frac{cN}{d}\right)(cNz+d)^{-k/2}|cNz+d|^{-s}\right\}\\&=y^{s/2}\left\{1+\sum_{c=1}^{\infty}(cN)^{-k/2-s}\sum_{d=1}^{cN}\omega(d)\varepsilon_d^k\left(\frac{cN}{d}\right)\right.\\&\quad\left.\times\sum_{n=-\infty}^{\infty}\left(z+\frac{d}{cN}+n\right)^{-k/2-s/2}\left(\overline{z}+\frac{d}{cN}+n\right)^{-s/2}\right\}\\&=y^{s/2}\left\{1+\sum_{n=-\infty}^{\infty}a_k(n,s,\omega,N)t_n(y,(k+s)/2,s/2)e(nx)\right\},\end{aligned}\tag{2.5}$$

where

$$a_k(n,s,\omega,N)=\sum_{c=1}^{\infty}(cN)^{-k/2-s}\sum_{d=1}^{cN}\omega(d)\varepsilon_d^k\left(\frac{cN}{d}\right)e\left(\frac{nd}{cN}\right).\tag{2.6}$$

For $\mathrm{Re}(s)>2-k/2$, define

$$E_k'(s,\omega,N)(z)=z^{-k/2}E_k(s,\omega,N)(-1/(Nz)).\tag{2.7}$$

Now assume that $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(N)$. Then by (2.3) we can verify easily that

$$E_k'(s,\omega,N)(\gamma(z))=\omega(d)\left(\frac{N}{d}\right)j(\gamma,z)^kE_k'(s,\omega,N)(z).\tag{2.8}$$

Now let $W'=\{(c,d)|c,d\in\mathbb{Z},\gcd(c,d)=1,N|c,d>0\}$. Then there exists a one-to-one correspondence between W' and the set of representatives of cosets of Γ_∞ in $\Gamma_0(N)$. Then we can similarly get that

$$E_k'(s,\omega,N)(z)=y^{s/2}N^{-s/2}\sum_{n=-\infty}^{\infty}b_k(n,s,\omega,N)t_n(y,(k+s)/2,s/2)e(nx),\tag{2.9}$$

where

$$b_k(n,s,\omega,N)=\sum_{d=1}^{\infty}\left(\frac{-N}{d}\right)\omega(d)\varepsilon_d^kd^{-s-k/2}\sum_{m=1}^{d}\left(\frac{m}{d}\right)e\left(\frac{nm}{d}\right).\tag{2.10}$$

Lemma 2.6 *Let ω_0 be a primitive character modulo r, ω be a character modulo rs, and $\omega(n)=\omega_0(n)$ for $\gcd(n,s)=1$. Then for any integer q we have*

$$\sum_{n=1}^{rs}\omega(n)e\left(\frac{nq}{rs}\right)=\sum_{m=1}^{r}\omega_0(m)e(m/r)\sum_{c|(s,q)}c\mu(s/c)\omega_0(s/c)\overline{\omega}_0(q/c).$$

Proof We have that

$$\begin{aligned}\sum_{n=1}^{rs}\omega(n)e\left(\frac{nq}{rs}\right)&=\sum_{n=1}^{rs}\omega_0(n)\sum_{d|(s,n)}\mu(d)e\left(\frac{nq}{rs}\right)\\&=\sum_{d|s}\mu(d)\sum_{n=1}^{rs/d}\omega_0(nd)e\left(\frac{ndq}{rs}\right)\\&=\sum_{d|s}\mu(d)\omega_0(d)\sum_{n=1}^{r}\omega_0(n)e\left(\frac{ndq}{rs}\right)\sum_{u=1}^{s/d}e\left(\frac{uq}{s/d}\right).\end{aligned}$$

Denote $c=s/d$, then the inner summation in the above formula is zero for all $c\nmid q$ and is c for $c|q$ which shows the lemma. □

Now let $d=ru^2$ be an odd positive integer with r square free. Taking $\omega=\left(\frac{\cdot}{d}\right)$, $\omega_0=\left(\frac{\cdot}{r}\right)$, $q=n, s=u^2$ in Lemma 2.6, we have

$$\sum_{m=1}^{d}\left(\frac{m}{d}\right)e\left(\frac{nm}{d}\right)=\varepsilon_r r^{1/2}\sum_{c|(u^2,n)}c\mu(u^2/c)\left(\frac{u^2/c}{r}\right)\left(\frac{n/c}{r}\right), \tag{2.11}$$

where we used the fact

$$\sum_{m=1}^{r}\left(\frac{m}{r}\right)e\left(\frac{m}{r}\right)=\varepsilon_r r^{1/2}.$$

Let $\lambda=(k-1)/2$ and n be an integer. We define a primitive character $\omega_k^{(n)}$ satisfying

$$\omega_k^{(n)}(d)=\left(\frac{(-1)^\lambda nN}{d}\right)\omega(d),\qquad \text{if } (d,nN)=1.$$

We also define a primitive character ω' satisfying

$$\omega'(d)=\omega^2(d),\qquad \text{if } (d,N)=1.$$

Suppose that χ is a character modulo a factor of N. Define

$$L_N(s,\chi)=\sum_{(n,N)=1}^{\infty}\chi(n)n^{-s}=\prod_{p\nmid N}(1-\chi(p)p^{-s})^{-1},$$

where p runs over all primes co-prime to N.

Proposition 2.1 *We have*

$$L_N(2s+2\lambda,\omega')b_k(0,s,\omega,N)=L_N(2s+2\lambda-1,\omega').$$

For $n\neq 0$, we have

$$L_N(2s+2\lambda,\omega')b_k(n,s,\omega,N)=L_N(s+\lambda,\omega_k^{(n)})\beta_k(n,s,\omega,N),$$

where

$$\beta_k(n,s,\omega,N)=\sum_{a,b}\mu(a)\omega_k^{(n)}(a)\omega'(b)a^{-s-\lambda}b^{-2s-2\lambda+1}, \tag{2.12}$$

where a, b run over all positive integers such that $(ab,N)=1$ and $(ab)^2|n$.

Proof For $n=0$, the inner summation of (2.10) is nonzero only for d a square. Therefore

$$\begin{aligned}
b_k(0,s,\omega,N)&=\sum_{u=1}^{\infty}\omega(u^2)u^{-2s-k}\varphi(u^2)\\
&=\prod_{p\nmid N}\left\{\sum_{i=0}^{\infty}\omega(p^{2i})p^{-(2s+k)i}\varphi(p^{2i})\right\}\\
&=\prod_{p\nmid N}\left\{1+\sum_{i=1}^{\infty}\left(1-\frac{1}{p}\right)(\omega(p^2)p^{-(2s+k-2)})^i\right\}\\
&=\prod_{p\nmid N}\frac{1-\omega(p^2)p^{-2s-k-1}}{1-\omega(p^2)p^{-2s-k+2}}\\
&=L_N(2s+2\lambda-1,\omega')L_N(2s+2\lambda,\omega')^{-1}.
\end{aligned}$$

Now assume that $n=tm^2\neq 0$, t square free. Since N is even, the summation in (2.10) is nonzero only for odd integer d. By (2.11) we get

$$\begin{aligned}
b_k(n,s,\omega,N)=&\sum_{r,u}\left(\frac{-N}{ru^2}\right)\varepsilon_r^{k+1}\omega(ru^2)(ru^2)^{-s-k/2}r^{1/2}\\
&\times\sum_{c|(u^2,n)}c\mu(u^2/c)\left(\frac{u^2/c}{r}\right)\left(\frac{n/c}{r}\right),
\end{aligned}$$

where r,u run over all positive integers with r square free. Denote $u^2=ac$, then $\mu(a)\neq 0$ only for a square free. So we can suppose $u=ab$. Then

$$c=ab^2,\quad u^2n/c^2=n/b^2,$$

hence

$$b_k(n,s,\omega,N)=\sum_{r,a,b}\mu(a)r^{-s-\lambda}a^{-2s-2\lambda}b^{-2s-2\lambda+1}\omega(ra^2b^2)\left(\frac{(-1)^{\lambda}nN/b^2}{r}\right),$$

where we used the fact

$$\varepsilon_r^{k+1} = \left(\frac{(-1)^{(k+1)/2}}{r} \right)$$

and r, a, b run over all positive integers such that $(rab, N) = 1$, $ab^2 \mid n$ and r square free. Since $ab^2 \mid n = tm^2$, we see that $b \mid m$. Let $m = bh$, then $a \mid th$, $n/b^2 = th^2$. Since

$$\omega(r) \left(\frac{(-1)^\lambda Nth^2}{r} \right) = \begin{cases} 0, & \text{if } (r, thN) > 1, \\ \omega_k^{(n)}(r), & \text{if } (r, thN) = 1, \end{cases}$$

we have

$$\begin{aligned} b_k(n, s, \omega, N) = &\sum_{b|m} \omega^2(b) b^{-2s-2\lambda+1} \sum_{a|th} \mu(a) \omega^2(a) a^{-2s-2\lambda} \\ &\times \sum_{(r,thN)=1} \mu^2(r) \omega_k^{(n)}(r) r^{-s-\lambda}. \end{aligned} \tag{2.13}$$

It is clear that

$$\sum_{a|th} \mu(a) \omega^2(a) a^{-2s-2\lambda} = \prod_{p|th, p\nmid N} \left(1 - \omega'(p) p^{-2s-2\lambda}\right) \tag{2.14}$$

and

$$\begin{aligned} \sum_{(r,thN)=1} \mu^2(r) \omega_k^{(n)}(r) r^{-s-\lambda} &= \prod_{p \nmid thN} \left(1 + \omega_k^{(n)}(p) p^{-s-\lambda}\right) \\ &= \prod_{p \nmid thN} \frac{1 - \omega'(p) p^{-2s-2\lambda}}{1 - \omega_k^{(n)}(p) p^{-s-\lambda}} \\ &= \frac{L_N(s+\lambda, \omega_k^{(n)})}{L_N(2s+2\lambda, \omega')} \prod_{p|th, p\nmid N} \frac{1 - \omega_k^{(n)}(p) p^{-s-\lambda}}{1 - \omega'(p) p^{-2s-2\lambda}}, \end{aligned} \tag{2.15}$$

For primes p such that $p \mid t, p \nmid N$, we have $\omega_k^{(n)}(p) = 0$. Inserting (2.14) and (2.15) into (2.13), we get

$$\begin{aligned} b_k(n, s, \omega, N) &= \frac{L_N(s+\lambda, \omega_k^{(n)})}{L_N(2s+2\lambda, \omega')} \sum_{b|m} \omega^2(b) b^{-2s-2\lambda+1} \prod_{p|h, p\nmid N} \left(1 - \omega_k^{(n)}(p) p^{-s-\lambda}\right) \\ &= \frac{L_N(s+\lambda, \omega_k^{(n)})}{L_N(2s+2\lambda, \omega')} \sum_{a,b} \mu(a) \omega_k^{(n)}(a) \omega'(b) a^{-s-\lambda} b^{-2s-2\lambda+1}, \end{aligned}$$

which completes the proof. □

Let n be any integer and χ_n a primitive character satisfying

$$\chi_n(d) = \left(\frac{n}{d} \right) \quad \text{for all } (d, 4n) = 1.$$

By Lemma 1.2, if $n = ab^2$ with a square free, then the conductor of χ_n is $|a|$ or $4|a|$ according to $a \equiv 1(4)$ or $a \equiv 2,3(4)$ respectively.

Proposition 2.2 *We have that*

$$a_k(n,s,\omega,N) = b_k(n,s,\omega\chi_N,N)c_k(n,s,\omega,N),$$

where

$$c_k(n,s,\omega,N) = \sum_{N|M|N^\infty}\sum_{d=1}^{M}\left(\frac{M}{d}\right)\omega(d)\varepsilon_d^k e\left(\frac{nd}{M}\right)M^{-s-k/2}. \tag{2.16}$$

And $c_k(n,s,\omega,N)$ is a finite series for all $n \neq 0$. ($M|N^\infty$ implies that every prime factor of M is also a factor of N.)

Proof Denote $cN = aM$ with $(a,N) = 1$ and $N|M|N^\infty$. Then

$$\sum_{d=1}^{cN}\omega(d)\varepsilon_d^k\left(\frac{cN}{d}\right)e\left(\frac{nd}{aM}\right) = \sum_{d_1=1}^{M}\sum_{d_2=1}^{a}\omega(d_1a+d_2M) \times \varepsilon_{d_1a}^k\left(\frac{aM}{d_1a+d_2M}\right)e\left(\frac{n(d_1a+d_2M)}{aM}\right).$$

For positive odds a, b, we have by Lemma 1.2 that

$$\left(\frac{a}{b}\right) = \left(\frac{b}{a}\right)\varepsilon_b^{-k}\varepsilon_a^{-k}\varepsilon_{ab}^k,$$

$$\left(\frac{aM}{d_1a+d_2M}\right) = \left(\frac{M}{d_1a+d_2M}\right)\left(\frac{a}{d_1a+d_2M}\right) = \left(\frac{M}{d_1a}\right)\left(\frac{d_2M}{a}\right)\varepsilon_a^{-k}\varepsilon_{d_1a}^{-k}\varepsilon_{d_1}^k. \tag{2.17}$$

Hence by (2.6) we have

$$a_k(n,s,\omega,N) = \sum_{a=1}^{\infty}\left(\frac{-1}{a}\right)\omega(a)\varepsilon_a^k a^{-s-k/2}\sum_{d=1}^{a}\left(\frac{d}{a}\right)e\left(\frac{nd}{a}\right) \times \sum_{N|M|N^\infty}\sum_{d=1}^{M}\left(\frac{M}{d}\right)\varepsilon_d^k\omega(d)e\left(\frac{nd}{M}\right)M^{-s-k/2} = b_k(n,s,\omega\chi_N,N)c_k(n,s,\omega,N).$$

For $k \equiv 1(4)$, we have

$$\varepsilon_d^k = \varepsilon_d = \frac{1}{2}\left(1+\left(\frac{-1}{d}\right)\right) + \frac{\mathrm{i}}{2}\left(1-\left(\frac{-1}{d}\right)\right).$$

Hence the coefficient of $M^{-s-k/2}$ in the inner summation of $c_k(n,s,\omega,N)$ is

$$\frac{1+\mathrm{i}}{2}\sum_{d=1}^{M}\left(\frac{M}{d}\right)\omega(d)e\left(\frac{nd}{M}\right)+\frac{1-\mathrm{i}}{2}\sum_{d=1}^{M}\left(\frac{-M}{d}\right)\omega(d)e\left(\frac{nd}{M}\right).$$

We employ Lemma 2.6 to the above sums. If M is sufficiently large, then $\mu(s/c)=0$ for any $c|(s,n)$(s is determined by M). This shows that $c_k(n,s,\omega,N)$ is a finite sum for $k\equiv 1(4)$. It can be proved similarly for the case $k\equiv 3(4)$, which completes the proof. □

In order to discuss the analytic continuation of $E_k(s,\omega,N)$, we also need the following two lemmas.

Lemma 2.7 *Let ω be a primitive character modulo r, $r\neq 1$, and*

$$R(s,\omega)=(r/\pi)^{(s+\nu)/2}\Gamma((s+\nu)/2)L(s,\omega),$$

where $\nu=0$ or 1 according to $\omega(-1)=1$ or -1 respectively. Then for any compact subset J of $\mathbb{R}$, there exists a constant c_J(it is independent on r and ω) such that

$$|R(s,\omega)|\leqslant c_J r^{|\sigma|/2+2},\quad \sigma=\mathrm{Re}(s)\in J.$$

Proof Put

$$g_\nu(t,\omega)=\sum_{n=-\infty}^{\infty}\omega(n)n^\nu \mathrm{e}^{-\pi n^2 t/r},\qquad \text{for } t>0.$$

Using

$$(n^2\pi/r)^{(-s+\nu)/2}\Gamma((s+\nu)/2)=\int_0^\infty \mathrm{e}^{-\pi n^2 t/r}t^{(s+\nu)/2-1}\mathrm{d}t,$$

we get

$$R(s,\omega)=\frac{1}{2}\int_0^\infty g_\nu(t,\omega)t^{(s+\nu)/2-1}\mathrm{d}t. \tag{2.18}$$

Taking the derivative with respect to x on both sides of (1.1), we have

$$\sum_{n=-\infty}^{\infty}(n+x)\mathrm{e}^{-\pi t(n+x)^2}=-\mathrm{i}t^{-3/2}\sum_{n=-\infty}^{\infty}n\mathrm{e}^{-\pi n^2/t+2\pi \mathrm{i}nx}.$$

Therefore

$$\begin{aligned}g_\nu(t^{-1},\omega)&=\sum_{d=1}^{r}\omega(d)r^\nu\sum_{m=-\infty}^{\infty}(m+d/r)^\nu \mathrm{e}^{-\pi r(m+d/r)^2/t}\\&=(-\mathrm{i})^\nu r^{-1/2}t^{\nu+1/2}\sum_{n=-\infty}^{\infty}n^\nu \mathrm{e}^{-\pi t n^2/r}\sum_{d=1}^{r}\omega(d)e\left(\frac{nd}{r}\right)\\&=(-\mathrm{i})^\nu r^{-1/2}t^{\nu+1/2}\sum_{d=1}^{r}\omega(d)e\left(\frac{d}{r}\right)\sum_{n=-\infty}^{\infty}\overline{\omega}(n)n^\nu \mathrm{e}^{-\pi t n^2/r}\\&=\varepsilon_\nu(\omega)t^{\nu+1/2}g_\nu(t,\overline{\omega}),\end{aligned} \tag{2.19}$$

where

$$\varepsilon_\nu(\omega) = (-\mathrm{i})^\nu r^{-1/2} \sum_{d=1}^{r} \omega(d) e(d/r),$$

whose absolute value equals to 1. By (2.18) and (2.19), we have

$$\begin{aligned} R(s,\omega) &= \frac{1}{2}\left(\int_1^\infty g_\nu(t,\omega) t^{(s+\nu)/2-1}\mathrm{d}t + \int_1^\infty g_\nu(t^{-1},\omega) t^{-(s+\nu)/2-1}\mathrm{d}t\right) \\ &= \frac{1}{2}\int_1^\infty g_\nu(t,\omega) t^{(s+\nu)/2-1}\mathrm{d}t + \frac{\varepsilon_\nu(\omega)}{2}\int_1^\infty g_\nu(t,\overline{\omega}) t^{(1-s+\nu)/2-1}\mathrm{d}t. \end{aligned} \quad (2.20)$$

Denote the first term in the formula above by $P(s,\omega)$. Then the second term is $\varepsilon_\nu(\omega)P(1-s,\overline{\omega})$. This shows that $R(s,\omega)$ can be analytically continued to a holomorphic function on the whole s-plane. And we have the following functional equation

$$R(1-s,\omega) = \varepsilon_\nu(\omega) R(s,\overline{\omega}). \quad (2.21)$$

For $\sigma > 1$, we have

$$|R(1-s,\omega)| = |R(s,\overline{\omega})| \leqslant (r/\pi)^{(\sigma+\nu)/2}\Gamma((\sigma+\nu)/2)\zeta(\sigma). \quad (2.22)$$

Now we only need to prove Lemma 2.7 for $-1 < \sigma \leqslant 2$ because of the functional equation. Since

$$|g_\nu(t,\omega)| \leqslant 2\sum_{n=1}^{\infty} n\mathrm{e}^{-\pi n t/r} = 2\mathrm{e}^{-\pi t/r}(1-\mathrm{e}^{-\pi t/r})^{-2},$$

hence

$$\begin{aligned} |P(s,\omega)| &\leqslant \int_1^\infty \mathrm{e}^{-\pi t/r}(1-\mathrm{e}^{-\pi t/r})^{-2} t^{(\sigma+\nu)/2-1}\mathrm{d}t \\ &= (r/\pi)^{(\sigma+\nu)/2}\int_{\pi/r}^\infty \mathrm{e}^{-t}(1-\mathrm{e}^{-t})^{-2} t^{(\sigma+\nu)/2-1}\mathrm{d}t. \end{aligned} \quad (2.23)$$

Without loss of generality, we can assume $r > \pi$. Then we divide the interval $(\pi/r,\infty)$ into $(\pi/r,1)$ and $(1,\infty)$. The integral on the interval $(1,\infty)$ is independent on r and ω. Since $t/(1-\mathrm{e}^{-t})$ is continuous on $(0,1)$, there exists a constant A such that $\mathrm{e}^{-t}(1-\mathrm{e}^{-t})^{-2} \leqslant At^{-2}$. This implies that for $-1 < \sigma < 2$ there exist constants B, C independent on r and ω such that

$$\int_{\pi/r}^\infty \mathrm{e}^{t}(1-\mathrm{e}^{-t})^{-2} t^{(\sigma+\nu)/2-1}\mathrm{d}t \leqslant A\int_{\pi/r}^\infty t^{(\sigma+\nu)/2-3}\mathrm{d}t \leqslant B + Cr^{2-(\sigma+\nu)/2}. \quad (2.24)$$

Inserting (2.24) into (2.23) we get

$$|P(s,\omega)| \leqslant Dr^2, \quad -1 < \sigma < 2$$

with a constant D. Therefore

$$|\mathrm{Re}(s,\omega)| \leqslant C_J r^2, \quad -1 < \sigma < 2. \tag{2.25}$$

Now (2.22) and (2.25) show our result. □

In the proof of Lemma 2.7 we really showed that $R(s,\omega)$ is holomorphic on the whole s-plane for any non-trivial character ω and got its functional equation (2.21). For the trivial character, let

$$\eta(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s), \quad \mathrm{Re}(s) > 1,$$

where $\zeta(s)$ is the Riemannian ζ-function. In a similar way we get for $\mathrm{Re}(s) > 1$:

$$\begin{aligned}\eta(s) &= \frac{1}{2}\int_0^\infty \left(\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi n^2 t} - 1\right) t^{s/2-1}\mathrm{d}t \\ &= \frac{1}{s(s-1)} + \frac{1}{2}\int_1^\infty \left(\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi n^2 t} - 1\right) t^{s/2-1}\mathrm{d}t \\ &\quad + \frac{1}{2}\int_1^\infty \left(\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi n^2 t} - 1\right) t^{(1-s)/2-1}\mathrm{d}t.\end{aligned}$$

This shows that $\xi(s) = s(s-1)\eta(s)$ is holomorphic on the whole s-plane and $\xi(s) = \xi(1-s)$.

Lemma 2.8 *Let K be a compact subset of $\mathbb{C}^2$, then there exist two constants A and B such that*

$$|y^\beta W(y,\alpha,\beta)| \leqslant A\max\{y^{-B}, 1\} \quad \textit{for all } (\alpha,\beta) \in K.$$

Proof In the proof of Lemma 2.4 we got

$$y^\beta W(y,\alpha,\beta) = \frac{1}{2\pi \mathrm{i}}\int_{-p-\mathrm{i}\infty}^{-p+\mathrm{i}\infty} \frac{\Gamma(-s)\Gamma(s+\beta)\Gamma(s+1-\alpha)}{\Gamma(-\alpha)\Gamma(\beta)} y^{-s}\mathrm{d}s,$$

where $0 < p < \min\{\mathrm{Re}(\beta), \mathrm{Re}(1-\alpha)\}$. Suppose $\mathrm{Re}(\beta) > -q, \mathrm{Re}(1-\alpha) > -q, q$ be a positive number. We move the integral line $\mathrm{Re}(s) = -p$ to $\mathrm{Re}(s) = q$. Since

$$\Gamma(-s) = \frac{\Gamma(-s+m+1)}{(-s)(-s+1)\cdots(-s+m)}, \quad m \geqslant 0,$$

the residue of $\Gamma(-s)$ at $s = m$ is $(-1)^m/m!$. Hence

$$\begin{aligned}y^\beta W(y,\alpha,\beta) = &\sum_{m=0}^{[q]} \frac{\Gamma(m+\beta)\Gamma(m+1-\alpha)}{\Gamma(m+1)\Gamma(1-\alpha)\Gamma(\beta)}(-y)^{-m} \\ &+ \frac{1}{2\pi \mathrm{i}}\int_{q-\mathrm{i}\infty}^{q+\mathrm{i}\infty} \frac{\Gamma(-s)\Gamma(s+\beta)\Gamma(s+1-\alpha)}{\Gamma(1-\alpha)\Gamma(\beta)} y^{-s}\mathrm{d}s.\end{aligned}$$

Since both terms of the above formula are holomorphic on $\mathbb{C}^2$, the above equality holds for all $(\alpha, \beta) \in \mathbb{C}^2$ and hence the lemma holds. □

Theorem 2.1 *Let $z \in \mathbb{H}, s \in \mathbb{C}$. Define*

$$F'_k(s,\omega,N)(z) = \Gamma\left(\frac{s+k}{2}\right)\Gamma\left(\frac{s+\lambda+\lambda_0}{2}\right)L_N(2s+2\lambda,\omega')E'_k(s,\omega,N)(z),$$

where $\lambda = (k-1)/2$ and $\lambda_0 = 0$ or 1 according to $2 \mid \lambda$ or $2 \nmid \lambda$ respectively. Then $(s+\lambda-1)F'_k$ can be continued to a holomorphic function on the s-plane. If $(k+1)/2$ is even or ω' is a non-trivial character, then F'_k can be continued to a holomorphic function on the s-plane.

Proof By (2.9) and Proposition 2.1 we have

$$(-\mathrm{i})^{k/2}(2\pi)^{-s-k/2}(N/y)^{s/2}F'_k(s,\omega,N)(z) = \sum_{n=-\infty}^{\infty} A(n,y,s)\mathrm{e}^{2\pi(\mathrm{i}nx-|n|y)}|n|^{s+k/2-1}, \tag{2.26}$$

where

$$\begin{aligned} A(n,y,s) = {} & L_N(s+\lambda,\omega_k^{(n)})\beta_k(n,s,\omega,N)\Gamma((s+\lambda+\lambda_0)/2) \\ & \times \begin{cases} W(4\pi ny,(s+k)/2,s/2), & \text{if } n>0, \\ \Gamma((s+k)/2)\Gamma(s/2)^{-1}W(4\pi|n|y,s/2,(s+k)/2), & \text{if } n<0. \end{cases} \end{aligned}$$

And

$$\begin{aligned} A(0,y,s) = {} & \Gamma(s/2)^{-1}\Gamma((s+\lambda+\lambda_0)/2)\Gamma(s+k/2-1) \\ & \cdot L_N(2s+2\lambda-1,\omega')(4\pi y)^{1-s-k/2}. \end{aligned}$$

We have

$$\Gamma((s+\lambda+\lambda_0)/2)\Gamma(s/2)^{-1} = 2^{-c}\prod_{a=1}^{c}(s+\lambda+\lambda_0-2a),$$

where $c = (\lambda+\lambda_0)/2$, and

$$\begin{aligned} & \Gamma(s+k/2-1)L_N(2s+k-2,\omega') \\ = {} & \Gamma(s+k/2-1)L(2s+k-2,\omega')\prod_{p|N}\left(1-\omega'(p)p^{2-2s-k}\right). \end{aligned}$$

This shows that $A(0,y,s)$ is meromorphic on the s-plane. If ω' is non-trivial, by (2.20) we know that $\Gamma(s+k/2-1)L(2s+k-2,\omega')$ is holomorphic on the s-plane. Hence $A(0,y,s)$ is holomorphic on the s-plane. If ω' is trivial, then $\Gamma(s+k/2-1)\zeta(2s+k-2)$ has two poles $s = 1-k/2, 1-\lambda$ with order 1. The first pole $s = 1-k/2$ can be cancelled by the factor $1-2^{2-2s-k}$. For odd λ, the second pole can be cancelled by the factor $s+\lambda-1$. Hence $(s+\lambda-1)A(0,y,s)$ is holomorphic on the s-plane. And if $\lambda+1=(k+1)/2$ is even or ω' is non-trivial, $A(0,y,s)$ is holomorphic on the s-plane.

If $n>0$, then $\beta_k(n,s,\omega,N)$ is holomorphic on the s-plane and

$$| \beta_k(n,s,\omega,N) | \leqslant \gamma |n|^{\delta\sigma+\varepsilon},$$

where constants $\gamma,\delta,\varepsilon$ is independent on n. $W(4\pi ny,(s+k)/2,s/2)$ is also holomorphic on the s-plane, and for any $s\in K\subset \mathbb{C}(K$ compact), by Lemma 2.8 we have

$$|W(4\pi ny,(s+k)/2,s/2)| \leqslant C(4\pi ny)^{-\delta/2}\max\{(4\pi ny)^{-B},1\},$$

where constants B,C are determined by K and independent on n. We also have

$$\begin{aligned}&\Gamma((s+\lambda+\lambda_0)/2)L_N(s+\lambda,\omega_k^{(n)})\\ &=\Gamma((s+\lambda+\lambda_0)/2)L(s+\lambda,\omega_k^{(n)})\prod_{p|N}\left(1-\omega_k^{(n)}(p)p^{-s-\lambda}\right).\end{aligned}$$

Since $\omega_k^{(n)}(-1)=-1$ and $\lambda_0=1$ for odd λ, $\omega_k^{(n)}(-1)=1$ and $\lambda_0=0$ for even λ, by Lemma 2.7 we know that $\Gamma((s+\lambda+\lambda_0)/2)L(s+\lambda,\omega_k^{(n)})$ is holomorphic on the s-plane for non-trivial $\omega_k^{(n)}$. And hence $A(n,y,s)$ is holomorphic on the s-plane. If $\omega_k^{(n)}$ is trivial (then λ is even), then $\Gamma((s+\lambda)/2)L(s+\lambda,\omega_k^{(n)})$ has two poles $s=-\lambda, s=1-\lambda$ with order 1. The first pole can be cancelled by the factor $1-2^{-s-\lambda}$. Therefore $(s+\lambda-1)A(n,y,s)$ is holomorphic on s-plane. By Lemma 2.7, we have that

$$|(s+\lambda-1)A(n,y,s)| \leqslant un^v(y^W+y^{-W}),\quad s\in K, \tag{2.27}$$

where constants u,v,W are determined by K and independent on n.

Now consider the case $n<0$. For odd or even λ, $\omega_k^{(n)}(-1)=1$ or -1 respectively. Let $\eta=0$ or 1 according to λ odd or even. Then

$$\begin{aligned}A(n,y,s)=\Gamma\left((s+\lambda+\eta)/2\right)L\left(s+\lambda,\omega_k^{(n)}\right)\times\prod_{p|N}\left(1-\omega_k^{(n)}(p)p^{-s-\lambda}\right)\beta_k(n,s,\omega,N)\\ \times W(4\pi|n|y,s/2,(s+k)/2)\times\Gamma((s+k)/2)\\ \times\Gamma((s+\lambda+\eta)/2)^{-1}\Gamma((s+\lambda+\lambda_0)/2)\Gamma(s/2)^{-1}.\end{aligned}$$

The product of the last four factors of the above equality is

$$2^{-c-d}\prod_{b=1}^{d}(s+k-2b)\prod_{a=1}^{c}(s+\lambda+\lambda_0-2a),$$

where $d=(\lambda-\eta+1)/2$. Proceeding similarly for the case $n>0$, we can prove that $A(n,y,s)$ is holomorphic on the s-plane, and for $s\in K$ we have

$$|A(n,y,s)| \leqslant u'n^{v'}(y^{W'}+y^{-W'}), \tag{2.28}$$

where constants u', v', W' are determined by K and independent on n.

By (2.27) and (2.28) we know that the series (2.26) multiplied by $(s+\lambda-1)$ is absolutely and uniformly convergent in K, which completes the proof. □

By Theorem 2.1, $E'_k(s,\, \omega,\, N)(z)$ can be continued to a meromorphic function on the s-plane. By (2.7), $E_k(s, \omega, N)(z)$ can be continued to a meromorphic function on the s-plane. And the transformation formulae (2.3) and (2.8) hold for all s. We want to calculate their values at $s = 0$.

Now suppose that $k \geqslant 3$, and ω is not a real character if $k = 3$. Then $L_N(\lambda, (\overline{\omega}\chi_N)_k^{(n)})$ is finite for any n. Define functions as follows

$$E_k(\omega, N)(z) = E_k(0, \overline{\omega}, N)(z),$$
$$E'_k(\omega, N)(z) = E'_k(0, \omega, N)(z).$$

Since $\Gamma(0)^{-1} = 0, W(4\pi ny, k/2, 0) = 1$, by Proposition 2.1, Proposition 2.2, Lemma 2.5 and equality (2.5) we have

$$E_k(\omega, N)(z) = 1 + \frac{(-2\pi \mathrm{i})^{k/2}}{\Gamma(k/2)} \sum_{n=1}^{\infty} \frac{L_N\left(\lambda, (\overline{\omega}\chi_N)_k^{(n)}\right)}{L_N\left(2\lambda, \overline{\omega}'\right)} \times \beta_k(n, 0, \overline{\omega}\chi_N, N) c_k(n, 0, \overline{\omega}, N) n^{k/2-1} e(nz). \tag{2.29}$$

Similarly, by (2.9) we can get

$$E'_k(\omega, N)(z) = \frac{(-2\pi \mathrm{i})^{k/2}}{\Gamma(k/2)} \sum_{n=1}^{\infty} \frac{L_N\left(\lambda, \omega_k^{(n)}\right)}{L_N\left(2\lambda, \omega'\right)} \times \beta_k(n, 0, \omega, N) n^{k/2-1} e(nz). \tag{2.30}$$

Denote $n = tm^2$ with t a square free positive integer. By Lemma 2.7 we know

$$\left| L_N\left(\lambda, (\overline{\omega}\chi_N)_k^{(n)}\right) \right| \leqslant \rho t^2,$$

where ρ is a constant independent on n. For $n \neq 0$, by Proposition 2.2 we know that $c_k(n, 0, \omega, N)$ is a series with finite terms. Hence

$$|E_k(\omega, N)(z)| \leqslant 1 + \rho \sum_{n=1}^{\infty} n^{k/2+1} \mathrm{e}^{-2\pi ny} \leqslant 1 + \rho y^{-(k+5)/2}, \tag{2.31}$$

where ρ may be a different constant. Therefore we know that $E_k(\omega, N)(z)$ is a holomorphic function on $\mathbb{H}$. Similarly we can prove that $E'_k(\omega, N)(z)$ is also a holomorphic function on $\mathbb{H}$.

By (2.3) and (2.8), we have

$$E_k(\omega,N)(\gamma(z))=\omega(d_\gamma)j(\gamma,z)^k E_k(\omega,N)(z),$$
$$E'_k(\omega,N)(\gamma(z))=\omega(d_\gamma)\left(\frac{N}{d_\gamma}\right)j(\gamma,z)^k E'_k(\omega,N)(z), \tag{2.32}$$

where $\gamma\in\Gamma_0(N)$.

Proposition 2.3 *Let ν be a positive integer, p an odd prime. Put*

$$a_k(2^\nu,n)=\sum_{d=1}^{2^\nu}\left(\frac{2^\nu}{d}\right)\varepsilon_d^k e(nd/2^\nu),\quad \nu\geqslant 2,$$

$$a_k(p^\nu,n)=\varepsilon_{p^\nu}^{-k}\sum_{d=1}^{p^\nu}\left(\frac{d}{p^\nu}\right)e(nd/p^\nu).$$

Then

$$c_k(n,s,\mathrm{id.},N)=\prod_{p|N}\sum_{\nu=N(p)}^{\infty}p^{-(s+k/2)\nu}a_k(p^\nu,n),$$

where id. *is the identity character, $N(p)$ satisfies that $p^{N(p)}\parallel N$.*

Proof By (2.16), we have

$$c_k(n,s,\mathrm{id.},N)=\sum_{N|M|N^\infty}\sum_{d=1}^{M}\left(\frac{M}{d}\right)\varepsilon_d^k e(nd/M)M^{-s-k/2}.$$

Put $M=2^eM_1, e\geqslant 2$, with M_1 odd. Then

$$\begin{aligned}
&\sum_{d=1}^{M}\left(\frac{M}{d}\right)\varepsilon_d^k e(nd/M)\\
&=\sum_{d_1=1}^{M_1}\sum_{d_2=1}^{2^e}\left(\frac{2^eM_1}{2^ed_1+M_1d_2}\right)\varepsilon_{M_1d_2}^k e(n(2^ed_1+M_1d_2)/(2^eM_1))\\
&=\sum_{d_2=1}^{2^e}\left(\frac{2^e}{M_1d_2}\right)\varepsilon_{M_1d_2}^k e(nd_2/2^e)\sum_{d_1=1}^{M_1}\left(\frac{2^ed_1}{M_1}\right)\varepsilon_{d_2}^k\varepsilon_{M_1}^{-k}\varepsilon_{M_1d_2}^{-k}e(nd_1/M_1)\\
&=a_k(2^e,n)\varepsilon_{M_1}^{-k}\sum_{d_1=1}^{M_1}\left(\frac{d_1}{M_1}\right)e(nd_1/M_1),
\end{aligned}$$

where we used (2.17). Furthermore put $M_1=M_2M_3$ with M_2 and M_3 co-prime. Then

$$\begin{aligned}
&\varepsilon_{M_1}^{-k}\sum_{d_1=1}^{M_1}\left(\frac{d_1}{M_1}\right)e(nd_1/M_1)\\
&=\varepsilon_{M_1}^{-k}\sum_{d_2=1}^{M_2}\sum_{d_3=1}^{M_3}\left(\frac{d_2M_3}{M_2}\right)\left(\frac{d_3M_2}{M_3}\right)e\left(n(d_2M_3+d_3M_2)/(M_2M_3)\right)\\
&=\varepsilon_{M_2}^{-k}\sum_{d_2=1}^{M_2}\left(\frac{d_2}{M_2}\right)e(nd_2/M_2)\varepsilon_{M_3}^{-k}\sum_{d_3=1}^{M_3}\left(\frac{d_3}{M_3}\right)e(nd_3)/M_3),
\end{aligned}$$

from which we can prove the proposition. □

Lemma 2.9 *Let $\nu \geqslant 2$ be a positive integer. Then*

$$a_k(2^\nu,n)=\begin{cases}2^{\nu-3/2}\mathrm{e}^{\pi\mathrm{i}(l/2+(-1)^\lambda/4)}, & \text{if } n=2^{\nu-2}l, 2|l, 2|\nu,\\ 2^{\nu-3/2}\mathrm{e}^{\pi\mathrm{i}(l/2-(-1)^\lambda/4)}, & \text{if } n=2^{\nu-2}l, 2\nmid l, 2|\nu,\\ 2^{\nu-1}\delta((u-(-1)^\lambda)/4)e^{\pi\mathrm{i}u/4}, & \text{if } n=2^{\nu-3}u, 2\nmid u, 2\nmid\nu,\\ 0, & \text{otherwise,}\end{cases}$$

where $\delta(x)=1$ or 0 according to x an integer or not respectively.

Proof If ν is even, then

$$a_k(2^\nu,n)=(e(n/2^\nu)+\mathrm{i}^ke(3n/2^\nu))\sum_{d=0}^{2^{\nu-2}-1}e(nd/2^{\nu-2}).$$

If $2^{\nu-2}\nmid n$, the above summation is zero. If $2^{\nu-2}|n$, the above summation is $2^{\nu-2}$. Suppose that $n=2^{\nu-2}l$, then

$$e(n/2^\nu)+\mathrm{i}^ke(3n/2^\nu)=e(l/4)(1+\mathrm{i}^k\mathrm{e}^{\pi\mathrm{i}l})=\begin{cases}\sqrt{2}\mathrm{e}^{\pi\mathrm{i}(l/2+(-1)^\lambda/4)}, & \text{if } 2|l,\\ \sqrt{2}\mathrm{e}^{\pi\mathrm{i}(l/2-(-1)^\lambda/4)}, & \text{if } 2\nmid l.\end{cases}$$

If ν is odd, then $\nu\geqslant 3$ and

$$a_k(2^\nu,n)=\left(e(n/2^\nu)-\mathrm{i}^ke(3n/2^\nu)-e(5n/2^\nu)+\mathrm{i}^ke(7n/2^\nu)\right)\sum_{d=0}^{2^{\nu-3}-1}e(nd/2^{\nu-3}).$$

If $2^{\nu-3}\nmid n$, the above summation is zero. If $2^{\nu-3}\mid n$, the above summation is $2^{\nu-3}$. Suppose that $n=2^{\nu-3}u$. If $2\mid u$, then

$$e(u/8)=e(5u/8),\quad e(3u/8)=e(7u/8),$$

which implies that the first factor of $a_k(2^\nu,n)$ is zero. If $2\nmid u$, the factor is

$$2(e(u/8)-\mathrm{i}^ke(3u/8))=2e(u/8)(1-\mathrm{i}^{k+u})=\begin{cases}0, & \text{if } k+u\equiv 0(4),\\ 4e(u/8), & \text{if } k+u\equiv 2(4).\end{cases}$$

But $\delta((u+k+2)/4)=\delta((u-(-1)^\lambda)/4)$ which implies the lemma. □

Lemma 2.10 *Let ν be a positive integer, p an odd prime. Then*

$$a_k(p^\nu, n) = \begin{cases} p^{\nu-1/2}\left(\dfrac{(-1)^\lambda np^{1-\nu}}{p}\right), & \text{if } p^{\nu-1} \mid n, p^\nu \nmid n, 2 \nmid \nu, \\ -p^{\nu-1}, & \text{if } p^{\nu-1} \mid n, p^\nu \nmid n, 2 \mid \nu, \\ \varphi(p^\nu), & \text{otherwise }. \end{cases}$$

Proof We know that

$$\begin{aligned} a_k(p^\nu, n) &= \varepsilon_{p^\nu}^{-k} \sum_{a=1}^{p-1} \sum_{b=1}^{p^{\nu-1}} \left(\frac{a+pb}{p^\nu}\right) e((na+nbp)/p^\nu) \\ &= \varepsilon_{p^\nu}^{-k} \sum_{a=1}^{p-1} \left(\frac{a}{p^\nu}\right) e(na/p^\nu) \sum_{b=1}^{p^{\nu-1}} e(nb/p^{\nu-1}). \end{aligned}$$

If $p^{\nu-1} \nmid n$, the inner summation of the above formula is zero. If $p^{\nu-1} \mid n$, it is $p^{\nu-1}$. Now suppose that $p^{\nu-1} \mid n$. If ν is odd, then

$$a_k(p^\nu, n) = \varepsilon_p^{-k} p^{\nu-1} \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) e(anp^{1-\nu}/p) = \begin{cases} 0, & \text{if } p^\nu \mid n, \\ p^{\nu-1/2}\varepsilon_p^{1-k}\left(\dfrac{np^{1-\nu}}{p}\right), & \text{if } p^\nu \nmid n. \end{cases}$$

But $\varepsilon_p^{1-k} = \left(\dfrac{(-1)^\lambda}{p}\right)$ which shows the case for odd ν. If ν is even, then

$$a_k(p^\nu, n) = p^{\nu-1} \sum_{a=1}^{p-1} e(anp^{1-\nu}/p) = \begin{cases} -p^{\nu-1}, & \text{if } p^\nu \nmid n, \\ p^{\nu-1}(p-1), & \text{if } p^\nu \mid n, \end{cases}$$

which completes the proof. □

Now define

$$\begin{aligned} A_k(2, n) &= \sum_{\nu=2}^{\infty} 2^{-\nu k/2} a_k(2^\nu, n), \\ A'_k(2, n) &= \sum_{\nu=3}^{\infty} 2^{-\nu k/2} a_k(2^\nu, n), \\ A_k(p, n) &= \sum_{\nu=1}^{\infty} p^{-\nu k/2} a_k(p^\nu, n). \end{aligned}$$

Let D be a square free positive odd integer. By Proposition 2.3, we have

$$\begin{aligned} c_k(n, 0, \text{id.}, 4D) &= A_k(2, n) \prod_{p \mid D} A_k(p, n), \\ c_k(n, 0, \text{id.}, 8D) &= A'_k(2, n) \prod_{p \mid D} A_k(p, n). \end{aligned}$$

Put

$$\lambda_k(n,4D) = \frac{L_{4D}(\lambda, \chi_{(-1)^\lambda n})}{L_{4D}(2\lambda, \mathrm{id.})}\beta_k(n,0,\chi_D,4D).$$

If $k>3$, by (2.29) and (2.30) we have

$$E_k(\mathrm{id.},4D)(z) = 1 + \frac{(-2\pi \mathrm{i})^{k/2}}{\Gamma(k/2)}\sum_{n=1}^{\infty}\lambda_k(n,4D)\prod_{p|2D}A_k(p,n)n^{k/2-1}e(nz), \tag{2.33}$$

$$E'_k(\chi_D,4D)(z) = \frac{(-2\pi \mathrm{i})^{k/2}}{\Gamma(k/2)}\sum_{n=1}^{\infty}\lambda_k(n,4D)n^{k/2-1}e(nz). \tag{2.34}$$

Lemma 2.11 *Let $\nu_2(n)$ be the integer such that $2^{\nu_2(n)} \parallel n$, then*

$$A_k(2,n) = \begin{cases} 2^{-k}(1+(-1)^\lambda \mathrm{i})\Big\{\dfrac{1-2^{(2-k)(\nu_2(n)-1)/2}}{1-2^{2-k}} - 2^{(2-k)(\nu_2(n)-1)/2}\Big\}, \\ \qquad\qquad\qquad\qquad \text{if } 2\nmid \nu_2(n), \\ 2^{-k}(1+(-1)^\lambda \mathrm{i})\Big\{\dfrac{1-2^{(2-k)\nu_2(n)/2}}{1-2^{2-k}} - 2^{(2-k)\nu_2(n)/2}\Big\}, \\ \qquad\qquad\qquad\qquad \text{if } 2\mid \nu_2(n), (-1)^\lambda n/2^{\nu_2(n)} \equiv -1(4), \\ 2^{-k}(1+(-1)^\lambda \mathrm{i})\Big\{\dfrac{1-2^{(2-k)(\nu_2(n)-1)/2}}{1-2^{2-k}} + 2^{(2-k)\nu_2(n)/2} \\ \quad \Big(1+2^{(3-k)/2}\Big(\dfrac{(-1)^\lambda n/2^{\nu_2(n)}}{2}\Big)\Big)\Big\}, \\ \qquad\qquad\qquad\qquad \text{if } 2\mid \nu_2(n), (-1)^\lambda n/2^{\nu_2(n)} \equiv 1(4) \end{cases}$$

and

$$A'_k(2,n) = \begin{cases} 0, & \text{if } (-1)^\lambda n \equiv 2,3(4), \\ A_k(2,n) - 2^{-k}(1+(-1)^\lambda \mathrm{i}), & \text{if } (-1)^\lambda n \equiv 0,1(4). \end{cases}$$

Proof In order to simplify the notation, we denote $\nu_2(n)$ by h. Suppose $2\nmid h$, by Lemma2.9, we have

$$\begin{aligned} A_k(2,n) &= \sum_{s=1}^{(h+1)/2} 2^{(2-k)s-3/2}\mathrm{e}^{\pi \mathrm{i}(n/2^{2s-1}+(-1)^\lambda/4)} \\ &= 4^{-1}(1+(-1)^\lambda \mathrm{i})\Big\{\sum_{s=1}^{(h-1)/2} 2^{(2-k)s} - 2^{(2-k)(h+1)/2}\Big\} \\ &= 2^{-k}(1+(-1)^\lambda \mathrm{i})\Big\{\frac{1-2^{(2-k)(h-1)/2}}{1-2^{2-k}} - 2^{(2-k)(h-1)/2}\Big\}. \end{aligned}$$

Suppose that $2 \mid h$, $n = 2^h u$. Then

$$
\begin{aligned}
A_k(2,n) = &\sum_{s=1}^{h/2} 2^{(2-k)/2-3/2} \mathrm{e}^{\pi \mathrm{i}(n/2^{2s-1}+(-1)^\lambda/4)} \\
&+ 2^{-k+(2-k)h/2+1/2} \mathrm{e}^{\pi \mathrm{i}(u/2-(-1)^\lambda/4)} \\
&+ 2^{(2-k)h/2-3k/2+1/2} \delta\left((u-(-1)^\lambda)/4\right) \mathrm{e}^{\pi \mathrm{i} u/4} \\
= &\, 2^{-k}(1+(-1)^\lambda \mathrm{i}) \frac{1-2^{(2-k)h/2}}{1-2^{2-k}} \\
&+ 2^{-k+(2-k)h/2+1/2} \mathrm{e}^{\pi \mathrm{i}(u/2-(-1)^\lambda/4)} \\
&+ 2^{(2-k)h/2-3k/2+1/2} \delta\left((u-(-1)^\lambda)/4\right) \mathrm{e}^{\pi \mathrm{i} u/4}.
\end{aligned}
$$

Now we can prove the results for $A_k(2,n)$ in the lemma by a direct computation. By Lemma 2.9, if $(-1)^\lambda n \equiv 2, 3(4)$, then $a_k(2^\nu, n) = 0$ for any $\nu \geqslant 3$. If $(-1)^\lambda n \equiv 0, 1(4)$, then $a_k(2^2, n) = 1 + (-1)^\lambda \mathrm{i}$. This implies the results for $A'_k(2,n)$ which completes the proof. □

By (2.29) and Lemma 2.11, we have

$$
\begin{aligned}
E_k(\mathrm{id.}, 8D)(z) = 1 + \frac{(-2\pi \mathrm{i})^{k/2}}{\Gamma(k/2)} &\sum_{n \geqslant 1, (-1)^\lambda n \equiv 0,1(4)} \lambda_k(n, 4D)(A_k(2,n) \\
&-2^{-k}(1+(-1)^\lambda \mathrm{i})) \prod_{p \mid 2D} A_k(p,n) n^{k/2-1} e(nz). \qquad (2.35)
\end{aligned}
$$

Lemma 2.12 *Let p be an odd prime, $p^{\nu_p(n)} \parallel n$. Then*

$$
A_k(p,n) = \begin{cases} \dfrac{(p-1)(1-p^{(2-k)(\nu_p(n)-1)/2})}{p(p^{k-2}-1)} - p^{(2-k)(\nu_p(n)+1)/2-1}, & \\ & \text{if } 2 \nmid \nu_p(n), \\ \dfrac{(p-1)(1-p^{(2-k)\nu_p(n)/2})}{p(p^{k-2}-1)} - \left(\dfrac{(-1)^\lambda n / p^{\nu_p(n)}}{p}\right) p^{(2-k)(\nu_p(n)+1)/2-1/2}, & \\ & \text{if } 2 \mid \nu_p(n). \end{cases}
$$

Proof Denote $\nu_p(n)$ by h. If $2 \nmid h$, by Lemma 2.10 we have

$$
\begin{aligned}
A_k(p,n) &= \sum_{s=1}^{(h-1)/2} p^{-ks} \varphi(p^{2s}) - p^{-k(h+1)/2+h} \\
&= \frac{(p-1)(1-p^{(2-k)(h-1)/2})}{p(p^{k-2}-1)} - p^{(2-k)(h+1)/2-1}.
\end{aligned}
$$

If $2 \mid h$, then

$$A_k(p,n) = \sum_{s=1}^{h/2} p^{-ks}\varphi(p^{2s}) + \left(\frac{(-1)^\lambda n/p^h}{p}\right) p^{-k(h+1)/2+h+1/2}$$
$$= \frac{(p-1)(1-p^{(2-k)(h-1)/2})}{p(p^{k-2}-1)} - \left(\frac{(-1)^\lambda n/p^h}{p}\right) p^{(2-k)(h+1)/2-1/2},$$

which completes the proof. □

Now we consider the values of $E_3(s, \mathrm{id.}, 4D)(z)$ and $E_3'(s, \mathrm{id.}, 4D)(z)$ at $s = 0$, where D is a square free positive odd integer. Suppose that $n = tm^2$ with t square free. Then it is easy to see that $(\chi_{4D})_3^{(n)} = \left(\frac{-t}{\cdot}\right)$. If n is negative and $-n$ is not a perfect square, then the term $e(nx)$ disappears in the expansions of $E_3(0, \mathrm{id.}, 4D)$ and $E_3'(0, \mathrm{id.}, 4D)$ since $t_n(y, 3/2, 0) = 0$ and $L_{4D}\left(1, \left(\frac{-n}{\cdot}\right)\right)$ is a finite number. If $n = -m^2$ is a negative perfect square, then $(\chi_{4D})_3^{(n)}$ is the trivial character. Since

$$\zeta(1+s)\Gamma^{-1}(s/2) = (s/2)\zeta(1+s)\Gamma^{-1}(1+s/2) \to 2^{-1}, \quad s \to 0,$$

the term $e(-m^2x)$ appears in the expansions of $E_3(0, \mathrm{id.}, 4D)$ and $E_3'(0, \mathrm{id.}, 4D)$.

By Proposition 2.3, Lemma 2.11 and Lemma 2.12 we have

$$c_3(-m^2, 0, \mathrm{id.}, 4D) = A_3(2, -m^2)\prod_{p\mid D} A_3(p, -m^2) = (4D)^{-1}(1-\mathrm{i}).$$

By (2.5), (2.9) and Proposition 2.2 we have

$$\begin{aligned} &E_3(0, \mathrm{id.}, 4D)(z) - (4D)^{-1}(1-\mathrm{i})E_3'(0, \mathrm{id.}, 4D)(z) \\ =&1 - 4\pi(1+\mathrm{i})\sum_{n=1}^{\infty} \lambda_3(n, 4D)\left(\prod_{p\mid D} A_3(p,n) - (4D)^{-1}(1-\mathrm{i})\right) n^{1/2} e(nz), \end{aligned} \tag{2.36}$$

which will be denoted by $f_1(\mathrm{id.}, 4D)(z)$.

For $l \mid D$, we get similarly that

$$\begin{aligned} &E_3(0, \chi_l, 4D)(z) - (4D)^{-1}(1-\mathrm{i})l^{1/2}E_3'(0, \chi_{D/l}, 4D)(z) \\ =&1 - 4\pi(1+\mathrm{i})l^{1/2}\sum_{n=1}^{\infty} \lambda_3(ln, 4D)\left(\prod_{p\mid D} A_3(p, ln) - (4D)^{-1}(1-\mathrm{i})\right) n^{1/2} e(nz), \end{aligned} \tag{2.37}$$

which will be denoted by $f_1(\chi_l, 4D)$. Likewise we have

$$c_3(-m^2, 0, \mathrm{id.}, 8D) = A_3'(2, -m^2)\prod_{p\mid D} A_3(p, -m^2) = (8D)^{-1}(1-\mathrm{i}).$$

For $l \mid 2D$, we have that

$$\begin{aligned} & E_3(0,\chi_l,8D)(z) - (8D)^{-1}(1-\mathrm{i})l^{1/2}E_3'(0,\chi_{2D/l},8D)(z) \\ =& 1 - 4\pi(1+\mathrm{i})l^{1/2}\sum_{n=1}^{\infty}\lambda_3(ln,8D) \\ & \times\left(A_3'(2,ln)\prod_{p\mid D}A_3(p,ln) - (8D)^{-1}(1-\mathrm{i})\right)n^{1/2}e(nz), \end{aligned} \tag{2.38}$$

which will be denoted by $f_1(\chi_l, 8D)$.

We consider furthermore the values of $E_3(s,\omega,N)$ and $E_3'(s,\omega,N)$ at $s=-1$. Put

$$f_2(\omega,N)(z) = -\frac{E_3(s,\overline{\omega},N)L_N(2s+2,\overline{\omega}')}{2\pi(1+i)L_N(2s+1,\overline{\omega}')}\bigg|_{s=-1}, \tag{2.39}$$

$$f_2^*(\omega,N)(z) = -\frac{E_3'(s,\omega\chi_N,N)L_N(2s+2,\omega')}{2\pi(1+i)N^{1/2}L_N(2s+1,\overline{\omega}')}\bigg|_{s=-1}. \tag{2.40}$$

If ω is a non-trivial even character, then $L(0,\omega)=0$ (see Lemma 2.7). Hence

$$L_N(1+s,\omega)\mid_{s=-1} = L(1+s,\omega)\prod_{p\mid N}\left(1-\frac{\omega(p)}{p^{1+s}}\right)\bigg|_{s=-1} = 0.$$

If ω is trivial, then the product in the above equality is zero. Therefore terms $e(nz)$ with $n<0$ disappear in the expansion of $f_2(\omega,N)$. If $n>0$, by Lemmas 2.3, 2.4 and 2.5 we have

$$\begin{aligned} t_n(y,1,-1/2) &= (2\pi)^{1/2}\mathrm{i}^{-3/2}\mathrm{e}^{-2\pi ny}n^{-1/2}W(4\pi ny,1,-1/2) \\ &= -2\pi(1+\mathrm{i})y^{1/2}\mathrm{e}^{-2\pi ny} \end{aligned}$$

and

$$t_0(y,1,-1/2) = -2\pi(1+\mathrm{i})y^{1/2}.$$

So we have

$$\begin{aligned} f_2(\omega,N)(z) = & c_3(0,-1,\overline{\omega},N) + \sum_{n=1}^{\infty}\frac{L_N(0,(\overline{\omega}\chi_N)_3^{(n)})}{L_N(-1,\overline{\omega}')} \\ & \times\beta_3(n,-1,\overline{\omega}\chi_N,N)c_3(n,-1,\overline{\omega},N)e(nz), \end{aligned} \tag{2.41}$$

where $c_3(0,-1,\overline{\omega},N)$ is the value at $s=-1$ of the analytic continuation of the series (2.16) with respect to s.

Similarly we get

$$f_2^*(\omega,N)(z) = 1 + \sum_{n=1}^{\infty}\frac{L_N(0,(\omega\chi_N)_3^{(n)})}{L_N(-1,\omega')}\times\beta_3(n,-1,\omega\chi_N,N)e(nz). \tag{2.42}$$

For the sake of our applications, we rewrite the $A_3(p,n)$ in Lemma 2.11, 2.12 as follows

$$A_3(2,n)=\begin{cases}4^{-1}(1-\mathrm{i})(1-3\cdot 2^{-(1+\nu_2(n))/2}), & \text{if } 2\nmid \nu_2(n),\\ 4^{-1}(1-\mathrm{i})(1-3\cdot 2^{-(1+\nu_2(n)/2)}), & \text{if } 2\mid \nu_2(n), n/2^{\nu_2(n)}\equiv 1(4),\\ 4^{-1}(1-\mathrm{i})(1-2^{-\nu_2(n)/2}), & \text{if } 2\mid \nu_2(n), n/2^{\nu_2(n)}\equiv 3(4),\\ 4^{-1}(1-\mathrm{i}), & \text{if } 2\mid \nu_2(n), n/2^{\nu_2(n)}\equiv 7(8).\end{cases} \tag{2.43}$$

If $p\neq 2$, then

$$A_3(p,n)=\begin{cases}p^{-1}-(1+p)p^{-(3+\nu_p(n))/2}, & \text{if } 2\nmid \nu_p(n),\\ p^{-1}-2p^{-1-\nu_p(n)/2}, & \text{if } 2\mid \nu_p(n), \left(\dfrac{-n/p^{\nu_p(n)}}{p}\right)=-1,\\ p^{-1}, & \text{if } 2\mid \nu_p(n), \left(\dfrac{-n/p^{\nu_p(n)}}{p}\right)=1.\end{cases} \tag{2.44}$$

Finally we have

Lemma 2.13 *Let m be a positive factor of D, then*

$$f_2^*(\mathrm{id.},4m)(z)=1-4\pi(1+\mathrm{i})\sum_{n=1}^{\infty}\lambda_3(n,4D)(A_3(2,n)-4^{-1}(1-\mathrm{i}))$$
$$\times\prod_{p\mid m}\left(A_3(p,n)-p^{-1}\right)\prod_{p\mid D/m}(1+A_3(p,n))\,n^{1/2}e(nz) \tag{2.45}$$

and

$$-2^{-1}(1+\mathrm{i})\mu(m)f_2(\mathrm{id.},8m)(z)$$
$$=1-4\pi(1+\mathrm{i})\sum_{n\equiv 0,3(4)}^{\infty}\lambda_3(n,4D)\left(A_3(2,n)-4^{-1}(1-\mathrm{i})\right)$$
$$\times\prod_{p\mid m}\left(A_3(p,n)-p^{-1}\right)\prod_{p\mid D/m}(1+A_3(p,n))\,n^{1/2}e(nz). \tag{2.46}$$

We omit the proof.

2.2 Eisenstein Series with Integral Weight

In this section we always assume that N and k are positive integers. Let ω be a character modulo N with $\omega(-1)=(-1)^k$. Put

$$\Gamma_\infty=\left\{\pm\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix}\Big|m\in\mathbb{Z}\right\}$$

and

$$W=\left\{\begin{pmatrix}* & *\\ mN & n\end{pmatrix}\in SL_2(\mathbb{Z})\Big|m\geqslant 0 \text{ and } n=1 \text{ if } m=0\right\}.$$

It is easy to check that W is a complete set of representatives of $\Gamma_\infty \setminus \Gamma_0(N)$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, set $J(\gamma, z) = cz + d$. Define

$$E_k(z, s, \omega, N) = y^s \sum_{\gamma \in \Gamma_\infty \setminus \Gamma_0(N)} \overline{\omega}(d_\gamma) J(\gamma, z)^{-k} |J(\gamma, z)|^{-2s}$$
$$= \frac{y^s}{2} \sum_{(m,n)=1} \overline{\omega}(n)(mNz + n)^{-k} |mNz + n|^{-2s},$$

where s is a complex variable, m, n run over the set of all co-prime pairs of integers (pairs of positive and negative integers). It is clear that for $\mathrm{Re}(s) > 2 - k$ the above series is absolutely convergent, so a holomorphic function with respect to the variable s. It is easy to verify that for any $\gamma \in \Gamma_0(N)$

$$E_k(\gamma(z), s, \omega, N) = \omega(d_\gamma) J(\gamma, z)^k E_k(z, s, \omega, N). \tag{2.47}$$

Using Lemma 2.5 we have

$$\begin{aligned} E_k(z, s, \omega, N) &= \frac{y^s}{2} L_N^{-1}(k + 2s, \overline{\omega}) {\sum_{m,n}}' \overline{\omega}(n)(mNz + n)^{-k} |mNz + n|^{-2s} \\ &= y^s + y^s N^{-k-2s} L_N^{-1}(k + 2s, \overline{\omega}) \sum_{m=1}^{\infty} \sum_{a=1}^{N} \overline{\omega}(a) \\ &\quad \times \sum_{j=-\infty}^{\infty} (mz + aN^{-1} + j)^{-k-s} (m\overline{z} + aN^{-1} + j)^{-s} \\ &= y^s + \mathrm{i}^{-k} (2\pi N^{-1})^{k+2s} y^s L_N^{-1}(k + 2s, \overline{\omega}) \\ &\quad \times \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} t_n(my, k + s, s) \sum_{a=1}^{N} \overline{\omega}(a) e(nmx + anN^{-1}), \end{aligned} \tag{2.48}$$

where $\sum'$ means summation over all $(m, n) \neq (0, 0)$. Similar to the case of half integral weight, $E_k(z, s, \omega, N)$ can be continued to a meromorphic function on the whole s-plane and (2.47) holds also for the continued function. For $k \neq 2$ or $k = 2, \omega \neq \mathrm{id}.$ we define

$$E_k(z, \omega, N) = E_k(z, 0, \omega, N).$$

Since $\Gamma(s)^{-1} \to 0 (s \to 0)$ and $W(y, \alpha, 0) = 1$, the terms corresponding to $n < 0$ of the expansion (2.48) of $E_k(z, 0, \omega, N)$ will disappear. Therefore

$$E_k(z, \omega, N) = 1 + \frac{(-2\pi \mathrm{i})^k}{N^k (k-1)! L_N(k, \omega)} \sum_{n=1}^{\infty} \left\{ \sum_{d|n} d^{k-1} \sum_{a=1}^{N} \overline{\omega}(a) e(ad/N) \right\} e(nz). \tag{2.49}$$

For $k = 2$ and $\omega = \text{id.}$, we see that

$$E_2(z, 0, \text{id.}, N) = 1 - \frac{\pi\varphi(N)}{2yN^2 L_N(2, \text{id.})} - \frac{4\pi^2}{N^2 L_N(2, \text{id.})} \times \sum_{n=1}^{\infty} \left\{ \sum_{d|n} \sum_{(a,N)=1, a=1}^{N} e(ad/N) \right\} e(nz). \tag{2.50}$$

Let f be a function on the upper half plane $\mathbb{H}$, $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$ and k a positive integer. Define

$$f|[\sigma]_k = \det(\sigma)^{k/2} J(\sigma, z)^{-k} f(\sigma(z)),$$

where $J(\sigma, z) = cz + d, \sigma(z) = \dfrac{az + b}{cz + d}$.

Let Q be a positive integer with $Q|N$ and $(Q, N/Q) = 1$, put

$$W(Q) = \begin{pmatrix} Qj & l \\ -N & Q \end{pmatrix} \in GL_2^+(\mathbb{Z})$$

with $jQ + lN/Q = 1$. It is clear that $W(Q)\Gamma_0(N)W(Q)^{-1} = \Gamma_0(N)$. We want to compute the Fourier expansion of $E_k(z, \omega, N)|[W(Q)]_k$. We first have that

$$\begin{aligned}
&L_N(k + 2s, \overline{\omega}) E_k(z, s, \omega, N)|[W(Q)] \\
&= \frac{Q^{-k/2-2s} y^s}{2} \sum_{m,n}{}' \overline{\omega}(n)((mjQ - n)NQ^{-1}z + lmNQ^{-1} + n)^{-k} \\
&\quad \times |(mjQ - n)NQ^{-1}z + lmNQ^{-1} + n|^{-2s} \\
&= \frac{Q^{-k/2-2s} y^s}{2} \sum_{m,n}{}' \overline{\omega}_1(-m)\overline{\omega}_2(n)(mNQ^{-1}z + n)^{-k}|mNQ^{-1}z + n|^{-2s} \\
&= N^{-k-2s} Q^{k/2} y^s \sum_{m=1}^{\infty} \overline{\omega}_1(-m) \sum_{a=1}^{N/Q} \overline{\omega}_2(a) \\
&\quad \times \sum_{u=-\infty}^{\infty} (mz + aQN^{-1} + u)^{-k-s}(m\overline{z} + aQN^{-1} + u)^{-s} \\
&= \mathrm{i}^k (2\pi N^{-1})^{k+2s} Q^{k/2} y^s \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} \overline{\omega}_1(-m) \\
&\quad \times t_n(my, k + s, s) \sum_{a=1}^{N/Q} \overline{\omega}_2(a) e(anQN^{-1} + nmx),
\end{aligned} \tag{2.51}$$

where ω_1, ω_2 are characters modulo Q and N/Q respectively with $\omega = \omega_1\omega_2$.

Therefore we get for $k \geqslant 3$ or $k = 2, \omega \neq \text{id.}$ that

$$\begin{aligned} E_k(z,\omega,N)|[W(Q)] &= E_k(z,0,\omega,N)|[W(Q)] \\ &= \frac{(-2\pi \mathrm{i})^k Q^{k/2}}{N^k(k-1)!L_N(k,\overline{\omega})} \sum_{n=1}^{\infty} \Big\{ \sum_{d|n} \overline{\omega}_1(-n/d) d^{k-1} \\ &\quad \times \sum_{a=1}^{N/Q} \overline{\omega}_2(a) e(adQ/N) \Big\} e(nz). \end{aligned} \tag{2.52}$$

For $k = 1, \omega_2 \neq \text{id.}$, we have that

$$\begin{aligned} & E_1(z,\omega,N)|[W(Q)] \\ &= E(z,0,\omega,N)|[W(Q)] \\ &= \frac{-2\pi \mathrm{i}\sqrt{Q}}{NL_N(1,\overline{\omega})} \sum_{n=1}^{\infty} \Big\{ \sum_{d|n} \overline{\omega}_1(-n/d) \sum_{a=1}^{N/Q} \overline{\omega}_2(a) e(adQ/N) \Big\} e(nz). \end{aligned} \tag{2.53}$$

If $k = 1, \omega_2 = \text{id.}$, then the term corresponding to $n = 0$ appears, we have that

$$\begin{aligned} E_1(z,\omega,N)|[W(Q)] &= E_1(z,0,\omega,N)|[W(Q)] \\ &= \frac{\pi \mathrm{i} L_Q(0,\overline{\omega})}{\sqrt{Q}L_N(1,\overline{\omega})} \prod_{p|N/Q} \left(1 - \frac{1}{p}\right) - \frac{2\pi \mathrm{i}\sqrt{Q}}{NL_N(1,\overline{\omega})} \\ &\quad \times \sum_{n=1}^{\infty} \Big\{ \sum_{d|n} \overline{\omega}(-n/d) \sum_{(a,N/Q)=1,a=1}^{N/Q} e(adQ/N) \Big\} e(nz). \end{aligned} \tag{2.54}$$

Finally, we see that for $k = 2, \omega = \text{id.}$

$$\begin{aligned} & E_2(z,0,\omega,N)|[W(Q)] \\ &= -\frac{\pi\varphi(N)}{2yN^2L_N(2,\text{id.})} - \frac{4\pi^2 Q}{N^2 L_N(2,\text{id.})} \\ &\quad \times \sum_{n=1}^{\infty} \Big\{ \sum_{d|n,(n/d,Q)=1} \sum_{(a,N/Q)=1,a=1}^{N/Q} e(adQ/N) \Big\} e(nz). \end{aligned} \tag{2.55}$$

Assume that ω is a primitive character modulo N and Q meets the conditions given before. Put

$$b_k(n) = \sum_{d|n} \omega_1(-n/d) d^{k-1} \sum_{a=1}^{N/Q} \overline{\omega}_2(a) e(adQ/N),$$

where ω_1, ω_2 are characters modulo Q and N/Q respectively satisfying $\omega_1\omega_2 = \omega$. It is clear that ω_2 is a primitive character modulo N/Q. Since the inner summation of the above formula is zero for any $(d, N/Q) > 1$, we see that

$$b_k(n) = \sum_{a=1}^{N/Q} \overline{\omega}_2(a) e(aQ/N) \sum_{d|n} \overline{\omega}_1(-n/d)\omega_2(d)d^{k-1}.$$

Let p be a prime. If $p \nmid N$ and $p \nmid n$, then it is clear that

$$b_k(pn) = (\omega_1(p) + \omega_2(p)p^{k-1})b_k(n),$$

if $p \nmid N$ and $p|n$, write $n = p^l n_1$, with $p \nmid n_1$. Then we have that

$$\begin{aligned} b_k(pn) &= \omega_1(p)b_k(n) + \omega_2(p)p^{k-1} \sum_{a=1}^{N/Q} \overline{\omega}_2(a)e(aQ/N) \sum_{d|n_1} \omega_1(-n_1/d)\omega_2(p^l d)(p^l d)^{k-1} \\ &= \omega_1(p)b_k(n) + \omega_2(p)p^{k-1}(b_k(n) - \omega_1(p)b_k(n/p)) \\ &= (\omega_1(p) + \omega_2(p)p^{k-1})b_k(n) - \omega(p)p^{k-1}b_k(n/p), \end{aligned}$$

if $p|Q$, then it is easy to see that

$$b_k(pn) = \omega_2(p)p^{k-1}b_k(n);$$

if $p|N/Q$, then

$$b_k(pn) = \omega_1(p)b_k(n).$$

Collecting the above discussions we obtain

$$(\omega_1(p) + \omega_2(p)p^{k-1})b_k(n) = b_k(pn) + \omega(p)p^{k-1}b_k(n/p),$$

where we put that $b_k(n/p) = 0$ for any $p \nmid n$. Therefore we see that

$$\begin{aligned} \sum_{n=1}^{\infty} b_k(n)n^{-s} &= b_k(1) \prod_{p|Q} \left(1 - \omega_2(p)p^{k-1-s}\right)^{-1} \prod_{p|N/Q} \left(1 - \omega_1(p)p^{-s}\right)^{-1} \\ &\quad \times \prod_{p\nmid N} \left(1 - (\omega_1(p) + \omega_2(p)p^{k-1})p^{-s} + \omega(p)p^{k-1-2s}\right)^{-1} \\ &= b_k(1) \prod_{p\nmid Q} \left(1 - \omega_1(p)p^{-s}\right)^{-1} \prod_{p\nmid N/Q} \left(1 - \omega_2(p)p^{k-1-s}\right)^{-1} \\ &= b_k(1)L(s, \omega_1)L(s - k + 1, \omega_2). \end{aligned}$$

For $k \neq 2$ or $k = 2, \omega \neq \mathrm{id.}$ (i.e. $N > 1$), set

$$E_k(z, \omega_1, \omega_2) = \frac{N^k(k-1)!L_N(k, \omega_1\overline{\omega}_2)}{(-2\pi \mathrm{i})^k Q^{k/2} \omega_1(-1) \sum_{a=1}^{N/Q} \overline{\omega}_2(a)e(aQ/N)} E_k(z, \overline{\omega}_1\omega_2, N)|[W(Q)].$$

It is easy to check that

$$E_k(z,\omega_1,\omega_2) = (b_k(1))^{-1}\sum_{n=1}^{\infty} b_k(n)e(nz)$$

and hence

$$L(s, E_k(z,\omega_1,\omega_2)) = L(s,\omega_1)L(s-k+1,\omega_2), \tag{2.56}$$

where for $f(z) = \sum_{n=0}^{\infty} a(n)e(nz)$ we define $L(s,f) = \sum_{n=1}^{\infty} a(n)n^{-s}$ and $L(s,f)$ is called the zeta function of f.

Let t be a square free positive integer with $t|N$. Define

$$g_t(z) = \sum_{Q|t} \mu(Q)E_2(z,0,\text{id.},t)|[W(t/Q)].$$

By (2.50) and (2.55), we see that

$$g_t(z) = \mu(t) - \frac{4\pi^2}{tL_t(2,\text{id.})}\sum_{Q|t}\frac{\mu(Q)}{Q}\sum_{n=1}^{\infty}\left\{\sum_{d|n,(t/Q,n/d)=1} d \sum_{a=1,(a,Q)=1}^{Q} e(ad/Q)\right\} e(nz).$$

Write $n = n'\prod_{p|t} p^{\nu_p(n)}$ with $(n',t)=1$ and put $Q^* = \prod_{p|Q} p^{\nu_p(n)}$. Then

$$\begin{aligned}
&\sum_{Q|t}\frac{\mu(Q)}{Q}\sum_{d|n,(t/Q,n/d)=1} d \sum_{a=1,(a,Q)=1}^{Q} e(ad/Q)\\
&= \sum_{Q|t}\frac{\mu(Q)}{Q}\prod_{p|t/Q} p^{\nu_p(n)}\sum_{d|Q^*} d\sum_{d'|n'} d' \sum_{a=1,(a,Q)=1}^{Q} e(ad/Q)\\
&= \sum_{d'|n'} d' \sum_{Q|t}\frac{\mu(Q)}{Q}\prod_{p|t/Q} p^{\nu_p(n)}\prod_{p|Q}\sum_{d|p^{\nu_p(n)}} d \sum_{(a,p)=1,a=1}^{p} e(ad/p)\\
&= \sum_{d'|n'} d'\prod_{p|t}\left(p^{\nu_p(n)} - p^{-1}\left(-1+(p-1)\sum_{i=1}^{\nu_p(n)} p^i\right)\right)\\
&= \sum_{d'|n'} d'\prod_{p|t}(1+p^{-1}).
\end{aligned}$$

Put

$$g_t^*(z) = \left(\frac{-4\pi^2}{tL_t(2,\mathrm{id.})} \prod_{p|t} (1+p^{-1}) \right)^{-1} g_t(z)$$

$$= -\frac{\prod_{p|t}(1+p^{-1})}{24} + \sum_{n=1}^{\infty} \left(\sum_{d|n,(d,t)=1} d \right) e(nz). \tag{2.57}$$

It is easy to see that

$$L(s, g_t^*) = \zeta(s)L(s-1, 1_t), \tag{2.58}$$

where 1_t is the trivial character modulo t.

Chapter 3

The Modular Group and Its Subgroups

Let

$$SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}.$$

For any $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, define a transformation on the whole complex plane as follows

$$\sigma(z) = \frac{az + b}{cz + d}.$$

It is easy to prove

$$\mathrm{Im}(\sigma(z)) = \frac{\mathrm{Im}(z)}{|cz + d|^2}.$$

So σ induces a transformation on the upper-half plane $\mathbb{H}$. Since $\pm\sigma$ induce the same transformation on $\mathbb{H}$, we get a transformation group $SL_2(\mathbb{R})/\pm I$ of $\mathbb{H}$.

The fixed points of the transformation $z \to \sigma(z)$ are roots of the equation

$$cz^2 + (d - a)z - b = 0.$$

If $c \neq 0$, then it has two roots $(a - d \pm \sqrt{(a + d)^2 - 4})/(2c)$. If $c = 0$, then $\sigma(\infty) = \infty$. And multiplied by a, the above equation becomes

$$(1 - a^2)z - ab = 0.$$

So if $c = 0, a^2 = 1$, the transformation has a unique fixed point ∞; if $c = 0, a^2 \neq 1$, then the transformation has two fixed points ∞ and $ab/(1 - a^2)$.

Definition 3.1 *Let $\sigma \in SL_2(\mathbb{R})$, $\sigma \neq \pm I$. If the transformation $z \to \sigma(z)$ has one fixed point in $\mathbb{H}$, then σ is called an elliptic element; if the transformation $z \to \sigma(z)$ has a unique fixed point in $\mathbb{R} \bigcup \{\infty\}$, then σ is called a parabolic element; if the transformation $z \to \sigma(z)$ has two fixed points in $\mathbb{R} \bigcup \{\infty\}$, then σ is called a hyperbolic element.*

Put $\mathrm{tr}(\sigma) = a + d$. Then by the above discussions we get

Proposition 3.1 *Let $\sigma \in SL_2(\mathbb{R}), \sigma \neq \pm I$. Then σ is an elliptic(or parabolic, hyperbolic respectively) element if and only if $|\mathrm{tr}(\sigma)| < 2$ (or $= 2, > 2$ respectively).*

Form this proposition we know that if σ is an elliptic (or parabolic, hyperbolic respectively) element, then so is $\tau\sigma\tau^{-1}$ for any $\tau \in SL_2(\mathbb{R})$.

Let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$. If $\sigma(\mathrm{i}) = \mathrm{i}$, then $a = d, c = -b$, hence $a^2 + b^2 = 1$. So

$$\{\sigma \in SL_2(\mathbb{R}) \mid \sigma(\mathrm{i}) = \mathrm{i}\} = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \middle| 0 \leqslant \theta < 2\pi \right\}.$$

We denote this group by $SO(2)$. Let $z = x + \mathrm{i}y \in \mathbb{H}$. Then

$$\tau = \left(\begin{pmatrix} y^{1/2} & xy^{1/2} \\ 0 & y^{-1/2} \end{pmatrix}\right) \in SL_2(\mathbb{R})$$

and $\tau(\mathrm{i}) = z$. So

$$\{\sigma \in SL_2(\mathbb{R}) \mid \sigma(z) = z\} = \tau \cdot SO(2) \cdot \tau^{-1}.$$

Let $s \in \mathbb{R} \bigcup \{\infty\}$ and

$$F(s) = \{\sigma \in SL_2(\mathbb{R}) \mid \sigma(s) = s\},$$
$$P(s) = \{\sigma \in F(s) \mid \sigma \text{ is parabolic or } \pm I\}.$$

It is easy to see that

$$F(\infty) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| a, b \in \mathbb{R}, a \neq 0 \right\},$$
$$P(\infty) = \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \middle| h \in \mathbb{R} \right\}.$$

For any $s \in \mathbb{R}$, put

$$\tau = \begin{pmatrix} 0 & -1 \\ 1 & -s \end{pmatrix} \in SL_2(\mathbb{R}).$$

Since $\tau(s) = \infty$,

$$F(s) = \tau^{-1} F(\infty) \tau, \quad P(s) = \tau^{-1} P(\infty) \tau.$$

The topology of $\mathbb{R}$ induces a topology of $SL_2(\mathbb{R})$. Suppose that $\varGamma$ is a discrete subgroup of $SL_2(\mathbb{R})$, z is a point in $\mathbb{H}$. If there is an elliptic element $\sigma \in \varGamma$ such that $\sigma(z) = z$, then z is called an elliptic point of $\varGamma$. Suppose that $s \in \mathbb{R} \bigcup \{\infty\}$. If there is a parabolic element $\sigma \in \varGamma$ such that $\sigma(s) = s$, then s is called a cusp point of $\varGamma$. If ω is an elliptic (or a cusp respectively) point of $\varGamma$, then $\gamma(\omega)$ is also an elliptic (or a cusp respectively) point of $\varGamma$ for any $\gamma \in \varGamma$.

The modular group $SL_2(\mathbb{Z})$ is an important discrete subgroup of $SL_2(\mathbb{R})$. Let N be a positive integer, put

$$\varGamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| a \equiv d \equiv 1, b \equiv c \equiv 0 (N) \right\},$$

which is also a discrete subgroup of $SL_2(\mathbb{R})$.

If Γ is a subgroup of the modular group, and there exists a positive integer N such that $\Gamma(N) \subset \Gamma$, then Γ is called a congruence subgroup of the modular group. Hereafter congruence subgroups $\Gamma_0(N)$ and $\Gamma(N)$ will be our main research objects.

Proposition 3.2 *Suppose that Γ is a discrete subgroup of $SL_2(\mathbb{R})$, s a cusp point of Γ and z an elliptic point of Γ, then*

(1) $\Gamma_z := \{\sigma \in \Gamma \mid \sigma(z) = z\}$ is a finite cyclic group (in this case, we call $[\Gamma_z : \Gamma \bigcap \{\pm I\}]$ the order of the elliptic point z);

(2) $\Gamma_s/\Gamma \bigcap \{\pm I\}$ (where $\Gamma_s := \{\sigma \in \Gamma \mid \sigma(s) = s\}$) is isomorphic to $\mathbb{Z}$, and any element of Γ_s is $\pm I$ or parabolic.

Proof These are two well-known facts, therefore we omit the proof. □

Definition 3.2 *Let $w_1, w_2 \in \mathbb{H} \bigcup \mathbb{R} \bigcup \{\infty\}$. If there exists a $\tau \in \Gamma$ such that $\tau(w_1) = w_2$, then w_1 and w_2 are called Γ-equivalent.*

Now we discuss the cusp points and elliptic points of the modular group.

It is clear that ∞ is a fixed point of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. So ∞ is a cusp point of the modular group. Let $s \in \mathbb{R}$ be a cusp point of the modular group. Then there exists a parabolic matrix $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that s is its unique fixed point. Since $s \neq \infty$, then $c \neq 0$. So s is the unique solution of the following equation

$$cx^2 + (d - a)x - b = 0,$$

which implies that s is rational. Conversely if p/q is any rational number such that p, q are co-prime, then there exist integers u, t satisfying $pt - uq = 1$, i.e., $\sigma = \begin{pmatrix} p & u \\ q & t \end{pmatrix} \in SL_2(\mathbb{Z})$. Since $\sigma(\infty) = p/q$ and ∞ is a cusp point of the modular group, so is p/q. From above we know that $\mathbb{Q} \bigcup \{\infty\}$ are all cusp points of the modular group and all are equivalent to ∞.

Suppose that σ is an elliptic element of the modular group. By Proposition 3.1 we have that $tr(\sigma) = 0$ or 1 since σ is an integral matrix. Then the characteristic polynomial of σ is $x^2 + 1$ or $x^2 \pm x + 1$. Hence $\sigma^2 = -I$ or $\sigma^3 = \pm I$. But if $\sigma^3 = -I$, then $(-\sigma)^3 = I$. So we only need to consider the cases $\sigma^2 = -I$ and $\sigma^3 = I$.

Let $\sigma^2 = -I$. Put $\mathbb{Z}[\sigma] = \{a + b\sigma \mid a, b \in \mathbb{Z}\}$. Then $\mathbb{Z}[\sigma]$ is isomorphic to $\mathbb{Z}[\mathrm{i}]$. So it is an Euclidean ring. For any $\tau \in \mathbb{Z}[\sigma]$, we can define a transformation on $\mathbb{Z}^2$ as follows:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \tau \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2.$$

So $\mathbb{Z}^2$ is a $\mathbb{Z}[\sigma]$-module. If for any non-zero element $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2$, there exists $\tau = a+b\sigma$ such that $\tau \begin{pmatrix} x \\ y \end{pmatrix} = 0$, then

$$0 = (a - b\sigma)(a + b\sigma) \begin{pmatrix} x \\ y \end{pmatrix} = (a^2 + b^2) \begin{pmatrix} x \\ y \end{pmatrix}.$$

This shows that $a = b = 0$, i.e., $\tau = 0$. By the fundamental theorem for finitely generated modules over Euclidean rings, there exists a $u \in \mathbb{Z}^2$ such that

$$\mathbb{Z}^2 = \mathbb{Z}[\sigma]u = \mathbb{Z}u + \mathbb{Z}\sigma u.$$

Put $v = \sigma u$, then $\sigma v = -u$. So

$$\sigma(u, v) = (u, v) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where (u, v) represents the matrix with columns u, v. Since u, v consist of a basis of $\mathbb{Z}^2$, $\det(u, v) = \pm 1$. If $\det(u, v) = 1$, then $(u, v) \in SL_2(\mathbb{Z})$ and

$$\sigma = (u, v) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (u, v)^{-1},$$

If $\det(u, v) = -1$, then $(v, u) \in SL_2(\mathbb{Z})$ and

$$\sigma = (v, u) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (v, u)^{-1}.$$

This shows that σ is conjugate to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. But i is the fixed point in $\mathbb{H}$ of these two elements. So the fixed point of σ is equivalent to i which is an elliptic point with order 2.

Now let $\sigma^3 = I$. Then $\mathbb{Z}[\sigma]$ is also an Euclidean ring and $\mathbb{Z}^2$ is a $\mathbb{Z}[\sigma]$-module. If for any non-zero element $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2$, then there exists $\tau = a + b\sigma$ such that $\tau \begin{pmatrix} x \\ y \end{pmatrix} = 0$, then

$$0 = (a - b - b\sigma)(a + b\sigma) \begin{pmatrix} x \\ y \end{pmatrix} = (a^2 - ab + b^2) \begin{pmatrix} x \\ y \end{pmatrix}.$$

This shows that $a^2 - ab + b^2 = 0$, so $a = b = 0$, i.e., $\tau = 0$. By the fundamental theorem for finitely generated modules over Euclidean rings, there exists a $u \in \mathbb{Z}^2$ such that

$$\mathbb{Z}^2 = \mathbb{Z}[\sigma]u = \mathbb{Z}u + \mathbb{Z}\sigma u.$$

Put $v = \sigma u$, then $\sigma v = -\sigma u - u = -v - u$,

$$\sigma(u, v) = (u, v)\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

where (u, v) represents the matrix with columns u, v. Since u, v consist of a basis of $\mathbb{Z}^2$, $\det(u, v) = \pm 1$. If $\det(u, v) = 1$, then $(u, v) \in SL_2(\mathbb{Z})$ and

$$\sigma = (u, v)\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}(u, v)^{-1}.$$

If $\det(u, v) = -1$, then $(v, u) \in SL_2(\mathbb{Z})$ and

$$\sigma = (v, u)\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}(v, u)^{-1}.$$

So σ is conjugate to

$$\tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \text{ or } \tau^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

The fixed point in $\mathbb{H}$ of τ is the root $\rho = \mathrm{e}^{2\pi\mathrm{i}/3}$ of $z^2 - z + 1 = 0$. Hence the fixed point in $\mathbb{H}$ of σ is an elliptic point with order three and equivalent to ρ. Therefore we have the following

Theorem 3.1 $\mathbb{Q}\bigcup\{\infty\}$ *are all cusp points of the modular group. Every cusp point is equivalent to* ∞. *Any elliptic point of the modular group has order* 2 *or* 3. *All elliptic points with order* 2 (*or with order* 3 *respectively*) *are equivalent to* i (*or* ρ *respectively*).

Now we want to discuss the cusp and elliptic points of the congruence subgroup $\Gamma(N)$ and $\Gamma_0(N)$ with $N > 1$ (It is clear that $\Gamma(1) = \Gamma_0(1) = SL_2(\mathbb{Z})$).

By above discussions we know that all elliptic elements of the modular group are conjugate to one of the following elements:

$$\pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \pm\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

$\Gamma(N)$ is a normal subgroup of the modular group. All the above elements do not belong to $\Gamma(N)$ for $N > 1$. Hence $\Gamma(N)$ has no elliptic points. By Theorem 3.1, the elliptic points of $\Gamma_0(N)$ are only of order 2 or 3.

Theorem 3.2 *Let* ν_2 *and* ν_3 *be the numbers of the equivalence classes of the elliptic points with order* 2 *and* 3 *of* $\Gamma_0(N)$ *respectively. Then*

$$\nu_2 = \begin{cases} 0, & if\ 4 \mid N, \\ \prod\limits_{p \mid N}\left(1 + \left(\dfrac{-1}{p}\right)\right), & if\ 4 \nmid N, \end{cases}$$

$$\nu_3 = \begin{cases} 0, & if\ 9 \mid N, \\ \prod\limits_{p \mid N}\left(1 + \left(\dfrac{-3}{p}\right)\right), & if\ 9 \nmid N, \end{cases}$$

where

$$\left(\frac{-1}{p}\right)=\begin{cases}0, & if\ p=2,\\ 1, & if\ p\equiv 1(4),\\ -1, & if\ p\equiv 3(4),\end{cases}$$
$$\left(\frac{-3}{p}\right)=\begin{cases}0, & if\ p=3,\\ 1, & if\ p\equiv 1(3),\\ -1, & if\ p\equiv 2(3).\end{cases}$$

Proof We consider first elliptic points with order 2. Let z_1, z_2 be two elliptic points with order 2. Then

$$\Gamma_{z_1}=\{\sigma\in\Gamma_0(N)\mid\sigma(z_1)=z_1\}=\{\pm I,\pm\sigma_1\},$$
$$\Gamma_{z_2}=\{\sigma\in\Gamma_0(N)\mid\sigma(z_2)=z_2\}=\{\pm I,\pm\sigma_2\},$$

where σ_1,σ_2 are elliptic elements of $\Gamma_0(N)$ which can be assumed to be equivalent to $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ in the modular group. If z_1, z_2 are $\Gamma_0(N)$-equivalent, then there exists $\tau\in\Gamma_0(N)$ such that $\tau(z_1)=z_2$. Then $\tau^{-1}\sigma_2\tau\in\Gamma_{z_1}$. It can be shown that $\tau^{-1}\sigma_2\tau$ must be σ_1. So z_1, z_2 are $\Gamma_0(N)$-equivalent if and only if σ_1,σ_2 are $\Gamma_0(N)$-conjugate. This means that ν_2 is the number of the conjugate classes of the elements in the set

$$\Sigma=\left\{T^{-1}\begin{pmatrix}0&-1\\1&0\end{pmatrix}T\in\Gamma_0(N)\middle| T\in SL_2(\mathbb{Z})\right\}$$

in $\Gamma_0(N)$.

Suppose $\sigma=T^{-1}\begin{pmatrix}0&-1\\1&0\end{pmatrix}T\in\Sigma$, and put

$$(\omega_1,\omega_2)=(1,\mathrm{i})T.$$

Then $\{\omega_1,\omega_2\}$ is a basis of $\mathbb{Z}[\mathrm{i}]$ as a $\mathbb{Z}$-module. And

$$(\mathrm{i}\omega_1,\mathrm{i}\omega_2)=(1,\mathrm{i})\begin{pmatrix}0&-1\\1&0\end{pmatrix}T=(\omega_1,\omega_2)\sigma. \tag{3.1}$$

Put

$$J=\{a\omega_1+bN\omega_2\mid a,b\in\mathbb{Z}\}\subset\mathbb{Z}[\mathrm{i}].$$

Then the equality (3.1) shows that J is an ideal of $\mathbb{Z}[\mathrm{i}]$ and J satisfies the following two properties

(1) The Norm $N(J)$ of the ideal J is $[\mathbb{Z}[\mathrm{i}]:J]=N$;

(2) For any integer $q\neq\pm1$, J is not contained in the ideal (q) generated by q(since $\omega_1\notin(q)=\{aq\omega_1+bq\omega_2\mid a,b\in\mathbb{Z}\}$).

Conversely, if J is an ideal of $\mathbb{Z}[\mathrm{i}]$ with the above properties, then by property (1) we can find a basis ω_1,ω_2 of $\mathbb{Z}[\mathrm{i}]$ such that $\varepsilon_1\omega_1,\varepsilon_2\omega_2(\varepsilon_1,\varepsilon_2\in\mathbb{Z})$ is a basis of J and

$\varepsilon_1 \mid \varepsilon_2, \varepsilon_1\varepsilon_2 = N$. In this case we have that $J \subset (\epsilon_1)$. By property (2) we know that $\varepsilon_1 = 1, \varepsilon_2 = N$. If necessary, substituting ω_2 by $-\omega_2$, we can assume

$$(\omega_1, \omega_2) = (1, \mathrm{i})T, \quad T \in SL_2(\mathbb{Z}).$$

Therefore

$$(\mathrm{i}\omega_1, \mathrm{i}\omega_2) = (\omega_1, \omega_2)T^{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}T.$$

Since $\mathrm{i}\omega_1 \in J$, so $T^{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}T \in \Sigma$.

Now we can prove that there is a bijection between the set of the conjugate classes of the elements of Σ in $\Gamma_0(N)$ and the set of the ideals of $\mathbb{Z}[\mathrm{i}]$ with properties (1) and (2). Let

$$\sigma = T^{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}T \in \Sigma, \quad \sigma_1 = T_1^{-1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}T_1 \in \Sigma.$$

Define

$$(\omega_1, \omega_2) = (1, \mathrm{i})T, \quad (\omega_1', \omega_2') = (1, \mathrm{i})T_1$$

and

$$J = \{a\omega_1 + bN\omega_2 \mid a, b \in \mathbb{Z}\},$$
$$J_1 = \{a\omega_1' + bN\omega_2' \mid a, b \in \mathbb{Z}\}.$$

If $J = J_1$, since $(\omega_1, \omega_2) = (\omega_1', \omega_2')T_1^{-1}T$ and $\omega_1 \in J_1$, we have that

$$T_1^{-1}T = \tau \in \Gamma_0(N).$$

Hence $\sigma = \tau^{-1}\sigma_1\tau$ which means that σ, σ_1 are conjugate in $\Gamma_0(N)$. Conversely, if σ, σ_1 are conjugate in $\Gamma_0(N)$, suppose that $\sigma = \tau^{-1}\sigma_1\tau$ with $\tau \in \Gamma_0(N)$. Put

$$(\omega_1'', \omega_2'') = (\omega_1', \omega_2')\tau,$$

we have that

$$(\mathrm{i}\omega_1, \mathrm{i}\omega_2) = (\omega_1, \omega_2)\sigma, \quad (\mathrm{i}\omega_1'', \mathrm{i}\omega_2'') = (\omega_1'', \omega_2'')\sigma.$$

(ω_1, ω_2) and (ω_1'', ω_2'') are the solution of

$$\begin{cases} (\sigma_{11} - \mathrm{i})x + \sigma_{21}y = 0, \\ \sigma_{12}x + (\sigma_{22} - \mathrm{i})y = 0, \end{cases}$$

where $\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$.

Therefore there is a $\lambda \in \mathbb{Q}(\mathrm{i})$ such that $\omega_1 = \lambda\omega_1'', \omega_2 = \lambda\omega_2''$. Since $\{\omega_1'', \omega_2''\}$ is a basis of $\mathbb{Z}[\mathrm{i}]$, there exist integers n, m such that $n\omega_1'' + m\omega_2'' = 1$ and hence $n\omega_1 + m\omega_2 = \lambda$, i.e., $\lambda \in \mathbb{Z}[\mathrm{i}]$. Since $\{\omega_1, \omega_2\}$ is a basis of $\mathbb{Z}[\mathrm{i}]$, we can similarly prove that $\lambda^{-1} \in \mathbb{Z}[\mathrm{i}]$. So λ is an invertible element of $\mathbb{Z}[\mathrm{i}]$ and

$$J = \{a\omega_1'' + bN\omega_2'' \mid a, b \in \mathbb{Z}\} = J_1.$$

It is well-known that $\mathbb{Z}[i]$ is a principal ideal domain. If an ideal $J = (x + iy)$ has properties (1) and (2), then

$$x^2 + y^2 = N, \quad (x, y) = 1.$$

The number of solutions of this equation is (please see, e.g., Hua Luokong: An introduction to number theory, §7, Chapter 6)

$$\begin{cases} 4\prod_{p|N}\left(1 + \left(\frac{-1}{p}\right)\right), & \text{if } 4 \nmid N; \\ 0, & \text{if } 4 \mid N. \end{cases}$$

Since $\pm(x + iy), \pm(-y + ix)$ generate the same ideal, we get the result for ν_2.

Now we consider the elliptic points with order 3 of $\Gamma_0(N)$. Let z_1, z_2 be two elliptic points with order 3 of $\Gamma_0(N)$. Then

$$\Gamma_{z_1} = \{\pm I, \pm\sigma_1, \pm\sigma_1^2\}, \quad \Gamma_{z_2} = \{\pm I, \pm\sigma_2, \pm\sigma_2^2\}.$$

We can assume that σ_1, σ_2 are conjugate to $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ in the modular group. Then we can prove that z_1, z_2 are $\Gamma_0(N)$-equivalent if and only if σ_1, σ_2 are $\Gamma_0(N)$-conjugate. In fact, if z_1, z_2 are $\Gamma_0(N)$-equivalent, then there exists a $\tau \in \Gamma_0(N)$ such that $\tau(z_1) = z_2$. Hence $\tau^{-1}\sigma_2\tau \in \Gamma_{z_1}$. We want to show that $\tau^{-1}\sigma_2\tau$ must be σ_1. In order to prove this, we only need to show that

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

are not conjugate to each other in the modular group. Their characteristic polynomials are respectively

$$\lambda(\lambda + 1) + 1, \quad \lambda(\lambda + 1) + 1, \quad \lambda(\lambda - 1) + 1, \quad \lambda(\lambda - 1) + 1.$$

So it is only possible that the first and the second are conjugate, the third and the fourth are conjugate. Let α be $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. Then $\alpha^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. Suppose that there is a $\gamma \in SL_2(\mathbb{Z})$ such that $\gamma\alpha\gamma^{-1} = \alpha^{-1}$. We know that there exists a $\tau \in SL_2(\mathbb{R})$ such that $\tau\alpha\tau^{-1} = \begin{pmatrix} p & q \\ -q & p \end{pmatrix} \in SO(2)$. Put

$$\tau\gamma\tau^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since $\gamma\alpha = \alpha^{-1}\gamma$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p & q \\ -q & p \end{pmatrix} = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} ap-bq & bp+aq \\ cp-dq & dp+cq \end{pmatrix} = \begin{pmatrix} ap-cq & bp-dq \\ cp+aq & dp+bq \end{pmatrix}.$$

It shows that $a=-d, b=c$ and hence

$$\det\gamma = ad-bc = -a^2-d^2 < 0,$$

which is impossible. Therefore α, α^{-1} are not conjugate in the modular group. Similarly we can prove that $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ are not conjugate in the modular group. Therefore z_1, z_2 are $\Gamma_0(N)$-equivalent if and only if σ_1, σ_2 are $\Gamma_0(N)$-conjugate. So ν_3 is the number of the conjugate classes of the elements of the set

$$\left\{T^{-1}\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}T \in \Gamma_0(N) \middle| T \in SL_2(\mathbb{Z})\right\}$$

in $\Gamma_0(N)$. By a similar reasoning for elliptic points of order 2, but substituting $\mathbb{Z}[\mathrm{i}]$ by $\mathbb{Z}[\rho]$, we can show that $6\nu_3$ is the number of solutions of the following equation:

$$x^2-xy+y^2=N, \quad (x,y)=1.$$

By referring to the classical result for the number of solutions of the equation, we are finished with the proof of the theorem. □

Lemma 3.1 *We have that*

$$[SL_2(\mathbb{Z}):\Gamma(N)] = N^3\prod_{p|N}(1-p^{-2}),$$
$$[SL_2(\mathbb{Z}):\Gamma_0(N)] = N\prod_{p|N}(1+p^{-1}).$$

Proof Let $\Gamma = SL_2(\mathbb{Z})$ and define a homomorphism

$$f:\Gamma \to SL_2(\mathbb{Z}/N\mathbb{Z}),$$
$$\alpha \to \alpha \bmod N.$$

Then the kernel of the map is $\Gamma(N)$. We now show that f is an epimorphism. For any 2×2 integral matrix A with $\det A \equiv 1(N)$, it is well known that there exist $U, V \in \Gamma$ such that

$$UAV = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

So $a_1a_2 = 1+rN$ with r an integer. Put

$$B' = \begin{pmatrix} a_1+xN & yN \\ N & a_2 \end{pmatrix}.$$

Since a_2, N are co-prime, there are two integers x, y such that $r + a_2x - yN = 0$. Thus

$$\det B' = a_1a_2 + a_2xN - yN^2 = 1,$$

i.e., $B' \in \Gamma$. It is easy to see that $UAV \equiv B'(N)$. Taking $B = U^{-1}B'V^{-1}$, then $B \in \Gamma$ and $A \equiv B(N)$ which means that f is an epimorphism. And so

$$[\Gamma : \Gamma(N)] = [SL_2(\mathbb{Z}/N\mathbb{Z}) : 1].$$

Let $N = \prod p^e$ be the standard factorization of N, by Chinese Remainder Theorem, we have that

$$[SL_2(\mathbb{Z}/N\mathbb{Z}) : 1] = \prod_{p|N}[SL_2(\mathbb{Z}/p^e\mathbb{Z}) : 1]. \tag{3.2}$$

Consider the map

$$\begin{aligned} h \quad &: GL_2(\mathbb{Z}/p^e\mathbb{Z}) \to GL_2(\mathbb{Z}/p\mathbb{Z}), \\ &\alpha \bmod p^e \qquad \to \alpha \bmod p. \end{aligned}$$

Then the kernel of h is

$$X = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}/p^e\mathbb{Z}) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv I(p) \right\}.$$

It is easy to see that $[X : 1] = p^{4(e-1)}$. It is well-known that

$$[GL_2(\mathbb{Z}/p\mathbb{Z}) : 1] = (p^2 - 1)(p^2 - p),$$

so

$$[GL_2(\mathbb{Z}/p^e\mathbb{Z}) : 1] = p^{4e}(1 - p^{-1})(1 - p^{-2}).$$

Consider the map: $\alpha \mapsto \det \alpha$ from $GL_2(\mathbb{Z}/p^e\mathbb{Z})$ to $(\mathbb{Z}/p^e\mathbb{Z})^*$. The map is an epimorphism and the kernel is $SL_2(\mathbb{Z}/p^e\mathbb{Z})$. So

$$[SL_2(\mathbb{Z}/p^e\mathbb{Z}) : 1] = [GL_2(\mathbb{Z}/p^e\mathbb{Z}) : 1]/\varphi(p^e) = p^{3e}(1 - p^{-2}).$$

By (3.2) we get the result for $[\Gamma : \Gamma(N)]$. The image of $\Gamma_0(N)$ under the homomorphism f is

$$\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in SL_2(\mathbb{Z}/N\mathbb{Z}) \middle| ad \equiv 1(N) \right\},$$

which has $N\varphi(N)$ elements. So

$$[\Gamma : \Gamma_0(N)] = \frac{[SL_2(\mathbb{Z}/N\mathbb{Z}) : 1]}{N\varphi(N)} = N \prod_{p|N} \left(1 + p^{-1}\right),$$

which completes the proof. □

Lemma 3.2 *Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$ and Γ' a subgroup of Γ. If $[\Gamma : \Gamma'] < \infty$, then Γ and Γ' have the same set of cusp points.*

Proof It is obvious that any cusp point of Γ' is also one of Γ. Conversely, let s be a cusp point of Γ. Then there exists a parabolic element $\sigma \in \Gamma$ such that $\sigma(s) = s$. Since $[\Gamma : \Gamma'] < \infty$, there is a positive integer n such that $\sigma^n \in \Gamma'$. But σ must be conjugate to an element $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ in $SL_2(\mathbb{R})$. Then σ^n and $\begin{pmatrix} 1 & nh \\ 0 & 1 \end{pmatrix}$ are conjugate. So σ^n is a parabolic element. But $\sigma^n(s) = s$ implies that s is a cusp point of Γ'.

By Lemma 3.1, Lemma 3.2 we know that $\Gamma(N)$ and $\Gamma_0(N)$ have the set $\mathbb{Q}\bigcup\{\infty\}$ of cusp points. That d/c is a cusp point implies that d is an integer and c a non negative integer with $(c, d) = 1$. If $c = 0$, then $d = 1$ and $1/0$ is ∞. □

Theorem 3.3 *Let $R_{c,N}$ be a reduced residue system modulo $(c, N/c)$,* i.e. $\{d + (c, N/c)\mathbb{Z} \mid d \in R_{c,N}\} = (\mathbb{Z}/(c, N/c)\mathbb{Z})^*$. *The set*

$$\{d/c \mid c \mid N, (c, d) = 1, d \in R_{c,N}\} \tag{3.3}$$

is a complete set of representatives of equivalence classes of the cusp points of $\Gamma_0(N)$. Hence the number of equivalence classes of the cusp points of $\Gamma_0(N)$ is equal to

$$\nu_\infty = \sum_{c|N} \varphi((c, N/c)).$$

Proof If d' is prime to $(c, N/c)$, put

$$d = d' + (c, N/c) \prod_{p|c, p\nmid d'} p.$$

It is clear that d is prime to c and $d \equiv d' \bmod (c, N/c)$. So the number of the cusp points in the set (3.3) is equal to

$$\sum_{c|N} \varphi((c, N/c)).$$

Now we need only to prove that any cusp point of $\Gamma_0(N)$ is equivalent to one in the set (3.3) and any two elements in the set (3.3) are not equivalent for $\Gamma_0(N)$. Let d/c, d_1/c be two cusp points and $c \mid N$, $d \equiv d_1 \bmod (c, N/c)$. Then there are two matrices

$$\begin{pmatrix} a & d \\ b & c \end{pmatrix}, \quad \begin{pmatrix} a_1 & d_1 \\ b_1 & c_1 \end{pmatrix} \in SL_2(\mathbb{Z}).$$

It is clear that

$$bd \equiv b_1 d_1 \equiv -1 \bmod (c, N/c),$$

Therefore $b \equiv b_1 \bmod (c, N/c)$. There exist integers m, n such that $b = b_1 + mc + nN/c$. Hence

$$\gamma = \begin{pmatrix} a - md & d \\ b - mc & c \end{pmatrix} \begin{pmatrix} c & -d_1 \\ -b_1 & a_1 \end{pmatrix} \in \Gamma_0(N)$$

satisfies $\gamma(d_1/c) = d/c$. This shows that d/c and d_1/c are equivalent each other for $\Gamma_0(N)$.

Let n/m be a cusp point and $(m, N) = c$. There exist integers α, β such that

$$\alpha m + \beta n N = c.$$

Put

$$\alpha' = \alpha + nN/c \prod_{p|N, p\nmid\alpha} p, \quad \beta' = \beta - m/c \prod_{p|N, p\nmid\alpha} p.$$

It is easy to see that $\alpha' m/c + \beta' nN/c = 1$ and α' is prime to $\beta' N$. Therefore there is

$$\sigma = \begin{pmatrix} * & * \\ \beta' N & \alpha' \end{pmatrix} \in \Gamma_0(N),$$

which satisfies $\sigma(n/m) = d/c$, $(c, d) = 1$. d/c is $\Gamma_0(N)$-equivalent to some cusp point in the set (3.3). And so is n/m.

Now we assume that d/c, d_1/c_1 are two points in the set (3.3) and that they are $\Gamma_0(N)$-equivalent. Then there exists

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma N & \delta \end{pmatrix} \in \Gamma_0(N)$$

such that

$$\alpha d + \beta c = d_1, \quad \gamma N d + \delta c = c_1. \tag{3.4}$$

By the second equality of (3.4) we have $c \mid c_1$. Because of symmetry we can show similarly that $c_1 \mid c$. Hence $c = c_1$ and so $\delta \equiv 1 (\text{mod} N/c)$. But $\alpha\delta \equiv 1(\text{mod} N)$, so $\alpha \equiv 1(\text{mod} N/c)$. By the first equality of (3.4) we know that $d \equiv d_1 \bmod (c, N/c)$ which means that d/c and d_1/c_1 are the same point in the set (3.3). This completes the proof. □

Lemma 3.3 *Let a, b, c, d be positive integers, $(a, b) = 1, (c, d) = 1$ and $a \equiv c, b \equiv d(\text{mod} N)$. Then there exists $\sigma \in \Gamma(N)$ such that*

$$\begin{pmatrix} a \\ b \end{pmatrix} = \sigma \begin{pmatrix} a \\ b \end{pmatrix}.$$

Proof We first consider the case $c = 1, d = 0$. Then $a \equiv 1, b \equiv 0(\text{mod} N)$. There exist integers p, q such that $ap - bq = (1 - a)/N$ and so

$$\sigma = \begin{pmatrix} a & Nq \\ b & 1 + Np \end{pmatrix} \in \Gamma(N), \quad \begin{pmatrix} a \\ b \end{pmatrix} = \sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We now consider the general case. There exists

$$\tau = \begin{pmatrix} c & * \\ d & * \end{pmatrix} \in SL_2(\mathbb{Z})$$

such that

$$\tau \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix} \equiv \begin{pmatrix} a \\ b \end{pmatrix} \quad (\mathrm{mod} N).$$

Then

$$\tau^{-1} \begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (\mathrm{mod} N).$$

By above discussion we know that there exists $\sigma \in \Gamma(N)$ such that

$$\tau^{-1} \begin{pmatrix} a \\ b \end{pmatrix} = \sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

hence

$$\begin{pmatrix} a \\ b \end{pmatrix} = \tau\sigma\tau^{-1} \begin{pmatrix} c \\ d \end{pmatrix}.$$

It is easy to see that $\tau\sigma\tau^{-1} \in \Gamma(N)$ which implies the lemma. □

Theorem 3.4 *Let $s = a/b$, $s' = c/d$ be cusp points. Then s, s' are $\Gamma(N)$-equivalent if and only if $\pm \begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} c \\ d \end{pmatrix} (\mathrm{mod} N)$. The number of the equivalence classes of cusp points of $\Gamma(N)$ is*

$$\nu_\infty = \begin{cases} \dfrac{N^2}{2} \prod\limits_{p|N} (1 - p^{-2}), & if \ N > 2, \\ 3, & if \ N = 2. \end{cases}$$

Proof Assume that $\pm \begin{pmatrix} a \\ b \end{pmatrix} \equiv \begin{pmatrix} c \\ d \end{pmatrix} (\mathrm{mod} N)$. By Lemma 3.3 there is $\sigma \in \Gamma(N)$ such that $\sigma(s) = s'$. Conversely, if there is $\sigma \in \Gamma(N)$ such that $\sigma(s) = s'$, then $\sigma \begin{pmatrix} a \\ b \end{pmatrix} = m \begin{pmatrix} c \\ d \end{pmatrix}$ with an integer m. But $(a, b) = (c, d) = 1$ implies that $m = \pm 1$, and so $\begin{pmatrix} a \\ b \end{pmatrix} \equiv \pm \begin{pmatrix} c \\ d \end{pmatrix} (\mathrm{mod} N)$.

Let

$$J = \{(a_1, a_2) \mid 1 \leqslant a_1, a_2 \leqslant N, (a_1, a_2, N) = 1\}.$$

Let $s = c/d$ be a cusp point of $\Gamma(N)$. Put $a_1 \equiv c, a_2 \equiv d (\mathrm{mod} N)$ and $1 \leqslant a_1, a_2 \leqslant N$. Since $(a_1, a_2, N) \mid (c, d) = 1$, then $(a_1, a_2) \in J$. Thus each cusp point s corresponds to an element of J as shown above. If another cusp point s' corresponds to an element $(a_1', a_2') \in J$, then by the first result of the theorem, s, s' are $\Gamma(N)$-equivalent if and only if $a_1 = a_1', a_2 = a_2'$ or $a_1 = N - a_1', a_2 = N - a_2'$. Conversely, if (a_1, a_2) is any element of J, it is easy to see that a_2 and $c = a_1 + N \prod\limits_{p|a_2, p\nmid a_1} p$ are co-prime. Hence the cusp point c/a_2 corresponds to (a_1, a_2) according to the above definition. Therefore for $N > 2$, $N = \prod_p p^e$, we have

$$\begin{aligned}
\nu_\infty = \#J/2 &= \frac{1}{2}\sum_{a=1}^{N} \varphi((a,N))N/(a,N) \\
&= \frac{N}{2}\prod_{p|N}\sum_{a=1}^{p^e} \varphi((a,p^e))/(a,p^e) \\
&= \frac{N}{2}\prod_{p|N}\sum_{i=1}^{e} \varphi(p^i)\varphi(p^{e-i})/p^i \\
&= \frac{N^2}{2}\prod_{p|N}(1-p^{-2}).
\end{aligned}$$

A direct computation shows that $\nu_\infty = \#J = 3$ for $N = 2$, which completes the proof. □

Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$. A domain F in $\mathbb{H}$ is called a fundamental domain for Γ if

(I) F is a connected open set;

(II) any two points in F are Γ-inequivalent;

(III) any point in $\mathbb{H}$ is Γ-equivalent to a point of the closure $\overline{F}$ of F.

Lemma 3.4 *The following set*

$$F = \{z \in \mathbb{H} \mid -1/2 < \mathrm{Re}(z) < 1/2, |z| > 1\}$$

is a fundamental domain of the modular group.

Proof It is clear that F satisfies the first condition (I). Let z_1, $z_2 \in F$. If they are $SL_2(\mathbb{Z})$–equivalent, then there exists $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that $\sigma(z_2) = z_1$. Without loss of generality, we can assume that $\mathrm{Im}(z_2) \leqslant \mathrm{Im}(z_1) = \dfrac{\mathrm{Im}(z_2)}{|cz_2 + d|^2}$. Then

$$|c|\mathrm{Im}(z_2) \leqslant |cz_2 + d| \leqslant 1. \tag{3.5}$$

If $c = 0$, then $a = d = \pm 1$, $z_1 = z_2 \pm b, b$ is an integer which is impossible. So $c \neq 0$. Since $z_2 \in F$, $\mathrm{Im}(z_2) \geqslant \sqrt{3}/2$. By (3.5) we have $|c| = 1$ and $|z_2 \pm d| \leqslant 1$ which is also impossible. Therefore the second condition (II) is satisfied. Let z be any point of $\mathbb{H}$. $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Because

$$\mathrm{Im}(\sigma(z)) = \frac{\mathrm{Im}(z)}{|cz+d|^2},$$

we know that $\mathrm{Im}(\sigma(z))$ will get its maximum when σ runs over the modular group. Let $\mathrm{Im}(\sigma_0(z))$ be the maximum. Write $w = \sigma_0(z) = x + \mathrm{i}y, \gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$. Then

$$\mathrm{Im}(\gamma\sigma_0(z)) = \mathrm{Im}(-1/w) = y/|w|^2 \leqslant y.$$

This shows that $|w| \geqslant 1$. Put $\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$. Then for any integer h

$$\tau^h(\sigma_0(z)) = x + h + \mathrm{i}y,$$

which implies that $\mathrm{Im}(\tau^h\sigma_0(z)) = \mathrm{Im}(\sigma_0(z))$. So $|\tau^h\sigma_0(z)| \geqslant 1$ for any h. A suitable h will assure that $\tau^h\sigma_0(z) \in \overline{F}$ which shows that the third condition (III) is satisfied. □

Now Put

$$F' = F\bigcup\{z \in \mathbb{H} \mid |z| \geqslant 1, \mathrm{Re}(z) = -1/2\}\bigcup\{z \in \mathbb{H} \mid |z| = 1, -1/2 < \mathrm{Re}(z) \leqslant 0\}.$$

It is clear that F' is a complete set of representatives of $SL_2(\mathbb{Z}) \setminus \mathbb{H}$. Put

$$\mathbb{H}^* = \mathbb{H}\bigcup\mathbb{Q}\bigcup\{\infty\}.$$

$\mathbb{Q}\bigcup\{\infty\}$ are all cusp points of the modular group. By Theorem 3.1 we have

$$SL_2(\mathbb{Z}) \setminus \mathbb{H}^* = (SL_2(\mathbb{Z}) \setminus \mathbb{H})\bigcup\{\infty\}.$$

In general, let $G = SL_2(\mathbb{R})$. The topology of $\mathbb{R}^4$ induces a topology on G, and so G becomes a topological group. Let Γ be a discrete subgroup of G. Put

$$\mathbb{H}^* = \mathbb{H}\bigcup\{\text{ all cusp points of } \Gamma\}.$$

We introduce a topology on $\Gamma \setminus \mathbb{H}^*$. We first introduce a topology on $\mathbb{H}^*$. If $z \in \mathbb{H}$, then all the neighbors of z in $\mathbb{H}$ are all the ones of z in $\mathbb{H}^*$. If ∞ is a cusp point of Γ, define the following sets

$$\{\infty\}\bigcup\{z \in \mathbb{H} \mid \mathrm{Im}(z) > c > 0\} \tag{3.6}$$

as the system of open neighbors of ∞. If $s \in \mathbb{R}$ is a cusp point of Γ, define the system of open neighbors of s as follows

$$\{s\}\bigcup\{\text{ the inner of a disc in } \mathbb{H} \text{ tangent to the real axis at } s\}.$$

It can be verified directly that $\mathbb{H}^*$ becomes a topological space under the above definition. It can also be verified that each element of Γ defines a homeomorphism of

$\mathbb{H}^*$. Then the topology over $\Gamma \setminus \mathbb{H}^*$ is defined as the quotient topology of $\mathbb{H}^*$ with respect to Γ.

It can be proved that for any $v \in \mathbb{H}^*$ there is a neighbor U of v such that

$$\{\sigma \in \Gamma \mid \sigma(U)\bigcap U \neq \varnothing\} = \{\sigma \in \Gamma \mid \sigma(v) = v\} = \Gamma_v.$$

That is, $\Gamma_v \setminus U$ can be imbedded into $\Gamma \setminus \mathbb{H}^*$. Let φ be the natural map: $\mathbb{H}^* \to \Gamma \setminus \mathbb{H}^*$. If $v \in \mathbb{H}$ is not an elliptic point, then $\Gamma_v = \Gamma \bigcap \{\pm I\}$. Then $\varphi : U \to \Gamma_v \setminus U$ is a homeomorphism. Let $(\Gamma_v \setminus U, \varphi^{-1})$ be an element of the complex structure of $\Gamma \setminus \mathbb{H}^*$. If $v \in \mathbb{H}$ is an elliptic point, then $\overline{\Gamma_v} = \Gamma_v/(\Gamma_v \bigcap \{\pm I\})$ is a finite cyclic group with order e. Let

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_v$$

correspond to a generator of $\overline{\Gamma}_v$. Put

$$\lambda(z) = \frac{z - v}{z - \overline{v}},$$

whose matrix $\begin{pmatrix} 1 & v \\ 1 & -\overline{v} \end{pmatrix}$ is also denoted by λ. Then

$$\lambda\sigma\lambda^{-1} = \begin{pmatrix} c\overline{v} + d & 0 \\ 0 & cv + d \end{pmatrix}.$$

Denote $\xi = cv + d$. Then $\xi\overline{\xi} = 1$. Let e be the smallest positive integer such that $\sigma^e = \pm I$, i.e., $\xi^e = \pm 1$. If e is even, it must be that $\xi^e = -1$. So ξ is a primitive root of the unit with degree $2e$. Anyway, ξ^2 is always a primitive root of the unit with degree e. Put $\zeta = \xi^{-2}$. Then $\lambda\overline{\Gamma_v}\lambda^{-1}$ is the set of the following transformations:

$$z \mapsto \zeta^i z, \quad i = 1, 2, \cdots, e.$$

The transformation $z \mapsto \lambda(z)$ maps Γ_v-equivalent points in U to $\lambda\Gamma_v\lambda^{-1}$-equivalent points in $\lambda(U)$. That is, $\boldsymbol{\lambda}$ induces a bijection from $\Gamma_v \setminus U$ to $\lambda\Gamma_v\lambda^{-1} \setminus \lambda(U)$. Two points $w_1, w_2 \in \lambda(U)$ are $\lambda\Gamma_v\lambda^{-1}$-equivalent if and only if $w_1^e = w_2^e$. Define a map:

$$\begin{aligned} p: \quad \Gamma_v \setminus U &\to \mathbb{C}, \\ \varphi(z) &\mapsto \lambda(z)^e, \quad z \in U. \end{aligned}$$

We regard $(\Gamma_v \setminus U, p)$ as an element of the complex structure on $\Gamma \setminus \mathbb{H}^*$ which is a homeomorphism from $\Gamma_v \setminus U$ into $\mathbb{C}$.

If v is a cusp point of Γ, then there exists $\rho \in G$ such that $\rho(v) = \infty$. So

$$\rho\Gamma_v\rho^{-1} \cdot \{\pm I\} = \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^m \middle| m \in \mathbb{Z} \right\}, \quad h > 0.$$

Define a homeomorphism p from $\Gamma_v \setminus U$ into $\mathbb{C}$: $p(\varphi(z)) = \mathrm{e}^{2\pi \mathrm{i}\rho(z)/h}$. We consider $(\Gamma_v \setminus U, p)$ as an element of the complex structure on $\Gamma \setminus \mathbb{H}^*$.

We can prove that $\Gamma \setminus \mathbb{H}^*$ becomes a Riemann surface. In general it is locally compact. If it is compact, then the discrete subgroup Γ is called a Fuchsian group of the first kind.

We can show

Lemma 3.5 *$\Gamma \setminus \mathbb{H}^*$ is a compact* Riemann *surface if and only if there is a compact subset C of $\mathbb{H}^*$ such that $\mathbb{H}^* = \Gamma C$.*

Let

$$\overline{F} = \{\infty\} \bigcup \{z \in \mathbb{H} \mid |z| \geqslant 1, -1/2 \leqslant \mathrm{Re}(z) \leqslant 1/2\}.$$

Then $\overline{F}$ is a compact subset of $\mathbb{H}^*$ and $\mathbb{H}^* = SL_2(\mathbb{Z}) \cdot \overline{F}$. By the above lemma we know that $SL_2(\mathbb{Z}) \setminus \mathbb{H}^*$ is a compact Riemann surface and hence the modular group is a Fuchsian group of the first kind.

Let Γ be a Fuchsian group of the first kind. Γ' is a subgroup of Γ with $n = [\Gamma : \Gamma'] < \infty$. Then $\Gamma = \bigcup\limits_{i=1}^{n} \Gamma' \sigma_i$. By Lemma 3.5 there exists a compact subset C of $\mathbb{H}^*$ such that $\mathbb{H}^* = \Gamma C$. And hence

$$\mathbb{H}^* = \Gamma' \left(\bigcup_{i=1}^{n} \sigma_i C \right),$$

which implies that Γ' is of the first kind since $\bigcup\limits_{i=1}^{n} \sigma_i C$ is a compact subset of $\mathbb{H}^*$. This means that $\Gamma(N)$ and $\Gamma_0(N)$ are Fuchsian groups of the first kind.

Let Γ, Γ' be as above. Then they have the same cusp points and define the same $\mathbb{H}^*$. Let $v \in \mathbb{H}^*$. The set of the points Γ-equivalent to v are divided into finite Γ'-equivalence classes. Suppose the number of the Γ'-equivalence classes is h and $\omega_i (1 \leqslant i \leqslant h)$ is a system of representatives. Let φ' be the natural map $\mathbb{H}^* \to \Gamma' \setminus \mathbb{H}^*$. Then we can get a covering map f from $\Gamma' \setminus \mathbb{H}^*$ to $\Gamma \setminus \mathbb{H}^*$ as follows

$$\begin{array}{ccc} \mathbb{H}^* & \xrightarrow{\text{id.}} & \mathbb{H}^* \\ \varphi' \downarrow & & \downarrow \varphi \\ \Gamma' \setminus \mathbb{H}^* & \xrightarrow[f]{} & \Gamma \setminus \mathbb{H}^* \end{array}$$

It is clear that f is a holomorphic map. f maps $\varphi'(\omega_i)$ to $\varphi(v)$. Denote $q_i = \varphi'(\omega_i) \in \Gamma' \setminus \mathbb{H}^*$. Let u be the local coordinate at q_i and t the local coordinate at $\varphi(v)$. If

$$t(f(q)) = a_e(u(q))^e + a_{e+1}(u(q))^{e+1} + \cdots, \quad a_e \neq 0$$

for any point q in a neighborhood of q_i, then e is called the ramification index of f at q_i.

Since ω_i is Γ-equivalent to v, then there exists $\sigma_i \in \Gamma$ such that $\omega_i = \sigma_i(v)$. Denote $\overline{\Gamma} = \Gamma/(\Gamma \bigcap \{\pm I\})$.

Lemma 3.6 *Notations as above. Then the ramification index of f at q_i is*

$$e_i = [\overline{\Gamma}_{\omega_i} : \overline{\Gamma}'_{\omega_i}] = [\overline{\Gamma}_v : \sigma_i^{-1}\overline{\Gamma}'\sigma_i \bigcap \overline{\Gamma}_v], \quad 1 \leqslant i \leqslant h.$$

And $e_1 + e_2 + \cdots + e_h = [\overline{\Gamma} : \overline{\Gamma}']$. *That is, f is a covering with degree* $[\overline{\Gamma} : \overline{\Gamma}']$. *In particular, if* $\overline{\Gamma}'$ *is a normal subgroup of* $\overline{\Gamma}$, *then* $e_1 = e_2 = \cdots = e_h$ *and* $[\overline{\Gamma} : \overline{\Gamma}'] = e_1 h$.

Proof By the definition of the complex manifold, we know that $\lambda_i(z)^{[\overline{\Gamma}'_{\omega_i}:1]}$ is a local coordinate at q_i of $\Gamma \setminus \mathbb{H}^*$, where $\lambda_i(z) = (z-\omega_i)/(z-\overline{\omega_i})$. Similarly $\lambda_i(z)^{[\overline{\Gamma}_{\omega_i}:1]}$ is a local coordinate at $\varphi(\omega_i)$ of $\Gamma \setminus \mathbb{H}^*$. So the ramification index of f at q_i is $[\overline{\Gamma}_{\omega_i} : 1]/[\overline{\Gamma}'_{\omega_i} : 1] = e_i$. But $\overline{\Gamma}_{\omega_i} = \sigma_i \overline{\Gamma}_v \sigma_i^{-1}$, $\overline{\Gamma}'_{\omega_i} = \sigma_i \overline{\Gamma}_v \sigma_i^{-1} \bigcap \overline{\Gamma}'$. So

$$e_i = [\overline{\Gamma}_v : \overline{\Gamma}_v \bigcap \sigma_i^{-1}\overline{\Gamma}'\sigma_i].$$

We have a double coset decomposition of $\overline{\Gamma}$ as follows:

$$\overline{\Gamma} = \bigcup_{i=1}^{h} \overline{\Gamma}'\sigma_i\overline{\Gamma}_v.$$

In fact, for any $\sigma \in \overline{\Gamma}$, $\sigma(v)$ must be Γ'-equivalent to some ω_i. That is, there exist σ_i, $\sigma' \in \overline{\Gamma}'$ such that $\sigma(v) = \sigma'\sigma_i(v)$. So $(\sigma'\sigma_i)^{-1}\sigma \in \overline{\Gamma}_v$. Hence $\sigma \in \sigma'\sigma_i\overline{\Gamma}_v$. If $i \neq j$ and there is an element belonging to $\overline{\Gamma}'\sigma_i\overline{\Gamma}_v$ and $\overline{\Gamma}'\sigma_j\overline{\Gamma}_v$, then there are $\gamma_1, \gamma_2 \in \Gamma'$, $\delta_1, \delta_2 \in \overline{\Gamma}_v$ such that

$$\gamma_1\sigma_i\delta_1 = \gamma_2\sigma_j\delta_2.$$

Then

$$\gamma_1(\omega_i) = \gamma_1\sigma_i\delta_1(v) = \gamma_2\sigma_j\delta_2(v) = \gamma_2(\omega_j),$$

which is impossible because ω_i, ω_j are not Γ'-equivalent each other. This shows that $\overline{\Gamma}$ has such a decomposition. Now consider the number of the right cosets of $\overline{\Gamma}'$ in $\overline{\Gamma}'\sigma_i\overline{\Gamma}_v$. Let δ_1, $\delta_2 \in \overline{\Gamma}_v$. Then there exists $\gamma \in \overline{\Gamma}'$ such that $\sigma_i\delta_1 = \gamma\sigma_i\delta_2$ if and only if $\delta_1\delta_2^{-1} \in \sigma_i^{-1}\overline{\Gamma}'\sigma_i \bigcap \overline{\Gamma}_v$. Hence there are $[\overline{\Gamma}_v : \sigma_i^{-1}\overline{\Gamma}'\sigma_i \bigcap \overline{\Gamma}_v] = e_i$ right cosets of $\overline{\Gamma}'$ in $\overline{\Gamma}'\sigma_i\overline{\Gamma}_v$. Therefore

$$[\overline{\Gamma} : \overline{\Gamma}'] = e_1 + e_2 + \cdots + e_h,$$

which completes the proof. □

Lemma 3.7 *Let f be a covering with degree n from a compact* Riemann *surface R' to another compact* Riemann *surface R. Suppose that the genuses of R' and R are g' and g respectively. Then*

$$2g' - 2 = n(2g-2) + \sum_{z \in R'} (e_z - 1),$$

where e_z is the ramification index of f at $z \in R'$.

Proof This is the so-called Hurwitz formula. □

Theorem 3.5 *Let Γ be a subgroup of the modular group and $\mu = [\overline{SL_2(\mathbb{Z})} : \overline{\Gamma}]$. Denote the numbers of the equivalence classes of elliptic points with order 2 and 3 of Γ by ν_2 and ν_3 respectively. Let ν_∞ be the number of the equivalence classes of the cusp points of Γ. Then the genus of $\Gamma \setminus \mathbb{H}^*$ is*

$$g = 1 + \frac{\mu}{12} - \frac{\nu_2}{4} - \frac{\nu_3}{3} - \frac{\nu_\infty}{2}.$$

Proof Consider the ramification covering f with degree μ defined in Lemma 3.6: $f : \Gamma \setminus \mathbb{H}^* \to SL_2(\mathbb{Z}) \setminus \mathbb{H}^*$. If the ramification indexes of the inverse images in $\Gamma \setminus \mathbb{H}^*$ of $\varphi(\mathrm{e}^{2\pi\mathrm{i}/3}) \in SL_2(\mathbb{Z}) \setminus \mathbb{H}^*$ are $e_1, e_2, \cdots, e_t$ respectively, then $e_1 + e_2 + \cdots + e_t = \mu$. Each e_i is equal to 1 or 3. And ν_3 is just the number of $e_i = 1$. Put $\nu_3' = t - \nu_3$. By $\nu_3 + 3\nu_3' = \mu$, we have

$$\sum_{i=1}^{t} (e_i - 1) = 2\nu_3' = 2(\mu - \nu_3)/3.$$

Similarly, if the ramification indexes of the inverse images in $\Gamma \setminus \mathbb{H}^*$ of $\varphi(i) \in SL_2(\mathbb{Z}) \setminus \mathbb{H}^*$ are $e_1', e_2', \cdots, e_h'$ respectively, then $e_1' + e_2' + \cdots + e_h' = \mu$. Each e_i is equal to 1 or 2. And ν_2 is just the number of $e_i = 1$. And the others are of index 2. Hence

$$\sum_{i=1}^{h} (e_i' - 1) = (\mu - \nu_2)/2.$$

ν_∞ is the number of the inverse images of $\varphi(\infty)$ under f. Let their ramification indexes be e_1'', $\cdots$, e_{ν_∞}'' respectively. Then

$$\sum_{i=1}^{\nu_\infty} (e_i'' - 1) = \mu - \nu_\infty.$$

But $SL_2(\mathbb{Z}) \setminus \mathbb{H}^*$ is a sphere with genus 0, by Lemma 3.7 we have

$$2g - 2 = -2\mu + 2(\mu - \nu_3)/3 + (\mu - \nu_2)/2 + \mu - \nu_\infty,$$

which implies the result of the theorem. □

Example Let $N > 2$, then $\Gamma(N)$ has no elliptic points and $-I \notin \Gamma(N)$. So

$$[\overline{SL_2(\mathbb{Z})} : \overline{\Gamma(N)}] = [SL_2(\mathbb{Z}) : \Gamma(N)]/2.$$

By Lemma 3.1 we have

$$\mu_N = [\overline{SL_2(\mathbb{Z})} : \overline{\Gamma(N)}] = \begin{cases} \dfrac{N^3}{2} \displaystyle\prod_{p|N} (1 - p^{-2}), & \text{if } N > 2, \\ 6, & \text{if } N = 2. \end{cases}$$

By Theorem 3.4 we know that $\nu_\infty = \mu_N/N$. So the genus of $\Gamma(N)\setminus\mathbb{H}^*$ is

$$1 + \mu_N(N-6)/(12N), \quad N > 1. \tag{3.7}$$

For $\Gamma_0(N)$ we have

$$[\overline{SL_2(\mathbb{Z})} : \overline{\Gamma_0(N)}] = [SL_2(\mathbb{Z}) : \Gamma_0(N)] = N\prod_{p|N}(1+p^{-1}).$$

By Theorems 3.2, 3.3, 3.5, we can compute the genus of $\Gamma_0(N)\setminus\mathbb{H}^*$.

Chapter 4

Modular Forms with Integral Weight or Half-integral Weight

4.1 Dimension Formula for Modular Forms with Integral Weight

Let Γ be a Fuchsian group of the first kind. Then $M = \Gamma \backslash \mathbb{H}^*$ is a compact Riemann surface. Denote by K the field of all meromorphic functions on M. It is well-known that K is an algebraic function field over $\mathbb{C}$. Let $\varphi : \mathbb{H}^* \to M$ be the natural map. For $g \in K$ we call $f(z) = g(\varphi(z))$ an automorphic function on $\mathbb{H}$ which is a meromorphic function on $\mathbb{H}$. It is clear that $f(\gamma(z)) = f(z)$ for any $\gamma \in \Gamma$. Now we introduce a more wide range of functions on $\mathbb{H}$.

Let $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$. Put $J(\sigma, z) = cz + d$ for any $z \in \mathbb{H}$. It can be easily verified that

$$J(\sigma\sigma', z) = J(\sigma, \sigma'(z))J(\sigma', z)$$

for any $\sigma, \sigma' \in GL_2(\mathbb{R})$. For any integer k, $\sigma \in GL_2^+(\mathbb{R})$, any function f on $\mathbb{H}$ we define an operator as follows

$$f|[\sigma]_k = \det(\sigma)^{k/2} J(\sigma, z)^{-k} f(\sigma(z)).$$

It is clear that

$$f|[\sigma\sigma']_k = (f|[\sigma]_k)|[\sigma']_k, \quad \sigma, \sigma' \in GL_2^+(\mathbb{R}).$$

Definition 4.1 *Let k be an integer, f a complex function on $\mathbb{H}$. We call f an automorphic form with weight k for Γ if it satisfies the following three conditions*:

(1) *f is meromorphic on $\mathbb{H}$;*

(2) *for any $\gamma \in \Gamma$, we have $f|[\gamma]_k = f$;*

(3) *f is meromorphic at each cusp point of Γ.*

The set of all automorphic forms with weight k for Γ is denoted by $A_k(\Gamma)$ which is a vector space over $\mathbb{C}$.

We need to give some explanation for the third condition: let s be a cusp point, then there is $\rho \in SL_2(\mathbb{R})$ such that $\rho(s) = \infty$. Then

$$\rho \Gamma_s \rho^{-1} \cdot \{\pm I\} = \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^m \middle| m \in \mathbb{Z} \right\}, \quad h > 0,$$

where h is a positive real number and

$$\Gamma_s = \{\gamma \in \Gamma \mid \gamma(s) = s\}.$$

By the second condition we know that $f|[\rho^{-1}]_k$ is invariant under the action of $[\sigma]_k$ with $\sigma \in \rho\Gamma_s\rho^{-1}$. Put $w = \rho(z), g(w) = (f|[\rho^{-1}]_k)(w)$. Then

$$g \left| \left[\pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \right]_k \right. = (\pm 1)^k g(w + h) = g(w). \tag{4.1}$$

If k is even, by (4.1) we have

$$g(w + h) = g(w).$$

In this case, the third condition means that there exists a meromorphic function $\Phi(q), q = \mathrm{e}^{2\pi i w/h}$ at a neighbor of zero such that $g(w) = \Phi(q)$. If k is odd and $-I \in \Gamma$, then the second condition implies $f = 0$, there are no non-zero automorphic forms with weight k. So we always assume that $-I \notin \Gamma$ if k is odd. In this case, one of $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, -\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ does not belong to $\rho\Gamma_s\rho^{-1}$. If $\rho\Gamma_s\rho^{-1}$ is generated by $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$, s is called a regular cusp point. If $\rho\Gamma_s\rho^{-1}$ is generated by $-\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$, s is called an irregular cusp point. If s is regular, the meaning of the third condition is the same as the one for k even. If s is irregular, by (4.1) we have

$$g(w + 2h) = g(w).$$

In this case the third condition means that there is an odd meromorphic function ψ at a neighbor of zero such that

$$g(w) = \psi(\mathrm{e}^{\pi i w/h}).$$

It is easy to show that the above definition is independent on the choices of ρ. By the discussion above, $f|[\rho^{-1}]_k$ is a power series of $\mathrm{e}^{2\pi i w/h}$ or $\mathrm{e}^{\pi i w/h}$:

$$f|[\rho^{-1}]_k = \sum_{n \geqslant n_0} c_n \mathrm{e}^{2\pi i n w/h} \text{ or } \sum_{n \geqslant n_0} c_n \mathrm{e}^{\pi i n w/h},$$

which is called the Fourier expansion of f at the cusp point s. c_n is called its n-th Fourier coefficient. If $n_0 = 0$, then c_0 is called the value of f at the cusp point s which is independent on the choices of ρ.

$A_0(K)$ is just the field K of functions on M. If an automorphic form f is holomorphic on $\mathbb{H}$, and the Fourier coefficients of f at all cusp points satisfy that $c_n = 0$ for all $n < 0$, then f is called a holomorphic automorphic form. In particular, f is called a cusp form if the Fourier coefficients of f at all cusp points vanish for all $n \leqslant 0$. We denote by $G_k(\Gamma)$(or $S_k(\Gamma)$ respectively) the set of all holomorphic forms (or cusp forms respectively). If Γ is a congruence subgroup of the modular group, then an automorphic form of Γ is called a modular form.

It is clear that $fg \in A_{m+n}(\Gamma)$ if $f \in A_m(\Gamma)$ and $g \in A_n(\Gamma)$. Similar results hold for $G_n(\Gamma), S_n(\Gamma)$. Hence if $f, g \in A_n(\Gamma)$ and $g \neq 0$, then $f/g \in A_0(\Gamma) = K$. So $A_n(\Gamma)$ is a vector space over K with dimension one if $A_n(\Gamma) \neq 0$.

For any meromorphic function $f \in K$ on the Riemann surface M, define a divisor associated with f as follows

$$\mathrm{div}(f) = \sum_{p \in M} \nu_p(f) p,$$

where $\nu_p(f)$ is the order of f at p, and $\nu_p(f)$ is positive(or negative) if p is a zero (or pole) of f. Otherwise $\nu_p(f) = 0$.

For $F \in A_k(\Gamma)$, we denote by $\nu_{z-z_0}(F)$ the degree of the leading term of the expansion of F at $z_0 \in \mathbb{H}$ with respect to $z - z_0$. Put $p = \varphi(z_0)$. If p is not an elliptic point, let $\nu_p(F) = \nu_{z-z_0}(F)$. If p is an elliptic point with order e, put

$$\lambda(z) = \frac{z - z_0}{z - \overline{z_0}}.$$

Then $\lambda(z)^e$ is a local coordinate at p, so we put

$$\nu_p(F) = \nu_{z-z_0}(F)/e.$$

Now let $p = \varphi(s)$ be a cusp point and

$$F|[\rho^{-1}]_k = \begin{cases} \psi(q^{1/2}), & \text{if } k \text{ is odd and } s \text{ is irregular,} \\ \Phi(q), & \text{otherwise ,} \end{cases}$$

where $q = \mathrm{e}^{2\pi \mathrm{i} w/h}$ (with the definitions of w, h as above.) is a local coordinate at p. Put

$$\nu_p(F) = \begin{cases} \nu_t(\psi)/2, & \text{if } k \text{ is odd and } s \text{ is irregular,} \\ \nu_q(\Phi), & \text{otherwise ,} \end{cases}$$

where $t = q^{1/2}$. Since ψ is an odd function, $\nu_t(\psi)$ is an odd integer.

Let $\mathbb{D}$ be the group of all divisors of M. Put $\mathbb{D}_{\mathbb{Q}} = \mathbb{D} \otimes_{\mathbb{Z}} \mathbb{Q}$. For any $F \in A_k(\Gamma)$, define a divisor of $\mathbb{D}_{\mathbb{Q}}$ as follows

$$\mathrm{div}(F) = \sum_{p \in M} \nu_p(F) p \in \mathbb{D}_{\mathbb{Q}},$$

which is a finite sum because of the compactness of M.

Let $f \in A_0(\Gamma) = K$ not be a constant. Then

$$f(\gamma(z)) = f(z)$$

for all $\gamma \in \Gamma$. Taking the derivative with respect to z on both sides, we get

$$\frac{\mathrm{d}f}{\mathrm{d}z}(z) = J(\gamma, z)^{-2}\frac{\mathrm{d}f}{\mathrm{d}\gamma(z)}(\gamma(z)).$$

Let

$$F(z) = \frac{\mathrm{d}f}{\mathrm{d}z}(z).$$

Then we have

$$F|[\gamma]_2 = F.$$

If s is a cusp point of Γ, k is even(or odd but regular), we have a meromorphic function $\Phi(q)$ at $q = 0$ such that $f(\rho^{-1}(w)) = \Phi(q)$. Taking the derivative with respect to w on both sides, we get

$$\Phi'(q)q \cdot 2\pi\mathrm{i}/h = \frac{\mathrm{d}f}{\mathrm{d}z}(\rho^{-1}(w))\frac{\mathrm{d}\rho^{-1}(w)}{\mathrm{d}w} = F|[\rho^{-1}]_2.$$

If k is odd and s is irregular, we can get an expansion of $F|[\rho^{-1}]_2$ similarly. These show that $F \in A_2(\Gamma)$. $\mathrm{d}f$ is a meromorphic differential which is represented formally by $F(z)\mathrm{d}z$. Conversely, for any $F_1(z) \in A_2(\Gamma)$, we can regard $F_1(z)\mathrm{d}z$ as a meromorphic differential on M since

$$F_1(z)\mathrm{d}z = F_1(z)\left(\frac{\mathrm{d}f}{\mathrm{d}z}\right)^{-1}\mathrm{d}f = \frac{F_1(z)}{F(z)}\mathrm{d}f$$

and $F_1/F \in K, \mathrm{d}f$ is a meromorphic differential. Denote by $\mathrm{Dif}(M)$ the set of all meromorphic differentials on M which is a vector space over K with dimension 1. Let $\omega \in \mathrm{Dif}(M)$, then there exists $g \in K$ such that $\omega = g\mathrm{d}f = gF(z)\mathrm{d}z$ with $gF \in A_2(\Gamma)$, which shows that $F(z) \mapsto F(z)\mathrm{d}z$ is an isomorphism from $A_2(\Gamma)$ to $\mathrm{Dif}(M)$ as vector spaces over K.

For any meromorphic differential $\omega \in \mathrm{Dif}(M)$, define its divisor as follows:

$$\mathrm{div}(\omega) = \sum_{p \in M} \nu_p(\omega)p,$$

where $\nu_p(\omega) = \nu_t(\omega/\mathrm{d}t)$ if t is a local coordinate at p.

Define a graded algebra $\mathcal{D}$ with degree as follows

$$\mathcal{D} = \sum_{n=-\infty}^{\infty} \mathrm{Dif}^n(M)$$

satisfying the following conditions

(1) $\mathrm{Dif}^0(M) = K$, $\mathrm{Dif}^1(M) = \mathrm{Dif}(M)$;

(2) $\mathrm{Dif}^n(M)$ is a vector space over K with dimension 1 for any $n \in \mathbb{Z}$;

(3) $0 \neq \alpha\beta \in \mathrm{Dif}^{m+n}(M)$ for any $0 \neq \alpha \in \mathrm{Dif}^m(M), 0 \neq \beta \in \mathrm{Dif}^n(M)$.

It can be proved that the conditions determine uniquely the algebra $\mathcal{D}$. Taking $0 \neq \omega \in \mathrm{Dif}(M)$, each element of $\mathrm{Dif}^n(M)$ can be represented as $\xi = f\omega^n$ with $f \in K$. If $f \neq 0$, we define

$$\nu_p(\xi) = \nu_p(f) + n\nu_p(\omega) = \nu_p(\xi/\mathrm{d}t^n),$$

where t is a local coordinate at p. So for any $0 \neq \xi \in \mathrm{Dif}^n(M)$ we define the divisor of ξ as follows

$$\mathrm{div}(\xi) = \sum_{p\in M} \nu_p(\xi)p = \mathrm{div}(f) + n\mathrm{div}(\omega).$$

It is clear $\mathrm{div}(\xi\eta) = \mathrm{div}(\xi) + \mathrm{div}(\eta)$ for any ξ, $\eta \in \mathcal{D}$. Suppose the genus of M is g. Then it is well known that $\deg(\mathrm{div}(\omega)) = 2g - 2$, $\deg(\mathrm{div}(f)) = 0$. So $\deg(\mathrm{div}(\xi)) = n(2g - 2)$ if $0 \neq \omega \in \mathrm{Dif}^n(M)$.

Let $f \in K$ not be a constant. If $F(z) \in A_{2n}(\Gamma)$, then $F/(f')^n \in K$. So

$$F(z)(\mathrm{d}z)^n = (F/(f')^n)(\mathrm{d}f)^n \in \mathrm{Dif}^n(M).$$

Conversely, if $\eta \in \mathrm{Dif}^n(M)$, then there exists $g \in K$ such that $\eta = g\omega^n$ with $\omega = F_1(z)\mathrm{d}z$ and $F_1(z) \in A_2(\Gamma)$. Hence

$$\eta = gF_1^n(z)(\mathrm{d}z)^n$$

and $gF_1^n \in A_{2n}(\Gamma)$ which shows that $F(z) \to F(z)(\mathrm{d}z)^n$ is an isomorphism from $A_{2n}(\Gamma)$ to $\mathrm{Dif}^n(M)$. Let F_1, F_2 be two automorphic forms, then

$$\mathrm{div}(F_1F_2) = \mathrm{div}(F_1) + \mathrm{div}(F_2).$$

Let $D_1 = \sum\limits_p a_1(p)p$, $D_2 = \sum\limits_p a_2(p)p$ be two divisors of $\mathbb{D}_{\mathbb{Q}}$. Then we define that $D_1 \geqslant D_2$ if $a_1(p) \geqslant a_2(p)$ for every $p \in M$.

After introducing divisors associated with automorphic forms, we can give some equivalent definitions for holomorphic forms and cusp forms:

$$G_k(\Gamma) = \{F \in A_k(\Gamma) | \mathrm{div}(F) \geqslant 0\}$$

and

$$S_k(\Gamma) = \begin{cases} \left\{F \in A_k(\Gamma) \middle| \mathrm{div}(F) \geqslant \sum\limits_{j=1}^{u} Q_j + \sum\limits_{j=1}^{u'} Q_j'\right\}, & \text{if } k \text{ is even,} \\ \left\{F \in A_k(\Gamma) \middle| \mathrm{div}(F) \geqslant \sum\limits_{j=1}^{u} Q_j + \frac{1}{2}\sum\limits_{j=1}^{u'} Q_j'\right\}, & \text{if } k \text{ is odd,} \end{cases}$$

where $Q_1, \cdots, Q_u$ are all regular cusp points of Γ, $Q'_1, \cdots, Q'_{u'}$ are all irregular cusp points of Γ.

Lemma 4.1 *Let $P_1, \cdots, P_r$ be all elliptic points of $M = \Gamma\backslash\mathbb{H}^*$ with order $e_1, \cdots, e_r$ respectively, and Q_i, Q'_i as above. Let $0 \neq F \in A_k(\Gamma)$ (k is even). Put*

$$\eta = F(z)(\mathrm{d}z)^{k/2} \in \mathrm{Dif}^{k/2}(M).$$

Then

$$\mathrm{div}(F) = \mathrm{div}(\eta) + \frac{k}{2}\left(\sum_{i=1}^{r}(1-e_i^{-1})p_i + \sum_{j=1}^{u} Q_j + \sum_{j=1}^{u'} Q'_j\right),$$

$$\deg(\mathrm{div}(F)) = \frac{k}{2}\left(2g-2+\sum_{i=1}^{r}(1-e_i^{-1}) + u + u'\right).$$

And the second equality above holds also for k odd.

Proof Now we assume that k is even and $P \in M$. If $p = \varphi(z_0), z_0 \in \mathbb{H}$ and z_0 is not an elliptic point, then z is a local coordinate at p. So

$$\nu_p(\eta) = \nu_{z-z_0}(F(z)(\mathrm{d}z/\mathrm{d}t)^{k/2}) = \nu_p(F).$$

If z_0 is an elliptic point of Γ with order e, then

$$t = \lambda(z)^e = \left(\frac{z-z_0}{z-\overline{z_0}}\right)^e$$

is a local coordinate at p, and

$$\begin{aligned}\nu_p(\eta) &= \nu_t(F(z)(\mathrm{d}z/\mathrm{d}t)^{k/2}) = \nu_p(F(z)) - \frac{k}{2}\nu_t(\mathrm{d}t/\mathrm{d}z)\\ &= \nu_p(F) - \frac{k}{2}\nu_t\left(e\lambda(z)^{e-1}(z_0-\overline{z_0})(z-\overline{z_0})^{-2}\right)\\ &= \nu_p(F) + \frac{k}{2}(e^{-1}-1).\end{aligned}$$

If $p = \varphi(s), s$ is a cusp point of Γ, then $q = \mathrm{e}^{2\pi\mathrm{i}w/h}$ is a local coordinate at p with $w = \rho(z), \rho(s) = \infty$. We have

$$\begin{aligned}F(z)(\mathrm{d}z)^{k/2} &= F(\rho^{-1}(w))(\mathrm{d}z/\mathrm{d}w)^{k/2}(\mathrm{d}q/\mathrm{d}w)^{-k/2}(\mathrm{d}q)^{k/2}\\ &= F|[\rho^{-1}]_k(q\cdot 2\pi\mathrm{i}/k)^{-k/2}(\mathrm{d}q)^{k/2}\\ &= \Phi(q)(2\pi\mathrm{i}q/k)^{-k/2}(\mathrm{d}q)^{k/2}.\end{aligned}$$

Hence

$$\nu_p(\eta) = \nu_q(F(z)(\mathrm{d}z/\mathrm{d}q)^{k/2}) = \nu_q(\Phi(q)q^{-k/2}) = \nu_p(F) - k/2,$$

which implies the first result of the lemma. If k is even, by

$$\deg(\mathrm{div}(\eta)) = \frac{k}{2}(2g-2),$$

we get the second equality from the first one. If k is odd, applying the first equality to F^2 and noting that $\mathrm{div}(F) = 2^{-1}\mathrm{div}(F^2)$, we get the second equality for odd k. □

Now we introduce the definition of modular forms with half integral weight. We introduce an extension of the group $GL_2^+(\mathbb{R})$ as follows. Let

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}).$$

Take any holomorphic function $\varphi(z)$ on $\mathbb{H}$ such that

$$\varphi^2(z) = t\det(\alpha)^{-1/2}(cz+d),$$

where t is any complex number satisfying $|t| = 1$. Consider all pairs $\{\alpha, \varphi(z)\}$ and define a multiplication for these pairs:

$$\{\alpha_1, \varphi_1(z)\}\{\alpha_2, \varphi_2(z)\} = \{\alpha_1\alpha_2, \varphi_1(\alpha_2(z))\varphi_2(z)\}. \tag{4.2}$$

It is easy to verify that the set of all such pairs with the above multiplication forms a group which is denoted by $\widehat{G}$. There exists a natural projection P from $\widehat{G}$ to $GL_2^+(\mathbb{R})$:

$$P : \{\alpha, \varphi(z)\} \mapsto \alpha.$$

It is clear that $\mathrm{Ker}(P) = \{(I,t)||t| = 1\}$. For any odd integer k, any function $f(z)$ on $\mathbb{H}$ and any $\xi = \{\alpha, \varphi(z)\} \in \widehat{G}$, we define an operator

$$f|[\xi]_k = f(\alpha(z))\varphi(z)^{-k}.$$

It is easy to verify that

$$f|[\xi\eta]_k = (f|[\xi]_k)|[\eta]_k. \tag{4.3}$$

Let $\det\xi = \det\alpha$, define a subgroup $\widehat{G_1}$ of $\widehat{G}$:

$$\widehat{G_1} = \{\xi \in G | \det\xi = 1\}.$$

A subgroup Δ of $\widehat{G_1}$ is called a Fuchsian group of the first kind if it satisfies the following three conditions:

(1) $P(\Delta)$ is a discrete subgroup of $SL_2(\mathbb{R})$ and $P(\Delta)\setminus\mathbb{H}^*$ is a compact Riemann surface;

(2) P is a bijection from Δ to $P(\Delta)$, i.e., there is no element of the form $\{I, t\}$ ($|t| = 1$) in Δ except $\{I, 1\}$;

(3) If $-I \in P(\Delta)$, then $\{-I, 1\} \in \Delta$.

Let Δ be a Fuchsian group of the first kind. A meromorphic (or holomorphic respectively) function $f(z)$ on $\mathbb{H}$ is called an automorphic (or holomorphic respectively) form with weight $k/2$ for the group Δ if

(1) $f|[\xi]_k = f$ for all $\xi \in \Delta$;

(2) f is meromorphic (or holomorphic respectively) at all cusp points of $P(\Delta)$.

The set of all automorphic (or holomorphic respectively) forms is denoted by $A_{k/2}(\Delta)$ (or $G_{k/2}(\Delta)$).

Now we need to explain the meaning of the second condition. Let $\xi = \{\alpha, \varphi\} \in \Delta$, s be a cusp point of $P(\Delta)$. Put $\xi(s) = \alpha(s)$ and

$$\Delta_s = \{\xi \in \Delta | \xi(s) = s\}.$$

By Proposition 3.2, Δ_s is an infinite cyclic group or the product of an infinite cyclic group and $\{-I, 1\}$. Now let η be the generator of the cyclic group. Choose $\rho \in \widehat{G_1}$ such that $\rho(s) = \infty$. Since $P(\rho\eta\rho^{-1})$ is a parabolic element, we have

$$\rho\eta\rho^{-1} = \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, t \right\}, \quad |t| = 1.$$

Without loss of generality, we can assume that $h > 0$ (otherwise we substitute η by η^{-1}).

It is easy to verify that t is independent on choices of ρ. If s and s_1 are $P(\Delta)$-equivalent, let $s = \gamma(s_1)$, then $\gamma^{-1}\eta\gamma$ is a generator of the infinite cyclic part of Δ_{s_1} and $\rho\gamma(s_1) = s_1$. Since $\rho\gamma \cdot \gamma^{-1}\eta\gamma \cdot (\rho\gamma)^{-1} = \rho\eta\rho^{-1}$, t is independent of the choice of the representative of the equivalence class of the cusp point. By (4.3), we have

$$(f|[\rho^{-1}]_k)| \left[\left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, t \right\} \right]_k = f|[\rho^{-1}]_k,$$

that is, $f|[\rho^{-1}]_k(z+h) = t^k f|[\rho^{-1}]_k$. Hence $f|[\rho^{-1}]_k$ has the following expansion

$$f|[\rho^{-1}]_k = \sum_n c_n e((n+r)z/h),$$

where $e(r) = t^k$ $(0 \leqslant r < 1)$. Now the meaning of the condition (2) is that f is meromorphic (or holomorphic respectively) if and only if $c_n \neq 0$ for finitely many $n < 0$ (or $c_n = 0$ for all $n < 0$ respectively). Let $\nu_s(f)$ be the exponent $n + r$ of the leading term of the expansion above. Similarly to the case for integral weight, we can define divisors associated with modular forms with half integral weight. Let N be a positive integer with $4|N$. Define a map from $\Gamma_0(N)$ to $\widehat{G_1}$:

$$L : \gamma \mapsto \{\gamma, j(\gamma, z)\},$$

where $j(\gamma, z)$ is defined as in Chapter 1. It is clear that L is an imbedding from $\Gamma_0(N)$ into $\widehat{G_1}$ and $j(-I, z) = 1$. So $L(\Gamma_0(N))$ is a Fuchsian subgroup of $\widehat{G_1}$ with the first kind which is denoted by $\Delta_0(N)$. Put

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \middle| a \equiv d \equiv 1(N) \right\}.$$

It is clear that $\Delta_1(N) := L(\Gamma_1(N))$, $\Delta(N) := L(\Gamma(N))$ are Fuchsian groups of the first kind.

Let k be an integer, ω a character modulo N and $\omega(-1) = (-1)^k$. Let $A(N, k, \omega)$ be the set of functions on $\mathbb{H}$ satisfying

(1) f is meromorphic on $\mathbb{H}$;

(2) $f|[\gamma]_k = \omega(d)f$ for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$;

(3) f is meromorphic at each cusp point of $\Gamma_0(N)$.

Such a function f is called a modular form of weight k and with character (or Neben-type) ω for $\Gamma_0(N)$. Denote by $G(N, k, \omega)$ (and $S(N, k, \omega)$ respectively) the set of holomorphic (and cusp respectively) modular forms in $A(N, k, \omega)$.

In the remaining part of this chapter, we shall give some dimension formulae for $G(N, k, \omega)$, $S(N, k, \omega)$ with the aid of the Riemann-Roch Theorem which is formulated as follows.

Suppose that A is a divisor of a compact Riemann surface M, K is the field of meromorphic functions on M. Define

$$L(A) = \{f \in K | f = 0 \text{ or } \operatorname{div}(f) \geqslant -A\},$$

which is a vector space with finite dimension $l(A)$.

Theorem 4.1 (Riemann-Roch Theorem) *Let M be a compact* Riemann *surface with genus g, ω a non-zero differential on M. Then for any divisor A of M, we have*

$$l(A) = \deg(A) - g + 1 + l(\operatorname{div}(\omega) - A).$$

Let $f(z) \in G(N, k, \omega)$. It is easy to show that $\overline{f(-\overline{z})} \in G(N, k, \overline{\omega})$. So $G(N, k, \omega)$ and $G(N, k, \overline{\omega})$ have the same dimension. Also $S(N, k, \omega)$ and $S(N, k, \overline{\omega})$ have the same dimension. If $f \in A(N, k, \omega), g \in A(N, 2-k, \overline{\omega})$, then $fg \in A_2(\Gamma_0(N))$. Hence $\alpha = fg\mathrm{d}z$ is a differential on $\Gamma_0(N) \setminus \mathbb{H}^*$. By Lemma 4.1, we have

$$\operatorname{div}(\alpha) = \operatorname{div}(f) + \operatorname{div}(g) - \sum_p \left(1 - e_p^{-1}\right) p. \tag{4.4}$$

where p runs over $\Gamma_0(N) \setminus \mathbb{H}^*$, $e_p = \infty$ if p is cusp point.

For any $g' \in A(N, 2-k, \overline{\omega}), g/g' \in A_0(\Gamma_0(N))$, so $\nu_p(g) - \nu_p(g')$ is an integer for any p. Hence there exist μ'_p such that

$$0 \leqslant \mu'_p < 1, \quad \nu_p(g') \equiv \mu'_p \bmod \mathbb{Z}.$$

If p is an elliptic point, then $e_p\mu'_p$ is an integer, $\mu'_p \leqslant 1 - e_p^{-1}$. Put $\mu_p = 1 - e_p^{-1} - \mu'_p$, by (4.4) we have

$$0 \leqslant \mu_p \leqslant 1, \quad \nu_p(f) \equiv \mu_p \bmod \mathbb{Z}$$

for any $f \in A(N, k, \omega)$.

If p is a cusp point with $\mu'_p = 0$, then p is called a regular cusp point. Otherwise p is an irregular cusp point. This definition is relative to k. It is a generalization of the concept of regular cusp points in Definition 4.1.

Define two divisors in $\mathbb{D}_{\mathbb{Q}}$:

$$D_1 = -\sum_p \mu_p p, \quad D_2 = -\sum_p \mu'_p p.$$

By (4.4) we have

$$D_1 + \mathrm{div}(f) + D_2 + \mathrm{div}(g) = \mathrm{div}(\alpha). \tag{4.5}$$

$D_1 + \mathrm{div}(f)$ and $D_2 + \mathrm{div}(g)$ are divisors in $\mathbb{D}$. By the definitions of holomorphic and cusp forms we have

$$\dim G(N, 2-k, \omega) = l(D_2 + \mathrm{div}(g)), \quad \dim S(N, k, \omega) = l(D_1 + \mathrm{div}(f)).$$

By Riemann-Roch Theorem and (4.5) we have

$$\begin{aligned}
&\dim S(N, k, \omega) - \dim G(N, 2-k, \omega) \\
=&\deg(D_1 + \mathrm{div}(f)) - g + 1 \\
=&\frac{k-1}{2}(2g - 2 + \sum_p (1 - e_p^{-1})) + \sum_p \Big(\frac{1 - e_p^{-1}}{2} - \mu_p\Big) \\
=&\frac{k-1}{2}\mu(\Gamma_0(N) \setminus \mathbb{H}^*) + \sum_p \Big(\frac{1 - e_p^{-1}}{2} - \mu_p\Big),
\end{aligned} \tag{4.6}$$

where we used the following fact

$$\mu\big(\Gamma_0(N) \setminus \mathbb{H}^*\big) = \iint_{\Gamma_0(N)\setminus\mathbb{H}^*} y^{-2} \mathrm{d}x\mathrm{d}y = 2g - 2 + \sum_p (1 - e_p^{-1}),$$

whose proof can be found in G. Shimura, 1971.

Theorem 4.2 (Dimension formula for integral weight) *Let ω be a character modulo $N = \prod p^{r_p}$ and $\omega(-1) = (-1)^k$. Suppose that $F = \prod p^{s_p}$ is the conductor of ω. Then*

$$\begin{aligned}&\dim S(N,k,\omega) - \dim G(N,2-k,\omega)\\ &= \frac{(k-1)N}{12}\prod_{p|N}(1+p^{-1}) - \frac{1}{2}\prod_{p|N}\lambda(r_p,s_p,p)\\ &+ \nu_k \sum_{\substack{x \bmod N,\\ x^2\equiv -1(N)}} \omega(x) + \mu_k \sum_{\substack{x \bmod N,\\ x^2+x+1\equiv 0(N)}} \omega(x)\end{aligned}$$

where

$$\lambda(r_p,s_p,p) = \begin{cases} p^{r'}+p^{r'-1}, & if\ 2s_p \leqslant r_p = 2r'(r'\in\mathbb{Z}),\\ 2p^{r'}, & if\ 2s_p \leqslant r_p = 2r'+1(r'\in\mathbb{Z}),\\ 2p^{r_p-s_p}, & if\ 2s_p > r_p,\end{cases}$$

$$\nu_k = \begin{cases} 0, & if\ 2\nmid k,\\ -\dfrac{1}{4}, & if\ k\equiv 2(4),\\ \dfrac{1}{4}, & if\ k\equiv 0(4),\end{cases} \qquad \mu_k = \begin{cases} 0, & if\ k\equiv 1(3),\\ -\dfrac{1}{3}, & if\ k\equiv 2(3),\\ \dfrac{1}{3}, & if\ k\equiv 0(3).\end{cases}$$

Proof Since $\Gamma_0(1)\setminus\mathbb{H}^*$ has genus 0, one cusp point, one elliptic point with order 2, one elliptic point with order 3,

$$\mu\big(\Gamma_0(1)\setminus\mathbb{H}^*\big) = -2+1+(1-1/2)+(1-1/3) = 1/6.$$

Hence by Lemma 3.1 we have

$$\mu\big(\Gamma_0(N)\setminus\mathbb{H}^*\big) = [\Gamma_0(1):\Gamma_0(N)]\mu(\Gamma_0(1)\setminus\mathbb{H}^*) = \frac{N}{6}\prod_{p|N}(1+p^{-1})$$

and obtain the first term of the dimension formula by equality (4.6). Consider the second summation on the right side of (4.6). For the remaining part of this proof we write $\Gamma = \Gamma_0(N)$. Let $p = \varphi(s)$ with $s = d/c$ be a cusp point, where φ is the natural map from $\mathbb{H}^*$ to $\Gamma\setminus\mathbb{H}^*$. By Theorem 3.3 we can assume that c is a divisor of N and $(d, N/c) = 1$. Let $c = \prod p^{c_p}$ be the standard factorization of c. There exists $\rho = \begin{pmatrix} a & b\\ c & -d\end{pmatrix} \in \Gamma_0(1)$ such that $\rho(s) = \infty$. Take $\delta \in \Gamma_s$ corresponding to a generator of $\overline{\Gamma_s}$. Since $-I \in \Gamma$, we can assume

$$\rho\delta\rho^{-1} = \begin{pmatrix} 1 & h\\ 0 & 1\end{pmatrix}, \quad h > 0,$$

which is a generator of $\rho\overline{\Gamma_s}\rho^{-1}$. Hence

$$\delta = \rho^{-1}\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}\rho = \begin{pmatrix}1-hcd & hd^2\\ -hc^2 & 1+hcd\end{pmatrix} \in \varGamma.$$

h should be the smallest positive integer such that $N|hc^2$. So $h = \dfrac{N}{c(c,N/c)}$. Since

$$(f|[\rho^{-1}]_k)|\left[\begin{pmatrix}1 & h\\ 0 & 1\end{pmatrix}\right]_k = (f|[\delta]_k)|[\rho^{-1}]_k = \omega(1+hcd)f|[\rho^{-1}]_k,$$

then

$$f|[\rho^{-1}]_k = c_n \mathrm{e}^{2\pi \mathrm{i}(n+r)z/h} + \cdots, \quad c_n \neq 0,$$

where $\omega(1+hcd) = \mathrm{e}^{2\pi \mathrm{i} r}, 0 < r \leqslant 1$. Hence $\mu_p = r$.

For any factor c of N, put

$$f_c = \sum_{s=d/c}\left(\frac{1}{2} - \mu_{\varphi(s)}\right),$$

where d runs over $(\mathbb{Z}/(c,N/c)\mathbb{Z})^*$ and $(d,c)=1$.

If $F|N/(c,N/c)$, then

$$\omega(1+hcd) = \omega(1+dN/(c,N/c)) = 1$$

and

$$\mu_{\varphi(d/c)} = 1, \quad f_c = -2^{-1}\varphi((c,N/c)).$$

If $F \nmid N/(c,N/c)$ and $(d,(c,N/c)) = 1$, then

$$\omega(1+dN/(c,N/c)) \neq 1.$$

In fact, if otherwise, there exists d_0 such that $(d_0,(c,N/c)) = 1$ and

$$\omega(1+d_0N/(c,N/c)) = 1.$$

Since $(c,N/c)^2|N$, for any integer m, we have

$$(1+d_0N/(c,N/c))^m \equiv 1 + md_0N/(c,N/c) \bmod N.$$

So $\omega(1+md_0N/(c,N/c)) = 1$. Since $(d_0,(c,N/c)) = 1$, there exists m_0 such that $m_0d_0 \equiv 1 \bmod (c,N/c)$. This means that for any integer m, we have

$$\omega(1+mN/(c,N/c)) = 1,$$

which induces $F|N/(c,N/c)$. This contradicts the assumption $F \nmid N/(c,N/c)$.

Now take d' such that $(d',c) = 1$, $d' \equiv -d \pmod{(c,N/c)}$. Put $p' = \varphi(d'/c)$. Then

$$\omega(1+d'N/(c,N/c)) = \overline{\omega}(1+dN/(c,N/c)) \neq 1.$$

If $(c, N/c) \neq 2$, then p and p' are different cusp points on $\Gamma \setminus \mathbb{H}^*$, $\mu_p + \mu_{p'} = 1$ and $f_c = 0$. If $(c, N/c) = 2$, then $\omega(1 + N/2) = -1$, $\mu_p = 1/2$ and $f_c = 0$. Hence we have $f_c = 0$ if $F \nmid N/(c, N/c)$. Therefore if p runs over all cusp points, we have

$$\sum_{p:\text{cusp points}} \left(\frac{1}{2} - \mu_p\right) = -\frac{1}{2} \sum_{(c,N/c)|N/F} \varphi((c, N/c))$$
$$= -\frac{1}{2} \prod_{p|N} \left(\sum_{\substack{c_p=0,\\ \min\{c_p, r_p - c_p\} \leqslant r_p - s_p}}^{r_p} \varphi((p^{c_p}, p^{r_p - c_p})) \right). \tag{4.7}$$

If $s_p \leqslant r_p/2$, the summation in the product of (4.7) is

$$\sum_{c_p=0}^{r_p} \varphi((p^{c_p}, p^{r_p - c_p})) = \begin{cases} p^{r'} + p^{r'-1}, & \text{if } r_p = 2r', r' \in \mathbb{Z}, \\ 2p^{r'}, & \text{if } r_p = 2r' + 1, r' \in \mathbb{Z}. \end{cases}$$

If $s_p > r_p/2$, then the summation is

$$\sum_{c_p=0}^{r_p - s_p} \varphi(p^{c_p}) + \sum_{c_p=s_p}^{r_p} \varphi(p^{r_p - c_p}) = 2 \sum_{c_p=0}^{r_p - s_p} \varphi(p^{c_p}) = 2p^{r_p - s_p},$$

which gives the second term of the dimension formula.

Now suppose that p is an elliptic point with order e. Let $z_0 \in \mathbb{H}$ with $p = \varphi(z_0)$. There exists $\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ such that $\beta(z_0) = z_0$ and β corresponds to a generator of $\overline{\Gamma}_{z_0}$. Take

$$\lambda = \begin{pmatrix} 1 & -z_0 \\ 1 & -\overline{z}_0 \end{pmatrix}.$$

Then $\lambda(z_0) = 0$ and

$$\lambda \beta \lambda^{-1} = \begin{pmatrix} c\overline{z}_0 + d & 0 \\ 0 & cz_0 + d \end{pmatrix}. \tag{4.8}$$

Since e is the smallest positive integer such that $\beta^e = \pm I$, so $(cz_0 + d)^2$ is an e-th primitive root of unity.

Let

$$f(z) = c_n (z - z_0)^n + \cdots, \quad c_n \neq 0$$

be the expansion of $f(z) \in A(N, k, \omega)$ at $z = z_0$. Noting that

$$\beta(z) - z_0 = \beta(z) - \beta(z_0) = \frac{z - z_0}{(cz + d)(cz_0 + d)}$$

and

$$f(\beta(z)) = \omega(d)(cz + d)^k f(z),$$

we have

$$c_n(cz+d)^{-n}(cz_0+d)^{-n}(z-z_0)^n+\cdots=\omega(d)(cz+d)^k c_n(z-z_0)^n+\cdots.$$

So

$$\omega(d)(cz_0+d)^k=(cz_0+d)^{-2n}=(cz_0+d)^{-2e\mu_p},\tag{4.9}$$

where we used the facts that $\nu_p(f)=n/e\equiv\mu_p \bmod \mathbb{Z}$ and $(cz_0+d)^2$ is a root of unity with degree e.

$\Gamma_0(N)$ has only elliptic points of order 2 or 3. We first assume that $e=2$ and β is conjugate to $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ in the modular group. That is, there exists $\gamma\in SL_2(\mathbb{Z})$ such that

$$\beta=\gamma\begin{pmatrix}0&-1\\1&0\end{pmatrix}\gamma^{-1},$$

It is clear that $\gamma(\mathrm{i})=z_0$ which is the fixed point of β in $\mathbb{H}$. Hence $\lambda\gamma(\mathrm{i})=0,\lambda\gamma(-\mathrm{i})=\infty$. So

$$\lambda\gamma=\begin{pmatrix}u&0\\0&v\end{pmatrix}\begin{pmatrix}1&-\mathrm{i}\\1&\mathrm{i}\end{pmatrix},\quad u,v\in\mathbb{C}$$

and

$$\lambda\beta\lambda^{-1}=\begin{pmatrix}-\mathrm{i}&0\\0&\mathrm{i}\end{pmatrix}.$$

By (4.8), we see that $cz_0+d=\mathrm{i}$. Since

$$-I=\beta^2=\begin{pmatrix}a^2+bc&ab+cd\\ac+dc&bc+d^2\end{pmatrix},$$

we know that $d^2+1\equiv 0(\mathrm{mod}N)$. Thus

$$\omega(d)^2=\omega(-1)=(-1)^k.\tag{4.10}$$

Let z_0' be another elliptic point with order 2 of Γ, $p'=\varphi(z_0')$, $\beta'=\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}$ corresponds to a generator of $\overline{\Gamma}_{z_0'}$ and β' conjugate to $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ in the modular group. We can similarly prove that $(d')^2+1\equiv 0(\mathrm{mod}N)$ and $c'z_0'+d'=\mathrm{i}$. If z_0',z_0 are Γ-equivalent, by the proof of Theorem 3.2, we know that β,β' are conjugate in Γ which implies that $d\equiv d'(\mathrm{mod}N)$. That is, they correspond to the same solution of the congruence equation:

$$x^2+1\equiv 0(\mathrm{mod}N).\tag{4.11}$$

The number ν_2 of elliptic points with order 2 on $\Gamma\setminus\mathbb{H}^*$ is just the one of solutions of (4.11). So there is a bijection between the set of elliptic points on $\Gamma\setminus\mathbb{H}^*$ and the set of solutions of the equation (4.11).

We first consider the case that k is odd. If $N \leqslant 2$, (4.11) has only one solution $d \equiv 1(\mathrm{mod} N)$. By (4.10) it is impossible for k to be an odd. So we have $N > 2$. Suppose that d is a solution of (4.11). Put $d' \equiv -d(\mathrm{mod} N)$. d' is also a solution of (4.11) and d, d' correspond to different elliptic points p, p'. By (4.10), without loss of generality, we can assume that $\omega(d) = \mathrm{i}, \omega(d') = -\mathrm{i}$. By (4.9), we have

$$\mathrm{i}^{k+1} = (-1)^{2\mu_p}, \quad -\mathrm{i}^{k+1} = (-1)^{2\mu_{p'}}.$$

These imply that $\mu_p = 0, \mu_{p'} = 1/2$ or $\mu_p = 1/2$, $\mu_{p'} = 0$. So the two terms in the summation of (4.6) corresponding to p, p' counteract each other. If k is even, (4.10) means that $\omega(d) = \pm 1$. By (4.9), if $\omega(d) = 1$, then

$$\mu_p = \begin{cases} 0, & \text{if } k \equiv 0(4), \\ 1/2, & \text{if } k \equiv 2(4). \end{cases}$$

If $\omega(d) = -1$, then

$$\mu_p = \begin{cases} 1/2, & \text{if } k \equiv 0(4), \\ 0, & \text{if } k \equiv 2(4). \end{cases}$$

So $1/4 - \mu_p = \nu_k \omega(d)$ which gives the third term of the equality of Theorem 4.2.

Finally we consider the case $e = 3$. By Theorem 3.2, $9 \nmid N$. Suppose that β is conjugate to $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ in the modular group. Then there is $\gamma \in SL_2(\mathbb{Z})$ such that $\beta = \gamma \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \gamma^{-1}$. Put $\rho = \mathrm{e}^{2\pi \mathrm{i}/3}$. It is easy to see $\gamma(-\rho) = z_0$ and $\lambda\gamma(-\rho) = 0$, $\lambda\gamma(-\overline{\rho}) = \infty$. Hence

$$\lambda\gamma = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 1 & \rho \\ 1 & \overline{\rho} \end{pmatrix}, \quad u, v \in \mathbb{C}$$

and

$$\lambda\beta\lambda^{-1} = \begin{pmatrix} \overline{\rho^2} & 0 \\ 0 & \rho^2 \end{pmatrix}.$$

By (4.8) we have that $cz_0 + d = \rho^2$. By $\beta^3 = I$ we get $d^3 \equiv 1(\mathrm{mod} N)$. We want to prove that d satisfies the following congruence equation:

$$x^2 + x + 1 \equiv 0(\mathrm{mod} N). \tag{4.12}$$

Suppose that q is a prime factor of $(d-1, N)$. Since $ad \equiv 1(\mathrm{mod} q), \mathrm{tr}(\beta) = a+d = \pm 1$, then $a + d \equiv 2 \equiv \pm 1(\mathrm{mod} q)$ and q must be 3. It is clear that $d^2 + d + 1 \equiv 0(\mathrm{mod} 3)$ holds for all d prime to 3. Since $9 \nmid N$, it shows that d is a solution of (4.12). The number of elliptic points with order 3 on $\Gamma \setminus \mathbb{H}^*$ is equal to the number of solutions of (4.12). Similar to the proof for the case $e = 2$, we can prove that there is a bijection between the set of elliptic points of order 3 on $\Gamma \setminus \mathbb{H}^*$ and the set of solutions of

(4.11). Let d be a solution of (4.12). Put $d' \equiv d^{-1}(\mathrm{mod} N)$. If $d \equiv d'(\mathrm{mod} N)$, then $d^3 \equiv d^2 \equiv 1(\mathrm{mod} N)$. So $d \equiv 1(\mathrm{mod} N)$ which shows that $N = 1$ or 3 by (4.9) . It is obvious that $\omega(d) = 1$. By (4.9) we get $\rho^{2k} = \rho^{6\mu_p}$. p is the unique elliptic point with order 3, so

$$\mu_p = \begin{cases} 0, & \text{if } k \equiv 0(3), \\ 1/3, & \text{if } k \equiv 1(3), \\ 2/3, & \text{if } k \equiv 2(3), \end{cases}$$

which implies that $1/3 - \mu_p = \mu_k$.

Now let $N \neq 1$ or 3. Then $d \not\equiv d'(\mathrm{mod} N)$. Suppose that the elliptic points corresponding to d, d' are p, p' respectively. Without loss of generality, we assume that $\omega(d) = \rho, \omega(d') = \rho^2$. By (4.9) we get

$$\rho^{2k+1} = \rho^{6\mu_p}, \quad \rho^{2k+2} = \rho^{6\mu_{p'}}.$$

Hence

$$\mu_p = \begin{cases} 2/3, & \text{if } k \equiv 0(3), \\ 0, & \text{if } k \equiv 1(3), \\ 1/3, & \text{if } k \equiv 2(3) \end{cases}$$

and

$$\mu_{p'} = \begin{cases} 1/3, & \text{if } k \equiv 0(3), \\ 2/3, & \text{if } k \equiv 1(3), \\ 0, & \text{if } k \equiv 2(3). \end{cases}$$

Therefore

$$(1/3 - \mu_p) + (1/3 - \mu_{p'}) = -\mu_k = \mu_k(\omega(d) + \omega(d')),$$

which completes the proof of the theorem. □

Proposition 4.1 *Let k be a negative integer, Γ be a Fuchsian group of the first kind. Then*

$$\dim G_k(\Gamma) = 0.$$

Proof Take a non-zero element $F_0 \in A_k(\Gamma)$, then

$$G_k(\Gamma) = \{fF_0 | f \in A_0(\Gamma), \mathrm{div}(fF_0) \geqslant 0\}.$$

If $\mathrm{div}(F_0) = \sum \nu_p p \in \mathbb{D}_{\mathbb{Q}}$, define divisor $[\mathrm{div}(F_0)] := \sum [\nu_p] p$. Then

$$\dim G_k(\Gamma) = l([\mathrm{div}(F_0)]).$$

By Lemma 4.1 and

$$\mu(\Gamma \setminus \mathbb{H}^*) = 2g - 2 + \sum_p (1 - e_p^{-1}),$$

we have

$$\deg([\mathrm{div}(F_0)]) \leqslant \deg(\mathrm{div}(F_0)) = \mu(\Gamma \setminus \mathbb{H}^*) \cdot k/2 < 0.$$

Therefore $\dim G_k(\Gamma) = 0$. □

By Theorem 4.2 and Proposition 4.1, we can get the formulae for $G(N,k,\omega)$, $S(N,k,\omega)$ for all $k \geqslant 2$ since $G(N,k,\omega) \subset G_k(\Gamma(N))$ and $\dim G(N,0,\text{id.}) = \dim G_0(\Gamma_0(N)) = 1$. And we can get also

$$\dim G(N,k,\omega) - \dim S(N,k,\omega) = \sum_{(c,N/c)|N/F} \varphi((c,N/c)), \text{ if } k \geqslant 3 \text{ or } k=2, \omega \neq \text{id.}, \tag{4.13}$$

$$\dim G(N,2,\text{id.}) - \dim S(N,2,\text{id.}) = \sum_{(c,N/c)|N/F} \varphi((c,N/c)) - 1 \tag{4.14}$$

and

$$\dim G(N,1,\omega) - \dim S(N,1,\omega) = \frac{1}{2} \sum_{(c,N/c)|N/F} \varphi((c,N/c)). \tag{4.15}$$

4.2 Dimension Formula for Modular Forms with Half-Integral Weight

For the remaining part of this chapter, we consider the dimension formula for modular forms with half integral weight.

Let k be an odd integer, N a positive integer with $4|N$ and ω a character modulo N. A holomorphic function on $\mathbb{H}$ is called a holomorphic modular form of $\Gamma_0(N)$ with weight $k/2$ and character ω if

(1) for any $\xi = \{\gamma, j(\gamma,z)\} \in \Delta_0(N)$, we have

$$f|[\xi]_k = \omega(d_\gamma)f, \quad \gamma = \begin{pmatrix} * & * \\ * & d_\gamma \end{pmatrix} \in \Gamma_0(N);$$

(2) $f(z)$ is holomorphic at all cusp points of $\Gamma_0(N)$.

The set of all such modular forms is denoted by $G(N,k/2,\omega)$. The constant term of the expansion of f at a cusp point p is called the value of f at p. $f(z) \in G(N,k/2,\omega)$ is called a cusp form if $\nu_s(f) > 0$ for any cusp point s of $\Gamma_0(N)$. The set of all such cusp forms is denoted by $S(N,k/2,\omega)$. We shall compute the dimensions of $G(N,k/2,\omega), S(N,k/2,\omega)$.

Since $\{-I,1\} \in \Delta_0(N)$, if ω is an odd character modulo N, then

$$f|[\{-I,1\}]_k = \omega(-1)f,$$

which implies that $f = 0$. So we must assume that ω is an even character modulo N. From the proof of (4.6) we know that the equality (4.6) holds also if the weight k is substituted by $k/2$. Since $4|N$, we know that $\Gamma_0(N)$ has no elliptic points by Theorem 3.2. So we have

$$\dim S(N,k/2,\omega) - \dim G(N,2-k/2,\omega) = \frac{k-2}{4}\mu(\Gamma_0(N)\backslash\mathbb{H}^*) + \sum_p (1/2 - \mu_p), \tag{4.16}$$

where p runs over all cusp points on $\Gamma_0(N)\setminus\mathbb{H}^*$. For any $f\in G(N,k/2,\omega)$ we have that $\nu_p(f)\equiv\mu_p \pmod{\mathbb{Z}}$ and $0<\mu_p\leqslant 1$.

Let F be the conductor of ω, $N=\prod p^{r_p}, F=\prod p^{s_p}$ be the standard factorizations of N, F respectively. We define the following condition:

$$\text{there is a prime factor } p \text{ of } N \text{ such that } p\equiv 3(4),\ r_p \text{ is odd or } 0<r_p<2s_p. \tag{4.17}$$

If (4.17) does not hold, and p is a prime factor of N with $p\equiv 3(4)$, then r_p must be an even integer and $r_p\geqslant 2s_p$.

Lemma 4.2 *Let n,p,q be positive integers with $n>1, p<q$. Then*

$$\sum_{r=0,(r,n)=1}^{n-1}\left\{\frac{p}{q}+\frac{r}{n}\right\}=\frac{\varphi(n)}{2}-\sum_{d|n}\mu(d)\left\{\frac{(q-p)n}{qd}\right\},$$

where $\{x\}$ is the fractional part of x, i.e., $\{x\}=x-[x]$.

Proof We have

$$\begin{aligned}\sum_{\substack{r=0,\\(r,n)=1}}^{n-1}\left\{\frac{p}{q}+\frac{r}{n}\right\}&=\sum_{r=0}^{n-1}\left\{\frac{p}{q}+\frac{r}{n}\right\}\sum_{d|(r,n)}\mu(d)\\
&=\sum_{d|n}\mu(d)\sum_{r=0}^{n/d-1}\left\{\frac{p}{q}+\frac{rd}{n}\right\}\\
&=\sum_{d|n}\mu(d)\left[\sum_{r=0}^{n/d-1}\left(\frac{p}{q}+\frac{rd}{n}\right)-\left(\frac{n}{d}-1-\frac{(q-p)n}{qd}+\left\{\frac{(q-p)n}{qd}\right\}\right)\right]\\
&=\sum_{d|n}\mu(d)\left[\frac{1}{2}(n/d+1)-\left\{\frac{(q-p)n}{qd}\right\}\right]\\
&=\frac{\varphi(n)}{2}-\sum_{d|n}\mu(d)\left\{\frac{(q-p)n}{qd}\right\},\end{aligned}$$

which completes the proof. □

For an odd integer n we define $\chi_2(n)=\left(\dfrac{-1}{n}\right)$.

Lemma 4.3 *Let n, k be positive odd integers. Suppose n has ν prime factors which are all congruent to 3 modulo 4. Then*

$$\sum_{d|n}\mu(d)\left\{\frac{kn}{4d}\right\}=-2^{\nu-2}\chi_2(kn).$$

Proof We only consider the case $k \equiv n \equiv 1(4)$. Other cases can be proved similarly. We have

$$\sum_{d|n} \mu(d) \left\{ \frac{kn}{4d} \right\} = \frac{1}{4} - \frac{3}{4}\binom{\nu}{1} + \frac{1}{4}\binom{\nu}{2} - \frac{3}{4}\binom{\nu}{3} + \cdots$$
$$= -\frac{1}{2}\left[1 + \binom{\nu}{2} + \binom{\nu}{4} + \cdots\right] = -2^{\nu-2},$$

which completes the proof. □

Theorem 4.3 (Dimension Formula for Half Integral Weight) *We have*

$$\dim S(N, k/2, \omega) - \dim G(N, 2-k/2, \omega) = \frac{k-2}{24} N \prod_{p|N} (1+p^{-1}) - \frac{\zeta}{2} \prod_{p|N, p\neq 2} \lambda(r_p, s_p, p),$$

where $\lambda(r_p, s_p, p)$ *is defined as in* Theorem 4.2, *and* ζ *is defined as follows*: *if* $r_2 \geqslant 4, \zeta = \lambda(r_2, s_2, 2)$; *if* $r_2 = 3, \zeta = 3$; *if* $r_2 = 2$ *and the condition* (4.17) *holds*, $\zeta = 2$; *if* $r_2 = 2$ *and the condition* (4.17) *does not hold*, *then*

$$\zeta = \begin{cases} 3/2, & \text{if } s_2 = 0 \textit{ and } k \equiv 1(4), \\ 5/2, & \text{if } s_2 = 2 \textit{ and } k \equiv 1(4), \\ 5/2, & \text{if } s_2 = 0 \textit{ and } k \equiv 3(4), \\ 3/2, & \text{if } s_2 = 2 \textit{ and } k \equiv 3(4). \end{cases}$$

Proof We only need to calculate the sum in the equality (4.16). Let M be the sum, $s = d/c$ a cusp point of $\Gamma_0(N)$ and c a positive factor of N. Put

$$f_c = \sum_{s=d/c} (1/2 - \mu_{\varphi(s)}),$$

where d runs over $(\mathbb{Z}/(c, N/c)\mathbb{Z})^*$ and $(d, c) = 1$, φ is the natural map $\mathbb{H}^* \to \Gamma_0(N) \setminus \mathbb{H}^*$. Hence

$$M = \sum_{c|N} f_c.$$

Take

$$\rho = \begin{pmatrix} a & b \\ c & -d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

It is clear that $\rho(d/c) = \infty$. Let δ be a generator of $\overline{\Gamma_s}$ where

$$s = d/c, \quad \Gamma_s = \{\gamma \in \Gamma_0(N) | \gamma(s) = s\}.$$

Since $-I \in \Gamma_0(N)$, we can assume that

$$\rho\delta\rho^{-1} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, \quad h > 0.$$

Hence

$$\delta = \rho^{-1} \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \rho = \begin{pmatrix} 1 - hcd & hd^2 \\ -hc^2 & 1 + hcd \end{pmatrix} \in \Gamma_0(N).$$

This implies that $h = N/(c(c, N/c))$ since h is the smallest positive integer such that $N|hc^2$. Put

$$\rho^* = \{\rho, (cz - d)^{1/2}\} \in \widehat{G_1}.$$

Then

$$\rho^* L(\delta)(\rho^*)^{-1} = \left\{ \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, \varepsilon_{1+hcd}^{-1} \left(\frac{-h}{1 + hcd} \right) \right\}.$$

Suppose $f \in G(N, k/2, \omega)$. Since

$$(f|[(\rho^*)^{-1}]_k)|[\rho^* L(\delta)(\rho^*)^{-1}]_k = \omega(1 + hcd) f|[(\rho^*)^{-1}]_k,$$

we know that $\nu_p(f) \equiv \mu_p \bmod \mathbb{Z}, 0 < \mu_p \leqslant 1$ where $p = \varphi(s)$ and μ_p is determined by

$$e(\mu_p) = \omega(1 + hcd)\varepsilon_{1+hcd}^{-k} \left(\frac{-h}{1 + hcd} \right).$$

We denote by $\psi(d/c)$ the right side of the above equality. Let $c = \prod p^{c_p}$ be the standard factorization of c. A direct computation shows that

$$\varepsilon_{1+hcd}^{-k} = \begin{cases} i^{-k}, & \text{if } r_2 = 2, c_2 = 1, \\ 1, & \text{otherwise,} \end{cases}$$

and

$$\left(\frac{-h}{1 + hcd} \right) = \begin{cases} 1, & \text{if } r_2 \geqslant 4, \\ 1, & \text{if } r_2 = 3, c_2 = 0, 2, 3, \\ -1, & \text{if } r_2 = 3, c_2 = 1, \\ 1, & \text{if } r_2 = 2, c_2 = 0, 2, \\ -1, & \text{if } r_2 = 2, c_2 = 1, h \equiv 1(4), \\ 1, & \text{if } r_2 = 2, c_2 = 1, h \equiv 3(4). \end{cases}$$

We now calculate M according to different cases:

(1) If $r_2 \geqslant 4$, then $\psi(d/c) = \omega(1 + hcd)$, similar to the proof of (4.7), we have

$$f_c = \begin{cases} 0, & \text{if } (c, N/c) \nmid N/F, \\ -\dfrac{1}{2}\varphi(c, N/c), & \text{if } (c, N/c)|N/F. \end{cases} \tag{4.18}$$

Hence

$$M = -\frac{1}{2} \sum_{c|N,(c,N/c)|N/F} \varphi(c, N/c) = -\frac{1}{2} \prod_{p|N} \lambda(r_p, s_p, p)$$

just as in the proof of Theorem 4.2.

(2) Suppose $r_2 = 3$. If $c_2 = 0, 2, 3$, then $\psi(d/c) = \omega(1 + hcd)$ and (4.18) holds yet. If $c_2 = 1$, then $\psi(d/c) = -\omega(1 + hcd)$. If $F|N/(c, N/c)$, then $\psi(d/c) = -\omega(1 + hcd) = -1$ and $f_c = 0$. Now suppose $F \nmid N/(c, N/c)$. If $\psi(d/c) = 1$, i.e., $\omega(1 + dN/(c, N/c)) = -1$, then

$$\omega(1 + 2dN/(c, N/c)) = 1,$$

which implies that $F|2N/(c, N/c)$ since $(d, (c, N/c)) = 1$. Hence $(c, N/c)|2N/F$. But $(c, N/c) \nmid N/F$, so $2 \nmid N/F$. If $2 \nmid N/F$ and $2^{-1}(c, N/c)|N/F$, then for any d we have $\psi(d/c) = 1$, and hence in this case

$$f_c = -\frac{1}{2}\varphi(c, N/c).$$

If $2|N/F$, taking d' such that $d' \equiv -d \pmod{(c, N/c)}$ and $(d', c) = 1$, we see that $\psi(d/c) = \overline{\psi}(d'/c) \neq 1$. Since $2|N/F$ and $(c, N/c) \nmid N/F$, we see that $(c, N/c) \neq 2$ and $\varphi(d/c), \varphi(d'/c)$ are different cusp points on $\Gamma_0(N) \setminus \mathbb{H}^*$. Hence $f_c = 0$. Therefore we have

$$\begin{aligned} M &= \sum_{\substack{(c,N/c)|N/F,\\ c_2=0,2,3}} f_c + \sum_{\substack{2^{-1}(c,N/c)|N/F,\\ c_2=1}} f_c \\ &= -\frac{3}{2} \prod_{p|N, p\neq 2} \lambda(r_p, s_p, p), \qquad \text{if } 2 \nmid N/F, \end{aligned}$$

and

$$M = \sum_{\substack{(c,N/c)|N/F,\\ c_2=0,2,3}} f_c = -\frac{3}{2} \prod_{p|N, p\neq 2} \lambda(r_p, s_p, p), \quad \text{if } 2|N/F.$$

(3) Suppose $r_2 = 2$. If $c_2 = 0, 2$, then $\psi(d/c) = \omega(1+hcd)$ and (4.18) holds. Hence

$$\sum_{c|N, c_2=0,2} f_c = \sum_{\substack{(c,N/c)|N/F,\\ c_2=0,2}} f_c = -\prod_{p|N, p\neq 2} \lambda(r_p, s_p, p). \tag{4.19}$$

If $c_2 = 1$, we have to discuss the following three cases:

(1) N has a prime factor $p \equiv 3 \pmod 4$. If r_p is odd, for any $c|N$, put $c' = cp^{r_p - 2c_p}$. Then $c'|N$ and

$$\frac{N}{c(c, N/c)} \equiv -\frac{N}{c'(c', N/c')} \pmod 4.$$

So we have

$$\psi(d/c) = \overline{\psi}\left(\frac{(c', N/c') - d}{c'}\right).$$

Hence $f_c + f_c' = 0$. By (4.19) we have

$$M = \sum_{c|N, c_2=0,2} f_c + \sum_{c|N, c_2=1} f_c = -\prod_{p|N, p\neq 2} \lambda(r_p, s_p, p).$$

Now we assume that r_p is even for any prime factor $p \equiv 3 \pmod 4$ of N. Then for any $c|N$, we have $h = N/(c(c, N/c)) \equiv 1 \pmod 4$. Hence

$$\psi(d/c) = \mathrm{e}^{k\pi \mathrm{i}/2}\omega(1 + dN/(c, N/c)).$$

Put

$$n_c = \prod_p p^{s_p - r_p + \min\{r_p - c_p, c_p\}},$$

where p runs over the set of all odd prime factors q of N satisfying $r_q - \min\{r_q - c_q, c_q\} < s_q$. It is easy to see that $2^{-1}(c, N/c)|N/F$ if and only if $n_c = 1$. Suppose $s_2 = 0$. If $n_c = 1$, then $\psi(d/c) = \mathrm{e}^{k\pi\mathrm{i}/2}$ and

$$\begin{aligned}\sum_{\substack{(c,N/c)|N/F,\\ c_2=1}} f_c &= \left(\frac{1}{2} - \left\{\frac{k}{4}\right\}\right) \sum_{\substack{(c,N/c)|N/F,\\ c_2=1}} \varphi((c, N/c)) \\ &= \frac{\chi_2(k)}{4} \prod_{p|N, p\neq 2} \lambda(r_p, s_p, p). \end{aligned} \tag{4.20}$$

If $n_c \neq 1$, then $\omega(1 + dN/(c, N/c))$ is a n_c-th primitive root of unity. Since

$$\omega(1 + d_1N/(c, N/c))\omega(1 + d_2N/(c, N/c)) = \omega(1 + (d_1 + d_2)N/(c, N/c)),$$

we can assume that $\omega(1+N/(c, N/c)) = \mathrm{e}^{2\pi\mathrm{i}/n_c}$. Hence $\psi(d/c) = \mathrm{e}^{2\pi\mathrm{i}(k/4+d/n_c)}$ where n_c is a factor of $(c, N/c)$. If d runs over $(\mathbb{Z}/(c, N/c)\mathbb{Z})^*$, then it runs over $(\mathbb{Z}/n_c\mathbb{Z})^*$ for $\varphi(c, N/c)/\varphi(n_c)$ times. By Lemma 4.2 we have

$$\begin{aligned} f_c &= \frac{1}{2}\varphi(c, N/c) - \sum_{d=0,(d,n_c)=1}^{n_c-1} \left\{\frac{k}{4} + \frac{d}{n_c}\right\} \frac{\varphi((c, N/c))}{\varphi(n_c)} \\ &= \frac{\varphi((c, N/c))}{\varphi(n_c)} \sum_{d|n_c} \mu(d) \left\{\frac{(4-k)n_c}{4d}\right\}. \end{aligned} \tag{4.21}$$

Suppose $s_2 = 2$. Then $\omega(1 + dN/(c, N/c))$ is a $2n_c$-th primitive root of the unity. If $n_c = 1$, then

$$\psi(d/c) = \mathrm{e}^{2\pi\mathrm{i}(2+k)/4}.$$

Hence

$$\begin{aligned}\sum_{\substack{2^{-1}(c,N/c)|N/F,\\ c_2=1}} f_c &= \left(\frac{1}{2} - \left\{\frac{2+k}{4}\right\}\right) \sum_{\substack{2^{-1}(c,N/c)|N/F,\\ c_2=1}} \varphi((c, N/c)) \\ &= -\frac{\chi_2(k)}{4} \prod_{p|N, p\neq 2} \lambda(r_p, s_p, p). \end{aligned} \tag{4.22}$$

If $n_c \neq 1$, without loss of generality, we can assume

$$\omega(1+dN/(c,N/c)) = e^{2\pi i d/2n_c} = -e^{2\pi i d'/n_c},$$

where $2d' \equiv d \pmod{n_c}$. Then

$$\psi(d/c) = e^{2\pi i(\frac{2+k}{4}+\frac{d'}{n_c})}$$

and

$$\begin{aligned} f_c &= \frac{1}{2}\varphi((c,N/c)) - \frac{\varphi((c,N/c))}{\varphi(n_c)} \sum_{\substack{d=0,\\ (d,n_c)=1}}^{n_c-1} \left\{\frac{2+k}{4}+\frac{d'}{n_c}\right\} \\ &= \frac{\varphi((c,N/c))}{\varphi(n_c)} \sum_{d|n_c} \mu(d) \left\{\frac{kn_c}{4d}\right\}. \end{aligned} \tag{4.23}$$

(2) We assume that r_p is even and $r_p \geqslant 2s_p$ for any prime factor $p \equiv 3 \pmod 4$ of N, i.e., the condition (4.17) does not hold. Since

$$r_p - \min\{r_p - c_p, c_p\} \geqslant r_p/2 \geqslant s_p,$$

n_c has no prime factors congruent to 3 modulo 4. If $n_c \neq 1$, then

$$\sum_{d|n_c} \mu(d) \left\{\frac{kn_c}{4d}\right\} = \left\{\frac{k}{4}\right\} \sum_{d|n_c} \mu(d) = 0.$$

Gathering (4.19), (4.20), (4.21), (4.22), (4.23), we get the desired result.

(3) We assume that r_p is even for any prime factor $p \equiv 3 \pmod 4$ of N, but there is at least one of these prime factors p such that $0 < r_p < 2s_p$. Put

$$R = \{p | p \equiv 3 \pmod 4, p|N, 0 < r_p < 2s_p\}.$$

If n_c has a prime factor congruent to 1 modulo 4, let $n_c = n_c' n_c''$, such that each prime factor of n_c is congruent to 1 modulo 4 and each one of n_c'' is congruent to 3 modulo 4. Since $n_c' \neq 1$,

$$\sum_{d|n_c} \mu(d) \left\{\frac{kn_c}{4d}\right\} = \sum_{d'|n_c'} \mu(d') \sum_{d''|n_c''} \mu(d'') \left\{\frac{kn_c''}{4d''}\right\} = 0.$$

So the corresponding $f_c = 0$. This shows that f_c may be non-zero only if all prime factors of n_c are congruent to 3 modulo 4. In this case, each prime factor of n_c belongs to the set R. For any subset R' of R, put

$$c(R') = \{c | c_2 = 1, c|N, \text{ the set of prime factors of } n_c \text{ is } R'\}.$$

Suppose $s_2 = 0$. By (4.21) and Lemma 4.3, we get

$$\begin{aligned}\sum_{n_c\neq 1, c_2=1} f_c &= \sum_{R'\subset R}\sum_{c\in c(R')}\prod_{p|N, p\nmid n_c}\varphi(p^{\min\{r_p-c_p,c_p\}})\prod_{p|n_c}2p^{r_p-s_p}\chi_2(kn_c)/4 \\ &= \frac{\chi_2(k)}{4}\sum_{R'\subset R}\prod_{p\in R'}\sum_{c_p=r_p-s_p+1}^{s_p-1}\chi_2(p^{s_p-r_p+\min\{r_p-c_p,c_p\}})\prod_{p|N,p\neq 2}\lambda(r_p,s_p,p) \\ &= \frac{\chi_2(k)}{4}\sum_{R'\subset R}(-1)^{|R'|}\prod_{p|N,p\neq 2}\lambda(r_p,s_p,p) \\ &= -\frac{\chi_2(k)}{4}\prod_{p|N,p\neq 2}\lambda(r_p,s_p,p).\end{aligned} \tag{4.24}$$

By (4.19), (4.20) and (4.24) we get the desired result. If $s_2 = 2$, by (4.19), (4.22), (4.23) and Lemma 4.3 we can get the result similarly. This completes the proof. □

By Proposition 4.1 we have $\dim G(N, k/2, \omega) = 0$ for any $k < 0$. In fact, for any $f \in G(N, k/2, \omega)$, we have $f^2 \in G(N, k, \omega^2) \subset G_k(\Gamma_0(N)) = 0$ by Proposition 4.1. Hence we shall get an expression for $\dim S(N, k/2, \omega)$ for $k \geqslant 5$ from Theorem 4.3. Similarly we can get an expression for $\dim G(N, k/2, \omega)$ if $k \geqslant 5$. But for $k = 1, 3$ we only get some expressions for

$$\dim S(N, 1/2, \omega) - \dim G(N, 3/2, \omega)$$

and

$$\dim S(N, 3/2, \omega) - \dim G(N, 1/2, \omega)$$

respectively. So if we want to know the dimension of $S(N, 3/2, \omega)$ (or $G(N, 3/2, \omega)$ respectively) we have to know the dimension of $G(N, 1/2, \omega)$(or $S(N, 1/2, \omega)$ respectively) which was found by J.P.Serre and H.M.Stark.

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Chapter 5

Operators on the Space of Modular Forms

5.1 Hecke Rings

Let G be a group, Γ, Γ' be subgroups of G. Then Γ and Γ' are commensurable if $\Gamma \bigcap \Gamma'$ is of finite index in Γ and in Γ'. We write $\Gamma \sim \Gamma'$ if Γ and Γ' are commensurable. For any subgroup Γ of G, put

$$\widetilde{\Gamma} = \{\alpha \in G | \alpha \Gamma \alpha^{-1} \sim \Gamma\}.$$

It is easy to see that $\widetilde{\Gamma}$ is a subgroup of G containing Γ and the center of G. Moreover, if Γ' is commensurable with Γ, then $\widetilde{\Gamma} = \widetilde{\Gamma'}$. We call $\widetilde{\Gamma}$ the commensurator of Γ in G.

Lemma 5.1 *Let Γ_1 and Γ_2 be two subgroups of G. For any $\alpha \in G$, put $d = [\Gamma_2 : \Gamma_2 \bigcap \alpha^{-1}\Gamma_1\alpha]$, $e = [\Gamma_1 : \Gamma_1 \bigcap \alpha\Gamma_2\alpha^{-1}]$, then we have disjoint coset decompositions*

$$\Gamma_1\alpha\Gamma_2 = \bigcup_{i=1}^{d} \Gamma_1\alpha_i, \quad \Gamma_1\alpha\Gamma_2 = \bigcup_{j=1}^{e} \beta_j\Gamma_2.$$

Proof Consider a disjoint coset decomposition

$$\Gamma_2 = \bigcup_{i=1}^{d} (\Gamma_2 \bigcap \alpha^{-1}\Gamma_1\alpha)\delta_i,$$

where $\delta_i \in \Gamma_2$. Therefore

$$\Gamma_1\alpha\Gamma_2 = \bigcup_{i=1}^{d} \Gamma_1\alpha\delta_i.$$

If $\Gamma_1\alpha\delta_i = \Gamma_1\alpha\delta_j$, then there is a $\gamma \in \Gamma_1$ such that $\alpha\delta_i = \gamma\alpha\delta_j$. Hence

$$\delta_i\delta_j^{-1} = \alpha^{-1}\gamma\alpha \in \Gamma_2 \bigcap \alpha^{-1}\Gamma_1\alpha,$$

which implies that $i = j$. This completes the proof. □

Let Γ be a subgroup of G, Δ be a semigroup in G such that $\Gamma \subseteq \Delta \subseteq \widetilde{\Gamma} \subseteq G$. Put

$$R(\Gamma, \Delta) = \Big\{ \sum c_i \Gamma\alpha_i\Gamma | \alpha_i \in \Delta, c_i \in \mathbb{Z} \Big\}.$$

We shall now introduce an addition and a multiplication on $R(\Gamma, \Delta)$. The addition is given by adding formally. We consider now the multiplication of two double cosets as follows: First consider disjoint coset decompositions

$$\Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha_i, \quad \Gamma\beta\Gamma = \bigcup_j \Gamma\beta_j$$

with α and $\beta \in \Delta$. Then $\Gamma\alpha\Gamma\beta\Gamma = \bigcup_j \Gamma\alpha\Gamma\beta_j = \bigcup_{i,j} \Gamma\alpha_i\beta_j$. Therefore $\Gamma\alpha\Gamma\beta\Gamma$ is a finite union of double cosets of the form $\Gamma\xi\Gamma$. We define the multiplication of $u := \Gamma\alpha\Gamma$ and $v := \Gamma\beta\Gamma$ to be the element

$$\sum_\xi c_\xi \Gamma\xi\Gamma \in R(\Gamma, \Delta),$$

where

$$c_\xi = \#\{(i,j) | \Gamma\alpha_i\beta_j = \Gamma\xi\}.$$

To make this definition meaningful, we have to show that c_ξ depends only on u, v and $w := \Gamma\xi\Gamma$, and not on the choice of representatives α_i, β_j, ξ. We see that $\Gamma\alpha_i\beta_j = \Gamma\xi$ if and only if $\Gamma\alpha_i = \Gamma\xi\beta_j^{-1}$. Further, for a given j, the last equality holds for exactly one i. Therefore

$$\begin{aligned} c_\xi &= \#\{(i,j)|\Gamma\alpha_i\beta_j = \Gamma\xi\} = \#\{j|\xi\beta_j^{-1} \in \Gamma\alpha\Gamma\} \\ &= \#\{j|\beta_j \in \Gamma\alpha^{-1}\Gamma\xi\} = \#\{j|\Gamma\beta_j \subset \Gamma\alpha^{-1}\Gamma\xi\} \\ &= \text{the number of right cosets of } \Gamma \text{ in } \Gamma\beta\Gamma \bigcap \Gamma\alpha^{-1}\Gamma\xi. \end{aligned}$$

The last number is obviously independent of the choice of α_i, β_j. Now, if $\Gamma\xi\Gamma = \Gamma\eta\Gamma$, then $\xi = \delta\eta\delta'$ with $\delta, \delta' \in \Gamma$, hence

$$\Gamma\beta\Gamma \bigcap \Gamma\alpha^{-1}\Gamma\xi = (\Gamma\beta\Gamma \bigcap \Gamma\alpha^{-1}\Gamma\eta)\delta'.$$

Therefore the number c_ξ is independent of the choice of ξ.

We can now define the multiplication by extending $\mathbb{Z}$-linearly the map $(u, v) \mapsto u \cdot v$ in an obvious way.

Definition 5.1 *The degree of $\Gamma\alpha\Gamma$ is defined to be the number of right cosets of Γ in $\Gamma\alpha\Gamma$ which is denoted by* $\deg(\Gamma\alpha\Gamma)$. *And* $\deg\left(\sum c_\xi \Gamma\xi\Gamma\right) = \sum c_\xi \deg(\Gamma\xi\Gamma)$.

Lemma 5.2 *If $\Gamma\alpha\Gamma \cdot \Gamma\beta\Gamma = \sum c_\xi \Gamma\xi\Gamma$, then*

$$c_\xi \deg(\Gamma\xi\Gamma) = \#\{(i,j)|\Gamma\alpha_i\beta_j\Gamma = \Gamma\xi\Gamma\}.$$

Proof Let $\Gamma\xi\Gamma = \bigcup_{k=1}^{f} \Gamma\xi_k$ be a disjoint coset decomposition. Then $\Gamma\alpha_i\beta_j\Gamma = \Gamma\xi\Gamma$ if and only if $\Gamma\alpha_i\beta_j = \Gamma\xi_k$ for exactly one k. Hence we have

$$\begin{aligned}\#\{(i,j)|\Gamma\alpha_i\beta_j\Gamma = \Gamma\xi\Gamma\} &= \sum_{k=1}^{f} \#\{(i,j)|\Gamma\alpha_i\beta_j = \Gamma\xi_k\} \\ &= c_\xi f = c_\xi \deg(\Gamma\xi\Gamma),\end{aligned}$$

where we used the fact that c_ξ is independent of the choice of the representative ξ(so $c_\xi = c_{\xi_k}$). This completes the proof. □

Lemma 5.3 *Let* $x,\ y \in R(\Gamma, \Delta)$, *then*

$$\deg(x)\deg(y) = \deg(xy).$$

Proof We only need to show the formula for $x = \Gamma\alpha\Gamma, y = \Gamma\beta\Gamma$ by linearity. Put

$$xy = \sum_\xi c_\xi \Gamma\xi\Gamma,$$

then by Lemma 5.2 we get

$$\begin{aligned}\deg(xy) &= \sum_\xi c_\xi \deg(\Gamma\xi\Gamma) \\ &= \sum_\xi \#\{(i,j)|\Gamma\alpha_i\beta_j\Gamma = \Gamma\xi\Gamma\} \\ &= \#\{(i,j)\} = \deg(x)\deg(y).\end{aligned}$$

This completes the proof. □

Lemma 5.4 *The above multiplication is associative in the sense that* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *for any* $x, y, z \in R(\Gamma, \Delta)$.

Proof Put

$$M = \left\{ \sum c_i \Gamma\eta_i | c_i \in \mathbb{Z}, \eta_i \in \widetilde{\Gamma} \right\},$$

which is a $\mathbb{Z}$-module of all formal finite sums $\sum c_i\Gamma\eta_i$. Let $u = \Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha_i$ (disjoint). Define a $\mathbb{Z}$-linear map of M as follows:

$$u \cdot \sum c_i \Gamma\eta_i = \sum_{i,j} c_i \Gamma\alpha_j\eta_i.$$

It is easy to see that this does not depend on the choice of α_j, η_i. By linearity we get a map from $R(\Gamma, \Delta)$ to $\mathrm{Hom}(M, M)$. We emphasize that this map is injective. In fact, if $\sum_{\alpha} c_\alpha(\Gamma\alpha\Gamma)\cdot\Gamma\eta = 0$ is a non-trivial cancellation, we have $\Gamma\alpha_1\Gamma = \Gamma\alpha_2\Gamma$ for some α_1, α_2. But this implies that $\Gamma\alpha_1\Gamma = \Gamma\alpha_2\Gamma$, hence it is impossible. Therefore the map is injective. Now consider disjoint coset decompositions:

$$\Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha_i, \quad \Gamma\beta\Gamma = \bigcup_j \Gamma\beta_j.$$

Put

$$\Gamma\alpha\Gamma\cdot\Gamma\beta\Gamma = \sum_t c_{\xi_t}\Gamma\xi_t\Gamma, \quad \Gamma\xi_t\Gamma = \bigcup_k \Gamma\xi_{t,k}.$$

Then we have

$$\begin{aligned}\Gamma\alpha\Gamma(\Gamma\beta\Gamma\cdot\Gamma\eta) &= \Gamma\alpha\Gamma\sum_j \Gamma\beta_j\eta\\ &= \sum_{i,j}\Gamma\alpha_i\beta_j\eta = \sum_{t,k} c_{\xi_t}\Gamma\xi_{t,k}\eta\\ &= (\Gamma\alpha\Gamma\cdot\Gamma\beta\Gamma)\cdot\Gamma\eta.\end{aligned}$$

This implies that $(yz)a = y(za)$ for any $y, z \in R(\Gamma,\Delta)$ and $a \in M$. Let now $x, y, z \in R(\Gamma,\Delta)$, then

$$((xy)z)a = (xy)(za) = x(y(za)) = x((yz)a) = (x(yz))a.$$

By the injectivity proved above, we get $(xy)z = x(yz)$. This completes the proof. □

By Lemma 5.4 we know that $R(\Gamma,\Delta)$ is an algebra. It is called the Hecke algebra for Γ and Δ.

Lemma 5.5 *Assume $\alpha \in \widetilde{\Gamma}$ such that $d = e$ (see Lemma 5.1 for the definitions of d, e). Then we can find $\{\alpha_i\}_{i=1}^d$ such that*

$$\Gamma\alpha\Gamma = \bigcup_{i=1}^d \Gamma\alpha_i = \bigcup_{i=1}^d \alpha_i\Gamma.$$

Proof Let

$$\Gamma\alpha\Gamma = \bigcup_{i=1}^d \Gamma\beta_i = \bigcup_{i=1}^d \beta_i'\Gamma$$

be decompositions of $\Gamma\alpha\Gamma$. Since $\beta_i \in \Gamma\alpha\Gamma = \Gamma\beta_i'\Gamma$, there are two elements $\delta, \epsilon \in \Gamma$ such that $\beta_i = \delta\beta_i'\epsilon$. Put $\alpha_i = \delta^{-1}\beta_i = \beta_i'\epsilon$. Then

$$\Gamma\alpha_i = \Gamma\beta_i, \quad \alpha_i\Gamma = \beta_i'\Gamma.$$

This completes the proof. □

Lemma 5.6 *If G has an anti-automorphism $\alpha \mapsto \alpha^*$ such that $\Gamma = \Gamma^*$ and $\Gamma\alpha\Gamma = (\Gamma\alpha\Gamma)^*$ for every $\alpha \in \Delta$, then $R(\Gamma,\Delta)$ is commutative.*

Proof Applying the anti-automorphism $*$ to $\Gamma\alpha\Gamma$, we find that $d = e$. Therefore, by Lemma 5.5, for any $\alpha, \beta \in \Delta$, we have disjoint decompositions

$$\Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha_i = \bigcup \alpha_i\Gamma, \quad \Gamma\beta\Gamma = \bigcup_j \Gamma\beta_j = \bigcup_j \beta_j\Gamma.$$

Then

$$\Gamma\alpha\Gamma = \Gamma\alpha^*\Gamma = \bigcup_i \Gamma\alpha_i^*, \quad \Gamma\beta\Gamma = \Gamma\beta^*\Gamma = \bigcup_j \Gamma\beta_j^*.$$

If $\Gamma\alpha\Gamma\beta\Gamma = \bigcup \Gamma\xi\Gamma$, then

$$\Gamma\beta\Gamma\alpha\Gamma = \Gamma\beta^*\Gamma\alpha^*\Gamma = (\Gamma\alpha\Gamma\beta\Gamma)^* = \bigcup_\xi \Gamma\xi\Gamma.$$

Then we have

$$(\Gamma\alpha\Gamma)(\Gamma\beta\Gamma) = \sum_\xi c_\xi(\Gamma\xi\Gamma), \quad (\Gamma\beta\Gamma)(\Gamma\alpha\Gamma) = \sum_\xi c'_\xi(\Gamma\xi\Gamma)$$

with the same components $\Gamma\xi\Gamma$. By Lemma 5.2 we have

$$\begin{aligned} c_\xi \deg(\Gamma\xi\Gamma) &= \#\{(i,j)|\Gamma\alpha_i\beta_j\Gamma = \Gamma\xi\Gamma\} \\ &= \#\{(i,j)|\Gamma\beta_j^*\alpha_i^*\Gamma = \Gamma\xi\Gamma\} \\ &= c'_\xi \deg(\Gamma\xi\Gamma), \end{aligned}$$

which shows that $c_\xi = c'_\xi$. This completes the proof. □

Let $G = GL_2^+(\mathbb{Q})$ and $\Gamma = \Gamma(1) = SL_2(\mathbb{Z})$. Then we have

Lemma 5.7 $\widetilde{\Gamma} = G$.

Proof For any $\boldsymbol{\alpha} \in G$, there exist $c \in \mathbb{Q}$, $\beta \in M_2(\mathbb{Z})$ such that $\alpha = c\beta$. We have that $\alpha\Gamma\alpha^{-1} = \beta\Gamma\beta^{-1}$. Put $b = \det(\beta)$ and $\Gamma_b = \Gamma(b)$. Since

$$b\beta^{-1}\Gamma_b\beta \equiv 0 \pmod b,$$

we see that $\beta^{-1}\Gamma_b\beta \in \Gamma$. Hence $\Gamma_b \subset \Gamma \bigcap \boldsymbol{\beta}\Gamma\boldsymbol{\beta}^{-1}$, and

$$[\Gamma : \Gamma\bigcap\boldsymbol{\beta}\Gamma\boldsymbol{\beta}^{-1}] < [\Gamma : \Gamma_b] < +\infty.$$

But

$$[\beta^{-1}\Gamma\beta : \beta^{-1}\Gamma\beta\bigcap\Gamma] = [\Gamma : \Gamma\bigcap\beta\Gamma\beta^{-1}],$$

we get, by substituting β^{-1} by β,

$$[\beta\Gamma\beta^{-1} : \Gamma\bigcap\beta\Gamma\beta^{-1}] < +\infty.$$

That is, $\alpha = c\beta \in \widetilde{\Gamma}$. This completes the proof. □

We choose now $\Delta = \{\alpha \in M_2(\mathbb{Z}) | \det(\alpha) > 0\}$ and consider the Hecke ring $R(\Gamma, \Delta)$. For any $\alpha \in \Delta$, there is a unique pair (a, b) with a, b positive integers and $a|b$ such that $\Gamma\alpha\Gamma = \Gamma \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \Gamma$. Hence we put, for any pair (a, b) of positive integers with $a|b$,

$$\mathrm{T}(a, b) = \Gamma \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \Gamma.$$

Theorem 5.1 *$R(\Gamma, \Delta)$ is commutative.*

Proof It is clear that the transposition on G is an anti-automorphism such that $\mathrm{T}(a, b)$ is invariant for any (a, b). So we get the theorem by Lemma 5.6. □

Lemma 5.8 *Let a_1, a_2, b_1, b_2 be positive integers such that $a_1|a_2$, $b_1|b_2$, $(a_2, b_2) = 1$, then*

$$\mathrm{T}(a_1, a_2)\mathrm{T}(b_1, b_2) = \mathrm{T}(a_1b_1, a_2b_2).$$

Proof Let

$$\alpha = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \quad \beta = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}.$$

It is clear that $\Gamma\alpha\beta\Gamma \subset \Gamma\alpha\Gamma\beta\Gamma$. For any $\gamma \in \Gamma$, consider elementary divisors of $\alpha\gamma\beta$. Since any entry of α is divisible by a_1, and any one of $\gamma\beta$ is divisible by b_1, any entry of $\alpha\gamma\beta$ is divisible by a_1b_1. In fact a_1b_1 is the maximal positive integer with this property. Hence $\alpha\gamma\beta \in \Gamma\alpha\beta\Gamma$. This implies that $\Gamma\alpha\beta\Gamma = \Gamma\alpha\Gamma\beta\Gamma$. We have disjoint decompositions

$$\Gamma\alpha\Gamma = \bigcup_{s_1, s_2, u} \Gamma \begin{pmatrix} s_1 & u \\ 0 & s_2 \end{pmatrix} a_1,$$

$$\Gamma\beta\Gamma = \bigcup_{t_1, t_2, v} \Gamma \begin{pmatrix} t_1 & v \\ 0 & t_2 \end{pmatrix} b_1.$$

where

$$s_1s_2 = a_2/a_1, \quad 0 \leqslant u < s_2, \quad (s, s_2, u) = 1,$$

and

$$t_1t_2 = b_2/b_1, \quad 0 \leqslant v < t_2, \quad (t_1, t_2, v) = 1.$$

If

$$\Gamma \begin{pmatrix} s_1 & u \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} t_1 & v \\ 0 & t_2 \end{pmatrix} a_1a_2 = \Gamma \begin{pmatrix} a_1b_1 & 0 \\ 0 & a_2b_2 \end{pmatrix},$$

it is easy to see that $s_1 = t_1 = 1$, $s_2 = a_2/a_1$, $t_2 = b_2/b_1$, $u = v = 0$. Therefore

$$\Gamma\alpha\Gamma \cdot \Gamma\beta\Gamma = \Gamma\alpha\beta\Gamma,$$

which completes the proof. □

Similar to the proof above, we have the following:

Lemma 5.9 $\mathrm{T}(c,c)\mathrm{T}(a,b) = \mathrm{T}(ac,bc)$.

By Lemma 5.8 and Lemma 5.9, we know that every $\mathrm{T}(a,b)$ can be represented as a polynomial of some $\mathrm{T}(p,p), \mathrm{T}(1,p^k)$ with p primes and k positive integers.

Let n be a positive integer, define

$$\mathrm{T}(n) = \sum_{ad=n, a|d} \mathrm{T}(a,d),$$

that is, $\mathrm{T}(n)$ is the sum of all double cosets $\Gamma\alpha\Gamma$ with $\det(\alpha) = n, \alpha \in \Delta$. Then by Lemma 5.8 we have

$$\mathrm{T}(m)\mathrm{T}(n) = \mathrm{T}(mn) \tag{5.1}$$

for any m, n with $(m,n) = 1$.

It is easy also to show that

Lemma 5.10 *We have*

$$\mathrm{T}(n) = \sum_{\substack{ad=n, d>0, \\ b \pmod d}} \Gamma \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

That is, we can choose

$$\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| ad = n, d > 0, b \pmod d \right\}$$

as a complete set of representatives of right cosets of Γ *in* $\mathrm{T}(n)$.

Lemma 5.11 *For any* $0 \leqslant m \leqslant n$, *we have*

$$\mathrm{T}(p^m)\mathrm{T}(p^n) = \sum_{i=0}^{m} p^i \mathrm{T}(p,p)^i \mathrm{T}(p^{m+n-2i}).$$

Proof We shall prove this lemma by induction. It is clear for $m = 0$ since $\mathrm{T}(1) = \mathrm{T}(1,1)$ is the identity of $R(\Gamma, \Delta)$.

For any prime p, by the definition of $\mathrm{T}(n)$, we know that

$$\mathrm{T}(p^n) = \sum_{2i \leqslant n} \mathrm{T}(p^i, p^{n-i}).$$

Hence, by Lemma 5.9, we get

$$\mathrm{T}(p^n) = \mathrm{T}(1,p^n) + \mathrm{T}(p,p)\mathrm{T}(p^{n-2}). \tag{5.2}$$

By Lemma 5.10 we see that the following sets

$$X := \left\{ \begin{pmatrix} 1 & s \\ 0 & p \end{pmatrix} \middle| s \pmod{p} \right\} \bigcup \left\{ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

$$Y := \left\{ \begin{pmatrix} p^i & t \\ 0 & p^{n-i} \end{pmatrix} \middle| 0 \leqslant i \leqslant n, t \pmod{p^{n-i}} \right\}$$

are complete sets of representatives of right cosets of Γ in $\mathrm{T}(p)$ and $\mathrm{T}(p^n)$ respectively. Then the set of products of elements in X and elements in Y is

$$\begin{aligned} &\left\{ \begin{pmatrix} p^i & t + sp^{n-i} \\ 0 & p^{n+1-i} \end{pmatrix} \middle| 0 \leqslant i \leqslant n, t \pmod{p^{n-i}}, s \pmod{p} \right\} \\ &\bigcup \left\{ \begin{pmatrix} p^{i+1} & pt \\ 0 & p^{n-i} \end{pmatrix} \middle| 0 \leqslant i \leqslant n, t \pmod{p^{n-i}} \right\} \\ = &\left\{ \begin{pmatrix} p^i & t \\ 0 & p^{n+1-i} \end{pmatrix} \middle| 0 \leqslant i \leqslant n+1, t \pmod{p^{n+1-i}} \right\} \\ &\bigcup \left\{ \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} p^i & t \\ 0 & p^{n-1-i} \end{pmatrix} \middle| 0 \leqslant i \leqslant n-1, t \pmod{p^{n-i}} \right\}. \end{aligned}$$

By Lemma 5.10, we know that $\left\{ \begin{pmatrix} p^i & t \\ 0 & p^{n+1-i} \end{pmatrix} \middle| 0 \leqslant i \leqslant n+1, t \pmod{p^{n+1-i}} \right\}$ is a complete set of representatives of right cosets of Γ in $\mathrm{T}(p^{n+1})$, and every element in $\mathrm{T}(p^{n-1})$ appears repeatedly p times in the following set

$$\left\{ \begin{pmatrix} p^i & t \\ 0 & p^{n-1-i} \end{pmatrix} \middle| 0 \leqslant i \leqslant n-1, t \pmod{p^{n-i}} \right\}$$

and above set has no other elements. So we get

$$\mathrm{T}(p)\mathrm{T}(p^n) = \mathrm{T}(p^{n+1}) + p\mathrm{T}(p,p)\mathrm{T}(p^{n-1}), \tag{5.3}$$

which shows the lemma for $m = 1$. We now assume that $m > 1$, then by (5.2), we see that

$$\mathrm{T}(p^m) = \mathrm{T}(p)\mathrm{T}(p^{m-1}) - p\mathrm{T}(p,p)\mathrm{T}(p^{m-2}).$$

Then by induction hypothesis we get

$$\begin{aligned} \mathrm{T}(p)\mathrm{T}(p^{m-1})\mathrm{T}(p^n) &= \mathrm{T}(p) \sum_{i=0}^{m-1} p^i \mathrm{T}(p,p)^i \mathrm{T}(p^{m+n-1-2i}) \\ &= \sum_{i=0}^{m-1} p^i \mathrm{T}(p,p)^i (\mathrm{T}(p^{m+n-2i}) + p\mathrm{T}(p,p)\mathrm{T}(p^{m+n-2-2i})) \\ &\quad - p\mathrm{T}(p,p)\mathrm{T}(p^{m-2})\mathrm{T}(p^n) \\ &= -p\mathrm{T}(p,p) \sum_{i=0}^{m-2} p^i \mathrm{T}(p,p)^i \mathrm{T}(p^{m+n-2-2i}). \end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{T}(p^m)\mathrm{T}(p^n) &= \sum_{i=0}^{m-1} p^i\mathrm{T}(p,p)^i\mathrm{T}(p^{m+n-2i}) + p^m\mathrm{T}(p,p)^m T(p^{n-m}) \\ &= \sum_{i=0}^{m} p^i\mathrm{T}(p,p)^i\mathrm{T}\left(p^{m+n-2i}\right).\end{aligned}$$

This completes the proof. □

By Lemma 5.11 and the equality (5.1) we get

$$\mathrm{T}(m)\mathrm{T}(n) = \sum_{d|(m,n),d>0} d\mathrm{T}(d,d)\mathrm{T}\left(\frac{mn}{d^2}\right). \tag{5.4}$$

Theorem 5.2 *Let p be a prime. Denote by R_p the subalgebra of $R(\Gamma,\Delta)$ generated by $\Gamma\alpha\Gamma$ with $\alpha \in \Delta$ and $\det(\alpha)$ a power of p. Then R_p is the polynomial algebra over $\mathbb{Z}$ generated by $\mathrm{T}(p)$ and $\mathrm{T}(p,p)$.*

Proof It is clear that R_p is generated by $\mathrm{T}(p^m,p^n)$ with $m \leqslant n$. By Lemma 5.9 we know that

$$\mathrm{T}(p^m,p^n) = (\mathrm{T}(p,p))^m\mathrm{T}(1,p^{n-m}).$$

By (5.2) we see that $\mathrm{T}(1,p^l) = \mathrm{T}(p^l) - \mathrm{T}(p,p)\mathrm{T}(p^{l-2})$ for any $l \geqslant 2$. Hence we know that $\mathrm{T}(p^n)$ is a polynomial of $\mathrm{T}(p)$ and $\mathrm{T}(p,p)$. This shows that R_p is generated by $\mathrm{T}(p)$ and $\mathrm{T}(p,p)$. We need to show that $\mathrm{T}(p)$ and $\mathrm{T}(p,p)$ are algebraically independent. Otherwise, put $\mathcal{I}_p = \mathrm{T}(p,p)R_p$. Then $\mathcal{I}_p$ is an ideal of R_p. By (5.3) we have

$$\mathrm{T}(p)\mathrm{T}(p^n) \equiv \mathrm{T}(p^{n+1}) \pmod{\mathcal{I}_p}.$$

Hence

$$\mathrm{T}(p)^n \equiv \mathrm{T}(p^n) \pmod{\mathcal{I}_p}.$$

And therefore

$$\mathrm{T}(p)^n \equiv \mathrm{T}(1,p^n) \pmod{\mathcal{I}_p}.$$

It is easy to see that $\mathrm{T}(1,p^n)$ $(n = 0,\ 1,\ 2,\ \cdots)$ are linearly independent modulo $\mathcal{I}_p$. So are $\mathrm{T}(p)^n$ $(n = 0,\ 1,\ 2,\ \cdots)$ modulo $\mathcal{I}_p$. Now let $f(x,y)$ be the polynomial with the lowest degree such that $f(\mathrm{T}(p),\mathrm{T}(p,p)) = 0$. Put

$$f(x,y) = f_0(x) + yf_1(x,y), \quad f_0(x) \in \mathbb{Z}[x], f_1(x,y) \in \mathbb{Z}[x,y].$$

Then by above discussion we get $f_0 = 0$. Hence

$$\mathrm{T}(p,p)f_1(\mathrm{T}(p),\mathrm{T}(p,p)) = 0.$$

But we see that $\mathrm{T}(p,p)$ is not a zero divisor by Lemma 5.8 and Lemma 5.9. So $f_1(\mathrm{T}(p),\mathrm{T}(p,p)) = 0$ which contradicts the assumption on the degree of f. This completes the proof. □

Corollary 5.1 *$R(\Gamma, \Delta)$ is the polynomial algebra generated by $\mathrm{T}(p), \mathrm{T}(p,p)$. ($p$ runs over the set of all primes.)*

Theorem 5.3 *The formal power series*

$$D(s) = \sum_{n=1}^{\infty} \mathrm{T}(n) n^{-s}$$

has the following infinite product expression

$$D(s) = \prod_p (1 - \mathrm{T}(p)p^{-s} + \mathrm{T}(p,p)p^{1-2s})^{-1},$$

where p runs over all primes.

Proof By (5.1) we get

$$D(s) = \prod_p \sum_{n=0}^{\infty} \mathrm{T}(p^n) p^{-ns}.$$

So we only need to show that

$$(1 - \mathrm{T}(p)p^{-s} + \mathrm{T}(p,p)p^{1-2s}) \left(\sum_{n=0}^{\infty} \mathrm{T}(p^n) p^{-ns} \right) = 1.$$

By (5.3) we obtain

$$\begin{aligned} \mathrm{T}(p)p^{-s} \sum_{n=0}^{\infty} \mathrm{T}(p^n) p^{-ns} &= \mathrm{T}(p)p^{-s} + \sum_{n=1}^{\infty} (\mathrm{T}(p^{n+1}) + p\mathrm{T}(p,p)\mathrm{T}(p^{n-1}))p^{-ns-s} \\ &= \sum_{n=1}^{\infty} \mathrm{T}(p^n)p^{-ns} + \mathrm{T}(p,p)p^{1-2s} \sum_{n=0}^{\infty} \mathrm{T}(p^n)p^{-ns} \\ &= -1 + \left(1 + \mathrm{T}(p,p)p^{1-2s}\right) \sum_{n=0}^{\infty} \mathrm{T}(p^n)p^{-ns}. \end{aligned}$$

This completes the proof. □

The product in Theorem 5.3 is called Euler product of $D(s)$. When a representation of $R(\Gamma, \Delta)$ is given, we can get the product property of the representation by $D(s)$. For example, since $\Gamma\alpha\Gamma \mapsto \deg(\Gamma\alpha\Gamma)$ is a representation of $R(\Gamma, \Delta)$ (see Lemma 5.3) and $\deg(\mathrm{T}(p)) = 1 + p, \deg(\mathrm{T}(p,p)) = 1$, we obtain

$$\begin{aligned} \sum_{m=1}^{\infty} \deg(\mathrm{T}(n))n^{-s} &= \prod_p (1 - (1+p)p^{-s} + p^{1-2s})^{-1} \\ &= \prod_p (1 - p^{-s})^{-1}(1 - p^{1-s})^{-1} = \zeta(s)\zeta(s-1). \end{aligned}$$

Hence we get

$$\deg(\mathrm{T}(n)) = \sum_{d|n} d.$$

From now on we study the Hecke algebra of a congruence subgroup of the full modular group. Let N be a positive integer, to simplify symbols, put $\Gamma_N = \Gamma(N)$. Assume that Γ' is a congruence subgroup such that $\Gamma_N \subset \Gamma' \subset \Gamma$.

Lemma 5.12 *Let a, b be positive integers and $c = (a, b)$. Then $\Gamma_c = \Gamma_a\Gamma_b$.*

Proof It is clear that $\Gamma_a\Gamma_b \subset \Gamma_c$. Now let α be an element of Γ_c. By the Chinese Remainder Theorem, we can find $\beta \in M_2(\mathbb{Z})$ such that

$$\beta \equiv 1 \pmod{a}, \quad \beta \equiv \alpha \pmod{b}.$$

Hence $\det(\beta) \equiv 1 \pmod{ab/c}$. Therefore there is a $\gamma \in \Gamma$ such that $\gamma \equiv \beta \pmod{ab/c}$. This shows that $\gamma \equiv 1 \pmod{a}$, $\gamma^{-1}\alpha \equiv 1 \pmod{b}$. That is, $\gamma \in \Gamma_a$, $\gamma^{-1}\alpha \in \Gamma_b$. Hence $\alpha = \gamma \cdot \gamma^{-1}\alpha \in \Gamma_a\Gamma_b$. This completes the proof. □

Let $\alpha \in M_2(\mathbb{Z})$. Define $\lambda_N(\alpha) \equiv \alpha \pmod{N} \in M_2(\mathbb{Z}/N\mathbb{Z})$. Put

$$\Delta_N = \{\alpha \in M_2(\mathbb{Z}) | \det(\alpha) > 0, (\det(\alpha), N) = 1\}$$

and

$$\Phi = \{\alpha \in \Delta_N | \lambda_N(\Gamma'\alpha) = \lambda_N(\alpha\Gamma')\}.$$

It is clear that $\lambda_N(\alpha) \in GL_2(\mathbb{Z}/N\mathbb{Z})$ for any $\alpha \in \Delta_N$ and $\Phi = \Delta_N$ if and only if $\Gamma' = \Gamma_N$.

Lemma 5.13 *Let notations be as above and $\alpha, \beta \in \Delta_N$. Then the following assertions hold:*

(1) $\Gamma\alpha\Gamma = \Gamma\alpha\Gamma_N = \Gamma_N\alpha\Gamma = \Gamma\alpha\Gamma' = \Gamma'\alpha\Gamma$;

(2) $\Gamma'\alpha\Gamma' = \{\xi \in \Gamma\alpha\Gamma | \lambda_N(\xi) \in \lambda_N(\Gamma'\alpha)\}$ *if* $\alpha \in \Phi$;

(3) $\Gamma_N\alpha\Gamma_N = \Gamma_N\beta\Gamma_N$ *if and only if* $\Gamma\alpha\Gamma = \Gamma\beta\Gamma$ *and* $\alpha \equiv \beta \pmod{N}$;

(4) $\Gamma'\alpha\Gamma' = \Gamma'\alpha\Gamma_N = \Gamma_N\alpha\Gamma'$ *if* $\boldsymbol{\alpha} \in \Phi$;

(5) *If* $\alpha \in \Phi$ *and* $\Gamma'\alpha\Gamma' = \bigcup_i \Gamma'\alpha_i$ *is a disjoint union, then* $\Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha_i$ *is a disjoint union.*

Proof Put $a = \det(\alpha)$. By Lemma 5.12, since $(a, N) = 1$, we have that $\Gamma = \Gamma_a\Gamma_N$. But $\alpha\Gamma_a\alpha^{-1} \subset \Gamma$. So $\Gamma = \Gamma_a\Gamma_N \subset \alpha^{-1}\Gamma\alpha\Gamma_N$, and then

$$\alpha^{-1}\Gamma\alpha\Gamma = \alpha^{-1}\Gamma\alpha\Gamma_a\Gamma_N \subset \alpha^{-1}\Gamma(\alpha\Gamma_a\alpha^{-1})\alpha\Gamma_N$$
$$\subset \alpha^{-1}\Gamma\Gamma\alpha\Gamma_N \subset \alpha^{-1}\Gamma\alpha\Gamma_N.$$

Hence $\Gamma\alpha\Gamma \subset \Gamma\alpha\Gamma_N \subset \Gamma\alpha\Gamma'$. Since the opposite inclusion is obvious, we get (1). To see (2), let $\xi \in \Gamma\alpha\Gamma$ and $\lambda_N(\xi) \in \lambda_N(\Gamma'\alpha)$. Then $\xi \equiv \gamma\alpha \pmod{N}$ with $\gamma \in \Gamma'$.

By (1), $\xi \in \Gamma\alpha\Gamma_N$, hence $\xi = \delta\alpha\epsilon$ with $\delta \in \Gamma$, $\epsilon \in \Gamma_N$. Then $\gamma \equiv \delta \pmod N$ and $\delta\gamma^{-1} \in \Gamma_N$. Since $\Gamma_N \subset \Gamma'$, we see that $\delta = (\delta\gamma^{-1})\gamma \in \Gamma_N\Gamma' \subset \Gamma'$, hence $\xi \in \Gamma'\alpha\Gamma_N \subset \Gamma'\alpha\Gamma'$. Conversely if $\xi \in \Gamma'\alpha\Gamma'$, we have clearly $\xi \in \Gamma\alpha\Gamma$, and by the definition of Φ, $\lambda_N(\xi) \in \lambda_N(\Gamma'\alpha)$. This completes (2). At the same time, we have proved that $\Gamma'\alpha\Gamma' \subset \Gamma'\alpha\Gamma_N$. Since the opposite inclusion is clear, we get (4). The assertion (3) is a special case of (2). Finally we want to prove (5). Let $\alpha \in \Phi$, and $\Gamma'\alpha\Gamma' = \bigcup_i \Gamma'\alpha_i$. Then $\Gamma\alpha\Gamma = \Gamma\alpha\Gamma' = \bigcup_i \Gamma\alpha_i$ by (1). Assume $\Gamma\alpha_i = \Gamma\alpha_j$. Then $\alpha_i = \boldsymbol{\gamma}\alpha_j$ with $\boldsymbol{\gamma} \in \Gamma$. By (2), since $\alpha_i, \alpha_j \in \Gamma'\alpha\Gamma'$, we have that $\alpha_i \equiv \delta\alpha_j \pmod N$ with $\delta \in \Gamma'$. Then $\gamma \equiv \delta \pmod N$, and so $\gamma\delta^{-1} \in \Gamma_N$. Since $\Gamma_N \subset \Gamma'$, we get $\gamma = (\gamma\delta^{-1})\delta \in \Gamma_N\Gamma' = \Gamma'$, so that $\Gamma'\alpha_i = \Gamma'\alpha_j$, and hence $i = j$. This completes the proof. □

Lemma 5.14 *The correspondence $\Gamma'\alpha\Gamma' \mapsto \Gamma\alpha\Gamma$, with $\alpha \in \Phi$, defines a homomorphism of $R(\Gamma', \Phi)$ into $R(\Gamma, \Delta)$.*

Proof We only need to show that the correspondence preserves the multiplications of $R(\Gamma', \Phi)$ and $R(\Gamma, \Delta)$. Let α, $\beta \in \Phi$, and let

$$\Gamma'\alpha\Gamma' = \bigcup \Gamma'\alpha_i, \quad \Gamma'\beta\Gamma' = \bigcup \Gamma'\beta_i$$

be disjoint unions. By (5) of Lemma 5.13, we have that

$$\Gamma\alpha\Gamma = \bigcup \Gamma\alpha_i, \quad \Gamma\beta\Gamma = \bigcup \Gamma\beta_i$$

are disjoint unions. Put

$$\Gamma'\alpha\Gamma' \cdot \Gamma'\beta\Gamma' = \sum_\xi c'_\xi \Gamma'\xi\Gamma',$$

where

$$c'_\xi = \#\{(i,j) | \Gamma'\alpha_i\beta_j = \Gamma'\xi\}.$$

By (1) of Lemma 5.13, we get

$$\Gamma\alpha\Gamma\beta\Gamma = \Gamma\alpha\Gamma'\beta\Gamma' = \bigcup_\xi \Gamma\xi\Gamma' = \bigcup_\xi \Gamma\xi\Gamma$$

and different ξ'*s* correspond to different double cosets. Otherwise, if $\Gamma\xi\Gamma = \Gamma\xi'\Gamma$, since

$$\lambda_N(\xi) \in \lambda_N(\Gamma'\alpha\beta), \quad \lambda_N(\xi') \in \lambda_N(\Gamma'\alpha\beta),$$

we get $\Gamma'\xi\Gamma' = \Gamma'\xi'\Gamma'$. Therefore put

$$\Gamma\alpha\Gamma \cdot \Gamma\beta\Gamma = \sum_\xi c_\xi \Gamma\xi\Gamma,$$

then

$$c_\xi = \#\{(i,j) | \Gamma\alpha_i\beta_j = \Gamma\xi\}.$$

We want to show that $c_\xi = c'_\xi$. That is, to show that $\Gamma'\alpha_i\beta_j = \Gamma'\xi$ if and only if $\Gamma\alpha_i\beta_j = \Gamma\xi$. Assume $\Gamma\alpha_i\beta_j = \Gamma\xi$. Then $\xi = \gamma\alpha_i\beta_j$ with $\gamma \in \Gamma$. Since $\xi \in \Gamma'\alpha\Gamma'\beta\Gamma'$, then

$$\lambda_N(\xi) \in \lambda_N(\Gamma'\alpha_i\beta_j).$$

Hence we have that $\xi \equiv \delta\alpha_i\beta_j \pmod N$ with $\delta \in \Gamma'$. Then $\delta \equiv \gamma \pmod N$, hence $\gamma \in \Gamma'$. Therefore $\Gamma'\alpha_i\beta_j = \Gamma'\xi$. Since the converse is obvious, this completes the proof. □

Let t be a positive divisor of N and $\mathcal{I}$ a subgroup of $(\mathbb{Z}/N\mathbb{Z})^*$. Put

$$\Gamma' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| a \in \mathcal{I},\ t|b,\ N|c \right\},$$

$$\Delta' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2^+(\mathbb{Z}) \middle| a \in \mathcal{I},\ t|b,\ N|c \right\}.$$

It is easy to see that for $\mathcal{I} = 1$, $t = N$, we have $\Gamma' = \Gamma_N$; for $\mathcal{I} = (\mathbb{Z}/N\mathbb{Z})^*$, $t = 1$ we have $\Gamma' = \Gamma_0(N)$. It is obvious that $\Gamma_N \subset \Gamma' \subset \Gamma$. Similar to the proof of Lemma 5.7, we can show that $\Delta' \subset \widetilde{\Gamma'}$. We discuss now the Hecke ring $R(\Gamma', \Delta')$. Put

$$\Delta'_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta' \middle| (d, N) = 1 \right\},$$

$$\Delta^*_N = \left\{ \boldsymbol{\alpha} \in M_2^+(\mathbb{Z}) \middle| \lambda_N(\boldsymbol{\alpha}) = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}, (d, N) = 1 \right\}.$$

It is clear that $\Delta^*_N \subset \Delta'_N \subset \Delta'$.

Lemma 5.15 *We have that*

$$\Delta'_N = \Delta^*_N\Gamma' = \Gamma'\Delta^*_N, \quad \Delta'_N \subset \Phi.$$

Proof Let $\alpha \in \Delta'_N$, $d = \det(\alpha)$, then

$$\det\left[\begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} \alpha\right] \equiv 1 \pmod N,$$

where $ed \equiv 1 \pmod N$. Hence we can find a $\gamma \in \Gamma$ such that

$$\gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix} \alpha \pmod N.$$

This implies that $\gamma \in \Gamma'$ and

$$\alpha\gamma^{-1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \pmod N,$$

then $\alpha = \alpha\gamma^{-1}\cdot\gamma \in \Delta_N^*\Gamma'$, so that $\Delta_N' = \Delta_N^*\Gamma'$. Similarly, we have that $\Delta_N' = \Gamma'\Delta_N^*$. Hence for any $\alpha \in \Delta_N'$ we can find a $\alpha' \in \Delta_N^*$ such that $\Gamma'\alpha\Gamma' = \Gamma'\alpha'\Gamma'$, so that

$$R(\Gamma', \Delta_N') = R(\Gamma', \Delta_N^*).$$

For any $\alpha \in \Delta_N'$, if $\alpha \in \Gamma'\alpha'$, $\lambda_N(\alpha') = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$, then

$$\Gamma'\alpha \equiv \Gamma' \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \Gamma' \equiv \alpha\Gamma' \pmod N.$$

Therefore $\Delta_N' \subset \Phi$. This completes the proof. □

Theorem 5.4 *The correspondence $\Gamma'\alpha\Gamma' \mapsto \Gamma\alpha\Gamma$, with $\alpha \in \Delta_N'$, defines an isomorphism of $R(\Gamma', \Delta_N')$ onto $R(\Gamma, \Delta_N)$.*

Proof By Lemma 5.14, it is sufficient to show that the map is injective and surjective. Let $\eta \in \Delta_N$, $d = \det(\eta)$. Then similar to the proof of Lemma 5.15 we can find a $\gamma \in \Gamma$ such that

$$\eta\gamma^{-1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \pmod N,$$

that is, $\eta\gamma^{-1} \in \Delta_N^*$. Hence

$$\Gamma'\eta\gamma^{-1}\Gamma' \mapsto \Gamma\eta\gamma^{-1}\Gamma = \Gamma\eta\Gamma,$$

which implies the surjectivity. Let α, $\beta \in \Delta_N^*$ (By Lemma 5.15, for any $\alpha \in \Delta_N'$, we can find $\alpha' \in \Delta_N^*$, $\gamma \in \Gamma'$ such that $\alpha = \alpha'\gamma$, so that $\Gamma'\alpha\Gamma' = \Gamma'\alpha'\Gamma'$. Hence we can assume $\alpha \in \Delta_N^*$) . Put

$$\lambda_N(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}, \quad \lambda_N(\beta) = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}.$$

If $\Gamma\alpha\Gamma = \Gamma\beta\Gamma$, then $c \equiv \det(\alpha) = \det(\beta) \equiv d \pmod N$. By (3) of Lemma 5.13, we get $\Gamma_N\alpha\Gamma_N = \Gamma_N\beta\Gamma_N$, hence $\Gamma'\alpha\Gamma' = \Gamma'\beta\Gamma'$. This completes the proof. □

Let p be a prime, put $E_p = GL_2(\mathbb{Z}_p)$. For any $\alpha, \beta \in \Delta$, $E_p\alpha E_p = E_p\beta E_p$ if and only if the p-part of the elementary divisors of α is equal to the ones of β.

Lemma 5.16 *Let $\alpha \in \Delta'$, $\det(\alpha) = mq$, $m|N^\infty$, $(q, N) = 1$. Then the following assertions hold:*

(1) $\Gamma'\alpha\Gamma' = \{\beta \in \Delta' | \det(\beta) = mq, E_p\alpha E_p = E_p\beta E_p$ *for all prime factors p of q* $\}$.

(2) *There exists an element $\xi \in \Delta_N^*$ such that* $\det(\xi) = q$ *and $E_p\alpha E_p = E_p\xi E_p$ for any prime factor p of q.*

(3) *Let $\eta = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$ and ξ be as in* (2), *then*

$$\Gamma'\alpha\Gamma' = \Gamma'\xi\Gamma' \cdot \Gamma'\eta\Gamma' = \Gamma'\eta\Gamma' \cdot \Gamma'\xi\Gamma'.$$

Proof Let $X(\alpha)$ denote the set defined by the right hand side of (1). It is clear that $\Gamma'\alpha\Gamma' \subset X(\alpha)$ (if $\beta \in \Gamma'\alpha\Gamma'$, then

$$\det(\beta) = \det(\alpha) = mq,$$

β and α have the same elementary divisors, so that $\beta \in X(\alpha)$). To prove the opposite inclusion, let $\beta = \begin{pmatrix} a & * \\ * & * \end{pmatrix} \in X(\alpha)$. Since $(a, mN) = 1$, $ae \equiv 1 \pmod{mN}$ for some $e \in \mathbb{Z}$. Hence there exists an element $\gamma \in SL_2(\mathbb{Z})$ such that $\gamma \equiv \begin{pmatrix} e & 0 \\ 0 & a \end{pmatrix}$ $\pmod{mN}$. Since $\beta \in \Delta'$, we have that $\gamma \in \Gamma'$, and $\gamma\beta \equiv \begin{pmatrix} 1 & tb \\ fN & * \end{pmatrix} \pmod{mN}$ with integers f and b. Put $\delta = \begin{pmatrix} 1 & 0 \\ -fN & 1 \end{pmatrix}$. Then $\delta \in \Gamma'$, and $\delta\gamma\beta \equiv \begin{pmatrix} 1 & tb \\ 0 & g \end{pmatrix}$ $\pmod{mN}$ with $g \in \mathbb{Z}$. Taking the determinant, we get $mq \equiv g \pmod{mN}$, so that $\delta\gamma\beta \equiv \begin{pmatrix} 1 & tb \\ 0 & mq \end{pmatrix} \pmod{mN}$. Put $\eta = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$, $\epsilon = \begin{pmatrix} 1 & tb \\ 0 & 1 \end{pmatrix}$, $\xi = \delta\gamma\beta\epsilon^{-1}\eta^{-1}$. Then $\det(\xi) = q, \xi \equiv \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \pmod{N}$. Therefore $\xi \in \Delta_N^*$. Moreover, we see that $\beta \in \Gamma'\xi\eta\Gamma'$. For any prime factor p of q, since δ, γ, ϵ, $\eta \in E_p$, we have that

$$E_p\xi E_p = E_p\beta E_p = E_p\alpha E_p,$$

which shows (2). The element ξ may depend on $\boldsymbol{\beta}$. We want to show that $\Gamma'\xi\eta\Gamma'$ is determined only by α and independent of the choice of $\boldsymbol{\beta}$. If so, then we have $X(\alpha) \subset \Gamma'\xi\eta\Gamma'$, that is, (1) holds. Let now β_1 be an element of $X(\alpha)$. In the same way as above we can find $\xi_1 \in \Delta_N^*$ such that $\det(\xi_1) = q$ and $E_p\xi_1 E_p = E_p\alpha E_p$ for any prime factor p of q. Then ξ and ξ_1 have the same elementary divisors, hence $\Gamma\xi\Gamma = \Gamma\xi_1\Gamma$. Since $\xi \equiv \xi_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \pmod{N}$, we have $\Gamma_N\xi\Gamma_N = \Gamma_N\xi_1\Gamma_N$ by (3) of Lemma 5.13. Hence $\xi_1 = \phi\xi\psi$ with ϕ and ψ in Γ_N. By the Chinese remainder theorem, we can find an element $\theta \in M_2(\mathbb{Z})$ such that

$$\begin{aligned} &\theta \equiv I \pmod{mN}, \\ &\theta \equiv \eta^{-1}\psi^{-1}\eta \pmod{qM_2(\mathbb{Z}_p)} \quad \text{for all } p \text{ dividing } q\ . \end{aligned}$$

Then $\det(\theta) \equiv 1 \pmod{mNq}$, we can assume that $\theta \in SL_2(\mathbb{Z})$, and hence $\theta \in \Gamma_{mN}$ and $\psi\eta\theta\eta^{-1} \in \Gamma_q$ by the first and second congruence relation respectively. Put $\omega = \xi\psi\eta\theta(\xi\eta)^{-1}$. Then $\det(\omega) = 1$ and

$$\begin{aligned} &\omega \equiv 1 \pmod{NM_2(\mathbb{Z}_p)} \quad \text{for all } p \text{ dividing } N, \\ &\omega \equiv 1 \pmod{M_2(\mathbb{Z}_p)} \quad \text{for all } p \text{ dividing } q. \end{aligned}$$

Therefore $\omega \in M_2(\mathbb{Z}_p)$ for all p, so that $\omega \in M_2(\mathbb{Z})$ and $\omega \in \Gamma_N$. Since $\xi\psi\eta = \omega\xi\eta\theta^{-1}$, we see that $\Gamma'\xi_1\eta\Gamma' = \Gamma'\xi\psi\eta\Gamma' = \Gamma'\xi\eta\Gamma'$, which shows that $\Gamma'\xi\eta\Gamma'$ is determined only by α. Moreover, we have that $\Gamma'\alpha\Gamma' \subset X(\alpha) \subset \Gamma'\xi\eta\Gamma'$. Therefore these three sets must coincide, which shows (1).

We want now to prove (3). By the above proof for (1), we have that

$$X(\alpha) = \Gamma'\alpha\Gamma' = \Gamma'\xi\Gamma'\eta\Gamma' = \Gamma'\eta\Gamma'\xi\Gamma'.$$

Suppose that $\Gamma'\xi\Gamma' = \bigcup \Gamma'\xi_i$. Since $\xi \in \Delta_N^* \subset \Phi$, by (5) of Lemma 5.13, we get $\Gamma\xi\Gamma = \bigcup \Gamma\xi_i$. We have the following disjoint union (see Lemma 5.17):

$$\Gamma'\eta\Gamma' = \bigcup_{r=0}^{m-1} \Gamma' \begin{pmatrix} 1 & tr \\ 0 & m \end{pmatrix} := \bigcup_{r=0}^{m-1} \Gamma'\eta_r.$$

It is easy to verify that

$$\Gamma \begin{pmatrix} 1 & tr \\ 0 & m \end{pmatrix}, \quad r = 0, 1, \cdots, m-1$$

are different right cosets of $\Gamma\alpha\Gamma$. Since $\det(\xi) = q$ is prime to $\det(\boldsymbol{\eta}) = m$, by Lemma 5.8, we have

$$\Gamma\xi\Gamma \cdot \Gamma\eta\Gamma = \Gamma\xi\eta\Gamma = \Gamma\alpha\Gamma.$$

So the number of (i, r) such that $\Gamma\xi_i\eta_r = \Gamma\alpha$ is at most one, hence the number of (i, r) such that $\Gamma'\xi_i\eta_r = \Gamma'\alpha$ is at most one. This shows that $\Gamma'\alpha\Gamma' = \Gamma'\xi\Gamma' \cdot \Gamma\eta\Gamma'$. The product $\Gamma\eta\Gamma' \cdot \Gamma'\xi\Gamma'$ can be treated in the same way. This completes the proof. □

Lemma 5.17 *Let* $\alpha \in \Delta'$, $\det(\alpha) = m$, $m|N^\infty$. *Then we have the following disjoint union*:

$$\Gamma'\alpha\Gamma' = \{\beta \in \Delta' | \det(\beta) = m\} = \bigcup_{r=0}^{m-1} \Gamma' \begin{pmatrix} 1 & tr \\ 0 & m \end{pmatrix}.$$

Proof The first equality is a special case of Lemma 5.16. We prove now the second equality. Let $\beta \in \Delta'$, $\det(\beta) = m$. By the proof of Lemma 5.16, there exist $\delta \in \Gamma'$, $\gamma \in \Gamma'$ and $\xi \in \Gamma_N$ such that

$$\delta\gamma\beta = \xi \begin{pmatrix} 1 & tb \\ 0 & m \end{pmatrix}.$$

Write $b = r + mh$ with $0 \leqslant r \leqslant m-1$, we have that

$$\begin{pmatrix} 1 & tb \\ 0 & m \end{pmatrix} = \begin{pmatrix} 1 & th \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & tr \\ 0 & m \end{pmatrix},$$

i.e.,

$$\beta \in \Gamma' \begin{pmatrix} 1 & tr \\ 0 & m \end{pmatrix}.$$

It is not difficult to show that

$$\Gamma' \begin{pmatrix} 1 & tr \\ 0 & m \end{pmatrix} \quad (0 \leqslant r \leqslant m-1)$$

are different right cosets in $\Gamma'\alpha\Gamma'$. This completes the proof. □

For any positive integer n, let $\mathrm{T}'(n)$ be the sum of all $\Gamma'\alpha\Gamma'$ with $\alpha \in \Delta'$ and $\det(\alpha) = n$. By Lemma 5.17 we see that $\deg(\mathrm{T}'(m)) = m$. For any positive integers a, d with $a|d$, $(d, N) = 1$, we define $\mathrm{T}'(a, d) \in R(\Gamma', \Delta'_N)$ as the image of $\mathrm{T}(a, d)$ under the isomorphism in Theorem 5.4. We have that

$$\mathrm{T}'(a,d) = \Gamma'\sigma_a \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma',$$

where $\sigma_a \in \Gamma$, $\sigma_a \equiv \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \pmod N$. Therefore $\sigma_a \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \Delta_N^*$.

Theorem 5.5 (1) *$R(\Gamma', \Delta')$ is the polynomial ring over $\mathbb{Z}$ generated by*

$$\mathrm{T}'(p), \quad \forall p|N; \quad \mathrm{T}'(1,p), \quad \mathrm{T}'(p,p), \quad \forall p \nmid N;$$

(2) *Every double coset $\Gamma'\alpha\Gamma'$ with $\alpha \in \Delta'$ is uniquely expressed as a product*

$$\Gamma'\alpha\Gamma' = \mathrm{T}'(m) \cdot \mathrm{T}'(a,d) = \mathrm{T}'(a,d) \cdot \mathrm{T}'(m)$$

with $m|N^\infty$, $a|d$, $(d, N) = 1$;
(3) $\mathrm{T}'(m)\mathrm{T}'(n) = \mathrm{T}'(mn)$ *for all $m|N^\infty$, $n|N^\infty$;*
(4) $\mathrm{T}'(n_1)\mathrm{T}'(n_2) = \mathrm{T}'(n_1 n_2)$ *if $(n_1, n_2) = 1$.*

Proof The assertion (2) follows from Lemma 5.16 and Lemma 5.17. By Lemma 5.17, we see that

$$\mathrm{T}'(m)\mathrm{T}'(n) = c\mathrm{T}'(mn)$$

with a positive integer c. Taking deg, we get

$$\deg(\mathrm{T}'(m))\deg(\mathrm{T}'(n)) = c\deg(\mathrm{T}'(mn)) = c\deg(\mathrm{T}'(m))\deg(\mathrm{T}'(n)).$$

Hence $c = 1$. This shows (3). (1) follows from (2),(3) and Theorem 5.4. (4) follows from (3) and Lemma 5.8. □

By Theorem 5.4, we have that

$$p\mathrm{T}'(p,p) = \mathrm{T}'(p)^2 - \mathrm{T}'(p^2)$$

for all primes p not dividing N. Thus the multiplication of elements $\mathrm{T}'(n)$ can be reduced to that of $\mathrm{T}'(p^k)$ with primes p. If $p|N$, we have $\mathrm{T}'(p^k) = \mathrm{T}'(p)^k$. If $(p, N) = 1$, the elements $\mathrm{T}'(p^k)$ satisfy the formulae satisfied by $\mathrm{T}(p^k)$ by Theorem 5.4. We can express these facts as

Theorem 5.6 *$R(\Gamma', \Delta')$ is a homomorphic image of $R(\Gamma, \Delta)$ through the map:*

$$\begin{aligned} \mathrm{T}(n) &\mapsto \mathrm{T}'(n), && \textit{for all positive integers } n, \\ \mathrm{T}(p,p) &\mapsto \mathrm{T}'(p,p), && \textit{for all primes } p \textit{ prime to } N, \\ \mathrm{T}(p,p) &\mapsto 0, && \textit{for all primes } p \textit{ dividing} N. \end{aligned}$$

Therefore we have that

$$\mathrm{T}'(m)\mathrm{T}'(n) = \sum_{\substack{d|(m,n),\\ (d,N)=1}} d\mathrm{T}'(d,d)\mathrm{T}'(mn/d^2)$$

by equality (5.4).

Moreover, defining a formal Dirichlet series

$$D'(s) := \sum_{n=1}^{\infty} \mathrm{T}'(n)n^{-s},$$

we obtain

$$\begin{aligned} D'(s) &= \prod_p \left(\sum_{k=0}^{\infty} \mathrm{T}'(p^k)p^{-ks} \right) \\ &= \prod_{p|N} (1 - \mathrm{T}'(p)p^{-s})^{-1} \times \prod_{p\nmid N} (1 - \mathrm{T}'(p)p^{-s} + \mathrm{T}'(p,p)p^{1-2s})^{-1}. \end{aligned} \tag{5.5}$$

By the definition, we have

$$\mathrm{T}'(p) = \Gamma' \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma'$$

for every prime p. For any positive integer q prime to N, there exists an element $\sigma_q \in SL_2(\mathbb{Z})$ such that

$$\lambda_N(\sigma_q) = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \pmod{N}.$$

Therefore $\lambda_N(q\sigma_q) = \begin{pmatrix} 1 & 0 \\ 0 & q^2 \end{pmatrix}$, and $\Gamma q \cdot \sigma_q \Gamma = \mathrm{T}(q,q)$, so that

$$\mathrm{T}'(q,q) = \Gamma' q \cdot \sigma_q \Gamma'.$$

Lemma 5.18 *Let σ_a $((a,N)=1)$ be defined as above. Then, for any positive integer n, we have the following disjoint union:*

$$\mathrm{T}'(n) = \{\alpha \in \Delta' \,|\, \det(\alpha) = n\} = \bigcup_{\substack{ad=n,\\ (d,N)=1}} \bigcup_{b=0}^{d-1} \Gamma' \sigma_a \begin{pmatrix} a & tb \\ 0 & d \end{pmatrix}.$$

Proof The right hand side is clearly contained in the left hand side. To show the disjointness of the right hand side, assume $\gamma\sigma_a \begin{pmatrix} a & tb \\ 0 & d \end{pmatrix} = \sigma_u \begin{pmatrix} u & tv \\ 0 & w \end{pmatrix}$ with $\gamma \in \Gamma'$. Put $\sigma_u^{-1}\gamma\sigma_a = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then $\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & tb \\ 0 & d \end{pmatrix} = \begin{pmatrix} u & tv \\ 0 & w \end{pmatrix}$, so that $g = 0$. Since $\det(\sigma_u^{-1}\gamma\sigma_a) = 1$ and $au > 0$, we see that $e = h = 1$, hence $a = u$, $d = w$, and $vt = bt + fd$. Since $\gamma \in \Gamma'$, $f = f't$ with some $f' \in \mathbb{Z}$. Therefore $v = b + f'd$, so that $v = b$ which shows the disjointness. Now let $n = mq$, $m|N^\infty$, $(q, N) = 1$, then

$$\deg(\mathrm{T}'(n)) = \deg(\mathrm{T}'(m))\deg(\mathrm{T}'(q)) = m\sum_{d|q} d.$$

This shows that $\deg(\mathrm{T}'(n))$ coincides with the number of the cosets of the right hand side which implies the equality desired. This completes the proof. □

Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C})$, put

$$\alpha^\tau = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Then it is clear that

$$\begin{aligned} (\alpha+\beta)^\tau &= \alpha^\tau + \beta^\tau, \\ (\alpha\beta)^\tau &= \beta^\tau\alpha^\tau, \\ (c\alpha)^\tau &= c\alpha^\tau (c \in \mathbb{C}), \\ \alpha + \alpha^\tau &= \mathrm{Tr}(\alpha)\cdot I, \\ \alpha\alpha^\tau &= \det(\alpha)\cdot I. \end{aligned}$$

The map τ is called the main involution of $M_2(\mathbb{C})$. For any $\alpha \in \Delta_N^*$ with $\det(\alpha) = q$, we can find a $\sigma_a \in \Gamma$ such that

$$\sigma_q \equiv \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} \pmod N,$$

hence

$$\alpha \equiv \sigma_q\alpha^\tau \equiv \alpha^\tau\sigma_q \pmod N,$$

by $\lambda_N(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}$, $\lambda_N(\alpha^\tau) = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$. Since α and α^τ have the same elementary divisors, by (3) of Lemma 5.13, we have

$$\Gamma'\alpha\Gamma' = \Gamma'\sigma_q\alpha^\tau\Gamma' = \Gamma'\alpha^\tau\sigma_q\Gamma'.$$

It is easy to verify that $\Gamma'\sigma_q = \sigma_q\Gamma'$, so that $\Gamma'\sigma_q\Gamma' = \Gamma'\sigma_q$. Let $\Gamma'\alpha^\tau\Gamma' = \bigcup \Gamma'\alpha_i$ be a disjoint union. We can verify that $\Gamma'\sigma_q\alpha_i = \Gamma'\sigma_q\alpha_j$ if and only if $i = j$ (if

$\Gamma'\sigma_q\alpha_i = \Gamma'\sigma_q\alpha_j$, then $\sigma_q\Gamma'\alpha_i = \sigma_q\Gamma'\alpha_j$ since $\Gamma'\sigma_q = \sigma_q\Gamma'$, so that $\Gamma'\alpha_i = \Gamma'\alpha_j$). Therefore

$$\Gamma'\alpha\Gamma' = \Gamma'\sigma_q\Gamma' \cdot \Gamma'\alpha^\tau\Gamma' = \Gamma'\alpha^\tau\Gamma' \cdot \Gamma'\sigma_q\Gamma'.$$

So that we obtain

$$\Gamma'\alpha\Gamma' \cdot \Gamma'\alpha^\tau\Gamma' = \Gamma'\alpha^\tau\Gamma' \cdot \Gamma'\alpha\Gamma'.$$

This showed the following

Lemma 5.19 *$\Gamma'\alpha\Gamma'$ commutes with $\Gamma'\alpha^\tau\Gamma'$ if $\alpha \in \Delta_N^*$.*

The following lemmas will be useful in Section 9.3.

Lemma 5.20 *Let $C_n = \begin{pmatrix} 4 & 1 \\ 0 & 4n^2 \end{pmatrix}$. Then a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ with $ad - bc = 16n^2$ is in $\Gamma_0(4N)C_n\Gamma_0(4N)$ if and only if $c \equiv 0 \pmod{16N}$, $a \equiv d \equiv 0 \pmod 4$, $(a/c, N) = 1$ and $(a, b, c, d) = 1$.*

Proof It can easily be checked by using the decomposition

$$\Gamma_0(4N)C_n\Gamma_0(4N) = \Gamma_0(4N)\begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix}\Gamma_0(4N) \cdot \Gamma_0(4N)\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}\Gamma_0(4N)$$

and two identities

$$\Gamma_0(4N)\begin{pmatrix} 1 & 0 \\ 0 & n^2 \end{pmatrix}\Gamma_0(4N)$$
$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{4N}, (a, 4N) = 1, (a, b, c, d) = 1, ad - bc = n^2 \right\}$$

and

$$\Gamma_0(4N)\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}\Gamma_0(4N) = \sum_{\mu \bmod 4} \Gamma_0(4N)\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 4N\mu & 1 \end{pmatrix}.$$

□

Let N be a positive integer, ω an even Dirichlet character modulo $4N$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$. Then we put $A^* = \{A, \phi(z)\}$ where

$$\phi(z) = \omega(d)\left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-\frac{1}{2}}(cz+d)^{\frac{1}{2}}.$$

Lemma 5.21 *Let C_n be as in* Lemma 5.20 *and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)C_n\Gamma_0(4N)$. Suppose that $(b, d) = 1$. Then*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (4n)^{-1/2}\left(\frac{\operatorname{sgn} d}{-\operatorname{sgn} c}\right)\left(\frac{d}{b}\right)\left(\frac{-4}{b}\right)^{-\frac{1}{2}}(cz+d)^{\frac{1}{2}} \right\};$$

here $\operatorname{sgn} x = \dfrac{x}{|x|}$ *for $x \neq 0$ and* $\operatorname{sgn} 0 = 1$.

Proof Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in $\Gamma_0(4N)C_n\Gamma_0(4N)$ and satisfies $(b,d) = 1$. First assume $d \geqslant 0$. Note that $a \equiv d \equiv 0 \pmod 4$, $c \equiv 0 \pmod{16N}$ and $b \equiv 1 \pmod 2$. From $(b,d) = 1$ and $ad - bc = 16n^2$ it follows that $(b, 4N^2) \mid \frac{a}{4}$ and $(d, 4n^2) \mid \frac{c}{4}$. Thus there exists an integer w such that

$$\frac{a}{4} - Nwb - b \equiv \frac{c}{4} - Nwd - d \equiv 0 \pmod{4n^2}. \tag{5.6}$$

Thus the matrix

$$C = \begin{pmatrix} \frac{a}{4} - Nwb & \dfrac{-\frac{a}{4} + Nwb + b}{4n^2} \\ \frac{c}{4} - Nwd & \dfrac{-\frac{c}{4} + Nwd + d}{4n^2} \end{pmatrix}$$

is in $\Gamma_0(4N)$, and we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = C \begin{pmatrix} 4 & 1 \\ 0 & 4n^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 4Nw & 1 \end{pmatrix},$$

hence

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* &= C^*(C_n, n^{\frac{1}{2}}) \begin{pmatrix} 1 & 0 \\ 4Nw & 1 \end{pmatrix}^* \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (4n)^{-\frac{1}{2}} \left(\frac{\frac{c}{4} - Nwd}{\frac{a}{4} - Nwb} \right) \left(\frac{-4}{\frac{a}{4} - Nwb} \right)^{-\frac{1}{2}} \right. \\ &\qquad \left. \cdot \left(\frac{cz+d}{4Nwz+1} \right)^{\frac{1}{2}} (4Nwz+1)^{\frac{1}{2}} \right\}. \end{aligned}$$

By (5.6)

$$\left(\frac{-4}{\frac{a}{4} - Nwb} \right) = \left(\frac{-4}{b} \right).$$

Furthermore using (5.6) and the conditions $d \geqslant 0$, $(b,d) = 1$ one checks that

$$\left(\frac{\frac{c}{4} - Nwd}{\frac{a}{4} - Nwb} \right) \left(\frac{cz+d}{4Nwz+1} \right)^{\frac{1}{2}} (4Nwz+1)^{\frac{1}{2}} = \left(\frac{d}{b} \right) (cz+d)^{\frac{1}{2}}.$$

Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (4n)^{-\frac{1}{2}} \left(\frac{d}{b} \right) \left(\frac{-4}{b} \right)^{-\frac{1}{2}} (cz+d)^{\frac{1}{2}} \right\} \tag{5.7}$$

as we claimed.

If $d < 0$ we write $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$, use (5.7) and

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^* = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 1 \right\}.$$

This completes the proof of Lemma 5.21. □

Lemma 5.22 *Let C_n be as in* Lemma 5.20. *Every elliptic or hyperbolic conjugate class in $\Gamma_0(4N)C_n\Gamma_0(4N)$ contains an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $d > 0$, $(b,d) = 1$ and $\left(\frac{b}{f}, \frac{t^2 - 64n^2}{f^2}\right) = 1$ where $t = a + d$, $f = (d - a, b, c)$.*

Proof First recall that for t and f fixed there is a bijective correspondence between the set $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}) \middle| ad - bc = 16n^2, a + d = t, (d - a, b, c) = f \right\}$ and the set of integral binary primitive quadratic forms with discriminant $(t^2 - 64n^2)/f^2$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{1}{f}(cX^2 + (d - a)XY - bY^2)$$

conjugation by $\Gamma(1)$ corresponding to the usual action of $\Gamma(a)$ on quadratic forms. We need the following:

Sublemma Let $F(X,Y) = \alpha X^2 + \beta XY + \gamma Y^2$ be an integral binary primitive quadratic form with γ odd. Let M be a non-zero integer. Then there exists $(x,y) \in \mathbb{Z}^2$ with $(x,y) = (y,M) = 1$ and $(F(x,y),M) = 1$.

This can be seen as follows. Let p be a prime factor of M and suppose $p \mid F(x,y)$ for all $(x,y) \in \mathbb{Z}^2$ with $p \nmid y$. Because of $F(0,1) = \gamma \equiv 1 \pmod 2$, the prime p must be odd. But from

$$\begin{aligned} F(0,1) &= \gamma \equiv 0 \pmod p, \\ F(1,1) &= \alpha + \beta + \gamma \equiv 0 \pmod p, \\ F(2,1) &= 4\alpha + 2\beta + \gamma \equiv 0 \pmod p, \end{aligned}$$

we then conclude $p \mid (\alpha, \beta, \gamma)$, a contradiction, since F is primitive. Now for each prime factor p_ν of M choose a pair $(x_\nu, y_\nu) \in \mathbb{Z}^2$ with $p_\nu \nmid y_\nu$ and $p_\nu \nmid F(x_\nu, y_\nu)$. Determine x, $y \in \mathbb{Z}$ with $x \equiv x_\nu \pmod p$, $y \equiv y_\nu \pmod p$ for all ν. If we put $x' = x/(x,y)$, $y' = y/(x,y)$ we have $(x',y') = 1 = (y',M)$ and $(F(x',y'),M) = 1$.

Now let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an elliptic or hyperbolic element in $\Gamma_0(4N)C_n\Gamma_0(4N)$ and $F(X,Y) = \frac{1}{f}(cX^2 + (d - a)XY - bY^2)$ its associated quadratic form. Note that b is odd and $(t^2 - 64n^2)/f^2 \neq 0$. Applying the Sublemma with $M = 4N(t^2 - 64n^2)/f^2$

we see that there exists $(x, y) \in \mathbb{Z}$ and $z \in \mathbb{Z}$ such that $(x, y) = (y, 4N) = 1$, $F(x, y) = z$ and $(z, (t^2 - 64n^2)/f^2) = 1$. Consequently we may transform F with a matrix $\begin{pmatrix} * & x \\ * & y \end{pmatrix} \in \Gamma_0(4N)$ into a form whose coefficient at Y^2 is z, i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\Gamma_0(4N)$-conjugate to an element $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ with $(b'/f, (t^2 - 64n^2)/f^2) = 1$.

Now observe

$$\begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & b'' \\ * & d'' \end{pmatrix}$$

with $b'' = a'm + b' - m(c'm + d')$ and $d'' = c'm + d'$; hence if we choose $m \in \mathbb{Z}$ in such a way that m is prime to (b', d') and divisible by all prime factors of $16n^2$, which do not divide (b', d'), and divisible by all primes dividing $(t^2 - 64n^2)/f^2$ (note that $(t^2 - 64n^2)/f^2$ is prime to (b', d')), then $(b'', d'') = 1$ and $(b''/f, (t^2 - 64n^2)/f^2) = 1$. Moreover, if we choose $|m|$ big with $\operatorname{sgn}(m) = \operatorname{sgn}(c)$ we have $d'' > 0$ which completes the proof. □

Lemma 5.23 *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ *be an elliptic or hyperbolic matrix with* $\det A = 16n^2$, $t \equiv 0 \pmod 4$ *and* f *odd. Then for* N *odd and square-free there are* $\mu(t/4, f, n^2, N)$ *matrices* $B \in \Gamma(1)/\Gamma_0(4N)$ *with* $B^{-1}AB \in \Gamma_0(4N)C_n\Gamma_0(4N)$ *where* t, f *were defined as in Lemma* 5.22 *and*

$$\begin{aligned} \mu(t, f, n, M) = \prod_{p|(M,f)} (1 + p) \times \#\{x \in \mathbb{Z} \mid 1 \leqslant x \leqslant M, \\ (x, M) = 1, \quad x^2 - tx + n \equiv 0 \pmod{(fM, M^2)}\}. \end{aligned}$$

Proof Denote by n_A the number of matrices $B \in \Gamma(1)/\Gamma_0(4N)$ with $B^{-1}AB \in \Gamma_0(4N)C_n\Gamma_0(4N)$. For $x \in \mathbb{Z}$ with $1 \leqslant x \leqslant N, (x, N) = 1$ put

$$V_x = \left\{ B \in \Gamma(1) \middle| B^{-1}AB \equiv \begin{pmatrix} 4x + 4N\nu & * \\ 0 & * \end{pmatrix} \pmod{16N} \text{ for some } \nu \in \mathbb{Z} \right\},$$

$$V_{x,1} = \left\{ B \in \Gamma(1) \middle| B^{-1}AB \equiv \begin{pmatrix} 4x & * \\ 0 & * \end{pmatrix} \pmod{16} \text{ for some } \nu \in \mathbb{Z} \right\}$$

and

$$V_{x,2} = \left\{ B \in \Gamma(1) \middle| B^{-1}AB \equiv \begin{pmatrix} 4x + 4N\nu & * \\ 0 & * \end{pmatrix} \pmod{16} \text{ for some } \nu \in \mathbb{Z} \right\}.$$

The group $\Gamma_0(4N)$ resp. $\Gamma_0(N)$ operates on V_x resp.$V_{x,1}$ by multiplication from right. Noticing $t \equiv 0 \pmod 4$ and Lemma 5.20 one sees that

$$\mathrm{n}_A = \sum_{\substack{1 \leqslant x \leqslant N, \\ (x,N)=1}} \#\{V_x/\Gamma_0(4N)\}.$$

We wish to show that

$$\#\{V_x/\Gamma_0(4N)\} = \#\{V_{x,1}/\Gamma_0(N)\}. \tag{5.8}$$

The latter number is equal to $\prod_{p|(N,f)} (1+p)$ or 0 according to

$$(4x)^2 - 4tx + 16n^2 \equiv 0 \pmod{(fN, N^2)}$$

or not by J. Oesterlé, 1977. Noticing that N and f are odd, we will then prove the lemma.

Now if $(B_1, B_2) \in V_{x,1} \times V_{x,2}$ choose $B \in \Gamma(1)$ with $B \equiv B_1 \pmod N$, $B \equiv B_2 \pmod{16}$ and define $\phi(B_1, B_2)$ to be the class of B mod $\Gamma_0(4N)$. Then ϕ is a well-defined map from $V_{x,1} \times V_{x,2}$ onto $V_x/\Gamma_0(4N)$, and $\phi(B_1, B_2) = \phi(B_1', B_2')$ if and only if B_1 and B_1' are equivalent under $\Gamma_0(N)$ and B_2 and B_2' are equivalent under $\Gamma_0(4)$.

To prove (5.8) it remains to show that modulo multiplication from the right there is exactly one equivalence class in $V_{x,2}$. Fix $\nu \in \mathbb{Z}$. If $B = \begin{pmatrix} u & w \\ v & z \end{pmatrix}$, the condition

$$B^{-1}AB \equiv \begin{pmatrix} 4x + 4N\nu & * \\ 0 & * \end{pmatrix} \pmod{16} \tag{5.9}$$

is equivalent to

$$\begin{pmatrix} a - 4(x+N\nu) & b \\ c & d - 4(x+N\nu) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \equiv 0 \pmod{16}. \tag{5.10}$$

From $ad - bc = 16n^2$ and $4|t$ we see that $(a - 4(x+N\nu))(d - 4(x+N\nu)) - bc \equiv 0 \pmod{16}$, and from $2 \nmid f$ it follows that $(a - 4(x+N\nu),\, b,\, c,\, d - 4(x+N\nu),\, 16) = 1$. Thus the system (5.10) has exactly one solution (u mod 16, v mod 16) in $P(\mathbb{Z}/16\mathbb{Z})$ (the projective line over $\mathbb{Z}/16\mathbb{Z}$), hence exactly one solution in $P(\mathbb{Z}/4\mathbb{Z})$. Since $\Gamma(1)/\Gamma_0(4)$ is one-to-one correspondence with $P(\mathbb{Z}/4\mathbb{Z})$ under the map $\begin{pmatrix} u & w \\ v & z \end{pmatrix} \mapsto$ (u mod 4, v mod 4), we conclude that modulo $\Gamma_0(4)$ the congruence (5.10) has exactly one solution.

Now if m is an odd integer we can choose $a', b' \in \mathbb{Z}$ with $a'Nm - 4b' = 1$. Then $\begin{pmatrix} a' & b' \\ 4 & Nm \end{pmatrix}$ is in $\Gamma_0(4)$ and

$$\begin{pmatrix} a' & b' \\ 4 & Nm \end{pmatrix}^{\nu-\nu'} \begin{pmatrix} 4x+4N\nu & m \\ 0 & 4y \end{pmatrix} \begin{pmatrix} a' & b' \\ 4 & Nm \end{pmatrix}^{\nu'-\nu}$$
$$\equiv \begin{pmatrix} 4x+4N\nu' & * \\ 0 & * \end{pmatrix} \pmod{16}.$$

This shows that there is indeed only one equivalence class in $V_{x,2}$ which completes the proof of Lemma 5.23. □

5.2 A Representation of the Hecke Ring on the Space of Modular Forms

In this section we shall consider a representation of the Hecke ring on the space of modular forms. We recall first some notations:

$$j(\sigma, z) = cz + d, \quad \forall z \in \mathbb{H}, \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}),$$
$$f|[\sigma]_k = \det(\sigma)^{k/2} f(\sigma(z)) j(\sigma, z)^{-k}$$

for a function on $\mathbb{H}$.

Let now $G = GL_2^+(\mathbb{R})$, Γ a Fuchsian group (of the first kind) of $SL_2(\mathbb{R})$ and $\widetilde{\Gamma}$ the commensurator of Γ in $GL_2^+(\mathbb{R})$. For $\alpha \in \widetilde{\Gamma}$, put $\Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha_i$ (disjoint union). We define a linear operator on $A_k(\Gamma)$:

$$f|[\Gamma\alpha\Gamma]_k = \det(\alpha)^{k/2-1} \sum_i f|[\alpha_i]_k, \quad \forall f \in A_k(\Gamma).$$

It is clear that the definition of $f|[\Gamma\boldsymbol{\alpha}\Gamma]_k$ is independent of the choice of the representatives α_i.

Lemma 5.24 *$[\Gamma\alpha\Gamma]_k$ maps $A_k(\Gamma)$, $G_k(\Gamma)$, $S_k(\Gamma)$ into $A_k(\Gamma)$, $G_k(\Gamma)$, $S_k(\Gamma)$ respectively.*

Proof Let $f \in A_k(\Gamma)$, then $f|[\alpha_i]_k \in A_k(\alpha_i^{-1}\Gamma\alpha_i)$. Put

$$\Gamma_1 = \bigcap_i (\alpha_i^{-1}\Gamma\alpha_i \bigcap \Gamma).$$

It is clear that $f|[\Gamma\boldsymbol{\alpha}\Gamma]_k \in A_k(\Gamma_1)$. Since $[\Gamma : \alpha_i^{-1}\Gamma\alpha_i \bigcap \Gamma] < \infty$, it is easy to show that $[\Gamma : \Gamma_1] < \infty$ and Γ, Γ_1 have the same set of cusp points. For any $\delta \in \Gamma$, the set $\{\Gamma\alpha_i\delta\}$ is a permutation of $\{\Gamma\alpha_i\}$, so that

$$f|[\Gamma\alpha\Gamma]_k|[\delta]_k = \det(\alpha)^{k/2-1} \sum_i f|[\alpha_i\delta]_k = \det(\alpha)^{k/2-1} \sum_i f|[\alpha_i]_k = f|[\Gamma\alpha\Gamma]_k.$$

So $f|[\Gamma\alpha\Gamma] \in A_k(\Gamma)$. By the above proof, we have also that $[\Gamma\alpha\Gamma]_k$ sends $G_k(\Gamma)$ and $S_k(\Gamma)$ into $G_k(\Gamma), S_k(\Gamma)$ respectively. This completes the proof. □

For any $X = \sum c_\xi \Gamma\xi\Gamma \in R(\Gamma, \widetilde{\Gamma})$, we define

$$f|[X]_k = \sum c_\xi f|[\Gamma\xi\Gamma]_k, \quad \forall f \in A_k(\Gamma).$$

Lemma 5.25 *Let $X, Y \in R(\Gamma, \widetilde{\Gamma})$. Then*

$$[X \cdot Y]_k = [X]_k \cdot [Y]_k.$$

Proof It is sufficient to show that

$$[\Gamma\alpha\Gamma]_k \cdot [\Gamma\beta\Gamma]_k = [\Gamma\alpha\Gamma \cdot \Gamma\beta\Gamma]_k, \quad \forall \alpha, \beta \in \widetilde{\Gamma}.$$

Let $\Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha_i$, $\Gamma\beta\Gamma = \bigcup_j \Gamma\boldsymbol{\beta}_j$ and

$$\Gamma\alpha\Gamma \cdot \Gamma\beta\Gamma = \sum_\xi c_\xi \Gamma\xi\Gamma, \quad \Gamma\xi\Gamma = \bigcup_h \Gamma\xi_h$$

be disjoint unions. By the definition of multiplication, we see that

$$\sum_{i,j} \Gamma\alpha_i\beta_j = \sum_{\xi,h} c_\xi \Gamma\xi_h.$$

Therefore we have

$$\begin{aligned}(f|[\Gamma\alpha\Gamma]_k)|[\Gamma\beta\Gamma]_k &= \det(\alpha\beta)^{\frac{k}{2}-1} \sum_{i,j} f|[\alpha_i\beta_j]_k \\ &= \det(\alpha\beta)^{\frac{k}{2}-1} \sum_{\xi,h} c_\xi f|[\xi_h]_k \\ &= \sum_\xi c_\xi f|[\Gamma\xi\Gamma]_k = f|[\Gamma\alpha\Gamma \cdot \Gamma\beta\Gamma]_k.\end{aligned}$$

This completes the proof. □

By the above lemmas we see that the action of $R(\Gamma, \widetilde{\Gamma})$ on $A_k(\Gamma)$ (resp. $G_k(\Gamma)$, $S_k(\Gamma)$) defines a representation of the Hecke ring on the space of modular forms.

Let $f, g \in G_k(\Gamma)$. Then $f(z)\overline{g(z)}y^k$ and $y^{-2}\mathrm{d}x\mathrm{d}y$ is invariant under Γ. Therefore the following integral is well-defined if it is convergent:

$$\int_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)}y^{k-2}\mathrm{d}x\mathrm{d}y.$$

We consider now the convergence of the integral. Since f, g are holomorphic on $\mathbb{H}$, it is sufficient to consider the convergence of it at cusp points of Γ. Let s be a cusp point of Γ, and $\rho \in SL_2(\mathbb{R})$ such that $\rho(s) = \infty$. Put $w = \rho(z)$, $q = \mathrm{e}^{\pi i w/h}$ with $h > 0$ defined as in Chapter 3, then we have

$$f|[\rho^{-1}]_k = \phi(q), \quad g|[\rho^{-1}]_k = \psi(q)$$

and ϕ, ψ are holomorphic at $q = 0$. Then

$$\begin{aligned}f(z)\overline{g(z)}y^k &= f(\rho^{-1}w)\overline{g(\rho^{-1}w)}(\mathrm{Im}(w))^k |J(\rho^{-1}, w)|^{-2k} \\ &= \phi(q)\overline{\psi(q)}(\mathrm{Im}(w))^k.\end{aligned}$$

If either f or g is in $S_k(\Gamma)$, then $\phi(0)\overline{\psi(0)} = 0$, so that the integral is convergent at $w = \infty$, and hence it is convergent at $z = s$. Put

$$\langle f, g\rangle = \frac{1}{\mu(D)} \int_D f(z)\overline{g(z)} y^{k-2} \mathrm{d}x\mathrm{d}y,$$

where D is a fundamental domain of Γ and

$$\mu(D) = \int_D y^{-2} \mathrm{d}x\mathrm{d}y.$$

We call $\langle f, g\rangle$ the Petersson inner product of f and g. It is easy to verify that $S_k(\Gamma)$ is a Hilbert space under the Petersson inner product. If Γ' is a subgroup of Γ and $[\Gamma : \Gamma'] < \infty$, then the Petersson inner product of f and g on Γ is equal to the one of f and g on Γ'.

Let $\alpha \in GL_2^+(\mathbb{R})$. Then $f|[\alpha]_k$, $g|[\alpha]_k \in A_k(\alpha^{-1}\Gamma\alpha)$. Denote by D' a fundamental domain of $\alpha^{-1}\Gamma\alpha$, then $\alpha(D')$ is a fundamental domain of Γ. So that

$$\begin{aligned}\langle f|[\alpha]_k, g|[\alpha]_k\rangle &= \det(\alpha)^k (\mu(D'))^{-1} \int_{D'} f(\alpha(z))\overline{g(\alpha(z))}|J(\alpha, z)|^{-2k} y^{k-2} \mathrm{d}x\mathrm{d}y \\ &= (\mu(D'))^{-1} \int_{\alpha(D')} f(z)\overline{g(z)} y^{k-2} \mathrm{d}x\mathrm{d}y = \langle f, g\rangle. \end{aligned} \tag{5.11}$$

Lemma 5.26 *Let f, $g \in S_k(\Gamma)$, $\alpha \in \widetilde{\Gamma}$, and $\alpha^\tau = \det(\alpha)\alpha^{-1}$ be the main involution of α. Then we have*

$$\langle f|[\Gamma\alpha\Gamma]_k, g\rangle = \langle f, g|[\Gamma\alpha^\tau\Gamma]_k\rangle.$$

Proof Let D be a fundamental domain of Γ and $\Gamma = \bigcup_i (\Gamma \bigcap \alpha^{-1}\Gamma\alpha)\epsilon_i$ (disjoint union) with $\epsilon_i \in \Gamma$. Then $\Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha\epsilon_i$ and $D_1 = \bigcup_i \epsilon_i(D)$ is a fundamental domain of $\Gamma \bigcap \alpha^{-1}\Gamma\alpha$. By (5.11), we have

$$\begin{aligned}\mu(D)\langle f|[\Gamma\alpha\Gamma]_k, g\rangle &= \det(\alpha)^{k/2-1} \sum_i \int_D (f|[\alpha\epsilon_i]_k)(\overline{g}|[\epsilon_i]_k) y^{k-2} \mathrm{d}x\mathrm{d}y \\ &= \det(\alpha)^{k/2-1} \sum_i \int_{\epsilon_i(D)} (f|[\alpha]_k)\overline{g} y^{k-2} \mathrm{d}x\mathrm{d}y,\end{aligned}$$

$$\begin{aligned}\det(\alpha)^{k/2-1} \int_{D_1} (f|[\alpha]_k)\overline{g} y^{k-2} \mathrm{d}x\mathrm{d}y &= \det(\alpha)^{k/2-1} \int_{\alpha(D_1)} f \cdot (\overline{g}|[\alpha^{-1}]_k) y^{k-2} \mathrm{d}x\mathrm{d}y \\ &= \det(\alpha)^{k/2-1} \int_{\alpha(D_1)} f \cdot (\overline{g}|[\alpha^\tau]_k) y^{k-2} \mathrm{d}x\mathrm{d}y,\end{aligned}$$

where we used the fact $g|[\alpha^{-1}]_k = g|[\alpha^\tau]_k$.

Let $\Gamma=\bigcup_j(\Gamma\bigcap(\alpha^\tau)^{-1}\Gamma\alpha^\tau)\epsilon'_j$. Then $\Gamma\alpha^\tau\Gamma=\bigcup_j\Gamma\alpha^\tau\epsilon'_j$ and $\bigcup_j\epsilon'_j(D)$ is a fundamental domain of $\Gamma\bigcap\alpha\Gamma\alpha^{-1}$ (since $\alpha\Gamma\alpha^{-1}=(\alpha^\tau)^{-1}\Gamma\alpha^\tau$). Therefore we have

$$\begin{aligned}\mu(D)\langle f,g|[\Gamma\alpha^\tau\Gamma]_k\rangle&=\det(\alpha)^{k/2-1}\sum_j\int_D f\cdot(\overline{g}|[\alpha^\tau\epsilon'_j]_k y^{k-2}\mathrm{d}x\mathrm{d}y\\&=\det(\alpha)^{k/2-1}\sum_j\int\limits_{\epsilon'_j(D)} f\cdot\overline{g}|[\alpha^\tau]_k y^{k-2}\mathrm{d}x\mathrm{d}y\\&=\det(\alpha)^{k/2-1}\int\limits_{\alpha(D_1)} f\cdot(\overline{g}|[\alpha^\tau]_k)y^{k-2}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

This completes the proof. □

In fact, by the above proof, we see that the lemma holds if either f or g is in $S_k(\Gamma)$.

We consider now the case $\Gamma=SL_2(\mathbb{Z})$. Let $\alpha\in\Delta=M_2^+(\mathbb{Z})$. Since α and α^τ have the same elementary divisors, we have that $\Gamma\alpha\Gamma=\Gamma\alpha^\tau\Gamma$, so that

$$\langle f|[\Gamma\alpha\Gamma]_k,g\rangle=\langle f,g|[\Gamma\alpha\Gamma]_k\rangle,\quad f,g\in S_k(\Gamma).$$

This shows that the operators $\{[\Gamma\alpha\Gamma]_k\}_{\alpha\in\Delta}$ are commutative and self-associated operators on $S_k(\Gamma)$. Therefore there exists a basis of $S_k(\Gamma)$ whose every element is a common eigenfunction of these operators and the corresponding eigenvalues are real numbers. For any $f\in S_k(\Gamma)$, we have

$$f|[\mathrm{T}(p,p)]_k=p^{k-2}f.$$

Since R_p is generated by $\mathrm{T}(p,p)$ and $\mathrm{T}(p)$, f is a common eigenfunction of all operators $\{[\Gamma\alpha\Gamma]_k\}_{\alpha\in\Delta}$ if and only if f is a common eigenfunction of $[\mathrm{T}(p)]_k$ for any prime p.

Theorem 5.7 *Let* $f=\sum_{n=0}^{\infty}c(n)e(nz)\in G_k(\Gamma)$ *not be a constant. Assume that for any positive integer* n, *we have*

$$f|[\mathrm{T}(n)]_k=\lambda_n f,\quad \lambda_n\in\mathbb{R}.$$

Then $c(1)\neq 0$, $c(n)=\lambda_n c(1)$, *and we have formally*

$$\sum_{n=1}^{\infty}\lambda_n n^{-s}=\prod_p(1-\lambda_p p^{-s}+p^{k-1-2s})^{-1}.\tag{5.12}$$

Conversely, if we have formally that

$$\sum_{n=1}^{\infty}c(n)n^{-s}=\prod_p(1-c(p)p^{-s}+p^{k-1-2s})^{-1},\tag{5.13}$$

then $f|[\mathrm{T}(n)]_k=c(n)f$ *for any positive integer* n.

Proof By Lemma 5.10, we have

$$\begin{aligned}\lambda_n f = f|[\mathrm{T}(n)]_k &= n^{k/2-1} \sum_{ad=n, a>0} \sum_{b=0}^{d-1} f\Big| \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}_k \\ &= n^{k-1} \sum_{ad=n} \sum_{b=0}^{d-1} \sum_{m=0}^{\infty} c(m) e(m(az+b)/d) d^{-k} \\ &= \sum_{ad=n} a^{k-1} d^{-1} \sum_{m=0}^{\infty} c(m) e(maz/d) \sum_{b=0}^{d-1} e(mb/d) \\ &= \sum_{t=1}^{\infty} \sum_{a|n} a^{k-1} c(nt/a) e(taz) \\ &= \sum_{m=0}^{\infty} \sum_{a|(n,m)} a^{k-1} c(mn/a^2) e(mz). \end{aligned} \tag{5.14}$$

Comparing the coefficient of $e(z)$ of both sides, we get

$$\lambda_n c(1) = c(n).$$

Since f is not a constant, $c(1) \neq 0$. Now Theorem 5.3 gives equality (5.12). The convergence of the series will be proved in next section.

We assume now that (5.13) holds. Put

$$\begin{aligned}\sum_{r=0}^{\infty} b(p^r) p^{-rs} &= (1 - c(p)p^{-s} + p^{k-1-2s})^{-1} = (1 - Ap^{-s})^{-1}(1 - Bp^{-s})^{-1} \\ &= \sum_{r=0}^{\infty} \frac{A^{r+1} - B^{r+1}}{A - B} p^{-rs},\end{aligned}$$

where A, B satisfy that $A + B = c(p)$, $AB = p^{k-1}$. Hence

$$b(p^r) = \frac{A^{r+1} - B^{r+1}}{A - B} = \sum_{i=0}^{r} A^{r-i} B^i.$$

For any $r \leqslant l$, we have

$$\begin{aligned} b(p^l)b(p^r) &= (A^{l+1} b(p^r) - B^{l+1} b(p^r))(A - B)^{-1} \\ &= \left(A^{l+1} \sum_{i=0}^{r} A^{r-i} B^i - B^{l+1} \sum_{i=0}^{r} A^i B^{r-i} \right)(A - B)^{-1} \\ &= \sum_{i=0}^{r} A^i B^i (A^{l+1+r-2i} - B^{l+1+r-2i})(A - B)^{-1} \\ &= \sum_{i=0}^{r} p^{i(k-1)} b(p^{l+r-2i}) = \sum_{a|(p^l, p^r)} a^{k-1} b(p^{l+r}/a^2). \end{aligned}$$

Let $0 \neq n = \prod p^{n_p}$ be the standard factorization of n. By (5.13), we obtain that

$$c(n) = \prod_{p|n} b(p^{n_p}).$$

Let $m = \prod p^{m_p}$. Then we have

$$\begin{aligned} c(n)c(m) &= \prod_p b(p^{n_p})b(p^{m_p}) \\ &= \prod_p \sum_{a|(p^{n_p}, p^{m_p})} a^{k-1} b\left(\frac{p^{n_p+m_p}}{a^2}\right) = \sum_{a|(m,n)} a^{k-1} c(mn/a^2). \end{aligned}$$

By the above equality and (5.14), we obtain that $f|[\mathrm{T}(n)]_k = c(n)f$. This completes the proof. □

We consider now the case that Γ' is a congruence subgroup of Γ. For any $\alpha \in \widetilde{\Gamma'}$, we define a linear operator $[\Gamma'\alpha\Gamma']_k$ on the space $A_k(\Gamma')$. Since $R(\Gamma', \Delta')$ is a commutative ring, the elements of $R(\Gamma', \Delta')$ give linear operators on $A_k(\Gamma')$ which commute one another. And by Lemma 5.19, we see that $[\Gamma'\alpha\Gamma']_k$ with $\alpha \in \Delta_N^*$ is a normal operator on $A_k(\Gamma')$. If $\alpha \in \Delta'$ is such that $\det(\alpha) = m|N^\infty$, then by Lemma 5.17, we see that $\Gamma'\alpha\Gamma' = \Gamma\alpha^\tau\Gamma'$. This implies that $[\Gamma'\alpha\Gamma']_k$ is a self-associated operator on $A_k(\Gamma')$. Since $S_k(\Gamma')$ is a finite dimensional vector space, we see that there exists a basis of $S_k(\Gamma')$ whose every element is a common eigenfunction of the operators $[\Gamma'\alpha\Gamma']_k$ for all $\alpha \in \Delta'$.

For a fixed positive divisor t of N, put

$$\begin{aligned} \Gamma_0' &= \left\{\gamma \in SL_2(\mathbb{Z}) \middle| \lambda_N(\gamma) = \begin{pmatrix} a & tb \\ 0 & a^{-1} \end{pmatrix}, a \in (\mathbb{Z}/N\mathbb{Z})^*, b \in \mathbb{Z}/N\mathbb{Z}\right\}, \\ \Gamma'' &= \left\{\gamma \in \Gamma_0' \middle| \lambda_N(\gamma) = \begin{pmatrix} 1 & tb \\ 0 & 1 \end{pmatrix}, b \in \mathbb{Z}/N\mathbb{Z}\right\}, \\ \Delta_0' &= \left\{\alpha \in \Delta \middle| \lambda_N(\alpha) = \begin{pmatrix} a & tb \\ 0 & d \end{pmatrix}, a \in (\mathbb{Z}/N\mathbb{Z})^*, b, d \in \mathbb{Z}/N\mathbb{Z}\right\}, \\ \Delta'' &= \left\{\alpha \in \Delta \middle| \lambda_N(\alpha) = \begin{pmatrix} 1 & tb \\ 0 & d \end{pmatrix}, b, d \in \mathbb{Z}/N\mathbb{Z}\right\}. \end{aligned}$$

It is clear that $\Gamma_0' = \Gamma_0(N)$ if $t = 1$. We denote by $\Gamma_1(N)$ the group Γ'' if $t = N$.

It is clear that Γ'' is a normal subgroup of Γ_0' and $\Gamma_0'/\Gamma'' \simeq (\mathbb{Z}/N\mathbb{Z})^*$. For any $f \in G_k(\Gamma'')$, $\gamma \in \Gamma_0'$, we see that $f|[\gamma]_k \in G_k(\Gamma'')$. Hence we get a representation of Γ_0' on $G_k(\Gamma'') : f \mapsto f|[\gamma]_k$. If $\gamma \in \Gamma''$, then $f|[\gamma]_k = f$, so that the representation induces a representation of Γ_0'/Γ'' on the space $G_k(\Gamma'')$. This implies that the space $G_k(\Gamma'')$ is a direct sum of the spaces $G_k(\Gamma_0', \psi)$, where ψ is a character modulo N

with $\psi(-1) = (-1)^k$ and

$$G_k(\Gamma_0', \psi) = \left\{ f \in G_k(\Gamma'') \middle| f|[\gamma]_k = \psi(d_\gamma) f \text{ for all } \gamma = \begin{pmatrix} * & * \\ * & d_\gamma \end{pmatrix} \in \Gamma_0' \right\}.$$

Let $\alpha \in \Delta_0'$ and $\Gamma_0' \alpha \Gamma_0' = \bigcup_v \Gamma_0' \alpha_v$ be a disjoint union. For $f \in G_k(\Gamma_0', \psi)$, put

$$f|[\Gamma_0' \alpha \Gamma_0']_{k,\psi} = (\det(\alpha))^{k/2-1} \sum_v \psi^{-1}(d_v) f|[\alpha_v]_k,$$

where $\alpha_v = \begin{pmatrix} * & * \\ * & d_v \end{pmatrix}$. It is easy to verify that $[\Gamma_0' \alpha \Gamma_0']_k$ is a linear operator on $G_k(\Gamma_0', \psi)$. Therefore we obtain a representation of $R(\Gamma_0', \Delta_0')$ on the space $G_k(\Gamma_0', \psi)$. Denote by $\mathrm{T}'(a,d)_{k,\psi}$ and $\mathrm{T}'(m)_{k,\psi}$ the actions of $\mathrm{T}'(a,d)$ and $\mathrm{T}'(m)$ on the space $G_k(\Gamma_0', \psi)$ respectively.

The subgroup fixing i∞ of Γ_0' is generated by $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. For any $f \in G_k(\Gamma_0')$, $f(z)$ has the following Fourier expansion at i∞:

$$f(z) = \sum_{n=0}^{\infty} c(n) e(nz/t).$$

Let m be a positive integer, put

$$g(z) := f|\mathrm{T}'(m)_{k,\psi} = \sum_{n=0}^{\infty} c'(n) e(nz/t).$$

By Lemma 5.18 and $\sigma_a \in \Gamma_0'$, we see that

$$\begin{aligned} g(z) &= \sum_{ad=m, a>0} \sum_{b=0}^{d-1} \psi(a) f\left(\frac{az+bt}{d}\right) d^{-k} \\ &= \sum_{n=0}^{\infty} c(n) \sum_{ad=m, a>0} a^{k-1} \psi(a) e(naz/\mathrm{d}t) d^{-1} \sum_{b=0}^{d-1} e(nb/d) \\ &= \sum_{n=0}^{\infty} \sum_{a|m} a^{k-1} \psi(a) c(nm/a) e(anz/t) \\ &= \sum_{n=0}^{\infty} \sum_{a|(n,m)} a^{k-1} \psi(a) c(nm/a^2) e(nz/t), \end{aligned}$$

so that,

$$c'(n) = \sum_{a|(n,m)} a^{k-1} \psi(a) c(nm/a^2).$$

If q is prime to N, then

$$f|\mathrm{T}'(q,q)_{k,\psi} = q^{k-2}\psi(q)f, \quad \forall f \in G_k(\Gamma_0', \psi).$$

By (5.5), we see that

$$\sum_{n=1}^{\infty} \mathrm{T}'(n)_{k,\psi} n^{-s} = \prod_p (1 - \mathrm{T}'(p)_{k,\psi} p^{-s} + \mathrm{T}'(p,p)_{k,\psi} p^{1-2s})^{-1}.$$

Therefore, similar to Theorem 5.7, we obtain the following:

Theorem 5.8 *Let* $0 \neq f = \sum_{n=0}^{\infty} c(n)e(nz/t) \in G_k(\Gamma_0', \psi)$ *be a common eigenfunction for all* Hecke *operators* $\mathrm{T}'(n)_{k,\psi}$:

$$f|\mathrm{T}'(n)_{k,\psi} = \lambda_n f.$$

Then $c(1) \neq 0, c(n) = \lambda_n c(1)$ *and we have formally*

$$\sum_{n=1}^{\infty} \lambda_n n^{-s} = \prod_p (1 - \lambda_p p^{-s} + \psi(p)p^{k-1-2s})^{-1}.$$

Conversely, if we have formally

$$\sum_{n=1}^{\infty} c(n) n^{-s} = \prod_p (1 - c(p)p^{-s} + \psi(p)p^{k-1-2s})^{-1},$$

then we have $f|\mathrm{T}'(n)_{k,\psi} = c(n)f$*, where* $\psi(p) = 0$ *if* $p|N$*.*

5.3 Zeta Functions of Modular Forms, Functional Equation, Weil Theorem

Let $f(z) = \sum_{n=0}^{\infty} c(n)q^n \in G(N,k,\omega)$. Define its Zeta function as follows:

$$L(s,f) = \sum_{n=1}^{\infty} c(n)n^{-s}, \quad s \in \mathbb{C}.$$

In this section we shall discuss the convergence, analytic continuation and functional equation of $L(s,f)$.

Lemma 5.27 *Let* $f(z)$ *be as above. Then there exists a constant* A *such that* $|f(z)| \leqslant A\mathrm{Im}(z)^{-k}$ *for* $z \in \mathbb{H}$ *and* $c(n) = O(n^k)$*.*

Proof Let $s = d/c$ be a cusp point of $\Gamma_0(N)$. Take

$$\rho = \begin{pmatrix} a & b \\ -c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

such that $\rho(s) = \mathrm{i}\infty$. By the definition of a holomorphic modular form, $f(\rho^{-1}(z))(cz+a)^{-k}$ is holomorphic at $z = \mathrm{i}\infty$. Therefore we see that

$$\lim_{z\to i\infty} f\left(\frac{dz-b}{cz+a}\right)(cz+a)^{-k} = \lim_{\tau\to 0}(-c\tau)^k f(\tau+s)$$

is a finite constant. Hence there exists a constant A_s such that

$$|f(z)| \leqslant A_s|z-s|^{-k} \leqslant A_s\mathrm{Im}(z)^{-k}$$

holds for all z nearby s. Since $\Gamma_0(N)\setminus\mathbb{H}^*$ is a compact Riemann surface, there exists a constant A such that

$$|f(z)| \leqslant A\mathrm{Im}(z)^{-k}$$

for any $z \in \mathbb{H}$. By the Cauchy integral theorem, we have

$$c(n) = \frac{1}{2\pi \mathrm{i}} \int\limits_{|q|=r} f(q)q^{-n-1}\mathrm{d}q.$$

Take $\mathrm{Im}(z) = \dfrac{1}{2\pi n}$, i.e., $r = \mathrm{e}^{-1/n}$, we obtain that

$$|c(n)| \leqslant \frac{1}{2\pi} \int\limits_{|q|=\mathrm{e}^{-1/n}} |f(q)|\mathrm{e}^{1+1/n}\mathrm{d}q \leqslant A(2\pi n)^k.$$

This completes the proof. □

Lemma 5.28 *Let* $f(z) = \sum_{n=1}^{\infty} c(n)q^n \in S(N,k,\omega)$. *Then there exists a constant* A *such that* $|f(z)| \leqslant \mathrm{Im}(z)^{-k/2}$ *with* $z \in \mathbb{H}$ *and* $c(n) = O(n^{k/2})$.

Proof Put $g(z) = f(z)\mathrm{Im}(z)^{k/2}$. Then

$$g(\gamma(z)) = g(z)$$

for any $\gamma \in \Gamma_0(N)$. Let s be a cusp point of $\Gamma_0(N)$. Take $\rho \in SL_2(\mathbb{Z})$ such that $\rho(s) = \mathrm{i}\infty$. By the definition of a cusp form, we see that $\lim_{z\to \mathrm{i}\infty} g(\rho^{-1}(z)) = 0$. Therefore $g(z)$ is a continuous function on the compact Riemann surface $\Gamma_0(N)\setminus\mathbb{H}^*$, so that it is bounded on $\Gamma_0(N)\setminus\mathbb{H}^*$. This shows that there exists A such that $|g(z)| \leqslant A$ for any $z \in \mathbb{H}$, i.e., $|f(z)| \leqslant A\mathrm{Im}(z)^{-k/2}$. The remaining part can be proved similarly in the way used in the proof of Lemma 5.27. This completes the proof. □

Lemma 5.29 *Let* $W(N) = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. *Then the map* : $f \mapsto f|[W(N)]_k$ *sends* $G(N, k, \omega)$ *into* $G(N, k, \overline{\omega})$.

Proof For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we have

$$W(N) \begin{pmatrix} a & b \\ c & d \end{pmatrix} W(N)^{-1} = \begin{pmatrix} d & -N^{-1}c \\ -Nb & a \end{pmatrix} \in \Gamma_0(N).$$

This shows the lemma. □

Theorem 5.9 *Let* $f(z) = \sum_{n=0}^{\infty} c(n)q^n \in G(N, k, \omega)$. *Put*

$$L(s, f) = \sum_{n=1}^{\infty} c(n)n^{-s},$$
$$R_N(s, f) = (2\pi)^{-s} N^s \Gamma(s) L(s, f).$$

Then $L(s, f)$ *is absolutely convergent for* $\mathrm{Re}(s) > 1 + k$. $R_N(s, f)$ *can be analytically continued to a meromorphic function on the* s*-plane with possible poles* $s = 0$ *and* $s = k$ *of order* 1, *and the residues are* $c(0)$ *and* $b(0)N^{-k/2}$ *respectively, where* $b(0)$ *is the constant term of the Fourier expansion of* $f|[W(N)]_k$ *at* $\mathrm{i}\infty$. *And* $R_N(s, f)$ *satisfies the following functional equation*:

$$R_N(s, f) = \mathrm{i}^k R_N(k - s, f|[W(N)]_k).$$

Proof For $\mathrm{Re}(s) > 1 + k$, by Lemma 5.27, we know that $L(s, f)$ is absolutely convergent. A formal computation shows that

$$\begin{aligned}\int_0^{\infty} (f(\mathrm{i}y) - c(0))y^{s-1}\mathrm{d}y &= \sum_{n=1}^{\infty} c(n) \int_0^{\infty} y^{s-1}\mathrm{e}^{-2\pi ny}\mathrm{d}y \\ &= (2\pi)^{-s}\Gamma(s)L(s, f).\end{aligned} \tag{5.15}$$

We verify now the rationality of (5.15). For positive real numbers ϵ, M, we have

$$\left| \int_M^{\infty} (f(\mathrm{i}y) - c(0))y^{s-1}\mathrm{d}y \right| \leqslant A \int_M^{\infty} \mathrm{e}^{-2\pi y} y^{\mathrm{Re}(s)-1}\mathrm{d}y \to 0, \quad M \to \infty,$$

with a constant A. Put

$$g = f|[W(N)]_k = \sum_{n=0}^{\infty} b(n)q^n.$$

Then we have, by Lemma 5.29 and Lemma 5.27,

$$\left| \int_0^{\epsilon} (f(\mathrm{i}y) - c(0))y^{s-1}\mathrm{d}y \right| = \left| \int_0^{\epsilon} \left(N^{k/2} y^{-k} g\left(\frac{\mathrm{i}}{yN}\right) - c(0) \right) y^{s-1}\mathrm{d}y \right| \to 0, \quad \epsilon \to 0.$$

Since $c(n) = O(n^k)$, we see that $\sum_{n=1}^{\infty} c(n)\mathrm{e}^{-2\pi ny}$ is absolutely convergent for $y \geqslant \epsilon$. Hence

$$\int_{\epsilon}^{M} (f(\mathrm{i}y) - c(0))y^{s-1}\mathrm{d}y = \sum_{n=1}^{\infty} c(n) \int_{\epsilon}^{M} \mathrm{e}^{-2\pi ny} y^{s-1}\mathrm{d}y.$$

For any given small number $\epsilon > 0$, $\eta > 0$, there exists a sufficiently large number n_0 such that

$$\begin{aligned}\left|\sum_{n>n_0} c(n) \int_{\epsilon}^{M} \mathrm{e}^{-2\pi ny} y^{s-1}\mathrm{d}y\right| &\leqslant \sum_{n>n_0} |c(n)| \int_{0}^{\infty} \mathrm{e}^{-2\pi ny} y^{t-1}\mathrm{d}y \\ &= (2\pi)^{-t}\Gamma(t) \sum_{n>n_0} |c(n)| n^{-t} < \eta,\end{aligned}$$

where $t = \mathrm{Re}(s)$. Therefore

$$\begin{aligned}&\left|\int_{0}^{\infty} (f(\mathrm{i}y) - c(0))y^{s-1}\mathrm{d}y - \sum_{n=1}^{n_0} c(n) \int_{0}^{\infty} \mathrm{e}^{-2\pi ny} y^{s-1}\mathrm{d}y\right| \\ &= \lim_{\substack{\epsilon\to 0,\\ M\to\infty}} \left|\int_{\epsilon}^{M} (f(\mathrm{i}y) - c(0))y^{s-1}\mathrm{d}y - \sum_{n=1}^{n_0} c(n) \int_{\epsilon}^{M} \mathrm{e}^{-2\pi ny} y^{s-1}\mathrm{d}y\right| < \eta.\end{aligned}$$

This proves (5.15). Taking $B = N^{-1/2}$, we have

$$\int_{0}^{\infty} (f(\mathrm{i}y)-c(0))y^{s-1}\mathrm{d}y = \int_{0}^{B} (f(\mathrm{i}y)-c(0))y^{s-1}\mathrm{d}y+\int_{B}^{\infty} (f(\mathrm{i}y)-c(0))y^{s-1}\mathrm{d}y. \quad (5.16)$$

The first integral is absolutely convergent for $\mathrm{Re}(s) > 1 + k$, the second one is convergent for any s. Substituting y by $\frac{1}{yN}$ in the first integral, we get

$$\begin{aligned}&\int_{0}^{B} (f(\mathrm{i}y) - c(0))y^{s-1}\mathrm{d}y \\ &= \int_{B}^{\infty} (\mathrm{i}^k N^{k/2} y^k g(\mathrm{i}y) - c(0))N^{-s} y^{-s-1}\mathrm{d}y \\ &= \mathrm{i}^k N^{k/2-s} \int_{B}^{\infty} (g(\mathrm{i}y) - b(0))y^{k-1-s}\mathrm{d}y - \frac{c(0)}{sN^{s/2}} - \frac{\mathrm{i}^k b(0)}{(k-s)N^{s/2}}, \end{aligned} \quad (5.17)$$

The last integral in (5.17) is convergent for any s. Inserting (5.17) into (5.16), we obtain

$$\begin{aligned}R_N(s, f) = &N^{s/2} \int_{B}^{\infty} (f(\mathrm{i}y) - c(0))y^{s-1}\mathrm{d}y \\ &+\mathrm{i}^k N^{k/2-s/2} \int_{B}^{\infty} (g(\mathrm{i}y) - b(0))y^{k-1-s}\mathrm{d}y - \frac{c(0)}{s} - \frac{\mathrm{i}^k b(0)}{k-s}. \end{aligned} \quad (5.18)$$

This shows that $R_N(s,f)$ can be analytically continued to a meromorphic function on the s-plane with two possible poles $s=0$ and $s=k$ of order 1, and with residues $c(0)$ and $b(0)$ respectively. Since $f=(-1)^k g|[W(N)]_k$, by changing f and g, we get

$$\begin{aligned}\mathrm{i}^k R_N(k-s,g) &= \mathrm{i}^k N^{k/2-s/2}\int_B^{\infty}(g(\mathrm{i}y)-b(0))y^{k-s-1}\mathrm{d}y \\ &\quad + N^{s/2}\int_B^{\infty}(f(\mathrm{i}y)-c(0))y^{s-1}\mathrm{d}y-\frac{\mathrm{i}^k b(0)}{k-s}-\frac{c(0)}{s} \\ &= R_N(s,f).\end{aligned}$$

This completes the proof. □

Lemma 5.30 (Phragmen-Lindelöf Theorem) *Let*

$$B=\{s\in\mathbb{C}|\sigma_1\leqslant \mathrm{Re}(s)\leqslant\sigma_2, \mathrm{Im}(s)\geqslant t_1\}.$$

Let $f(s)$ be a holomorphic function in an open set containing B such that

$$|f(s)|\leqslant C\mathrm{e}^{|s|^{\gamma}},\quad \forall s\in B,$$

where C,γ are positive constants. Assume furthermore that $|f(s)|\leqslant M$ for any s on the boundary of B. Then we have

$$|f(s)|\leqslant M,\quad \forall s\in B.$$

Proof It is well known as the Phragmen-Lindelöf Theorem. □

Theorem 5.10 *Let $\{a_n\},\{b_n\}$ be two complex series such that $a_n=O(n^{\sigma_0}), b_n=O(n^{\sigma_0})$ with σ_0 a positive constant. Put*

$$\begin{aligned}f(z)&=\sum_{n=0}^{\infty}a_n\mathrm{e}^{2\pi\mathrm{i}nz},\quad g(z)=\sum_{n=0}^{\infty}b_n\mathrm{e}^{2\pi\mathrm{i}nz},\quad z\in\mathbb{H},\\ D_f(s)&=\sum_{n=1}^{\infty}a_n n^{-s},\quad D_g(s)=\sum_{n=1}^{\infty}b_n n^{-s},\\ \varPhi_f(s)&=(2\pi)^{-s}\varGamma(s)D_f(s),\\ \varPhi_g(s)&=(2\pi)^{-s}\varGamma(s)D_g(s),\end{aligned}$$

where $\mathrm{Re}(s)>\sigma_0+1$. *If there exist constants $A>0,k>0,C\neq 0$ such that*

$$f(z)=CA^{k/2}\left(\frac{Az}{\mathrm{i}}\right)^{-k}g\big(-1/(Az)\big),\tag{5.19}$$

then $\varPhi_f(s)$, $\varPhi_g(s)$ can be analytically continued to a meromorphic function on the s-plane with two possible poles $s=0$, k of order 1, $\varPhi_f(s)$, $\varPhi_g(s)$ are bounded on any *domain* $\sigma_1 \leqslant \mathrm{Re}(s) \leqslant \sigma_2$ *not containing* $s = 0$, k, *and they satisfy the following functional equation*

$$\Phi_f(s) = CA^{-s+k/2}\Phi_g(k-s). \tag{5.20}$$

Conversely, if $\Phi_f(s)$, $\Phi_g(s)$ *have these properties, then the equality* (5.19) *holds and* $-a_0$, $-b_0$ *are the residues of* Φ_f, Φ_g *at* $s = 0$ *respectively.*

Proof The first part can be showed similarly as in the proof of Theorem 5.9. We prove now the second part of the theorem. Consider the following integral:

$$\mathrm{e}^{-x} = \frac{1}{2\pi \mathrm{i}} \int\limits_{\mathrm{Re}(s)=\sigma} \frac{\Gamma(s)}{x^s}\mathrm{d}s, \quad \mathrm{Re}(x) > 0,\, \sigma > 0. \tag{5.21}$$

Assume that $\Phi_f(s)$, $\Phi_g(s)$ have the properties stated in the theorem. Take

$$B = \{s|\sigma_1 \leqslant \mathrm{Re}(s) \leqslant \sigma_2, \mathrm{Im}(s) \geqslant t_1 > 0\}.$$

We want to show that there exists a constant a such that $D_f(s) = O(|s|^a)$ holds for any $s \in B$.

We take σ_1, σ_2 such that σ_1 is small enough and σ_2 is large enough. Then $D_f(s)$ and $D_g(k-s)$ are bounded on $\mathrm{Re}(s) = \sigma_2$ and $\mathrm{Re}(s) = \sigma_1$ respectively. (5.20) can be rewritten as

$$D_f(s) = CA^{-s+k/2}(2\pi)^{2s-k}\frac{\Gamma(k-s)}{\Gamma(k)}D_g(k-s).$$

By the Stirling formula, we see that

$$\Gamma(s) \sim \sqrt{2\pi}t^{\sigma-1/2}\mathrm{e}^{-\pi|t|/2}, \quad s = \sigma + \mathrm{i}t,\ |t| \to +\infty,$$

so that

$$\left|\frac{\Gamma(k-s)}{\Gamma(s)}\right| \sim |t|^{k-2\sigma_1}, \quad \mathrm{Re}(s) = \sigma_1,\ |t| \to +\infty.$$

Therefore $D_f(s) = O(|s|^{k-2\sigma_1})$ for any $\mathrm{Re}(s) = \sigma_1$. Let $a = \max\{0, k-2\sigma_1\}$. Then $s^{-a}D_f(s)$ is bounded on the boundary of B. By assumption, $\Phi_f(s)$ is bounded on B. The Stirling formula gives that $\Gamma(s)^{-1} = O(\mathrm{e}^{|s|\gamma})$ for any $\gamma > 1$. Hence we obtain that $D_f(s) = O(\mathrm{e}^{|s|\gamma})$, so that $s^{-a}D_f(s)$ is bounded on B by Lemma 5.30. Therefore $D_f(s) = O(|s|^a)$ holds on B. This shows that

$$\Phi_f(s) = O(|t|^{\sigma+a-1/2}\mathrm{e}^{-\pi|t|/2}), \quad |t| \to \infty,\ \mathrm{Re}(s) = \sigma \tag{5.22}$$

holds uniformly for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$.

For $\sigma > k$, put

$$F(x) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\mathrm{Re}(s)=\sigma} \Phi(s)x^{-s}\mathrm{d}s, \quad \mathrm{Re}(x) > 0. \tag{5.23}$$

Since $|x^{-s}| = \mathrm{e}^{-\sigma\log r + t\theta}$ ($r = |x|$, $\theta = \arg(x)$, $|\theta| < \pi/2$), by (5.22), we see that the integral in (5.23) is absolutely convergent for any $\sigma > k$. Therefore $F(x)$ is independent on σ by the holomorphy of $\Phi(s)$ (for $\mathrm{Re}(s) > k$), (5.22) and the Cauchy integral theorem. It is easy to verify that we can integrate (5.23) term by term for any σ large enough:

$$F(x) = \frac{1}{2\pi i}\sum_{n=1}^{\infty} a_n \int_{\mathrm{Re}(s)=\sigma} (2\pi)^{-s}\Gamma(s)x^{-s}n^{-s}\mathrm{d}s = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-2\pi nx},$$

by (5.21).

We can discuss $\Phi_g(s)$ similarly, i.e., for $\sigma > k$, put

$$G(x) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\sigma} \Phi_g(s)x^{-s}\mathrm{d}s,$$

then

$$G(x) = \sum_{n=1}^{\infty} b_n \mathrm{e}^{-2\pi nx}.$$

We consider now the integral of $\Phi(s)x^{-s}$ along the path: $\mathrm{Re}(s) = \sigma$, $\mathrm{Re}(s) = k - \sigma$, $\mathrm{Im}(s) = t$, $\mathrm{Im}(s) = -t$ with $t > 0$. By (5.22), we see that the integrals along $\mathrm{Im}(s) = \pm t$ converge to zero (for $t \to \infty$). Let $-a_0$, $-b_0$ be the residues of $\Phi_f(s)$ and $\Phi_g(s)$ at $s = 0$ respectively. Then (5.20) shows that $b_0CA^{-k/2}$ is the residue of $\Phi_f(s)$ at $s = k$, so that, by the residue theorem, we have

$$F(x) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=k-\sigma} \Phi_f(s)x^{-s}\mathrm{d}s - a_0 + b_0CA^{-\frac{k}{2}}x^{-k}.$$

By (5.20) again, we see that

$$F(x) + a_0 = CA^{-\frac{k}{2}}x^{-k}(G(1/(Ax)) + b_0).$$

That is,

$$\sum_{n=0}^{\infty} a_n \mathrm{e}^{-2\pi nx} = CA^{-\frac{k}{2}}x^{-k}\sum_{n=0}^{\infty} b_n \mathrm{e}^{-\frac{2\pi n}{(Ax)}}.$$

Let $x = -\mathrm{i}z$ in the above equality with $z \in \mathbb{H}$, then by the definition of f, g, we obtain

$$f(z) = CA^{\frac{k}{2}}\left(\frac{Az}{\mathrm{i}}\right)^{-k} g\left(-\frac{1}{(Az)}\right),$$

which is the desired result. This completes the proof. □

Remark 5.1 Let f, g be as in Theorem 5.9. Since $W(N)^2 = -N$, we see that $f = (-1)^k g|[W(N)]_k$, so that

$$f(z) = \mathrm{i}^k g\left(-\frac{1}{(Nz)}\right)\left(\frac{Nz}{\mathrm{i}}\right)^{-k} N^{\frac{k}{2}}.$$

Hence the functions f, g satisfy the conditions in Theorem 5.10, so that we get Theorem 5.9 from Theorem 5.10 (Take $A = N$, $C = \mathrm{i}^k$ in Theorem 5.10).

We consider the following:

Condition($\star$) Let $f(z)$ be a holomorphic function on $\mathbb{H}$ with generalized uniformly absolutely convergent Fourier expansion:

$$f(z) = \sum_{n=0}^{\infty} a_n \mathrm{e}^{2\pi \mathrm{i} n z}$$

and there exists a positive constant V such that

$$f(z) = O((\mathrm{Im}(z))^{-V}), \quad \mathrm{Im}(z) \to 0$$

holds uniformly for $\mathrm{Re}(z)$ (This is equivalent to $a_n = O(n^V)$).

By a "generalized uniformly" property we mean that there exists a positive number l such that the series has the property uniformly for $\mathrm{Im}(z) > l$. By Theorem 5.10, we have

Theorem 5.11 *Let k, N be positive integers, and let*

$$f(z) = \sum_{n=0}^{\infty} a_n \mathrm{e}^{2\pi \mathrm{i} n z}, \quad g(z) = \sum_{n=0}^{\infty} b_n \mathrm{e}^{2\pi \mathrm{i} n z}$$

satisfy the **Condition($\star$)**, *and $R_N(s, f)$, $R_N(s, g)$ defined as in* Theorem 5.9. *Then the following two assertions* (1) *and* (2) *are equivalent*:

(1) $g(z) = (f|[W(N)]_k)(z)$;

(2) *$R_N(s, f)$ and $R_N(s, g)$ can be continued analytically to a meromorphic function on the s-plane and satisfy the functional equation*:

$$R_N(s, f) = \mathrm{i}^k R_N(k - s, g)$$

and $R_N(s, f) + \dfrac{a_0}{s} + \dfrac{\mathrm{i}^k b_0}{k - s}$ *is a bounded holomorphic function in any domain* $\sigma_1 \leqslant \mathrm{Re}(s) \leqslant \sigma_2$.

Let $f(z)$ satisfy Condition($\star$) and ψ a primitive Dirichlet character with conductor $m = m_\psi$. Put

$$f_\psi(z) = \sum_{n=0}^{\infty} \psi(n) a_n \mathrm{e}^{2\pi \mathrm{i} n z},$$

$$L(s, f, \psi) = \sum_{n=1}^{\infty} \psi(n) a_n n^{-s},$$

$$R_N(s, f, \psi) = \left(\frac{m\sqrt{N}}{2\pi} \right)^s \Gamma(s) L(s, f, \psi).$$

It is clear that

$$L(s, f_\psi) = L(s, f, \psi), \quad R_{Nm^2}(s, f_\psi) = R_N(s, f, \psi). \tag{5.24}$$

By (5.24) and Theorem 5.11, we have

Lemma 5.31 *Let f, g be functions on $\mathbb{H}$ satisfying* Condition($\star$), *ψ a primitive* Dirichlet *character with conductor $m = m_\psi$. Then the following two assertions are equivalent*:

(A_ψ) $f_\psi|[W(Nm^2)]_k = C_\psi g_{\overline{\psi}}$;

(B_ψ) *$R_N(s, f, \psi)$ can be continued analytically to a holomorphic function on the s-plane, which is bounded in any domain $\sigma_1 \leqslant \mathrm{Re}(s) \leqslant \sigma_2$, and satisfies the following functional equation*:

$$R_N(s, f, \psi) = \mathrm{i}^k C_\psi R_N(k - s, g, \overline{\psi}).$$

For any real number a, put $\boldsymbol{\alpha}(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$.

Lemma 5.32 (1) *Let f, ψ be as in* Lemma 5.31. *Then for any positive integer k, we have*

$$f_\psi = W(\overline{\psi})^{-1} \sum_{u=1}^{m} \overline{\psi}(u) f| \left[\alpha\left(\frac{u}{m}\right)\right]_k, \tag{5.25}$$

where $W(\overline{\psi})$ is the Gauss sum of $\overline{\psi}$:

$$W(\overline{\psi}) = \sum_{u=1}^{m} \overline{\psi}(u) \mathrm{e}^{\frac{2\pi \mathrm{i} u}{m}}.$$

(2) *Let $f(z) \in G(N, k, \chi)$, m_χ the conductor of χ, M the least common multiple of N, m_ψ^2 and $m_\psi m_\chi$. Then $f_\psi \in G(M, k, \chi\psi^2)$, and f_ψ is a cusp form if f is a cusp form.*

Proof For any integer u, we see that

$$\left(f\middle| \left[\alpha\left(\frac{u}{m}\right)\right]_k\right)(z) = \sum_{n=0}^{\infty} a_n \mathrm{e}^{\frac{2\pi \mathrm{i} u n}{m}} \mathrm{e}^{2\pi \mathrm{i} n z}.$$

Hence

$$\sum_{u=1}^{m} \overline{\psi}(u) \left(f\middle| \left[\alpha\left(\frac{u}{m}\right)\right]_k\right)(z) = \sum_{n=0}^{\infty} \left(\sum_{u=1}^{m} \overline{\psi}(u) \mathrm{e}^{2\pi \mathrm{i} u n/m}\right) a_n \mathrm{e}^{2\pi \mathrm{i} n z} = W(\overline{\psi}) f_\psi(z).$$

This shows the first part of the lemma.

Suppose now that $f \in G(N, k, \chi)$. Since

$$\alpha(u/m)\Gamma(Nm^2)\alpha(u/m)^{-1} \subset \Gamma_0(N),$$

$f|[\alpha(u/m)]_k \in G(\Gamma(Nm^2), k)$. Hence $f_\psi \in G(\Gamma(Nm^2), k)$ by the first part of the lemma. So we only need to show that, for any $\gamma = \begin{pmatrix} * & * \\ * & d_\gamma \end{pmatrix} \in \Gamma_0(M)$,

$$f_\psi|[\gamma]_k = (\chi\psi^2)(d_\gamma) f_\psi$$

holds. For $\gamma = \begin{pmatrix} a & b \\ cM & d \end{pmatrix} \in \Gamma_0(M)$, put

$$\gamma' = \alpha(u/m)\gamma\alpha(d^2u/m)^{-1},$$

then $\gamma' \in \Gamma_0(M) \subset \Gamma_0(N)$. Denote $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, then

$$d' = d - \frac{cd^2uM}{m} \equiv d \pmod{m}.$$

Therefore

$$f|[\alpha(u/m)\gamma]_k = \chi(d) f|[\alpha(d^2u/m)]_k.$$

Multiplying on both sides by $\overline{\psi}(u)$ and adding them for u, by (5.25), we obtain

$$f_\psi|[\gamma]_k = \psi(d^2)\chi(d) f_\psi.$$

This completes the proof. □

Theorem 5.12 *Let $f \in G(N, k, \chi)$, ψ a primitive Dirichlet character with conductor m prime to N. Then*

$$f_\psi|[W(Nm^2)]_k = C_\psi g_{\overline{\psi}},$$

where $g = f|[W(N)]_k$ and

$$C_\psi = C_{N,\psi} = \chi(m)\psi(-N)\frac{W(\psi)}{W(\overline{\psi})} = \frac{\chi(m)\psi(N)W(\psi)^2}{m}.$$

Proof For any integer u prime to m, there exist integers n, v such that $nm - Nuv = 1$, then

$$\alpha(u/m)W(Nm^2) = (mI)W(N)\begin{pmatrix} m & -v \\ -uN & n \end{pmatrix}\alpha(v/m). \quad (5.26)$$

Since $g = f|[W(N)]_k \in G(N, k, \overline{\chi})$, we see that

$$f|[\alpha(u/m)W(Nm^2)]_k = \chi(m) g|[\alpha(v/m)]_k,$$

by (5.26). Now Lemma 5.32 gives

$$\begin{aligned} W(\overline{\psi}) f_\psi|[W(Nm^2)]_k &= \sum_u \overline{\psi}(u) f|[\alpha(u/m)W(Nm^2)]_k \\ &= \chi(m)\sum_v \psi(-Nv) g|[\alpha(v/m)]_k \\ &= \chi(m)\psi(-N)\sum_v \psi(v) g|[\alpha(v/m)]_k \\ &= \chi(m)\psi(-N)W(\psi) g_{\overline{\psi}}. \end{aligned}$$

This completes the proof. □

Theorem 5.13 *Let $f(z) \in S(N, k, \chi)$, ψ a primitive* Dirichlet *character with conductor m_ψ prime to N. Then $R_N(s, f, \psi)$ is a bounded holomorphic function in any domain $\sigma_1 \leqslant \mathrm{Re}(s) \leqslant \sigma_2$, and satisfies the functional equation*:

$$R_N(s, f, \psi) = \mathrm{i}^k C_\psi R_N(k - s, f|[W(N)]_k, \overline{\psi}),$$

where C_ψ is defined as in Theorem 5.12.

Proof This is a direct conclusion of Theorem 5.11, the equality (5.24) and Theorem 5.12. □

The remaining part of this section is dedicated to proving Weil's theorem which may be looked upon as the inverse proposition of Theorem 5.13 under some assumptions.

For any integers m, v with $(m, vN) = 1$, take integers n, u such that $mn - uvN = 1$, and put

$$\gamma(m, v) = \begin{pmatrix} m & -v \\ -uN & n \end{pmatrix} \in \Gamma_0(N).$$

It is clear that the choices of n, u are not determined uniquely by m, v. But $u \bmod m$ is unique and

$$\alpha(u/m)W(Nm^2) = (mI)W(N)\gamma(m, v)\alpha(v/m).$$

We can extend the action of $GL_2^+(\mathbb{R})$ on functions on the upper half plane to the group ring $\mathbb{C}[GL_2^+(\mathbb{R})]$: for any $\beta = \sum_\alpha a_\alpha \alpha \in \mathbb{C}[GL_2^+(\mathbb{R})]$, define

$$f|[\beta]_k = \sum_\alpha a_\alpha f|[\alpha]_k, \quad \forall f : \mathbb{H} \to \mathbb{C}.$$

Lemma 5.33 *Let k, N, f, g be as in* Theorem 5.11, *f, g satisfy the condition* (A) *in* Theorem 5.11 *and the condition* (A_ψ) *in* Lemma 5.31. *Let ψ be a primitive character with conductor $m > 4$ a prime. Put*

$$C_\psi = \chi(m)\psi(-N)\frac{W(\psi)}{W(\overline{\psi})}.$$

Then, for any integers u, v prime to m, we have

$$g|[(\chi(m) - \gamma(m, u))\alpha(u/m)]_k = g|[(\chi(m) - \gamma(m, v))\alpha(v/m)]_k.$$

Proof By the condition (A_ψ) and Lemma 5.32, we see that

$$\sum_u \overline{\psi}(u) f|[\alpha(u/m)W(Nm^2)]_k = \chi(m)\sum_u \psi(u) g|[\alpha(u/m)]_k. \tag{5.27}$$

For any u, take an integer v such that $-uvN \equiv 1 \pmod{m}$, by (5.26), we have

$$f|[\alpha(u/m)W(Nm^2)]_k = g|[\gamma(m,v)\alpha(v/m)]_k. \tag{5.28}$$

The left hand side of (5.28) is independent of the choice of the representatives of u mod m, so is the right one. Hence, by (5.27) and (5.28), we obtain

$$\sum_{v \bmod m} \psi(v)(g|[(\chi(m) - \gamma(m,v))\alpha(v/m)]_k = 0. \tag{5.29}$$

Let v_1, v_2 be integers prime to m. Multiplying the both sides of (5.29) by $(\psi(v_1) - \psi(v_2))$ and adding them for all primitive characters ψ modulo m, we get

$$g|[(\chi(m) - \gamma(m,v_1))\alpha(v_1/m)]_k = g|[(\chi(m) - \gamma(m,v_2))\alpha(v_2/m)]_k.$$

This completes the proof. □

Lemma 5.34 *Let k, N, f, g be as in* Lemma 5.33, $m > 4, n > 4$ *primes*, ψ *a primitive character* $\bmod m$ *or* $\bmod n$ *with conductor* m_ψ. *Put*

$$C_\psi = \chi(m_\psi)\psi(-N)\frac{W(\psi)}{W(\overline{\psi})}.$$

Then, for any $\gamma = \begin{pmatrix} m & -v \\ -uN & n \end{pmatrix} \in \Gamma_0(N)$, *we have*

$$g|[\gamma]_k = \overline{\chi}(\gamma)g.$$

Proof Put $\gamma' = \begin{pmatrix} m & v \\ uN & n \end{pmatrix}$, by Lemma 5.33, we have

$$g|[(\chi(m) - \gamma')\alpha(-v/m)]_k = g|[(\chi(m) - \gamma)\alpha(v/m)]_k.$$

Hence

$$g|[(\chi(m) - \gamma')\alpha(-2v/m)]_k = g|[\chi(m) - \gamma]_k. \tag{5.30}$$

Substituting γ by γ'^{-1}, in the same way, we obtain

$$g|[\chi(n) - \gamma'^{-1}]_k = g|[(\chi(n) - \gamma^{-1})\alpha(-2v/n)]_k. \tag{5.31}$$

Since $\chi(n)\chi(m) = I$,

$$\begin{aligned} \chi(n) - \gamma'^{-1} &= -\chi(n)(\chi(m) - \gamma')\gamma'^{-1}, \\ (\chi(n) - \gamma^{-1})\alpha(-2v/n) &= -\chi(n)(\chi(m) - \gamma)\gamma^{-1}\alpha(-2v/n). \end{aligned} \tag{5.32}$$

Inserting (5.32) into (5.31), we obtain

$$g|[\chi(m) - \gamma']_k = g|[(\chi(m) - \gamma)\gamma^{-1}\alpha(-2v/n)\gamma']_k. \tag{5.33}$$

Therefore, by (5.30) and (5.33), we see that

$$g|[(\chi(m)-\gamma)(I-\gamma^{-1}\alpha(-2v/n)\gamma'\alpha(-2v/m))]_k = 0.$$

Put $h = g|[\chi(m)-\gamma]_k = \chi(m)g - g|[\gamma]_k$, then h is an analytically function on $\mathbb{H}$ and satisfies

$$h|[\beta]_k = h, \tag{5.34}$$

where

$$\beta = \gamma^{-1}\alpha(-2v/n)\gamma'\alpha(-2v/m) = \begin{pmatrix} 1 & -2v/m \\ 2uN/n & -3+4/mn \end{pmatrix}.$$

Since $\mathrm{Tr}(\beta) = |-2+4/mn| < 2$, β is an elliptic element. It is clear that $|\mathrm{Tr}(\beta)| \neq 0, 1$. So that, the eigenvalues of β are not roots of the unity. Let $z_0 \in \mathbb{H}$ be a fixed point of β, put

$$\rho = (z_0 - \overline{z_0})^{-1}\begin{pmatrix} 1 & -z_0 \\ 1 & -\overline{z_0} \end{pmatrix} \in SL_2(\mathbb{C}).$$

For $z \in K := \{z \in \mathbb{C} \,|\, |z| \leqslant 1\}$, put

$$p(z) = (h|[\rho^{-1}]_k)(z) = j(\rho^{-1}, z)^{-k}h(\rho^{-1}z).$$

Then $p(z)$ is an analytical function on K. Let

$$\rho\beta\rho^{-1} = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix},$$

then, by (5.34), we get

$$p(\zeta^2 z) = \zeta^{-k}p(z).$$

Let $p(z) = \sum_{n=0}^{\infty} a_n z^n$ be the Taylor expansion of $p(z)$ at $z = 0$, then $a_n\zeta^{2n} = \zeta^{-k}a_n$. Since $\zeta^{2n+k} \neq 1$ we must have $a_n = 0$ and hence $h(z) = 0$. This shows that

$$g|[\gamma]_k = \overline{\chi}(\gamma)g.$$

This completes the proof. □

For any co-prime positive integers a, b, put $S(a,b) = \{a+bn | n \in \mathbb{Z}\}$. Let M be a set of some primes larger than 4, which satisfies the following two conditions:

(1) any element of M is prime to N;

(2) $M \bigcap S(a,b) \neq \varnothing$ for any $S(a,b)$.

There exists such set M, e.g., let M be the set of all primes which are prime to N and larger than 4, then M satisfies the conditions of Dirichlet's theorem about the existence of primes in an arithmetical progression.

We can now state and show the following important result:

Theorem 5.14 (A.Weil) *Let k, N be positive integers, χ a* Dirichlet *character modulo N and satisfying $\chi(-1)=(-1)^k$. Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two complex series satisfying $a_n=O(n^\mu)$ and $b_n=O(n^\mu)$ with μ a positive constant respectively. Put*

$$f(z)=\sum_{n=0}^{\infty}a_n\mathrm{e}^{2\pi inz},\quad g(z)=\sum_{n=0}^{\infty}b_n\mathrm{e}^{2\pi inz},\quad z\in\mathbb{H}.$$

If f and g satisfy the following two conditions:

(1) *$R_N(s,f)$ and $R_N(s,g)$ satisfy the condition* (B) *in* Theorem 5.11;

(2) *for any primitive Dirichlet character ψ with conductor $m_\psi\in M$, $R_N(s,\,f,\,\psi)$ and $R_N(s,\,g,\psi)$ satisfy the condition* (B_ψ) *in* Lemma 5.31 *with corresponding constant*

$$C_\psi=C_{N,\psi}=\chi(m_\psi)\psi(-N)\frac{W(\psi)}{W(\overline{\psi})},$$

then $f(z)\in G(N,k,\chi),g(z)\in G(N,k,\overline{\chi})$, and $g=f|[W(N)]_k$. If, moreover, there exists a positive number δ such that $L(s,f)$ is absolutely convergent at $s=k-\delta$, then f and g are cusp forms.

Proof Let $\gamma=\begin{pmatrix}a&b\\cN&d\end{pmatrix}\in\Gamma_0(N)$. If $c=0$, then $a=d=\pm1$, so that $g|[\gamma]_k=\overline{\chi}(d)g=\overline{\chi}(\gamma)g$ by $\chi(-1)=(-1)^k$. Assume now that $c\neq0$. By $(a,cN)=(d,cN)=1$ and the properties of M, there exist integers s,t such that $a+tcN$, $d+scN\in M$. Put $m=a+tcN$, $n=d+scN$, $u=-c$, $v=-(b+sm+bst+N+nt)$, then

$$\begin{pmatrix}a&b\\cN&d\end{pmatrix}=\begin{pmatrix}1&-t\\0&1\end{pmatrix}\begin{pmatrix}m&-v\\-uN&n\end{pmatrix}\begin{pmatrix}1&-s\\0&1\end{pmatrix}.$$

By Lemma 5.34, we see that

$$g|[\gamma]_k=\overline{\chi}(n)g=\overline{\chi}(d)g.$$

This shows that $g|[\gamma]_k=\overline{\chi}(\gamma)g$ for any $\gamma\in\Gamma_0(N)$. Since $b_n=O(n^\mu)$, $g(z)=O(y^{-1-\mu})$, hence $g\in G(N,k,\overline{\chi})$. By Theorem 5.11 and Lemma 5.29, we see that $f=(-1)^kg|[W(N)]_k\in G(N,k,\chi)$.

Assume now that $L(s,f)$ is absolutely convergent at $s=k-\delta$. Then $L(s,f)$ is absolutely convergent at $s=k$. By the functional equation, we see that $R_N(s,g)$ is holomorphic at $s=0$, so that $b_0=0$ since $-b_0$ is the residue of $R_N(s,g)$ at $s=0$. Hence $a_0=0$. Put $c(n)=\sum_{m=1}^{n}|a_m|$, then

$$c(n)=\sum_{m=1}^{n}|a_m|\leqslant n^{k-\delta}\sum_{m=1}^{n}|a_m|m^{-k+\delta}.$$

So that, $c(n) = O(n^{k-\delta})$. This implies that $\sum_{n=1}^{\infty} |c(n)| \mathrm{e}^{-2\pi n y}$ is convergent and

$$\sum_{n=1}^{\infty} |c(n)| \mathrm{e}^{-2\pi n y} = O(y^{-k+\delta-1}), \quad y \to 0.$$

Since $|a_n| = c(n) - c(n-1)$, we see that

$$|f(z)| \leqslant (1 - \mathrm{e}^{-2\pi y}) \sum_{n=1}^{\infty} c(n) \mathrm{e}^{-2\pi n y}.$$

Hence $f(z) = O(y^{-k+\delta})$ which implies that f is a cusp form. This completes the proof. □

5.4 Hecke Operators on the Space of Modular Forms with Half-Integral Weight

Let N be a positive integer with $4|N$, k an odd positive integer. Put

$$L : \gamma \mapsto \{\gamma, j(\gamma, z)\}$$

the map from $\Gamma_0(N)$ to $\widehat{G}$. Denote by $\Delta_0(N)$, $\Delta_1(N)$, $\Delta(N)$ the images of $\Gamma_0(N)$, $\Gamma_1(N)$, $\Gamma(N)$ under the map L respectively. Denote by $G(\Delta_0(N), k/2)$, $G(\Delta_1(N), k/2)$, $G(\Delta(N), k/2)$ the spaces of holomorphic modular forms with weight $k/2$ and the groups $\Delta_0(N)$, $\Delta_1(N)$, $\Delta(N)$ respectively. And let $S(\Delta_0(N), k/2)$, $S(\Delta_1(N), k/2)$, $S(\Delta(N), k/2)$ be the corresponding spaces of cusp forms. Let Δ be a Fuchsian subgroup of the first kind of $\widehat{G}$. For any f, $g \in G(\Delta, k/2)$ (at least one of them is in $S(\Delta, k/2)$), we can define the Petersson inner product

$$\langle f, g \rangle = \langle f, g \rangle_\Delta = \frac{1}{\mu(D)} \int_D f(z) \overline{g(z)} y^{k/2-2} \mathrm{d}x \mathrm{d}y,$$

where D is a fundamental domain of Δ and

$$\mu(D) = \int_D y^{-2} \mathrm{d}x \mathrm{d}y.$$

It is clear that $\langle f, g \rangle_\Delta = \langle f, g \rangle_{\Delta'}$ if $\Delta' \subset \Delta$ and $[\Delta : \Delta'] < \infty$.

It is obvious that $\Delta_1(N)$ is a normal subgroup of $\Delta_0(N)$. For any $\xi \in \Delta_0(N)$, we have a linear operator on $G(\Delta_1(N), k/2)$ as follows:

$$\xi : f \mapsto f|[\xi]_k, \quad f \in G(\Delta_1(N), k/2),$$

where the definition of $f|[\xi]_k$ is the same as the one for modular forms with half integral weight in Chapter 3. Hence we get a representation of the quotient group

$\varDelta_0(N)/\varDelta_1(N) \simeq (\mathbb{Z}/N\mathbb{Z})^*$ on $G(\varDelta_1(N), k/2)$. Since $(\mathbb{Z}/N\mathbb{Z})^*$ is an abelian group, the space $G(\varDelta_1(N), k/2)$ can be decomposed into a direct sum of some one dimensional representation spaces, so that

$$G(\varDelta_1(N), k/2) = \bigoplus_{\omega} G(N, k/2, \omega), \tag{5.35}$$

where ω runs over all even characters modulo N. Similarly, we have

$$S(\varDelta_1(N), k/2) = \bigoplus_{\omega} S(N, k/2, \omega). \tag{5.36}$$

Let $\mathcal{E}(\varDelta_1(N), k/2)$, $\mathcal{E}(\varDelta(N), k/2)$ and $\mathcal{E}(N, k/2, \omega)$ be the orthogonal complement spaces of $S(\varDelta_1(N), k/2)$, $S(\varDelta(N), k/2)$ and $S(N, k/2, \omega)$ in $G(\varDelta_1(N), k/2), G(\varDelta(N), k/2)$ and $G(N, k/2, \omega)$ with respect to Petersson inner product respectively. For any $f \in \mathcal{E}(N, k/2, \omega)$, $g \in S(N, k/2, \omega')$ with $\omega \neq \omega'$, we see that

$$\omega(d_\xi)\langle f, g\rangle = \langle f|[\xi]_k, g\rangle = \langle f, g|[\xi^{-1}]_k\rangle = \omega'(d_\xi)\langle f, g\rangle$$

for any $\xi \in \varDelta_0(N)$. Since d_ξ is any element of $(\mathbb{Z}/N\mathbb{Z})^*$, it shows that $\langle f, g\rangle = 0$, i.e., $f \in \mathcal{E}(\varDelta_1(N), k/2)$. Therefore

$$\mathcal{E}(\varDelta_1(N), k/2) = \bigoplus_{\omega} \mathcal{E}(N, k/2, \omega). \tag{5.37}$$

Lemma 5.35 *Let N, M be positive integers with $N|M$, ω an even character modulo N, k an odd positive integer. Then*

(1) $\mathcal{E}(\varDelta_1(N), k/2) = G(\varDelta_1(N), k/2) \bigcap \mathcal{E}(\varDelta(M), k/2)$;
(2) $\mathcal{E}(N, k/2, \omega) = G(N, k/2, \omega) \bigcap \mathcal{E}(\varDelta(M), k/2)$;
(3) *For any $f \in \mathcal{E}(\varDelta_1(N), k/2)$, $\alpha \in GL_2^+(\mathbb{Z})$, $\xi = \{\alpha, \phi(z)\} \in \widehat{G}$, we have*

$$f|[\xi]_k \in \mathcal{E}(\varDelta(N\det(\alpha)), k/2).$$

Proof It is clear that

$$G(\varDelta_1(N), k/2) \bigcap \mathcal{E}(\varDelta(M), k/2) \subset \mathcal{E}(\varDelta_1(N), k/2),$$

since $S(\varDelta_1(N), k/2) \subset S(\varDelta(M), k/2)$. Let

$$\varDelta_1(N) = \bigcup_{j=1}^{m} \varDelta(M)\xi_j$$

be a disjoint union, where $m = [\varGamma_1(N) : \varGamma(M)] = [\varDelta_1(N) : \varDelta(M)] < \infty$. For any $f \in \mathcal{E}(\varDelta_1(N), k/2)$, $g \in S(\varDelta(M), k/2)$, it is easy to see that

$$g' = \sum_{j=1}^{m} g|[\xi_j]_k \in S(\varDelta_1(N), k/2),$$

so that

$$0 = \langle f, g' \rangle_{\Delta_1(N)} = \sum_{j=1}^{m} \langle f, g|[\xi_j]_k \rangle_{\Delta(M)}$$
$$= \sum_{j=1}^{m} \langle f|[\xi_j^{-1}]_k, g \rangle_{\Delta(M)} = m \langle f, g \rangle_{\Delta(M)},$$

which shows that $f \in \mathcal{E}(\Delta(M), k/2)$. That is,

$$\mathcal{E}(\Delta_1(N), k/2) \subset \mathcal{E}(\Delta(M), k/2).$$

This shows the first assertion of the lemma. By (5.35), (5.37) and (1), we obtain (2). We want now to prove (3). By (2), it is clear that $f \in \mathcal{E}(\Delta(N \det^2(\alpha)), k/2)$. For any $g \in S(\Delta(N \det(\alpha)), k/2)$, then $g|[\xi^{-1}]_k \in S(\Delta(N \det^2(\alpha)), k/2)$, and

$$\langle f|[\xi]_k, g \rangle_{\Delta(N \det(\alpha))} = \langle f, g|[\xi^{-1}]_k \rangle_{\Delta(N \det^2(\alpha))} = 0.$$

This shows that $f|[\xi]_k \in \mathcal{E}(\Delta(N \det(\alpha)), k/2)$ and hence completes the proof. □

Let Δ be a Fuchsian subgroup of $\widehat{G}$ of the first kind. For $\xi \in \widehat{G}$, Δ and $\xi^{-1}\Delta\xi$ are commensurable, then we have a disjoint union:

$$\Delta\xi\Delta = \bigcup_{j=1}^{d} \Delta\xi_j.$$

For any $f \in G(\Delta, k/2)$, define

$$f|[\Delta\xi\Delta]_k = (\det(\xi))^{k/4-1} \sum_{j=1}^{d} f|[\xi_j]_k.$$

It is easy to see that $f|[\Delta\xi\Delta]_k \in G(\Delta, k/2)$. Let P be the projection from $\widehat{G}$ to $GL_2^+(\mathbb{R})$. Put $\Gamma = P(\Delta)$. For $\alpha \in GL_2^+(\mathbb{R})$, Γ and $\alpha^{-1}\Gamma\alpha$ are commensurable. Take a $\xi \in \widehat{G}$ such that $P(\xi) = \alpha$. For any $\gamma \in \Gamma \bigcap \alpha^{-1}\Gamma\alpha$, $P(\xi L(\gamma)\xi^{-1}) = \alpha\gamma\alpha^{-1} \in \Gamma$, then there exists a $t(\gamma)$ such that

$$L(\alpha\gamma\alpha^{-1}) = \xi L(\gamma)\xi^{-1}\{1, t(\gamma)\}, \quad \gamma \in \Gamma \bigcap \alpha^{-1}\Gamma\alpha.$$

The map $t : \gamma \mapsto t(\gamma)$ is a homomorphism from $\Gamma \bigcap \alpha^{-1}\Gamma\alpha$ to $T := \{z \in \mathbb{C} | |z| = 1\}$ which is independent on the choices of ξ.

Here and after, we write $f|[*]$ for $f|[*]_k$.

Lemma 5.36 *Let Δ, Γ, ξ, α and the map t be as above, then $L(\mathrm{Ker}(t)) = \Delta \bigcap \xi^{-1}\Delta\xi$. If $[\Gamma : \mathrm{Ker}(t)] < \infty$, then Δ and $\xi^{-1}\Delta\xi$ are commensurable. If $t^k \neq 1$, then $f|[\Delta\xi\Delta] = 0$ for any $f \in G(\Delta, k/2)$.*

Proof If $\gamma \in \mathrm{Ker}(t)$, then

$$L(\gamma) = \xi^{-1} L(\alpha\gamma\alpha^{-1})\xi \in \Delta \bigcap \xi^{-1}\Delta\xi.$$

Conversely, if $L(\gamma) \in \Delta \bigcap \xi^{-1}\Delta\xi$, then $\xi L(\gamma)\xi^{-1} \in \Delta$. Since $P(\xi L(\gamma)\xi^{-1}) = \alpha\gamma\alpha^{-1}$, we see that $L(\alpha\gamma\alpha^{-1}) = \xi L(\gamma)\xi^{-1}$, so that $t(\gamma) = 1$. This shows that $L(\mathrm{Ker}(t)) = \Delta \bigcap \xi^{-1}\Delta\xi$. This implies also that $[\Delta : \Delta \bigcap \xi^{-1}\Delta\xi] = [\Gamma : \mathrm{Ker}(t)]$. Since P is an isomorphism from $\xi^{-1}\Delta\xi$ to $\alpha^{-1}\Gamma\alpha$, we have

$$[\xi^{-1}\Delta\xi : \Delta \bigcap \xi^{-1}\Delta\xi] = [\alpha^{-1}\Gamma\alpha : \mathrm{Ker}(t)].$$

If $[\Gamma : \mathrm{Ker}(t)] < \infty$, since Γ and $\alpha^{-1}\Gamma\alpha$ are commensurable, we see that Δ and $\xi^{-1}\Delta\xi$ are commensurable.

Let $\Gamma \bigcap \alpha^{-1}\Gamma\alpha = \bigcup_i \mathrm{Ker}(t)\alpha_i$, $\Gamma = \bigcup_j (\Gamma \bigcap \alpha^{-1}\Gamma\alpha)\gamma_j$ be disjoint unions. Then

$$\Delta = \bigcup_j L(\Gamma \bigcap \alpha^{-1}\Gamma\alpha)L(\gamma_j) = \bigcup_{i,j} (\Delta \bigcap \xi^{-1}\Delta\xi)L(\alpha_i\gamma_j),$$

so that

$$\Delta\xi\Delta = \bigcup_{i,j} \Delta\xi \cdot L(\alpha_i\gamma_j). \tag{5.38}$$

Since $\alpha_i \in \Gamma \bigcap \alpha^{-1}\Gamma\alpha$, we see that

$$\xi L(\alpha_i) = L(\alpha\alpha_i\alpha^{-1})\xi\{1, t(\alpha_i)^{-1}\}.$$

For any $f \in G(\Delta, k/2)$, we have

$$\begin{aligned} f|[\Delta\xi\Delta] &= (\det(\xi))^{k/4-1} \sum_{i,j} f|[\xi L(\alpha_i\gamma_j)] \\ &= (\det(\xi))^{k/4-1} \sum_i t(\alpha_i)^k \sum_j f|[\xi L(\gamma_j)] = 0, \end{aligned}$$

where we used the fact that $\sum_i t(\alpha_i)^k = 0$. This completes the proof. □

Lemma 5.37 *Let the notations be as in* Lemma 5.36. *Then the following assertions are equivalent*:

(1) $L(\Gamma \bigcap \alpha^{-1}\Gamma\alpha) = \Delta \bigcap \xi^{-1}\Delta\xi$;

(2) $L(\alpha\gamma\alpha^{-1}) = \xi L(\gamma)\xi^{-1}$ *for any* $\gamma \in \Gamma \bigcap \alpha^{-1}\Gamma\alpha$;

(3) P *is a bijection from* $\Delta\xi\Delta$ *to* $\Gamma\alpha\Gamma$.

If the above conditions hold, then $\Delta\xi\Delta = \bigcup \Delta\xi_l$ *if and only if* $\Gamma\alpha\Gamma = \bigcup \Gamma \cdot P(\xi_l)$.

Proof The assertions (1) and (2) are both equivalent to $\mathrm{Ker}(t) = \Gamma \bigcap \alpha^{-1}\Gamma\alpha$. Since $\alpha\alpha_i\alpha^{-1} \in \Gamma$ (Using the notations in the proof of Lemma 5.36), P maps bijectively the right coset $\Delta\xi L(\alpha_i\gamma_j)$ in (5.38) to the right coset $\Gamma\alpha\gamma_j$. Since $\Gamma\alpha\Gamma = \bigcup_j \Gamma\alpha\gamma_j$ and (5.38), P is a bijection from $\Delta\xi\Delta$ to $\Gamma\alpha\Gamma$ if and only if $\mathrm{Ker}(t) = \Gamma \bigcap \alpha^{-1}\Gamma\alpha$. This shows that the assertion (3) is equivalent to the assertions (1) and (2). Finally, if the assertion (3) holds, then it is easy to see that $\Delta\xi\Delta = \bigcup \Delta\xi_l$ if and only if $\Gamma\alpha\Gamma = \bigcup \Gamma P(\xi_l)$. This completes the proof. □

Let $\Delta = \Delta_0(N)$ with $4|N$ and $\Gamma = \Gamma_0(N)$. Put

$$\alpha = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix},$$

with m, n positive integers. Take $\xi = \{\alpha, t(n/m)^{1/4}\} \in \widehat{G}$. For any $\gamma \in \Gamma_0(4)$, put

$$\gamma^* = \{\gamma, j(\gamma, z)\}.$$

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\bigcap\alpha^{-1}\Gamma\alpha$, then

$$\alpha\gamma\alpha^{-1} = \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} m^{-1} & 0 \\ 0 & n^{-1} \end{pmatrix} = \begin{pmatrix} a & bmn^{-1} \\ cnm^{-1} & d \end{pmatrix} \in \Gamma,$$

so that,

$$\begin{aligned}(\alpha\gamma\alpha^{-1})^* &= \left\{\alpha\gamma\alpha^{-1}, \epsilon_d^{-1}\left(\frac{cmn}{d}\right)(cnz/m + d)^{1/2}\right\} \\ &= \xi\gamma^*\xi^{-1}\left\{1, \left(\frac{mn}{d}\right)\right\}.\end{aligned}$$

This shows that $\gamma \mapsto \left(\frac{mn}{d}\right)$ is the map t, so that $f|[\Delta\xi\Delta] = 0$ for any $f \in G(\Delta, k/2)$ if $\left(\frac{mn}{d}\right)$ is not identical to 1.

Let χ_m denote the character $\left(\frac{m}{\cdot}\right)$.

Lemma 5.38 *Let m be a positive integer with $m|N^\infty$ and the conductor of the character χ_m is a divisor of N. Put $\Delta_1 = \Delta_1(N)$, $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$ and $\xi = \{\alpha, m^{1/4}\} \in \widehat{G}$. Then $[\Delta_1\xi\Delta_1]$ maps $G(N, k/2, \omega)$, $S(N, k/2, \omega)$ and $\mathcal{E}(N, k/2, \omega)$ into $G(N, k/2, \omega\chi_m)$, $S(N, k/2, \omega\chi_m)$ and $\mathcal{E}(N, k/2, \omega\chi_m)$ respectively. Suppose*

$$f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N, k/2, \omega),$$

then

$$f|[\Delta_1\xi\Delta_1] = \sum_{n=0}^{\infty} a(mn)e(nz).$$

Proof For any $f \in G(N, k/2, \omega)$, it is easy to see that

$$g = f|[\Delta_1\xi\Delta_1] \in G(\Delta_1, k/2).$$

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(mN)$ with $mN|b$, put $\delta = \gamma^*$, $\varepsilon = \xi\delta\xi^{-1}$. Since

$$\alpha\gamma\alpha^{-1} = \begin{pmatrix} a & bm^{-1} \\ cm & d \end{pmatrix} \in \Gamma_0(N),$$

we see that

$$\varepsilon = (\alpha\gamma\alpha^{-1})^*\left\{1, \left(\frac{m}{d}\right)\right\}.$$

Since $\delta\Delta_1\delta^{-1} = \varepsilon\Delta_1\varepsilon^{-1} = \Delta_1$, we have

$$\Delta_1\xi\Delta_1 \cdot \Delta_1\delta\Delta_1 = \Delta_1\xi\delta\Delta_1 = \Delta_1\varepsilon\xi\Delta_1 = \Delta_1\varepsilon\Delta_1 \cdot \Delta_1\xi\Delta_1.$$

And hence

$$g|[\delta] = f|[\Delta_1\xi\Delta_1] \cdot [\delta] = f|[\varepsilon] \cdot [\Delta_1\xi\Delta_1] = \omega(d)\left(\frac{m}{d}\right)g.$$

For any $\gamma' \in \Gamma_0(N)$, we can find an element $\beta \in \Gamma_1(N)$ such that $\beta\gamma' \in \Gamma_0(mN)$ and the upper right entry of $\beta\gamma'$ is divisible by mN, so that $g \in G(N, k/2, \omega\chi_m)$. Since the value of g at a cusp point is a linear combination of the values of f at some cusp points, $[\Delta_1\xi\Delta_1]$ maps $S(N, k/2, \omega)$ into $S(N, k/2, \omega\chi_m)$. If $f \in \mathcal{E}(N, k/2, \omega)$, by Lemma 5.35, $f \in \mathcal{E}(\Delta(mN), k/2)$. For any $g' \in S(\Delta(N), k/2)$, by Lemma 5.26 (it is clear that the lemma holds also for the half integral case), we have

$$\langle g, g' \rangle = \langle f, g'|[\Delta_1\xi\Delta_1] \rangle = 0,$$

since $g'|[\Delta_1\xi\Delta_1] \in S(\Delta(mN), k/2)$. This shows that $g \in \mathcal{E}(\Delta(N), k/2)$, so that $[\Delta_1\xi\Delta_1]$ sends $\mathcal{E}(N, k/2, \omega)$ into $\mathcal{E}(N, k/2, \omega\chi_m)$.

Denote by Γ_1 the group $\Gamma_1(N)$. By Lemma 5.17, we have

$$\Gamma_1\alpha\Gamma_1 = \bigcup_{b=1}^{m} \Gamma_1 \begin{pmatrix} 1 & b \\ 0 & m \end{pmatrix} = \bigcup_{b=1}^{m} \Gamma_1\alpha \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1 \bigcap \alpha^{-1}\Gamma_1\alpha$, then $\alpha\gamma\alpha^{-1} = \begin{pmatrix} a & bm^{-1} \\ cm & d \end{pmatrix} \in \Gamma_1$. Since $d \equiv 1 \pmod{N}$, we see that $(\alpha\gamma\alpha^{-1})^* = \xi\gamma^*\xi^{-1}$. By Lemma 5.37, we have

$$\Delta_1\xi\Delta_1 = \bigcup_{b=1}^{m} \Delta_1\xi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^*.$$

If $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N, k/2, \omega)$, then

$$\begin{aligned} f|[\Delta_1 \xi \Delta_1] &= m^{k/4-1} \sum_{b=1}^{m} f \Bigg| \left[\xi \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^* \right] \\ &= m^{-1} \sum_{b=1}^{m} f\left(\frac{z+b}{m}\right) \\ &= m^{-1} \sum_{n=0}^{\infty} a(n)e(nz/m) \sum_{b=1}^{m} e(nb/m) \\ &= \sum_{n=0}^{\infty} a(mn)e(nz). \end{aligned}$$

This completes the proof. □

Lemma 5.39 *Let m, n be square integers, and*

$$\alpha = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix},$$
$$\xi = \{\alpha, m^{\frac{1}{4}}\}, \quad \eta = \{\beta, n^{\frac{1}{4}}\}.$$

Suppose that $(m, n) = 1$ or $m|N^\infty$, and Δ is any one of $\Delta_0(N)$, $\Delta_1(N)$ and $\Delta(N)$, then

$$\Delta\xi\Delta \cdot \Delta\eta\Delta = \Delta\xi\eta\Delta = \Delta\eta\Delta \cdot \Delta\xi\Delta.$$

Proof Denote by Γ the group $P(\Delta)$. By Theorem 5.5, Lemma 5.8 and Theorem 5.4, we see that

$$\Gamma\alpha\Gamma \cdot \Gamma\beta\Gamma = \Gamma\alpha\beta\Gamma = \Gamma\beta\Gamma \cdot \Gamma\alpha\Gamma.$$

Let $\Gamma\alpha\Gamma = \bigcup \Gamma\alpha_i$, $\Gamma\beta\Gamma = \bigcup \Gamma\beta_j$, $\Gamma\alpha\beta\Gamma = \bigcup \Gamma\varepsilon_k$ be disjoint unions. Since mn is a square, by Lemma 5.37, we have

$$\Delta\xi\Delta = \bigcup \Delta\alpha_i', \quad \Delta\eta\Delta = \bigcup \Delta\beta_j', \quad \Delta\xi\eta\Delta = \bigcup \Delta\varepsilon_k'$$

with $P(\alpha_i') = \alpha_i$, $P(\beta_j') = \beta_j$, $P(\varepsilon_k') = \varepsilon_k$. Since there exists unique (i, j) such that $\Gamma\alpha_i\beta_j = \Gamma\alpha\beta$, there exists an unique (i, j) such that $\Delta\alpha_i'\beta_j' = \Delta\xi\eta$. This completes the proof. □

Let m be a square, $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}$ and $\xi = \{\alpha, m^{\frac{1}{4}}\}$. Put

$$\Gamma_0 = \Gamma_0(N), \quad \Delta_0 = \Delta_0(N), \quad \Delta_1 = \Delta_1(N).$$

Let $\Gamma_0\alpha\Gamma_0 = \bigcup \Gamma_0\alpha_i$, $\Delta_0\xi\Delta_0 = \bigcup \Delta_0\xi_i$ with $\alpha_i = P(\xi_i)$. We define the Hecke operator $\mathrm{T}_{N,k,\omega}(m)$ on $G(N, k/2, \omega)$ as follows:

$$f|\mathrm{T}_{N,k,\omega}(m) = m^{k/4-1}\sum_i \omega(a_i)f|[\xi_i],$$

where $\alpha_i = \begin{pmatrix} a_i & * \\ * & * \end{pmatrix}$. It is easy to verify that the actions of $\mathrm{T}_{N,k,\omega}$ on $G(N, k/2, \omega)$ coincides with the one of $[\Delta_1\xi\Delta_1]$, which sends $S(N, k/2, \omega)$ and $\mathcal{E}(N, k/2, \omega)$ into themselves respectively.

Theorem 5.15 *Let p be a prime, $f = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N, k/2, \omega)$. Put*

$$f|\mathrm{T}_{N,k,\omega}(p^2) = \sum_{n=0}^{\infty} b(n)e(nz).$$

Then

$$b(n) = a(p^2n) + \omega_1(p)\left(\frac{n}{p}\right)p^{\lambda-1}a(n) + \omega(p^2)p^{k-2}a(n/p^2), \tag{5.39}$$

where $\lambda = \dfrac{k-1}{2}$, $\omega_1 = \omega\left(\dfrac{(-1)^\lambda}{*}\right)$ and $a(n/p^2) = 0$ if $p^2 \nmid n$.

Proof If $p|N$, we have $b(n) = a(p^2n)$ by Lemma 5.38. So we assume that $p \nmid N$. Put $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}$ and $\xi = \{\alpha, p^{\frac{1}{2}}\}$. The following $p^2 + p$ elements consist of a complete set of representatives of right cosets of Γ_0 in $\Gamma_0\alpha\Gamma_0$:

$$\begin{aligned}
\alpha_b &= \begin{pmatrix} 1 & b \\ 0 & p^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad 0 \leqslant b < p^2,\\
\beta_h &= \begin{pmatrix} p & h \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ psN & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}\begin{pmatrix} p & h \\ -sN & r \end{pmatrix}, \quad 0 < h < p,\\
\sigma &= \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^2 & -t \\ N & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}\begin{pmatrix} p^2d & t \\ -N & 1 \end{pmatrix},
\end{aligned}$$

where for each h we choose r, s such that $pr + shN = 1$, and t, d satisfy $p^2d + tN = 1$. For γ, $\delta \in \Gamma_0$, we define $L(\gamma\alpha\delta) = \gamma^*\xi\delta^*$. By Lemma 5.29, this is a bijection from $\Gamma_0\alpha\Gamma_0$ to $\Delta_0\xi\Delta_0$, $L(\alpha_b)$ $(0 \leqslant b < p^2)$, $L(\beta_h)$ $(0 < h < p)$ and $L(\sigma)$ consist of a complete set of representatives of right cosets of Δ_0 in $\Delta_0\xi\Delta_0$. A direct computation shows that

$$L(\alpha_b) = \{\alpha_b, p^{\frac{1}{2}}\}, \quad L(\beta_h) = \left\{\beta_h, \varepsilon_p^{-1}\left(\frac{-h}{p}\right)\right\}, \quad L(\sigma) = \{\sigma, p^{-\frac{1}{2}}\}.$$

Therefore

$$f|\mathrm{T}_{N,k,\omega}(p^2) = p^{k/2-2}\biggl(\sum_b f|[L(\alpha_b)] + \omega(p)\sum_h f|[L(\beta_h)] + \omega(p^2)f|[L(\sigma)]\biggr). \quad (5.40)$$

But

$$\begin{aligned} p^{k/2-2}\sum_b f|[L(\alpha_b)] &= p^{k/2-2}\sum_b f((z+b)/p^2)p^{-k/2} \\ &= p^{-2}\sum_{n=0}^{\infty} a(n)e(nz/p^2)\sum_{b=0}^{p^2-1} e(bn/p^2) \\ &= \sum_{n=0}^{\infty} a(p^2n)e(nz), \end{aligned} \quad (5.41)$$

$$\begin{aligned} p^{k/2-2}\sum_h f|[L(\beta_h)] &= p^{k/2-2}\varepsilon_p^k\sum_h \left(\frac{-h}{p}\right)f(z+h/p) \\ &= p^{k/2-2}\varepsilon_p^k\left(\frac{-1}{p}\right)\sum_{n=0}^{\infty} a(n)e(nz)\sum_{h=1}^{p-1}\left(\frac{h}{p}\right)e(nh/p) \\ &= p^{\lambda-1}\left(\frac{(-1)^{\lambda}}{p}\right)\sum_{n=0}^{\infty}\left(\frac{n}{p}\right)a(n)e(nz), \end{aligned} \quad (5.42)$$

where we used the Gauss sum

$$\sum_{h=1}^{p-1}\left(\frac{h}{p}\right)e(h/p) = \varepsilon_p p^{\frac{1}{2}}.$$

It is clear that

$$p^{k/2-2}f|[L(\sigma)] = p^{k-2}\sum_{n=0}^{\infty} a(n)e(np^2z). \quad (5.43)$$

Inserting (5.41)–(5.43) into (5.40), we obtain the desired result. This completes the proof. □

Let m be any positive integer, we define a translation operator $V(m)$ as follows:

$$f|V(m) = f(mz), \quad \forall f \in G(N, k/2, \omega).$$

Theorem 5.16 *If $f \in G(N,\ k/2,\ \omega)$ (or $S(N,\ k/2,\ \omega)$, $\mathcal{E}(N,\ k/2,\ \omega)$ respectively), then $f|V(m) \in G(mN, k/2, \omega\chi_m)$ (or $S(mN, k/2, \omega\chi_m)$, $\mathcal{E}(mN, k/2, \omega\chi_m)$ respectively).*

Proof Put $\xi = \left\{\begin{pmatrix} m & 0 \\ 0 & 1\end{pmatrix}, m^{-1/4}\right\}$. Then

$$f|V(m) = m^{-k/4}f|[\xi].$$

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(mN)$, we see that

$$\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} m^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & bm \\ cm^{-1} & d \end{pmatrix} \in \Gamma_0(N),$$

so that,

$$\gamma^* = \xi^{-1} \begin{pmatrix} a & bm \\ cm^{-1} & d \end{pmatrix}^* \xi \left\{ 1, \left(\frac{m}{d} \right) \right\}.$$

Hence

$$f|[\xi][\gamma^*] = \omega(d) \left(\frac{m}{d} \right) f|[\xi],$$

which shows that $f|V(m) \in G(mN, k/2, \omega\chi_m)$. The still open assertion can be proved along similar lines as used in the proof of Lemma 5.38. This completes the proof. □

We introduce now the Fricke operator $W(Q)$. Let Q be a positive divisor of N such that $(Q, N/Q) = 1$. Take integers u, v such that $vQ + uN/Q = 1$, then $\begin{pmatrix} Q & -1 \\ uN & vQ \end{pmatrix}$ satisfies

$$\begin{pmatrix} Q & -1 \\ uN & vQ \end{pmatrix} \Gamma_0(N) \begin{pmatrix} Q & -1 \\ uN & vQ \end{pmatrix}^{-1} = \Gamma_0(N).$$

If $2 \nmid Q$, put

$$\begin{aligned} W(Q) &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & Q \end{pmatrix}, Q^{\frac{1}{4}} \right\} \begin{pmatrix} Q & -1 \\ uN/Q & v \end{pmatrix}^* \\ &= \left\{ \begin{pmatrix} Q & -1 \\ uN & vQ \end{pmatrix}, \varepsilon_Q^{-1} Q^{\frac{1}{4}} (uNQ^{-1}z + v)^{\frac{1}{2}} \right\}. \end{aligned}$$

If $4|Q$, put

$$\begin{aligned} W(Q) &= \left\{ \begin{pmatrix} 0 & -1 \\ Q & 0 \end{pmatrix}, Q^{\frac{1}{4}} (-\mathrm{i}z)^{\frac{1}{2}} \right\} \begin{pmatrix} uN/Q & v \\ -Q & 1 \end{pmatrix}^* \\ &= \left\{ \begin{pmatrix} Q & -1 \\ uN & vQ \end{pmatrix}, \mathrm{e}^{-\pi\mathrm{i}/4} Q^{\frac{1}{4}} (uNQ^{-1}z + v)^{\frac{1}{2}} \right\}. \end{aligned}$$

It is clear that $W(Q) \in \widehat{G}$ is dependent on the choices of u, v.

Theorem 5.17 *Let $f \in G(N,\, k/2,\, \omega_1\omega_2)$ with ω_1 and ω_2 characters modulo Q and N/Q respectively. Then $g = f|[W(Q)] \in G(N,\, k/2,\, \overline{\omega_1}\omega_2\chi_Q)$ is independent on the choices of u, v. And the operator $[W(Q)]$ sends $S(N,\, k/2,\, \omega_1\omega_2)$ and $\mathcal{E}(N,\, k/2,\, \omega_1\omega_2)$ into $S(N,\, k/2,\, \overline{\omega_1}\omega_2\chi_Q)$ and $\mathcal{E}(N,\, k/2,\, \overline{\omega_1}\omega_2\chi_Q)$ respectively.*

Proof Suppose Q is an odd(we can similarly prove the theorem if $4|Q$) and u_1, v_1 satisfy $v_1Q + u_1N/Q = 1$ also. Then

$$\begin{aligned}&\left\{\begin{pmatrix}1 & 0\\ 0 & Q\end{pmatrix}, Q^{\frac{1}{4}}\right\}\begin{pmatrix}Q & -1\\ uN/Q & v\end{pmatrix}^*\begin{pmatrix}v_1 & 1\\ -u_1N/Q & Q\end{pmatrix}^*\left\{\begin{pmatrix}1 & 0\\ 0 & Q^{-1}\end{pmatrix}, Q^{-\frac{1}{4}}\right\}\\ &=\begin{pmatrix}1 & 0\\ (uv_1-vu_1)N & 1\end{pmatrix}^*.\end{aligned}$$

This shows that g is independent on the choices of u, v since $\begin{pmatrix}1 & 0\\ (uv_1-vu_1)N & 1\end{pmatrix}^* \in \Delta_0(N)$. Let $\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma_0(N)$, $\alpha = \begin{pmatrix}Q & -1\\ uN/Q & v\end{pmatrix}$, then

$$\gamma_0 = \begin{pmatrix}a_0 & b_0\\ c_0 & d_0\end{pmatrix} = \begin{pmatrix}Q & -1\\ uN & vQ\end{pmatrix}\gamma\begin{pmatrix}Q & -1\\ uN & vQ\end{pmatrix}^{-1} \in \Gamma_0(N)$$

and

$$d_0 = auN/Q + buN + cv + dvQ.$$

Since $uN/Q + vQ = 1$, $ad \equiv 1 \pmod N$, we get $d_0 \equiv a \pmod{4Q}$ and $d_0 \equiv d \pmod{N/Q}$. But

$$\alpha\gamma\alpha^{-1} = \begin{pmatrix}a_0 & b_0Q\\ c_0/Q & d_0\end{pmatrix},$$

so that

$$\begin{aligned}&W(Q)\gamma^*W(Q)^{-1}\\ &=\left\{\begin{pmatrix}1 & 0\\ 0 & Q\end{pmatrix}, Q^{\frac{1}{4}}\right\}(\alpha\gamma\alpha^{-1})^*\left\{\begin{pmatrix}1 & 0\\ 0 & Q^{-1}\end{pmatrix}, Q^{-\frac{1}{4}}\right\}\\ &=\left\{\begin{pmatrix}1 & 0\\ 0 & Q\end{pmatrix}, Q^{\frac{1}{4}}\right\}\left\{\begin{pmatrix}1 & 0\\ 0 & Q^{-1}\end{pmatrix}\gamma_0, \varepsilon_{d_0}^{-1}\left(\frac{c_0Q}{d_0}\right)(c_0z+d_0)^{\frac{1}{2}}Q^{-\frac{1}{4}}\right\}\\ &=\gamma_0^*\left\{1, \left(\frac{Q}{d_0}\right)\right\} = \gamma_0^*\left\{1, \left(\frac{Q}{d}\right)\right\},\end{aligned}$$

since $\left(\frac{Q}{d_0}\right) = \left(\frac{Q}{a}\right) = \left(\frac{Q}{d}\right)$. Therefore we have

$$g|\gamma^* = f|[W(Q)\gamma^*] = (\omega\chi_Q)(d_0)g = (\overline{\omega_1}\omega_2\chi_Q)(d)g,$$

where $\omega = \omega_1\omega_2$. This shows that $[W(Q)]$ sends $G(N,\, k/2,\, \omega_1\omega_2)$ into $G(N,\, k/2,\, \overline{\omega_1}\omega_2\chi_Q)$. It is easy to see that $[W(Q)]$ sends also $S(N,\, k/2,\, \omega_1\omega_2)$ into $S(N,\, k/2,\, \overline{\omega_1}\omega_2\chi_Q)$. Then, by Lemma 5.35, we see that $[W(Q)]$ sends also $\mathcal{E}(N,\, k/2,\, \omega_1\omega_2)$ into $\mathcal{E}(N,\, k/2,\, \overline{\omega_1}\omega_2\chi_Q)$. This completes the proof. □

It is easy to verify that $W(Q)^2$ is the identity operator. If $Q=N$, since $\begin{pmatrix} u & -1 \\ vN & 1 \end{pmatrix}^* \in \Delta_0(N)$ and

$$\begin{aligned}
&\begin{pmatrix} u & -1 \\ vN & 1 \end{pmatrix}^* \left\{ \begin{pmatrix} N & -1 \\ uN & vN \end{pmatrix}, \mathrm{e}^{-\pi \mathrm{i}/4} N^{\frac{1}{4}} (uz+v)^{\frac{1}{2}} \right\} \\
&= \left\{ \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}, N^{\frac{1}{4}} (-\mathrm{i}z)^{\frac{1}{2}} \right\},
\end{aligned}$$

we can take $W(N) = \left\{ \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}, N^{\frac{1}{4}} (-\mathrm{i}z)^{\frac{1}{2}} \right\}$.

Let $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N, k/2, \omega)$, and ψ a primitive character modulo m. Put

$$\begin{aligned}
\mathrm{Twist}(f) &= \sum_{u=1}^{m} \overline{\psi}(u) f\left(z + \frac{u}{m}\right) \\
&= \sum_{u=1}^{m} \overline{\psi}(u) e\left(\frac{u}{m}\right) \sum_{n=0}^{\infty} \psi(n) a(n) e(nz).
\end{aligned}$$

Theorem 5.18 *Let s be the conductor of ω. Then* $\mathrm{Twist}(f) \in G(N^*, k/2, \omega\psi^2)$ *with N^* the least common multiple of N, sm, $4m$ and m^2. If $f \in S(N,\ k/2,\ \omega)$ or $f \in \mathcal{E}(N,\ k/2,\ \omega)$, then* $\mathrm{Twist}(f) \in S(N^*,\ k/2,\ \omega\psi^2)$ *or* $\mathrm{Twist}(f) \in \mathcal{E}(N^*,\ k/2,\ \omega\psi^2)$ *respectively.*

Proof Let $\gamma = \begin{pmatrix} a & b \\ cN^* & d \end{pmatrix} \in \Gamma_0(N^*)$ and

$$\begin{aligned}
a' &= a + cuN^*/m, \\
b' &= b + du(1-ad)/m - cd^2u^2N^*/m^2, \\
d' &= d - cd^2uN^*/m.
\end{aligned}$$

It is clear that a', b', d' are integers. It is easy to verify that

$$\left\{ \begin{pmatrix} 1 & u/m \\ 0 & 1 \end{pmatrix}, 1 \right\} \gamma^* = \begin{pmatrix} a' & b' \\ cN^* & d' \end{pmatrix}^* \left\{ \begin{pmatrix} 1 & \frac{d^2u}{m} \\ 0 & 1 \end{pmatrix}, 1 \right\},$$

where we used the facts: $d \equiv d' \pmod 4$ and $\left(\frac{cN^*}{d'}\right) = \left(\frac{cN^*}{d}\right)$. Hence

$$\begin{aligned}
\mathrm{Twist}(f)|\gamma^* &= \sum_{u=1}^{m} \overline{\psi}(u) f \left| \left[\left\{ \begin{pmatrix} 1 & \frac{u}{m} \\ 0 & 1 \end{pmatrix}, 1 \right\} \gamma^* \right] \right. \\
&= \omega(d') \sum_{u=1}^{m} \overline{\psi}(u) f \left| \left[\left\{ \begin{pmatrix} 1 & \frac{d^2u}{m} \\ 0 & 1 \end{pmatrix}, 1 \right\} \right] \right. \\
&= (\omega\psi^2)(d) \mathrm{Twist}(f),
\end{aligned}$$

where we used the fact $sm|N^*$. This shows that $\mathrm{Twist}(f) \in G(N^*,\ k/2,\ \omega\psi^2)$. The rest results can be proved similarly as in the proof of Theorem 5.17. This completes the proof. □

The operator Twist(·) is called the twist operator.

Let $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N,\ k/2,\ \omega)$. We define the conjugate operator H as follows:

$$(f|H)(z) = \overline{f(-\overline{z})} = \sum_{n=0}^{\infty} \overline{a(n)}e(nz).$$

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we have

$$(f|H)(\gamma(z)) = \overline{f\left(\frac{a(-\overline{z}) - b}{-c(-\overline{z}) + d}\right)} = \overline{\omega}(d)\varepsilon_d^k\left(\frac{-c}{d}\right)(cz+d)^{k/2}\overline{f(-\overline{z})}.$$

That is, $f|H \in G(N,\ k/2,\ \overline{\omega})$. It is easy to verify that $f|H \in S(N,\ k/2,\ \overline{\omega})$ (resp. $\in \mathcal{E}(N,\ k/2,\ \overline{\omega})$) if $f \in S(N,\ k/2,\ \omega)$ (resp. $\in \mathcal{E}(N,\ k/2,\ \omega)$).

Theorem 5.19 *Let $f \in G(N,\ k/2,\ \omega)$. Then*

$$\begin{aligned}(f|V(m))|\mathrm{T}_{mN,k,\omega\chi_m}(p^2) &= (f|\mathrm{T}_{N,k,\omega}(p^2))|V(m), \quad p \nmid m,\\ (f|H)|\mathrm{T}_{N,k,\overline{\omega}}(p^2) &= (f|\mathrm{T}_{N,k,\omega}(p^2))|H,\\ (f|[W(N)])|\mathrm{T}_{N,k,\overline{\omega}\chi_N}(p^2) &= \overline{\omega}(p^2)(f|\mathrm{T}_{N,k,\omega}(p^2))|[W(N)], \quad p \nmid N.\end{aligned}$$

Proof The first two equalities can be deduced by (5.39) and the definitions of $V(m)$ and H respectively. By (5.40), we have

$$\begin{aligned}(f|[W(N)])|\mathrm{T}_{N,k,\overline{\omega}\chi_N}(p^2)|[W(N)]^{-1} = p^{k/2-2}\Bigg(&\sum_{b=0}^{p^2-1} f|[W(N)L(\alpha_b)W(N)^{-1}]\\ &+\overline{\omega}(p)\left(\frac{N}{p}\right)\sum_{h=1}^{p-1} f|[W(N)L(\beta_h)W(N)^{-1}]\\ &+\overline{\omega}(p^2)f|[W(N)L(\sigma)W(N)^{-1}]\Bigg). \qquad (5.44)\end{aligned}$$

Write $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}$, $\beta = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$, then $\beta\alpha\beta^{-1} = \sigma$. It is easy to see that

$$W(N)L(\alpha)W(N)^{-1} = L(\sigma).$$

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, it is easy to verify that

$$W(N)L(\gamma)W(N)^{-1} = L(\beta\gamma\beta^{-1})\left\{1, \left(\frac{N}{d}\right)\right\}.$$

For any $\delta = \gamma_1\alpha\gamma_2 = \begin{pmatrix} * & * \\ * & d \end{pmatrix}$, $\gamma_1, \gamma_2 \in \Gamma_0(N)$, then

$$\begin{aligned} W(N)L(\delta)W(N)^{-1} &= W(N)L(\gamma_1)W(N)^{-1} \\ &\quad \times W(N)L(\alpha)W(N)^{-1}W(N)L(\gamma_2)W(N)^{-1} \\ &= L(\beta\delta\beta^{-1})\{1, (\frac{N}{d})\} \end{aligned} \tag{5.45}$$

If $p \nmid b$, taking integers s, t such that $sp^2 + tbN = 1$, we have

$$\beta\alpha_b\beta^{-1} = \begin{pmatrix} p^2 & 0 \\ -bN & 1 \end{pmatrix} = \begin{pmatrix} p^2 & t \\ -bN & s \end{pmatrix} \begin{pmatrix} 1 & -t \\ 0 & p^2 \end{pmatrix}.$$

If $p|b \neq 0$, taking integers s', t' such that $s'p^2 + t'bN = p$, we have

$$\beta\alpha_b\beta^{-1} = \begin{pmatrix} p^2 & 0 \\ -bN & 1 \end{pmatrix} = \begin{pmatrix} p & t' \\ -bN/p & s' \end{pmatrix} \begin{pmatrix} p & -t' \\ 0 & p \end{pmatrix}.$$

For any h $(0 < h < p)$, taking integers s'', t'' such that $s''p + t''hN = 1$, we have

$$\beta\beta_h\beta^{-1} = \begin{pmatrix} p & 0 \\ -hN & p \end{pmatrix} = \begin{pmatrix} p & t'' \\ -hN & s'' \end{pmatrix} \begin{pmatrix} 1 & -t''p \\ 0 & p^2 \end{pmatrix}.$$

By (5.45), we see that the right hand side of (5.44) equals to

$$\begin{aligned} p^{k/2-1}(f|[L(\sigma)] &+ \overline{\omega}(p^2) \sum_{0<b<p^2, p\nmid b} f|[L(\alpha_b)] \\ &+ \overline{\omega}(p) \sum_{0<h<p} f|[L(\beta_h)] + \overline{\omega}(p^2) \sum_{0<b<p^2, p|b} f|[L(\alpha_b)] \\ &+ \overline{\omega}(p^2) f|[L(\alpha_0)]) = \overline{\omega}(p^2) f|\mathrm{T}_{N,k,\omega}(p^2). \end{aligned}$$

This completes the proof. □

Let p_0 be a prime with $p_0|N/4$. Then $\Gamma_0(N)$ is a subgroup of $\Gamma_0(N/p_0)$. Denote by u the index of $\Gamma_0(N)$ in $\Gamma_0(N/p_0)$. Let

$$\Gamma_0(N/p_0) = \bigcup_{j=1}^{u} \Gamma_0(N)A_j$$

be a disjoint union, where $A_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in \Gamma_0(N/p_0)$. Let $f \in G(N, k/2, \omega)$. We define the trace operator $\mathrm{Tr}(\omega)$ as follows:

$$f|\mathrm{Tr}(\omega) = \sum_{j=1}^{u} \omega(a_j) f|[A_j^*],$$

which is independent on the choices of $\{A_j\}_{j=1}^{u}$. If ω is a character modulo N/p_0, then $f|\mathrm{Tr}(\omega) \in \Gamma_0(N/p_0, k/2, \omega)$. If $f \in G(N/p_0, k/2, \omega)$, then $f|\mathrm{Tr}(\omega) = uf$.

Theorem 5.20 *Let $f \in G(N, k/2, \omega)$. Assume ω is a character modulo N/p_0. Then for any prime p prime to N, we have*

$$(f|\mathrm{Tr}(\omega))|\mathrm{T}_{N/p_0,k,\omega}(p^2) = (f|\mathrm{T}_{N,k,\omega}(p^2))|\mathrm{Tr}(\omega).$$

Proof For any integers a, c, we can find two integers s, t such that $(s,t) = 1$, $p^2|sa + tc$ and $N|s$. This implies that if necessary, we can left multiply A_j by an element γ of $\Gamma_0(N)$ such that the lower left entry of γA_j is divisible by p^2. So we can assume that $p^2|c_j$. Put

$$\xi = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}, p^{1/2} \right\}, \quad \Gamma_0(N)\xi\Gamma_0(N) = \bigcup_i \Gamma_0(N)\xi\alpha_i^*, \quad \alpha_i \in \Gamma_0(N).$$

For any i, we have

$$\sum_{j=1}^{u} \omega(a_j)A_j^*\xi\alpha^* = \xi\alpha_i^* \sum_{j=1}^{u} \omega(a_j)(\alpha_i^{-1})^* \begin{pmatrix} a_j & b_jp^2 \\ c_jp^{-2} & d_j \end{pmatrix}^* \alpha_i^*. \tag{5.46}$$

For any two positive integers j, j' satisfying $1 \leqslant j < j' \leqslant u$, if there exists $\gamma \in \Gamma_0(N)$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}^{-1} A_j \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix} = \gamma \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}^{-1} A_{j'} \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix},$$

then

$$A_j = \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix} \gamma \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}^{-1} A_{j'},$$

and hence

$$\begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix} \gamma \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}^{-1} = A_j A_{j'}^{-1} \in \Gamma_0(N/p_0).$$

Since $\gamma \in \Gamma_0(N)$, we see that

$$\begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix} \gamma \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}^{-1} \in \Gamma_0(N).$$

This contradicts the fact that A_j and $A_{j'}$ belong to different right cosets of $\Gamma_0(N)$ in $\Gamma_0(N/p_0)$. Therefore

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}^{-1} A_j \begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}, 1 \leqslant j \leqslant u \right\}$$

is a complete set of representatives of right cosets of $\Gamma_0(N)$ in $\Gamma_0(N/p_0)$. Therefore

$$\left\{ \alpha_i^{-1} \begin{pmatrix} a_j & b_jp^2 \\ c_jp^{-2} & d_j \end{pmatrix} \alpha_i, 1 \leqslant j \leqslant u \right\}$$

is also a complete set of representatives of right cosets of $\Gamma_0(N)$ in $\Gamma_0(N/p_0)$. Hence the equality in the theorem holds due to (5.46) and the definitions of $\mathrm{Tr}(\omega)$ and $\mathrm{T}_{*,k,\omega}$. This completes the proof. $\square$

Put

$$S(\omega) := S(\omega, N, p_0) := p_0^{k/4} u^{-1}[W(N)]\mathrm{Tr}(\overline{\omega}\chi_N)[W(N/p_0)].$$

Theorem 5.21 *Let $\omega\chi_{p_0}$ be well-defined modulo N/p_0. Then*

(1) *$S(\omega)$ sends $G(N, k/2, \omega)$ into $G(N/p_0, k/2, \omega\chi_{p_0})$;*

(2) *If $(m, p_0) = 1$ and $f \in G(N, k/2, \omega)$, then*

$$f|S(\omega, N, p_0) = f|S(\omega, mN, p_0);$$

(3) *If $p \nmid N$ and $f \in G(N, k/2, \omega)$, then*

$$(f|S(\omega))|\mathrm{T}_{N/p_0,k,\omega\chi_{p_0}}(p^2) = (f|\mathrm{T}_{N,k,\omega}(p^2))|S(\omega);$$

(4) *If $g \in G(N/p_0, k/2, \omega\chi_{p_0})$, then $g|V(p_0) \in G(N, k/2, \omega)$ and*

$$(g|V(p_0))|S(\omega, N, p_0) = g.$$

(5) *Let p be a prime with $4p|N, p \neq p_0$, and $\omega\chi_p$ well-defined modulo N/p, if $g \in G(N/p,\ k/2,\ \omega\chi_p)$, then*

$$(g|V(p))|S(\omega, N, p_0) = (g|S(\omega\chi_p, N/p, p_0))|V(p).$$

Proof If $\omega\chi_{p_0}$ is well-defined modulo N/p_0, then $\overline{\omega}\chi_N = \overline{\omega}\chi_{p_0}\chi_{N/p_0}$ is too, so that (1) can be deduced from Theorem 5.17. Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(mN/p_0)$$

with $p_0 \nmid m$. Since

$$W(mN)\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* W(mN/p_0) = \{mI, 1\}W(N)\begin{pmatrix} a & bm \\ cm^{-1} & d \end{pmatrix}^* W(N/p_0),$$

we see that (2) holds. If $p \nmid N$, then Theorem 5.19 and Theorem 5.20 give (3). Since

$$\left\{\begin{pmatrix} p_0 & 0 \\ 0 & 1 \end{pmatrix}, p_0^{-\frac{1}{4}}\right\}W(N) = \{p_0 I, 1\}W(N/p_0),$$

we have

$$(g|V(p_0))|[W(N)] = p_0^{-k/4} g|[W(N/p_0)].$$

Since $g|[W(N/p_0)] \in G(N/p_0,\ k/2,\ \overline{\omega}\chi_N)$, it is fixed by $u^{-1}\mathrm{Tr}(\overline{\omega}\chi_N)$, (4) holds by $[W(N/p_0)]^2 = I$.

Finally, since $4pp_0|N$, $\omega\chi_{pp_0}$ is well-defined modulo N/pp_0 and

$$\left\{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p^{-\frac{1}{4}}\right\}W(N) = \{pI, 1\}W(N/p),$$

$$W(N/p_0) = W(N/pp_0)\left\{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, p^{-\frac{1}{4}}\right\},$$

$$\overline{\omega}\chi_N = \overline{\omega\chi_p}\chi_{N/p},$$

(5) holds. This completes the proof. □

Finally, we discuss now Zeta functions and Euler products of modular forms with half integral weight.

Theorem 5.22 *Let* $f(z) = \sum_{n=0}^{\infty} c(n)q^n \in G(N, k/2, \omega)$. *Put*

$$L(s,f) = \sum_{n=1}^{\infty} c(n)n^{-s},$$
$$R_N(s,f) = (2\pi)^{-s}N^{s/2}\Gamma(s)L(s,f).$$

Then $L(s,f)$ *is absolutely convergent for* $\mathrm{Re}(s) > 1+k/2$. $R_N(s,f)$ *can be analytically continued to a meromorphic function on the* s*-plane with possible poles* $s=0$ *and* $s=k/2$ *of order* 1, *and the residues of* $R_N(s,f)$ *at* $s=0$ *and* $k/2$ *are* $c(0)$ *and* $b(0)N^{-k/4}$ *respectively, where* $b(0)$ *is the constant term of the* Fourier *expansion of* $f|[W(N)]$ *at* $\mathrm{i}\infty$. *And* $R_N(s,f)$ *satisfies the following functional equation*:

$$R_N(s,f) = R_N(k/2-s, f|[W(N)]).$$

Proof This can be proved completely similarly as done in the proof of Theorem 5.9. □

Lemma 5.40 *Let* t *be a positive integer*, p *a prime. Let*

$$f(z) = \sum_{n=0}^{\infty} c(n)e(nz) \in G(N, k/2, \omega)$$

be an eigenfunction of $\mathrm{T}_{N,k,\omega}(p^2)$ *with eigenvalue* λ_p. *Suppose that* $p|N$ *or* $p^2 \nmid t$. *Then*

(1) $\lambda_p c(t) = c(p^2t) + \omega_1(p)\left(\frac{t}{p}\right)p^{\lambda-1}c(t)$;

(2) $\lambda_p c(p^{2m}t) = c(p^{2m+2}t) + \omega_1(p^2)p^{k-2}c(p^{2m-2}t)$ *for any positive integer* m, *and*

$$\sum_{n=1}^{\infty} c(tn^2)n^{-s} = \left(\sum_{(p,n)=1} c(tn^2)n^{-s}\right)\left(1-\omega_1(p)\left(\frac{t}{p}\right)p^{\lambda-1-s}\right)$$
$$\times (1-\lambda_p p^{-s} + \omega(p^2)p^{k-2-2s})^{-1},$$

where $\lambda = \dfrac{k-1}{2}$, $\omega_1 = \omega\left(\dfrac{(-1)^{\lambda}}{\cdot}\right)$.

Proof By Theorem 5.15 and $f|\mathrm{T}_{N,k,\omega}(p^2) = \lambda_p f$, we obtain immediately, if $(n,p)=1$, that

$$\lambda_p c(tn^2) = c(tp^2n^2) + \omega_1(p)\left(\frac{t}{p}\right)p^{\lambda-1}c(tn^2), \tag{5.47}$$
$$\lambda_p c(tp^{2m}n^2) = c(tp^{2m+2}n^2) + \omega(p^2)p^{k-2}c(tp^{2m-2}n^2), \quad m>0. \tag{5.48}$$

This shows the first two equalities (1) and (2). Put

$$H_n(x) = \sum_{m=0}^{\infty} c(tn^2p^{2m})x^m.$$

Multiplying both sides of (5.47) and (5.48) by x and x^{m+1} respectively, we get

$$\lambda_p x H_n(x) = H_n(x) - c(tn^2) + \omega_1(p)\left(\frac{t}{p}\right)p^{\lambda-1}c(tn^2)x + \omega(p^2)p^{k-2}x^2H_n(x),$$

so that

$$H_n(x) = c(tn^2)\left(1 - \omega_1(p)\left(\frac{t}{p}\right)p^{\lambda-1}x\right)(1 - \lambda_p x + \omega(p^2)p^{k-2}x^2)^{-1}.$$

Since $\sum_{n=1}^{\infty} c(tn^2)n^{-s} = \sum_{(p,n)=1} H_n(p^{-s})n^{-s}$, we see that

$$\begin{aligned}
&\sum_{n=1}^{\infty} c(tn^2)n^{-s} \\
&= \sum_{(p,n)=1} n^{-s}c(tn^2)\left(1 - \omega_1(p)\left(\frac{t}{p}\right)p^{\lambda-1}p^{-s}\right)(1 - \lambda_p p^{-s} + \omega(p^2)p^{k-2}p^{-2s})^{-1} \\
&= \left(\sum_{(p,n)=1} c(tn^2)n^{-s}\right)\left(1 - \omega_1(p)\left(\frac{t}{p}\right)p^{\lambda-1-s}\right)(1 - \lambda_p p^{-s} + \omega(p^2)p^{k-2-2s})^{-1}.
\end{aligned}$$

This completes the proof. □

Theorem 5.23 *Let* $f(z) = \sum_{n=0}^{\infty} c(n)e(nz) \in G(N,\ k/2,\ \omega)$ *satisfy* $f|\mathrm{T}_{N,k,\omega}(p^2) = \lambda_p f$ *for any prime* p. *Suppose that* t *is a square free positive integer and prime to* N. *Then we have the following* Euler *product*:

$$\sum_{n=1}^{\infty} c(tn^2)n^{-s} = c(t)\prod_p \left(1-\omega_1(p)\left(\frac{t}{p}\right)p^{\lambda-1-s}\right)(1-\lambda_p p^{-s}+\omega(p^2)p^{k-2-2s})^{-1}. \quad (5.49)$$

Proof This is a direct conclusion of Lemma 5.40. □

Remark 5.2 We recall the Euler product of modular forms with integral weight k (see Theorem 5.8). Let $g = \sum_{n=0}^{\infty} c(n)e(nz) \in G(N, l, \psi)$ not be a constant with $c(1) = 1$. Assume that for any positive integer n, we have

$$g|[\mathrm{T}(n)]_l = \lambda_n g, \quad \lambda_n \in \mathbb{R}.$$

Then $c(n) = \lambda_n$, and we have

$$\sum_{n=1}^{\infty} \lambda_n n^{-s} = \prod_p \left(1 - c(p)p^{-s} + \psi(p)p^{k-1-2s}\right)^{-1}.$$

So the denominator of (5.49) is very similar to an Euler product (take $l = k - 1$, $\psi = \omega^2$). Put

$$\sum_{n=1} A(n)n^{-s} = \prod_p \left(1 - \lambda_p p^{-s} + \omega(p^2)p^{k-2-2s}\right)^{-1},$$

$$F(z) = \sum_{n=1}^{\infty} A(n)e(nz).$$

Shimura showed that $F(z)$ is a modular form with weight $k - 1$, character ω^2 and level N' if $f \in S(N,\ k/2,\ \omega)$, $k \geqslant 3$. We call the map from $f \in S(N, k/2, \omega)$ to $F(z) \in G(N', k-1, \omega^2)$ the Shimura lifting. We will discuss the map in detail later.

References

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Chapter 6

New Forms and Old Forms

6.1 New Forms with Integral Weight

Let N, k positive integers, χ a character modulo N. We know that the Hecke operators $\mathrm{T}(n), (n, N) = 1$ can be diagonalized simultaneously in the space $S(N, k, \chi)$. On the other hand, if f is an eigenfunction of all Hecke operators $\mathrm{T}(n)$, then $L(s, f)$ has an Euler product. So we want to ask the following question: Can all Hecke operators $\mathrm{T}(n)$ be diagonalized simultaneously in the space $S(N, k, \chi)$. The following example gives a counterexample to the question:

Example 6.1 Consider the space $V = S(2, 12, \mathrm{id.})$ which has dimension 2. Then

$$
\begin{aligned}
f_1(z) &= \Delta(z) := \frac{64\pi^{12}}{27}((E_4(z))^3 - (E_6(z))^2) \in V, \\
f_2(z) &= \Delta(2z) \in V.
\end{aligned}
$$

For any odd prime p, they have the same eigenvalue for $\mathrm{T}(p)$. If there exists a basis $\{g_1, g_2\}$ of V such that g_1, g_2 are eigenfunctions of all Hecke operators $\mathrm{T}(p)$ for any prime p, then by the properties of f_1, f_2, we see that $(g_1 - g_2)|\mathrm{T}(p) = 0$ for any odd prime p. Hence $(g_1 - g_2)|\mathrm{T}(n) = 0$ if n has an odd divisor. That is, the n-th Fourier coefficient $c(n)$ of $g_1 - g_2$ is equal to 0 if n has an odd divisor. This implies that $g_1 - g_2 = 0$ by the following Lemma 6.1. This contradicts the assertion. □

Lemma 6.1 (1) *Let* $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$ *with* $(a, b, c, d) = 1, \det(\alpha) = n > 1$, $(n, N) = 1$. *Assume that* $f \in G_k(\Gamma(N))$ *and* $f|[\alpha]_k \in G_k(\Gamma(N))$, *then* $f = 0$.

(2) *Let* $p \nmid N$ *be a prime and* $f(z) = \sum_{n=0}^{\infty} c(n) e^{2\pi i n z/N} \in G_k(\Gamma(N))$ *satisfy*

$$
c(n) = 0, \quad \text{for all } n \not\equiv 0 \pmod{p}.
$$

Then $f = 0$.

(3) *Let* p *and* f *be as above. If*

$$
c(n) = 0, \quad \text{for all } n \equiv 0 \pmod{p},
$$

then $f = 0$.

Proof Since $\Gamma(N)$ is a normal subgroup of $\Gamma(1)$, we may assume that $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$. Put $\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $f|[\alpha]_k[\tau^N]_k = f|[\alpha]_k$, i.e., $f|[\alpha\tau^N\alpha^{-1}]_k = f$. But $\alpha\tau^N\alpha^{-1} = n^{-1}\begin{pmatrix} n & N \\ 0 & n \end{pmatrix}$, so that

$$f\left|\left[\begin{pmatrix} n & N \\ 0 & n \end{pmatrix}\right]_k = f. \quad (6.1)$$

Take $\gamma \in \Gamma(1)$ such that $\gamma \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \pmod n, \gamma \equiv I \pmod N$. Then $\gamma \in \Gamma(N)$. Put

$$\beta = \gamma\begin{pmatrix} n & N \\ 0 & n \end{pmatrix} \equiv N\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \pmod n.$$

Then $\beta^l \equiv N^l\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \pmod n$ and $\det(\beta^l) = n^{2l}$ for any positive integer l. This implies that β^l is primitive (i.e., the entries of β^l are co-prime.). By (6.1), we have $f|[\beta]_k = f$ and hence

$$f|[\beta^l]_k = f$$

for any positive integer l. Take a positive integer l such that $n^l \equiv 1 \pmod N$, then

$$\beta^l \equiv \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}^l \equiv I \pmod N.$$

Since β^l is primitive, its elementary divisors are $\{1, n^{2l}\}$. Therefore there exist $\delta, \epsilon \in \Gamma(1)$ such that $\beta^l = \delta\begin{pmatrix} 1 & 0 \\ 0 & n^{2l} \end{pmatrix}\epsilon = \delta\alpha^{2l}\epsilon$. By the choice of l, we see that $\delta\epsilon \equiv \epsilon\delta \equiv I \pmod N$, i.e., $\delta\epsilon$, $\epsilon\delta \in \Gamma(N)$, so that

$$f|[\delta]_k[\alpha^{2l}]_k = f|[\delta]_k. \quad (6.2)$$

Put $g = f|[\delta]_k$, then $g \in G_k(\Gamma(N))$. Let

$$g(z) = \sum_{s=0}^{\infty} a(s)\mathrm{e}^{2\pi \mathrm{i} sz/N}$$

be the Fourier expansion of g at $\mathrm{i}\infty$. Then by (6.2) we see that $g\left(\frac{z}{r}\right) = r^{m/2}g(z)$ with $r = n^{2l}$, so that

$$a(s) = 0, \quad \forall r \nmid s, \quad a(sr) = r^{m/2}a(s).$$

This implies that $a(s) = 0$ for all $s \geqslant 1$, so that $g = 0$ and $f = 0$. This shows (1).

By the assumption of (2), we see that $f(z + N/p) = f(z)$, so that $f \in G_k(\Gamma(N))$

and $f|[\alpha]_k = f \in G_k(\Gamma(N))$ with $\alpha = \begin{pmatrix} p & N \\ 0 & p \end{pmatrix}$. Since α is primitive, we obtain (2) by (1).

By Lemma 5.18, we have

$$p^{1-k/2} f|\mathrm{T}(p) = f\left|\left[\sigma_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]_k + \sum_{b=0}^{p-1} f\right|\left[\begin{pmatrix} 1 & bt \\ 0 & p \end{pmatrix}\right]_k,$$

where $t|N$. By the assumption of (3), we see that

$$\begin{aligned} \sum_{b=0}^{p-1} f\left|\left[\begin{pmatrix} 1 & bt \\ 0 & p \end{pmatrix}\right]_k\right. &= p^{-k/2} \sum_{b=0}^{p-1} \left(\frac{z+bt}{p}\right) \\ &= p^{-k/2} \sum_{n=0, p\nmid n}^{\infty} c(n) \mathrm{e}^{2\pi i n z/p} \sum_{b=0}^{p-1} \mathrm{e}^{2\pi i n t b/p} = 0, \end{aligned}$$

where we used the fact $\sum_{b=0}^{p-1} \mathrm{e}^{2\pi i n t b/p} = 0$ (since $p \nmid nt$). Therefore

$$f\left|\left[\sigma_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right]_k\right. = p^{1-k/2} f|\mathrm{T}(p) \in G_k(\Gamma(N)).$$

Since $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ is primitive, we see that $f|[\sigma]_k = 0$ by (1), so that $f = 0$. This completes the proof. □

Let k, l be positive integers, put $\delta_l = \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}$. It is clear that, for any function f on $\mathbb{H}$, we have

$$f(lz) = l^{-k/2}(f|[\delta_l]_k)(z).$$

For any element $\gamma = \begin{pmatrix} a & b \\ clN & d \end{pmatrix} \in \Gamma_0(lN)$, we have

$$\delta_l \gamma \delta_l^{-1} = \begin{pmatrix} a & bl \\ cN & d \end{pmatrix} \in \Gamma_0(N).$$

For any $f \in G(N, k, \chi)$, put $g = f|[\delta_l]_k$. Then

$$g|[\gamma]_k = (f|[\delta_l \gamma \delta_l^{-1}]_k)|[\delta_l]_k = \chi(d) f|[\delta_l]_k = \chi(d) g,$$

so that we have the following:

Lemma 6.2 *Let $f \in G(N, k, \chi)$. Then, for any positive integer l, we have*

$$f(lz) = l^{-k/2}(f|[\delta_l]_k)(z) \in G(Nl, k, \chi).$$

Furthermore, $f(lz)$ is a cusp form if f is a cusp form.

Remark 6.1 We denote by $V(l)$ the operator in Lemma 6.2 and call it translation operator. It is clear that it is an analog of the translation operator for modular forms with half integral weight (see Theorem 5.16). Similar to Theorem 5.19, we can prove the following:

Lemma 6.3 *Let $f \in G(N,k,\chi)$, l a positive integer. Then we have*

$$(f|V(l))|\mathrm{T}(n) = (f|\mathrm{T}(n))|V(l), (n,l) = 1.$$

Let χ be a primitive character modulo m with $m|N$. Then $S(N,k,\chi)$ contains the following set

$$\left\{ f(z), f(lz) \middle| f(z) \in S(L,k,\chi), m|L, L|N, l \middle| \frac{N}{L} \right\}. \tag{6.3}$$

The functions f_1, f_2 are in the corresponding set (6.3) of $S(2,12,\mathrm{id.})$. We shall show that all Hecke operators can be diagonalized in the orthogonal complement of the space spanned by (6.3) in $S(N,k,\chi)$ with respect to Petersson inner product.

Put

$$\Delta_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \middle| c \equiv 0 \pmod N, (a,N) = 1, ad - bc > 0 \right\},$$
$$\Delta_0^*(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \middle| c \equiv 0 \pmod N, (d,N) = 1, ad - bc > 0 \right\}.$$

Lemma 6.4 *Let $\alpha \in \Delta_0(N)$ or $\in \Delta_0^*(N)$ respectively. Then there exist positive integers l, m satisfying $l|m$, $(l,N) = 1$ such that*

$$\Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N)\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}\Gamma_0(N)$$

or

$$\Gamma_0(N)\alpha\Gamma_0(N) = \Gamma_0(N)\begin{pmatrix} m & 0 \\ 0 & l \end{pmatrix}$$

respectively.

Proof Let $\alpha = \begin{pmatrix} a & b \\ cN & d \end{pmatrix}$, $a' = (a,c)$. Then $(a,cN) = a'$. Let u, v be integers such that $(u,v) = 1$, $au + cNv = a'$. Then $\begin{pmatrix} u & v \\ -cN/a' & a/a' \end{pmatrix} \in \Gamma_0(N)$ and

$$\begin{pmatrix} u & v \\ -cN/a' & a/a' \end{pmatrix}\begin{pmatrix} a & b \\ cN & d \end{pmatrix} = \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \in \Delta_0(N).$$

It is clear that $0 < a' \leqslant |a|$, and $0 < a' < |a|$ if $a \nmid c$. Put $a_1 = (a', b')$, then $0 < a_1 \leqslant a'$, and $0 < a_1 < a'$ if $a' \nmid b'$. It is easy to see that $(a', b'N) = a_1$. Let u_1, v_1

be integers such that $(u_1, v_1) = 1, a'u_1 + b'Nv_1 = a_1$, then $\begin{pmatrix} u_1 & -b'/a_1 \\ v_1N & a'/a_1 \end{pmatrix} \in \Gamma_0(N)$ and

$$\begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} \begin{pmatrix} u_1 & -b'/a_1 \\ v_1N & a'/a_1 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ c_1N & d_1 \end{pmatrix} \in \Delta_0(N).$$

The above process shows that, if $a \nmid b$ or c, then there exist $\gamma_1, \gamma_2 \in \Gamma_0(N)$ such that $\gamma_1\alpha\gamma_2 \in \Delta_0(N)$ and the upper left entry a_1 of $\gamma_1\alpha\gamma_2$ satisfies $1 \leqslant |a_1| < |a|$. Repeating the above process, we may assume that $\alpha \in \Delta_0(N)$ satisfies $a|(b,c)$. Then $\begin{pmatrix} 1 & 0 \\ -cN/a & 1 \end{pmatrix} \in \Gamma_0(N), \begin{pmatrix} 1 & -b/a \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$ and

$$\begin{pmatrix} 1 & 0 \\ -cN/a & 1 \end{pmatrix} \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \begin{pmatrix} 1 & -b/a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d_1 \end{pmatrix} \in \Delta_0(N).$$

Put $l = (a,\ d_1)$, then $l = (a,\ d_1N)$. Take integers $a_2,\ c_2$ such that $(a_2,\ c_2) = 1$, $a_2a - c_2Nd_1 = l$, then $\begin{pmatrix} 1 & -1 \\ -d_1c_2N/l & aa_2/l \end{pmatrix} \in \Gamma_0(N)$, $\begin{pmatrix} a_2 & d_1/l \\ c_2N & a/l \end{pmatrix} \in \Gamma_0(N)$ and

$$\begin{pmatrix} 1 & -1 \\ -d_1c_2N/l & aa_2/l \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & d_1/l \\ c_2N & a/l \end{pmatrix} = \begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix} \in \Delta_0(N).$$

Taking determinants, we obtain that $ad_1 = lm = \det(\alpha)$, so that $m > 0$, $l|m$ since $l = (a, d_1)$. This shows the assertion for $\Delta_0(N)$. We can prove the assertion for $\Delta_0^*(N)$ similarly. This completes the proof. □

Lemma 6.5 *Let $f \in G(N, k, \chi)$. Let $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Delta_0(N)$ satisfy*

(1) $\det(\alpha) > 1$;
(2) $(\det(\alpha), N) = 1$;
(3) $(a, b, c, d) = 1$.

If $f|[\alpha^{-1}]_k \in G(N, k, \chi)$, then $f = 0$.

Proof By (2), we see that $\alpha \in \Delta_0^*(N)$, by Lemma 6.4, there exist $\gamma_1,\ \gamma_2 \in \Gamma_0(N)$ such that $\gamma_1\alpha\gamma_2 = \begin{pmatrix} m & 0 \\ 0 & l \end{pmatrix}$ with $l|m,\ l,\ m > 0$. By (3), $(l, m) = 1$, so that $l = 1$. By (1), $m > 1$ and

$$\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ N/m & 1 \end{pmatrix} \notin \Gamma_0(N),$$

hence $\alpha\Gamma_0(N)\alpha^{-1} \not\subset \Gamma_0(N)$. Take $\gamma \in \Gamma_0(N)$ such that $\alpha\gamma\alpha^{-1} \notin \Gamma_0(N)$. Since $\det(\alpha)\alpha^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \in \Delta_0(N)$, $\det(\alpha)\alpha\gamma\alpha^{-1} \in \Delta_0(N)$, by Lemma 6.4, there exist $\gamma_3,\ \gamma_4 \in \Gamma_0(N)$ such that

$$\det(\alpha)\gamma_3\alpha\gamma\alpha^{-1}\gamma_4 = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \quad u|v, u, v > 0. \tag{6.4}$$

Taking the determinants, we have $(\det(\alpha))^2 = uv$. If $u = v$, then $\alpha\gamma\alpha^{-1} = \gamma_3^{-1}\gamma_4^{-1} \in \Gamma_0(N)$ which is impossible. Therefore, $h = v/u > 1$. Considering the action of both sides of (6.4) on $g = f|[\alpha^{-1}]_k$, we obtain that

$$g(z/h) = (\det(\alpha))^{-k}v^k\chi(\gamma_3)\chi(\gamma)\chi(\gamma_4)g(z) := cg(z).$$

Let $g(z) = \sum_{n=0}^{\infty} a(n)e(nz)$ be the Fourier expansion of g. Then, for any positive integer s, we have

$$a(n) = c^{-1}a(n/h) = c^{-s}a(n/h^s),$$

so that $a(n) = 0$ for any $n \geqslant 0$ since $k > 0$ and $|c| = h^{k/2} > 1$. Therefore $g = 0$ and hence $f = 0$. This completes the proof. □

Theorem 6.1 *Let l be a positive integer, f a function on $\mathbb{H}$ satisfying*:

(i) $f(z+1) = f(z)$;

(ii) $f(lz) \in G(N, k, \chi)$.

Then the following two assertions hold:

(1) $f(z) \in G(N/l, k, \chi)$ *if* $lm_\chi | N$;

(2) $f(z) = 0$ *if* $lm_\chi \nmid N$,

where m_χ *is the conductor of* χ. *Furthermore*, $f(z) \in S(N/l, k, \chi)$ *if* $f(lz) \in S(N, k, \chi)$.

Proof We need only to show the theorem for l a prime since we can apply induction on the number of prime factors of l. So we assume now that l is a prime. Because of the assumptions in the theorem, we have

$$\begin{aligned} G(N, k, \chi) \ni f(lz)|\mathrm{T}(l) &= l^{k/2-1}\left(f(lz)\left|\left[\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}\right]_k + \sum_{m=0}^{l-1} f(lz)\right|\left[\begin{pmatrix} l & m \\ 0 & l \end{pmatrix}\right]_k\right) \\ &= l^{k-1}f(l^2z) + \frac{1}{l}\sum_{m=0}^{l-1} f(l(z+m/l)) \\ &= l^{k-1}f(l^2z) + f(lz). \end{aligned}$$

Hence $f(l^2z) \in G(N, k, \chi)$ since $f(lz) \in G(N, k, \chi)$. If $l \nmid N$, taking $\alpha = \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}$ in Lemma 6.5, we see that $f(l^2z) = 0$, so that $f(z) = 0$. Therefore we assume now $l|N$.

We consider first the case $lm_\chi \nmid N$. For any element $\gamma_1 = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$, owing to the assumptions in the theorem, we see that

$$f\left|\left[\begin{pmatrix} a & bl \\ N/l & d \end{pmatrix}\right]_k\right. = f|[\delta_l\gamma_1\delta_l^{-1}] = \chi(d)f. \tag{6.5}$$

For any given positive integers m, n, put

$$\gamma = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ N/l & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+mN/l & m+n(1+mN/l) \\ N/l & 1+nN/l \end{pmatrix} \in \Gamma_0(N/l). \tag{6.6}$$

In particular, if n, m are chosen such that $nN/l+1 \not\equiv 0 \pmod{l}$ and

$$n(1+mN/l)+m = n+(nN/l+1)m \equiv 0 \pmod{l}, \tag{6.7}$$

then, by (6.6) and (6.7), we have

$$\begin{pmatrix} 1+mN/l & l^{-1}(m+n(1+mN/l)) \\ N & 1+nN/l \end{pmatrix} \in \Gamma_0(N).$$

Then we obtain

$$f|[\gamma]_k = \chi(1+nN/l)f$$

by (6.5). But $\delta_l \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \delta_l^{-1} = \begin{pmatrix} 1 & 0 \\ N/l & 1 \end{pmatrix}$, so by assumptions (i) and (ii), we see that

$$f|[\gamma]_k = f\left|\left[\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ N/l & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\right]_k\right. = f.$$

This shows that $\chi(1+nN/l)=1$ for any $(1+nN/l,l)=1$ if $f \neq 0$. This implies that the conductor m_χ of χ satisfies $m_\chi|N/l$. This contradicts $lm_\chi \nmid N$. Hence we have $f=0$ if $lm_\chi \nmid N$.

We now assume that $lm_\chi|N$. For any $\gamma = \begin{pmatrix} a & b \\ cN/l & d \end{pmatrix} \in \Gamma_0(N/l)$, we can find an m satisfying $l \nmid (a+mcN/l)$ since $(a,cN/l)=1$, then take an n such that $l|(a+mcN/l)n+b+md$, so that

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ cN/l & d \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a' & b'l \\ c'N/l & d' \end{pmatrix}$$

with a', b', c', d' integers. Hence $\begin{pmatrix} a' & b' \\ c'N & d' \end{pmatrix} \in \Gamma_0(N)$ and $d' \equiv d \pmod{N/l}$. Put $z=lw$, $g(w)=f(lw)$, by (i), (ii) and $m_\chi|N/l$, we have

$$\begin{aligned}
(f|[\gamma]_k)(z+n) &= \left(f\left|\left[\begin{pmatrix} a' & b'l \\ c'N/l & d' \end{pmatrix}\right]_k\right.\right)(z) \\
&= (c'Nz/l+d')^{-k} f\left(\frac{a'z+b'l}{c'Nz/l+d'}\right) \\
&= (c'Nw+d')^{-k} f\left(\frac{l(a'w+b')}{c'Nw+d'}\right) \\
&= \left(g\left|\left[\begin{pmatrix} a' & b' \\ c'N & d' \end{pmatrix}\right]_k\right.\right)(w) \\
&= \chi(d')g(w) = \chi(d)f(z).
\end{aligned}$$

This shows that $f| \in G(N/l,k,\chi)$. It is clear that $f(z) \in S(N/l,k,\chi)$ if $f(lz) \in S(N,k,\chi)$. This completes the proof. □

Lemma 6.6 *Let* $f = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N, k, \chi)$ *and* L *a positive integer. Put* $g(z) = \sum_{(n,L)=1} a(n)e(nz)$. *Then* $g(z) \in G(M, k, \chi)$ *with* $M = N \prod_{p|L, p|N} p \prod_{q|L, q\nmid N} q^2$, *where* p, q *are primes. Furthermore,* $g(z)$ *is a cusp form if* $f(z)$ *is a cusp form.*

Proof We only need to show the lemma for L a prime since we can apply induction on the number of prime factors of L. So we assume now that L is a prime. Put

$$N' = \begin{cases} N, & \text{if } p|N, \\ pN, & \text{if } p \nmid N. \end{cases}$$

Then $p|N'$. By Lemma 5.17, we have

$$\Gamma_0(N') \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(N') = \bigcup_{m=0}^{p-1} \Gamma_0(N') \begin{pmatrix} 1 & m \\ 0 & p \end{pmatrix}. \tag{6.8}$$

Since $G(N, k, \chi) \subset G(N', k, \chi)$, we see that

$$f|\mathrm{T}(p) \in G(N', k, \chi)$$

holds in $G(N', k, \chi)$. By (6.8), we have

$$(f|\mathrm{T}(p))(z) = p^{-1} \sum_{n=0}^{\infty} a(n) \sum_{m=0}^{p-1} \mathrm{e}^{2\pi i n(z+m)/p} = \sum_{n=0}^{\infty} a(np)e(nz).$$

By Lemma 6.2, we see that

$$(f|\mathrm{T}(p))(pz) = \sum_{n=0}^{\infty} a(np)e(npz) \in G(N'p, k, \chi).$$

Put $M = N'p$, then

$$g(z) = f(z) - (f|\mathrm{T}(p))(pz) \in G(M, k, \chi).$$

This completes the proof. □

Lemma 6.7 *Let* N *be a positive integer,* p *a prime. Then*

$$\Gamma_0(pN) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(N)$$
$$= \begin{cases} \bigcup_{m=0}^{p-1} \Gamma_0(pN) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, & \text{if } p|N, \\ \Gamma_0(pN) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \sigma_p \bigcup_{m=0}^{p-1} \Gamma_0(pN) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, & \text{if } p \nmid N, \end{cases}$$

where σ is a matrix satisfying

$$\sigma_p \in \Gamma_0(N), \quad \sigma_p \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N, \quad \sigma_p \equiv \begin{pmatrix} 0 & -l \\ l' & 0 \end{pmatrix} \pmod p$$

with l any fixed integer such that $p \nmid l$ and l' an integer such that $ll' \equiv 1 \pmod p$.

Proof Assume first that $p|N$. Let $\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$. Then $(a, cN) = 1$ and hence $p \nmid a$. Take $0 \leqslant v \leqslant p-1$ with $av \equiv b \pmod p$. Put $b_1 = (b - av)/p$, $d_1 = d - vcN$. Then $\gamma_1 = \begin{pmatrix} a & b_1 \\ cpN & d_1 \end{pmatrix} \in \Gamma_0(pN)$ and

$$\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \begin{pmatrix} a & b_1 \\ cpN & d_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} = \gamma.$$

This shows the first case in the lemma.

Now assume that $p \nmid N$. For any $\gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma_0(N)$, if $p \nmid a$, then similar to the first case, there exists $\gamma_1 \in \Gamma_0(pN)$, $0 \leqslant v \leqslant p-1$ such that

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}.$$

If $p|a$, since $p \nmid N$, there exists a_1 such that $a_1 p \equiv 1 \pmod N$. Take c_1 such that $c_1 N \equiv l' \pmod p$ and $(c_1, a_1 p) = 1$ (since $p \nmid c_1$, if necessary, take an integer t such that $pt + c_1$ is a prime larger than a_1, then $(pt + c_1, a_1 p) = 1$). Then $(a_1 p^2, c_1 N^2) = 1$. Take b_1, $d_1 \in \mathbb{Z}$ such that $d_1 a_1 p^2 - b_1 c_1 N^2 = 1$, then $\sigma_p = \begin{pmatrix} a_1 p & b_1 N \\ c_1 N & d_1 p \end{pmatrix}$ satisfies the conditions in the lemma. And

$$\gamma \sigma_p^{-1} = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \begin{pmatrix} d_1 p & -b_1 N \\ -c_1 N & a_1 p \end{pmatrix} = \begin{pmatrix} a_2 & b_2 p \\ c_2 N & d_2 \end{pmatrix} \in \Gamma_0(N)$$

and a_2, b_2, c_2, $d_2 \in \mathbb{Z}$. Therefore $\begin{pmatrix} a_2 & b_2 \\ c_2 pN & d_2 \end{pmatrix} \in \Gamma_0(pN)$, and

$$\begin{pmatrix} 1 & 0 \\ 0 & p^{-1} \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 pN & d_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} a_2 & b_2 p \\ c_2 N & d_2 \end{pmatrix} = \gamma \sigma_p^{-1}.$$

This shows the second case in the lemma. This completes the proof. □

Lemma 6.8 *Let χ be a character modulo N, l a positive integer, $p \nmid l$ a prime. Put $M = lN$, then we have the following two commutative diagrams*:

(1)

$$\begin{array}{ccc} G(pN,k,\chi) & \xrightarrow{\Gamma_0(pN)\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_0(N)} & G(N,k,\chi) \\ \Big\downarrow \text{Embedding} & & \Big\downarrow \text{Embedding} \\ G(pM,k,\chi) & \xrightarrow[\Gamma_0(pM)\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_0(M)]{} & G(M,k,\chi) \end{array}$$

(2)

$$\begin{array}{ccc} G(pN,k,\chi) & \xrightarrow{\Gamma_0(pN)\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_0(N)} & G(N,k,\chi) \\ {\scriptstyle [\delta_l]_k}\Big\downarrow & & \Big\downarrow {\scriptstyle [\delta_l]_k} \\ G(pM,k,\chi) & \xrightarrow[\Gamma_0(pM)\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_0(M)]{} & G(M,k,\chi). \end{array}$$

And similar results hold for cusp forms.

Proof The diagram (1) is an immediate conclusion of Lemma 6.7. We show now the second diagram. Let $f(z) \in G(pN,k,\chi)$. Put $g(z) = f|[\delta_l]_k$. By Lemma 6.7, we have

$$g|\Gamma_0(pM)\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_0(M)$$

$$= \sum_{v=0}^{p-1} g\left|\left[\begin{pmatrix}1&0\\0&p\end{pmatrix}\begin{pmatrix}1&v\\0&1\end{pmatrix}\right]_k + g\right|\left[\begin{pmatrix}1&0\\0&p\end{pmatrix}\sigma_p\right]_k$$

(where the last term disappears if $p|M$).

$$= \sum_{v=0}^{p-1} f\left|\left[\delta_l\begin{pmatrix}1&0\\0&p\end{pmatrix}\begin{pmatrix}1&v\\0&1\end{pmatrix}\right]_k + f\right|\left[\delta_l\begin{pmatrix}1&0\\0&p\end{pmatrix}\sigma_p\right]_k$$

$$= \sum_{v=0}^{p-1} f\left|\left[\begin{pmatrix}1&0\\0&p\end{pmatrix}\delta_l\begin{pmatrix}1&v\\0&1\end{pmatrix}\delta_l^{-1}\delta_l\right]_k + f\right|\left[\begin{pmatrix}1&0\\0&p\end{pmatrix}\delta_l\sigma_p\delta_l^{-1}\delta_l\right]_k$$

$$= \sum_{v=0}^{p-1} f\left|\left[\begin{pmatrix}1&0\\0&p\end{pmatrix}\begin{pmatrix}1&vl\\0&1\end{pmatrix}\delta_l\right]_k + f\right|\left[\begin{pmatrix}1&0\\0&p\end{pmatrix}\widehat{\sigma_p}\delta_l\right]_k,$$

where $\widehat{\sigma_p} \in \Gamma_0(N)$ satisfies $\widehat{\sigma_p} \equiv \begin{pmatrix}1&0\\0&1\end{pmatrix} \pmod N$, and furthermore $\widehat{\sigma_p} \equiv \begin{pmatrix}0&-ml\\(ml)'&0\end{pmatrix}$ $\pmod p$ if $\sigma_p \equiv \begin{pmatrix}0&-m\\m'&0\end{pmatrix} \pmod p$. Hence, by Lemma 6.7, we see that

$$(f|[\delta_l]_k)|\Gamma_0(pM)\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_0(M) = \left(f|\Gamma_0(pN)\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_0(N)\right)|[\delta_l]_k.$$

This completes the proof. □

Lemma 6.9 *Let l be a square free positive integer, $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N,k,\chi)$ such that $a(n) = 0$ if $(n,l) = 1$. Then*

$$f(z) = \sum_{p|l} g_p(pz),$$

where $g_p(z) \in G(Nl^2, k, \chi)$ and moreover $g_p(z) \in G(Nl, k, \chi)$ if $l|N$. Furthermore, all g_p are cusp forms if $f(z)$ is a cusp form.

Proof We assume first that l is a prime. Put $g(z) = f(z/l)$. By Theorem 6.1, we see that $g(z) \in G(N/l, k, \chi)$ or $g(z) = 0$ if $lm_\chi|N$ or $lm_\chi \nmid N$ respectively. Anyway, $g(z) \in G(Nl, k, \chi)$ and $f(z) = g(lz)$, the lemma holds. Now assume that l is a composite and the lemma holds for any proper factor of l. Let p be a prime factor of l. Put $l' = l/p$ and $h(z) = \sum_{p\nmid n} a(n)e(nz)$. By Lemma 6.6, we see that $h(z) \in G(Np^2, k, \chi)$. Put $f(z) - h(z) = \sum_{n=0}^{\infty} b(n)e(nz)$. It is clear that $b(n) = 0$ if $p \nmid n$. Set $g_p(z) = f(z/p) - h(z/p)$, by Theorem 6.1, we have that $g_p(z) \in G(Np, k, \chi)$ and

$$f(z) = g_p(pz) + h(z).$$

Since $h(z), Np^2, l'$ satisfy the conditions in the lemma, by induction hypothesis, we have

$$h(z) = \sum_{q|l'} g_q(qz), g_q(z) \in G(Nl'^2, k, \chi) \subset G(Nl^2, k, \chi),$$

with q primes. It is clear that, by Lemma 6.6 and the above proof, $g_p \in G(Nl, k, \chi)$ if $l|N$. This completes the proof. □

Theorem 6.2 *Let $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N, k, \chi)$, l a positive integer. Assume that $a(n) = 0$ if $(l, n) = 1$. Then*

(1) $f(z) = 0$ if $(l, N/m_\chi) = 1$;

(2) if $(l, N/m_\chi) \neq 1$, then for any prime factor p of $(l, N/m_\chi)$ there exists $f_p(z) \in G(N/p, k, \chi)$ such that

$$f(z) = \sum_{p|(l,N/m_\chi)} f_p(pz),$$

where m_χ is the conductor of χ. Furthermore, all f_p are cusp forms if f is a cusp form.

Proof Without loss of generality, we may assume that l is square free. It is clear that, by Theorem 6.1, the theorem holds for l a prime. Now assume that l is a composite and the theorem holds for any proper factor of l. Let p be a prime factor of l and $l' = l/p$. Set

$$\begin{aligned} h(z) &= \sum_{(n,l')\neq 1} a(n)e(nz), \\ g(z) &= f(z) - h(z) = \sum_{(n,l')=1} a(n)e(nz). \end{aligned} \tag{6.9}$$

By Lemma 6.6, $g(z) \in G(Nl'^2, k, \chi)$ and so $h(z) \in G(Nl'^2, k, \chi)$. It is clear that the Fourier coefficient $a(n)$ of $g(z)$ must be zero if $p \nmid n$, so that $g_p(z+1) = g_p(z)$ where $g_p(z) = g(z/p)$. If $pm_\chi \nmid N$, then $pm_\chi \nmid Nl'^2$, and $g(z) = 0$ by Theorem 6.1. Therefore $f(z) = h(z) = \sum_{(n,l')\neq 1} a(n)e(nz)$. This shows that the theorem holds by the induction hypothesis. Now assume that $pm_\chi | N$. By Theorem 6.1, we see that $g_p(z) \in G(Nl'^2/p, k, \chi)$. Lemma 6.7 gives

$$\Gamma_0(Nl'^2)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_0(Nl'^2/p) = \Gamma_0(Nl'^2)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\sigma_p \bigcup_{v=0}^{p-1}\Gamma_0(Nl'^2)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix},$$

where the first term disappears if $p^2 | N$, so that,

$$\begin{aligned} &\left(g|\Gamma_0(Nl'^2)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_0(Nl'^2/p)\right)(z) \\ =& p^{k/2-1}\sum_{v=0}^{p-1}\left(g\left|\left[\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}\right]_k\right.\right)(z) + p^{k/2-1}\left(g\left|\left[\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\sigma_p\right]_k\right.\right)(z) \\ =& p^{-1}\sum_{v=0}^{p-1}\left(g_p\left|\left[\begin{pmatrix}1 & v\\ 0 & 1\end{pmatrix}\right]_k\right.\right)(z) + p^{-1}(g_p|[\sigma_p]_k)(z) \\ =& \frac{d}{p}g_p(z), \end{aligned}$$

where $d = \begin{cases} p, & \text{if } p^2|N, \\ p+1, & \text{if } p^2 \nmid N \end{cases}$. Therefore

$$g(z) = g_p(pz) = \frac{d}{p}\left(g\left|\Gamma_0(Nl'^2)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_0(Nl'^2/p)\right.\right)(pz). \tag{6.10}$$

Since

$$f_p(z) = \frac{d}{p}\left(f\left|\Gamma_0(N)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma_0(N/p)\right.\right)(z) \in G(N/p, k, \chi),$$

we have that, by Lemma 6.8,

$$f_p(z) = \frac{d}{p}\left(f\Big|\varGamma_0(Nl'^2)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\varGamma_0(Nl'^2/p)\right)(z). \tag{6.11}$$

We want to show that $f(z) - f_p(pz)$ satisfies the conditions in the theorem for l', and hence we can complete the proof by induction. It is clear that $f(z) - f_p(pz) \in G(N, k, \chi)$. By (6.9)–(6.11), we see that

$$\begin{aligned} f(z) - f_p(pz) &= f(z) - f_p(pz) - g(z) + g_p(pz) \\ &= h(z) - \frac{d}{p}\left(h\Big|\varGamma_0(Nl'^2)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\varGamma_0(Nl'^2/p)\right)(pz). \end{aligned} \tag{6.12}$$

Applying the induction hypothesis for $h(z)$, Nl'^2 and l', we have

$$h(z) = \sum_{q|l'} h_q(qz), \quad h_q(z) \in G(Nl'^2, k, \chi) \tag{6.13}$$

with q primes. By Lemma 6.8, for any prime factor q of l', we have

$$h\big|\varGamma_0(Nl'^2)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\varGamma_0(Nl'^2/p) = h\big|\varGamma_0(Nl'^3q)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\varGamma_0(Nl'^3q/p) \tag{6.14}$$

and this holds also if h is substituted by h_q. By (6.13), (6.14) and (2) of Lemma 6.8, we have

$$\begin{aligned} &\left(h\Big|\varGamma_0(Nl'^2)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\varGamma_0(Nl'^2/p)\right)(z) \\ &= \left(\sum_{q|l'}(q^{-k/2}h_q|[\delta_q]_k)\Big|\varGamma_0(Nl'^3q)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\varGamma_0(Nl'^3q/p)\right)(z) \\ &= \sum_{q|l'}\left(h_q\Big|\varGamma_0(Nl'^3)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\varGamma_0(Nl'^3/p)\right)(qz). \end{aligned}$$

This implies that the Fourier coefficient $b(n)$ of $\left(h\big|\varGamma_0(Nl'^2)\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\varGamma_0(Nl'^2/p)\right)(z)$ must be zero if $(n, l') = 1$, and hence, by (6.12) and (6.13), so is the Fourier coefficient $c(n)$ of $f(z) - f_p(pz)$. This shows that $f(z) - f_p(pz)$ satisfies the conditions in the theorem for l'. Hence

$$f(z) - f_p(pz) = \sum_{q|l'} f_q(qz), \quad f_q(z) \in G(N/q, k, \chi),$$

where q runs over all prime factors of $(l', N/m_\chi)$. This completes the proof. □

Definition 6.1 *Denote by* $S^{\text{old}}(N, k, \chi)$ *the subspace of* $S(N, k, \chi)$ *generated by*

$$\bigcup_{\substack{m_\chi|M|N,\\ M\neq N}} \bigcup_{l|N/M} \{f(lz)|f(z) \in S(M, k, \chi)\}.$$

And any modular form in $S^{\text{old}}(N, k, \chi)$ *is called an old form.*

Definition 6.2 *Denote by $S^{\text{new}}(N,k,\chi)$ the orthogonal complement subspace of $S^{\text{old}}(N,k,\chi)$ in $S(N,k,\chi)$ with respect to the* Petersson *inner product. And any modular form in $S^{\text{new}}(N,k,\chi)$ is called a new form.*

By the definitions, we have

Lemma 6.10 (1) *$S(N,k,\chi)=S^{\text{new}}(N,k,\chi)$ if χ is a primitive character modulo N;*

(2) *$S(M,k,\chi)\subset S^{\text{old}}(N,k,\chi)$ if $m_\chi|M|N$ and $M\neq N$;*

(3) *$S(N,k,\chi)$ is generated by $\bigcup\limits_{m_\chi|M|N}\bigcup\limits_{l|N/M}\{f(lz)|f(z)\in S^{\text{new}}(M,k,\chi)\}$.*

Lemma 6.11 *Let n be a positive integer with $(n,N)=1$. Then $\mathrm{T}(n)$ sends $S^{\text{old}}(N,k,\chi)$ (and $S^{\text{new}}(N,k,\chi)$* resp.) *into $S^{\text{old}}(N,k,\chi)$ (and $S^{\text{new}}(N,k,\chi)$ resp.).*

Proof Let $f(z)\in S^{\text{old}}(N,k,\chi)$. By the definition of old forms, we have

$$f(z)=\sum_v f_v(l_vz),\quad f_v\in S(M_v,k,\chi), l_vM_v|N, M_v\neq N.$$

Put $g_v(z)=f_v(l_vz)$. Since $\mathrm{T}(n)$ commutes with $[\delta_l]_k$ for any $(n,l)=1$, we see that

$$(f|\mathrm{T}(n))(z)=\sum_v(g_v|\mathrm{T}(n))(z)=\sum_v(f_v|\mathrm{T}(n))(l_vz).$$

Since $f_v\in S(M_v,k,\chi)$, we have that $f_v|\mathrm{T}(n)\in S(M_v,k,\chi)$, so that $f|\mathrm{T}(n)\in S^{\text{old}}(N,k,\chi)$. This shows that $\mathrm{T}(n)$ sends $S^{\text{old}}(N,k,\chi)$ into itself. The next lemma will show that $\overline{\chi}(n)\mathrm{T}(n)$ is the conjugate operator of $\mathrm{T}(n)$ on the space $S(N,k,\chi)$ with respect to the Petersson inner product, so that $\mathrm{T}(n)$ sends $S^{\text{new}}(N,k,\chi)$ into itself. This completes the proof. □

Lemma 6.12 *Let $f(z)=\sum\limits_{m=1}^{\infty}a(m)e(mz)\in S(N,k,\chi)$ and $f(z)|\mathrm{T}(n)=\sum\limits_{m=1}^{\infty}b(m)e(mz)\in S(N,k,\chi)$. Then*

(1) $b(m)=\sum\limits_{1\leqslant d|(m,n)}\chi(d)d^{k-1}a(mn/d^2)$;

(2) *the conjugate operator* $\mathrm{T}(n)^*$ *of* $\mathrm{T}(n)$ (*with respect to the* Petersson *inner product) is equal to $\overline{\chi}(n)\mathrm{T}(n)$ for any $(n,N)=1$.*

Proof (1) is a direct conclusion of (5.14).

(2) is a direct conclusion of Lemma 5.18 and Lemma 5.26. □

By Lemma 6.11, there is a basis in $S^{\text{new}}(N,k,\chi)$ (and in $S^{\text{old}}(N,k,\chi)$ resp.) whose elements are eigenfunctions of all Hecke operators $\mathrm{T}(n)$ with $(n,N)=1$.

Lemma 6.13 *Let L be a positive integer, and*

$$0 \neq f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in S^{\text{new}}(N,k,\chi)$$

an eigenfunction of all Hecke *operators* $\mathrm{T}(n)$ *with* $(n,L)=1$*. Then* $a_1 \neq 0$*.*

Proof Assume that $a_1 = 0$. If $a(n)=0$ for any $(n,L)=1$, then, by Theorem 6.2, $f(z) \in S^{\text{old}}(N,k,\chi)$ which is impossible. Hence

$$m = \min\{n|(n,L)=1, a(n) \neq 0\} > 1.$$

Let p be a prime factor of m. Then $f|\mathrm{T}(p) = c_p f$ with c_p a constant. By Lemma 6.12, we see that $c_p a(m/p) = a(m) + \chi(p)p^{k-1}a(m/p^2)$. By the definition of m, we have $a(m/p) = a(m/p^2) = 0$, so that $a(m)=0$, which is impossible. This completes the proof. □

Theorem 6.3 *Let L be a positive integer, f and $g \in S(N,k,\chi)$ such that $f|\mathrm{T}(n) = \lambda_n f$, $g|\mathrm{T}(n) = \lambda_n g$ for all $(n,L)=1$ with λ_n constants. Then $f = cg$ for a constant c if $0 \neq f \in S^{\text{new}}(N,k,\chi)$.*

Proof Let $f(z) = \sum_{n=1}^{\infty} a(n)e(nz)$. Without loss of generality, we can assume that $a(1)=1$ by Lemma 6.13. We may assume also that $N|L$. Set

$$g(z) = g^{(0)}(z) + g^{(1)}(z), \quad g^{(0)}(z) \in S^{\text{new}}(N,k,\chi), \quad g^{(1)}(z) \in S^{\text{old}}(N,k,\chi).$$

By Lemma 6.11, we see that

$$g^{(0)}|\mathrm{T}(n) = \lambda_n g^{(0)}, \quad g^{(1)}|\mathrm{T}(n) = \lambda_n g^{(1)}, \quad (n,L)=1.$$

Hence, by Lemma 6.13, $b(1) \neq 0$ if $g^{(0)}(z) = \sum_{n=1}^{\infty} b(n)e(nz) \neq 0$. By Lemma 6.12, we have

$$f|\mathrm{T}(n) = a(n)f, \quad g^{(0)}|\mathrm{T}(n) = \frac{b(n)}{b(1)}g^{(0)}, \quad (n,L)=1.$$

This shows that $a(n)b(1) = b(n)$ for all $(n,L)=1$. Put

$$g^{(0)} - b(1)f = \sum_{n=1}^{\infty} c(n)e(nz),$$

then $c(n)=0$ for all $(n,L)=1$, so that $g^{(0)} - b(1)f \in S^{\text{old}}(N,k,\chi)$ by Theorem 6.2. This implies that $g^{(0)} - b(1)f = 0$. We shall now prove that $g^{(1)} = 0$. If $m_\chi = N$, then $S^{\text{old}}(N,k,\chi) = 0$. So we may assume that $m_\chi \neq N$. Suppose that $g^{(1)} \neq 0$, then

$$g^{(1)}(z) = \sum_v h_v(l_v z), \quad h_v \in S^{\text{new}}(M_v,k,\chi), \quad l_v M_v|N, \quad M_v \neq N. \tag{6.15}$$

Since there is a basis in $S^{\text{new}}(M_v, k, \chi)$ whose elements are eigenfunctions for all $\mathrm{T}(n)$ $((n, M_v) = 1)$, we may assume that $h_v(z)$ is an eigenfunction of all $\mathrm{T}(n)$ $((n, M_v) = 1)$, so that, by Lemma 6.3, $h_v(l_v z)$ is an eigenfunction of all $\mathrm{T}(n)$ $((n, L) = 1)$. Since eigenfunctions corresponding to different eigenvalues are linearly independent, the sum of $h_v(l_v z)$ with eigenvalue different from $a(n)$ with respect to $\mathrm{T}(n)$ must be zero. Therefore every $h_v(z)$ on the right hand side of (6.15) must satisfy

$$h_v|\mathrm{T}(n) = a(n)h_v, \quad (n, L) = 1.$$

Denote by h any fixed one of these h_v. Let d be the first coefficient of the Fourier expansion of h, then $d \neq 0$ by Lemma 6.13. Put

$$h(z) - df(z) = \sum_{n=1}^{\infty} d(n)e(nz),$$

then $d(n) = 0$ for all $(n, L) = 1$, so that $h(z) - df(z) \in S^{\text{old}}(N, k, \chi)$ by Theorem 6.2. Therefore

$$f(z) = -\frac{1}{d}\left(h(z) - df(z)\right) + \frac{1}{d}h(z) \in S^{\text{old}}(N, k, \chi),$$

which implies that $f(z) = 0$ since $f(z) \in S^{\text{new}}(N, k, \chi)$. This contradicts the hypothesis $f \neq 0$. This completes the proof. □

Theorem 6.4 *Let $R_0(N)$ and $R_0^*(N)$ be the* Hecke *algebras $R(\Gamma_0(N), \Delta_0(N))$ and $R(\Gamma_0(N), \Delta_0^*(N))$ respectively. Then there is a basis in $S^{\text{new}}(N, k, \chi)$ whose elements are common eigenfunctions of $R_0(N)$ and $R_0^*(N)$.*

Proof By Theorem 5.5, $R_0(N)$ and $R_0^*(N)$ are commutative and $\mathrm{T}(n) \in R_0(N)$ for any $(n, N) = 1$. Let $\{f_1, f_2, \cdots, f_r\}$ be a basis of $S^{\text{new}}(N, k, \chi)$ such that every f_i is a common eigenfunction of $\mathrm{T}(n)$ for all $(n, N) = 1$. Put $f_i|\mathrm{T}(n) = a(n, i)f_i$, $(n, N) = 1$ with $a(n, i)$ a constant. For any $T \in R_0(N)$, since $\mathrm{T}(n)$ $((n, N) = 1)$ commutes with T, we see that

$$(f_i|T)|\mathrm{T}(n) = (f_i|\mathrm{T}(n))|T = a(n, i)f_i|T, \quad (n, N) = 1.$$

That is, $f_i|T$ is a common eigenfunction of all $\mathrm{T}(n)$ with eigenvalue $a(n, i)$. By Theorem 6.3, we have that $f_i|T = cf_i$ with a constant c. This shows that f_i is a common eigenfunction of $R_0(N)$. This shows the first part of the theorem. Since $\mathrm{T}(n)^* \in R_0^*(N)$ $((n, N) = 1)$ commutes with any $T \in R_0^*(N)$, and $\mathrm{T}(n)^* = \overline{\chi}(n)\mathrm{T}(n)$, $(n, N) = 1$, we see that $\mathrm{T}(n)$ commutes with $T \in R_0^*(N)$. Similar to the above process, $f_i|T = c'f_i$ with a constant c' for any $T \in R_0^*(N)$, so that, f_i is also a common eigenfunction of $R_0^*(N)$. Therefore f_i $(1 \leqslant i \leqslant r)$ are common eigenfunctions of $R_0(N)$ and $R_0^*(N)$. This completes the proof. □

Definition 6.3 $f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S(N, k, \chi)$ *is called a primitive cusp form* *if it satisfies the following two conditions*:

(1) $f \in S^{\text{new}}(N,k,\chi)$ *and it is a common eigenfunction of* $R_0(N)$;

(2) $a(1) = 1$.

By Theorem 6.4, a primitive cusp form is also a common eigenfunction of $R_0^*(N)$, and there exists a basis in $S^{\text{new}}(N,k,\chi)$ whose elements are primitive cusp forms.

Lemma 6.14 *Let* $f \in S(N,k,\chi)$ *be a common eigenfunction of all* $\mathrm{T}(n)$ *with* $(n,N)=1$, *and* $f|\mathrm{T}(n) = a(n)f$, $(n,N)=1$. *Then there exists a factor* M *of* N *and a primitive cusp form* g *of* $S^{\text{new}}(M,k,\chi)$ *such that*

$$g|\mathrm{T}(n) = a(n)g, \quad (n,N)=1.$$

Furthermore, we can take $M \neq N$ *if* $f \notin S^{\text{new}}(N,k,\chi)$.

Proof If $f \in S^{\text{new}}(N,k,\chi)$, the lemma is obvious. So assume $f \notin S^{\text{new}}(N,k,\chi)$. By the proof of Theorem 6.3, there exists $N \neq M|N$ and $h \in S^{\text{new}}(M,k,\chi)$ such that

$$h|\mathrm{T}(N) = a(n)h, (n,N)=1.$$

Take $g = \frac{1}{d}h$ with d the first Fourier coefficient of h. This completes the proof. □

Lemma 6.15 *Let* $f \in G(N,k,\chi)$. *Then*

$$(f|\mathrm{T}(l,m))|[W(N)]_k = (f|[W(N)]_k)|\mathrm{T}(m,l)^*,$$
$$(f|\mathrm{T}(n))|[W(N)]_k = (f|[W(N)]_k)|\mathrm{T}(n)^*.$$

Proof It is clear that we only need to show the first equality in the lemma. It is clear that the map: $\alpha \mapsto W(N)^{-1}\alpha W(N)$ is an isomorphism from $\varDelta_0(N)$ to $\varDelta_0^*(N)$, and $W(N)^{-1}\varGamma_0(N)\ W(N) = \varGamma_0(N)$. For any $\alpha \in \varDelta_0(N)$, we have

$$\chi(W(N)^{-1}\alpha W(N)) = \chi(\alpha)^{-1}.$$

Let $\varGamma_0(N)\begin{pmatrix} l & 0 \\ 0 & m \end{pmatrix}\varGamma_0(N) = \bigcup_v \varGamma_0(N)\alpha_v$ be a disjoint union, then

$$\varGamma_0(N)\begin{pmatrix} m & 0 \\ 0 & l \end{pmatrix}\varGamma_0(N) = \bigcup_v \varGamma_0(N)(W(N)^{-1}\alpha_v W(N)).$$

Hence, for any $g \in G(N,k,\overline{\chi})$, we have

$$\begin{aligned} & g|[W(N)^{-1}]_k \mathrm{T}(l,m)[W(N)]_k \\ =& (lm)^{k/2-1}\sum_v \chi(\alpha_v)^{-1} g|[W(N)^{-1}\alpha_v W(N)]_k \\ =& (lm)^{k/2-1}\sum_v \overline{\chi}(W(N)^{-1}\alpha_v W(N))^{-1} g|[W(N)^{-1}\alpha_v W(N)]_k \\ =& g|\mathrm{T}(m,l)^*. \end{aligned}$$

Since $W(N)$ is an isomorphism from $G(N,k,\chi)$ to $G(N,k,\overline{\chi})$, we see that the first equality holds in the lemma. This completes the proof. □

Theorem 6.5 (1) *The map:* $f \mapsto f|[W(N)]_k$ *induces the following isomorphisms.*

$$S^{\text{new}}(N,k,\chi) \simeq S^{\text{new}}(N,k,\overline{\chi}),$$
$$S^{\text{old}}(N,k,\chi) \simeq S^{\text{old}}(N,k,\overline{\chi});$$

(2) *Let*

$$f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S(N,k,\chi)$$

be a primitive cusp form, then

$$g(z) := \sum_{n=1}^{\infty} \overline{a(n)}e(nz)$$

is a primitive cusp form of $S(N,k,\overline{\chi})$, *and* $f|[W(N)]_k = cg$ *with a constant* c.

Proof (1) We show first that $[W(N)]_k$ sends $S^{\text{old}}(N,k,\chi)$ into $S^{\text{old}}(N,k,\overline{\chi})$. This is equivalent to show the following assertion: let $N \neq M|N$, $m_\chi|M$, $l|N/M$, and let $h \in S(M,k,\chi)$ such that $f(z) = h|[\delta_l]_k$, then $f|[W(N)]_k \in S^{\text{old}}(N,k,\overline{\chi})$. We show now the assertion. Put $l' = N/(lM)$. Then $\delta_l W(N)\delta_{l'}^{-1} = lW(M)$, so that

$$f|[W(N)]_k = h|[\delta_l W(N)\delta_{l'}^{-1}\delta_{l'}]_k = (h|[W(M)]_k)|[\delta_{l'}]_k.$$

Since $h|[W(M)]_k \in S(M,k,\overline{\chi})$, $f|[W(N)]_k \in S^{\text{old}}(N,k,\overline{\chi})$. Now suppose $f \in S^{\text{new}}(N,k,\chi)$. Then, for any $f_1 \in S^{\text{old}}(N,k,\overline{\chi})$, we have

$$\langle f|[W(N)]_k, f_1\rangle = \langle f, f_1|[W(N)^\tau]_k = (-1)^k\langle f, f_1|[W(N)]_k\rangle = 0,$$

since $f_1|[W(N)]_k \in S^{\text{old}}(N,k,\chi)$. Therefore $f|[W(N)]_k \in S^{\text{new}}(N,k,\overline{\chi})$. This shows (1).

(2) By (1), we have $f|[W(N)]_k \in S^{\text{new}}(N,k,\overline{\chi})$. By Lemma 6.15, we have

$$(f|[W(N)]_k)|\mathrm{T}(n) = (f|\mathrm{T}(n)^*)|[W(N)]_k = \overline{a(n)}f|[W(N)]_k$$

for any positive integer n. Hence $f|[W(N)]_k$ must be a constant multiple of some primitive cusp form g. Let $b(n)$ be the n-th Fourier coefficient of $f|[W(N)]_k$, then $b(n) = \overline{a(n)}b(1)$, so that

$$(f|[W(N)]_k)(z) = b(1)\sum_{n=1}^{\infty}\overline{a(n)}e(nz).$$

Since $a(1) = 1$ and the first Fourier coefficient of g is also equal to 1, we see that

$$g(z) = \sum_{n=1}^{\infty}\overline{a(n)}e(nz), \quad f|[W(N)]_k = b(1)g.$$

This completes the proof. □

Let $f(z)=\sum_{n=1}^{\infty}a(n)e(nz)\in S(N,k,\chi)$ be a primitive cusp form. Then

$$\begin{aligned}L(s,f)&=\sum_{n=1}^{\infty}a(n)n^{-s}=\prod_{p}\left(1-a(p)p^{-s}+\chi(p)p^{k-1-2s}\right)^{-1}\\&=\prod_{p\nmid N}\left(1-a(p)p^{-s}+\chi(p)p^{k-1-2s}\right)^{-1}\prod_{p|N}\left(1-a(p)p^{-s}\right)^{-1}.\end{aligned}$$

For any $p\nmid N$, by the Ramanujan-Petersson Conjecture (proved by Deligne), we have $|a(p)|\leqslant 2p^{(k-1)/2}$. We discuss now $a(p)$ for $p|N$. For any $p|N$, set $N=N_pN_p'$ with $p\nmid N_p'$, and χ_p the character modulo N_p induced from χ. Fix a prime factor q of N, put $\chi'=\prod_{p\neq q}\chi_p$. Let γ_q, $\gamma_q'\in SL_2(\mathbb{Z})$ satisfy

$$\gamma_q\equiv\begin{cases}\begin{pmatrix}0&-1\\1&0\end{pmatrix} & \pmod{N_q^2},\\ \begin{pmatrix}1&0\\0&1\end{pmatrix} & \pmod{(N/N_q)^2},\end{cases}\qquad \gamma_q'\equiv\begin{cases}\begin{pmatrix}1&0\\0&1\end{pmatrix} & \pmod{N_q^2},\\ \begin{pmatrix}0&-1\\1&0\end{pmatrix} & \pmod{(N/N_q)^2}.\end{cases}$$

Set

$$\eta_q=\gamma_q\begin{pmatrix}N_q&0\\0&1\end{pmatrix},\quad \eta_q'=\gamma_q'\begin{pmatrix}N/N_q&0\\0&1\end{pmatrix},$$

then

$$\eta_q\varGamma_0(N)\eta_q^{-1}=\varGamma_0(N),\quad \eta_q'\varGamma_0(N)\eta_q'^{-1}=\varGamma_0(N)$$

and for any $\gamma\in\varGamma_0(N)$, we have

$$\chi(\eta_q\gamma\eta_q^{-1})=(\chi'\overline{\chi_q})(\gamma),\quad \chi(\eta_q'\gamma\eta_q'^{-1})=(\overline{\chi'}\chi_q)(\gamma).$$

Hence we have the following two isomorphisms:

$$\begin{aligned}S(N,k,\chi)&\xrightarrow{[\eta_q]_k}S(N,k,\chi'\overline{\chi_q}),\\ S(N,k,\chi)&\xrightarrow{[\eta_q']_k}S(N,k,\overline{\chi'}\chi_q).\end{aligned}$$

And the following two diagrams are commutative:

$$\begin{array}{ccc}S(N,k,\chi) & \xrightarrow{\overline{\chi_q}(n)\mathrm{T}(n)} & S(N,k,\chi)\\ {\scriptstyle[\eta_q]_k}\big\downarrow & & \big\downarrow{\scriptstyle[\eta_q]_k}\\ S(N,k,\chi'\overline{\chi_q}) & \xrightarrow{\mathrm{T}(n)} & S(N,k,\chi'\overline{\chi_q})\end{array},\quad (n,N_q)=1;$$

$$\begin{array}{ccc}S(N,k,\chi) & \xrightarrow{\overline{\chi_q'}(n)\mathrm{T}(n)} & S(N,k,\chi)\\ {\scriptstyle[\eta_q']_k}\big\downarrow & & \big\downarrow{\scriptstyle[\eta_q']_k}\\ S(N,k,\overline{\chi'}\chi_q) & \xrightarrow{\mathrm{T}(n)} & S(N,k,\overline{\chi'}\chi_q)\end{array},\quad (n,N/N_q)=1.$$

These can be proved along similar lines as in the proof of Lemma 6.15. In particular, we see that $f|[\eta_q]_k \in S(N,k,\chi'\overline{\chi_q})$ and $f|[\eta'_q]_k \in S(N,k,\overline{\chi'}\chi_q)$ are common eigenfunctions of all $\mathrm{T}(n)$ $((n,N)=1)$ if $f \in S(N,k,\chi)$ is a common eigenfunction of all $\mathrm{T}(n)$ $((n,N)=1)$. Therefore we see that the assertion (1) of the following theorem holds:

Theorem 6.6 (1) *We have the following isomorphisms*:

$$\begin{aligned}
[\eta_q]_k &: S^{\mathrm{new}}(N,k,\chi) \simeq S^{\mathrm{new}}(N,k,\chi'\overline{\chi_q}),\\
[\eta_q]_k &: S^{\mathrm{old}}(N,k,\chi) \simeq S^{\mathrm{old}}(N,k,\chi'\overline{\chi_q}),\\
[\eta'_q]_k &: S^{\mathrm{new}}(N,k,\chi) \simeq S^{\mathrm{new}}(N,k,\overline{\chi'}\chi_q),\\
[\eta'_q]_k &: S^{\mathrm{old}}(N,k,\chi) \simeq S^{\mathrm{old}}(N,k,\overline{\chi'}\chi_q).
\end{aligned}$$

(2) *For any $f \in S(N,k,\chi)$, we have*

$$\begin{aligned}
f|[\eta_q^2]_k &= \chi_q(-1)\overline{\chi'}(N_q)f,\\
f|[{\eta'}_q^2]_k &= \chi'(-1)\overline{\chi_q}(N/N_q)f,\\
f|[\eta_q\eta'_q]_k &= \overline{\chi'}(N_q)f|[W(N)]_k.
\end{aligned}$$

(3) *If $f = \sum_{n=1}^{\infty} a(n)e(nz) \in S^{\mathrm{new}}(N,k,\chi)$ is a primitive cusp form, set*

$$f|[\eta_q]_k = c\sum_{n=1}^{\infty} b(n)e(nz), b(1)=1, \quad g_q(z) = \sum_{n=1}^{\infty} b(n)e(nz),$$

then $g_q(z)$ is a primitive cusp form of $S(N,k,\chi'\overline{\chi_q})$ and

$$b(p) = \begin{cases} \overline{\chi_q}(p)a(p), & \text{if } p \neq q,\\ \chi'(q)\overline{a(q)}, & \text{if } p = q. \end{cases}$$

Proof (2) Put $\eta_q^2 = N_q\gamma$, then $\gamma \in \Gamma(1)$ and

$$\gamma \equiv \begin{cases} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \pmod{N_q},\\ \begin{pmatrix} N_q & 0 \\ 0 & N_q^{-1} \end{pmatrix} \pmod{(N/N_q)}. \end{cases}$$

So that, $\gamma \in \Gamma_0(N)$, and hence $f|[\eta_q^2]_k = \chi_q(-1)\overline{\chi'}(N_q)f$. Similarly set ${\eta'}_q^2 = \dfrac{N}{N_q}\gamma_1$, then $\gamma_1 \in \Gamma_0(N)$ and

$$\gamma_1 \equiv \begin{cases} \begin{pmatrix} N/N_q & 0 \\ 0 & (N/N_q)^{-1} \end{pmatrix} \pmod{N/N_q},\\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \pmod{N_q}. \end{cases}$$

Hence

$$f|[\eta'^2_q]_k = \chi'(-1)\overline{\chi_q}(N/N_q)f.$$

Set $\gamma_2 = \eta_q\eta'_q W(N)^{-1}$, then $\gamma_2 \in \Gamma_0(N)$ and

$$\gamma_2 \equiv \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & (\text{mod } N_q), \\ \begin{pmatrix} N_q & 0 \\ 0 & N_q^{-1} \end{pmatrix} & (\text{mod } (N/N_q)). \end{cases}$$

Hence

$$f|[\eta_q\eta'_q]_k = \overline{\chi'}(N_q)f|[W(N)]_k.$$

(3) If $(n,q)=1$, then

$$(f|[\eta_q]_k)|\mathrm{T}(n) = \overline{\chi_q}(n)(f|\mathrm{T}(n))|[\eta_q]_k = \overline{\chi_q}(n)a(n)f|[\eta_q]_k. \tag{6.16}$$

If $(n, N/N_q)=1$, then

$$(f|[\eta'_q]_k)|\mathrm{T}(n) = \overline{\chi'}(n)a(n)f|[\eta'_q]_k. \tag{6.17}$$

Since $f|[\eta_q]_k \in S^{\mathrm{new}}(N,k,\chi'\overline{\chi_q})$ by (1), $f|[\eta_q]_k$ is a constant multiple of a primitive cusp form by Lemma 6.14, and by (6.16) we have

$$b(p) = \overline{\chi_q}(p)a(p), \qquad \text{if } p \neq q.$$

By (2), we see that $f|[\eta_q]_k = cf|[W(N)\eta'_q]_k$ with $c = \overline{\chi'}(-N_q)\overline{\chi_q}(N/N_q)$, so that

$$(f|[\eta_q]_k)|\mathrm{T}(n) = c((f|[W(N)]_k)|[\eta'_q]_k)|\mathrm{T}(n).$$

Since $f|[W(N)]_k \in S(N,k,\overline{\chi})$, we see that, by (6.17) and Lemma 6.15,

$$\begin{aligned}(f|[\eta_q]_k)|\mathrm{T}(n) &= c\chi'(q)((f|[W(N)]_k)|\mathrm{T}(n))|[\eta'_q]_k \\ &= c\chi'(q)\overline{a(n)}f|[W(N)\eta'_q]_k \\ &= \chi'(q)\overline{a(n)}f|[\eta_q]_k.\end{aligned}$$

Therefore $b(q) = \chi'(q)\overline{a(q)}$. This completes the proof. □

Theorem 6.7 *Let $f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S(N,k,\chi)$ be a primitive cusp form, m the conductor of χ. For any prime $q|N$, put $N = N_qN'_q$, $m = m_qm'_q$ with $q \nmid N'_q$ and $q \nmid m'_q$. Then*

(1) $|a_q| = q^{(k-1)/2}$, *if $N_q = m_q$;*

(2) $a_q^2 = \overline{\chi'}(q)q^{k-2}$, *if $N_q = q$ and $m_q = 1$;*

(3) $a_q = 0$, *if $q^2|N$ and $N_q \neq m_q$.*

Proof (1) Let γ_q, η_q be as above, a a positive integer prime to q. Take a positive integer b such that $ab+1 \equiv 0 \pmod{N_q}$ and $a \equiv b \pmod{N/N_q}$. Let γ be a matrix satisfying

$$\begin{pmatrix} 1 & a \\ 0 & q^e \end{pmatrix} \gamma_q = \gamma \begin{pmatrix} 1 & b \\ 0 & q^e \end{pmatrix}, \quad N_q = q^e,$$

then $\gamma \in SL_2(\mathbb{Z})$ and

$$\gamma \equiv \begin{pmatrix} a & * \\ 0 & -b \end{pmatrix} \pmod{N_q}, \quad \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N/N_q},$$

so that $\gamma \in \mathit{\Gamma}_0(N)$ and $\chi(\gamma) = \chi_q(-b)$. Therefore we obtain

$$f \left| \left[\begin{pmatrix} 1 & a \\ 0 & q^e \end{pmatrix} \gamma_q \right]_k \right. = \chi_q(-b) f \left| \left[\begin{pmatrix} 1 & b \\ 0 & q^e \end{pmatrix} \right]_k \right. .$$

Let a run over a reduced residue system modulo N_q, then we get

$$\begin{aligned}
& q^{e(k/2-1)} \sum_{(a,N_q)=1} \left(f \left| \left[\begin{pmatrix} 1 & a \\ 0 & q^e \end{pmatrix} \right]_k \right. \right) \Big| [\eta_q]_k \\
&= q^{e(k/2-1)} \left(\sum_{(b,N_q)=1} \chi_q(-b) f \left| \left[\begin{pmatrix} 1 & b \\ 0 & q^e \end{pmatrix} \right]_k \right. \right) \left| \left[\begin{pmatrix} q^e & 0 \\ 0 & 1 \end{pmatrix} \right]_k \right. \\
&= q^{e(k/2-1)} \chi_q(-1) \left(\sum_{n=1}^{\infty} \sum_{(b,N_q)=1} \chi_q(b) \mathrm{e}^{2\pi i n b/q^e} \right) a(n) e(nz) \\
&= q^{e(k/2-1)} W(\chi_q) \sum_{n=1}^{\infty} \overline{\chi_q}(-n) a(n) e(nz),
\end{aligned} \tag{6.18}$$

where $W(\chi_q)$ is the Gauss sum of χ_q. Since

$$f|\mathrm{T}(n) = n^{k/2-1} \sum_{\substack{ad=n, a>0, \\ (a,N)=1}} \sum_{b \bmod d} f \left| \left[\sigma_a \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right]_k \right. ,$$

we see that

$$\begin{aligned}
q^{e(k/2-1)} \sum_{(a,N_q)=1} f \left| \left[\begin{pmatrix} 1 & a \\ 0 & q^e \end{pmatrix} \right]_k \right. &= f|\mathrm{T}(q^e) - q^{k/2-1} (f|\mathrm{T}(q^{e-1})) \left| \left[\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \right]_k \right. \\
&= a(q^e) f - q^{k/2-1} a(q^{e-1}) f \left| \left[\begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \right]_k \right. .
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
& q^{e(k/2-1)} \sum_{(a,N_q)=1} \left(f \left| \left[\begin{pmatrix} 1 & a \\ 0 & q^e \end{pmatrix} \right]_k \right. \right) \Big| [\eta_q]_k \\
&= a(q^e) f | [\eta_q]_k - \chi'(q) q^{k/2-1} a(q^{e-1}) (f | [\eta_q]_k \left| \left[\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \right]_k \right. ,
\end{aligned} \tag{6.19}$$

where we used the facts: $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \eta_q \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}^{-1} \eta_q^{-1} \in SL_2(\mathbb{Z})$ and $\chi(\gamma) = \chi'(q)$. Let $g(z) = \sum_{n=1}^{\infty} b(n)e(nz)$ be as in (3) of Theorem 6.6, then $f|[\eta_q]_k = cg$ with a constant c. Comparing the coefficients of $e(z)$ and $e(qz)$ of (6.18), (6.19), we obtain

$$ca(q^e) = q^{e(k/2-1)}W(\chi_q), \quad ca(q^e)b(q) - c\chi'(q)q^{k-1}a(q^{e-1}) = 0.$$

Hence we have, by Theorem 6.6,

$$|a(q)|^2 = q^{k-1}, \quad c = W(\chi_q)q^{e(k/2-1)}a(q^e)^{-1}.$$

(2) By Lemma 5.17 and Lemma 6.8, since $N_q = q$, we see that

$$\Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_0(N/q) = \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_0(N) \bigcup \Gamma_0(N)\eta_q,$$

since we can take $\sigma_q = \gamma_q$ and $\gamma = \gamma_q \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \sigma_q^{-1} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}^{-1} \in \Gamma_0(N)$. Therefore

$$f\Big|\Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_0(N/q) = f|\mathrm{T}(q) + q^{k/2-1} f|[\eta_q]_k.$$

If $(n, N) = 1$, then $\mathrm{T}(n)$ commutes with $\mathrm{T}(q)$ and $[\eta_q]_k$, so that

$$g := f\Bigg|\Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_0(N/q) \in S(N/q, k, \chi)$$

is a common eigenfunction of all $\mathrm{T}(n)$, $(n, N) = 1$ and the eigenvalues are the same as the ones of f. By Theorem 6.3, g is a constant multiple of f. This implies that $g = 0$ since $g \in S(N/q, k, \chi)$ and f is a new form. So that, we get

$$q^{k/2-1} f|[\eta_q]_k = -a(q)f,$$

and hence, by (2) of Theorem 6.6, we have

$$q^{k/2-1}\chi_q(-1)\overline{\chi'}(q)f = q^{k/2-1} f|[\eta_q^2]_k = -a(q)f|[\eta_q]_k = q^{1-k/2}a(q)^2 f.$$

That is, $a(q)^2 = \chi_q(-1)\overline{\chi'}(q)q^{k-2}$. Since $m_q = 1$, $\chi_q(-1) = 1$, $a(q)^2 = \overline{\chi'}(q)q^{k-2}$.

(3) Similar to the proof of (2), we have

$$\Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_0(N/q) = \Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_0(N).$$

Hence we get, along similar arguments for the assertion (2),

$$f|\mathrm{T}(q) = f\Bigg|\left[\Gamma_0(N) \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \Gamma_0(N/q)\right]_k = 0.$$

This implies that $a(q) = 0$, which completes the proof. □

During the proof of Theorem 6.7, we have also shown the following:

Corollary 6.1 (1) *If* $N_q = m_q$, *then*

$$f|[\eta_q]_k = a(q^e)^{-1}q^{e(k/2-1)}W(\chi_q)g$$

with g *a primitive cusp form of* $S(N,k,\chi'\overline{\chi_q})$.

(2) *if* $N_q = q$, $m_q = 1$, *then*

$$f|[\eta_q]_k = -a(q)q^{1-k/2}f, \quad \overline{a(q)} = \chi'(q)a(q).$$

Theorem 6.8 *Let* $f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S(N,k,\chi)$ *be a common eigenfunction of* $R_0(N)$ *and* $R_0^*(N)$, $a(1) = 1$ *and* $g = \sum_{n=1}^{\infty} b(n)e(nz) \in S(M,k,\omega)$ *a primitive cusp form. Assume that there exists a positive integer* L *such that* $a(n) = b(n)$ *for all* $(n,L) = 1$. *Then* $N = M$, $\chi = \omega$ *and* $f = g$.

Proof Without loss of generality, we may assume that L is a common multiple of M and N. If $p \nmid L$, by Lemma 6.12, we have

$$p^{k-1}\chi(p) = a(p)^2 - a(p^2), \quad p^{k-1}\omega(p) = b(p)^2 - b(p^2).$$

But $b(p) = a(p)$ and $a(p^2) = b(p^2)$ for any $p \nmid L$, so that $\chi(p) = \omega(p)$ for any $p \nmid L$. Hence we obtain

$$\chi(n) = \omega(n), \quad \text{if } (n,L) = 1.$$

By the functional equation in Theorem 5.9, we see that

$$\frac{R_N(s,f)}{R_M(s,g)} = \frac{R_N(k-s, f|[W(N)]_k)}{R_M(k-s, g|[W(M)]_k)}. \tag{6.20}$$

Since $L_N(s,f)$ and $L_M(s,g)$ have Euler products for $\mathrm{Re}(s) > 1 + k/2$ respectively, we see that for $\mathrm{Re}(s) > 1 + k/2$

$$\frac{R_N(s,f)}{R_M(s,g)} = \left(\frac{\sqrt{N}}{\sqrt{M}}\right)^s \prod_{p|L} \frac{1 - b(p)p^{-s} + \omega(p)p^{k-1-2s}}{1 - a(p)p^{-s} + \chi(p)p^{k-1-2s}}. \tag{6.21}$$

By the analytic continuation principle, we know that (6.21) holds for all s. Similarly, by (2) of Theorem 6.5 and Lemma 6.15, we have

$$\frac{R_N(k-s, f|[W(N)]_k)}{R_M(k-s, g|[W(M)]_k)} = c\left(\frac{\sqrt{N}}{\sqrt{M}}\right)^{k-s} \prod_{p|L} \frac{1 - \overline{b(p)}p^{s-k} + \overline{\omega}(p)p^{2s-k-1}}{1 - \overline{a(p)}p^{s-k} + \overline{\chi}(p)p^{2s-k-1}} \tag{6.22}$$

with a constant c. Comparing (6.20)–(6.22), we obtain

$$\left(\frac{N}{M}\right)^s \prod_{p|L} \frac{1-b(p)p^{-s}+\omega(p)p^{k-1-2s}}{1-a(p)p^{-s}+\chi(p)p^{k-1-2s}} = c\left(\frac{\sqrt{N}}{\sqrt{M}}\right)^k \prod_{p|L} \frac{1-\overline{b(p)}p^{s-k}+\overline{\omega}(p)p^{2s-k-1}}{1-\overline{a(p)}p^{s-k}+\overline{\chi}(p)p^{2s-k-1}}. \tag{6.23}$$

Let M_p and N_p be the p-parts (i.e., $M_p = p^{\nu_p(M)}$ and $N_p = p^{\nu_p(N)}$, where $\nu_p(*)$ is the p-valuation.) of M and N respectively. By (6.23) and the uniqueness of Dirichlet series, for $p|L$ we have that

$$\left(\frac{N_p}{M_p}\right)^s \frac{1-b(p)p^{-s}+\omega(p)p^{k-1-2s}}{1-a(p)p^{-s}+\chi(p)p^{k-1-2s}} = c_p \frac{1-\overline{b(p)}p^{s-k}+\overline{\omega}(p)p^{2s-k-1}}{1-\overline{a(p)}p^{s-k}+\overline{\chi}(p)p^{2s-k-1}}$$

with c_p a constant. Set $x = p^{-s}$, then

$$1-a(p)p^{-s}+\chi(p)p^{k-1-2s} = 1-a(p)x+\chi(p)p^{k-1}x^2,$$
$$1-b(p)p^{-s}+\omega(p)p^{k-1-2s} = 1-b(p)x+\omega(p)p^{k-1}x^2.$$

Denote by u, v the degrees of the above polynomials with respect to x. It is clear that $0 \leqslant u, v \leqslant 2$.

(1) If $u = v = 0$, we see that $M_p = N_p$.

(2) If $u = 0$, $v = 1$, set $N_p/M_p = p^e$, then we see that

$$1-b(p)x = c_p x^e(1-\overline{b(p)}p^{-k}x^{-1}), b(p) \neq 0.$$

Therefore $|b(p)|^2 = p^k$ which contradicts Theorem 6.7, so that it is impossible that $u = 0$ and $v = 1$.

(3) If $u = 1$, $v = 0$, similar to (2), it is easy to see that $M_p = pN_p$.

(4) If $u = 0$, $v = 2$, set $N_p/M_p = p^e$, then

$$1-b(p)x+\omega(p)p^{k-1}x^2 = c_p x^e(1-\overline{b(p)}p^{-k}x^{-1}+\overline{\omega}(p)p^{-k-1}x^{-2}).$$

This implies that $e = 2$ and hence $|\omega(p)| = p$ which is impossible, so that it is impossible that $u = 0$, $v = 2$.

(5) If $u = 2$, $v = 0$, similar to (4), it is easy to see that $M_p = p^2 N_p$.

(6) If $u = 1$, $v = 2$, set $N_p/M_p = p^e$, then

$$\frac{1-b(p)x+\omega(p)p^{k-1}x^2}{1-a(p)x} = c_p x^e \frac{1-\overline{b(p)}p^{-k}x^{-1}+\overline{\omega}(p)p^{-k-1}x^{-2}}{1-\overline{a(p)}p^{-k}x^{-1}}.$$

This implies that $e = 1$, so that

$$\begin{aligned}&(1-b(p)x+\omega(p)p^{k-1}x^2)(x-\overline{a(p)}p^{-k})\\ &= c_p(1-a(p)x) \times (x^2-\overline{b(p)}p^{-k}x+\overline{\omega}(p)p^{-k-1}).\end{aligned} \tag{6.24}$$

By comparing the coefficients on both sides of (6.24), we obtain

$$|a(p)| = p^{k/2-1}, \quad |c_p| = p^{k/2}. \tag{6.25}$$

By (6.24) and (6.25), we see that $a(p)^{-1} = p^{-k+2}\overline{a(p)}$ should be a root of $1 - b(p)x + \omega(p)p^{k-1}x^2 = 0$, i.e.,

$$1 - b(p)p^{-k+2}\overline{a(p)} + \omega(p)p^{3-k}\overline{a(p)}^2 = 0,$$

so that,

$$b(p) = a(p) + \omega(p)p\overline{a(p)} = a(p) - c(p). \tag{6.26}$$

By (6.25) and (6.26), we have

$$\left|1 - |b(p)|p^{-k/2}\right| < p^{-1},$$

which contradicts Theorem 6.7, and it is impossible that $u = 1$, $v = 2$.

(7) If $u = 2$, $v = 1$, similar to (6), it is easy to see that $M_p = pN_p$.

(8) If $u = v = 2$, it is easy to see that $M_p = N_p$.

Anyway, we proved that $N|M$ and $\chi(n) = \omega(n)$ if $(n, M) = 1$. This implies that $S(N, k, \chi) \subset S(M, k, \omega)$. By Theorem 6.3, we have $f = g$, and hence $M = N$ in terms of Lemma 6.14. This completes the proof. □

By Lemma 6.14 and Theorem 6.8, it is easy to show the following:

Corollary 6.2 (1) *Let* $0 \neq f(z) \in S(N, k, \chi)$, *and*

$$f|\mathrm{T}(n) = a(n)f, \quad (n, N) = 1.$$

Then there exists a unique factor M *of* N *and a unique primitive cusp form* $g(z)$ *of* $S(M, k, \chi)$ *such that*

$$g|\mathrm{T}(n) = a(n)g, \quad (n, N) = 1.$$

(2) *Let* $f(z) \in S(N, k, \chi)$ *be a common eigenfunction of* $R_0(N)$ *and* $R_0^*(N)$. *Then* $f(z)$ *is a constant multiple of some primitive cusp form of* $S^{\mathrm{new}}(N, k, \chi)$.

6.2 New Forms with Half Integral Weight

In this section we discuss the Kohnen's theory of new forms with half integral weight. Here and after, we always assume that N is an odd square free positive integer, χ a quadratic character modulo N with conductor t. Put $\varepsilon = \chi(-1)$ and $\chi_1 = \left(\frac{4\varepsilon}{\cdot}\right)\chi$.

We define $S_{k+1/2}(N, \chi)$ as the space of cusp forms of weight $k + 1/2$ and character χ_1 on $\Gamma_0(4N)$ which have a Fourier expansion $\sum_{n=1}^{\infty} a(n)e(nz)$ with $a(n) = 0$ for $\varepsilon(-1)^k n \equiv 2, 3 \pmod 4$. We write $S_{k+1/2}(N)$ for $S_{k+1/2}(N, \mathrm{id.})$ and we call this space Kohnen's "+" space. It is clear that $S_{k+1/2}(N, \chi) \subset S(4N, k + 1/2, \chi_1)$.

Put

$$\xi := \xi_{k,\varepsilon} = \left\{ \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \varepsilon^{1/2} e^{\pi i/4} \right\},$$
$$Q := Q_{k,N,\chi_1} = [\Delta_0(4N,\chi_1)\xi_{k,\varepsilon}\Delta_0(4N,\chi_1)],$$

where $\Delta_0(M,\omega) := \left\{ (A,\phi) | A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M), \phi(z) = \omega(d) \left(\frac{c}{d}\right) \left(\frac{-4}{d}\right)^{-1/2} (cz + d)^{1/2} \right\}$. We usually omit the subscripts $k+1/2, 4N, \chi_1$ and write just ξ, Q.

Lemma 6.16 *The operator Q satisfies the quadratic equation $(Q-\alpha)(Q-\beta) = 0$ where $\alpha = (-1)^{[(k+1)/2]}\varepsilon 2\sqrt{2}$ and $\beta = -\dfrac{\alpha}{2}$. It is Hermitian, and its α eigenspace is just $S_{k+1/2}(N,\chi)$.*

Proof It is easy to check that

$$\xi^{\mp}\Delta_0(4N,\chi_1)\xi^{\pm} \bigcap \Delta_0(4N,\chi_1) = \Delta_0(16N,\chi_1).$$

Therefore

$$\Delta_0(4N,\chi_1)\xi^{\pm}\Delta_0(4N,\chi_1) = \bigcup \Delta_0(16N,\chi_1)\xi^{\pm}\xi_u \tag{6.27}$$

is a disjoint union, where $\{\xi_u\}$ is a set of representatives for $\Delta_0(4N,\chi_1)/\Delta_0(16N,\chi_1)$. For any $v \in \mathbb{Z}$, put $A_v = \begin{pmatrix} 1 & 0 \\ 4Nv & 1 \end{pmatrix}$. Then $\{A_v^* | v \bmod 4\}$ is a set of representatives for $\Delta_0(4N,\chi_1)/\Delta_0\ (16N,\chi_1)$, by (6.27), we see that

$$f|Q = \sum_{v \bmod 4} f|[\xi A_v^*],$$
$$f|Q^2 = \sum_{v \bmod 4} \sum_{u \bmod 4} f|[\xi A_v^* \xi A_u^*].$$

Now

$$\begin{aligned}
\xi A_0 \xi A_u &= \left\{ 8 \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \varepsilon i^{1/2} \right\} A_u^* \\
&= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \chi_1(-2Nu+1) \left(\frac{4Nu}{-2Nu+1} \right) \left(\frac{-4}{-2Nu+1} \right)^{1/2} \right\} \\
&\quad \times \begin{pmatrix} 1+2Nu & -Nu \\ 4Nu & 1-2Nu \end{pmatrix}^* \left\{ 8 \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \varepsilon i^{1/2} \right\}.
\end{aligned}$$

By the invariance of f under the operation of elements in $\Delta_0(4N,\chi_1)$ and the fact that

$$\sum_{u \bmod 4} \chi_1(-2Nu+1) \left(\frac{4Nu}{-2Nu+1} \right) \left(\frac{-4}{-2Nu+1} \right)^{-k-1/2} = 0,$$

we obtain that

$$\sum_{u \bmod 4} f|[\xi A_0^* \xi A_u^*] = 0.$$

Next we observe that

$$\xi A_{\pm 1}^* \xi = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \chi_1(1 \pm N + N^2) \left(\frac{-4}{1 \pm N + N^2} \right)^{-1/2} e^{\pi i/4} \right\} \times \begin{pmatrix} 1 \mp N + N^2 & \left(\dfrac{N \pm 1}{2} \right)^2 \\ -4N^2 & 1 \pm N + N^2 \end{pmatrix}^* \xi A_{\pm 1}^*,$$

hence

$$\sum_{u \bmod 4} f|[\xi A_{\pm 1}^* \xi A_u^*] = \chi_1(1 \pm N + N^2) \left(\frac{-4}{1 \pm N + N^2} \right)^{k-1/2} \varepsilon^{-k-1/2} e^{-(2k+1)\pi i/4} f|Q.$$

Since

$$\chi_1(1 + N + N^2) \left(\frac{-4}{1 + N + N^2} \right)^{k-1/2} + \chi_1(1 - N + N^2) \left(\frac{-4}{1 - N + N^2} \right)^{k-1/2} = 1 + \varepsilon(-1)^k i,$$

we obtain

$$\sum_{u \bmod 4} (f|[\xi A_1^* \xi A_u^*] + f|[\xi A_{-1}^* \xi A_u^*]) = (1 + \varepsilon(-1)^k i) \varepsilon^{-k-1/2} e^{-(2k+1)\pi i/4} f|Q.$$

Finally

$$\xi A_2^* \xi = \left\{ \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}, 1 \right\} \begin{pmatrix} 1 + 2N & \dfrac{1 + N}{2} \\ 8N & 1 + 2N \end{pmatrix}^*$$

and so

$$\sum_{u \bmod 4} f|[\xi A_2^* \xi A_u^*] = 4f.$$

Summarizing the facts above we showed that

$$Q^2 = (1 + \varepsilon(-1)^k i) \varepsilon^{-k-1/2} e^{-(2k+1)\pi i/4} Q + 4,$$

that is,

$$(Q - \alpha)(Q - \beta) = 0.$$

The adjoint operator of Q is given by

$$f|\widetilde{Q} = \sum_{\xi} f|[\xi],$$

where ξ runs through a set of representatives of the right cosets of $\Delta_0(4N, \chi_1)$ in $\Delta_0(4N, \chi_1)\ \xi' \Delta_0(4N, \chi_1)$ with $\xi' = \left\{ \begin{pmatrix} 4 & -1 \\ 0 & 4 \end{pmatrix}, \varepsilon^{-k-1/2} e^{-(2k+1)\pi i/4} \right\}$, but

$$\xi' = \begin{pmatrix} 1-2N & \dfrac{N-1}{2} \\ 8N & 1-2N \end{pmatrix}^* \xi \begin{pmatrix} 1 & 0 \\ -8N & 1 \end{pmatrix}^*,$$

so that Q is Hermitian.

Let $f = \sum_{n=1}^{\infty} a(n)e(nz)$ be an element of $S(4N, k+1/2, \chi_1)$. Then

$$\begin{aligned} f|[\xi+\xi'] &= \varepsilon^{-k-1/2}\mathrm{e}^{-(2k+1)\pi\mathrm{i}/4} f(z+1/4) + \varepsilon^{k+1/2}\mathrm{e}^{(2k+1)\pi\mathrm{i}/4} f(z-1/4) \\ &= \varepsilon^k \sum_{n=1}^{\infty} (\varepsilon^{-1/2}\mathrm{i}^{-k}\mathrm{e}^{-\pi\mathrm{i}/4}\mathrm{e}^{\pi\mathrm{i}n/2} + \varepsilon^{1/2}\mathrm{i}^{k}\mathrm{e}^{\pi\mathrm{i}/4}\mathrm{e}^{-\pi\mathrm{i}n/2}) a(n)e(nz) \end{aligned}$$

and hence

$$f|[\xi+\xi'] = (-1)^{[(k+1)/2]}\varepsilon\sqrt{2}\Bigg(\sum_{\varepsilon(-1)^k n \equiv 0,1 \bmod 4} a(n)e(nz) - \sum_{\varepsilon(-1)^k n \equiv 2,3 \bmod 4} a(n)e(nz) \Bigg). \tag{6.28}$$

This shows that f is in $S_{k+1/2}(N,\chi)$ if and only if $f|[\xi+\xi'] = \dfrac{\alpha}{2} f$. Now by the definition of the trace operator in Section 5.4, we see that, by (6.27),

$$f|Q = (f|[\xi])|\mathrm{Tr}, \quad f|\widetilde{Q} = (f|[\xi'])|\mathrm{Tr}, \tag{6.29}$$

where Tr is the trace operator from $S(16N, k+1/2, \chi_1)$ to $S(4N, k+1/2, \chi_1)$. Thus, if $f \in S_{k+1/2}(N,\chi)$, we see that

$$f|Q = \frac{1}{2} f|[Q+\widetilde{Q}] = \frac{1}{2}((f|[\xi])|\mathrm{Tr} + (f|[\xi'])|\mathrm{Tr}) = \frac{\alpha}{4} f|\mathrm{Tr} = \alpha f.$$

Conversely, suppose that $f|Q = \alpha f$. Then

$$(f|[\xi - \alpha/4])|\mathrm{Tr} = (f|[\xi' - \alpha/4])|\mathrm{Tr} = 0$$

and so

$$(f|[\xi+\xi'-\alpha/2])|\mathrm{Tr} = 0. \tag{6.30}$$

By the definition of Tr, the equation (6.30) implies that the function $f' := f|[\xi+\xi'-\alpha/2]$ is in the orthogonal complement of $S(4N, k+1/2, \chi_1)$ in $S(16N, k+1/2, \chi_1)$. In particular, we have

$$\langle f', f \rangle = 0.$$

Since $(f|[\xi+\xi'])|[\xi+\xi'] = 2f$, we see that

$$\langle f', f|[\xi+\xi'] \rangle = \langle f'|[\xi+\xi'], f \rangle = \Big\langle 2f - \frac{\alpha}{2} f|[\xi+\xi'], f \Big\rangle = -\frac{\alpha}{2}\langle f', f \rangle = 0.$$

Together with $\langle f', f \rangle = 0$, this implies that $\langle f', f' \rangle = 0$, i.e. $f|[\xi+\xi'] = \dfrac{\alpha}{2} f$. Therefore f is in $S_{k+1/2}(N,\chi)$. This completes the proof. □

For each prime divisor p of N, we defined an operator $W(p)$ in Section 5.4 by

$$W(p) = \left\{ \begin{pmatrix} p & a \\ 4N & pb \end{pmatrix},\ \varepsilon_p^{-1} p^{1/4} (4Nz + pb)^{1/2} \right\},$$

where a, b are integers with $p^2 b - 4Na = p$. Then $W(p)$ maps $S(4N, k+1/2, \chi_1)$ to $S\left(4N, k+1/2, \chi_1 \left(\frac{4p}{\cdot}\right)\right)$ and $\left(\frac{-4}{p}\right)^{-(2k+1)/4} W(p)$ acts as an unitary involution on the sum of these spaces (see Section 5.4).

Lemma 6.17 *$W(p)$ maps the space $S_{k+1/2}(N, \chi)$ isomorphically onto the space $S_{k+1/2}\left(N, \chi \left(\frac{\cdot}{p}\right)\right)$.*

Proof We must show that $S_{k+1/2}(N, \chi)|W(p) \subset S_{k+1/2}(N, \chi(\frac{\cdot}{p}))$. In view of Lemma 6.16 we only need to show that

$$(f|W(p))|Q_{k,N,\left(\frac{4p}{\cdot}\right)\chi_1} = \left(\frac{-4}{p}\right)(f|Q_{k,N,\chi_1})|W(p) \tag{6.31}$$

holds for $f \in S(4N, k+1/2, \chi_1)$. It is easy to verify that for every $v \in \mathbb{Z}$ there is some $\gamma_v \in \Gamma_0(4N)$ such that

$$W(p)\xi_{k,\left(\frac{-4}{p}\right)\varepsilon} A_v^* = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \left(\frac{-4}{p}\right) \right\} \gamma_v^* \xi_{k,\varepsilon} A_u^* W(p),$$

where u is determined mod 4 by $Nu \equiv -1 - b(1 + Nv) + N/p \pmod 4$. This implies (6.31) since $f|Q = \sum_{v \bmod 4} (f|[\xi_{k,\left(\frac{-4}{p}\right)\varepsilon}])|A_v^*$. This completes the proof. □

Let $m|N^\infty$ and $U(m)$ be the operator defined as in Lemma 5.38. For any prime divisor p of N, put

$$w := w_{p,k+1/2,N} := p^{-(2k-1)/4} U(p) W(p)$$

and define $S_{k+1/2}^{\pm p}(N)$ as the subspace of $S_{k+1/2}(N)$ consisting of forms whose n-th Fourier coefficients vanish for $\left(\frac{(-1)^k n}{p}\right) = \mp 1$. Then we set

$$w_{p,\chi} := w_{p,k+1/2,N,\chi} := U(t)^{-1} w_{p,k+1/2,N} U(t),$$
$$S_{k+1/2}^{\pm p}(N, \chi) = S_{k+1/2}^{\pm p}(N)|U(t),$$

where we used the fact that $U(t)$ is an isomorphism from $S_{k+1/2}(N)$ to $S_{k+1/2}(N, \chi)$ which will be proved in (1) of the following lemma.

Lemma 6.18 (1) *The operator $U(t)$ maps isomorphically $S_{k+1/2}(N)$ onto $S_{k+1/2}(N, \chi)$ where t is the conductor of χ.*

(2) *The operator $w_{p,k+1/2,N,\chi}$ is a Hermitian involution on $S_{k+1/2}(N,\chi)$ whose (± 1)-eigen -space is $S^{\pm p}_{k+1/2}(N,\chi)$. In particular, for any $p|N$, we have an orthogonal decomposition*

$$S_{k+1/2}(N,\chi) = S^{+p}_{k+1/2}(N,\chi) \oplus S^{-p}_{k+1/2}(N,\chi).$$

If $p \nmid t$, then $w_{p,\chi}$ coincides with the restriction of $\left(\frac{t}{p}\right) p^{-(2k-1)/4}U(p)W(p)$ to $S_{k+1/2}(N,\chi)$, and $S^{\pm p}_{k+1/2}(N,\chi)$ coincides with the subspace of $S_{k+1/2}(N,\chi)$ consisting of forms whose $n-th$ Fourier coefficients vanish for $\left(\frac{(-1)^k tn}{p}\right) = \mp 1$.

Proof We prove first the following assertion: suppose $p \nmid t$, then $p^{-(2k-1)/4}U(p)W(p)$ defines a Hermitian involution on $S_{k+1/2}(N,\chi)$ whose (±1)-eigenspace consists of those functions f which have a Fourier expansion $f = \sum_{n=1}^{\infty} a(n)e(nz)$ with $a(n) = 0$ for $\left(\frac{(-1)^k n}{p}\right) = \mp 1$.

In fact, by the definition of $U(p)$, we see that

$$f|U(p) = p^{(2k-3)/4} \sum_{v \bmod p} f \left| \left[\left\{ \begin{pmatrix} 1 & v \\ 0 & p \end{pmatrix}, p^{1/4} \right\} \right] \right.$$

and so

$$\begin{aligned} & f|p^{-(2k-1)/4}U(p)W(p) \\ = & p^{-1/2} \sum_{v \bmod p} f \left| \left[\left\{ \begin{pmatrix} p+4Nv & a+pbv \\ 4Np & p^2 b \end{pmatrix}, \left(\frac{-4}{p}\right)^{-1/2} (4Nz+pb)^{1/2} \right\} \right] \right. . \end{aligned}$$

If $1+4Nv/p \not\equiv 0 \pmod p$, then $4N$ and $1+4Nv/p$ are co-prime, and so we can find integers α, β such that $\alpha(-1-4Nv/p) - 4N\beta = 1$. Thus $\begin{pmatrix} \alpha & \beta \\ 4N & -1-4Nv/p \end{pmatrix} \in \Gamma_0(4N)$, by $f \in S_{k+1/2}(N,\chi)$ and $p \nmid t$, we see that

$$\begin{aligned} & f \left| \left[\left\{ \begin{pmatrix} p+4Nv & a+pbv \\ 4Np & p^2 b \end{pmatrix}, (4Nz+pb)^{1/2} \right\} \right] \right. \\ = & \left(\frac{N/p}{p}\right) f \left| \left[\left\{ \begin{pmatrix} p & -a\alpha \\ 0 & p \end{pmatrix}, \left(\frac{a\alpha}{p}\right) \right\} \right] \right. . \end{aligned}$$

Hence we have

$$f|p^{-(2k-1)/4}U(p)W(p) = \left(\frac{N/p}{p}\right)\left(\frac{-4}{p}\right)^{k+1/2} p^{-1/2} \sum_{\substack{\alpha \bmod p, \\ (\alpha,p)=1}} f \left| \left[\left\{ \begin{pmatrix} p & \alpha \\ 0 & p \end{pmatrix}, \left(\frac{-\alpha}{p}\right) \right\} \right] \right.$$

$$+p^{-1/2}\left(f\left|\left[\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}\right]\right)\right|W(p), \tag{6.32}$$

where v_0 is an integer with $1+4Nv_0/p \equiv 0 \pmod p$. Since

$$\sum_{\substack{\alpha \bmod p,\\ (\alpha,p)=1}} \left(f\left|\left[\left\{\begin{pmatrix}p & \alpha\\ 0 & p\end{pmatrix}, \left(\frac{-\alpha}{p}\right)\right\}\right]\right)\right|U(p)=0,$$

we see from (6.32) that

$$\begin{aligned}
&f|(p^{-(2k-1)/4}U(p)W(p))^2\\
=&p^{-(2k+1)/4}f\left|\left[\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}\right]\right|W(p)\Big|U(p)W(p)\\
=&\frac{1}{p}\sum_{u \bmod p}\left(f\left|\left[\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}\right]\right)\right|W(p)\left|\left[\left\{\begin{pmatrix}1 & u\\ 0 & p\end{pmatrix}, p^{1/4}\right\}\right]\right|W(p).
\end{aligned}$$

Since $p\nmid t$, it is easy to check that

$$\left\{\begin{pmatrix}p^{-2} & 0\\ 0 & p^{-2}\end{pmatrix},1\right\}\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}W(p)\left\{\begin{pmatrix}1 & u\\ 0 & p\end{pmatrix}, p^{1/4}\right\}W(p)\in \varDelta_0(4N,\chi),$$

so that, we have

$$f|(p^{-(2k-1)/4}U(p)W(p))^2=f.$$

Since the adjoint of $\left\{\begin{pmatrix}p & \alpha\\ 0 & p\end{pmatrix}, \left(\frac{-\alpha}{p}\right)\left(\frac{-4}{p}\right)^{-1/2}\right\}$ is $\left\{\begin{pmatrix}p & -\alpha\\ 0 & p\end{pmatrix}, \left(\frac{\alpha}{p}\right)\left(\frac{-4}{p}\right)^{-1/2}\right\}$, and the adjoint of $\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}W(p)$ can be written as $C^*\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}W(p)$ with $C\in \varGamma_0(4N)$, it follows that $p^{-(2k-1)/4}U(p)W(p)$ is Hermitian.

Finally, by Gauss sum and (6.32), we have

$$\begin{aligned}
f|p^{-(2k-1)/4}U(p)W(p)=&\left(\frac{N/p}{p}\right)\sum_{n=1}^{\infty}\left(\frac{(-1)^k n}{p}\right)a(n)e(nz)\\
&+p^{-1/2}f\left|\left[\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}\right]\right|W(p).
\end{aligned} \tag{6.33}$$

Therefore to complete the proof of our assertion we only need to show that

$$f|U(p)=\pm\left(\frac{-4}{p}\right)^{-k-1/2}p^{(2k-1)/4}f|W(p)$$

is equivalent to the identity

$$p^{-1/2}\left(f\left|\left[\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}\right]\right|W(p)\right)(z)=\pm\left(\frac{N/p}{p}\right)(f|U(p))(pz),$$

which can be derived from the following fact

$$\left\{\begin{pmatrix}1 & v_0\\ 0 & p\end{pmatrix}, p^{1/4}\right\}W(p) = \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \left(\frac{-4}{p}\right)^{1/2}\left(\frac{N/p}{p}\right)\right\}$$
$$C^*W(p)\left\{\begin{pmatrix}p & 0\\ 0 & 1\end{pmatrix}, p^{-1/4}\right\} \tag{6.34}$$

with $C \in \Gamma_0(4N)$, and hence the assertion is proved. Since we have the following commutation rule

$$f|U(t)W(p) = \left(\frac{t}{p}\right) f|W(p)U(t), \quad p \nmid t,$$

the assertions in (2) of the lemma will be clear once (1) will have been proved. By Lemma 6.17, we have that $\dim(S_{k+1/2}(N)) = \dim(S_{k+1/2}(N,\chi))$. So we only need to show that $U(t)$ is injective on $S_{k+1/2}(N)$. But we have shown above that $U(p)W(p)$ is injective on $S_{k+1/2}(N,\chi)$ for $p \nmid t$, so $U(p)$ is injective on $S_{k+1/2}(N,\chi)$ for $p \nmid t$, and hence we conclude by induction that $U(t)$ is injective on $S_{k+1/2}(N)$. This completes the proof. □

We introduce now the Hecke operators on $S_{k+1/2}(N,\chi)$. Let

$$\mathrm{pr} := \mathrm{pr}_{k,N,\chi_1} := \frac{1}{\alpha - \beta}(Q_{k,N,\chi_1} - \beta)$$

be the orthogonal projection onto $S_{k+1/2}(N,\chi)$. For a prime $p \nmid N$, we define $\mathrm{T}(p) := \mathrm{T}_{N,k,\chi}(p)$ as the restriction of

$$\nu_p p^{k-3/2}\left[\Delta_0(4N,\chi_1)\left\{\begin{pmatrix}1 & 0\\ 0 & p^2\end{pmatrix}, p^{1/2}\right\}\Delta_0(4N,\chi_1)\right]\mathrm{pr}$$

to $S_{k+1/2}(N,\chi)$, where $\nu_p = 1$ or $3/2$ according to $p \neq 2$ or $p = 2$. It is clear that for an odd p, $\mathrm{T}_{N,k,\chi}(p)$ is the restriction of the Hecke operator $\mathrm{T}_{N,k,\chi_1}(p^2)$. We write $\mathrm{T}_{N,k}(p)$ for $\mathrm{T}_{N,k,\mathrm{id.}}(p)$.

Lemma 6.19 *Let* $f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S_{k+1/2}(N,\chi)$. *Put* $f|\mathrm{T}_{N,k,\chi}(p) = \sum_{n=1}^{\infty} b(n)e(nz)$. *Then*

$$b(n) = \begin{cases} a(p^2n) + \chi(p)\left(\dfrac{\varepsilon(-1)^k n}{p}\right)p^{k-1}a(n) + a(n/p^2), & \text{if } \varepsilon(-1)^k n \equiv 0,1 \pmod 4,\\ 0, & \text{if } \varepsilon(-1)^k n \equiv 2,3 \pmod 4. \end{cases} \tag{6.35}$$

The operators $\mathrm{T}(p)$ *generate a commutative* $\mathbb{C}$*-algebra of Hermitian operators.*

Proof Since $\mathrm{T}(p)$ is just the Hecke operator $\mathrm{T}(p^2)$ for $p \neq 2$, so (6.35) is clear for p odd by Theorem 5.15. Let us now prove (6.35) for $p = 2$. We use the same notations as in the proof of Lemma 6.16. By the definition of $U(m)$, we see that

$$U(4) = 2^{k-3/2}\left[\Delta_0(4N,\chi_1)\left\{\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix}, 2^{1/2}\right\}\Delta_0(4N,\chi_1)\right].$$

By the definition of T(2) and (6.29), we have

$$f|\mathrm{T}(2) = \frac{1}{\alpha}\left((f|U(4))|[\xi]\right)|\mathrm{Tr} + \frac{1}{2}f|U(4) = f_1 + f_2 + f_3$$

with

$$\begin{aligned} f_1 &= \frac{1}{\alpha}((f|U(4))|[\xi])|[A_0^* + A_2^*] + \frac{1}{2}f|U(4),\\ f_2 &= \frac{1}{\alpha}((f|U(4))|[\xi])|[A_N^*],\\ f_3 &= \frac{1}{\alpha}((f|U(4))|[\xi])|[A_{-N^3}^*]. \end{aligned}$$

Since

$$A_0^* = \left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, 1\right\}, \quad \xi A_2^* = \begin{pmatrix}1+2N & \dfrac{N+1}{2}\\ 8N & 1+2N\end{pmatrix}^* \xi'$$

and $f \in S_{k+1/2}(N,\chi)$, we see that

$$f_1 = \frac{1}{\alpha}(f|U(4))|[\xi + \xi'] + \frac{1}{2}f|U(4).$$

By (6.28) and Lemma 5.38, we have

$$f_1 = \sum_{\varepsilon(-1)^k n \equiv 0,1 \bmod 4} a(4n)e(nz).$$

But we have also

$$f|U(4) = 2^{k-3/2}\sum_{v \bmod 4} f\left|\left[\left\{\begin{pmatrix}1 & v\\ 0 & 4\end{pmatrix}, 2^{1/2}\right\}\right]\right.,$$

so that

$$\begin{aligned} f_2 &= \frac{2^{k-3/2}}{\alpha}\sum_{v \bmod 4} f\left|\left[\left\{\begin{pmatrix}1 & v\\ 0 & 4\end{pmatrix}, 2^{1/2}\right\}\right]\right|A_N^*\\ &= 2^{k-3/2}\sum_{v \bmod 4} f\left|\left[\left\{\begin{pmatrix}4+4N^2(4v+1) & 4v+1\\ 64N^2 & 16\end{pmatrix}, \varepsilon^{1/2}e^{\pi i/4}(8N^2z+2)^{1/2}\right\}\right]\right.. \end{aligned}$$

For $v \in \mathbb{Z}$ we can find an integer a such that

$$-a(1+N^2(4v+1)) + 2(4v+1) \equiv 0 \pmod{16},$$

so that

$$\left\{\begin{pmatrix} 4+4N^2(4v+1) & 4v+1 \\ 64N^2 & 16 \end{pmatrix}, \varepsilon^{1/2}\mathrm{e}^{\pi i/4}(8N^2z+2)^{1/2}\right\}$$
$$= \begin{pmatrix} \dfrac{1+N^2(4v+1)}{2} & \dfrac{-a(1+N^2(4v+1))+2(4v+1)}{16} \\ 8N^2 & -aN^2+2 \end{pmatrix}^*$$
$$\times \left\{\begin{pmatrix} 8 & a \\ 0 & 8 \end{pmatrix}, \chi(2)\left(\frac{4\varepsilon}{a}\right)\left(\frac{8}{a}\right)\left(\frac{-4}{a}\right)^{-1/2}\right\}.$$

Moreover, if v runs through integers mod 4, a runs through a reduced residue system mod 8. Thus

$$f_2 = \chi(2)\frac{2^{k-3/2}}{\alpha}\sum_{\substack{a \bmod 8,\\ a \text{ odd}}} f\left|\left[\left\{\begin{pmatrix} 8 & a \\ 0 & 8 \end{pmatrix}, \varepsilon^{1/2}\mathrm{e}^{\pi i/4}\left(\frac{4\varepsilon}{a}\right)\left(\frac{8}{a}\right)\left(\frac{-4}{a}\right)^{-1/2}\right\}\right]\right..$$

From this equality, it is easy to verify that

$$f_2 = \chi(2)\sum_{n=1}^{\infty}\left(\frac{\varepsilon(-1)^k n}{2}\right)a(n)e(nz).$$

We want now to compute f_3. By the proof of Lemma 6.16, we know that

$$f|[\xi+\xi'] = \frac{\alpha}{2}f. \tag{6.36}$$

Since

$$\left\{\begin{pmatrix} 1 & \pm 1 \\ 0 & 4 \end{pmatrix}, 2^{1/2}\right\}\left\{\begin{pmatrix} -N^4+1 & 1 \\ -4N^4 & 4 \end{pmatrix}, \varepsilon^{1/2}\mathrm{e}^{\pi i/4}2^{1/2}(-N^4z+1)^{1/2}\right\}$$
$$= \begin{pmatrix} \mp N^4 + \dfrac{1-N^4}{4} & \pm 1 + \dfrac{(4\pm 1)(1-N^4)}{16} \\ -4N^4 & \mp N^4+4 \end{pmatrix}^* \xi^{\mp 1},$$

so (6.36) implies

$$\frac{\alpha}{2}f = \sum_{v=1,-1} f\left|\left[\left\{\begin{pmatrix} 1 & v \\ 0 & 4 \end{pmatrix}, 2^{1/2}\right\}\left\{\begin{pmatrix} -N^4+1 & 1 \\ -4N^4 & 4 \end{pmatrix}, \varepsilon^{1/2}\mathrm{e}^{\pi i/4}2^{1/2}(-N^4z+1)^{1/2}\right\}\right]\right.,$$

and hence

$$\sum_{v=1,-1} f\left|\left[\left\{\begin{pmatrix} 1 & v \\ 0 & 4 \end{pmatrix}, 2^{1/2}\right\}\right]\right.$$
$$= \frac{\alpha}{2}f\left|\left[\left\{\begin{pmatrix} 4 & -1 \\ 4N^4 & -N^4+1 \end{pmatrix}, \varepsilon^{-1/2}\mathrm{e}^{-\pi i/4}2^{-1/2}(4N^4z-N^4+1)^{1/2}\right\}\right]\right.. \tag{6.37}$$

Since $a(n)=0$ for $n \equiv 2 \pmod 4$, we have

$$\sum_{v=1,-1} f\left|\left[\left\{\begin{pmatrix} 1 & v \\ 0 & 4 \end{pmatrix}, 2^{1/2}\right\}\right]\right. = 2^{1/2-k}f|U(4).$$

From (6.37) we obtain

$$\begin{aligned} f|U(4) &= 2^{k-3/2}\alpha f\left|\left[\left\{\begin{pmatrix} 4 & -1 \\ 4N^4 & -N^4+1 \end{pmatrix}, \varepsilon^{-1/2}\mathrm{e}^{-\pi i/4}2^{-1/2}(4N^4z-N^4+1)^{1/2}\right\}\right]\right. \\ &= 2^{k-3/2}\alpha\left(f\left|\left[\left\{\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, 2^{-1/2}\right\}\right]\right.\right)\Big|[A^*_{N^3}]\Big|[\xi^{-1}]. \end{aligned}$$

Hence

$$\begin{aligned} f_3 &= \frac{1}{\alpha}((f|U(4))|[\xi])|[A^*_{-N^3}] = 2^{k-3/2}f\left|\left[\left\{\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, 2^{-1/2}\right\}\right]\right. \\ &= 2^{2k-1}\sum_{n=1}^{\infty} a(n/4)e(nz). \end{aligned} \tag{6.38}$$

Putting together all expansions for f_1, f_2 and f_3, we get (6.35) for $p = 2$. It is clear that the operators $\mathrm{T}_{N,k,\chi}(p)$ commute each other from (6.35). $\mathrm{T}_{N,k,\chi}(p)$ ($p \nmid 2N$) is Hermitian since the operator $\left[\left\{\Delta_0(4N,\chi_1)\begin{pmatrix} 1 & 0 \\ 0 & p^2 \end{pmatrix}, p^{1/2}\right\}\Delta_0(4N,\chi_1)\right]$ is Hermitian for $p \nmid N$. So we only need to show that $\mathrm{T}(2)$ is Hermitian. Let f, g be in $S_{k+1/2}(N,\chi)$. Then

$$\begin{aligned} \frac{2}{3}\langle f|\mathrm{T}(2), g\rangle &= \langle f|U(4)\mathrm{pr}, g\rangle = \langle f|U(4), g|\mathrm{pr}\rangle \\ &= \langle f|U(4), g\rangle = 2^{k-3/2}\sum_{v \bmod 4}\left\langle f\left|\left[\left\{\begin{pmatrix} 1 & v \\ 0 & 4 \end{pmatrix}, 2^{1/2}\right\}\right], g\right.\right\rangle \\ &= 2^{k-3/2}\sum_{v \bmod 4}\left\langle f, g\left|\left[\left\{\begin{pmatrix} 4 & -v \\ 0 & 1 \end{pmatrix}, 2^{-1/2}\right\}\right]\right.\right\rangle \\ &= 2^{k+1/2}\left\langle f, g\left|\left[\left\{\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, 2^{-1/2}\right\}\right]\right.\right\rangle. \end{aligned}$$

Now we have

$$\begin{aligned} \frac{1}{\alpha}((g|U(4))|[\xi])|[A^*_{-N^3}] &= 2^{k-3/2}g\left|\left[\left\{\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, 2^{-1/2}\right\}\right]\right. \\ &= \frac{1}{\alpha}((g|U(4))|[\xi^{-1}])|[A^*_{N^3}], \end{aligned} \tag{6.39}$$

and the first equality is derived from (6.38), and the second can be proved similarly. By (6.39), we see easily that

$$2^{k+1/2}g\left|\left[\left\{\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, 2^{-1/2}\right\}\right]\right. = \frac{2}{\alpha}(g|U(4))|[\xi A^*_{-N^3} + \xi^{-1}A^*_{N^3}].$$

Thus

$$\begin{aligned} \frac{2}{3}\langle f|\mathrm{T}(2), g\rangle &= \frac{2}{\alpha}\langle f, g|U(4))|[\xi A^*_{-N^3} + \xi^{-1}A^*_{N^3}]\rangle \\ &= \frac{2}{\alpha}\langle f|[A^*_{N^3}\xi^{-1} + A^*_{-N^3}\xi], g|U(4)\rangle \end{aligned}$$

$$= \frac{2}{\alpha}\langle f|[\xi+\xi^{-1}], g|U(4)\rangle$$
$$= \langle f, g|U(4)\rangle = \frac{2}{3}\langle f, g|\mathrm{T}(2)\rangle.$$

This completes the proof. □

For a positive divisor d of N we set $S_{k+1/2}(d,\chi) = S_{k+1/2}(d)|U(t)$. Put

$$S^{\text{old}}_{k+1/2}(N,\chi) = \sum_{N\neq d|N} \left(S_{k+1/2}(d,\chi) + S_{k+1/2}(d,\chi)|U(N^2/d^2)\right),$$

which is called the space of old forms in $S_{k+1/2}(N,\chi)$. And we define the space of new forms, denoted by $S^{\text{new}}_{k+1/2}(N,\chi)$, to be the orthogonal complement of the space of old forms in $S_{k+1/2}(N,\chi)$ with respect to the Petersson inner product. We write

$$S^{\text{new}}_{k+1/2}(N) = S^{\text{new}}_{k+1/2}(N, \text{id.}).$$

Lemma 6.20 *We have*

$$S^{\text{new}}_{k+1/2}(N,\chi) = S^{\text{new}}_{k+1/2}(N)|U(t).$$

Proof By Lemma 6.18 it suffices to show the inclusion

$$S^{\text{new}}_{k+1/2}(N)|U(t) \subset S^{\text{new}}_{k+1/2}(N,\chi).$$

Let $f \in S^{\text{new}}_{k+1/2}(N)$. We must show that

$$\langle g|U(t), f|U(t)\rangle = 0$$

for all old forms g in $S_{k+1/2}(N)$. Let $t = p_1 \cdots p_r$ be the standard factorization of t. Then we have

$$\langle g|U(t), f|U(t)\rangle = p_r^{k+1/2}\langle g|U(t/p_r), f|U(t/p_r)\rangle,$$

since $W(p_r)$ is unitary and $p_r^{-(2k+1)/4}U(p_r)W(p_r)$ is a Hermitian involution on $S_{k+1/2}(N)|U\left(\frac{t}{p_r}\right)$ (by the proof of Lemma 6.18). By induction, we see that

$$\langle g|U(t), f|U(t)\rangle = t^{k+1/2}\langle g, f\rangle = 0.$$

This completes the proof. □

We shall carry over the basic facts about the space of new forms $S^{\text{new}}(N,\ 2k)$ to $S^{\text{new}}_{k+1/2}(N,\ \chi)$. Recall that for every prime divisor p of N the operator $U(p)$ preserves $S^{\text{new}}(N,2k) \subset S(N,2k)$ and that $U(p) = -p^{k-1}W_{p,2k,N}$ on $S^{\text{new}}(N,2k)$, where $W_{p,2k,N}$ is the Atkin-Lehner involution on $S(N,2k)$ defined by

$$(f|W_{p,2k,N})(z) = p^k(4Nz+pb)^{-2k} f\left(\frac{pz+a}{4Nz+pb}\right), \quad a,b\in\mathbb{Z}, p^2b-4Na = p.$$

We shall now prove an analogous result for new forms of half integral weight.

Theorem 6.9 *For every prime $p|N$, the operators $U(p^2)$ and $w_{p,\chi} := w_{p,k,N,\chi}$ preserve the space of new forms. And we have $U(p^2) = -p^{k-1}w_{p,\chi}$ on $S^{\text{new}}_{k+1/2}(N,\chi)$.*

Proof We first show that $w_{p,\chi} := w_{p,k,N,\chi}$ maps new forms to new forms. Since $w_{p,\chi}$ is Hermitian it is sufficient to show that $w_{p,\chi}$ maps old forms to old forms. By the definitions we only need to show this for $\chi = \text{id}$. Now set $w_p := w_{p,k,N}$. We only need to show that w_p maps $S_{k+1/2}(N/l)$ and $S_{k+1/2}(N/l)|U(l^2)$ to old forms for every prime divisor l of N.

Let $f \in S_{k+1/2}(N/l)$. If $p \neq l$, by (2) of Lemma 6.18, $f|w_p$ is in $S_{k+1/2}(N/l)$ and so an old form. The same is true for $f|U(l^2)|w_p = f|w_p|U(l^2)$. Thus we assume that $p = l$. Let $f(z) = \sum_{n=1}^{\infty} a(n)e(nz)$. Then, by (6.34) and (6.35) in the proof of Lemma 6.18, we see that

$$\begin{aligned} f|w_p =& \left(\frac{N/p}{p}\right)\sum_{n=1}^{\infty}\left(\frac{(-1)^k n}{p}\right)a(n)e(nz) \\ &+\left(\frac{-4}{p}\right)^{-k-1/2}\left(\frac{N/p}{p}\right)p^{-1/2}(f|W(p))\left|\left[\left\{\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}, p^{-1/4}\right\}\right]\right.. \end{aligned}$$

Since $f \in S_{k+1/2}(N/p)$, we have

$$\begin{aligned} f|W(p) &= \left(f\left|\begin{pmatrix} -1 & 0 \\ 4N/p & -1\end{pmatrix}^*\right.\right)\Big|W(p) \\ &= f\left|\left[\left\{\begin{pmatrix} -p & -a \\ 0 & -1\end{pmatrix}, \left(\frac{-4}{p}\right)^{-1/2}p^{-1/4}\right\}\right]\right. \\ &= f\left|\left[\left\{\begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix}, \left(\frac{-4}{p}\right)^{-1/2}p^{-1/4}\right\}\right]\right.. \end{aligned}$$

Thus we obtain that

$$f|w_p = \left(\frac{N/p}{p}\right)\sum_{n=1}^{\infty}\left(\left(\frac{(-1)^k n}{p}\right)a(n) + p^k a(n/p^2)\right)e(nz),$$

i.e.

$$f|w_p = \left(\frac{N/p}{p}\right)p^{-k+1}(-f|U(p^2) + f|\mathrm{T}_{N/p,k}(p^2)). \tag{6.40}$$

This shows that $f|w_p$ is an old form. Moreover, applying w_p on both sides of (6.40) and noting $w_p^2 = \text{id}$. we see that $(f|U(p^2))|w_p$ is an old form. This shows that w_p maps old forms to old forms, and so that, new forms to new forms.

Finally, we must now prove that on $S^{\text{new}}_{k+1/2}(N,\chi)$

$$U(p^2) = -p^{k-1}w_{p,k,N,\chi}, \quad p \text{ prime }, p|N. \tag{6.41}$$

But Lemma 6.20 and the injectivity of $U(t)$ on $S_{k+1/2}(N)$ (see Lemma 6.18) allows us to assume $\chi = \text{id.}$ for the proof of (6.41). Denote by $\mathrm{Tr} := \mathrm{Tr}^N_{N/p} : S(N, k+1/2) \to S(N/p,\, k+1/2)$ the trace operator. It is easy to verify that $\mathrm{Tr}^N_{N/p}$ maps $S_{k+1/2}(N)$ to $S_{k+1/2}(N/p)$ by Lemma 6.16. Let $f \in S^{\mathrm{new}}_{k+1/2}(N)$. Since f is orthogonal to $S_{k+1/2}(N/p)$, it follows that $f|\mathrm{Tr} = 0$. On the other hand, $\begin{pmatrix} 1 & 0 \\ 4N/p & 1 \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ ($u \bmod p$) together with $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ form a complete set of representatives for $\Gamma_0(4N)/\Gamma_0(4N/p)$. Thus we have

$$f|\mathrm{Tr} = f + \sum_{u \bmod p} f \left| \left[\begin{pmatrix} 1 & 0 \\ 4N/p & 1 \end{pmatrix}^* \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}^* \right] \right. .$$

But

$$\begin{pmatrix} 1 & 0 \\ 4N/p & 1 \end{pmatrix}^* \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}^* = \left\{ \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, \left(\frac{-4}{p}\right)^{1/2} \right\} W(p) \left\{ \begin{pmatrix} 1 & u-a \\ 0 & 1 \end{pmatrix}, p^{1/4} \right\},$$

so that

$$f|\mathrm{Tr} = f + \left(\frac{-4}{p}\right)^{-k-1/2} p^{-k/2+3/4} f|W(p)U(p).$$

Since $f|\mathrm{Tr} = 0$, we obtain that

$$f|W(p)U(p) = -\left(\frac{-4}{p}\right)^{k+1/2} p^{(2k-3)/4} f.$$

By (2) of Lemma 6.18 and the fact that $w_{p,k,N,\chi}$ preserves the space of new forms, we see that $U(p)W(p)$ is an isomorphism of $S^{\mathrm{new}}_{k+1/2}(N)$. Thus replacing f with $f|U(p)W(p)$ in the above equality, we see that

$$\begin{aligned} \left(\frac{-4}{p}\right)^{k+1/2} f|U(p^2) &= f|U(p)W(p)W(p)U(p) \\ &= -\left(\frac{-4}{p}\right)^{k+1/2} p^{(2k-3)/4} f|U(p)W(p), \end{aligned}$$

i.e.

$$f|U(p^2) = -p^{k-1} f|w_p.$$

This completes the proof. □

Lemma 6.21 *Let* $f = \sum_{n=1}^{\infty} a(n)e(nz) \in S(4N, k+1/2, \chi_1)$ *satisfy that* $a(n) = 0$ *for* $n \equiv 2 \pmod 4$. *Then* f *is in* $S_{k+1/2}(N, \chi)$.

Proof The hypothesis $a(n) = 0$ for $n \equiv 2 \pmod 4$ is equivalent to

$$f\left|\left[\left\{\begin{pmatrix}4&1\\0&4\end{pmatrix},1\right\}\right]+f\right|\left[\left\{\begin{pmatrix}4&-1\\0&4\end{pmatrix},1\right\}\right]$$
$$=2^{-k+1/2}(f|U(4))\left|\left[\left\{\begin{pmatrix}4&0\\0&1\end{pmatrix},2^{-1/2}\right\}\right].\right.$$

Now apply the trace operator $\mathrm{Tr} := \mathrm{Tr}_{4N}^{16N}$ from $S(16N, k+1/2, \chi_1)$ to $S(4N, k+1/2, \chi)$ on both sides of the above equation. Because of the identity (6.29) and the fact that Q is Hermitian, we obtain that

$$\varepsilon(-1)^{[(k+1)/2]}\sqrt{2}f|Q = 2^{-k+1/2}\left((f|U(4))\left|\left[\left\{\begin{pmatrix}4&0\\0&1\end{pmatrix},2^{-1/2}\right\}\right]\right)\right|\mathrm{Tr}. \tag{6.42}$$

Since $U(4)$ and $\left\{\begin{pmatrix}4&0\\0&1\end{pmatrix},2^{-1/2}\right\}\Big|\mathrm{Tr}$ equal $2^{k-3/2}\left[\varDelta_0(4N,\chi_1)\left\{\begin{pmatrix}1&0\\0&4\end{pmatrix},2^{1/2}\right\}\varDelta_0(4N,\chi_1)\right]$ and $\left[\varDelta_0(4N,\chi_1)\left\{\begin{pmatrix}4&0\\0&1\end{pmatrix},2^{-1/2}\right\}\varDelta_0(4N,\chi_1)\right]$ respectively, and also since

$$\begin{aligned}
&\varDelta_0(4N,\chi_1)\left\{\begin{pmatrix}1&0\\0&4\end{pmatrix},2^{1/2}\right\}\varDelta_0(4N,\chi_1)\\
&\cdot\varDelta_0(4N,\chi_1)\left\{\begin{pmatrix}4&0\\0&1\end{pmatrix},2^{-1/2}\right\}\varDelta_0(4N,\chi_1)\\
=&4\varDelta_0(4N,\chi_1)\left\{\begin{pmatrix}4&0\\0&4\end{pmatrix},1\right\}\varDelta_0(4N,\chi_1)\\
&+\varDelta_0(4N,\chi_1)\left\{\begin{pmatrix}4&1\\0&4\end{pmatrix},1\right\}\varDelta_0(4N,\chi_1)\\
&+\varDelta_0(4N,\chi_1)\left\{\begin{pmatrix}4&-1\\0&4\end{pmatrix},1\right\}\varDelta_0(4N,\chi_1)\\
&+\varDelta_0(4N,\chi_1)\left\{\begin{pmatrix}4&2\\0&4\end{pmatrix},1\right\}\varDelta_0(4N,\chi_1),\\
&\varDelta_0(4N,\chi_1)\left\{\begin{pmatrix}4&2\\0&4\end{pmatrix},1\right\}\varDelta_0(4N,\chi_1)=0,
\end{aligned}$$

the right hand side of (6.42) equals

$$\frac{1}{2}\left(4f+\varepsilon(-1)^{[(k+1)/2]}\sqrt{2}f|Q\right),$$

so that

$$f|Q = \varepsilon(-1)^{[(k+1)/2]}2\sqrt{2}f$$

and hence f is in $S_{k+1/2}(N,\chi)$ by Lemma 6.16. This completes the proof. □

Lemma 6.22 *Let p be a prime and $0 \neq f = \sum_{n=1}^{\infty} a(n)e(nz) \in G(N, k/2, \omega)$. Assume* *that* $a(n) = 0$ *for all* n *with* $p \nmid n$. *Then* $p|N/4$, $\omega\chi_p$ *is well-defined modulo* N/p *and* $f = g|V(p)$ *with* $g \in G(N/p, k/2, \omega\chi_p)$ *where* $\chi_p = \left(\frac{p}{*}\right)$.

Proof Put

$$g(z) = f(z/p) = \sum_{n=0}^{\infty} a(np)e(nz) = p^{k/4} f \Bigg| \left[\left\{ \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, p^{1/4} \right\} \right]. \tag{6.43}$$

Set

$$N' = \begin{cases} N/p, & \text{if } p|N/4, \\ N, & \text{if } p \nmid N/4. \end{cases} \qquad \Gamma_0(N', p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N') \middle| p|b \right\}.$$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N', p)$, then $A_1 = \begin{pmatrix} a & b/p \\ cp & d \end{pmatrix} \in \Gamma_0(N)$ and we see that

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, p^{1/4} \right\} A^* = \{1, \chi_p(d)\} A_1^* \left\{ \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, p^{1/4} \right\}.$$

Hence

$$g|[A^*] = \omega(d)\chi_p(d)g. \tag{6.44}$$

By (6.43) we have

$$g \Bigg| \left[\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right\} \right] = g.$$

Since $\Gamma_0(N')$ can be generated by $\Gamma_0(N', p)$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we see that (6.44) holds for any $A \in \Gamma_0(N')$. We declare that $\omega\chi_p$ must be well-defined modulo N'. Otherwise, there exist integers a and d such that $ad \equiv 1 \pmod{N'}$ and $\omega\chi_p(a) \cdot \omega\chi_p(d) \neq 1$. Take

$$B = \begin{pmatrix} a & b \\ N' & d \end{pmatrix} \in \Gamma_0(N'),$$

we have that $g = g|[B^*(B^{-1})^*] = \omega\chi_p(a)\omega\chi_p(d)g$, which is impossible since $g \neq 0$. Therefore $\omega\chi_p$ must be well-defined modulo N', so that $p|N/4$ and $N' = N/p$. It is therefore clear that g is in $G(N/p, k/2, \omega\chi_p)$ and $f = g|V(p)$. This completes the proof. □

Lemma 6.23 *Let* m *be a positive integer, and*

$$f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N, k/2, \omega).$$

Suppose that $a(n) = 0$ *for any* n *with* $(n, m) = 1$. *Then*

$$f = \sum f_p|V(p), \quad f_p \in G(N/p, k/2, \omega\chi_p),$$

where the prime p *runs over the set of common factors of* m *and* $N/4$. *And* $\omega\chi_p$ *is well-defined modulo* N/p. f_p *can be chosen as cusp forms if* f *is a cusp form.* f_p *are eigenfunctions for almost all* Hecke *operators* $\mathrm{T}(p^2)$ *if* f *is an eigenfunction for almost all* Hecke *operators* $\mathrm{T}(p^2)$.

Proof We can assume that m is square-free. Let r be the number of different prime factors of m. If $r = 0$, then $f = 0$ and the lemma holds. If $r = 1$, this is the Lemma 6.22. We now prove the lemma by induction on r. Let $m = p_0 m_0$. Take a prime p and put $K(p) = 1 - \mathrm{T}(p, Np)V(p)$ where $\mathrm{T}(p, Np)$ is the Hecke operator $\mathrm{T}_{Np,k,\omega}(p)$ on the space $G(pN, k/2, \omega)$. By the properties of Hecke operators, we have

$$f|K(p) = \sum_{(n,p)=1} a(n)e(nz) \in G(p^2N, k/2, \omega).$$

So

$$h := \sum_{(n,m_0)=1} a(n)e(nz) = f\Big| \prod_{p|m_0} K(p) \in G(m_0^2N, k/2, \omega).$$

If $h = 0$, replacing m by m_0, we see that the lemma holds by induction hypothesis. Now suppose that $h \neq 0$. If $(n, m_0) = 1$ and $a(n) \neq 0$, then $p_0|n$. By Lemma 6.22, there is $g_{p_0} \in G(m_0^2N/p, k/2, \omega\chi_{p_0})$ such that $h = g_{p_0}|V(p_0)$, and $\omega\chi_{p_0}$ is well-defined modulo m_0^2N/p_0. Hence $p_0|N/4$ and $\omega\chi_{p_0}$ is well-defined modulo N/p_0. We have

$$f - h = f - g_{p_0}|V(p_0) = \sum_{n=0}^{\infty} b(n)e(nz).$$

Noting that $b(n) = 0$ if $(n, m_0) = 1$ and applying induction hypothesis, we have

$$f - g_{p_0}|V(p_0) = \sum_{p} g_p|V(p),$$

where p runs over the set of prime factors of m_0, and $\omega\chi_p$ is well-defined modulo m_0^2N/p. Therefore by Theorem 5.21, we see that

$$f|S(\omega) - g_{p_0} = \sum_{p} (g_p|S(\omega\chi_p, m_0^2N/p, p_0))|V(p).$$

Put $f_{p_0} = f|S(\omega)$. Then $f_{p_0} \in G(N/p_0, k/2, \omega\chi_{p_0})$. If we write

$$f_{p_0}|V(p_0) = \sum_{n=0}^{\infty} c(n)e(nz),$$

then the nth Fourier coefficient of $f_{p_0}|V(p_0) - g_{p_0}|V(p_0)$ is not zero only for $(n, m_0) \neq 1$. So we get $c(n) = a(n)$ for $(n, m_0) = 1$, and hence the nth Fourier coefficient of $f - f_{p_0}|V(p_0)$ is zero for $(n, m_0) = 1$. By the induction hypothesis we get the decomposition of f as stated in the lemma. The other results can be proved also by induction. This completes the proof. □

Corollary 6.3 *Let f be as in* Lemma 6.23. *If f is an eigenfunction of almost all Hecke operators, then $f \in G^{\mathrm{old}}(N, k/2, \omega)$.*

Theorem 6.10 *We have the following decomposition:*

$$S_{k+1/2}(N, \chi) = \bigoplus_{r,d \geqslant 1, rd|N} S_{k+1/2}^{\mathrm{new}}(d, \chi)|U(r^2).$$

Proof We now prove the decomposition for the case $N = q$ with q an odd prime. We can prove the general case by induction. First assume $\chi = 1$. Suppose that $f \in S_{k+1/2}(1)$ and $f|U(q^2) \in S_{k+1/2}(1)$. We may assume that f is an eigenfunction of all Hecke operators $\mathrm{T}(p) := \mathrm{T}_{1,k,1}(p)$. To prove the decomposition we must show that $f = 0$. If otherwise, since f and $f|U(q^2)$ have the same eigenvalues for all $\mathrm{T}(p)$ with $p \neq q$, we conclude that $f|U(q^2) = cf$ with some constant $c \in \mathbb{C}$ (in fact, by Theorem 6.3, a non-zero Hecke eigenform in $S(1, 2k, \text{id.})$ is completely determined up to a constant factor by prescribing all up to finitely many of its eigenvalues, so is also a non zero Hecke eigenform in $S_{k+1/2}(1)$ by Theorem 9.7).

Now let λ_q be the eigenvalue of f with respect to $\mathrm{T}(q)$ and write $f = \sum_{n=1}^{\infty} a(n)e(nz)$. Then, by the definition of $\mathrm{T}(q)$ and the fact that $f|U(q^2) = cf$, we have

$$\left(\lambda_q - c - \left(\frac{(-1)^k n}{q}\right) q^{k-1}\right) a(n) = q^{2k-1} a(n/q^2), \quad \forall n \in \mathbb{N}. \tag{6.45}$$

By Lemma 6.22 we can choose n' such that $q \nmid n'$ and $a(n') \neq 0$. We see then that

$$\lambda_q = c + \left(\frac{(-1)^k n'}{q}\right) q^{k-1}. \tag{6.46}$$

Substituting (6.46) into (6.45) we have

$$\left(\left(\frac{(-1)^k n'}{q}\right) - \left(\frac{(-1)^k n}{q}\right)\right) a(n) = q^k a(n/q^2), \quad \forall n \in \mathbb{N},$$

so that

$$f|U(q^2) = \left(\frac{(-1)^k n'}{q}\right) q^k f, \quad \forall n \equiv 0 \pmod{q^2},$$

i.e.,

$$c = \left(\frac{(-1)^k n'}{q}\right) q^k.$$

Thus by (6.46) we see that

$$|\lambda_q| = q^k + q^{k-1},$$

which is impossible by Ramanujan-Petersson-Deligne's Theorem. Thus we proved that

$$S_{k+1/2}(1) \bigcap S_{k+1/2}(1)|U(q^2) = \{0\}.$$

Hence by the definitions of new forms and old forms, we have

$$\begin{aligned} S_{k+1/2}(q) &= S^{\text{new}}_{k+1/2}(q) \oplus \left(S_{k+1/2}(1) + S_{k+1/2}(1)|U(q^2)\right) \\ &= S^{\text{new}}_{k+1/2}(q) \oplus S_{k+1/2}(1) \oplus S_{k+1/2}(1)|U(q^2) \\ &= S^{\text{new}}_{k+1/2}(q) \oplus S^{\text{new}}_{k+1/2}(1) \oplus S^{\text{new}}_{k+1/2}(1)|U(q^2). \end{aligned}$$

Thus the theorem is proved for $\chi = 1$. If χ is primitive modulo q, the theorem follows from the following facts (see Lemma 6.18 and Lemma 6.20) :

$$S_{k+1/2}(q)|U(q) = S_{k+1/2}(q,\chi), \quad S^{\text{new}}_{k+1/2}(q)|U(q) = S^{\text{new}}_{k+1/2}(q,\chi).$$

This completes the proof. $\square$

Theorem 6.11 (1) *The space* $S^{\text{new}}_{k+1/2}(N,\chi)$ *has an orthogonal basis of common eigenfunctions for all operators* $\mathrm{T}(p) := \mathrm{T}_{N,k,\chi}(p)$ *(*p *prime,* $p \nmid N$*) and* $U(p^2)$ *(*p *prime,* $p|N$*), uniquely determined up to multiplication with non-zero complex numbers, the eigenvalues corresponding to* $U(p^2)$ *with* $p|N$ *are* $\pm p^{k-1}$*. If* f *is such an eigenfunction and* λ_p *the eigenvalue corresponding to* $\mathrm{T}(p)$ *resp.* $U(p^2)$*, then there is an eigenfunction* $F \in S^{\text{new}}_{2k}(N)$*, uniquely determined up to multiplication with a non-zero complex number, which satisfies* $F|\mathrm{T}_{N,2k}(p) = \lambda_p F$ *resp.* $F|U(p^2) = \lambda_p F$ *for all primes* p *with* $p \nmid N$ *resp.* $p|N$*. Let* $f = \sum_{n=1}^{\infty} a(n)e(nz)$ *and* $F = \sum_{n=1}^{\infty} A(n)e(nz)$*, and* D *a fundamental discriminant with* $\varepsilon(-1)^k D > 0$*. Then we have*

$$L(s-k+1,\chi\chi_D)\sum_{n=1}^{\infty} a(|D|n^2)n^{-s} = a(|D|)\sum_{n=1}^{\infty} A(n)n^{-s}.$$

(2) *Let the map* $L_{D,N,k,\chi}$ *be defined by*

$$\sum_{n=1}^{\infty} b(n)e(nz) \mapsto \sum_{n=1}^{\infty}\Big(\sum_{d|n}\chi(d)\chi_D(d)d^{k-1}b(n^2|D|/d^2)\Big)e(nz).$$

Then $L_{D,N,k,\chi}$ *maps* $S_{k+1/2}(N,\chi)$ *to* $S(N,2k,\text{id.})$*,* $S^{\text{new}}_{k+1/2}(N,\chi)$ *to* $S^{\text{new}}(N,2k,\text{id.})$ *and* $S^{\pm p}_{k+1/2}(N,\chi) \bigcap S^{\text{new}}_{k+1/2}(N,\chi)$ *to* $S^{\pm p}(N,2k,\text{id.}) \bigcap S^{\text{new}}(N,2k,\text{id.})$ *with* p *any prime divisor of* N *where* $S^{\pm p}(N,2k,\text{id.}) = \{f \in S(N,2k,\text{id.}) \,|\, f|W_{p,2k,N} = \pm f\}$*. It satisfies*

$$\mathrm{T}_{N,k,\chi}(p)L_{D,N,k,\chi} = L_{D,N,k,\chi}\mathrm{T}_{N,2k,1}(p), \quad \forall p \nmid N,$$
$$U(p^2)L_{D,N,k,\chi} = L_{D,N,k,\chi}U(p), \quad \forall p|N.$$

There exists a linear combination of the $L_{D,N,k,\chi}$ *which maps* $S^{\text{new}}_{k+1/2}(N,\chi)$ *resp.*

$$S^{\pm p}_{k+1/2}(N,\chi) \bigcap S^{\text{new}}_{k+1/2}(N,\chi)$$

isomorphically onto $S^{\text{new}}(N,2k,\text{id.})$ *resp.* $S^{\pm p}(N,2k,\text{id.}) \bigcap S^{\text{new}}(N,2k,\text{id.})$*.*

Proof Since $\mathrm{T}(p)$ commutes with $U(d^2)$ for $d|N$, and since for $f \in S_{k+1/2}(N)$ we have

$$f|U(t)|\mathrm{T}(p) = f|\mathrm{T}(p)|U(t),$$

it follows that the Hecke operator $\mathrm{T}(p)$ preserves the space of old forms and so preserves also $S_{k+1/2}^{\mathrm{new}}(N,\chi)$. We now have that

$$\mathrm{Tr}(\mathrm{T}_{N,k,\chi}(n), S_{k+1/2}^{\mathrm{new}}(N,\chi)) = \mathrm{Tr}(\mathrm{T}_{N,2k}(n), S^{\mathrm{new}}(N,2k)) \tag{6.47}$$

for all $n \in \mathbb{N}$ with $(n, 2N) = 1$. In fact, this follows by induction from the decompositions:

$$S_{k+1/2}(N,\chi) = \bigoplus_{r,d \geqslant 1, rd|N} S_{k+1/2}^{\mathrm{new}}(d,\chi)|U(r^2),$$
$$S^{\mathrm{new}}(N) = \bigoplus_{r,d \geqslant 1, rd|N} S^{\mathrm{new}}(d,2k)|U(r)$$

and from the Theorem 9.7.

By (6.47) and the corresponding statement for $S^{\mathrm{new}}(N,2k)$ (see Section 6.1), we deduce that $S_{k+1/2}^{\mathrm{new}}(N,\chi)$ has an orthogonal basis of common eigenfunctions for all operators $\mathrm{T}_{N,k,\chi}(p)$ $(p \nmid 2N)$, uniquely determined up to multiplication with non-zero complex numbers. Since $\mathrm{T}_{N,k,\chi}(p)$ $(p \nmid 2N)$, $U(p^2)(p|N)$ and $\mathrm{T}_{N,k,\chi}(2)$ commute, so these functions are also eigenfunctions of $U(p^2)(p|N)$ and $\mathrm{T}_{N,k,\chi}(2)$. Furthermore, by Theorem 6.9 and in particular the fact that $w_{N,p,k+1/2,\chi}$ is an involution shows that the eigenvalues with respect to $U(p^2)(p|N)$ are $\pm p^{k-1}$. Now let $f = \sum_{n=1}^{\infty} a(n)e(nz)$ be an eigenfunction and assume that $f|\mathrm{T}(p) = \lambda_p f$ resp. $f|U(p^2) = \lambda_p f$ for $p \nmid N$ resp. $p|N$. Then a formal computation as in Lemma 5.40 and Theorem 5.23 shows that

$$L(s-k+1, \chi\chi_D) \sum_{n=1}^{\infty} a(|D|n^2) = a(|D|) \prod_p \left(1 - \lambda_p p^{-s} + \left(\frac{N}{p}\right)^2 p^{2k-1-2s}\right)^{-1}$$

for every fundamental discriminant D with $\varepsilon(-1)^k D > 0$.

Let us show the assertions about the maps $L_D := L_{D,N,k,\chi}$. Note that the Hecke operators $\mathrm{T}_{N,k,\chi}(p)$ and $\mathrm{T}_{N,2k,\mathrm{id.}}(p)$ act in a natural way on the formal power series in $q = e(z)$. It is clear that for a formal power series $f = \sum_{\varepsilon(-1)^k n \equiv 0,1 \bmod 4} a(n)q^n$, we have

$$f|\mathrm{T}_{N,k,\chi}(p)|L_D = f|L_D|\mathrm{T}_{N,2k,\mathrm{id.}}(p), \quad \forall p \nmid N,$$
$$f|U(p^2)|L_D = f|L_D|U(p), \quad \forall p|N,$$

by a formal computation.

The other assertions will be shown first under the assumption that $D \equiv 0 \pmod 4$. Write $D = 4t$ with t square free and $t \equiv 2, 3 \pmod 4$. For

$$f = \sum_{n=1}^{\infty} a(n)e(nz) \in S_{k+1/2}(N,\chi),$$

put

$$f|L_{t,4N,k,\chi_1} = \sum_{n=1}^{\infty} \left(\sum_{d|n} \left(\frac{4\varepsilon t}{d}\right) \chi(d) d^{k-1} a(n^2|t|/d^2) \right) e(nz).$$

Then $f|L_{t,4N,k,\chi_1}$ is a cusp form of weight $2k$ on $\Gamma_0(2N)$ by the results of Chapter 8. Since $f \in S_{k+1/2}(N,\chi)$, the nth Fourier coefficients of $f|L_{t,4N,k,\chi_1}$ are zero for any odd n. Hence the function $(f|L_{t,4N,k,\chi_1})|U(2) = f|L_{D,N,k,\chi}$ is in $S(N, 2k, \text{id.})$.

If $f \in S^{\text{new}}_{k+1/2}(N,\chi)$ is a Hecke eigenfunction, then from Theorem 6.9 we see that

$$f|U(p^2) = \pm p^{k-1} f, \quad \forall\ p|N.$$

Therefore $F = f|L_D$ is a Hecke eigenform in $S(N, 2k, \text{id.})$ with $F|U(p) = \pm p^{k-1} F$ for all $p|N$, and this implies that F must be in $S^{\text{new}}(N, 2k, \text{id.})$ by the results in Section 6.1.

That L_D maps $S^{\pm p}_{k+1/2}(N,\chi) \bigcap S^{\text{new}}_{k+1/2}(N,\chi)$ to $S^{\pm p}(N, 2k, \text{id.}) \bigcap S^{\text{new}}(N, 2k, \text{id.})$ follows from Theorem 6.9, the identity $U(p^2)L_D = L_D U(p)$ and the fact that $U(p) = -p^{k-1} W_{p,N,2k}$ on $S^{\text{new}}(N, 2k, \text{id.})$.

We shall now prove that there is a linear combination of L_D with $D \equiv 0 \pmod 4$ which gives an isomorphism of $S^{\text{new}}_{k+1/2}(N,\ \chi)$ onto $S^{\text{new}}(N,\ 2k,\ \text{id.})$. Now suppose that $f \in S^{\text{new}}_{k+1/2}(N,\ \chi)$ is a non-zero Hecke eigenfunction. We declare that there is a fundamental discriminant $D \equiv 0 \pmod 4$ with $\varepsilon(-1)^k D > 0$ such that the Fourier coefficient of f at $e(|D|z)$ is non-zero. Otherwise, then the n-th Fourier coefficients of $g = f|U(4)$ are zero for all $n \equiv 2 \pmod 4$, and so that g is in $S_{k+1/2}(N,\ \chi)$ by Lemma 6.21. It follows that $g = cf$ for some constant c. In fact, by Theorem 9.7 and identity (6.47), we see that there exists an isomorphism $\psi : S_{k+1/2}(N,\ \chi) \to S(N,\ 2k,$ id.) which maps new forms onto new forms and $\psi \mathrm{T}_{N,k+1/2,\chi}(p) = \mathrm{T}_{N,2k}(p)\psi$ for all primes $p \nmid 2N$. So $f|\psi$ is a new form with the same eigenvalues as $g|\psi$ for all Hecke operators $\mathrm{T}_{N,2k}(p)$ with $p \nmid 2N$, and so that $g|\psi \in \mathbb{C} f|\psi$ by the results in Section 6.1. This shows that $g = cf$ for some constant c. Now note that f is an eigenfunction of $\mathrm{T}_{N,k,\chi}(2)$. Denote by λ_2 the corresponding eigenvalue, then similar to the proof of Theorem 6.10, we have

$$|\lambda_2| = 2^k + 2^{k-1},$$

which contradicts the Ramanujan-Petersson-Deligne Theorem. Thus we proved the above claim.

Let $f_1, f_2, \cdots, f_r \in S_{k+1/2}(N,\chi)$ be an orthogonal basis of common eigenfunctions of the operators $\mathrm{T}_{N,k,\chi}(p)(p \nmid N)$ resp. $U(p^2)(p|N)$, and write $f_i = \sum_{n=1}^{\infty} a_i(n) e(nz)$. For every i find a fundamental discriminant $D_i \equiv 0 \pmod 4$ with $\varepsilon(-1)^k D_i > 0$ and $a_i(|D_i|) \neq 0$. Then the polynomial

$$P(x_1, x_2, \cdots, x_r) = \prod_{1 \leqslant i \leqslant r} (a_i(|D_1|)x_1 + \cdots + a_i(|D_r|)x_r)$$

is non-zero. Choose $c_1, \cdots, c_r \in \mathbb{C}$ such that $P(c_1, \cdots, c_r) \neq 0$ and put

$$L_{N,k,\chi} = \sum_i c_i L_{D_i,N,k,\chi}.$$

Then it is immediate that $L_{N,k,\chi}$ is an isomorphism of $S^{\text{new}}_{k+1/2}(N,\chi)$ onto $S^{\text{new}}(N, 2k, \text{id.})$. By Lemma 6.18 and the fact that $S^{\pm p}_{k+1/2}(N,\chi)$ is the (± 1)-eigenspace of the involution $w_{p,k+1/2,N,\chi}$, we see that $L_{N,k,\chi}$ maps $S^{\pm p}_{k+1/2}(N,\chi) \bigcap S^{\text{new}}_{k+1/2}(N,\chi)$ onto $S^{\pm p}(N, 2k, \text{id.}) \bigcap S^{\text{new}}(N, 2k, \text{id.})$.

Finally we must prove the assertions about $L_{D,N,k,\chi}$ for $D \equiv 1 \pmod 4$. It is enough to show that $L_{D,N,k,\chi}$ maps $S^{\text{new}}_{k+1/2}(N,\chi)$ to $S^{\text{new}}(N, 2k, \text{id.})$. In fact, for any prime divisor $l|N$, it is easy to verify that

$$L_{D,N/l,k,\chi} = L_{D,N,k,\chi}\left(1 - \left(\frac{D}{l}\right) l^{k-1} V(l)\right),$$
$$U(t) L_{D,N,k,\chi} = L_{D_0,N,k,\text{id.}} U((D,t)^2),$$

where $V(l)$ is the translation operator defined by $(f|V(l))(z) = f(lz)$ and $\left(\frac{D_0}{*}\right)$ is the primitive character induced by $\left(\frac{D}{*}\right)\chi$. It then follows inductively that $S_{k+1/2}(N,\chi)$ is mapped to $S(N, 2k, \text{id.})$. And the same argument as in the case $D \equiv 0 \pmod 4$ shows that $S^{\pm p}_{k+1/2}(N,\chi) \bigcap S^{\text{new}}_{k+1/2}(N,\chi)$ is mapped to $S^{\pm p}(N, 2k, \text{id.}) \bigcap S^{\text{new}}(N, 2k, \text{id.})$.

Now let F be a normalized eigenform in $S^{\text{new}}(N, 2k, \text{id.})$ with $F|\mathrm{T}_{N,2k}(p) = \lambda_p F$ resp. $F|U(p) = \lambda_p F$ for all primes $p \nmid N$ resp. $p|N$. Then $F = \sum_{n=1}^{\infty} \lambda_n e(nz)$ and λ_n is determined by

$$\sum_{n=1}^{\infty} \lambda_n n^{-s} = \prod_p (1 - \lambda_p p^{-s} + \chi_N(p)^2 p^{2k-1-2s})^{-1}.$$

Write $\phi_{N,k,\chi}$ for the inverse of $L_{N,k,\chi}$ and put $G = F|\phi_{N,k,\chi} L_{D,N,k,\chi}$. Then G is a power series in $q = e(z)$ which converges on $\mathbb{H}$ and satisfies $G|\mathrm{T}_{N,2k}(p) = \lambda_p G$ resp. $G|U(p) = \lambda_p G$ for all primes $p \nmid N$ resp. $p|N$. Hence it follows that the coefficient of G at $e(nz)$ equals $c\lambda_n$ with c the first Fourier coefficient of G. Thus we have that $(F|\phi_{N,k,\chi})|L_{D,N,k,\chi} = cF$. This shows that $L_{D,N,k,\chi}$ maps $S^{\text{new}}_{k+1/2}(N,\chi)$ to $S^{\text{new}}(N, 2k, \text{id.})$. This completes the proof. □

Corollary 6.4 *Let N_1 and N_2 be two square free positive integers, f_1 and f_2 two new forms in $S^{\text{new}}_{k+1/2}(N_1, \omega_1)$ and $S^{\text{new}}_{k+1/2}(N_2, \omega_2)$ respectively such that f_1 and f_2 have the same eigenvalues with respect to infinitely many operators $\mathrm{T}(p)$ for $(p, N_1N_2) = 1$. Then $N_1 = N_2$ and $f_1 = cf_2$ with some constant c.*

Proof This is a direct conclusion of Theorem 6.11 and Theorem 6.8. □

6.3 Dimension Formulae for the Spaces of New Forms

In this section we shall give some dimension formulae of the spaces of new forms. Recall first the following result:

Theorem 6.12 *Let k be any even positive integer and N a positive integer. Then we have*

$$d_0(N,k)=\frac{k-1}{12}Ns_0(N)-\frac{1}{2}\nu_\infty(N)+c_2(k)\nu_2(N)+c_3(N)\nu_3(N)+\delta_{1,k/2},$$

where $d_0(N,k)$ is the dimension of the space of cusp forms with weight k on the group $\Gamma_0(N)$, $\delta_{x,y}$ is zero or 1 according to $x=y$ or $x\neq y$ respectively, and the functions s_0, ν_∞, ν_2, ν_3, c_2 and c_3 are defined as follows:

s_0 : *the multiplicative function defined by* $s_0(p^t)=1+\dfrac{1}{p}$ *for all* $t\geqslant 1$;

ν_∞ : *the multiplicative function defined by*

$$\nu_\infty(p^t)=\begin{cases}2p^{(t-1)/2}, & \textit{if } t \textit{ is odd},\\ p^{t/2}+p^{t/2-1}, & \textit{if } t \textit{ is even}.\end{cases}$$

ν_2 : *the multiplicative function defined by*

$$\nu_2(p^t)=\begin{cases}1, & \textit{if } p=2, t=1,\\ 0, & \textit{if } p=2, t\geqslant 2,\\ 2, & \textit{if } p\equiv 1(4), t\geqslant 1,\\ 0, & \textit{if } p\equiv 3(4), t\geqslant 1\end{cases}$$

ν_3 : *the multiplicative function defined by*

$$\nu_3(p^t)=\begin{cases}1, & \textit{if } p=3, t=1,\\ 0, & \textit{if } p=3, t\geqslant 2,\\ 2, & \textit{if } p\equiv 1(3), t\geqslant 1,\\ 0, & \textit{if } p\equiv 2(3), t\geqslant 1.\end{cases}$$

c_2 : *the function defined by* $c_2(k)=\dfrac{1}{4}+\left\lfloor\dfrac{k}{4}\right\rfloor$;

c_3 : *the function defined by* $c_3(k)=\dfrac{1}{3}+\left\lfloor\dfrac{k}{3}\right\rfloor$.

Proof This is a direct conclusion of the dimension formula of the space of cusp forms with integral weight in Section 4.1. □

We now denote by $d_0^{\text{new}}(N,k)$ the dimension of the space of new forms with weight k on the group $\Gamma_0(N)$.

Then we have the following

Theorem 6.13 *Let k be any even positive integer and N a positive integer. Then*

$$\begin{aligned} d_0^{\text{new}}(N,k) = \frac{k-1}{12} N s_0^{\text{new}}(N) - \frac{1}{2}\nu_\infty^{\text{new}}(N) + c_2(k)\nu_2^{\text{new}}(N) \\ + c_3(k)\nu_3^{\text{new}}(N) + \delta_{1,k/2}\mu(N), \end{aligned}$$

where the function $c_2, c_3, \delta_{1,k/2}$ are as in Theorem 6.12, μ is the Moebius function and s_0^{new}, ν_∞^{new}, ν_2^{new}, ν_3^{new} are defined as follows:

s_0^{new} : *the multiplicative function defined by*

$$s_0^{\text{new}}(p^t) = \begin{cases} 1 - \dfrac{1}{p}, & \text{if } t = 1, \\ 1 - \dfrac{1}{p} - \dfrac{1}{p^2}, & \text{if } t = 2, \\ \left(1 - \dfrac{1}{p}\right)\left(1 - \dfrac{1}{p^2}\right), & \text{if } t \geqslant 3. \end{cases}$$

ν_∞^{new} : *the multiplicative function defined by*

$$\nu_\infty^{\text{new}} = \begin{cases} 0, & \text{if } t \text{ is odd}, \\ p-2, & \text{if } t = 2, \\ p^{t/2-2}(p-1)^2, & \text{if } t \geqslant 4 \text{ even}. \end{cases}$$

ν_2^{new} : *the multiplicative function defined by*

$$\nu_2^{\text{new}}(p^t) = \begin{cases} -1, & \text{if } p = 2, t = 1 \text{ or } 2, \\ 1, & \text{if } p = 2, t = 3, \\ 0, & \text{if } p = 2, t \geqslant 4, \\ 0, & \text{if } p \equiv 1(4), t = 1 \text{ or } t \geqslant 3, \\ -1, & \text{if } p \equiv 1(4), t = 2, \\ -2, & \text{if } p \equiv 3(4), t = 1, \\ 1, & \text{if } p \equiv 3(4), t = 2, \\ 0, & \text{if } p \equiv 3(4), t \geqslant 3. \end{cases}$$

ν_3^{new} : *the multiplicative function defined by*

$$\nu_3^{\text{new}}(p^t) = \begin{cases} -1, & \text{if } p = 3, t = 1 \text{ or } 2, \\ 1, & \text{if } p = 3, t = 3, \\ 0, & \text{if } p = 3, t \geqslant 4, \\ 0, & \text{if } p \equiv 1(3), t = 1 \text{ or } t \geqslant 3, \\ -1, & \text{if } p \equiv 1(3), t = 2, \\ -2, & \text{if } p \equiv 2(3), t = 1, \\ 1, & \text{if } p \equiv 2(3), t = 2, \\ 0, & \text{if } p \equiv 2(3), t \geqslant 3. \end{cases}$$

Proof We recall first the following facts about arithmetic functions: the set of arithmetic functions $f : \mathbb{N} \to \mathbb{C}$ forms a ring under the usual addition of functions and the Dirichlet convolution as the multiplication operation:

$$(f * g)(n) = \sum_{d|n} f(d)g(n/d) \tag{6.48}$$

for any two arithmetic functions f and g. And the function $\delta(n) := \delta_{1,n}$ is the multiplicative identity of the ring. And the set of all multiplicative functions f with $f(1) \neq 0$ forms a multiplicative subgroup under the Dirichlet convolution. In fact, if $f(1) \neq 0$, then the function g defined as follows:

$$g(n) = \begin{cases} \dfrac{1}{f(1)}, & \text{if } n = 1, \\ -\dfrac{1}{f(1)} \displaystyle\sum_{d|n, d \neq n} f(n/d)g(d), & \text{if } n > 1 \end{cases} \tag{6.49}$$

is the inverse of f. By Moebius inversion formula we see that the Moebius function μ is the inverse of the function $1(n)$ which takes the value 1 at all positive integers:

$$(\mu * 1)(n) = \sum_{d|N} \mu(d) = \delta(n).$$

And in general we use the following Moebius inversion formula: for any two arithmetic functions f and g, we have

$$f(n) = \sum_{d|n} g(d), \quad \forall\, n \in \mathbb{N}$$

if and only if

$$g(n) = \sum_{d|n} \mu(n/d) f(d), \quad \forall\, n \in \mathbb{N}.$$

In fact, we have

$$f(n) = \sum_{d|n} g(d) = (1 * g)(n)$$

if and only if

$$\begin{aligned} g(n) &= ((\mu * 1) * g)(n) \\ &= (\mu * (1 * g))(n) \\ &= (\mu * f)(n) \\ &= \sum_{d|n} \mu(n/d) f(d). \end{aligned}$$

From the results in Section 6.1 we have

$$S(N, k) = \bigoplus_{l|N} \bigoplus_{m|N/l} S^{\text{new}}(l, k)|V(m),$$

where $V(m)$ is the translation operator defined by $f|V(m) = f(mz)$ which is an injection from $S(l,k)$ to $S(N,k)$. Therefore we have

$$d_0(N,k) = \sum_{l|N} \sum_{m|N/l} d_0^{\text{new}}(l,k) = \sum_{l|N} d_0^{\text{new}}(l,k)\tau(N/l), \tag{6.50}$$

where $\tau(n) = \sum_{d|n} 1$ is the number of positive divisors of n. In terms of Dirichlet convolution, we see that from (6.50)

$$d_0 = d_0^{\text{new}} * \tau$$

holds for any fixed k. Let λ be the inverse of τ. Since $\tau = 1 * 1$, we see that

$$\lambda = \tau^{-1} = (1*1)^{-1} = 1^{-1} * 1^{-1} = \mu * \mu.$$

Hence, from (6.48), λ is the multiplicative function defined by

$$\lambda(p^t) = \begin{cases} -2, & \text{if } t = 1, \\ 1, & \text{if } t = 2, \\ 0, & \text{if } t \geqslant 3 \end{cases}$$

Therefore we see that $d_0^{\text{new}} = d_0 * \lambda$, and so that

$$\begin{aligned} d_0^{\text{new}}(N,k) = &\frac{k-1}{12}((i_0 s_0) * \lambda)(N) - \frac{1}{2}(\nu_\infty * \lambda)(N) \\ &+ c_2(k)(\nu_2 * \lambda)(N) + c_3(k)(\nu_3 * \lambda)(N) + \delta_{1,k/2}(1 * \lambda)(N) \end{aligned}$$

from Theorem 6.12 and the fact that the set of arithmetic functions forms a ring under the usual addition and the Dirichlet convolution, where $i_0(n) = n$ is the identity function on $\mathbb{N}$. But we see that $1 * \lambda = 1 * (\mu * \mu) = (1 * \mu) * \mu = \delta * \mu = \mu$, and $\nu_\infty * \lambda$, $\nu_2 * \lambda$, $\nu_3 * \lambda$ are multiplicative functions which equal ν_∞^{new}, ν_2^{new}, ν_3^{new} respectively by (6.48) and the definitions of ν_∞^{new}, ν_2^{new}, ν_3^{new}. Finally we see that

$$i_0(p^t)s_0(p^t) * \lambda(p^t) = \sum_{m=0}^{t} p^m s_0(p^m)\lambda(p^{t-m}) = p^t s_0^{\text{new}}(p^t),$$

i.e. the multiplicative function $((i_0 s_0) * \lambda)(N) = N s_0^{\text{new}}(N)$. This completes the proof. □

By Theorem 6.11, there exists a linear combination of the Shimura lifting $L_{D,N,k,\chi}$ which maps $S_{k+1/2}^{\text{new}}(N,\chi)$ isomorphically onto $S^{\text{new}}(N,2k)$, so that

$$\dim(S_{k+1/2}^{\text{new}}(N,\chi)) = \dim\left(S^{\text{new}}(N,2k)\right).$$

Hence by Theorem 6.13 we have the following:

Corollary 6.5 *Let k be a positive integer, N a square free positive integer and χ a quadratic character modulo N. Then*

$$\begin{aligned} d_0^{\mathrm{new}}(N, k+1/2) = & \frac{2k-1}{12} N s_0^{\mathrm{new}}(N) - \frac{1}{2}\nu_\infty^{\mathrm{new}}(N) \\ & + c_2(2k)\nu_2^{\mathrm{new}}(N) + c_3(2k)\nu_3^{\mathrm{new}}(N) + \delta_{1,k}\mu(N), \end{aligned}$$

where $d_0^{\mathrm{new}}(N, k+1/2) := \dim(S_{k+1/2}^{\mathrm{new}}(N, \chi))$.

Chapter 7

Construction of Eisenstein Series

7.1 Construction of Eisenstein Series with Weight $\geqslant 5/2$

In this section we study the following two problems: construct a basis of the Eisenstein space $\mathcal{E}(4N, k+1/2, \chi_l)$ which are eigenfunctions for all Hecke operators, and calculate their values at all cusp points.

Now we introduce some notations as in Chapter 2. For any odd positive integer k, let $\lambda = \dfrac{k-1}{2}$, and

$$\lambda_k(n, 4N) = L_{4N}(2\lambda, \mathrm{id.})^{-1} L_{4N}(\lambda, \chi_{(-1)^\lambda n}) \beta_k(n, \chi_N, 4N)$$

$$A_k(2,n) = \begin{cases} 2^{-k}(1+(-1)^\lambda \mathrm{i}) \left(\dfrac{1-2^{(2-k)(\nu_2(n)-1)/2}}{1-2^{2-k}} - 2^{(2-k)(\nu_2(n)-1)/2} \right), \\ \qquad \text{if } 2 \nmid \nu_2(n), \\ 2^{-k}(1+(-1)^\lambda \mathrm{i}) \left(\dfrac{1-2^{(2-k)\nu_2(n)/2}}{1-2^{2-k}} - 2^{(2-k)\nu_2(n)/2} \right), \\ \qquad \text{if } 2 | \nu_2(n), (-1)^\lambda n/2^{\nu_2(n)} \equiv -1 \pmod 4, \\ 2^{-k}(1+(-1)^\lambda \mathrm{i}) \left(\dfrac{1-2^{(2-k)\nu_2(n)/2}}{1-2^{2-k}} + 2^{(2-k)\nu_2(n)/2} \right. \\ \qquad \left. \left(1 + 2^{(3-k)/2} \left(\dfrac{(-1)^\lambda n/2^{\nu_2(n)}}{2} \right) \right) \right), \\ \qquad \text{if } 2 | \nu_2(n), (-1)^\lambda n/2^{\nu_2(n)} \equiv 1 \pmod 4, \end{cases}$$

$$A_k(p,n) = \begin{cases} \dfrac{(p-1)(1-p^{(2-k)(\nu_p(n)-1)/2})}{p(p^{k-2}-1)} \\ \qquad - p^{(2-k)(\nu_p(n)+1)/2-1}, & \text{if } 2 \nmid \nu_p(n), \\ \dfrac{(p-1)(1-p^{(2-k)\nu_p(n)/2})}{p(p^{k-2}-1)} \\ \qquad + \left(\dfrac{(-1)^\lambda n/p^{\nu_p(n)}}{p} \right) p^{(2-k)(\nu_p(n)+1)/2-1/2}, & \text{if } 2 | \nu_p(n), \end{cases}$$

$$L_N(s,\chi) = \sum_{(n,N)=1}^{\infty} \chi(n)n^{-s} = \prod_{p\nmid N}(1-\chi(p)p^{-s})^{-1},$$

$$\beta_k(n,\chi_N,4N) = \sum_{\substack{(ab)^2|n,(ab,2N)=1\\ a,b \text{ positive integers}}} \mu(a)\left(\frac{(-1)^\lambda n}{a}\right) a^{-\lambda}b^{2-k},$$

$$\lambda_k'(n,4N) = \frac{(-2\pi\mathrm{i})^{k/2}}{\Gamma(k/2)}\lambda_k(n,4N).$$

We define functions $g_k(\chi_l,4m,4N)(z)$ $(m|N)$ and $g_k(\chi_l,m,4N)(z)$ $(m|N)$ as follows: For $k \geqslant 5$,

$$g_k(\chi_l,4N,4N)(z) = 1 + \sum_{n=1}^{\infty}\lambda_k'(ln,4N)\prod_{p|2N}(A_k(p,ln)-\eta_p)(ln)^{k/2-1}q^n,$$

$$g_k(\chi_l,4m,4N)(z) = \sum_{n=1}^{\infty}\lambda_k'(ln,4N)\prod_{p|2m}(A_k(p,ln)-\eta_p)(ln)^{k/2-1}q^n, \quad \forall\ N \neq m|N,$$

$$g_k(\chi_l,m,4N)(z) = \sum_{n=1}^{\infty}\lambda_k'(ln,4N)\prod_{p|m}(A_k(p,ln)-\eta_p)(ln)^{k/2-1}q^n, \quad \forall\ m|N,$$

where $q = e(z) = \mathrm{e}^{2\pi\mathrm{i}z}$, $\eta_2 = \dfrac{1+(-1)^\lambda \mathrm{i}}{2^k-4}$ and $\eta_p = \dfrac{p-1}{p(p^{k-2}-1)}$ for $p \neq 2$.

Lemma 7.1 *Let k be a positive odd integer, n a positive integer and p a prime, D a square free positive integer and $m|D$. Then*

(I) $\lambda_k(n,4m) = \lambda_k(n,4D)\prod_{p|D/m}(1+A_k(p,n))$,

(II) $A_k(p,p^2n)-\eta_p = p^{k-2}(A_k(p,n)-\eta_p)$.

Proof The second equality is clear from the definition of $A_k(p,n)$. The first equality can be proved from the definition of $\lambda_k(n,4D)$ and the properties of $\beta_k(n,\chi_D,4D)$. We omit the details. □

Theorem 7.1 *Let $k \geqslant 5$ be an odd positive integer, D a square-free positive odd integer and l a divisor of D. Then the functions*

$$\{g_k(\chi_l,4m,4D), g_k(\chi_l,m,4D) \mid m|D\}$$

constitute a basis of $\mathcal{E}(4D,k/2,\chi_l)$ and are eigenfunctions for all Hecke operators, and

$$g_k(\chi_l,j,4D)(z)|T(p^2) = \begin{cases} g_k(\chi_l,j,4D)(z), & \text{if } p|j,\\ p^{k-2}g_k(\chi_l,j,4D)(z), & \text{if } p|8D/j,\\ (1+p^{k-2})g_k(\chi_l,j,4D)(z), & \text{if } p\nmid 2D,\end{cases}$$

where $j = m$ or $4m, m|D$.

Proof By the definition of a Hecke operator, we know that $g_k(\chi_l, j, 4D) = g_k(\text{id.}, j, 4D)|T(l)$. Hence we only need to prove Theorem 7.1 for $l = 1$. We first show that $g_k(\text{id.}, j, 4D)$ belongs to $\mathcal{E}(4D, k/2, \text{id.})$.

By Chapter 2, for square free odd positive integer D, the following functions belong to $\mathcal{E}(4D, k/2, \text{id.})$

$$\begin{aligned}
E_k(\text{id.}, 4D)(z) &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4D)} j(\gamma, z)^{-k} \\
&= 1 + \sum_{n=1}^{\infty} \lambda'_k(n, 4D) \prod_{p|2D} A_k(p, n) n^{k/2-1} q^n, \\
E'_k(\chi_D, 4D)(z) &= z^{-k/2} E_k(\chi_D, 4D)\left(-\frac{1}{4Dz}\right) \\
&= \sum_{n=1}^{\infty} \lambda'_k(n, 4D) n^{k/2-1} q^n,
\end{aligned}$$

We introduce the following functions:

$$\begin{aligned}
F_k(4D)(z) &= E_k(\text{id.}, 4D)(z) = 1 + \sum_{n=1}^{\infty} \lambda'_k(n, 4D) \prod_{p|2D} A_k(p, n) n^{k/2-1} q^n, \\
F_k(4m)(z) &= \sum_{n=1}^{\infty} \lambda'_k(n, 4D) \prod_{p|2m} A_k(p, n) n^{k/2-1} q^n, \\
F_k(m) &= \sum_{n=1}^{\infty} \lambda'_k(n, 4D) \prod_{p|m} A_k(p, n) n^{k/2-1} q^n.
\end{aligned} \tag{7.1}$$

Since Lemma 7.1, we see that for any $m|D$,

$$\begin{aligned}
E_k(\text{id.}, 4m)(z) &= 1 + \sum_{n=1}^{\infty} \lambda'_k(n, 4m) \prod_{p|2m} A_k(p, n) n^{k/2-1} q^n \\
&= 1 + \sum_{n=1}^{\infty} \lambda'_k(n, 4D) \prod_{p|2m} A_k(p, n) \prod_{p|D/m} (A_k(p, n) + 1) n^{k/2-1} q^n, \\
E'_k(\chi_m, 4m)(z) &= \sum_{n=1}^{\infty} \lambda'_k(n, 4m) n^{k/2-1} q^n \\
&= \sum_{n=1}^{\infty} \lambda'_k(n, 4D) \prod_{p|D/m} (A_k(p, n) + 1) n^{k/2-1} q^n.
\end{aligned} \tag{7.2}$$

Because

$$\begin{aligned}
\prod_{p|2m} A_k(p, n) &= \prod_{p|2m} A_k(p, n) \prod_{p|D/m} (1 + A_k(p, n) - A_k(p, n)) \\
&= \sum_{d|D/m} \mu(d) \prod_{p|2md} A_k(p, n) \prod_{p|D/(md)} (1 + A_k(p, n)),
\end{aligned}$$

$$\prod_{p|m} A_k(p,n) = \sum_{d|m} \mu(d) \prod_{p|m/d} (1 + A_k(p,n)). \tag{7.3}$$

By (7.1)–(7.3), we see that

$$F_k(4m) = \sum_{d|D/m} \mu(d) E_k(\text{id.}, 4md) \in E_{k/2}(4D, \text{id.}),$$
$$F_k(m) = \sum_{d|m} \mu(d) E_k'(\chi_{dD/m}, 4dD/m) \in E_{k/2}(4D, \text{id.}).$$

But

$$\begin{aligned} g_k(\text{id.}, 4m, 4D) &= \sum_{d|m} \mu(d) \prod_{p|d} \eta_p F_k(4m/d) - \sum_{d|m} \mu(d) \prod_{p|2d} \eta_{2p} F_k(m/d), \\ g_k(\text{id.}, m, 4D) &= \sum_{d|m} \mu(d) \prod_{p|d} \eta_p F_k(m/d), \end{aligned} \tag{7.4}$$

which implies that $g_k(\text{id.}, 4m, 4D)$ and $g_k(\text{id.}, m, 4D)$ belong to $\mathcal{E}(4D,\ k/2,\ \text{id.})$.

We now want to prove the equalities in Theorem 7.1. We recall the definition of Hecke operators: for any $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(4D, k/2, \omega)$, we have that $f(z)|\mathrm{T}(p^2) = \sum_{n=0}^{\infty} b(n)e(nz)$ where

$$b(n) = a(p^2n) + \omega(p)\left(\frac{(-1)^{\lambda}n}{p}\right)p^{\lambda-1}a(n) + \omega(p^2)p^{k-2}a(n/p^2),$$

where $a(n/p^2) = 0$ if $p^2 \nmid n$.

In particular, if $p|4D$, then $b(n) = a(p^2n)$. It is clear that $\beta_k(p^2n, \chi_D, 4D) = \beta_k(n, \chi_D, 4D)$ for any $p|2D$. So the first two equalities in Theorem 7.1 can easily be deduced from Lemma 7.1 (II) and the obvious fact that $A_k(p, qn) = A_k(p, n)$ if $p \nmid q$. So we only need to prove the third equality. So suppose that q is a prime with $q \nmid 2D$. We consider the action of $\mathrm{T}(q^2)$ on $f = g_k(\text{id.}, 4m, 4D)$. Denote

$$a(n) = \lambda_k'(n, 4D) \prod_{p|2m} (A_k(p,n) - \eta_p) n^{k/2-1}$$

and

$$f|\mathrm{T}(q^2) = \sum_{n=0}^{\infty} b(n)e(nz).$$

Since $q \nmid 2D$, then $A_k(p, q^2n) = A_k(p,n)$ and

$$L_{4D}(\lambda, \chi_{(-1)^{\lambda}lq^2n}) \prod_{p|2m} (A_k(p, lnq^2) - \eta_p) = L_{4D}(\lambda, \chi_{(-1)^{\lambda}n}) \prod_{p|2m} (A_k(p, ln) - \eta_p).$$

Now consider the term $\beta_k(ln, \chi_D, 4D)$. Denote $ln = \tau\sigma^2$ with τ a square free positive integer. Let $\nu_p(m)$ be the valuation of m with respect to p. Then we have that

$$\begin{aligned}\beta_k(\tau\sigma^2, \chi_D, 4D) &= \sum_{\substack{(ab)^2|\tau\sigma^2,(ab,2D)=1\\ a,b \text{ positive integers}}} \mu(a)\left(\frac{(-1)^\lambda ln}{a}\right) a^{-\lambda} b^{-k+2},\\ &= \prod_{p\nmid 2D, p|\tau} \sum_{t=0}^{(\nu_p(\tau\sigma^2)-1)/2} p^{(-k+2)t}\\ &\quad\times \prod_{p\nmid 2D\tau, p|\sigma}\left(\sum_{t=0}^{\nu_p(\tau\sigma^2)/2} p^{(-k+2)t} - \chi_{(-1)^\lambda ln}(p)p^{-\lambda}\sum_{t=0}^{\nu_p(\tau\sigma^2)/2-1} p^{(-k+2)t}\right).\end{aligned}$$

Therefore, if $\nu_q(ln) = 0$, i.e., $q \nmid ln$, then

$$\beta_k(\tau\sigma^2 q^2, \chi_D, 4D) = (1 + q^{-k+2} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda})\beta_k(\tau\sigma^2, \chi_D, 4D). \tag{7.5}$$

If $q|\tau$, then

$$\beta_k(\tau\sigma^2 q^2, \chi_D, 4D) = \left(\sum_{t=0}^{(\nu_q(\tau\sigma^2)+1)/2} q^{(-k+2)t}\right)\left(\sum_{t=0}^{(\nu_q(\tau\sigma^2)-1)/2} q^{(-k+2)t}\right)^{-1}\beta_k(\tau\sigma^2, \chi_D, 4D). \tag{7.6}$$

If $q \nmid \tau$, $q|\sigma$, then

$$\begin{aligned}\beta_k(\tau\sigma^2 q^2, \chi_D, 4D) &= \left(\sum_{t=0}^{\nu_q(\tau\sigma^2)/2+1} q^{(-k+2)t} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda}\sum_{t=0}^{\nu_q(\tau\sigma^2)/2} q^{(-k+2)t}\right)\\ &\quad\times\left(\sum_{t=0}^{\nu_q(\tau\sigma^2)/2} q^{(-k+2)t} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda}\sum_{t=0}^{\nu_q(\tau\sigma^2)/2-1} q^{(-k+2)t}\right)^{-1}\\ &\quad\times\beta_k(\tau\sigma^2, \chi_D, 4D)\\ a(n) &= \lambda_k'(n, 4D)\prod_{p|2m}(A_k(p,n) - \eta_p)n^{k/2-1}\\ &= \frac{(-2\pi i)^{k/2}}{\Gamma(k/2)}\frac{L_{4D}(\lambda, \chi_{(-1)^\lambda ln})}{L_{4D}(2\lambda, \text{id.})}\beta_k(ln, \chi_D, 4D)\\ &\quad\times\prod_{p|2m}(A_k(p, ln) - \eta_p)(ln)^{k/2-1}.\end{aligned} \tag{7.7}$$

Hence we know that the coefficient $b(n)$ of $f|\mathrm{T}(q^2)$ is

(1) If $\nu_q(ln) = 0$, then by equality (7.5)

$$\begin{aligned}b(n) &= a(q^2 n) + \chi_{(-1)^\lambda l}(q)\left(\frac{n}{q}\right)q^{\lambda-1}a(n) + q^{k-2}a(n/q^2)\\ &= (1 + q^{-k+2} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda})q^{k-2}a(n) + \chi_{(-1)^\lambda ln}(q)q^{\lambda-1}a(n)\\ &= (1 + q^{k-2})a(n).\end{aligned}$$

(2) If $\nu_q(ln) = 1$, i.e., $q|\tau$, $q \nmid \sigma$, we see by equality (7.6) that

$$\begin{aligned} b(n) &= a(q^2n) + \chi_{(-1)^\lambda l\tau}(q)q^{\lambda-1}a(n) + q^{k-2}a(n/q^2) \\ &= a(q^2n) + \chi_{(-1)^\lambda l\tau}(q)q^{\lambda-1}a(n) \\ &= a(q^2n) = (1+q^{-k+2})q^{k-2}a(n) = (1+q^{k-2})a(n). \end{aligned}$$

(3) If $q|\tau$, $q|\sigma$, then $\nu_q(ln) \geqslant 3$, we have by equality (7.6),

$$\begin{aligned} b(n) &= a(q^2n) + \chi_{(-1)^\lambda l\tau}(q)q^{\lambda-1}a(n) + q^{k-2}a(n/q^2) = a(q^2n) + q^{k-2}a(n/q^2) \\ &= \left(\sum_{s=0}^{(\nu_q(ln)+1)/2} q^{(-k+2)s}\right)\left(\sum_{s=0}^{(\nu_q(ln)-1)/2} q^{(-k+2)s}\right)^{-1} q^{k-2}a(n) \\ &\quad + q^{k-2}\left(\sum_{s=0}^{(\nu_q(ln)-3)/2} q^{(-k+2)s}\right)\left(\sum_{s=0}^{(\nu_q(ln)-1)/2} q^{(-k+2)s}\right)^{-1} a(n)q^{-(k-2)} \\ &= \left(q^{k-2}\sum_{s=0}^{(\nu_q(ln)+1)/2} q^{(-k+2)s} + \sum_{s=0}^{(\nu_q(ln)-3)/2} q^{(-k+2)s}\right) \\ &\quad \cdot \left(\sum_{s=0}^{(\nu_q(ln)-1)/2} q^{(-k+2)s}\right)^{-1} a(n) \\ &= \left(q^{k-2}\sum_{s=0}^{(\nu_q(ln)-1)/2} q^{(-k+2)s} + q^{(-k+2)(\nu_q(ln)-1)/2}\right. \\ &\quad \left. + \sum_{s=0}^{(\nu_q(ln)-1)/2} q^{(-k+2)s} - q^{(-k+2)(\nu_q(ln)-1)/2}\right) \\ &\quad \cdot \left(\sum_{s=0}^{(\nu_q(ln)-1)/2} q^{(-k+2)s}\right)^{-1} a(n) \\ &= (1+q^{k-2})a(n). \end{aligned}$$

Finally, if $q \nmid \tau, q|\sigma$, then by equality (7.7), we have that

$$\begin{aligned} b(n) &= a(q^2n) + \chi_{(-1)^\lambda l\tau}(q)q^{\lambda-1}a(n) + q^{k-2}a(n/q^2) \\ &= q^{k-2}a(n)\left(\sum_{t=0}^{\nu_q(ln)/2+1} q^{(-k+2)t} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda}\sum_{t=0}^{\nu_q(ln)/2} q^{(-k+2)t}\right) \\ &\quad \cdot \left(\sum_{t=0}^{\nu_q(ln)/2} q^{(-k+2)t} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda}\sum_{t=0}^{\nu_q(ln)/2-1} q^{(-k+2)t}\right)^{-1} \end{aligned}$$

$$+ a(n)\left(\sum_{t=0}^{\nu_q(ln)/2-1} q^{(-k+2)t} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda}\sum_{t=0}^{\nu_q(ln)/2-2} q^{(-k+2)t}\right)$$

$$\cdot\left(\sum_{t=0}^{\nu_q(ln)/2} q^{(-k+2)t} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda}\sum_{t=0}^{\nu_q(ln)/2-1} q^{(-k+2)t}\right)^{-1}$$

$$= q^{k-2}a(n)\left(1 + (q^{(-k+2)(\nu_q(ln)/2+1)} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda+(-k+2)\nu_q(ln)/2}\right)$$

$$\cdot\left(\sum_{t=0}^{\nu_q(ln)/2} q^{(-k+2)t} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda}\right)\left(\sum_{t=0}^{\nu_q(ln)/2-1} q^{(-k+2)t}\right)^{-1}$$

$$+ a(n)\left(1 - (q^{(-k+2)\nu_q(ln)/2} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda+(-k+2)(\nu_q(ln)/2-1)}\right)$$

$$\cdot\left(\sum_{t=0}^{\nu_q(ln)/2} q^{(-k+2)t} - \chi_{(-1)^\lambda l\tau}(q)q^{-\lambda}\sum_{t=0}^{\nu_q(ln)/2-1} q^{(-k+2)t}\right)^{-1}$$

$$= (1 + q^{k-2})a(n).$$

Hence we have proved that for any prime $q \nmid 2D$, $g(\chi_l, 4m, 4D)|\mathrm{T}(q^2) = (1+q^{k-2})g(\chi_l, 4m, 4D)$. Similarly, we can show that for any $q \nmid 2D$, $g(\chi_l, m, 4D)|\mathrm{T}(q^2) = (1 + q^{k-2})g(\chi_l, m, 4D)$.

Since the functions in Theorem 7.1 are eigenfunctions of Hecke operators with different eigenvalues, they are linearly independent. Thus they constitute a basis of $\mathcal{E}(4D,\ k/2,\ \chi_l)$ since the number of the functions is equal to the dimension of $\mathcal{E}(4D,\ k/2,\ \chi_l)$.

This completes the proof of Theorem 7.1. □

Theorem 7.2 *Let $k \geqslant 5$ be an odd positive integer, D a square-free positive odd integer, m, l be divisors of D, α be a divisor of m, $\delta_k = 1$ or -1 according to $k \equiv 1$ or $-1 \pmod 4$ respectively. Then*

$$V(g_k(\chi_l, 4m, 4D), 1/\alpha) = -\frac{1+\mathrm{i}^{-\delta_k}}{2^k - 4}\mu(m/\alpha)\eta_{m/\alpha}l^{k/2-1}(l,\alpha)^{-k/2+1}\varepsilon_{\alpha/(l,\alpha)}^{\delta_k}\left(\frac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right).$$

$$V(g(\chi_l, 4m, 4D), 1/(4\alpha)) = \mu(m/\alpha)\eta_{m/\alpha}l^{k/2-1}(l,\alpha)^{-k/2+1}\varepsilon_{l/(l,\alpha)}^{-\delta_k}\left(\frac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right).$$

$$V(g(\chi_l, 4m, 4D), p) = 0,\ \text{if } p \neq 1/\alpha \text{ or } 1/4\alpha (\alpha|D), p \text{ a cusp point.}$$

$$V(g(\chi_l, m, 4D), 1/\alpha) = \mathrm{i}^{-\delta_k}\mu(m/\alpha)\eta_{m/\alpha}l^{k/2-1}(l,\alpha)^{-k/2+1}\varepsilon_{\alpha/(l,\alpha)}^{\delta_k}\left(\frac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right).$$

$$V(g(\chi_l, m, 4D), p) = 0, \quad \text{if } p \neq 1/\alpha\, (\alpha|D),$$

where p is a cusp point and $V(f, p)$ is the value of f at the cusp point p, and $\eta_\alpha = \prod_{p|\alpha}\eta_p$.

Proof In order to calculate the values of functions at cusp points, we first remember the definition of the value of a function at a cusp point. Let $f(z) \in G(N,\ k/2,\ \chi_l)$, and $s = d/c$ be a cusp point of $\Gamma_0(N)$. Let $\rho = \begin{pmatrix} a & b \\ -c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, then $\rho(s) = \mathrm{i}\infty$. We call the constant term of the Fourier expansion at $z = \mathrm{i}\infty$ of $f|\rho^{-1}$ the value of f at the cusp point s. Denote it by $V(f,s)$. For $c \neq 0$, we have

$$\begin{aligned} V(f,s) &= \lim_{z\to i\infty} f\left(\frac{dz-b}{cz+a}\right)(cz+a)^{-k/2} \\ &= \lim_{z\to i\infty} f(-c^{-1}(cz+a)^{-1}+dc^{-1})(cz+a)^{-k/2} \\ &= \lim_{\tau\to 0} f(\tau+dc^{-1})(-c\tau)^{k/2}. \end{aligned} \tag{7.8}$$

In particular, for $s = 1/N$, we see that $V(f,1/N) = V(f,\mathrm{i}\infty) = \lim\limits_{z\to i\infty} f(z)$. □

An obvious, but useful fact is

Lemma 7.2 *Let $f \in G(N,k/2,\omega)$. Suppose cusp point $s_1 = d_1/c_1$ and $s_2 = d_2/c_2$ are equivalent for the group $\Gamma_0(N)$,* i.e., *there exists $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ such that $\rho(s_1) = s_2$, then*

$$V(f,s_2) = \bar{\omega}\chi_c(d)\varepsilon_d^{-k}V(f,s_1).$$

A classical result for the values of Eisenstein series $E_k(\omega,N)(z), E'_k(\omega,N)(z)$ is the following Lemma 7.3, which can be showed by the results in Chapter 2 and Lemma 7.2. Now we denote $S(N)$ a complete set of representatives of equivalence classes of cusp points for the group $\Gamma_0(N)$. In fact we can choose

$$S(N) = \{d/c \mid \ c|N, d \in (\mathbb{Z}/(c,N/c)\mathbb{Z})^* \text{ and } (d,c)=1\}.$$

Lemma 7.3 *Let $k \geqslant 5$ be an odd, ω a character modulo N. Then we have*

(1) $V(E'_k(\omega,N),1) = \mathrm{i}^{-k}$, *and for any $d/c \in S(N)$ with $c \neq 1, V(E'_k(\omega,N),d/c) = 0$;*

(2) $V(E_k(\omega,N),i\infty) = 1$, *and for any $d/c \in S(N)$ with $c \neq N, V(E_k(\omega,N),d/c) = 0$.*

We now return to our proof of Theorem 7.2. We need the following:

Lemma 7.4 *Let D be square free odd positive integer, m, l, and β are divisors of D, α a divisor of m. And suppose that $f \in G(8D,k/2,\chi_l)$ satisfies*

$$f|\mathrm{T}(p^2) = f \quad \textit{for all prime } p|m,$$
$$f|\mathrm{T}(p^2) = p^{k-2}f \quad \textit{for all prime } p|Dm^{-1}.$$

Then we have

$$V(f,1/\alpha) = \mu(\alpha)\eta_\alpha^{-1}(\alpha,l)^{-k/2+1}\varepsilon_{\alpha/(\alpha,l)}^{\delta_k}\left(\frac{l/(\alpha,l)}{\alpha/(\alpha,l)}\right)V(f,1),$$

$$V(f,1/(4\alpha)) = \mu(\alpha)\eta_\alpha^{-1}(\alpha,l)^{-k/2+1}\varepsilon_{l/(\alpha,l)}^{\delta_k}\varepsilon_l^{-1}\left(\frac{\alpha/(\alpha,l)}{l/(\alpha,l)}\right)V(f,1/4),$$
$$V(f,1/(8\alpha)) = \mu(\alpha)\eta_\alpha^{-1}(\alpha,l)^{-k/2+1}\varepsilon_{l/(\alpha,l)}^{\delta_k}\varepsilon_l^{-1}\left(\frac{2}{(\alpha,l)}\right)\left(\frac{\alpha/(\alpha,l)}{l/(\alpha,l)}\right)V(f,1/8),$$

where $\eta_\alpha = \prod_{p|\alpha}\eta_p$, $\delta_k = 1$ *or* -1 *according to* $k \equiv 1$ *or* $-1 \pmod 4$ *respectively. And for* $(\beta, D/m) \neq 1, r = 0,1,2,3$, *we have that* $V(f,1/(2^r\beta)) = 0$.

Proof We only prove the Lemma 7.4 for the case $k \equiv 3 \pmod 4$. For the case $k \equiv 1 \pmod 4$ it can be proved by a similar method. We first prove the last result. Suppose p prime, $p|(\beta, D/m)$. By our assumption in the lemma we have $f|\mathrm{T}(p^2) = p^{k-2}f$ and by the definition of Hecke operators, we see that

$$p^{k-2}f\left(z+\frac{1}{2^r\beta}\right) = p^{-2}\sum_{b=1}^{p^2} f\left(\frac{z}{p^2}+\frac{1+2^r\beta b}{2^r\beta p^2}\right).$$

Since $(1+2^r\beta b, 2^r\beta p^2) = 1$, the rational number $\dfrac{1+2^r\beta b}{2^r\beta p^2}$ is a cusp point. By equality (7.8), we know

$$p^{k-2}V\left(f,\frac{1}{2^r\beta}\right) = p^{-2}\sum_{b=1}^{p^2} V\left(f,\frac{1+2^r\beta b}{2^r\beta p^2}\right). \tag{7.9}$$

Since $(2^r\beta p^2, 8D) = 2^r\beta$ and $(2^r\beta, 8D/(2^r\beta)) = 1$ or 2 according to $r = 0$, 3 or $r = 1$, 2, we know that the cusp point $\dfrac{1+2^r\beta b}{2^r\beta p^2}$ is equivalent to the cusp point $\dfrac{1}{2^r\beta}$ for the group $\Gamma_0(8D)$. Therefore there exists a matrix $\begin{pmatrix} a & e \\ c & d\end{pmatrix} \in \Gamma_0(8D)$ such that

$$\begin{pmatrix} a & e \\ c & d\end{pmatrix}\begin{pmatrix} 1 \\ 2^r\beta\end{pmatrix} = \begin{pmatrix} 1+2^r\beta b \\ 2^r\beta p^2\end{pmatrix}.$$

Hence $a + 2^r\beta e = 1 + 2^r\beta b$, $c + 2^r\beta d = 2^r\beta p^2$. Noting $ad - ce = 1$ and $8D|c$, we have that $a \equiv d \equiv 1 \pmod{2^r\beta}$, and $d \equiv p^2 \pmod{8D/(2^r\beta)}$. This shows that for $r = 0$, 1, 2, 3, we have $\varepsilon_d = 1$ and

$$\left(\frac{c}{d}\right) = \left(\frac{2^r\beta p^2 - 2^r\beta d}{d}\right) = \left(\frac{2^r\beta}{d}\right) = 1.$$

By Lemma 7.2, we see

$$V\left(f,\frac{1+2^r\beta b}{2^r\beta p^2}\right) = V\left(f,\frac{1}{2^r\beta}\right).$$

By equality (7.9), we obtain

$$p^{k-2}V(f,1/(2^r\beta)) = p^{-2}\sum_{b=1}^{p^2} V\left(f,\frac{1+2^r\beta b}{2^r\beta p^2}\right)$$
$$= p^{-2}\sum_{b=1}^{p^2} V(f,1/(2^2\beta)) = V(f,1/(2^r\beta)),$$

which implies that $V(f,1/(2^r\beta)) = 0$. Now we begin to prove the first equality in Lemma 7.4. It is clear that the equality holds for $\alpha = 1$. We shall complete the proof by induction on the number of prime divisors of α. We assume that the equality holds for α with $\alpha \neq m$. We must prove that the equality holds for $V(f,1/(\alpha p))$ with p prime and satisfying $\alpha p|m$. Since $f|\mathrm{T}(p^2) = f$, we get

$$f(z+1/\alpha) = p^{-2}\sum_{b=1}^{p^2} f\left(\frac{z}{p^2}+\frac{1+b\alpha}{p^2\alpha}\right).$$

Because it is possible that $p|1+b\alpha$, in general the rational number $\dfrac{1+b\alpha}{p^2\alpha}$ is not reduced. We have to cancel the greatest common divisor in order to obtain a cusp point. Now there exists a unique integer b_1 such that $1 \leqslant b_1 \leqslant p$, $1+\alpha b_1 = pt_1$. Similarly, there exists a unique integer b_2 such that $1 \leqslant b_2 \leqslant p^2$, $1+b_2\alpha = p^2 t_2$, where t_1, t_2 are integers. Hence by the definition of values of a modular function at cusp points and equality (7.8), we obtain

$$V(f,1/\alpha) = p^{-2}\sum_{\substack{1\leqslant b\leqslant p^2\\ p\nmid 1+b\alpha}} V\left(f,\frac{1+b\alpha}{p^2\alpha}\right)$$
$$+p^{k/2-2}\sum_{\substack{1\leqslant b\leqslant p\\ p\nmid t_1+b\alpha}} V\left(f,\frac{t_1+b\alpha}{p\alpha}\right) + p^{k-2}V(f,t_2/\alpha). \qquad (7.10)$$

The cusp points $\dfrac{1+b\alpha}{p^2\alpha}$ $(p\nmid 1+b\alpha)$, $\dfrac{t_1+b\alpha}{p\alpha}$ $(p\nmid t_1+b\alpha)$ and t_2/α are equivalent to $\dfrac{1}{p\alpha}$, $\dfrac{1}{p\alpha}$ and $1/\alpha$ under the group $\Gamma_0(8D)$ respectively. We now consider the case $p\nmid l$. Let $\begin{pmatrix} a & e\\ c & d\end{pmatrix} \in \Gamma_0(8D)$ such that

$$\begin{pmatrix} a & e\\ c & d\end{pmatrix}\begin{pmatrix} 1\\ p\alpha\end{pmatrix} = \begin{pmatrix} 1+b\alpha\\ p^2\alpha\end{pmatrix}, \qquad (7.11)$$

which deduces that $a+ep\alpha = 1+b\alpha, c+dp\alpha = p^2\alpha$. But $ad-ce = 1$. So we obtain that $d \equiv a \equiv 1 \pmod{\alpha}$, $d \equiv p\left(\bmod\ \dfrac{c}{p\alpha}\right)$. Since $8D|c, p\nmid l$, then $d \equiv p \pmod{4l/(l,\alpha)}$. By Lemma 7.2, we obtain

$$\begin{aligned}
V\left(f, \frac{1+b\alpha}{p^2\alpha}\right) &= \left(\frac{lc}{d}\right)\varepsilon_d V(f, 1/(p\alpha)) \\
&= \left(\frac{l/(l,\alpha)}{d}\right)\left(\frac{c/(l,\alpha)}{d}\right)\varepsilon_d V(f, 1/(p\alpha)) \\
&= \left(\frac{l/(l,\alpha)}{p}\right)\left(\frac{d}{\alpha/(l,\alpha)}\right)\varepsilon_{d\alpha/(l,\alpha)}\varepsilon^{-1}_{\alpha/(\alpha,l)}\varepsilon_d^{-1} V(f, 1/(p\alpha)) \\
&= \left(\frac{l/(l,\alpha)}{p}\right)\varepsilon_{p\alpha/(l,\alpha)}\varepsilon^{-1}_{\alpha/(\alpha,l)} V(f, 1/(p\alpha)). \qquad (7.12)
\end{aligned}$$

Similarly, we can deduce

$$\begin{cases} V\left(f, \dfrac{t_1+b\alpha}{p\alpha}\right) = \left(\dfrac{t_1+b\alpha}{p}\right)\left(\dfrac{p}{\alpha/(l,\alpha)}\right) V(f, 1/(p\alpha)), \\ V(f, t_2/\alpha) = V(f, 1/\alpha). \end{cases} \qquad (7.13)$$

Inserting equalities (7.12) and (7.13) into (7.10), we see that the second sum in equality (7.10) is zero, and hence

$$\begin{aligned}
V(f, 1/\alpha) &= p^{-2} \sum_{\substack{1\leqslant b\leqslant p^2 \\ p\nmid 1+b\alpha}} \varepsilon_{\alpha p/(l,\alpha)}\varepsilon^{-1}_{\alpha/(l,\alpha)}\left(\frac{l/(l,\alpha)}{p}\right) V(f, 1/(p\alpha)) + p^{k-2}V(f, 1/\alpha) \\
&= p^{-2}(p^2-p)\varepsilon_{\alpha p/(l,\alpha)}\varepsilon^{-1}_{\alpha/(l,\alpha)}\left(\frac{l/(l,\alpha)}{p}\right) V(f, 1/(p\alpha)) + p^{k-2}V(f, 1/\alpha),
\end{aligned}$$

which implies, by the induction assumption,

$$\begin{aligned}
V(f, 1/(p\alpha)) &= -\frac{(p^{k-2}-1)p}{p-1}\varepsilon^{-1}_{\alpha p/(l,\alpha)}\varepsilon_{\alpha/(l,\alpha)}\left(\frac{l/(l,\alpha)}{p}\right) V(f, 1/\alpha) \\
&= -\eta_p^{-1}\varepsilon^{-1}_{\alpha p/(l,\alpha)}\varepsilon_{\alpha/(l,\alpha)}\left(\frac{l/(l,\alpha)}{p}\right) V(f, 1/\alpha) \\
&= -\eta_p^{-1}\varepsilon^{-1}_{\alpha p/(l,\alpha)}\varepsilon_{\alpha/(l,\alpha)}\left(\frac{l/(l,\alpha)}{p}\right)\mu(\alpha)\eta_\alpha^{-1}(\alpha, l)^{-k/2+1}\varepsilon^{-1}_{\alpha/(l,\alpha)} \\
&\quad \cdot \left(\frac{l/(l,\alpha)}{\alpha/(\alpha,l)}\right) V(f, 1) \\
&= \mu(p\alpha)\eta_{p\alpha}^{-1}(p\alpha, l)^{-k/2+1}\varepsilon^{-1}_{\alpha p/(l,\alpha)}\left(\frac{l/(l,p\alpha)}{p\alpha/(p\alpha,l)}\right) V(f, 1),
\end{aligned}$$

where we assumed $p \nmid l$. Therefore for $p \nmid l$ we have proved the result. Now suppose $p|l$. In this case, from equality (7.11), we see

$$d \equiv a \equiv 1 \pmod{\alpha}, \quad d \equiv p \pmod{4l/(l, p\alpha)}, \quad (1+b\alpha)d \equiv 1 \pmod{p}.$$

Hence by Lemma 7.2,

$$V\left(f, \frac{1+b\alpha}{p^2\alpha}\right) = \left(\frac{lc}{d}\right)\varepsilon_d V(f, 1/(p\alpha))$$

$$= \left(\frac{l/(l,p\alpha)}{d}\right)\left(\frac{p\alpha/(l,\alpha)}{d}\right)\varepsilon_d V(f,1/(p\alpha))$$
$$= \left(\frac{l/(l,p\alpha)}{p}\right)\left(\frac{d}{p\alpha/(l,\alpha)}\right)\varepsilon_{\alpha/(l,\alpha)}\varepsilon_{p\alpha/(l,\alpha)}^{-1}V(f,1/(p\alpha))$$
$$= \varepsilon_{\alpha/(l,\alpha)}\varepsilon_{p\alpha/(l,\alpha)}^{-1}\left(\frac{(1+b\alpha)l/(l,p\alpha)}{p}\right)V(f,1/(p\alpha)).$$

Similarly we can show

$$V\left(f,\frac{t_1+b\alpha}{p\alpha}\right) = \left(\frac{p}{\alpha/(\alpha,l)}\right)V(f,1/(p\alpha)),$$
$$V(f,t_2/\alpha) = V(f,1/\alpha).$$

Inserting these results into the equality (7.10), we get that the first sum in the equality is zero, and hence

$$V(f,1/\alpha) = p^{k/2-2}\sum_{\substack{1\leqslant b\leqslant p\\ p\nmid t_1+b\alpha}}\left(\frac{p}{\alpha/(\alpha,l)}\right)V(f,1/(p\alpha)) + p^{k-2}V(f,1/\alpha)$$
$$= p^{k/2-2}(p-1)\left(\frac{p}{\alpha/(\alpha,l)}\right)V(f,1/(p\alpha)) + p^{k-2}V(f,1/\alpha),$$

which implies, by the induction assumption,

$$V(f,1/(p\alpha)) = -\frac{p^{k-2}-1}{p^{k/2-2}(p-1)}\left(\frac{p}{\alpha/(\alpha,l)}\right)V(f,1/\alpha)$$
$$= -\eta_p^{-1}p^{-k/2+1}\left(\frac{p}{\alpha/(\alpha,l)}\right)V(f,1/\alpha)$$
$$= -\eta_p^{-1}p^{-k/2+1}\left(\frac{p}{\alpha/(\alpha,l)}\right)\mu(\alpha)\eta_\alpha^{-1}(\alpha,l)^{-k/+1}\varepsilon_{\alpha/(\alpha,l)}^{-1}\left(\frac{l/(\alpha,l)}{\alpha/(\alpha,l)}\right)V(f,1)$$
$$= \mu(p\alpha)\eta_{p\alpha}^{-1}(p\alpha,l)^{-k/2+1}\varepsilon_{p\alpha/(p\alpha,l)}^{-1}\left(\frac{l/(p\alpha,l)}{p\alpha/(p\alpha,l)}\right)V(f,1),$$

where we assumed $p|l$. Hence for the case $p|l$ the first equality in the Lemma 7.4 holds. By induction, we know that this equality holds for any $\alpha|m$. The other two equalities in the Lemma 7.4 can be proved by a similar method which we omit. This completes the proof of Lemma 7.4.

Now we can prove Theorem 7.2 as follows.

Noting that $g_k(\text{id.},j,4D)|\mathrm{T}(l) = g_k(\chi_l,j,4D)$, we first consider the case $l=1$, i.e., $\chi_l = \text{id}$. For this case, by the equality (7.4), we have

$$g_k(\text{id.},4m,4D) = \sum_{d|m}\mu(d)\eta_d F_k(4m/d) - \sum_{d|m}\mu(d)\eta_{2d}F_k(m/d),$$

where

$$F_k(4D) = E_k(\text{id.},4D)(z),$$

$$F_k(4m) = \sum_{d|D/m} \mu(d) E_k(\mathrm{id.}, 4md),$$
$$F_k(m) = \sum_{d|m} \mu(d) E_k'(\chi_{dD/m}, 4dD/m).$$

By Lemma 7.3, we have

$$\begin{aligned} V(F_k(4D), 1) &= V(E_k(\mathrm{id.}, 4D), 1) = 0, \\ V(F_k(4m), 1) &= \sum_{d|D/m} \mu(d) V(E_k(\mathrm{id.}, 4md), 1) = 0, \\ V(F_k(m), 1) &= \sum_{d|m} \mu(d) V(E_k'(\chi_{dD/m}, 4dD/m), 1) = \sum_{d|m} \mu(d) \mathrm{i}^{-k} \\ &= \mathrm{i}^{-k} \text{ or } 0 \text{ according to } m = 1 \text{ or } \neq 1. \end{aligned}$$

Hence

$$\begin{aligned} V(g_k(\mathrm{id.}, 4m, 4D), 1) &= \sum_{d|m} \mu(d) \eta_d V(F_k(4m/d), 1) - \sum_{d|m} \mu(d) \eta_{2d} V(F_k(m/d), 1) \\ &= -\mu(m) \eta_{2m} \mathrm{i}^{-k} = -\mathrm{i}^{-\delta_k} \mu(m) \eta_{2m}. \end{aligned}$$

We now show that for any $\beta | D$, $V(g_k(\mathrm{id.}, 4m, 4D), 1/(2\beta)) = 0$. In fact, since

$$g_k(\mathrm{id.}, 4m, 4D) | \mathrm{T}(4) = g_k(\mathrm{id.}, 4m, 4D),$$

we know

$$g_k(\mathrm{id.}, 4m, 4D)(z + 1/(2\beta)) = 4^{-1} \sum_{b=1}^{4} g_k(\mathrm{id.}, 4m, 4D) \left(z/4 + \frac{1 + 2\beta b}{8\beta} \right).$$

Because $(1+2\beta b, 8\beta) = 1$, $\dfrac{1 + 2\beta b}{8\beta}$ is a cusp point equivalent to the cusp point $1/(4\beta)$ for the group $\Gamma_0(4D)$. Therefore there exists a matrix $\begin{pmatrix} a_b & e_b \\ c_b & d_b \end{pmatrix} \in \Gamma_0(4D)$ such that

$$\begin{pmatrix} a_b & e_b \\ c_b & d_b \end{pmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 + 2b\beta \\ 8\beta \end{pmatrix},$$

which implies that $a_b + 4\beta_b = 1 + 2\beta b$, $c_b + 4\beta d_b = 8\beta$, $d_b(1 + 2\beta b) \equiv 1 \pmod{4\beta}$. By equality (7.8) and Lemma 7.2, we obtain

$$\begin{aligned} V(g_k(\mathrm{id.}, 4m, 4D), 1/2\beta) &= 4^{-1} \sum_{b=1}^{4} \left(\frac{c_b}{d_b} \right) \varepsilon_{d_b}^{-k} V(g_k(\mathrm{id.}, 4m, 4D), 1/4\beta) \\ &= 4^{-1} \sum_{b=1}^{4} \left(\frac{8\beta - 4\beta d_b}{d_b} \right) \varepsilon_{1+2\beta b}^{-k} V(g_k(\mathrm{id.}, 4m, 4D), 1/4\beta) \\ &= 4^{-1} \sum_{b=1}^{4} \left(\frac{2\beta}{1 + 2\beta b} \right) \varepsilon_{1+2\beta b}^{-k} V(g_k(\mathrm{id.}, 4m, 4D), 1/4\beta). \end{aligned}$$

Since $\left(\frac{2\beta}{a+4\beta b}\right) = -\left(\frac{2\beta}{a}\right)$, it is clear that the above is equal to zero.

In order to compute the value of $g_k(\mathrm{id.}, 4m, 4D)$ at the cusp point $1/4$, we use the fact

$$g_k(\mathrm{id.}, 4m, 4D)|\mathrm{T}(4) = g_k(\mathrm{id.}, 4m, 4D),$$

Then

$$g_k(\mathrm{id.}, 4m, 4D)(z) = 4^{-1}\sum_{b=1}^{4} g_k(\mathrm{id.}, 4m, 4D)(z/4 + b/4).$$

Since $V(g_k(\mathrm{id.}, 4m, 4D), 1/2) = 0$, we see

$$\begin{aligned} &V(g_k(\mathrm{id.}, 4m, 4D), 1) \\ =&4^{-1}V(g_k(\mathrm{id.}, 4m, 4D), 1/4) + 4^{-1}V(g_k(\mathrm{id.}, 4m, 4D), 3/4) \\ &+2^{k-2}V(g_k(\mathrm{id.}, 4m, 4D), 1) \end{aligned} \tag{7.14}$$

But the cusp point $3/4$ is equivalent to $1/4$ for the group $\Gamma_0(4D)$. Therefore there exists a matrix $\begin{pmatrix} a & e \\ c & d \end{pmatrix} \in \Gamma_0(4D)$ such that

$$\begin{pmatrix} a & e \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Hence by Lemma 7.2, we have

$$\begin{aligned} V(g_k(\mathrm{id.}, 4m, 4D), 3/4) &= \left(\frac{c}{d}\right)\varepsilon_d^{-\delta_k} V(g_k(\mathrm{id.}, 4m, 4D), 1/4) \\ &= \mathrm{i}^{-\delta_k} V(g_k(\mathrm{id.}, 4m, 4D), 1/4). \end{aligned}$$

Combining with equality (7.14), we have

$$\begin{aligned} V(g_k(\mathrm{id.}, 4m, 4D), 1/4) &= -\frac{2^k - 4}{1 + \mathrm{i}^{-\delta_k}} V(g_k(\mathrm{id.}, 4m, 4D), 1) \\ &= -\frac{2^k - 4}{1 + \mathrm{i}^{-\delta_k}}(-\mathrm{i}^{-\delta_k}\mu(m)\eta_{2m}) \\ &= \mu(m)\eta_m. \end{aligned}$$

By the above discussions, we know that

$$\begin{aligned} V(g_k(\mathrm{id.}, 4m, 4D), 1) &= -\mathrm{i}^{-\delta_k}\mu(m)\eta_{2m} = -\frac{1 + \mathrm{i}^{-\delta_k}}{2^k - 4}\mu(m)\eta_m, \\ V(g_k(\mathrm{id.}, 4m, 4D), 1/4) &= \mu(m)\eta_m, \\ V(g_k(\mathrm{id.}, 4m, 4D), 1/2\beta) &= 0, \text{ for any } \beta | D. \end{aligned}$$

Hence by Theorem 7.1 and Lemma 7.4, we have proved that the first two equalities in Theorem 7.2 hold for $l = 1$. Now we consider the function $g_k(\mathrm{id.}, m, 4D)$. By Theorem 7.1, we have

$$g_k(\text{id.}, m, 4D)|\mathrm{T}(p^2) = g_k(\text{id.}, m, 4D) \quad \text{for all } p|m,$$
$$g_k(\text{id.}, m, 4D)|\mathrm{T}(p^2) = p^{k-2} g_k(\text{id.}, m, 4D) \quad \text{for all } p|2D/m.$$

In particular, we see

$$g_k(\text{id.}, m, 4D)|\mathrm{T}(4) = 2^{k-2} g_k(\text{id.}, m, 4D).$$

Noting that the cusp point $(1 + 4b\beta)/(16\beta)$ is equivalent to $1/(4\beta)$ for the group $\Gamma_0(4D)$, by equality (7.8) and Lemma 7.2, we see

$$\begin{aligned} 2^{k-2} V(g_k(\text{id.}, m, 4D), 1/4\beta) &= 4^{-1} \sum_{b=1}^{4} V\left(g_k(\text{id.}, m, 4D), \frac{1+4b\beta}{16\beta}\right) \\ &= V(g_k(\text{id.}, m, 4D), 1/4\beta), \end{aligned}$$

which implies that $V(g_k(\text{id.}, m, 4D), 1/4\beta) = 0$. In the same way, by equality (7.8), we have

$$2^{k-2} V(g_k(\text{id.}, m, 4D), 1/2\beta) = 4^{-1} \sum_{b=1}^{4} V\left(g_k(\text{id.}, m, 4D), \frac{1+2b\beta}{8\beta}\right).$$

Since the cusp point $(1 + 2b\beta)/(8\beta)$ is equivalent to $1/(4\beta)$ for the group $\Gamma_0(4D)$, the right hand side of the above equality is zero. So by Lemma 7.4, we only need to calculate the value of $g_k(\text{id.}, m, 4D)$ at the cusp point 1. But we know from the proof of Theorem 7.2,

$$g_k(\text{id.}, m, 4D) = \sum_{d|m} \mu(d)\eta_d F_k(m/d).$$

Noting that $V(F_k(m), 1) = \mathrm{i}^{-\delta_k}$ or 0 according to $m = 1$ or $m \neq 1$ respectively, we have

$$\begin{aligned} V(g_k(\text{id.}, m, 4D), 1) &= \sum_{d|m} \mu(d)\eta_d V(F_k(m/d), 1) \\ &= \mathrm{i}^{-\delta_k} \mu(m)\eta_m. \end{aligned}$$

Hence by Theorem 7.1 and Lemma 7.4, we have proved the claim for the values of $g_k(\text{id.}, m, 4D)$.

Now we consider the case $l \neq 1$. In this case we have

$$\begin{aligned} g_k(\chi_l, 4m, 4D)(z) &= g_k(\text{id.}, 4m, 4D)(z)|\mathrm{T}(l) \\ &= l^{-1} \sum_{b=1}^{l} g_k(\text{id.}, 4m, 4D)\left(\frac{z+b}{l}\right). \end{aligned}$$

Hence by the equality (7.8) and Lemma 7.2, we see

$$V(g_k(\chi_l, 4m, 4D), 1) = l^{-1} \sum_{d|l} d^{k/2} \sum_{\substack{b=1 \\ (b,l/d)=1}}^{l/d} V(g_k(\text{id.}, 4m, 4D), b/(ld^{-1}))$$

$$\begin{aligned}
&= l^{-1}\sum_{d|l} d^{k/2}\sum_{b=1}^{l/d}\left(\frac{b}{ld^{-1}}\right)V(g_k(\mathrm{id.},4m,4D),1/(ld^{-1}))\\
&= l^{-1}\sum_{d|l} d^{k/2}V(g_k(\mathrm{id.},4m,4D),1/(ld^{-1}))\sum_{b=1}^{l/d}\left(\frac{b}{ld^{-1}}\right)\\
&= l^{-1}l^{k/2}V(g_k(\mathrm{id.},4m,4D),1)\\
&= l^{k/2-1}(-\mathrm{i}^{-\delta_k}\mu(m)\eta_{2m})\\
&= -\frac{1+\mathrm{i}^{-\delta_k}}{2^k-4}\mu(m)\eta_m l^{k/2-1}.
\end{aligned}$$

Similar to the case $l = 1$, we can prove $V(g_k(\chi_l, 4m, 4D), 1/2\beta) = 0$ for any $\beta|D$. Since $g_k(\chi_l, 4m, 4D)|\mathrm{T}(4) = g_k(\chi_l, 4m, 4D)$, we have

$$\begin{aligned}
V(g_k(\chi_l,4m,4D),1) = {}& 4^{-1}V(g_k(\chi_l,4m,4D),1/4)+4^{-1}V(g_k(\chi_l,4m,4D),3/4)\\
&+2^{k-2}V(g_k(\chi_l,4m,4D),1),
\end{aligned}\tag{7.15}$$

where we used the fact $V(g_k(\chi_l, 4m, 4D), 1/2\beta) = 0$ for any $\beta|D$. Because the cusp point $3/4$ is equivalent to $1/4$, so there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4D)$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix},$$

which implies $d \equiv 3 \pmod 4$, $d \equiv 1 \pmod l$, $c \equiv 4 \pmod d$. By Lemma 7.2, we have

$$\begin{aligned}
V(g_k(\chi_l,4m,4D),3/4) &= \left(\frac{l}{d}\right)\mathrm{i}^{-\delta_k}V(g_k(\chi_l,4m,4D),1/4)\\
&= \mathrm{i}^{-\delta_k}\varepsilon_l^{d-1}\left(\frac{d}{l}\right)V(g_k(\chi_l,4m,4D),1/4)\\
&= \mathrm{i}^{-\delta_k}\varepsilon_l^{2}V(g_k(\chi_l,4m,4D),1/4).
\end{aligned}$$

Inserting this into equality (7.15), we obtain

$$\begin{aligned}
V(g_k(\chi_l,4m,4D),1/4) &= -\frac{2^k-4}{1+\mathrm{i}^{-\delta_k}\varepsilon_l^2}V(g_k(\chi_l,4m,4D),1)\\
&= -\frac{2^k-4}{1+\mathrm{i}^{-\delta_k}\varepsilon_l^2}\left(-\frac{1+\mathrm{i}^{-\delta_k}}{2^k-4}\mu(m)\eta_m l^{k/2-1}\right)\\
&= \mu(m)\eta_m l^{k/2-1}\varepsilon_l^{-\delta_k}.
\end{aligned}$$

Similarly we can prove that $V(g_k(\chi_l, m, 4D), 1/2\beta) = V(g_k(\chi_l, m, 4D), 1/4\beta) = 0$ for any $\beta|D$ and $V(g_k(\chi_l, 4m, 4D), 1) = \mathrm{i}^{-\delta_k}\mu(m)\eta_m l^{k/2-1}$. Collecting all the above and Lemma 7.2 we proved our Theorem 7.2 for $l \neq 1$. This completes the whole proof for Theorem 7.2. □

7.2 Construction of Eisenstein Series with Weight 1/2

Let ψ be a primitive character modulo r with $\psi(-1) = (-1)^v$ ($v = 0$ or 1). Put

$$\theta_\psi(z) = \sum_{n=-\infty}^{\infty} \psi(n) n^v e(n^2 z), \quad z \in \mathbb{H}.$$

Then it is easy to see that

$$\theta_\psi(z) = \sum_{k=1}^{r} \psi(k)\theta(2rz; k, r),$$

where

$$\theta(z; k, r) = \sum_{m \equiv k \bmod r} m^v e(zm^2/(2r)), \quad z \in \mathbb{H}.$$

Lemma 7.5 *We have the following transformation formula:*

$$\theta(-1/z; k, r) = (-1)^v r^{-1/2}(-\mathrm{i}z)^{(1+2v)/2} \sum_{j=1}^{r} e(jk/r)\theta(z; j, r).$$

Proof Set

$$g(x) = \sum_{m=-\infty}^{\infty} (x+m)^v e(\mathrm{i}rt(x+m)^2/2).$$

It is obvious that $g(x+1) = g(x)$. So by some computation we have a Fourier expansion:

$$g(x) = \sum_{m=-\infty}^{\infty} a(m)e(mx)$$

with

$$a(m) = (-\mathrm{i})^v (rt)^{-(1+2v)/2} \mathrm{e}^{-\pi m^2/(rt)},$$

so that

$$g(x) = (-\mathrm{i})^v (rt)^{-(1+2v)/2} \sum_{m=-\infty}^{\infty} \mathrm{e}^{2\pi \mathrm{i} mx - \pi m^2/(rt)}.$$

It is easy to see that

$$\theta(\mathrm{i}t; k, r) = r^v g(k/r) = (-\mathrm{i})^v r^{-1/2} t^{-(1+2v)/2} \sum_{j=1}^{r} e(jk/r)\theta(-1/(\mathrm{i}t); j, r),$$

which implies the lemma. This completes the proof. □

Lemma 7.6 *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *with* b *even and* $c \equiv 0 \pmod{2r}$. *Then*

$$\theta(\gamma(z); k, r) = e(abk^2/(2r))\varepsilon_d^{-1}\left(\frac{2cr}{d}\right)(cz+d)^{(1+2v)/2}\theta(z; ak, r).$$

Proof Assume that $c > 0$. By Lemma 7.5, we have

$$\begin{aligned}\theta(\gamma(z); k, r) &= \sum_{n\equiv k \bmod r} n^v e\left(n^2\left(a - \frac{1}{cz+d}\right)/(2cr)\right) \\ &= (-\mathrm{i})^v (cr)^{-1/2}(-\mathrm{i}(cz+d))^{(1+2v)/2} \sum_{t \bmod cr} \Phi(k,t) \\ &\quad \sum_{n\equiv t \bmod cr} n^v e(n^2 z/(2r)),\end{aligned}$$

where

$$\Phi(k,t) = \sum_{\substack{g \bmod cr,\\ g\equiv k \bmod r}} e((\alpha g^2 + tg + \delta t^2)/(cr))$$

and α, δ are integers such that $a \equiv 2\alpha \pmod{cr}$, $d \equiv 2\delta \pmod{cr}$. The remaining part of this proof is completely similar to the proof of Proposition 1.2. This completes the proof. □

Theorem 7.3 *$\theta_\psi(z)$ is in $G(4r^2, 1/2, \psi)$ if $v = 0$ and $\theta_\psi(z)$ is in $S(4r^2, 3/2, \psi\chi_{-1})$ if $v = 1$.*

Proof Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4r^2)$. By Lemma 7.6, we see that

$$\begin{aligned}\theta_\psi(\gamma(z)) &= \sum_{k=1}^{r} \psi(k)\theta\left(\frac{2rza + 2rb}{2rz(c/(2r)) + d}; k, r\right) \\ &= \varepsilon_d^{-1}\left(\frac{c}{d}\right)(cz+d)^{(1+2v)/2} \sum_{k=1}^{r} \psi(k)\theta(2rz; ak, r) \\ &= \psi(d)\varepsilon_d^2 j(\gamma, z)^{1+2v}\theta_\psi(z).\end{aligned}$$

Consider the holomorphy of $\theta_\psi(z)$ at cusp points. Let $\rho = \begin{pmatrix} a & b \\ -c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with $c > 0$. Then we see that

$$|\theta_\psi(z)| \leqslant 1 - v + 2\sum_{n=1}^{\infty} n^v \mathrm{e}^{-2\pi n^2 y} < 1 - v + Cy^{-(1+v/2)}, \quad y \to \infty,$$

where C is a constant. So that

$$\begin{aligned}|\theta_\psi(\rho^{-1}(z))(cz+a)^{-(1+2v)/2}| &\leqslant (1 - v + Cy^{-(1+v/2)}|cz+a|^{v+2})|cz+a|^{-(1+2v)/2} \\ &\leqslant (1 - v + C'y^{1+v/2})y^{-(1+2v)/2}, \quad y \to \infty,\end{aligned}$$

which implies that $\theta_\psi(z) \in G(4r^2, 1/2, \psi)$ or $S(4r^2, 3/2, \psi\chi_{-1})$ according to $v = 0$ or 1 respectively. This completes the proof. □

Let now t be a positive integer, ψ a primitive even character modulo r. Put

$$\theta_{\psi,t}(z) = \sum_{n=-\infty}^{\infty} \psi(n)e(tn^2 z), \quad z \in \mathbb{H},$$

which is equal to $\theta_\psi|V(t)$, so $\theta_{\psi,t}(z)$ is in $G(4r^2t, 1/2, \psi\chi_t)$. Let ω be an even character modulo N, ψ a primitive even character modulo $r(\psi)$, t a positive integer. We denote by $\Omega(N,\omega)$ the set of pairings (ψ, t) satisfying the following conditions:

(1) $4(r(\psi))^2 t|N$;

(2) $\omega(n) = \psi(n)\chi_t(n)$ for any integer n prime to N.

Let $\psi = \prod_{p|r(\psi)} \psi_p$ with ψ_p the p-part of the character ψ. If every ψ_p is an even character, then ψ is called a totally even character. Denote by $\Omega_{\mathrm{e}}(N,\omega)$ the set of all parings (ψ, t) in $\Omega(N,\omega)$ where ψ is totally even. Set $\Omega_{\mathrm{c}}(N,\omega) = \Omega(N,\omega) - \Omega_{\mathrm{e}}(N,\omega)$. The following is our main result in this section.

Theorem 7.4 (1) *The set* $\{\theta_{\psi,t}|(\psi,t) \in \Omega(N,\omega)\}$ *is a basis of* $G(N, 1/2, \omega)$;

(2) *The set* $\{\theta_{\psi,t}|(\psi,t) \in \Omega_{\mathrm{c}}(N,\omega)\}$ *is a basis of* $S(N, 1/2, \omega)$, *and the set* $\{\theta_{\psi,t}|(\psi,t) \in \Omega_{\mathrm{e}}(N,\omega)\}$ *is a basis of the orthogonal complement of* $S(N, 1/2, \omega)$ *in* $G(N, 1/2, \omega)$.

To show Theorem 7.4 we need some lemmas.

Lemma 7.7 (1) *There exists a basis in* $G(N, k/2, \omega)$ *such that all Fourier coefficients of every function in the basis belong to some algebraic number field*;

(2) *let* $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N, k/2, \omega)$ *with* $a(n)$ *all algebraic numbers for* $n \geqslant 0$. *Then there exists an integer* D *such that* $Da(n)$ *are all algebraic integers for all* $n \geqslant 0$.

Proof Put

$$f_0(z) = \theta(z)^{3k} = 1 + 6ke(z) + \cdots .$$

Define a map $\phi:\ f \mapsto ff_0$. Then ϕ maps $G(N, k/2, \omega)$ into $G(N, 2k, \omega)$. If f has algebraic coefficients, so does ff_0. (2) holds for ff_0 (Please compare Theorem 3.52 of G. Shimura, 1971), so does (2) for f. Now show (1). $\theta(z)$ has no zeros in $\mathbb{H}$, and it is zero only at the cusp point $1/2 \in S(4) = \{1,\ 1/2,\ 1/4\}$. A function $g \in G(N,\ 2k,\ \omega)$ is an image of ϕ (i.e., $g/f_0 \in G(N,\ k/2,\ \omega)$) if and only if g has high enough orders of zeros at all cusp points in $S(N)$ which are $\Gamma_0(N)$-equivalent to $1/2$. We know that the theorem we want to show holds for the spaces of modular forms integral weights. So there exists a basis $\{g_i\}$ in $G(N,\ 2k,\ \omega)$ such that the Fourier coefficients of g_i at every cusp point are algebraic numbers. g is a linear combination of $\{g_i\}$, and g gets value zero with some orders at part of cusp points. This implies that the coefficients of the linear combination satisfy a system of some linear equations with algebraic numbers as the coefficients of these linear equations. Hence there exists a basis in $G(N, k/2, \omega)$ whose every element has algebraic coefficients. This completes the proof. $\square$

Lemma 7.8 *Let* $0 \neq f(z) = \sum_{n=0}^{\infty} a(n)e(nz)$ *be in* $G(N, 1/2, \omega), p \nmid N$ *a prime and* $f|\mathrm{T}(p^2) = c_p f$. *Assume that* m *is a positive integer with* $p^2 \nmid m$. *Then*

(1) $a(mp^{2n}) = a(m)\omega(p)^n \left(\frac{m}{p}\right)^n$ *for any* $n \geqslant 0$;

(2) *if* $a(m) \neq 0$, *then* $p \nmid m$ *and* $c_p = \omega(p)\left(\frac{m}{p}\right)(1 + p^{-1})$.

Proof Since $\mathrm{T}(p^2)$ maps a modular form with algebraic coefficients to one of the same kind, by Lemma 7.7, we see that the eigenvalue c_p of $\mathrm{T}(p^2)$ is an algebraic number and the corresponding eigenspace has a basis with algebraic coefficients. Without loss of generality, we may assume that the coefficients of f are algebraic. Put

$$A(T) = \sum_{n=0}^{\infty} a(mp^{2n})T^n.$$

By Lemma 5.40 we have

$$A(T) = a(m)\frac{1 - \alpha T}{(1 - \beta T)(1 - \gamma T)},$$

where $\alpha = \omega(p)p^{-1}\left(\frac{m}{p}\right)$, $\beta + \gamma = c_p$, $\beta\gamma = \omega(p^2)p^{-1}$. Assume $a(m) \neq 0$. Then $A(T)$ is a non-zero rational function. We may think $A(T)$ as a p-adic T function, i.e., think the coefficients of $A(T)$ as elements in some algebraic extension of the p-adic number field $\mathbb{Q}_p$. By Lemma 7.7 the p-adic absolute value of $a(mp^{2n})$ $(n \geqslant 0)$ are bounded. Therefore $A(T)$ is convergent for all $|T|_p < 1$. $A(T)$ has no poles in the unit disc $U = \{T \mid |T|_p < 1\}$. If $(1 - \beta T)(1 - \gamma T)$ is prime to $1 - \alpha T$, then $|\beta|_p < 1$, $|\gamma|_p < 1$. But $|\beta\gamma|_p = |\omega(p^2)p^{-1}|_p > 1$. So we see that one of β and γ must be α. We may assume that $\beta = \alpha$ and hence $A(T) = a(m)/(1 - \gamma T)$, $a(mp^{2n}) = \gamma^n a(m)$. Since $\beta\gamma \neq 0$, we see that $\alpha \neq 0$, so $p \nmid m$ and

$$\gamma = \beta\gamma/\alpha = \frac{\omega(p^2)p^{-1}}{\omega(p)p^{-1}\left(\frac{m}{p}\right)} = \omega(p)\left(\frac{m}{p}\right).$$

This shows that $a(mp^{2n}) = a(m)\omega(p)^n\left(\frac{m}{p}\right)^n$ which is (1). And $c_p = \beta + \gamma = \alpha + \gamma = \omega(p)\left(\frac{m}{p}\right)(1 + p^{-1})$ which is (2). This completes the proof. $\square$

Lemma 7.9 *Let* $0 \neq f(z) = \sum_{n=0}^{\infty} a(n)e(nz)$ *be in* $G(N, 1/2, \omega)$, N' *a multiple of* N.

Assume that $f|\mathrm{T}(p^2) = c_p f$ *for any* $p \nmid N'$. *Then there exists a unique square-free positive integer* t *such that* $a(n) = 0$ *if* n/t *is not a square and*

(1) $t|N'$;

(2) $c_p = \omega(p)\left(\frac{t}{p}\right)(1 + p^{-1})$ *for any* $p \nmid N'$;

(3) $a(nu^2) = a(n)\omega(u)\left(\frac{t}{u}\right)$ *for any* $u \geqslant 1$ *with* $(u, N') = 1$.

Proof Let m, m' be any positive integers with $a(m) \neq 0$ and $a(m') \neq 0$, P the set of primes satisfying $p \nmid N'mm'$. For any $p \notin P$, by Lemma 7.8 we see that

$$\omega(p)\left(\frac{m}{p}\right)(1 + p^{-1}) = \omega(p)\left(\frac{m'}{p}\right)(1 + p^{-1}),$$

so that $\left(\frac{mm'}{p}\right) = 1$. This implies that mm' must be a square. Therefore there exists a square-free positive integer t with $m = tv^2, m' = t(v')^2$ which implies the first part of the lemma. Let now p be any prime with $p \nmid N'$. Write $v = p^n u$, $p \nmid u$. Since $0 \neq a(m) = a(tp^{2n}u^2)$, we see that $a(tu^2) \neq 0$ by the part (1) of Lemma 7.8, so that $p \nmid t$ and $c_p = \omega(p)\left(\frac{t}{p}\right)(1 + p^{-1})$ by the part (2) of Lemma 7.8. This showed (2) and (1) since t is square-free. For the proof of the part (3), we only need to consider the case that $u = p, p \nmid N'$, then we can write $n = mp^{2a}$ with $p^2 \nmid m$. It is then clear that (3) can be deduced from the part (2) of Lemma 7.8. This completes the proof. □

Corollary 7.1 *Let the assumptions be as in* Lemma 7.9. *And assume furthermore* $a(1) \neq 0$. *Then* $t = 1$ *and* $c_p = \omega(p)(1 + p^{-1})$ *for any* $p \nmid N'$. *This implies that the character* ω *is determined uniquely by the set of eigenvalues* c_p.

Corollary 7.2 *Under the assumptions of* Lemma 7.9 *we have that*

$$\sum_{n=1}^{\infty} a(n)n^{-s} = t^{-s}\left(\sum_{n|N'^\infty} a(tn^2)n^{-2s}\right)\prod_{p\nmid N'}\left(1 - \omega(p)\left(\frac{t}{p}\right)p^{-2s}\right)^{-1}.$$

Proof This is a direct conclusion of the parts (1) and (3) of Lemma 7.9. □

From now on we always assume that

$$f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N, 1/2, \omega)$$

is a new form.

Lemma 7.10 *Let* $f(z) = \sum_{n=0}^{\infty} a(n)e(nz)$ *be a new form in* $G(N, 1/2, \omega)$ *which is an eigenfunction of* $\mathrm{T}(p^2)$ *for almost all primes* p. *Then* $a(1) \neq 0$ *and* $t = 1$.

Proof If $a(1) = 0$, then $a(n) = 0$ for any n with $(n, N') = 1$ by Lemma 7.9. By Corollary 6.3 we see that f is in $G^{\text{old}}(N, 1/2, \omega)$ which is impossible, so that $a(1) \neq 0$ and hence $t = 1$ by Corollary 7.1. This completes the proof. □

From now on we always assume that $a(1) = 1$. In this case f is called a normalized new form.

Lemma 7.11 *Let $g \in G(N, 1/2, \omega)$ be an eigenfunction of $\mathrm{T}(p^2)$ for almost all primes p and whose eigenvalues are equal to the ones of f. Then $g = cf$ with a constant c.*

Proof Let c be the coefficient of $e(z)$ of the Fourier expansion of g. Then the coefficient of $e(z)$ of the Fourier expansion of $h = g - cf$ is zero. If $h \neq 0$, then h is an eigenfunction of almost all Hecke operators. By Corollary 7.2 we can find N' such that the coefficient of $e(nz)$ of the Fourier expansion of h is zero for all n with $(n, N') = 1$. By Corollary 6.3 we know that $h \in G^{\text{old}}(N, 1/2, \omega)$. Hence there exists a factor N_1 of N, a character ψ modulo N_1 and a normalized new form g_1 in $G(N_1, 1/2, \psi)$ such that g_1, f and h have the same eigenvalues for almost all Hecke operators. But the character ψ is determined uniquely by the set of all eigenvalues c_p by Corollary 7.1. Hence $\psi = \omega$ and $g_1 \in G^{\text{old}}(N, 1/2, \omega)$. Similarly we have that $f - g_1 \in G^{\text{old}}(N, 1/2, \omega)$, so $f = g_1 + (f - g_1) \in G^{\text{old}}(N, 1/2, \omega)$ which contradicts that f is a new form. This implies that $h = 0$, i.e., $g = cf$. This completes the proof. □

Lemma 7.12 *Let f be a new form in $G(N, 1/2, \omega)$ and be an eigenfunction of almost all* Hecke *operators. Then f is an eigenfunction of all* Hecke *operators* $\mathrm{T}(p^2)$. *Assume that $f|\mathrm{T}(p^2) = c_p f$. Then*

$$\sum_{n=1}^{\infty} a(n)n^{-s} = \prod_{p|N}(1 - c_p p^{-2s})^{-1} \prod_{p\nmid N}(1 - \omega(p)p^{-2s})^{-1}$$

and $c_p = 0$ if $4p|N$.

Proof Let p be any prime. Put $g = f|\mathrm{T}(p^2)$. By the assumptions of the lemma we know that g and f have the same eigenvalues with respect to the Hecke operators $\mathrm{T}(q^2)$ for almost all primes q. By Lemma 7.11 we have $g = cf$. This shows that f is an eigenfunction of all Hecke operators. The Euler product can be deduced by Corollary 7.2. Assume that $4p|N$, then by Lemma 7.9 we see that $f|\mathrm{T}(p) \in G(N, 1/2, \omega\chi_p)$ and

$$f|\mathrm{T}(p) = \sum_{n=0}^{\infty} a(np)e(nz) = \sum_{m=0}^{\infty} a(m^2p^2)e(pm^2z) = (f|\mathrm{T}(p^2))|V(p) = c_p f|V(p).$$

If $c_p \neq 0$, applying Lemma 6.22 to $f|\mathrm{T}(p)$ we know that ω is well-defined modulo N/p and there exists a $g \in G(N/p, 1/2, \omega)$ such that $f|\mathrm{T}(p) = g|V(p)$. Hence $g = c_p f$ which contradicts the fact that f is a new form, so that $c_p = 0$. This completes the proof. □

Lemma 7.13 *Let the assumptions be the same as in* Lemma 7.12. *Then N is a square and $f|W(N) = cf|H$ with a constant c.*

Proof Let $p \nmid N$ be a prime. Then $f|\mathrm{T}(p^2) = c_p f$ and $c_p = \omega(p)(1 + p^{-1})$. By Theorem 5.19 we see that

$$f|W(N)\mathrm{T}(p^2) = \overline{\omega}(p^2)c_p f|W(N) = \overline{c_p} f|W(N), \quad f|H\mathrm{T}(p^2) = (c_p f)|H = \overline{c_p} f|H.$$

Since $W(N), H$ send new forms to new forms, $f|W(N)$ is a new form in $G(N, 1/2, \overline{\omega}\chi_N)$ and $f|H$ a new form in $G(N, 1/2, \overline{\omega})$. Since they have the same eigenvalues with respect to $\mathrm{T}(p^2)$ for all $p \nmid N$, and the set of eigenvalues $\overline{c_p}$ determines uniquely the corresponding character, we know that $\overline{\omega}\chi_N = \overline{\omega}$. This shows that N is a square. Lemma 7.11 implies that $f|W(N) = cf|H$ with a constant c. This completes the proof. □

Theorem 7.5 *Let $f \in G(N, 1/2, \omega)$ be a normalized new form which is an eigenfunction of almost all* Hecke *operators. Denote by r the conductor of ω. Then $N = 4r^2$, $f = \dfrac{1}{2}\theta_\omega$.*

Proof Put

$$F(s) := \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_{p|N}(1 - c_p p^{-2s})^{-1} \prod_{p\nmid N}(1 - \omega(p)p^{-2s})^{-1},$$

$$\overline{F}(s) := \sum_{n=1}^{\infty} \overline{a(n)}n^{-s}.$$

By Theorem 5.22 we know that the above series is absolutely convergent for $\mathrm{Re}(s) > 3/2$ and we have the following functional equation:

$$(2\pi)^{-s}\Gamma(s)F(s) = c_1\left(\frac{2\pi}{N}\right)^{s-1/2}\Gamma(1/2 - s)\overline{F}(1/2 - s), \tag{7.16}$$

where we used the fact that $f|W(N) = cf|H$, c_1 and the following c_2, c_3, c_4 are all constants. Set

$$G(s) = L(2s, \omega) = \prod_{p\nmid r}(1 - \omega(p)p^{-2s})^{-1},$$

$$\overline{G}(s) = L(2s, \overline{\omega}).$$

Then we have

$$(2\pi)^{-s}\Gamma(s)G(s) = c_2\left(\frac{2\pi}{4r^2}\right)^{s-1/2}\Gamma(1/2 - s)\overline{G}(1/2 - s). \tag{7.17}$$

From (7.16) and (7.17) we see that

$$\prod_{p|m}\frac{1 - c_p p^{-2s}}{1 - \omega(p)p^{-2s}} = c_3\left(\frac{N}{4r^2}\right)^{s-1/2}\prod_{p|m}\frac{1 - \overline{c_p}p^{2s-1}}{1 - \overline{\omega}_p p^{2s-1}}, \tag{7.18}$$

where m is the product of all prime divisors p of N with $c_p \neq \omega(p)$. If there exists a $p|m$ with $\omega(p) \neq 0$, then the function on the left (resp. right) hand side of (7.18) has infinite (resp. no) poles on the line $\mathrm{Re}(s) = 0$. Hence $\omega(p) = 0$ (i.e., $p|r$) for any $p|m$. In this case we have $c_p \neq 0$ since $c_p \neq \omega(p)$,

$$\prod_{p|m}(1 - c_p p^{-2s}) = c_4 \left(\frac{Nm^2}{4r^2}\right)^s \prod_{p|m}(1 - c_p' p^{-2s}),$$

where $c_p' = p/\overline{c_p}$. Considering the zeros of the functions on both sides of the above equality we know that $c_p = c_p'$ for any $p|m$, so that $|c_p|^2 = p$ and hence $c_4 = 1$, $Nm^2 = 4r^2$. By Lemma 7.12 we know that $c_p = 0$ if $4p|N$. This implies that $m = 1$ or $m = 2$ by the definition of m. If $m = 1$, then $N = 4r^2$. If $m = 2$, then $c_2 \neq 0$, so $8 \nmid N$. But $\omega(2) = 0$, so $4|r$ which contradicts the fact that $4N = 4r^2$ and $8 \nmid N$. We have shown that $N = 4r^2$ and $F(s) = G(s)$. Thus for any $n \geqslant 1$ the coefficients of $e(nz)$ in the Fourier expansions of f and $\frac{1}{2}\theta_\omega$ coincide with each other, i.e., $f - \frac{1}{2}\theta_\omega \in G(N, 1/2, \omega)$ is a constant, so that it must be zero. This completes the proof. □

Lemma 7.14 *Let ω be an even character with conductor r. Then $\frac{1}{2}\theta_\omega \in G(4r^2, 1/2, \omega)$ is a normalized new form.*

Proof We know that θ_ω is in $G(4r^2, 1/2, \omega)$. By Theorem 5.15 we see that

$$\theta_\omega|\mathrm{T}(p^2) = \omega(p)(1 + p^{-1})\theta_\omega, \quad \forall\, p \nmid 4r^2.$$

If θ_ω is not a new form in $G(4r^2, 1/2, \omega)$, then there exists a proper divisor N_1 of $4r^2$, a character ψ modulo N_1 and a new form f in $G(N_1, 1/2, \psi)$ such that f and θ_ω have the same eigenvalues $\psi(p)(1 + p^{-1}) = \omega(p)(1 + p^{-1})$ for almost all Hecke operators $\mathrm{T}(p^2)$. Therefore $\omega = \psi$ and $N_1 = 4r^2$ by Theorem 7.5. This contradicts $N_1 < 4r^2$, hence $\theta_\omega \in G(4r^2, 1/2, \omega)$ is a new form. This completes the proof. □

Let

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N \right\}.$$

Suppose that $f(z) = \sum_{n=0}^{\infty} a(n)e(nz)$ is a modular form of weight $k/2$ for the group $\Gamma_1(N)$. Let ε be a periodic function on $\mathbb{Z}$ with period M. Put

$$(f * \varepsilon)(z) = \sum_{n=0}^{\infty} a(n)\varepsilon(n)e(nz).$$

The Fourier transformation of ε is

$$\hat{\varepsilon}(m) = M^{-1} \sum_{n=1}^{M} \varepsilon(n) e(-nm/M),$$

by the inverse Fourier transformation we have

$$\varepsilon(n) = \sum_{m=1}^{M} \hat{\varepsilon}(m) e(nm/M).$$

Hence we obtain that

$$(f * \varepsilon)(z) = \sum_{m=1}^{M} \hat{\varepsilon}(m) f(z + m/M),$$

It is clear that the function $f(z+m/M)$ is a modular form of weight $k/2$ for the group $\Gamma_1(NM^2)$.

Lemma 7.15 *The following two assertions are equivalent*:

(1) *the values of f at all cusp points m/M ($m \in \mathbb{Z}$) are equal to zero* (*where m and M may not be co-prime to each other*);

(2) *for every periodic function ε with period M, the function*

$$L(f * \varepsilon, s) = \sum_{n=1}^{\infty} a(n)\varepsilon(n)n^{-s}$$

is holomorphic at $s = k/2$.

The similar result holds also for modular forms of integral weights and the proof is completely similar to the following one.

Proof The assertion (1) is equivalent to the fact that for any periodic function ε with period M the function $f * \varepsilon$ takes value 0 at the cusp point $s = 0$. By Theorem 5.22 the assertion (2) is equivalent to the fact that the function $f * \varepsilon | W(NM^2)$ takes value 0 at $\mathrm{i}\infty$. But the value of $f * \varepsilon | W(NM^2)$ at $\mathrm{i}\infty$ differs from the one of $f * \varepsilon$ at the cusp point $s = 0$ by a constant multiple, so the lemma holds. This completes the proof. □

Corollary 7.3 *f is a cusp form if and only if $L(f * \varepsilon, s)$ is holomorphic at $s = k/2$ for any periodic function ε on $\mathbb{Z}$.*

Since every cusp point is $\Gamma_0(N)$-equivalent to some cusp point m/N, (m and N may not be co-prime to each other), we only need to consider periodic functions with period N for $f \in G(N, 1/2, \omega)$.

Lemma 7.16 *Let ψ be an even character but not totally even. Then θ_ψ is a cusp form.*

Proof Let ε be any periodic function on $\mathbb{Z}$ with period N. Without loss of generality, we may assume that N is a multiple of the conductor $r(\psi)$ of ψ. By Corollary 7.3, we only need to show that

$$F_\varepsilon(s) = 2\sum_{n=1}^{\infty} \varepsilon(n^2)\psi(n)n^{-2s}$$

is holomorphic at $s = 1/2$. We have

$$F_\varepsilon(s) = 2\sum_{m=1}^{N} \varepsilon(m^2)\psi(m)F_{m,N}(2s),$$

where

$$F_{m,N}(s) = \sum_{\substack{n\equiv m \bmod N,\\ n\geqslant 1}} n^{-s}.$$

It is well known that $F_{m,N}(s)$ has a simple pole at $s = 1$ with residue $1/N$. Hence the residue of $F_\varepsilon(s)$ at $s = 1/2$ is equal to $R(\varepsilon,\psi)/N$ with $R(\varepsilon,\psi) = \sum_{m=1}^{N} \varepsilon(m^2)\psi(m)$. We now only need to show that $R(\varepsilon,\psi) = 0$. Since ψ is not totally even, there exists a prime divisor l of $r(\psi)$ such that the l-part ψ_l of ψ is odd. Write $N = l^a N'$ with $l \nmid N'$. Take an integer l' such that $l' \equiv -1 \pmod{l^a}$, $l' \equiv 1 \pmod{N'}$. It is clear that l' is invertible in $\mathbb{Z}/N\mathbb{Z}$ and $l'^2 \equiv 1(N), \psi(l') = -1$. Therefore

$$R(\varepsilon,\psi) = \sum_{m \bmod N} \varepsilon((l'm)^2)\psi(l'm) = -\sum_{m \bmod N} \varepsilon(m^2)\psi(m) = -R(\varepsilon,\psi),$$

i.e., $R(\varepsilon,\psi) = 0$. This completes the proof. □

Lemma 7.17 *Let ψ be a totally even character, T a finite set of positive integers. If $f = \sum_{t\in T} c_t\theta_{\psi,t} (c_t \in \mathbb{C})$ is a cusp form, then $c_t = 0$ for all t.*

Proof Otherwise, let t_0 be the smallest number in T such that $c_{t_0} \neq 0$. Take a positive integer M such that M is a common multiple of $2r(\psi)$ and all numbers of T. Since ψ is totally even, there exists a character α modulo M with $\alpha^2 = \psi$. Define a periodic function ε on $\mathbb{Z}$ as follows:

$$\varepsilon(n) = \begin{cases} \overline{\alpha}(n/t_0), & \text{if } t_0|n \text{ and } (n/t_0, M) = 1,\\ 0, & \text{otherwise.}\end{cases}$$

We see that

$$\varepsilon(t_0n^2) = \begin{cases} \overline{\psi}(n), & \text{if } (n, M) = 1,\\ 0, & \text{otherwise}\end{cases}$$

and

$$\varepsilon(tn^2) = 0, \quad \text{if } t \in T, t > t_0,$$

(since $(tn^2, M) \geqslant t > t_0$). Therefore

$$L(f * \varepsilon, s) = 2c_{t_0} \sum_{(n,M)=1, n\geqslant 1} \overline{\psi}(n)\psi(n)(t_0 n^2)^{-s} = 2c_{t_0} t_0^{-s} \sum_{(n,M)=1, n\geqslant 1} n^{-2s}$$

whose residue at $s = 1/2$ is

$$c_{t_0} t_0^{-1/2} \varphi(M)/M \neq 0.$$

By Corollary 7.3 we see that f is not a cusp form which is impossible. This completes the proof. □

Proof of Theorem 7.4 (1) We first prove that $\{\theta_{\psi,t} | (\psi, t) \in \Omega(N, \omega)\}$ are linearly independent. Since ψ is determined uniquely by ω and t, t appears only one time as the second entry of a paring (ψ, t) in $\Omega(N, \omega)$. Assume

$$\sum_{i=1}^{m} \lambda_i \theta_{\psi_i, t_i} = 0,$$

where $t_1 < t_2 < \cdots < t_m$, $\lambda_i \neq 0$ $(1 \leqslant i \leqslant m)$. The coefficient of $e(t_1 z)$ of the Fourier expansion of θ_{ψ_1,t_1} is equal to 2, and the ones of θ_{ψ_i,t_i} $(i \geqslant 2)$ are equal to 0. This shows that $\lambda_1 = 0$ which contradicts $\lambda_1 \neq 0$.

We now show that $\{\theta_{\psi,t} | (\psi, t) \in \Omega(N, \omega)\}$ generate $G(N, 1/2, \omega)$. Let $f, g \in G(N, 1/2, \omega)$. For any $p \nmid N$, using Lemma 5.26 we have

$$\langle f | \mathrm{T}(p^2), g\rangle = \omega(p^2)\langle f, g | \mathrm{T}(p^2)\rangle,$$

which shows that $\overline{\omega}\mathrm{T}(p^2)$, $p \nmid N$ are Hermitian and commute each other. So there is a basis of $G(N, 1/2, \omega)$ whose every element is an eigenfunction of $\mathrm{T}(p^2)$, $p \nmid N$. Hence we only need to show that if f is an eigenfunction of $\mathrm{T}(p^2)$ $(p \nmid N)$ then f is a linear combination of $\{\theta_{\psi,t} | (\psi, t) \in \Omega(N, \omega)\}$. We apply induction on N. If f is a new form, Theorem 7.5 gives what we want. If f is an old form, then f is either in $G(N/p, 1/2, \omega)$ and ω is well-defined for modulo N/p, or $f = g|V(p)$ with $g \in G(N/p, 1/2, \omega\chi_p)$ and $\omega\chi_p$ well-defined modulo N/p. In the first case, f is a linear combination of $\{\theta_{\psi,t} | (\psi, t) \in \Omega(N/p, \omega)\}$ by the induction hypothesis. It is clear that $\Omega(N/p, \omega) \subset \Omega(N, \omega)$. For the second case, g is a linear combination of $\{\theta_{\psi,t} | (\psi, t) \in \Omega(N/p, \omega\chi_p)\}$ due to the induction hypothesis, hence f is a linear combination of $\{\theta_{\psi,t} | (\psi, t) \in \Omega(N, \omega)\}$. This completes the proof of the part (1).

(2) We only need to show the following three assertions: ① if $(\psi, t) \in \Omega_{\mathrm{c}}(N, \omega)$, then $\theta_{\psi,t}$ is a cusp form; ② any non-zero linear combination of $\{\theta_{\psi,t} | (\psi, t) \in \Omega_{\mathrm{e}}(N, \omega)\}$ is not a cusp form; ③ if $(\psi, t) \in \Omega_{\mathrm{c}}(N, \omega)$, $(\psi', t') \in \Omega_{\mathrm{e}}(N, \omega)$, then $\theta_{\psi,t}$ is orthogonal with $\theta_{\psi',t'}$ under the Petersson inner product.

The assertion ① is deduced from Lemma 7.16. Let now V be the intersection of the set of linear combinations of $\{\theta_{\psi,t} | (\psi, t) \in \Omega_{\mathrm{e}}(N, \omega)\}$ and the space of cusp forms.

If $V \neq 0$, since V is an invariant space for the Hecke operators $\mathrm{T}(p^2)(p \nmid N)$, there exists a $0 \neq f \in V$ which is an eigenfunction of all $\mathrm{T}(p^2)(p \nmid N)$. But $\psi(p)(1+p^{-1})$ is the eigenvalue of $\theta_{\psi,t}$ with respect to $\mathrm{T}(p^2)$. Hence f is a linear combination of some $\theta_{\psi,t}$ with the same ψ. This contradicts Lemma 7.17 and hence $V = 0$ which shows the assertion ②. Finally we prove the assertion ③. Since $\overline{\psi'}\omega^2$ is a totally even character, we see that $\overline{\psi'}\omega^2 \neq \psi$. So there exists a prime p with $\psi(p) \neq \overline{\psi'}\omega^2(p)$. Then $\psi(p)(1+p^{-1})$ and $\psi'(p)(1+p^{-1})$ are the eigenvalues of $\theta_{\psi,t}$ and $\theta_{\psi',t'}$ respectively with respect to $\mathrm{T}(p^2)$. By the properties of Petersson inner product we have

$$\langle \theta_{\psi,t}|\mathrm{T}(p^2), \theta_{\psi',t'}\rangle = \omega^2(p)\langle \theta_{\psi,t}, \theta_{\psi',t'}|\mathrm{T}(p^2)\rangle,$$

thus

$$\psi(p)\langle \theta_{\psi,t}, \theta_{\psi',t'}\rangle = \overline{\psi'}\omega^2(p)\langle \theta_{\psi,t}, \theta_{\psi',t'}\rangle,$$

i.e.,

$$\langle \theta_{\psi,t}, \theta_{\psi',t'}\rangle = 0,$$

which showed ③. This completes the proof of Theorem 7.4. □

7.3 Construction of Eisenstein Series with Weight 3/2

In this section we shall construct a basis of the Eisenstein space of weight 3/2 for a modular group $\Gamma_0(4N)$ with N a square-free odd positive integer. The content of this section is due to D. Y. Pei, 1982. Considering the Eisenstein series in Chapter 2, we have

Theorem 7.6 *For any $k > 3$ and ω not a real character, $E_k(\omega, N)$ and $E'_k(\overline{\omega}\chi_N, N)$ belong to $\mathcal{E}(N, k/2, \omega)$. The functions $f_2^*(\omega, N)$ and $f_2(\omega, N)$ belong to $\mathcal{E}(N, 3/2, \omega)$. If D is a square-free odd positive integer, then the functions $f_1(\mathrm{id.}, 4D)$ and $f_1(\mathrm{id.}, 8D)$ belong to $\mathcal{E}(4D, 3/2, \mathrm{id.})$ and $\mathcal{E}(8D, 3/2, \mathrm{id.})$ respectively.*

Proof We only prove the theorem for $E_k(\omega, N)$ since the other assertion can be proved similarly. In Chapter 2 we proved that $E_k(\omega, N)$ is a holomorphic function on $\mathbb{H}$. We prove that it is also holomorphic at each cusp point. It is clear that $E_k(\omega, N)$ is holomorphic at i∞. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with $c \neq 0$, we have

$$\begin{aligned} |E_k(\omega, N)(\gamma(z))(cz+d)^{-k/2}| &\leqslant \left(1 + \rho y^{-(k+5)/2}|cz+d|^{k+5}\right)|cz+d|^{-k/2} \\ &\leqslant \rho' y^{5/2} \quad (y \to \infty) \end{aligned}$$

by equality (2.31).

This shows that $E_k(\omega, N)$ is holomorphic at all cusp points which means that $E_k(\omega, N)$ belongs to $G(N, k/2, \omega)$. Now, we want to prove $E_k(\omega, N)$ is orthogonal to cusp forms. Let

$$f(z) = \sum_{n=1}^{\infty} c(n)e(nz) \in S(N, k/2, \omega)$$

and $\gamma \in \Gamma_0(N)$. Since $\displaystyle\int_0^1 \overline{f}(z)\mathrm{d}x = 0$ and

$$\overline{f}\big(\gamma(z)\big)\mathrm{Im}\big(\gamma(z)\big)^{(s+k)/2} = \overline{\omega}(d_\gamma)j(\gamma,z)^{-k}|j(\gamma,z)|^{-2s}\overline{f}(z)y^{(s+k)/2},$$

we have

$$\begin{aligned}0 &= \int_0^\infty y^{(s+k)/2-2}\int_0^1 \overline{f}(z)\mathrm{d}x\mathrm{d}y = \int_{\Gamma_\infty\backslash\mathbb{H}} \overline{f}(x+\mathrm{i}y)y^{(s+k)/2-2}\mathrm{d}x\mathrm{d}y\\ &= \iint\limits_{\Gamma_0(N)\backslash\mathbb{H}} E_k(s,\overline{\omega},N)(x+\mathrm{i}y)\overline{f}(x+\mathrm{i}y)y^{k/2-2}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

To take $s=0$ gives the orthogonality. □

We can compute the values of $E_3'(\omega,N)$, $E_3(\omega,N)$, $f_1(\mathrm{id.},4D)$, $f_2^*(\mathrm{id.},4D)$, $f_2^*(\mathrm{id.},8D)$ and $f_2(\mathrm{id.},8D)$ at cusp points similarly as is done in Section 7.1.

Lemma 7.18 (1) *Let* $\omega^2 \neq \mathrm{id.}$, *then* $V(E_3'(\omega,N),1) = \mathrm{i}$. *For any* $d/c \in S(N)$ *and* $c \neq 1$, *we have* $V(E_3'(\omega,N),d/c) = 0$.

(2) *Let* $\omega^2 \neq \mathrm{id.}$, *then* $V(E_3(\omega,N),\mathrm{i}\infty) = 1$. *For any* $d/c \in S(N)$ *and* $c \neq N$, *we have* $V(E_3(\omega,N),d/c) = 0$.

Proof (1) By (2.7) we have

$$(-z)^{3/2}E_3'(\omega,N)(z) = \mathrm{i}E_3(\omega,N)(-1/(Nz)). \tag{7.19}$$

Hence, $V(E_3'(\omega,N),1) = \mathrm{i}V(E_3(\omega,N),\mathrm{i}\infty) = \mathrm{i}$.

The other assertion can be proved by a method similar to the proof of Theorem 7.2.

(2) The first result is obvious and the second one is obvious from (7.19). □

Lemma 7.19 *We have*

$$\begin{aligned}V(f_1(\mathrm{id.},4D),1) &= -(1+\mathrm{i})(4D)^{-1},\\ V(f_1(\mathrm{id.},8D),1) &= -(1+\mathrm{i})(8D)^{-1}.\end{aligned}$$

Proof By the definition of $f_1(\mathrm{id.},4D)$, we have

$$f_1(\mathrm{id.},4D)(z) = E_3(0,\mathrm{id.},4D)(z) - (1-\mathrm{i})(4D)^{-1}z^{-3/2}E_3'(0,\chi_D,4D)(-(4Dz)^{-1}).$$

Therefore,

$$\begin{aligned}z^{-3/2}f_1(\mathrm{id.},4D)(-(4Dz)^{-1}) &= E_3'(0,\mathrm{id.},4D)(z) - 2D^{1/2}(1+\mathrm{i})E_3(0,\chi_D,4D)(z)\\ &= -2D^{1/2}(1+\mathrm{i})f_1(\chi_D,4D)(z).\end{aligned}$$

By the definition of $V(f_1(\mathrm{id.},4D),1)$ and (2.37), we have

$$\begin{aligned}V(f_1(\text{id.},4D),1) &= \lim_{z\to \mathrm{i}\infty}(4Dz)^{-3/2}f_1(\text{id.},4D)(-(4Dz)^{-1})\\ &= -(1+\mathrm{i})(4D)^{-1}.\end{aligned}$$

And the second result can be proved similarly. □

Lemma 7.20 *We have*

$$\begin{aligned}V(f_2^*(\text{id.},4D),1/\beta) &= -4^{-1}(1+\mathrm{i})\mu(D/\beta)\beta/(D\varepsilon_\beta),\\ V(f_2^*(\text{id.},4D),1/(2\beta)) &= 0,\\ V(f_2^*(\text{id.},4D),1/(4\beta)) &= \mu(D/\beta)\beta/D.\end{aligned}$$

Proof We know that $f_2^*(\text{id.}, 4D) \in G(4D, 3/2, \text{id.})$ and for any prime factor $p|2D$, $f_2^*|\mathrm{T}(p^2) = f_2^*$ (This can be proved by (2.42)).

In particular, $f_2^*|\mathrm{T}(4) = f_2^*$. Hence

$$f_2^*(\text{id.},4D)\left(z+\frac{1}{2\beta}\right) = 4^{-1}\sum_{k=1}^{4} f_2^*(\text{id.},4D)\left(\frac{z}{4}+\frac{1+2k\beta}{8\beta}\right).$$

But $(1+2\beta k)/(8\beta)$ and $1/(4\beta)$ are $\Gamma_0(4D)$-equivalent. So we have

$$\begin{aligned}V(f_2^*(\text{id.},4D),1/(2\beta)) &= 4^{-1}\sum_{k=1}^{4} V\left(f_2^*(\text{id.},4D),\frac{1+2\beta k}{8\beta}\right)\\ &= 4^{-1}\sum_{k=1}^{4}\left(\frac{2\beta}{1+2\beta k}\right)\varepsilon_{1+2k}V(f_2^*(\text{id.},4D),1/(4\beta)) = 0,\end{aligned}$$

where we used the fact $\left(\frac{2\beta}{a+4\beta}\right) = -\left(\frac{2\beta}{a}\right)$. Since $V(f_2^*(\text{id.},4D),1/(4D)) = 1$, by Lemma 7.1, we have $V(f_2^*(\text{id.},4D),1/4) = \mu(D)D^{-1}$. Hence we get the third equality by the second equality of Lemma 7.1. Using

$$f_2^*(\text{id.},4D)(z) = 4^{-1}\sum_{k=1}^{4} f_2^*(\text{id.},4D)\left(\frac{z}{4}+\frac{k}{4}\right)$$

and

$$V(f_2^*(\text{id.},4D),1/2) = 0,$$

we get

$$V(f_2^*(\text{id.},4D),1) = 4^{-1}(1+\mathrm{i})V(f_2^*(\text{id.},4D),1/4) + 2V(f_2^*(\text{id.},4D),1).$$

Since $3/4$ and $1/4$ are $\Gamma_0(4D)$-equivalent, we get

$$V(f_2^*(\text{id.},4D),1) = -4^{-1}(1+\mathrm{i})\mu(D)D^{-1}.$$

This proves the first assertion in Lemma 7.20 from Lemma 7.1. This completes the proof. □

Lemma 7.21 *Let m, β, l be factors of D. Let $f(z) \in G(8D, 3/2, \chi_{2l})$ satisfy*

$$\begin{aligned} f|\mathrm{T}(p^2) &= f, \quad \forall\ p|m, \\ f|\mathrm{T}(p^2) &= pf, \quad \forall\ p|Dm^{-1}. \end{aligned}$$

Then

$$\begin{aligned} V(f, 1/(2^r\alpha)) &= \mu(\alpha)\alpha(\alpha, l)^{-1/2}\varepsilon_{\alpha/(\alpha,l)}^{-1}\left(\frac{2^{1-r}l/(\alpha, l)}{\alpha/(\alpha, l)}\right)V(f, 1/2^r), \quad r = 0, 1, \\ V(f, 1/(8\alpha)) &= \mu(\alpha)\alpha(\alpha, l)^{-1/2}\varepsilon_{l/(\alpha,l)}\varepsilon_l^{-1}\left(\frac{\alpha/(\alpha, l)}{l/(\alpha, l)}\right)V(f, 1/8), \\ V(f, 1/(2^r\beta)) &= 0, \quad r = 0, 1, 3 \text{ and } (\beta, D/m) \neq 1. \end{aligned}$$

Proof This can be proved in a similar way as in the proof of Lemma 7.4. □

Lemma 7.22 *Let β be any factor of D. Then we have*

$$\begin{aligned} V(f_2^*(\chi_{2D}, 8D), 1/\beta) &= -2^{-3/2}(1+\mathrm{i})\mu(D/\beta)\beta^{1/2}D^{-1/2}, \\ V(f_2^*(\chi_{2D}, 8D), 1/(2\beta)) &= 2^{-1}(1+\mathrm{i})\mu(D/\beta)\beta^{1/2}D^{-1/2}, \\ V(f_2^*(\chi_{2D}, 8D), 1/(4\beta)) &= 0, \\ V(f_2^*(\chi_{2D}, 8D), 1/(8\beta)) &= \mu(D/\beta)\beta^{1/2}D^{-1/2}\varepsilon_{D/\beta}. \end{aligned}$$

Proof Put $h = f_2^*(\chi_{2D}, 8D)$. Then $h \in G(8D, 3/2, \chi_{2D})$ and $h|\mathrm{T}(p^2) = h$ for any prime factor $p|2D$. Using $h|\mathrm{T}(4) = h$ and $V(h, 1/(8D)) = 1$, we can prove $V(h, 1/(4\beta)) = 0$ for any $\beta|D$ and

$$\begin{aligned} V(h, 1) &= -2^{-3/2}(1+\mathrm{i})\mu(D)D^{-1/2}, \\ V(h, 1/2) &= 2^{-1}(1+\mathrm{i})\mu(D)D^{-1/2}, \\ V(h, 1/8) &= \mu(D)D^{-1/2}\varepsilon_D. \end{aligned}$$

Now taking $l = D$ in Lemma 7.21 gives Lemma 7.22. □

Lemma 7.23 *Let β be any factor of D. Then we have*

$$\begin{aligned} -2^{-1}(1+\mathrm{i})\mu(D)V\big(f_2(\mathrm{id.}, 8D), 1/\beta\big) &= -16^{-1}(1+\mathrm{i})\mu(D/\beta)\beta D^{-1}\varepsilon_\beta^{-1}, \\ -2^{-1}(1+\mathrm{i})\mu(D)V\big(f_2(\mathrm{id.}, 8D), 1/(2\beta)\big) &= 0, \\ -2^{-1}(1+\mathrm{i})\mu(D)V\big(f_2(\mathrm{id.}, 8D), 1/(4\beta)\big) &= -2^{-1}\mu(D/\beta)\beta D^{-1}, \\ -2^{-1}(1+\mathrm{i})\mu(D)V\big(f_2(\mathrm{id.}, 8D), 1/(8\beta)\big) &= \mu(D/\beta)\beta D^{-1}. \end{aligned}$$

Proof By the definition of $f_2^*(\chi_{2D}, 8D)(z)$ and $f_2(\mathrm{id.}, 8D)(z)$, we have

$$f_2^*(\chi_{2D}, 8D)(-1/(8Dz))z^{-3/2} = 8\mathrm{i}Df_2(\mathrm{id.}, 8D)(z).$$

Let c be a divisor of $8D$. Since

$$(-cz)^{3/2}f_2(\mathrm{id.},8D)(z+c^{-1}) = -\mathrm{i}(8D)^{-1}c^{3/2}f_2^*(\chi_{2D},8D) \times \left(\frac{cz}{8D(z+c^{-1})}-\frac{c}{8D}\right)\left(-\frac{z}{z+c^{-1}}\right)^{3/2}.$$

We have

$$V(f_2(\mathrm{id.},8D),1/c) = -\mathrm{i}(8D)^{-1}c^{3/2}V(f_2^*(\chi_{2D},8D),-c/(8D)).$$

Since the cusp points $-c/(8D)$ and $c/(8D)$ are $\Gamma_0(8D)$-equivalent, we get the lemma by Lemma 7.22. □

Lemma 7.24 *Let $f \in G(N,3/2,\omega)$ be zero at all cusp points of $S(N)$ except $1/N$. Then $g = f|W(Q)$ is zero at all cusp points of $S(N)$ except $1/(NQ^{-1})$.*

Proof It is clear that the transformation $z \to \dfrac{Qz-1}{uNz+vQ}$ induces a permutation of the equivalent classes of cusp points of $\Gamma_0(N)$ and

$$\left.\frac{Qz-1}{uNz+vQ}\right|_{z=QN^{-1}} = \frac{Q-N/Q}{(u+v)N},$$

which is $\Gamma_0(N)$-equivalent to $1/N$. These two facts imply Lemma 7.24. □

Let $N=2^rN'$, $r \geqslant 2$, $2\nmid N'$ and ω be an even character modulo N with conductor $r(\omega)$. Then by the dimension formula, we have

$$\dim\mathcal{E}(N,3/2,\omega) = \begin{cases} 2\displaystyle\sum_{\substack{c|N'\\(c,N'/c)|N/r(\omega)}} \phi((c,N'/c)) - \dim\mathcal{E}(N,1/2,\omega), & \text{if } r=2,\\ 3\displaystyle\sum_{\substack{c|N'\\(c,N'/c)|N/r(\omega)}} \phi((c,N'/c)) - \dim\mathcal{E}(N,1/2,\omega), & \text{if } r=3,\\ \displaystyle\sum_{\substack{c|N\\(c,N/c)|N/r(\omega)}} \phi((c,N/c)) - \dim\mathcal{E}(N,1/2,\omega), & \text{if } r\geqslant 4. \end{cases}$$

By Theorem 7.4, we know that $\dim\mathcal{E}(N,1/2,\omega)$ is the number of pairs (ψ,t) of $\Omega_{\mathrm{e}}(N,\omega)$.

Now we always assume that D is an odd square-free positive integer, m,l and β are divisors of D, α is a divisor of m and v is the number of prime divisors of D. Since $\Omega_{\mathrm{e}}(4D,\chi_l)=\{(\mathrm{id.},l)\}$, $\Omega_{\mathrm{e}}(8D,\chi_l)=\{(\mathrm{id.},l)\}$, $\Omega_{\mathrm{e}}(8D,\chi_{2l})=\{(\mathrm{id.},2l)\}$, we have

$$\dim\mathcal{E}(4D,3/2,\chi_l) = 2^{v+1}-1,$$
$$\dim\mathcal{E}(8D,3/2,\chi_l) = \dim\mathcal{E}(8D,3/2,\chi_{2l}) = 3\cdot 2^v - 1.$$

We shall construct a basic of $\mathcal{E}(4D, 3/2, \chi_l)$, $\mathcal{E}(8D, 3/2, \chi_l)$ and $\mathcal{E}(8D, 3/2, \chi_{2l})$ respectively. Since only Eisenstein series of weight 3/2 are considered, we shall omit all Subscripts 3. E.g., we define

$$\lambda(n, 4D) = \lambda_3(n, 4D) = L_{4D}(2, \mathrm{id.})^{-1} L_{4D}(1, \chi_{-n})\beta_3(n, 0, \chi_D, 4D)$$

and

$$A(p, n) = A_3(p, n), \quad \text{etc.}$$

Define functions

$$\begin{aligned}
g(\chi_l, 4D, 4D)(z) =& 1 - 4\pi(1+\mathrm{i})l^{1/2} \sum_{n=1}^{\infty} \lambda(ln, 4D)(A(2, ln) - 4^{-1}(1-\mathrm{i})) \\
& \times \prod_{p|D} (A(p, ln) - p^{-1}) n^{1/2} e(nz), \\
g(\chi_l, 4m, 4D)(z) =& -4\pi(1+\mathrm{i})l^{1/2} \sum_{n=1}^{\infty} \lambda(ln, 4D)(A(2, ln) - 4^{-1}(1-\mathrm{i})) \\
& \times \prod_{p|m} (A(p, ln) - p^{-1}) n^{1/2} e(nz), \quad \forall m \neq D, \\
g(\chi_l, m, 4D)(z) =& 2\pi l^{1/2} \sum_{n=1}^{\infty} \lambda(ln, 4D) \prod_{p|m} (A(p, ln) - p^{-1}) n^{1/2} e(nz), \quad \forall\ m \neq 1.
\end{aligned}$$

Theorem 7.7 (1) *The functions* $g(\chi_l, 4m, 4D)$, $(\forall m|D)$ *and* $g(\chi_l, m, 4D)$ $(\forall\ 1 \neq m|D)$ *constitute a basis of* $\mathcal{E}(4D, 3/2, \chi_l)$.

(2) *For any* $p \in S(4D)$, *we have*

$$V(g(\chi_l, 4m, 4D), p) = \begin{cases} -4^{-1}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1} l^{1/2}(l, \alpha)^{-1/2} \varepsilon_{\alpha/(l,\alpha)}^{-1} \left(\dfrac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right), \\ \qquad\qquad \text{if } p = 1/\alpha,\ \alpha|m, \\ \mu(m/\alpha)\alpha m^{-1} l^{1/2}(l,\alpha)^{-1/2} \varepsilon_{l/(l,\alpha)} \left(\dfrac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right), \\ \qquad\qquad \text{if } p = 1/(4\alpha),\ \alpha|m, \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

(3) *For any* $p \in S(4D)$, *we have*

$$V(g(\chi_l, m, 4D), p) = \begin{cases} -4^{-1}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1} l^{1/2}(l, \alpha)^{-1/2} \varepsilon_{\alpha/(l,\alpha)}^{-1} \left(\dfrac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right), \\ \qquad\qquad \text{if } p = 1/\alpha,\ \alpha|m, \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

Proof We first prove (2) for $l = 1$. By equality (2.45), we have $g(\mathrm{id.}, 4D, 4D) = f_2^*(\mathrm{id.}, 4D)$. Hence the theorem holds for $g(\mathrm{id.}, 4D, 4D)$ by Theorem 7.6 and Lemma 7.20. Now suppose $m \neq D$. We have

$$\begin{aligned} g(\text{id.}, 4m, 4D) &= -4\pi(1+\mathrm{i}) \prod_{p|D/m} p(1+p)^{-1} \sum_{n=1}^{\infty} \lambda(n, 4D)(A(2,n) - 4^{-1}(1-\mathrm{i})) \\ &\quad \times \prod_{p|m} (A(p,n) - p^{-1}) \prod_{p|D/m} \left\{1 + A(p,n) - (A(p,n) - p^{-1})\right\} n^{1/2} e(nz) \\ &= \prod_{p|D/m} p(1+p)^{-1} \sum_{d|D/m} \mu(d) f_2^*(\text{id.}, 4md). \end{aligned}$$

Therefore $g(\text{id.}, 4m, 4D) \in \mathcal{E}(4D, 3/2, \text{id.})$. But

$$\begin{aligned} A(2, 4n) - 4^{-1}(1-\mathrm{i}) &= 2^{-1}(A(2,n) - 4^{-1}(1-\mathrm{i})), \\ A(p, p^2 n) - p^{-1} &= p^{-1}(A(p,n) - p^{-1}), \quad p \neq 2 \end{aligned} \tag{7.20}$$

implies that

$$\begin{aligned} g(\text{id.}, 4m, 4D)|\mathrm{T}(p^2) &= g(\text{id.}, 4m, 4D), \quad p|2m \\ g(\text{id.}, 4m, 4D)|\mathrm{T}(p^2) &= p g(\text{id.}, 4m, 4D), \quad p|D/m. \end{aligned} \tag{7.21}$$

By Lemma 7.20, we have

$$\begin{aligned} V(g(\text{id.}, 4m, 4D), 1) &= \prod_{p|D/m} p(1+p)^{-1} \sum_{d|D/m} \mu(d) V(f_2^*(\text{id.}, 4md), 1) \\ &= -4^{-1}(1+\mathrm{i}) \prod_{p|D/m} p(1+p)^{-1} \sum_{d|D/m} \mu(d)\mu(md)(md)^{-1} \\ &= -4^{-1}(1+\mathrm{i})\mu(m)m^{-1}. \end{aligned}$$

Using $g(\text{id.}, 4m, 4D)|\mathrm{T}(4) = g(\text{id.}, 4m, 4D)$ and the method for showing Lemma 7.20, we can prove that

$$V(g(\text{id.}, 4m, 4D), 1/(2p)) = 0$$

and

$$V(g(\text{id.}, 4m, 4D), 1/4) = -4(1+\mathrm{i})^{-1} V(g(\text{id.}, 4m, 4D), 1) = \mu(m)m^{-1}.$$

By Lemma 7.4 we get part (2) of the theorem for $l = 1$.

For $l \neq 1$, we have

$$g(\chi_l, 4m, 4D)(z) = g(\text{id.}, 4m, 4D)(z)|\mathrm{T}(l) = l^{-1} \sum_{k=1}^{l} g(\text{id.}, 4m, 4D)\left(\frac{z+k}{l}\right).$$

Hence $g(\chi_l, 4m, 4D) \in \mathcal{E}(4D, 3/2, \chi_l)$ and we have

$$V(g(\chi_l, 4m, 4D), 1) = l^{-1} \sum_{d|l} d^{3/2} \sum_{\substack{k=1 \\ (k,l/d)=1}}^{l/d} V(g(\text{id.}, 4m, 4D), k/(ld^{-1}))$$

$$= l^{-1} \sum_{d|l} l^{3/2} \sum_{k=1}^{l/d} \left(\frac{k}{ld^{-1}} \right) V(g(\mathrm{id.}, 4m, 4D), 1/(ld^{-1}))$$
$$= -4^{-1}(1+\mathrm{i})\mu(m)m^{-1}l^{1/2}$$

by Lemma 7.2. Since (7.21) holds also for $g(\chi_l, 4m, 4D)$, we can prove that the part (2) of the theorem holds also for $g(\chi_l, 4m, 4D)$. This completes the proof of the part (2).

Now we prove part (3) of the theorem. Similar to the above, we only need to consider the case $l = 1$. Suppose $g(\mathrm{id.}, m, 4D) \in \mathcal{E}(4D, 3/2, \mathrm{id.})$, then by (7.20) we have

$$\begin{aligned} &g(\mathrm{id.}, m, 4D)|\mathrm{T}(p^2) = g(\mathrm{id.}, m, 4D), \quad \forall\, p|m, \\ &g(\mathrm{id.}, m, 4D)|\mathrm{T}(p^2) = pg(\mathrm{id.}, m, 4D), \quad \forall\, p|2D/m. \end{aligned} \tag{7.22}$$

Using (7.22) for $p = 2$, we have

$$\begin{aligned} 2V(g(\mathrm{id.}, m, 4D), 1/(4\beta)) &= 4^{-1} \sum_{k=1}^{4} V\left(g(\mathrm{id.}, m, 4D), \frac{1+4\beta k}{4\beta} \right) \\ &= V(g(\mathrm{id.}, m, 4D), 1/(4\beta)), \end{aligned}$$

which implies $V(g(\mathrm{id.}, m, 4D), 1/(4\beta)) = 0$.

Using again (7.22) for $p = 2$, we have also

$$2V(g(\mathrm{id.}, m, 4D), 1/(2\beta)) = 4^{-1} \sum_{k=1}^{4} V\left(g(\mathrm{id.}, m, 4D), \frac{1+2\beta k}{8\beta} \right) = 0.$$

So if $V(g(\mathrm{id.}, m, 4D), 1)$ is known, then the values of $g(\mathrm{id.}, m, 4D)$ at all cusp points can be computed by Lemma 7.4. Put

$$f_3(\mathrm{id.}, 4D)(z) = 2\pi \sum_{n=1}^{\infty} \lambda(n, 4D) \left(\prod_{p|D} A(p, n) - D^{-1} \right) n^{1/2} e(nz).$$

Then

$$\begin{aligned} f_1(\mathrm{id.}, 4D) = &-f_3(\mathrm{id.}, 4D) + 1 - 4\pi(1+\mathrm{i}) \sum_{n=1}^{\infty} \lambda(n, 4D) \big(A(2, n) - 4^{-1}(1-\mathrm{i}) \big) \\ &\times \prod_{p|D} A(p, n) n^{1/2} e(nz) \\ = &\, D^{-1} \sum_{m|D} mg(\mathrm{id.}, 4m, 4D) - f_3(\mathrm{id.}, 4D), \end{aligned}$$

which implies that $f_3(\mathrm{id.}, 4D) \in \mathcal{E}(4D, 3/2, \mathrm{id.})$ and

$$V(f_3(\mathrm{id.}, 4D), 1) = D^{-1} \sum_{m|D} mV(g(\mathrm{id.}, 4m, 4D), 1) - V(f_1(\mathrm{id.}, 4D), 1)$$

$$
\begin{aligned}
&= -4^{-1}(1+\mathrm{i})D^{-1}\sum_{m|D}\mu(m) + (1+\mathrm{i})(4D)^{-1}\\
&= (1+\mathrm{i})(4D)^{-1}.
\end{aligned} \tag{7.23}
$$

We shall prove that $g(\mathrm{id.}, m, 4D) \in \mathcal{E}(4D, 3/2, \mathrm{id.})$ and calculate $V(g(\mathrm{id.}, m, 4D), 1)$ by induction, and hence will complete the proof of part (3).

If $D = p$ is a prime, then $g(\mathrm{id.}, p, 4p) = f_3(\mathrm{id.}, 4p) \in \mathcal{E}(4p, 3/2, \mathrm{id.})$ and then (7.23) implies the part (3). Now we use induction on the number of prime divisors of D. Since

$$
\begin{aligned}
&\prod_{p|\beta}(1+p)^{-1}\prod_{p|D}(A(p,n)-p^{-1})\\
&= \prod_{p|D/\beta}(A(p,n)-p^{-1})\prod_{p|\beta}\left\{(1+A(p,n))(1+p)^{-1}-p^{-1}\right\}\\
&= \sum_{d|\beta}\mu(\beta|d)d\beta^{-1}\prod_{p|D/\beta}(A(p,n)-p^{-1})\prod_{p|d}(1+A(p,n))(1+p)^{-1},
\end{aligned}
$$

we get

$$
\begin{aligned}
&\sum_{D\neq\beta|D}\mu(\beta)\prod_{p|\beta}(1+p)^{-1}\prod_{p|D}(A(p,n)-p^{-1})\\
&= \prod_{p|D}A(p,n) - D^{-1} + \sum_{D\neq\beta|D}\ \sum_{1\neq d|\beta}\mu(d)d\beta^{-1}\\
&\quad \prod_{p|D/\beta}(A(p,n)-p^{-1})\prod_{p|d}(1+A(p,n))(1+p)^{-1}.
\end{aligned}
$$

But

$$
\lambda_k(n, 4m) = \lambda_k(n, 4D)\prod_{p|D/m}(1+A_k(p,n)),
$$

we get

$$
\begin{aligned}
&\sum_{D\neq\beta|D}\mu(\beta)\prod_{p|\beta}(1+p)^{-1}g(\mathrm{id.}, D, 4D)\\
&= f_3(\mathrm{id.}, 4D) + \sum_{D\neq\beta|D}\ \sum_{1\neq d|\beta}\mu(d)d\beta^{-1}g(\mathrm{id.}, D/\beta, 4D/d).
\end{aligned}
$$

By induction hypothesis, we get $g(\mathrm{id.}, D, 4D) \in \mathcal{E}(4D, 3/2, \mathrm{id.})$ and

$$
\begin{aligned}
&\sum_{D\neq\beta|D}\mu(\beta)\prod_{p|\beta}(1+p)^{-1}V(g(\mathrm{id.}, D, 4D), 1)\\
&= (1+\mathrm{i})(4D)^{-1} + \sum_{D\neq\beta|D}\ \sum_{1\neq d|\beta}\mu(d)d\beta^{-1}\prod_{p|d}(1+p)^{-1}\left(-4^{-1}(1+\mathrm{i})\mu(D/\beta)\beta D^{-1}\right)\\
&= -(4D)^{-1}(1+\mathrm{i})\mu(D)\sum_{D\neq\beta|D}\mu(\beta)\prod_{p|\beta}(1+p)^{-1}.
\end{aligned}
$$

Therefore, $V(g(\mathrm{id.}, D, 4D), 1) = -(4D)^{-1}(1+\mathrm{i})\mu(D)$, which completes the proof of part (3) for $m = D$.

For $m|D$, by the method used in the proof of the part (2), we get

$$g(\mathrm{id.}, m, 4D) = \prod_{p|D/m} p(1+p)^{-1} \sum_{d|D/m} \mu(d) g(\mathrm{id.}, md, 4md).$$

Using the induction hypothesis and the above result, $g(\mathrm{id.}, md, 4md) \in \mathcal{E}(4D, 3/2, \mathrm{id.})$, and hence $g(\mathrm{id.}, m, 4D) \in \mathcal{E}(4D, 3/2, \mathrm{id.})$ and as well as

$$\begin{aligned} V(g(\mathrm{id.}, m, 4D), 1) &= \prod_{p|D/m} p(1+p)^{-1} \sum_{d|D/m} \mu(d) V(g(\mathrm{id.}, md, 4md), 1) \\ &= -4^{-1}(1+\mathrm{i}) \prod_{p|D/m} p(1+p)^{-1} \sum_{d|D/m} \mu(d)\mu(md)(md)^{-1} \\ &= -(4m)^{-1}(1+\mathrm{i})\mu(m), \end{aligned}$$

we complete the proof of part (3).

Finally we prove part (1). For each prime divisor p of D, we define

$$\begin{aligned} G(\chi_l, p, 4D) &= 2(\mathrm{i}-1) l^{-l/2} (l,p)^{1/2} \varepsilon_{p/(l,p)} \left(\frac{l/(l,p)}{p/(l,p)}\right) g(\chi_l, p, 4D), \\ G(\chi_l, 4, 4D) &= l^{-1/2} \varepsilon_l^{-1} g(\chi_l, 4, 4D). \end{aligned}$$

We define the following function by induction on the number of prime factors of m:

$$\begin{aligned} G(\chi_l, 4m, 4D) = {} & l^{-l/2}(l,m)^{1/2} \varepsilon_{l/(l,m)}^{-1} \left(\frac{m/(l,m)}{l/(l,m)}\right) \Big\{ g(\chi_l, 4m, 4D) - g(\chi_l, m, 4D) \\ & - \mu(m) m^{-1} l^{1/2} \sum_{m \neq \alpha | m} \mu(\alpha)\alpha(l,\alpha)^{-1/2} \varepsilon_{l/(l,\alpha)} \\ & \times \left(\frac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right) G(\chi_l, 4\alpha, 4D) \Big\} \end{aligned}$$

and

$$\begin{aligned} G(\chi_l, m, 4D) = {} & 2(\mathrm{i}-1) l^{-1/2} (l,m)^{1/2} \varepsilon_{m/(l,m)} \left(\frac{l/(l,m)}{m/(l,m)}\right) \\ & \times \Big\{ g(\chi_l, m, 4D) + (1+\mathrm{i})(4m)^{-1} \\ & \times \mu(m) \sum_{1, m \neq \alpha | m} \mu(\alpha)\alpha l^{1/2} (l,\alpha)^{-1/2} \varepsilon_{\alpha/(l,\alpha)}^{-1} \\ & \times \left(\frac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right) G(\chi_l, \alpha, 4D) \Big\}. \end{aligned}$$

We can prove that for $r = 0$ or 2, $V(G(\chi_l, 2^r m, 4D), p) = 0$ for all $p \in S(4D)$ except for $p = 1$ and $1/(2^r m)$ and

$$V(G(\chi_l, 4m, 4D), 1/(4m)) = V(G(\chi_l, m, 4D), 1/m) = 1,$$
$$V(G(\chi_l, 4m, 4D), 1) = -(4m)^{-1}(1+\mathrm{i})(l,m)^{1/2}\varepsilon_{l/(l,m)}^{-1}\left(\frac{m/(l,m)}{l/(l,m)}\right),$$
$$V(G(\chi_l, m, 4D), 1) = -m^{-1}(l,m)^{1/2}\varepsilon_{m/(l,m)}\left(\frac{l/(l,m)}{m/(l,m)}\right).$$

These equalities imply that $G(\chi_l, 4m, 4D)\,(\forall m|D)$ and $G(\chi_l, m, 4D)\,(1 \neq m|D)$ are linearly independent. But the number of these functions is equal to the dimension of $\mathcal{E}(4D, 3/2, \chi_l)$. So they constitute a basis of $\mathcal{E}(4D, 3/2, \chi_l)$, so do $g(\chi_l, 4m, 4D)$ and $g(\chi_l, m, 4D)$. This completes the proof of the theorem. □

We shall construct a basis of $\mathcal{E}(8D, 3/2, \chi_l)$ and $\mathcal{E}(8D, 3/2, \chi_{2l})$ respectively. Put

$$R = \{n \in \mathbb{Z} | n \geqslant 1, n \equiv 1 \text{ or } 2 \pmod 4\}.$$

Define

$$f_4(\mathrm{id.}, 4D) = 2\pi \sum_{n\in R} \lambda(n, 4D) \prod_{p|D}(A(p,n) - p^{-1})n^{1/2}e(nz).$$

Then

$$f_2^*(\mathrm{id.}, 4D) + 2^{-1}(1+\mathrm{i})\mu(D)f_2(\mathrm{id.}, 8D) = \frac{3}{2}f_4(\mathrm{id.}, 8D),$$

where we used the fact $A(2,n) - 4^{-1}(1-\mathrm{i}) = \dfrac{3}{8}(\mathrm{i}-1)$ for $n \in R$. It follows that $f_4(\mathrm{id.}, 8D) \in \mathcal{E}(8D, 3/2, \mathrm{id.})$. By Lemma 7.21 and Lemma 7.23, we get

$$\begin{aligned} V(f_4(\mathrm{id.}, 8D), 1/(8\beta)) &= V(f_4(\mathrm{id.}, 8D), 1/(2\beta)) = 0, \\ V(f_4(\mathrm{id.}, 8D), 1/\beta) &= -8^{-1}(1+\mathrm{i})\mu(D/\beta)\beta D^{-1}\varepsilon_\beta^{-1}, \\ V(f_4(\mathrm{id.}, 8D), 1/(4\beta)) &= \mu(D/\beta)\beta D^{-1}. \end{aligned} \tag{7.24}$$

For any $m|D$, define

$$g(\chi_l, 4m, 8D) = 2\pi l^{1/2} \sum_{ln\in R} \lambda(ln, 4D) \prod_{p|m}(A(p,ln) - p^{-1})n^{1/2}e(nz).$$

Theorem 7.8 (1) *The functions* $g(\chi_l, 4m, 8D)$ $(\forall\ m|D), g(\chi_l, 4m, 4D)$ $(\forall\ m|D)$ $g(\chi_l, m, 4D)$ $(\forall\ 1 \neq m|D))$ *constitute a basis of* $\mathcal{E}(8D, 3/2, \chi_l)$.

(2)

$$V(g(\chi_l, 4m, 8D), p) = \begin{cases} -8^{-1}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{\alpha/(l,\alpha)}^{-1}\left(\dfrac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right), \\ \qquad\qquad \text{if } p = 1/\alpha,\ \alpha|m, \\ \mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{l/(l,\alpha)}\left(\dfrac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right), \\ \qquad\qquad \text{if } p = 1/(4\alpha),\ \alpha|m, \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

Proof We first prove (2). Since $g(\chi_l, 4m, 8D) = g(\mathrm{id.}, 4m, 8D)|\mathrm{T}(l)$. So we only need to prove (2) for $l = 1$. We can get

$$g(\mathrm{id.}, 4m, 8D) = \prod_{p|D/m} p(1+p)^{-1} \sum_{d|D/m} \mu(d) f_4(\mathrm{id.}, 8md) \in \mathcal{E}(8D, 3/2, \mathrm{id.})$$

by a similar method used in the proof of theorem 7.7. By (7.24) we have

$$\begin{aligned} V(g(\mathrm{id.}, 4m, 8D), 1/(8\beta)) &= V(g(\mathrm{id.}, 4m, 8D), 1/(2\beta)) = 0, \\ V(g(\mathrm{id.}, 4m, 8D), 1) &= -8^{-1}(1+\mathrm{i})\mu(m)m^{-1}, \\ V(g(\mathrm{id.}, 4m, 8D), 1/4) &= \mu(m)m^{-1}. \end{aligned}$$

But

$$g(\mathrm{id.}, 4m, 8D)|\mathrm{T}(p^2) = \begin{cases} g(\mathrm{id.}, 4m, 8D), & \forall p|m, \\ pg(\mathrm{id.}, 4m, 8D), & \forall p|D/m \end{cases}$$

implies (2) by Lemma 7.4.

Now we prove (1) by a method similar to the proof of Theorem 7.7. Since $\dfrac{1}{8\alpha}$ and $\dfrac{1}{4\alpha}$ are $\Gamma_0(4D)$-equivalent, we have

$$V(g(\chi_l, 4m, 4D), 1/(8\alpha)) = \mu(m/\alpha)\alpha m^{-1} l^{1/2} (l,\alpha)^{-1/2} \varepsilon_{l/(l,\alpha)} \left(\frac{2\alpha(l,\alpha)}{l/(l,\alpha)}\right).$$

Define

$$\begin{aligned} G(\chi_l, 4, 8D) &= l^{-1/2}\varepsilon_l^{-1} g(\chi_l, 4, 8D), \\ G(\chi_l, 8, 8D) &= l^{-1/2}\varepsilon_l^{-1}\left(\frac{2}{l}\right)\{g(\chi_l, 4, 4D) - g(\chi_l, 4, 8D)\}. \end{aligned}$$

Then we define by induction

$$\begin{aligned} G(\chi_l, 8m, 8D) = & l^{-1/2}(l,m)^{1/2}\varepsilon_{l/(l,m)}^{-1}\left(\frac{2m/(l,m)}{l/(l,m)}\right)\Big\{g(\chi_l, 4m, 4D) \\ & - g(\chi_l, 4m, 8D) - 2^{-1}g(\chi_l, m, 4D) \\ & - \mu(m)m^{-1}l^{1/2}\sum_{m\neq\alpha|m}\mu(\alpha)\alpha(l,\alpha)^{-1/2} \\ & \times \varepsilon_{l/(l,\alpha)}\left(\frac{2\alpha/(l,\alpha)}{l/(l,\alpha)}\right)G(\chi_l, 8\alpha, 8D)\Big\} \end{aligned}$$

and

$$\begin{aligned} G(\chi_l, 4m, 8D) = & l^{-1/2}(l,m)^{1/2}\varepsilon_{l/(l,m)}^{-1}\left(\frac{m/(l,m)}{l/(l,m)}\right)\Big\{g(\chi_l, 4m, 8D) \\ & - 2^{-1}g(\chi_l, m, 4D) - \mu(m)m^{-1}l^{1/2}\sum_{m\neq\alpha|m}\mu(\alpha)\alpha(l,\alpha)^{-1/2} \\ & \times \varepsilon_{l/(l,\alpha)}\left(\frac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right)G(\chi_l, 4\alpha, 4D)\Big\}. \end{aligned}$$

We define also $G(\chi_l, m, 8D) = G(\chi_l, m, 4D)$ for $m \neq 1$. We can prove that for $r = 0, 2, 3, V(G(\chi_l, 2^r m, 8D), p) = 0$ for all $p \in S(8D)$ except $p = 1$ and $1/(2^r m)$ by induction, and

$$V(G(\chi_l, m, 8D), 1/m) = 1, \quad m \neq 1,$$
$$V(G(\chi_l, 4m, 8D), 1/(4m)) = V(G(\chi_l, 8m, 8D), \quad 1/(8m)) = 1,$$
$$V(G(\chi_l, m, 8D), 1) = -m^{-1}(l,m)^{1/2}\varepsilon_{m/(l,m)}\left(\frac{l/(l,m)}{m/(l,m)}\right),$$
$$V(G(\chi_l, 4m, 8D), 1) = -8^{-1}(1+\mathrm{i})m^{-1}(l,m)^{1/2}\varepsilon_{l/(l,m)}^{-1}\left(\frac{m/(l,m)}{l/(l,m)}\right),$$
$$V(G(\chi_l, 8m, 8D), 1) = -8^{-1}(1+\mathrm{i})m^{-1}(l,m)^{1/2}\varepsilon_{l/(l,m)}^{-1}\left(\frac{2m/(l,m)}{l/(l,m)}\right).$$

Gathering the values of $G(\chi_l, m, 4D)$ at $1/m$ and 1 computed in the proof of Theorem 7.7, we know that $G(\chi_l, 8m, 8D)$ ($\forall\ m|D$), $G(\chi_l, 4m, 4D)$ ($\forall\ m|D$) and $G(\chi_l, m, 8D)$ ($\forall\ 1 \neq m|D$) constitute a basis of $\mathcal{E}(8D, 3/2, \chi_l)$. This completes the proof. □

Finally we consider $\mathcal{E}(8D, 3/2, \chi_{2l})$. Define

$$g(\chi_{2l}, m, 8D) = g(\chi_l, m, 4D)|\mathrm{T}(2), \quad \forall\ 1 \neq m|D,$$
$$g(\chi_{2l}, 2m, 8D) = g(\chi_l, 4m, 8D)|\mathrm{T}(2), \quad \forall\ m|D,$$
$$g(\chi_{2l}, 8m, 8D) = g(\chi_l, 4m, 4D)|\mathrm{T}(2), \quad \forall\ m|D.$$

Then we have

Theorem 7.9 (1) *The functions* $g(\chi_{2l}, m, 8D)$ ($\forall\ 1 \neq m|D$), $g(\chi_{2l}, 2m, 8D)$ ($\forall\ m|D$) *and* $g(\chi_{2l}, 8m, 8D)$ ($\forall\ m|D$) *constitute a basis of* $\mathcal{E}(8D, 3/2, \chi_{2l})$.

(2) *For* $p \in S(8D)$, *we have*

$$V(g(\chi_{2l},\ m, 8D), p) = \begin{cases} -2^{-3/2}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{\alpha/(l,\alpha)}^{-1}\left(\frac{2l/(l,\alpha)}{\alpha/(l,\alpha)}\right), \\ \qquad \text{if } p = 1/\alpha,\ \alpha|m, \\ 0, \qquad \text{otherwise,} \end{cases}$$

$$V(g(\chi_{2l}, 2m, 8D), p) = \begin{cases} -2^{-5/2}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{\alpha/(l,\alpha)}^{-1}\left(\frac{2l/(l,\alpha)}{\alpha/(l,\alpha)}\right), \\ \qquad \text{if } p = 1/\alpha,\ \alpha|m, \\ -2^{-1}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{\alpha/(l,\alpha)}^{-1}\varepsilon_l^{-1}\left(\frac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right), \\ \qquad \text{if } p = 1/(2\alpha),\ \alpha|m, \\ 0, \qquad \text{otherwise,} \end{cases}$$

$$V(g(\chi_{2l}, 8m, 8D), p) = \begin{cases} -2^{-3/2}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{\alpha/(l,\alpha)}^{-1}\left(\dfrac{2l/(l,\alpha)}{\alpha/(l,\alpha)}\right), \\ \qquad\qquad \text{if } p = 1/\alpha,\ \alpha|m, \\ -2^{-1}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{\alpha/(l,\alpha)}^{-1}\varepsilon_l^{-1}\left(\dfrac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right), \\ \qquad\qquad \text{if } p = 1/(2\alpha),\ \alpha|m, \\ \mu(m/\alpha)\alpha m^{-1}l^{1/2}(l,\alpha)^{-1/2}\varepsilon_{l/(l,\alpha)}\left(\dfrac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right), \\ \qquad\qquad \text{if } p = 1/(8\alpha),\ \alpha|m, \\ 0, \qquad\qquad \text{otherwise.} \end{cases}$$

Proof Since $\dim\mathcal{E}(8D, 3/2, \chi_{2l}) = \dim\mathcal{E}(8D, 3/2, \chi_l)$ and T(2) is a linear operator from $\mathcal{E}(8D, 3/2, \chi_l)$ to $\mathcal{E}(8D, 3/2, \chi_{2l})$, we get the part (1) by Theorem 7.8. The part (2) can be proved by Theorem 7.7, Theorem 7.8 and the definitions of $g(\chi_{2l}, 2^r m, 8D)$ ($r = 0, 1, 3$). □

Several applications of the basis given in Theorems 7.1–7.9 will be described in the rest part of the book:

(1) Construct certain generalization of Cohen-Eisenstein (Section 7.4);

(2) Prove Siegel theorem for positive definite ternary quadratic forms (Section 10.1);

(3) Determine the eligible numbers of certain positive definite ternary quadratic forms (Section 10.3).

It is worth mentioning one more application briefly, which is due to G. Shimura, [S5] here. Let

$$f(z) = \sum_{n=1}^{\infty} a(n)\exp\{2\pi\mathrm{i}nz\}, \quad g(z) = \sum_{n=0}^{\infty} b(n)\exp\{2\pi\mathrm{i}nz\}$$

be a cusp form with the weight $k/2$ and a modular form with the weight $l/2$ respectively, where k and $l (l < k)$ are positive odd numbers and the Fourier coefficients $a(n)$ and $b(n)$ are algebric numbers. Define Zeta function

$$D(s, f, g) = \sum_{n=1}^{\infty} a(n)b(n)n^{-s}.$$

Shimura proved that the number $D(t/2, f, g)$, where $1 \leqslant t \leqslant k-2$, multiplied by the number $\pi^{-r}u_-(F)$ is a algebric number, where the integer r is determined by t, l, k and $u_-(F)$ is the period of a modular form F determined by f with the weight $k-1$. In the Shimura's proof of the above result the basis constructed in Theorems 7.7–7.9 were used when $k = 3$.

7.4 Construction of Cohen-Eisenstein Series

Let χ be a Dirichlet character modulo N, and denote by $L(s,\chi)$ the associated L-series

$$L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s}.$$

For a positive integer k we have that $L(1-k,\chi)=-\frac{B_{k,\chi}}{k}$, where the numbers $B_{k,\chi}$ are defined by

$$\sum_{a=1}^{N}\frac{\chi(a)t\mathrm{e}^{at}}{\mathrm{e}^{Nt}-1}=\sum_{k=0}^{\infty}B_{k,\chi}\frac{t^k}{k!}.$$

Fix an integer $k\geqslant 2$ and define rational numbers $H(k,n)$ by

$$H(k,n):=\begin{cases}\zeta(1-2k), & \text{if } n=0,\\ L(1-k,\chi_D)\sum_{d|f}\mu(d)\chi_D(d)d^{k-1}\sigma_{2k-1}(f/d), & \text{if } (-1)^k n=Df^2,\\ 0, & \text{otherwise,}\end{cases}$$

where ζ denotes the Riemann ζ-function, μ the Moebius function, D a fundamental discriminant, χ_D the quadratic character associated with $\mathbb{Q}(\sqrt{D})$ and the arithmetical function σ_r is defined by $\sigma_r(m)=\sum_{d|m}d^r$. H.Cohen introduced the rational numbers $H(k,n)$ and proved that

$$H_k(z):=\sum_{n=0}^{\infty}H(k,n)\exp(2\pi\mathrm{i}nz) \tag{7.25}$$

is a modular form of half-integral weight $k+1/2$ for $\Gamma_0(4)$ in [C] which is now named Cohen-Eisenstein series. For $k=1$ and group $\Gamma_0(4p)$ with p a prime, Cohen-Eisenstein series is defined by

$$H_{1,p}(z):=\sum_{n=0}^{\infty}H(n)_p\exp(2\pi\mathrm{i}nz), \tag{7.26}$$

where $H(n)_p:=H(p^2n)-pH(n)$ with $H(n)$ (for $n>0$) the number of classes of positive definite binary quadratic forms of discriminant $-n$ (where forms equivalent to a multiple of x^2+y^2 or x^2+xy+y^2 are counted with multiplicity $1/2$ or $1/3$ respectively) and $H(0)=-1/12$. $H_{1,p}$ is a modular form of weight $3/2$ on $\Gamma_0(4p)$.

We shall construct some explicit modular forms in the space $E^{+}_{k+1/2}(4N,\chi_l)$ with $k\geqslant 1$ which can be viewed as a generalization of Cohen-Eisenstein series and constitute a basis of $E^{+}_{k+1/2}(4N,\chi_l)$.

Let $B_{k,\chi}$ be the generalized Bernoulli number defined by

$$\sum_{a=1}^{N}\frac{\chi(a)t\mathrm{e}^{at}}{\mathrm{e}^{Nt}-1}=\sum_{k=0}^{\infty}B_{k,\chi}\frac{t^k}{k!},$$

where N is a square free odd positive integer and χ is a Dirichlet character modulo N. And let $M^+_{k+1/2}(4N,\chi_l)$ be Kohnen's "+ space" defined by

$$M^+_{k+1/2}(4N,\chi_l) := \Big\{ f(z) = \sum_{n=0}^{\infty} a(n)q^n | f \in G(4N, k+1/2, \chi_l) \\ \text{with } a(n) = 0 \text{ whenever } \varepsilon(-1)^k n \equiv 2,3 \pmod 4 \Big\},$$

$S^+_{k+1/2}(4N,\chi_l)$ the Kohnen's "space" defined by

$$S^+_{k+1/2}(4N,\chi_l) := \Big\{ f(z) = \sum_{n=0}^{\infty} a(n)q^n | f \in S(4N, k+1/2, \chi_l) \\ \text{with } a(n) = 0 \text{ whenever } \varepsilon(-1)^k n \equiv 2,3 \pmod 4 \Big\},$$

$E^+_{k+1/2}(4N,\chi_l)$ the Kohnen's "space" defined by

$$E^+_{k+1/2}(4N,\chi_l) := \Big\{ f(z) = \sum_{n=0}^{\infty} a(n)q^n | f \in \mathcal{E}(4N, k+1/2, \chi_l) \\ \text{with } a(n) = 0 \text{ whenever } \varepsilon(-1)^k n \equiv 2,3 \pmod 4 \Big\}.$$

We define the following rational numbers $H(k,l,N,N;n)$ and $H(k,l,m,N;n)$ with $N \neq m|N$:

$$H(k,l,N,N;n) := \begin{cases} L_N(1-2k, \text{id.}), & \text{if } n = 0, \\ L_N(1-k, \chi_{D'_n}) \sum_{d|f_n} \mu(d)\chi'_l(d)\chi_{D_n}(d)d^{k-1}\sigma_{N,2k-1}(f_n/d), \\ \qquad\qquad \text{if } \varepsilon(-1)^k n = D_n f_n^2 \text{ and } (-1)^k ln = D'_n(f'_n)^2, \\ 0, & \text{otherwise,} \end{cases}$$

where $\sigma_{N,2k-1}$ is the arithmetical function defined by $\sigma_{N,2k-1}(t) := \sum_{d|t,(d,N)=1} d^{2k-1}$, and

$$H(k,l,m,N;n) := \begin{cases} 0, & \text{if } n = 0, \\ L_m(1-k,\chi_{D'_n}) \prod_{p|N/m} \dfrac{1-p^{-k}\left(\frac{D'_n}{p}\right)}{1-p^{-2k}} \left(\dfrac{(l,D_n)}{(l,D_n,m)}\right)^{2k-1} \\ \qquad \times \sum_{d|f_n} \mu(d)\chi'_l(d)\chi_{D_n}(d)d^{k-1}\sigma_{m,N,2k-1}(f_n/d), \\ \qquad\qquad \text{if } \varepsilon(-1)^k n = D_n f_n^2 \text{ and } (-1)^k ln = D'_n(f'_n)^2, \\ 0, & \text{otherwise,} \end{cases}$$

where $\sigma_{m,N,2k-1}$ is the arithmetical function defined by

$$\sigma_{m,N,2k-1}(t) := \sum_{\substack{d|t,(d,m)=1,\\(t/d,N/m)=1}} d^{2k-1}.$$

Note that $H(k,1,1,1;n) = H(k,n)$ are just the rational numbers defined by H.Cohen.

Theorem 7.10 *Let N be a square-free odd positive integer and l a divisor of N. Then*

(1) *If $k=1$ and $N>1$, then the functions defined by*

$$H_1(\chi_l,N,N)(z) := \sum_{n=0}^{\infty} H(1,l,N,N;n)q^n,$$

$$H_1(\chi_l,m,N)(z) := \sum_{n=0}^{\infty} H(1,l,m,N;n)q^n \quad \textit{for all } m \textit{ with } 1,N \neq m|N$$

belong to $E^+_{3/2}(4N,\chi_l)$ and constitute a basis of the space $E^+_{3/2}(4N,\chi_l)$.

(2) *If $k \geqslant 2$, then the functions defined by*

$$H_k(\chi_l,N,N)(z) := \sum_{n=0}^{\infty} H(k,l,N,N;n)q^n,$$

$$H_k(\chi_l,m,N)(z) := \sum_{n=0}^{\infty} H(k,l,m,N;n)q^n \quad \textit{for all } m \textit{ with } N \neq m|N$$

belong to $E^+_{k+1/2}(4N,\chi_l)$ and constitute a basis of the space $E^+_{k+1/2}(4N,\chi_l)$.

Remark 7.1 $H_k(\mathrm{id.},1,1)(z)$ is just the Cohen-Eisenstein series $H_k(z)$. Since

$$L_N(-1,\mathrm{id.}) = -\frac{1}{12}\prod_{p|N}(1-p)$$

and

$$H(n) = \frac{h(D)}{w(D)}\sum_{d|f}\mu(d)\left(\frac{D}{d}\right)\sigma_1(f/d),$$

where $-n = Df^2$ with D a negative fundamental discriminant, $w(D)$ half the number of units in $\mathbb{Q}(\sqrt{D})$, we see that $H_1(\mathrm{id.},p,p)$ is just the Cohen-Eisenstein series $H_{1,p}(z)$ by class number formula.

We need the following:

Lemma 7.25 *Let n be a positive integer with $(-1)^k n = D(2^r f)^2$ where D is a fundamental discriminant, f is a positive odd integer and $r \geqslant -1$ is an integer. Then*

$$(A_k(2,n)-\eta_2)2^{k-2}(1-(-1)^\lambda \mathrm{i})(1-2^{k-2})\left(1-2^{-\lambda}\left(\frac{D}{2}\right)\right)(1-2^{1-k})^{-1}$$

$$= 2^{-r(k-2)}\left(1 - 2^{\lambda-1}\left(\frac{D}{2}\right)\right),$$

$$(A_k(p,n) - \eta_p)p^{\nu_p(f)(k-2)}(1-p^{k-2})\left(1 - p^{-\lambda}\left(\frac{D}{p}\right)\right)(1-p^{1-k})^{-1}$$

$$= 1 - p^{\lambda-1}\left(\frac{D}{p}\right), \quad p \textit{ is an odd prime },$$

where $\lambda = (k-1)/2$ *for an odd integer* k.

Proof The lemma can be proved by the definitions and some direct calculations. □

Proof of Thorem 7.10 (1) We know that the dimension of $E^{+}_{3/2}(4N, \chi)$ is $2^{t(N)} - 1$. So we only need to prove that $H_1(\chi_l, m, N)(z)$ $(1 \neq m|N)$ belong to $E^{+}_{3/2}(4N, \chi)$ and are linearly independent.

By the results in Section 7.3 we know that the following functions

$$H'_1(\chi_l, m, N) := g(\chi_l, 4m, 4N) - \frac{3}{2}g(\chi_l, m, 4N), \quad \forall\ 1 \neq m|N \tag{7.27}$$

belong to $\mathcal{E}(4N, 3/2, \chi_l)$ and are linearly independent. We now prove that $H'_1(\chi_l, m, N)$ belongs to $E^{+}_{3/2}(4N, \chi_l)$ and is a non-zero multiple of $H_1(\chi_l, m, N)$ with $1 \neq m|N$. By the definition, we see that

$$\begin{aligned} H'_1(\chi_l, m, N) :=& \sum_{n=1}^{\infty} a_m(n)q^n = -4\pi(1+\mathrm{i})\sum_{n=1}^{\infty}\lambda_3(ln, 4N)(A_3(2, ln) + 2^{-3}(1-\mathrm{i})) \\ &\times \prod_{p|m}(A_3(p, ln) - p^{-1})(ln)^{1/2}q^n, \quad \forall\ m|N, m \neq 1, N, \end{aligned} \tag{7.28}$$

$$\begin{aligned} H'_1(\chi_l, N, N) :=& \sum_{n=1}^{\infty} a_N(n)q^n = 1 - 4\pi(1+\mathrm{i})\sum_{n=1}^{\infty}\lambda_3(ln, 4N)(A_3(2, ln) + 2^{-3}(1-\mathrm{i})) \\ &\times \prod_{p|N}(A_3(p, ln) - p^{-1})(ln)^{1/2}q^n. \end{aligned}$$

Denote

$$I(l, n) := A_3(2, ln) + 2^{-3}(1-\mathrm{i}). \tag{7.29}$$

By the definition of $A(2, ln)$, we see easily that $I(l, n) = 0$ if $ln \equiv 1, 2 \pmod 4$ and hence $a_m(n) = 0$, $a_N(n) = 0$ if $ln \equiv 1, 2 \pmod 4$. This implies that $H'_1(\chi_l, m, N) \in E^{+}_{3/2}(4N, \chi_l)$. When $ln \equiv 0, 3 \pmod 4$, $\varepsilon = (-1)^{\frac{l-1}{2}} \equiv l \pmod 4$ which implies that $\varepsilon n \equiv 0, 3 \pmod 4$. Hence we can suppose that $-\varepsilon n = D_n f_n^2$ and $-ln = D'_n(f'_n)^2$ with D_n and D'_n fundamental discriminants, f_n and f'_n positive integers. It is clear that $D'_n = \varepsilon l D_n/(l, D_n)^2$, $f'_n = (l, D_n)f_n$. From these we see that if $p \nmid N$ then $p|D_n$ if and only if $p|D'_n$ and $\nu_p(f_n) = \nu_p(f'_n)$. By the definition of $A_3(p, ln)$ and some calculations we have that

$$I(l,n)=\begin{cases} 4^{-1}(1-\mathrm{i})\left(1+\frac{1}{2}\left(\frac{D_n'}{2}\right)\right), & \text{if } ln\equiv 3 \pmod 4,\\ \frac{3}{16}(1-\mathrm{i})\sum_{t=0}^{\nu_2(f_n')}2^{-t}, & \text{if } ln\equiv 0 \pmod 4 \text{ and } 2\nmid \nu_2(ln),\\ 4^{-1}(1-\mathrm{i})\left(1+\frac{1}{2}\left(\frac{D_n'}{2}\right)\right)\left(\sum_{t=0}^{\nu_2(f_n')}2^{-t}-\frac{1}{2}\left(\frac{D_n'}{2}\right)\sum_{t=0}^{\nu_2(f_n')-1}2^{-t}\right), & \\ & \text{if } ln\equiv 0 \pmod 4 \text{ and } 2|\nu_2(ln), 2\nmid D_n',\\ \frac{3}{16}(1-\mathrm{i})\sum_{t=0}^{\nu_2(f_n')}2^{-t}, & \text{if } ln\equiv 0 \pmod 4 \text{ and } 2|\nu_2(ln), 2|D_n'. \end{cases} \tag{7.30}$$

By Lemma 7.25 we obtain that for $ln\equiv 0,3 \pmod 4$

$$\begin{aligned}\prod_{p|m}(A_3(p,ln)-p^{-1})(ln)^{1/2} &= | D_n' |^{1/2}\prod_{p|m}\left(1-\left(\frac{D_n'}{p}\right)\right)(1-p)^{-1}\\ &\quad\times\left(1-p^{-1}\left(\frac{D_n'}{p}\right)\right)^{-1}(1-p^{-2})\prod_{p\nmid m}p^{\nu_p(f_n')}\\ &= | D_n' |^{1/2}\,\frac{(l,D_n)}{(l,D_n,m)}\prod_{p|m}\left(1-\left(\frac{D_n'}{p}\right)\right)(1-p)^{-1}\\ &\quad\times\left(1-p^{-1}\left(\frac{D_n'}{p}\right)\right)^{-1}(1-p^{-2})\prod_{p\nmid m}p^{\nu_p(f_n)}. \end{aligned}\tag{7.31}$$

We also have that

$$\begin{aligned}&\beta_3(ln,\chi_N,4N)\\ &=\sum_{\substack{(ab)^2|ln,(ab,2N)=1\\ a,b \text{ positive integers}}}\mu(a)\left(\frac{-ln}{a}\right)(ab)^{-1}\\ &=\prod_{p|D_n',p\nmid 2N}\sum_{t=0}^{\nu_p(f_n')}p^{-t}\prod_{p\nmid 2ND_n'}\left(\sum_{t=0}^{\nu_p(f_n')}p^{-t}-p^{-1}\left(\frac{D_n'}{p}\right)\sum_{t=0}^{\nu_p(f_n')-1}p^{-t}\right)\\ &=\prod_{p|D_n,p\nmid 2N}\sum_{t=0}^{\nu_p(f_n)}p^{-t}\prod_{p\nmid 2ND_n}\left(\sum_{t=0}^{\nu_p(f_n)}p^{-t}-p^{-1}\chi_l'(p)\left(\frac{D_n}{p}\right)\sum_{t=0}^{\nu_p(f_n)-1}p^{-t}\right), \end{aligned}\tag{7.32}$$

where we have used the fact that $p|D_n$ if and only if $p|D_n'$ and $\nu_p(f_n)=\nu_p(f_n')$ for $p\nmid N$. By the functional equation of L-functions we see that

$$\frac{-4\pi(1+\mathrm{i})L_{4N}\left(1,\left(\frac{D'_n}{\cdot}\right)\right)}{L_{4N}(2,\mathrm{id.})}$$
$$= 2(1+\mathrm{i}) \mid D'_n \mid^{-1/2} \frac{L\left(0,\left(\frac{D'_n}{\cdot}\right)\right)}{\zeta(-1)} \prod_{p\mid 2N} \frac{\left(1-p^{-1}\left(\frac{D'_n}{p}\right)\right)}{1-p^{-2}}. \tag{7.33}$$

Using these equalities (7.28)–(7.33), we finally find that for $1, N \neq m|N$ and $n \geqslant 1$

$$a_m(n) = \frac{L_m\left(0,\left(\frac{D'_n}{\cdot}\right)\right)}{L_m(-1,\mathrm{id.})} \frac{(l,D_n)}{(l,D_n,m)} \prod_{p\mid N/m} \frac{\left(1-p^{-1}\left(\frac{D'_n}{p}\right)\right)}{1-p^{-2}}$$
$$\times \sum_{d\mid f_n} \mu(d)\chi'_l(d)\left(\frac{D_n}{d}\right) \sum_{\substack{e\mid f_n/d,(e,m)=1\\ (f_n/de,N/m)=1}} e,$$

where we used the fact that

$$\prod_{p\nmid m} p^{\nu_p(f_n)} \prod_{p\mid D_n, p\nmid N} \sum_{t=0}^{\nu_p(f_n)} p^{-t} \prod_{p\nmid ND_n} \left(\sum_{t=0}^{\nu_p(f_n)} p^{-t} - p^{-1}\chi'_l(p)\left(\frac{D_n}{p}\right) \sum_{t=0}^{\nu_p(f_n)-1} p^{-t}\right)$$
$$= \prod_{p\mid N/m} p^{\nu_p(f_n)} \prod_{p\mid D_n, p\nmid N} \sum_{t=0}^{\nu_p(f_n)} p^{t} \prod_{p\nmid ND_n} \left(\sum_{t=0}^{\nu_p(f_n)} p^{t} - p^{-1}\chi'_l(p)\left(\frac{D_n}{p}\right) \sum_{t=0}^{\nu_p(f_n)-1} p^{t}\right)$$
$$= \sum_{d\mid f_n} \mu(d)\chi'_l(d)\left(\frac{D_n}{d}\right) \sum_{\substack{e\mid f_n/d,(e,m)=1,\\ (f_n/de,N/m)=1}} e.$$

Similarly we have that

$$a_N(n) = \frac{L_N\left(0,\left(\frac{D'_n}{\cdot}\right)\right)}{L_N(-1,\mathrm{id.})} \sum_{d\mid f_n} \mu(d)\chi'_l(d)\left(\frac{D_n}{d}\right) \sum_{\substack{e\mid f_n/d\\ (e,N)=1}} e.$$

These show that

$$H'_1(\chi_l,N,N) = 1 + \sum_{\substack{n>0,\\ ln\equiv 0,3 \pmod 4}} \left\{ \frac{L_N\left(0,\left(\frac{D'_n}{\cdot}\right)\right)}{L_N(-1,\mathrm{id.})} \sum_{d\mid f_n} \mu(d)\chi'_l(d)\left(\frac{D_n}{d}\right) \sum_{\substack{e\mid f_n/d\\ (e,N)=1}} e \right\} q^n,$$

$$H'_1(\chi_l, m, N) = \sum_{\substack{n>0,\\ ln\equiv 0,3 (\mathrm{mod}\ 4)}} \left\{ \frac{L_m\left(0, \left(\frac{D'_n}{\cdot}\right)\right)}{L_m(-1, \mathrm{id.})} \frac{(l, D_n)}{(l, D_n, m)} \right.$$
$$\left. \times \prod_{p|N/m} \frac{\left(1 - p^{-1}\left(\frac{D'_n}{p}\right)\right)}{1-p^{-2}} \sum_{d|f_n} \mu(d)\chi'_l(d)\left(\frac{D_n}{d}\right) \sum_{\substack{e|f_n/d, (e,m)=1\\ (f_n/de, N/m)=1}} e \right\} q^n.$$

Comparing the coefficients of $H_1(\chi_l, m, N)$ and $H'_1(\chi_l, m, N)$, we find that

$$\begin{aligned} H_1(\chi_l, m, N) &= L_m(-1, \mathrm{id.}) H'_1(\chi_l, m, N) \\ &= -\frac{1}{12}\prod_{p|m}(1-p) H'_1(\chi_l, m, N) \end{aligned}$$

for all $1 \neq m|N$. This completes the proof of (1).

(2) We define the following functions

$$H'_k(\chi_l, m, N) := g_{2k+1}(\chi_l, 4m, 4N) + (2^{-2k-1}(1 + (-1)^k i) + \eta_2) g_{2k+1}(\chi_l, m, 4N).$$

Similar to the proof of (1), we want to prove that $H'_k(\chi_l, m, N)$ with $m|N$ constitute a basis of $E^+_{k+1/2}(4N, \chi_l)$ and is a non-zero multiple of $H_k(\chi_l, m, N)$. Since the dimension of $E^+_{k+1/2}(4N, \chi_l)$ is equal to the number of positive divisors of N, by Theorem 7.1 we only need to show that $H'_k(\chi_l, m, N) \in E^+_{k+1/2}(4N, \chi_l)$ and is a non-zero multiple of $H_k(\chi_l, m, N)$. By results in Section 7.1 we see that

$$\begin{aligned} H'_k(\chi_l, m, N) :=& \sum_{n=1}^{\infty} a_m(n) q^n \\ =& \sum_{n=1}^{\infty} \lambda'_{2k+1}(ln, 4N)(A_{2k+1}(2, ln) + 2^{-2k-1}(1 + (-1)^k \mathrm{i})) \\ & \times \prod_{p|m}(A_{2k+1}(p, ln) - \eta_p)(ln)^{k-1/2} q^n, \quad \forall m|N, m \neq N, \\ & H'_k(\chi_l, N, N) := \sum_{n=1}^{\infty} a_N(n) q^n \\ =& 1 + \sum_{n=1}^{\infty} \lambda'_{2k+1}(ln, 4N)(A_{2k+1}(2, ln) \\ & + 2^{-2k-1}(1 + (-1)^k \mathrm{i})) \prod_{p|N}(A_{2k+1}(p, ln) - \eta_p)(ln)^{k-1/2} q^n. \end{aligned} \tag{7.34}$$

Let

$$I_k(l, n) := A_{2k+1}(2, ln) + 2^{-2k-1}(1 + (-1)^k \mathrm{i}).$$

By the definition of $A_k(2, ln)$, we see that $I_k(l,n) = 0$ if $(-1)^k ln \equiv 2,3 \pmod 4$. This shows that $a_m(n) = 0$ and $a_N(n) = 0$ whenever $(-1)^k ln \equiv 2,3 \pmod 4$ and hence $H'_k(\chi_l, m, N) \in E^+_{k+1/2}(4N, \chi_l)$. Now we must compute the coefficients $a_m(n)$ of $H'_k(\chi_l, m, N)$ for all $m|N$. When $(-1)^k ln \equiv 0,1 \pmod 4$, we denote that $\varepsilon = (-1)^{\frac{l-1}{2}} \equiv l \pmod 4$, $(-1)^k \varepsilon n = D_n f_n^2$ and $l(-1)^k ln = D'_n (f'_n)^2$ with D_n, D'_n fundamental discriminants, f_n, f'_n positive integers. It is clear that $D'_n = \varepsilon l D_n/(l, D_n)^2$ and $f'_n = (l, D_n) f_n$.

By the definition of $A_k(p, ln)$ and some calculations we have that

$$I_k(l,n) = \begin{cases} 2^{-2k}(1+(-1)^k \mathrm{i})\left(1+2^{-k}\left(\dfrac{D'_n}{2}\right)\right), & \text{if } (-1)^k ln \equiv 1 (\mathrm{mod}\ 4), \\ 2^{-2k}(1+(-1)^k \mathrm{i})(1-2^{-2k}) \displaystyle\sum_{t=0}^{\nu_2(f'_n)} 2^{(1-2k)t}, & \\ \qquad \text{if } (-1)^k ln \equiv 0 (\mathrm{mod}\ 4) \text{ and } 2 \nmid \nu_2(ln), & \\ 2^{-2k}(1+(-1)^k \mathrm{i})\left(1+2^{-k}\left(\dfrac{D'_n}{2}\right)\right) & \\ \quad \times\left(\displaystyle\sum_{t=0}^{\nu_2(f'_n)} 2^{(1-2k)t} - 2^{-k}\left(\dfrac{D'_n}{2}\right)\sum_{t=0}^{\nu_2(f'_n)-1} 2^{(1-2k)t}\right), & \\ \qquad \text{if } (-1)^k ln \equiv 0 (\mathrm{mod}\ 4), 2|\nu_2(ln) \text{ and } 2 \nmid D'_n, & \\ 2^{-2k}(1+(-1)^k \mathrm{i})(1-2^{-2k}) \displaystyle\sum_{t=0}^{\nu_2(f'_n)} 2^{(1-2k)t}, & \\ \qquad \text{if } ln \equiv 0 (\mathrm{mod}\ 4), 2|\nu_2(ln) \text{ and } 2|D'_n. & \end{cases} \tag{7.35}$$

By Lemma 7.25 we obtain that for $(-1)^k ln \equiv 0,1 \pmod 4$

$$\begin{aligned} &\prod_{p|m}(A_{2k+1}(p, ln) - \eta_p)(ln)^{k-1/2} \\ &= | D'_n |^{k-1/2} \prod_{p|m}\left(1-p^{k-1}\left(\frac{D'_n}{p}\right)\right)(1-p^{2k-1})^{-1} \\ &\quad \times \left(1-p^{-k}\left(\frac{D'_n}{p}\right)\right)^{-1}(1-p^{-2k})\prod_{p\nmid m} p^{(2k-1)\nu_p(f'_n)} \\ &= | D'_n |^{k-1/2}\left(\frac{(l, D_n)}{(l, D_n, m)}\right)^{2k-1}\prod_{p|m}\left(1-p^{k-1}\left(\frac{D'_n}{p}\right)\right)(1-p^{2k-1})^{-1} \\ &\quad \times \left(1-p^{-k}\left(\frac{D'_n}{p}\right)\right)^{-1}(1-p^{-2k})\prod_{p\nmid m} p^{(2k-1)\nu_p(f_n)}. \end{aligned} \tag{7.36}$$

We also have that

$$
\begin{aligned}
&\beta_{2k+1}(ln, \chi_N, 4N) \\
=& \sum_{\substack{(ab)^2|ln,(ab,2N)=1 \\ a,b \text{ positive integers}}} \mu(a)\left(\frac{(-1)^k ln}{a}\right) a^{-k} b^{1-2k} \\
=& \prod_{p|D_n', p\nmid 2N} \sum_{t=0}^{\nu_p(f_n')} p^{(1-2k)t} \prod_{p\nmid 2ND_n'} \left(\sum_{t=0}^{\nu_p(f_n')} p^{(1-2k)t} - p^{-k}\left(\frac{D_n'}{p}\right) \sum_{t=0}^{\nu_p(f_n')-1} p^{(1-2k)t} \right) \\
=& \prod_{p|D_n, p\nmid 2N} \sum_{t=0}^{\nu_p(f_n)} p^{(1-2k)t} \prod_{p\nmid 2ND_n} \left(\sum_{t=0}^{\nu_p(f_n)} p^{(1-2k)t} - p^{-k}\chi_l'(p)\left(\frac{D_n}{p}\right) \sum_{t=0}^{\nu_p(f_n)-1} p^{(1-2k)t} \right),
\end{aligned} \tag{7.37}
$$

where we have used the fact that $p|D_n$ if and only if $p|D_n'$ and $\nu_p(f_n) = \nu_p(f_n')$ for $p \nmid N$. By the functional equation of L-function we see that

$$
\lambda_k'(ln, 4N) = 2^{2k-1}(1-(-1)^k \mathrm{i}) \mid D_n' \mid^{1/2-k} \frac{L\left(1-k, \left(\frac{D_n'}{\cdot}\right)\right)}{\zeta(1-2k)} \times \prod_{p|2N} \frac{\left(1-p^{-k}\left(\frac{D_n'}{p}\right)\right)}{1-p^{-2k}}. \tag{7.38}
$$

Using these equalities (7.33)–(7.37), we finally find that for $N \neq m|N$ and $n \geqslant 1$

$$
\begin{aligned}
a_m(n) =& \frac{L_m\left(1-k, \left(\frac{D_n'}{\cdot}\right)\right)}{L_m(1-2k, \mathrm{id.})} \left(\frac{(l, D_n)}{(l, D_n, m)}\right)^{2k-1} \prod_{p|N/m} \frac{\left(1-p^{-k}\left(\frac{D_n'}{p}\right)\right)}{1-p^{-2k}} \\
& \times \sum_{d|f_n} \mu(d)\chi_l'(d)\left(\frac{D_n}{d}\right) d^{k-1} \sum_{\substack{e|f_n/d,(e,m)=1 \\ (f_n/de, N/m)=1}} \mathrm{e}^{2k-1}
\end{aligned}
$$

where we used the fact that

$$
\begin{aligned}
& \prod_{p\nmid m} p^{(2k-1)\nu_p(f_n)} \prod_{p|D_n, p\nmid N} \sum_{t=0}^{\nu_p(f_n)} p^{(1-2k)t} \\
& \times \prod_{p\nmid ND_n} \left(\sum_{t=0}^{\nu_p(f_n)} p^{(1-2k)t} - p^{-k}\chi_l'(p)\left(\frac{D_n}{p}\right) \sum_{t=0}^{\nu_p(f_n)-1} p^{(1-2k)t} \right) \\
=& \prod_{p|N/m} p^{(2k-1)\nu_p(f_n)} \prod_{p|D_n, p\nmid N} \sum_{t=0}^{\nu_p(f_n)} p^{(2k-1)t}
\end{aligned}
$$

$$\times \prod_{p \nmid ND_n} \left(\sum_{t=0}^{\nu_p(f_n)} p^{(2k-1)t} - p^{k-1}\chi_l'(p) \left(\frac{D_n}{p}\right) \sum_{t=0}^{\nu_p(f_n)-1} p^{(2k-1)t} \right)$$

$$= \sum_{d|f_n} \mu(d)\chi_l'(d) \left(\frac{D_n}{d}\right) d^{k-1} \sum_{\substack{e|f_n/d,(e,m)=1 \\ (f_n/de,N/m)=1}} \mathrm{e}^{2k-1}.$$

Similarly we have that

$$a_N(n) = \frac{L_N\left(1-k, \left(\frac{D_n'}{\cdot}\right)\right)}{L_N(1-2k,\mathrm{id.})} \sum_{d|f_n} \mu(d)\chi_l'(d) \left(\frac{D_n}{d}\right) d^{k-1} \sum_{e|f_n/d,(e,N)=1} \mathrm{e}^{2k-1}.$$

These show that

$$H_k'(\chi_l, N, N) = 1 + \sum_{\substack{n>0, \\ (-1)^k ln \equiv 0,1 \pmod 4}} \left\{ \frac{L_N\left(1-k, \left(\frac{D_n'}{\cdot}\right)\right)}{L_N(1-2k,\mathrm{id.})} \right.$$

$$\left. \times \sum_{d|f_n} \mu(d)\chi_l'(d) \left(\frac{D_n}{d}\right) d^{k-1} \sum_{\substack{e|f_n/d \\ (e,N)=1}} \mathrm{e}^{2k-1} \right\} q^n;$$

$$H_k'(\chi_l, m, N) = \sum_{\substack{n>0, \\ (-1)^k ln \equiv 0,1 \pmod 4}} \left\{ \frac{L_m\left(1-k, \left(\frac{D_n'}{\cdot}\right)\right)}{L_m(1-2k,\mathrm{id.})} \left(\frac{(l,D_n)}{(l,D_n,m)}\right)^{2k-1} \right.$$

$$\times \prod_{p|N/m} \frac{\left(1-p^{-k}\left(\frac{D_n'}{p}\right)\right)}{1-p^{-2k}} \sum_{d|f_n} \mu(d)\chi_l'(d) \left(\frac{D_n}{d}\right) d^{k-1}$$

$$\left. \times \sum_{\substack{e|f_n/d,(e,m)=1 \\ (f_n/de,N/m)=1}} \mathrm{e}^{2k-1} \right\} q^n$$

Comparing the coefficients of $H_k(\chi_l, m, N)$ and $H_k'(\chi_l, m, N)$ show that $H_k(\chi_l, m, N) = L_m(1-2k,\mathrm{id.})H_k'(\chi_l, m, N) = -\frac{B_{2k}}{2k}H_k'(\chi_l, m, N)$ for all $m|N$ where $B_r := B_{r,\mathrm{id.}}$ is the r-th Bernoulli number. This completes the proof of (2). □

7.5 Construction of Eisenstein Series with Integral Weight

Let N and k be positive integers, ω a character modulo N with $\omega(-1) = (-1)^k$. Take a positive integer Q such that $Q|N$ and $(Q, N/Q) = 1$. Define a matrix

$$W(Q) = \begin{pmatrix} Qs & t \\ Nu & Qv \end{pmatrix} \in GL_2^+(\mathbb{Z}), \quad \det(W(Q)) = Q.$$

We see that $W(Q)\mathit{\Gamma}_0(N)W(Q)^{-1} = \mathit{\Gamma}_0(N)$.

Lemma 7.26 *Let $W(Q)$ be as above, $\omega = \omega_1\omega_2$, where ω_1 and ω_2 are characters modulo Q and N/Q respectively. If $f \in G(N,k,\omega)$ (resp. $\mathcal{E}(N,k,\omega)$), then $g = f|[W(Q)]_k \in G(N,k,\overline{\omega_1}\omega_2)$ (resp. $\mathcal{E}(N,k,\overline{\omega_1}\omega_2)$).*

Proof Take any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathit{\Gamma}_0(N)$, set $W(Q)\gamma W(Q)^{-1} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$. It is easy to check that $c_0 \equiv 0 \pmod N$, $d_0 \equiv a \pmod Q$, $d_0 \equiv d \pmod{(N/Q)}$. Hence we see that

$$g|[\gamma] = f|[W(Q)\gamma W(Q)^{-1}W(Q)] = \omega(d_0)f|[W(Q)] = \omega(d_0)g,$$

i.e., $g \in G(N,k,\overline{\omega_1}\omega_2)$. Similar to Lemma 5.35, we have for $N|M$ that

$$\mathcal{E}(N,k,\omega) = G(N,k,\omega) \bigcap \mathcal{E}(\mathit{\Gamma}(M),k),$$

from which the last conclusion of the lemma can be deduced. This completes the proof. □

Let now $E_k(z,\omega_1,\omega_2)$ be as in Section 2.2. By the computation in Section 2.2 we see that $E_k(z,\omega_1,\omega_2)$ is a common eigenfunction of all Hecke operators and

$$E_k(z,\omega_1,\omega_2)|\mathrm{T}(p) = (\omega_1(p) + p^{k-1}\omega_2(p))E_k(z,\omega_1,\omega_2).$$

Similar to Theorem 5.18 we have the following:

Lemma 7.27 *Let $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N,k,\omega)$. Assume that t is the conductor of ω and ψ is a primitive character modulo r. Put*

$$h(z) = \sum_{u=1}^{r} \overline{\psi}(u)f(z+u/r) = \sum_{u=1}^{r} \overline{\psi}(u)e(u/r)\sum_{n=1}^{\infty} \psi(n)a(n)e(nz),$$

then $h(z) \in G(M,k,\omega\psi^2)$ with $M = [N,rt,r^2]$. If $f(z) \in S(N,k,\omega)$ (resp. $\mathcal{E}(N,k,\omega)$), then $h(z) \in S(M,k,\omega\psi^2)$ (resp. $\mathcal{E}(M,k,\omega\psi^2)$).

Let $E_k(z,\omega,N)$ be as in Section 2.2. From the transformation formula of $E_k(z,\omega,N)$ and a standard method invented by Petersson we know that $E_k(z,\omega,N) \in \mathcal{E}(N,k,\omega)$ for $k \neq 2$ or $k=2$, $\omega \neq \mathrm{id}$. Hence we know that $E_k(z,\omega,N)|[W(Q)] \in \mathcal{E}(N,k,\overline{\omega_1}\omega_2)$ from Lemma 7.26. Let now $\omega = \omega_1\omega_2$. Assume that r_1 and r_2 are the conductors of ω_1 and ω_2 respectively. Write

$$r_1 = \prod_{i=1}^{m} p_i^{\alpha_i}, r_2 = \prod_{i=1}^{m} p_i^{\beta_i}, \quad \omega_1 = \prod_{i=1}^{m} \omega_{1,i}, \omega_2 = \prod_{i=1}^{m} \omega_{2,i},$$

where $\omega_{1,i}$ and $\omega_{2,i}$ have conductors $p_i^{\alpha_i}$ and $p_i^{\beta_i}$ respectively. Without loss of generality, we may assume that there is a positive integer m_1 such that $\alpha_i \geqslant \beta_i$ for $1 \leqslant i \leqslant m_1 \leqslant m$ and $\alpha_i < \beta_i$ for $m_1 < i \leqslant m$. In terms of Lemma 7.26, we know that there is a $\widetilde{E}_k(z)$ such that

$$\begin{aligned}\widetilde{E}_k(z) &= E_k\left(z, \prod_{i=1}^{m_1} \omega_{1,i}\overline{\omega}_{2,i}, \prod_{i=m_1+1}^{m} \overline{\omega}_{1,i}\omega_{2,i}\right) \\ &\in \mathcal{E}\left(\prod_{i=1}^{m_1} p_i^{\alpha_i} \prod_{i=m_1+1}^{m} p_i^{\beta_i}, k, \prod_{i=1}^{m_1} \omega_{1,i}\overline{\omega}_{2,i} \prod_{i=m_1+1}^{m} \overline{\omega}_{1,i}\omega_{2,i}\right).\end{aligned}$$

Put $\psi = \prod_{i=1}^{m_1} \omega_{2,i} \prod_{i=m_1+1}^{m} \omega_{1,i}$, then the conductor of ψ is $r = \prod_{i=1}^{m_1} p_i^{\beta_i} \prod_{i=m_1+1}^{m} p_i^{\alpha_i}$. Set

$$E_k(z, \omega_1, \omega_2) = \left(\sum_{u=1}^{r} \overline{\psi}(u) e(u/r)\right)^{-1} \sum_{u=1}^{r} \overline{\psi}(u) \widetilde{E}_k(z + u/r), \tag{7.39}$$

then $E_k(z, \omega_1, \omega_2) \in \mathcal{E}(r_1 r_2, k, \omega)$ by Lemma 7.27. And we have also that

$$\begin{aligned}L(s, E_k(z, \omega_1, \omega_2)) &= L\left(s, \psi \prod_{i=1}^{m_1} \omega_{1,i}\overline{\omega}_{2,i}\right) L\left(s - k + 1, \psi \prod_{i=m_1+1}^{m} \overline{\omega}_{1,i}\omega_{2,i}\right) \\ &= L(s, \omega_1) L(s - k + 1, \omega_2).\end{aligned}$$

Let l be a positive integer, ω a character modulo N with conductor r, ω_1 and ω_2 two primitive characters modulo r_1 and r_2 respectively. Denote by $A(N, r)$ the number of (l, ω_1, ω_2) satisfying

$$\omega = \omega_1\omega_2, l r_1 r_2 | N. \tag{7.40}$$

For any such (l, ω_1, ω_2) there is a function

$$E_k(lz, \omega_1, \omega_2) \in \mathcal{E}(l r_1 r_2, k, \omega) \subset \mathcal{E}(N, k, \omega)$$

such that

$$L(s, E_k(lz, \omega_1, \omega_2)) = l^{-s} L(s, \omega_1) L(s - k + 1, \omega_2).$$

Lemma 7.28 *We have that*

$$A(N, r) = \sum_{c|N, (c, N/c)|N/r} \varphi((c, N/c)).$$

Proof Let $B(N, r)$ be the right hand side of the above equality. If $N = N_1 N_2$, $r = r_1 r_2$ with $(N_1, N_2) = 1$, $r_1 | N_1$, $r_2 | N_2$, then we see that $A(N, r) = A(N_1, r_1) A(N_2, r_2)$, $B(N, r) = B(N_1, r_1) B(N_2, r_2)$. Hence we only need to show the lemma for the case

$N = p^a$, $r = p^b$ with $b \leqslant a$. If $(p^i, \omega_1, \omega_2)$ satisfies (7.40), then one of the r_1 and r_2 must be a multiple of r, so $0 \leqslant i \leqslant a-b$. If one of the r_1 and r_2 is larger than r, then $r_1 = r_2$. Since $\omega_2 = \omega\overline{\omega_1}$, we see that ω_2 is determined by ω_1.

We assume first that $2b \leqslant a$. If $0 \leqslant i \leqslant a-2b$, the maximal possible value of r_1 is $p^{[(a-i)/2]}$. We see that $[(a-i)/2] \geqslant b$ and ω_1 can be any character modulo $p^{[(a-i)/2]}$. If $a-2b+1 \leqslant i \leqslant a-b$, then $b \geqslant 1$, $2b+i > a$ and it is impossible that $p^b|r_1$ and $p^b|r_2$. But one of r_1 and r_2 must be p^b, so ω_1 can be χ or $\omega\chi$ where χ is any character modulo p^{a-b-i}. Hence we see that

$$A(p^a, p^b) = 2\sum_{i=0}^{b-1} \varphi(p^i) + \sum_{i=0}^{a-2b} \varphi(p^{[(a-i)/2]})$$
$$= \begin{cases} 2\displaystyle\sum_{i=0}^{a/2-1} \varphi(p^i) + \varphi(p^{a/2}) = B(p^a, p^b), & \text{if } 2|a, \\ 2\displaystyle\sum_{i=0}^{(a-1)/2} \varphi(p^i) = B(p^a, p^b), & \text{if } 2\nmid a. \end{cases}$$

Assume now $a < 2b$. Then one of r_1 and r_2 must be p^b and ω_1 can be χ or $\omega\chi$ with χ any character modulo p^{a-b-i}. Therefore

$$A(p^a, p^b) = 2\sum_{i=0}^{a-b} \varphi(p^i) = B(p^a, p^b).$$

This completes the proof. □

By Theorem 5.9 we see that $-L(0, \omega_1)L(1-k, \omega_2)$ is the constant term of the Fourier expansion at ∞ of $E_k(lz, \omega_1, \omega_2)$. And if ω is a primitive character modulo $r \neq 1$ with $\omega(-1) = (-1)^\nu$ ($\nu = 0$ or 1), then the function

$$R(s, \omega) := (r/\pi)^{(s+\nu)/2}\Gamma\left(\frac{s+\nu}{2}\right)L(s, \omega)$$

is holomorphic on the whole s-plane. It is well known that the function

$$\pi^{-s/2}s(s-1)\Gamma(s/2)\zeta(s)$$

is holomorphic on the whole s-plane. Since $s = 0$ and negative integers are poles of $\Gamma(s)$ with order 1, we know that $L(0, \omega) = 0$ (resp.$L(1-k, \omega) = 0$) if ω is a non-trivial even character(resp. if $k > 1$ is odd and ω is even or k is even and ω is odd.). Hence

$$-L(0, \omega_1)L(1-k, \omega_2) = \begin{cases} 0, & \text{if } k \neq 1 \text{ and } \omega_1 \text{ is nontrivial,} \\ & \text{or both } \omega_1 \text{ and } \omega_2 \text{ are non-trivial,} \\ \dfrac{L(1-k, \omega)}{2}, & \text{otherwise,} \end{cases}$$

where we used the fact that $\zeta(0) = -1/2$.

Let $N = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ be a positive integer. We introduce an order in the set of all factors of N as follows: if $l = p^{\beta_1} \cdots p_n^{\beta_n}$ and $l' = p_1^{\gamma_1} \cdots p_n^{\gamma_n}$ are two divisors of N, then we define $l \succ l'$ if there exist i with $0 \leqslant i \leqslant n$ such that $\beta_j = \gamma_j$ for $1 \leqslant j \leqslant i$ and $\beta_{i+1} > \gamma_{i+1}$.

Theorem 7.11 *Let ω, ω_1, ω_2, r_1, r_2 be as above. Then*

(1) *For $k \geqslant 3$ or $k = 2, \omega \neq \mathrm{id.}$, the functions*

$$E_k(lz, \omega_1, \omega_2) = -L(0, \omega_1)L(1-k, \omega_2) + \sum_{n=1}^{\infty} \left(\sum_{d|n} \omega_1(n/d)\omega_2(d)d^{k-1} \right) e(lnz),$$

constitute a basis of $\mathcal{E}(N, k, \omega)$ where (l, ω_1, ω_2) runs over all triples satisfying (7.40).

(2) *The functions*

$$E_1(lz, \omega_1, \omega_2) = -L(0, \omega_1)L(0, \omega_2) + \sum_{n=1}^{\infty} \left(\sum_{d|n} \omega_1(n/d)\omega_2(d) \right) e(lnz)$$

constitute a basis of $\mathcal{E}(N, 1, \omega)$ where (l, ω_1, ω_2) runs over all triples satisfying (7.40) *but only one of (l, ω_1, ω_2) and (l, ω_2, ω_1) can be taken.*

Proof (1) It is clear that $E_k(lz, \omega_1, \omega_2) \in \mathcal{E}(N, k, \omega)$. By dimension formula and Lemma 7.28 we have that $\dim(\mathcal{E}(N, k, \omega)) = A(N, r)$. Hence it is sufficient to show that the functions are linearly independent. Assume

$$0 = \sum_{n=0}^{\infty} b(n)e(nz) = \sum_{(l, \omega_1, \omega_2)} c(l, \omega_1, \omega_2)E_k(lz, \omega_1, \omega_2),$$

where (l, ω_1, ω_2) runs over the set of triples satisfying (7.40). Let 1_N be the trivial character modulo N. For any given $(1, \omega_1, \omega_2)$ satisfying (7.40), we see that

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} 1_N \overline{\omega_2}(n) b(n) n^{-s} \\ &= c(1, \omega_1, \omega_2) L(s, \omega_1 \overline{\omega_2} 1_N) L(s-k+1, 1_N) \\ &\quad + \sum_{\omega_2' \neq \omega_2} c(1, \omega_1', \omega_2') L(s, \omega_1' \overline{\omega_2} 1_N) L(s-k+1, \omega_2' \omega_2 1_N), \end{aligned} \tag{7.41}$$

where the last summation is taken for triples (l, ω_1, ω_2') satisfying (7.40) but $\omega_2 \neq \omega_2'$. The first term on the right hand side of (7.41) has a pole at $s = k$ with order 1 and the others have no poles at $s = k$. Hence $c(1, \omega_1, \omega_2) = 0$ for any $(1, \omega_1, \omega_2)$. Assume that $c(l', \omega_1, \omega_2) = 0$ for all $l' \prec l$ and that (l, ω_1, ω_2) satisfies (7.40), we see that

$$\begin{aligned} 0 &= \sum_{n=1}^{\infty} 1_N \overline{\omega_2}(n) b(ln) n^{-s} \\ &= c(l, \omega_1, \omega_2) L(s, \omega_1 \overline{\omega_2} 1_N) L(s-k+1, 1_N) \\ &\quad + \sum_{\omega_2' \neq \omega_2} c(l, \omega_1', \omega_2') L(s, \omega_1' \overline{\omega_2} 1_N) L(s-k+1, \omega_2' \overline{\omega_2} 1_N), \end{aligned}$$

so $c(l,\omega_1,\omega_2)=0$ by a similar argumentation. By induction we see that $c(l,\omega_1,\omega_2)=0$ for any (l,ω_1,ω_2).

(2) It is clear that $E_1(lz,\omega_1,\omega_2)\in\mathcal{E}(N,1,\omega)$. By the dimension formula we see that $\dim(\mathcal{E}(N,1,\omega))=\frac{1}{2}A(N,r)$. Therefore we only need to show that the functions are linearly independent. But this can be done similarly as we did in the proof of (1). This completes the proof. □

Recall the definition of the function $g_t^*(z)$ in Section 2.2:

$$g_t^*(z)=-\frac{1}{24}\prod_{p|t}(1-p)+\sum_{n=1}^{\infty}\Bigg(\sum_{d|n,(d,t)=1}d\Bigg)e(nz).$$

It is easy to show that $g_t^*\in\mathcal{E}(t,2,\mathrm{id.})$. For any positive integer l, put $t(l)=\prod_{p|l}p$. For $l\neq 1$ we define

$$E_2(lz,\mathrm{id.},\mathrm{id.})=g_{t(l)}^*(lz/t(l))\in\mathcal{E}(l,2,\mathrm{id.}).$$

It is easy to see that

$$L(s,E_2(lz,\mathrm{id.},\mathrm{id.}))=(l/t(l))^{-s}\zeta(s)L(s-1,1_{t(l)}).$$

It should be noticed that the symbol $E_2(z,\mathrm{id.},\mathrm{id.})$ is not defined. If ω_1 is non-trivial but $\omega_1^2=\mathrm{id.}$, we define

$$E_2(z,\omega_1,\omega_2)=\Bigg(\sum_{u=1}^{r_1}\omega_1(u)e(u/r_1)\Bigg)^{-1}\sum_{u=1}^{r_1}\omega_1(u)g_{t(r_1)}^*(z+u/r_1),$$

then $E_2(z,\omega_1,\omega_2)\in\mathcal{E}(r_1^2,2,\mathrm{id.})$ by Lemma 7.27, and

$$L(s,E_2(z,\omega_1,\omega_2))=L(s,\omega_1)L(s-1,\omega_2).$$

If $\omega_1^2\neq\mathrm{id.}$, we define

$$E_2(z,\omega_1,\omega_2)=\Bigg(\sum_{u=1}^{r_1}\overline{\omega_1}(u)e(u/r_1)\Bigg)^{-1}\sum_{u=1}^{r_1}\overline{\omega_1}(u)E_2(z+u/r_1,\mathrm{id.},\overline{\omega_1}^2),$$

where $E_2(z,\mathrm{id.},\overline{\omega_1}^2)$ is well defined as in (7.39) since $\omega_1^2\neq\mathrm{id.}$. It is not difficult to show that $E_2(z,\omega_1,\omega_2)\in\mathcal{E}(r_1^2,2,\mathrm{id.})$ and

$$L(s,E_2(z,\omega_1,\omega_2))=L(s,\omega_1)L(s-1,\omega_2).$$

So we have a function $E_2(lz,\omega_1,\omega_2)\in\mathcal{E}(N,2,\mathrm{id.})$ for every triple (l,ω_1,ω_2) satisfying

$$\omega_1\omega_2=\mathrm{id.},\quad lr_1r_2|N\quad\text{and}\quad l\neq 1\quad\text{if}\quad r_1=r_2=1. \tag{7.42}$$

Let $a_0(l,\omega_1,\omega_2)$ be the constant term of the Fourier expansion of $E_2(lz,\omega_1,\omega_2)$. It is easy to see that

$$a_0(l,\omega_1,\omega_2)=\begin{cases}0, & \text{if } \omega_1 \text{ is non-trivial,}\\ -\dfrac{1}{24}\prod\limits_{p|l}(1-p), & \text{if } \omega_1 \text{ is trivial,}\end{cases}$$

Theorem 7.12 *The functions*

$$E_2(lz,\omega_1,\omega_2)=a_0(l,\omega_1,\omega_2)+\sum_{n=1}^{\infty}\left(\sum_{d|n}\omega_1(n/d)\omega_2(d)d\right)e(lnz)$$

constitute a basis of $\mathcal{E}(N,2,\text{id.})$, *where* (l,ω_1,ω_2) *runs over the set of triples* (l,ω_1,ω_2) *satisfying* (7.42).

Proof We only need to show that the functions are linearly independent. Assume

$$\sum c(l,\omega_1,\omega_2)E_2(lz,\omega_1,\omega_2)=0, \tag{7.43}$$

where the summation was taken over all triples (l,ω_1,ω_2) satisfying (7.42).

Let $f(z)=\sum\limits_{n=0}^{\infty}a(n)e(nz)\in G(N,k,\omega)$, $r|N$ and ψ any character modulo N. Define

$$L(s,f,\psi,r)=\sum_{n=1}^{\infty}\psi(n)a(rn)n^{-s}.$$

We have that $L(s,E_2(lz,\text{id.},\text{id.}),\psi,r)=0$ if $l/t(l)\nmid r$. If $l/t(l)|r$, then

$$\begin{aligned}L(s,E_2(lz,\text{id.},\text{id.}),\psi,r)&=\sum_{n=1}^{\infty}\psi(n)\left(\sum_{\substack{d|nrt(l)/l,\\(d,l)=1}}d\right)n^{-s}\\&=\prod_{p|r,p\nmid l}(1+p+\cdots+p^{\nu_p(r)})L(s,\psi)L(s-1,\psi),\end{aligned}$$

where $\nu_p(r)$ is the p-adic valuation of r. If ψ is non-trivial, then $L(s,E_2(lz,\text{id.},\text{id.}),\psi,r)$ is holomorphic at $s=2$, by the same argumentation as in the proof of Theorem 7.11 and (7.43) we know that $c(l,\omega_1,\omega_2)=0$ if ω_2 is a non-trivial character.

Denote by f the left hand side of (7.43). It is clear that $L(s,f,1_N,r)$ has no pole at $s=2$. Hence

$$A_r=\sum_{\substack{l|N,l\neq 1,\\l/t(l)|r}}\prod_{\substack{p|r,\\p\nmid l}}(1+p+\cdots+p^{\nu_p(r)})c(l)=0,\quad N\neq r|N, \tag{7.44}$$

where $c(l)=c(l,\text{ id., id.})$. The equality (7.44) is a system of linear equations with respect to $\{c(l)|1\neq l|N\}$. We shall prove the system has only zero as solution which implies the theorem. If $N = p^n$ with p a prime, it is then clear that $A_1 = 0$, $A_p = 0$, $\cdots$, $A_{p^{n-1}} = 0$, so $c(p) = 0$, $c(p^2) = 0$, $\cdots$, $c(p^n) = 0$. We apply induction to the number of prime factors of N: let $N = p_1^n N_1$ with $(p_1, N_1) = 1$, suppose that (7.44) has only zero as solution if $N = N_1$. Now suppose that $r_1|N_1$, then

$$A_{p_1^n r_1} - A_{p_1^{n-1} r_1} = p_1^n \sum_{\substack{1\neq l|N,\\ l/t(l)|r_1}} \prod_{\substack{p|r_1,\\ p\nmid l}} (1 + p + \cdots + p^{\nu_p(r_1)}) c(l) = 0, \quad N_1 \neq r_1|N_1.$$

By induction hypothesis we see that $c(l) = 0$ if $p_1 \nmid l$. But p_1 can be any prime factor of N, we see that $c(l) = 0$ if there exists some prime factor p of N such that $p \nmid l$. Hence

$$A_{r_1} = \sum_{\substack{1\neq l|N_1,\\ l/t(l)|r_1}} \prod_{\substack{p|r_1,\\ p\nmid l}} (1 + p + \cdots + p^{\nu_p(r_1)}) c(p_1 l) = 0, \quad N_1 \neq r_1|N_1.$$

By induction hypothesis again we see that $c(p_1 l) = 0$ for $l|N_1$. Similarly using the fact that $A_{p_1 r_1} = 0, A_{p_1^2 r_1} = 0, \cdots, A_{p_1^{n-1} r_1} = 0$ for $N_1 \neq r_1|N_1$, we obtain that $c(p_1^2 l) = 0, \cdots, c(p_1^n l) = 0$ for $l|N_1$. This shows that the system (7.44) has only zero solution. This completes the proof. □

Theorem 7.13 *Let* $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in G(N, k, \omega)$. *Then* $f(z)$ *is a cusp form if and only if the function* $L(s, f, \psi, r)$ *is holomorphic at* $s = k$ *for any proper divisor* r *of* N *and any character* ψ *modulo* N.

Proof The necessity can be deduced from Lemma 7.15. We now assume that the function $L(s, f, \psi, r)$ is holomorphic at $s = k$. Since $G(N, k, \omega) = \mathcal{E}(N, k, \omega) \oplus S(N, k, \omega)$, we have

$$f(z) = \sum c(l, \omega_1, \omega_2) E_k(lz, \omega_1, \omega_2) + g(z),$$

where the summation was taken over the set of triples satisfying the conditions in Theorem 7.11 or Theorem 7.12 according to $k \neq 2, k = 2, \omega \neq \text{id.}$ or $k = 2, \omega = \text{id.}$ respectively, and $g(z) \in S(N, k, \omega)$. By the holomorphy of $L(s, f, \psi, r)$ at $s = k$ and applying the similar argumentation used in the proofs of Theorem 7.11 and Theorem 7.12, we can prove that $c(l, \omega_1, \omega_2) = 0$. Hence $f(z) \in S(N, k, \omega)$. This completes the proof. □

Remark 7.2 The hypothesis in Theorem 7.13 can be represented as follows: $L(s, f, \psi, r)$ is holomorphic at $s = k$ for any proper divisor r of N and any primitive character ψ induced from any character modulo N. The necessity can be deduced from Lemma 7.15. We now assume the above condition is satisfied. Let χ be any character modulo N and ψ the primitive character induced by χ. Then

$$\begin{aligned} L(s,f,\chi,r) &= \sum_{n=1}^{\infty} \chi(n)a(rn)n^{-s} = \sum_{n=1}^{\infty} \psi(n) \sum_{d|(n,N)} \mu(d)a(rn)n^{-s} \\ &= \sum_{d|N} \psi(d)d^{-s} L(s,f,\psi,rd), \end{aligned}$$

which implies the holomorphy of $L(s, f, \chi, r)$ at $s = k$. Hence f is a cusp form by Theorem 7.13. Also the condition can be represented as follows: $L(s, f, \psi, r)$ is holomorphic at $s = k$ for any positive integer $r|N$ and any primitive character ψ.

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Chapter 8

Weil Representation and Shimura Lifting

8.1 Weil Representation

Let V be an n-dimensional real vector space and V^* be the dual space of V. Denote by B a bilinear form on $(V \times V^*) \times (V \times V^*)$ given by $B(z_1, z_2) = (v_1, v_2^*) = v_2^*(v_1)$ for $z_1 = (v_1, v_1^*)$ and $z_2 = (v_2, v_2^*)$. Let $A(V)$ be the Lie group with underlying manifold $V \times V^* \times T$ whose multiplication is given by

$$(z, t)(z', t') = (z + z', tt'e(B(z, z'))), \quad \forall z, z' \in V \times V^*, t, t' \in T,$$

where $T = \{z \in \mathbb{C} || z| = 1\}$ and $e(z) = \mathrm{e}^{2\pi \mathrm{i} z}$.

We fix a Euclidean measure $\mathrm{d}x$ on V and denote by $\mathrm{d}x^*$ the Euclidean measure which is dual to $\mathrm{d}x$. Namely, the Fourier transformation

$$f^*(x^*) \mapsto \int_{V^*} f^*(x^*) e((x, x^*)) \mathrm{d}x^*$$

gives an isometric mapping from $L^2(V^*, \mathrm{d}x^*)$ onto $L^2(V, \mathrm{d}x)$. We denote by U a unitary representation of $A(V)$ on $L^2(V)$ given by

$$\{U(z, t)f\}(x) = te((x, v^*))f(x + v), \quad \forall x \in V, z = (v, v^*) \in V \times V^*, t \in T.$$

Then U is irreducible and $\phi(V)$, the space of rapidly decreasing functions over V, is a dense invariant subspace of $L^2(V)$. A linear transformation of $V \times V^*$ is said to be sympletic if it leaves the alternating form $A(z_1, z_2) = B(z_1, z_2) - B(z_2, z_1)$ invariant. We denote by $S_p(V \times V^*)$ the group of symplectic linear transformations of $V \times V^*$. For $\sigma \in S_p(V, V^*)$ and $z = (v, v^*) \in V \times V^*$, we write

$$\sigma(z) = (v, v^*) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a(v) + c(v^*), b(v) + d(v^*)),$$

where a, b, c and d are linear mappings from V to V, from V to V^*, from V^* to V and from V^* to V^* respectively. In the following we often identity σ with the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. For $\sigma \in S_p(V \times V^*)$ and $z \in V \times V^*$. Put

$$F_\sigma(z) = \exp\left(\pi \mathrm{i} B(\sigma(z), \sigma(z))\right) / \exp(\pi \mathrm{i} B(z, z)).$$

It is easy to see that

$$\begin{aligned} F_\sigma(z+z') &= F_\sigma(z)F_\sigma(z')e(B(\sigma(z),\sigma(z'))-B(z,z')),\\ F_{\sigma\tau}(z) &= F_\tau(\sigma(z))F_\sigma(z). \end{aligned} \tag{8.1}$$

This shows that the group $S_p(V\times V^*)$ acts on $A(V)$ as a group of automomorphisms via the mapping:

$$w \mapsto w^\sigma = (\sigma(z), tF_\sigma(z)), \quad \forall w=(z,t)\in A(V).$$

Set $U^\sigma(w)=U(w^\sigma)$, then U^σ is an irreducible unitary representation of $A(V)$ which is equivalent to U. Namely, there is a unitary operator $r(\sigma)$ on $L^2(V)$ which satisfies

$$U(w^\sigma) = r(\sigma)^{-1}U(w)r(\sigma), \quad \forall w\in A(V). \tag{8.2}$$

The operator $r(\sigma)$ is unique up to a multiplication by a complex number of modulus 1. Furthermore, the mapping $\sigma\to\gamma(\sigma)$ gives rise to a projective unitary representation of $S_p(V\times V^*)$ on $L^2(V)$. In other words, for each pair $(\sigma,z)\in S_p(V\times V^*)\times S_p(V\times V^*)$, there is a constant $c(\sigma,z)$ which satisfies

$$r(\sigma z) = c(\sigma,z)r(\sigma)r(z). \tag{8.3}$$

This projective unitary representation is called the Weil representation of $S_p(V\times V^*)$. If the entry c of σ is either non-singular or zero, we may normalize $r(\sigma)$ as follows:

$$r(\sigma)f(v) = \begin{cases} |c|^{1/2}\displaystyle\int_{V^*} F_\sigma(v,v^*))f(a(v)+c(v^*))\mathrm{d}v^*, & \text{if } c \text{ is non-singular,}\\ |a|^{1/2}e\left(\dfrac{1}{2}(a(v),b(v))\right)f(a(v)), & \text{if } c=0, \end{cases} \tag{8.4}$$

where $\sigma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, $\mathrm{d}(c(x^*))=|c|d^*x^*$ and $\mathrm{d}(a(x))=|a|\mathrm{d}x$.

Let L be a lattice in V and L^* be the dual lattice of L in V^*. Let M^* be a sublattice of L^* and M the dual lattice of M^* in V. Denote by $S_p(L\times M^*)$ the subgroup of $S_p(V\times V^*)$ consisting of linear transformations which leave the lattice $L\times M^*$ invariant. For a character χ of $L\times M^*$ and for a $\sigma\in S_p(L\times M^*)$, we set

$$\chi^\sigma(\lambda) = \chi(\sigma^{-1}(\lambda))F_{\sigma^{-1}}(\lambda), \quad \forall\lambda\in L\times M^*.$$

Then χ^σ is also a character of $L\times M^*$ and $\chi^{\sigma\tau}=(\chi^\sigma)^\tau$.

We denote also by χ the character of a subgroup $L\times M^*\times T$ of $A(V)$ given by

$$\chi\big((z,t)\big) = t\chi(z), \quad \forall\, z\in L\times M^*.$$

Then there exists a $(v_\chi, v_\chi^*)\in V\times V^*$ satisfying

$$\chi(\lambda,\mu^*) = e\big((v_\chi,\mu^*)-(\lambda,v_\chi^*)\big), \quad \forall\,(\lambda,\mu^*)\in L\times M^*.$$

The map $\chi \to (v_\chi, v_\chi^*)$ gives an isomorphism between the character group of $L \times M^*$ and the additive group $V/M \times V^*/L^*$. For a $\mu \in M/L$, we denote by $\chi(\mu)$ the character of $L \times L^*$ corresponding to $(v_\chi + \mu, v_\chi^*)$ of $V/L \times V^*/L^*$. Any extension of χ to a character of $L \times L^*$ coincides with $\chi(\mu)$ for a suitable $\mu \in M/L$. We denote by $T_\chi(L \times M^*)$ the unitary representation of $A(V)$ induced from the character χ of $L \times M^* \times T$ as follows: the representation space $\Theta_\chi(L \times M^*)$ is the Hilbert space of measurable functions $\theta(z)$ on $V \times V^*$ satisfying the following conditions:

$$e\big(B(\lambda, z)\big)\theta(\lambda + z) = \chi(\lambda)\theta(z), \quad \forall\, \lambda \in L \times M^*,\ z \in V \times V^*,$$
$$||\theta||^2 = \int\limits_{V/L \times V^*/M^*} |\theta(x, x^*)|^2 \mathrm{d}x\mathrm{d}x^* < +\infty$$

and $T_\chi(L \times M^*)$ is given by

$$T_\chi(L, M^*)\big((w, t)\big)\theta(z) = te\big(B(z, w)\big)\theta(z + w).$$

It is easy to see that the space $\Theta_{\chi(\mu)}(L \times L^*)$ ($\forall\, \mu \in M/L$) is a closed invariant subspace of $\Theta_\chi(L \times M^*)$ and

$$\Theta_\chi(L \times M^*) = \bigoplus_{\mu \in M/L} \Theta_{\chi(\mu)}(L \times L^*).$$

Put

$$\Theta_\chi = \Theta_\chi(L \times M^*), \quad \Theta_{\chi(\mu)} = \Theta_{\chi(\mu)}(L \times L^*), \quad T_\chi = T_\chi(L \times M^*).$$

For an $f \in \phi(V)$(where $\phi(V)$ is the space of rapidly decreasing functions on V, for the definition, please compare [**?**]), we define

$$\theta_{\chi(\mu)}(f)(x, x^*) = \big(\sqrt{\mathrm{vol}(V^*/M^*)}\big)^{-1} \sum_{l \in L} e\big((l + \mu + v_\chi, x^*) + (l, v_\chi^*)\big) f(x + l + \mu + v_\chi),$$

where $\mathrm{vol}(V^*/M^*) = \int\limits_{V^*/M^*} \mathrm{d}x^*$.

It is clear that $\theta_{\chi(\mu)}(f)$ depends on the choice of a representative of $(v_\chi + \mu) \in V/L$ in V. Here and after we choose representatives for $(v_\chi + \mu)$ ($\mu \in M/L$) and fix them. Then $\theta_{\chi(\mu)}(f)$ is a smooth function in $\Theta_{\chi(\mu)}$ and

$$\theta_{\chi(\mu)}\big(U(g)f\big) = T_\chi(g)\theta_{\chi(\mu)}(f), \quad \forall\, g \in A(V),$$
$$||\theta_{\chi(\mu)}(f)||^2 = ||f||^2 = \int\limits_V |f(x)|^2 \mathrm{d}x.$$

Conversely, for a smooth function $\theta \in \Theta_{\chi(\mu)}$, the following function

$$f_\theta(x) = \left(\sqrt{\mathrm{vol}(V^*/M^*)}\right)^{-1} \int\limits_{V^*/M^*} \theta(x-\mu-v_\chi, x^*)e\big(-(\mu+v_\chi, x^*)\big)\mathrm{d}x^* \qquad (8.5)$$

belongs to $\phi(V)$ and $\theta_{\chi(\mu)}(f) = \theta$. Thus $\theta_{\chi(\mu)}$ gives a norm preserving linear map from $\phi(V)$ onto the space of smooth functions in $\Theta_{\chi(\mu)}$ which commutes with the action of $A(V)$. The inverse of $\theta_{\chi(\mu)}$ is given by (8.5). These show that $\theta_{\chi(\mu)}$ is extended to linear isometric map from $L^2(V)$ onto $\Theta_{\chi(\mu)}$ which gives an equivalence of two unitary representations $(U, L^2(V))$ and $(T_\chi, \Theta_{\chi(\mu)})$ for any $\mu \in M/L$. Since $(U, L^2(V))$ is irreducible and (T_χ, Θ_χ) is a direct sum of $(T_\chi, \Theta_{\chi(\mu)})$ $(\mu \in M/L)$, any bounded linear map of $L^2(V)$ into Θ_χ is a linear combination of $\theta_{\chi(\mu)}$ $(\mu \in M/L)$ if it commutes with the action of $A(V)$. Finally, put

$$\theta(f, \chi(\mu)) = \theta_{\chi(\mu)}(f)(0,0).$$

All the above results and their proofs can be found in André Weil, 1964.

Proposition 8.1(Generalized Poisson Summation Formula) (1) *Let* $r(\sigma)(\sigma \in S_p(L\times M^*))$ *be the unitary operator in* $L^2(V)$ *which satisfies* (8.2). *There exist constants* $C_\sigma^\chi(u,v)$ $(u, v \in M/L)$ *which satisfy*

$$\theta(r(\sigma)f, \chi(u)) = \sum_{v\in M/L} C_\sigma^\chi(u,v)\theta(f, \chi^\sigma(v)), \quad \forall f \in \phi(V).$$

(2) *Denote by* C_σ^χ *the matrix of size* $[M:L]$ *whose* (u,v)*-entry* $(u, v \in M/L)$ *is* $C_\sigma^\chi(u,v)$. *Then* C_σ^χ *is a unitary matrix and* $C_{\sigma\tau}^\chi = c(\sigma,\tau)C_\sigma^\chi C_\tau^{\chi^\sigma}$ *where* $c(\sigma,\tau)$ *is a complex number of modulus* 1 *defined in* (8.3).

(3) *Set* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and assume c is non-singular and* $r(\sigma)$ *is nomalized by the formula* (8.4). *Then the constant* $C_\sigma^\chi(u,v)$ *is given by*

$$\mathrm{vol}(V^*/M^*)|c|^{1/2}C_\sigma^\chi(u,v) = \sum_{l\in L/c^*(M^*)} e\bigg(\frac{1}{2}(l+u'), c^{-1}a(l+u')\bigg) - (l+u', c^{-1}(v')) + \frac{1}{2}(v', dc^{-1}(v')) + (l, v_\chi^*)),$$

where $u' = u + v_\chi$ *and* $v' = v + v_{\chi^\sigma}$.

Proof For the details, see T. Shintani, 1975. □

From now on, we set $V = \mathbb{R}^n$. Take a non-degenerate symmetric $n\times n$ matrix Q and identify V with its dual by setting $(x,y) = y^{\mathrm{T}}Qx$. We put $\mathrm{d}x = \mathrm{d}x_1\cdots\mathrm{d}x_n$. Then the dual measure $\mathrm{d}x^*$ is given by $\mathrm{d}x^* = |\mathrm{det}Q|\mathrm{d}x$. We denote by $r(\cdot, Q)$ the Weil representation of $S_p(V\times V^*)$ on $L^2(V)$, to emphasize its dependence on Q. Identify the group $SL_2(\mathbb{R})$ with a subgroup of $S_p(V\times V^*)$ by settings

$$\sigma(x,y) = (ax+cy, bx+dy), \quad \forall x, y \in V, \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R}).$$

By (8.4), we have the following expression for $r(\sigma)=r(\sigma, Q)$ $(\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in SL_2(\mathbb{R}))$:

$$(r(\sigma,Q)f)(x)=\begin{cases} |c|^{-n/2}\sqrt{|\det Q|}\displaystyle\int_V e\left(\frac{a(x,x)-2(x,y)+d(y,y)}{2c}\right)f(y)\mathrm{d}y, & \text{if } c\neq 0,\\ |a|^{n/2}e\left(\dfrac{ab(x,x)}{2}\right)f(ax), & \text{if } c=0. \end{cases}$$

The group $GL_n(\mathbb{R})$ acts on $L^2(V)$, as a group of unitary operators if we put

$$(Tf)(x) = \sqrt{|\det T|^{-1}}f(T^{-1}x). \tag{8.6}$$

It is clear to verify that

$$r(\sigma, (T^{-1})^{\mathrm{T}}QT^{-1})\cdot T = T\cdot r(\sigma, Q), \quad \forall \sigma \in SL_2(\mathbb{R}), T \in GL_n(\mathbb{R}). \tag{8.7}$$

We are going to determine the constant $c(\sigma,\tau)$ in (8.3) for σ, $\tau \in SL_2(\mathbb{R})$.

Denote by $\mathbb{H}$ the complex upper half plane. For $\sigma \in SL_2(\mathbb{R})$, set

$$\varepsilon(\sigma) = \begin{cases} \sqrt{\mathrm{i}}, & \text{if } c>0,\\ \mathrm{i}^{(1-\mathrm{sgn}(d))/2}, & \text{if } c=0,\\ \sqrt{\mathrm{i}}^{-1}, & \text{if } c<0. \end{cases}$$

Take a positive definite symmetrix R such that $RQ^{-1}R = Q$. For $z = u+\mathrm{i}v \in \mathbb{H}$, put

$$Q_z = uQ + \mathrm{i}vR.$$

Let $P_v(x)$ be a homogeneous polynomial of degree v which has the following expression:

$$P_v(x) = \begin{cases} 1, & \text{if } v=0,\\ (r,x),\ (r\in\mathbb{C}^n, Qr=Rr), & \text{if } v=1,\\ \sum c_r(r,x)^v, c_r\in\mathbb{C}, r\in\mathbb{C}^n, Qr=Rr, (r,r)=0, & \text{if } v\geqslant 2 \end{cases}$$

(if $\mathrm{rank}(Q-R)=1$, we assume $v\leqslant 1$).

Lemma 8.1 *Assume Q has p positive and q negative eigenvalues ($p+q=n, p>0$). Set*

$$F_z(x) = e\left(\frac{1}{2}Q_z(x)\right)P_v(x).$$

Then

$$r(\sigma,Q)F_z(x) = \varepsilon(\sigma)^{p-q}\sqrt{J(\sigma,z)}^{q-p}|J(\sigma,z)|^{-q}J(\sigma,z)^{-v}F_{\sigma(z)}(x)$$

for any $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, *and where* $J(\sigma, z) = cz + d$.

Proof There exists a $T \in GL_n(\mathbb{R})$ such that $T^{\mathrm{T}}QT = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$ and $T^{\mathrm{T}}RT = I_n$. By (8.7), it is sufficient to show the lemma under the additional assumption that

$$Q = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}, \quad R = I_n.$$

Put $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $c = 0$, the lemma is clear. If $c \neq 0$, by a direct computation, we have

$$r(\sigma, Q)F_z(x) = |c|^{-n/2}\sqrt{v - \mathrm{i}u - \mathrm{i}d/c}^{-p}\sqrt{v + \mathrm{i}u + \mathrm{i}d/c}^{-q}J(\sigma, z)^{-v}F_{\sigma(z)}(x).$$

Now the lemma follows from the definitions of $\varepsilon(\sigma)$ and $J(\sigma, z)$. This completes the proof. □

By Lemma 8.1, we have

$$\begin{aligned} c(\sigma, \tau) &= \left(\frac{\varepsilon(\sigma\tau)}{\varepsilon(\sigma)\varepsilon(\tau)}\right)^{p-q} c_0(\sigma, \tau)^{q-p}, \\ c_0(\sigma, \tau) &= \frac{\sqrt{J(\sigma\tau, i)}}{\sqrt{J(\sigma, \tau(i))}\sqrt{J(\tau, i)}}. \end{aligned} \tag{8.8}$$

For $\sigma \in SL_2(\mathbb{R})$, set

$$r_0(\sigma, Q) = \varepsilon(\sigma)^{q-p} r(\sigma, Q). \tag{8.9}$$

Let G_1 be the Lie group with the underlying manifold $SL_2(\mathbb{R}) \times T$ and the multiplication given by

$$(\sigma, t)(\sigma', t') = (\sigma\sigma', tt'c_0(\sigma, \sigma')).$$

Then the subgroup $\{(\sigma, \pm 1) | \sigma \in SL_2(\mathbb{R})\}$ of G_1 is isomorphic to the two-fold covering group of $SL_2(\mathbb{R})$. For a $\widetilde{\sigma} = (\sigma, t) \in G_1$, set $r_0(\widetilde{\sigma}, Q) = t^{p-q} r_0(\sigma, Q)$. The following lemma is now immediate to see.

Lemma 8.2 (1) *The mapping:* $\widetilde{\sigma} \mapsto r_0(\widetilde{\sigma}, Q)$ *gives a unitary representation of* G_1 *on* $L^2(V)$. *The space* $\phi(V)$ *is a dense invariant subspace.*

(2) *For any* $f \in \phi(V)$, *the mapping* $\widetilde{\sigma} \mapsto r_0(\widetilde{\sigma}, Q)f$ *is a smooth mapping from* G_1 *into* $\phi(\mathbb{R}^n)$;

It is clear that the mapping $\sigma \mapsto (\sigma, 1)$ gives a locally isomorphic imbedding of $SL_2(\mathbb{R})$ into G_1. Hence, for any element u of the universal enveloping algebra of the Lie algebra of $SL_2(\mathbb{R})$, $r_0(u, Q)$ has an obvious meaning as a differential operator on V. In particular set

$$\begin{aligned} C_Q &= r_0(C, Q), \quad C = 2XY + 2YX + H^2, \\ X &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{8.10}$$

Then C_Q commutes with $r_0(\widetilde{\sigma}, Q)$ for any $\widetilde{\sigma} \in G_1$.

For $\theta \in \mathbb{R}$, let $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and $\Omega = \{(k_\theta, \varepsilon) | \theta \in \mathbb{R}, \varepsilon = \pm 1\}$. Put

$$\chi_m((k_\theta, \varepsilon)) = \left(\sqrt{\mathrm{e}^{-\mathrm{i}\theta}}\right)^{-m} \varepsilon^m.$$

Then χ_m is a character of Ω and for any $f \in \phi(V)$ we have

$$r_0(k, Q) f = \chi_m(k) f, \quad \forall k \in \Omega. \tag{8.11}$$

Lemma 8.3 *For $z = u + \mathrm{i}v \in \mathbb{H}$, set*

$$\sigma_z = \begin{pmatrix} \sqrt{v} & \sqrt{v}^{-1} u \\ 0 & \sqrt{v}^{-1} \end{pmatrix}.$$

Then

$$r_0(\sigma_z, Q) C_Q f = 4v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} - 2\mathrm{i}mv \frac{\partial}{\partial u} \right) r_0(\sigma_z, Q) f.$$

Proof See I. Gelfand. □

Let G be the connected component of the identity element of the group $O(Q)$ of real linear transformations which leave the quadratic form Q invariant. Then (8.6) gives a unitary representative of G on $L^2(V)$ which commutes with $r(\widetilde{\sigma}, Q)$ for any $\widetilde{\sigma} \in G_1$. Take a $T \in GL_n(\mathbb{R})$ satisfying $T^{\mathrm{T}} Q T = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ and set

$$\begin{aligned} X_{ij} &= T(e_{ij} - e_{ji}) T^{-1}, \quad 1 \leqslant i < j \leqslant p \ \text{ or } \ p < i < j \leqslant n, \\ Y_{kl} &= T(e_{kl} + e_{lk}) T^{-1}, \quad 1 \leqslant k \leqslant p < l \leqslant n. \end{aligned}$$

Then X_{ij} and Y_{kl} form a base of the Lie algebra of G. Put

$$L_Q = - \sum_{\substack{1 \leqslant i < j \leqslant p \text{ or} \\ p < i < j \leqslant n}} X_{ij}^2 + \sum_{1 \leqslant k \leqslant p < l \leqslant n} Y_{kl}^2. \tag{8.12}$$

Then L_Q is the Casimir operator on G. The representation (8.6) of G maps L_Q to a second order differential operator on $\mathbb{R}^n$ which is also denoted by L_Q.

Lemma 8.4 *For any $F \in \phi(V)$, we have*

$$C_Q F = (L_Q + n(n-4)/4) F.$$

Proof By (8.7), we may assume that $Q = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$. In this case, a simple computation shows that

$$\begin{aligned} r(H,Q)F &= \sum_{x=1}^{n} x_i \frac{\partial F}{\partial x_i} + \frac{n}{2}F, \\ r(X,Q)F &= \pi \mathrm{i}(x,x)F, \\ r(Y,Q)F &= \mathrm{i}(4\pi)^{-1}\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{n}\frac{\partial^2}{\partial x_j^2}\right)F, \end{aligned}$$

where $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus

$$\begin{aligned} C_Q F = &-(x,x)\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{n}\frac{\partial^2}{\partial x_j^2}\right)F + \sum_{i=1}^{n} x_i^2 \frac{\partial^2 F}{\partial x_i^2} \\ &+ (n-1)\sum_{i=1}^{n} x_i \frac{\partial F}{\partial x_i} + \left(\frac{n^2}{4} - n\right)F + 2\sum_{1 \leqslant i < j \leqslant n} x_i x_j \frac{\partial^2 F}{\partial x_i \partial x_j}. \end{aligned}$$

On the other hand,

$$\begin{aligned} L_Q F = &-\sum \left(x_j \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x_j}\right)^2 F + \sum \left(x_i \frac{\partial}{\partial x_k} + x_k \frac{\partial}{\partial x_i}\right)^2 F \\ = &-(x,x)\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{n}\frac{\partial^2}{\partial x_j^2}\right)F + 2\sum_{1 \leqslant i < j \leqslant n} x_i x_j \frac{\partial^2 F}{\partial x_i \partial x_j} \\ &+ (n-1)\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} x_i^2 \frac{\partial^2 F}{\partial x_i^2}. \end{aligned}$$

Therefore, $C_Q = L_Q + n(n-4)/4$. □

Here and after, we assume Q to be a rational symmetric matrix with $p\ (>0)$ positive and $q\ (= n-p)$ negative eigenvalues. Let L be a lattice of V, and L^* be the dual of L in V, i.e.,

$$L^* = \{x \in V | (x,y) = x^{\mathrm{T}} Q y \in \mathbb{Z}, \forall y \in L\}.$$

We always assume $L \subset L^*$. Let $v(L)$ be the volume of the fundamental parallelotop of L in V:

$$v(L) = \int_{\mathbb{R}^n/L} \mathrm{d}x.$$

For any $f \in \phi(V)$ and $h \in L^*/L$, put $\theta(f,h) = \sum_{l \in L} f(l+h)$.

Proposition 8.2 *Let* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *satisfy the following condition*

$$ab(x,x) \equiv cd(y,y) \equiv 0 \pmod 2, \quad \forall x, y \in L. \tag{8.13}$$

Then we have

(1) $\theta(r(\sigma,Q)f,h) = \sum_{k\in L^*/L} c(h,k)_\sigma \theta(f,k), \quad \forall f \in \phi(V)$, *where*

$$c(h,k)_\sigma =$$

$$\begin{cases} \delta_{h,ak} e\left(\frac{ab(h,h)}{2}\right), & \text{if } c=0, \\ \sqrt{|\det Q|}^{-1} v(L)^{-1} |c|^{-n/2} \sum_{r\in L/cL} e\left(\frac{a(h+r,h+r)-2(k,h+r)+d(k,k)}{2c}\right), & \text{if } c\neq 0. \end{cases}$$

(2) *Further assume that* c *is even,* $cL^* \subset L$, $cd \neq 0$ *and* $c(x,x) \equiv 0 \pmod 2$ *for any* $x \in L^*$. *Let* $\{\lambda_1, \cdots, \lambda_n\}$ *be a* $\mathbb{Z}$*–base of* L *and set* $D = \det((\lambda_i,\lambda_j))$. *Then*

$$\sqrt{\mathrm{i}}^{-(p-q)\mathrm{sgn}(cd)} c(h,k)_\sigma = \begin{cases} \delta_{h,dk} e\left(\frac{ab(h,h)}{2}\right) \varepsilon_d^{-n} (\mathrm{sgn}(c)\mathrm{i})^n \left(\frac{2c}{d}\right)^n \left(\frac{D}{-d}\right), & \text{if } d<0, \\ \delta_{h,dk} e\left(\frac{ab(h,h)}{2}\right) \varepsilon_d^{n} \left(\frac{-2c}{d}\right)^n \left(\frac{D}{d}\right), & \text{if } d>0, \end{cases}$$

where $\varepsilon_d = 1$ *or* i *according to* $d \equiv 1$ *or* $3 \pmod 4$ *respectively.*

Proof (1) We note that the group $SL_2(\mathbb{Z})$ is mapped into a subgroup of $S_p(L\times L)$ by our embedding of $SL_2(\mathbb{R})$ into $S_p(V\times V^*)$. Thus, the result in (1) is an immediate consequence of Proposition 8.1.

(2) Let e_0 be the index of L in L^*. Denote by C_σ the matrix of size e_0 whose (h,k) entry is $c(h,k)_\sigma$ $(h,k \in L^*/L)$. If σ, σ' and $\sigma\sigma'$ all satisfy the condition (8.13), it follows from the second statement of Proposition 8.1 that

$$C_{\sigma\sigma'} = c(\sigma,\sigma') C_\sigma C_{\sigma'}.$$

Put $\sigma' = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}$ and $\omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then σ', ω and $\sigma = \sigma'\omega$ all satisfy the condition (8.13). By (8.8) we have

$$c(\sigma',\omega) = \sqrt{\mathrm{i}}^{(p-q)\mathrm{sgn}(cd)}.$$

Hence

$$\begin{aligned} c(h,k)_\sigma = & \sqrt{\mathrm{i}}^{(p-q)\mathrm{sgn}(cd)} |\det(Q)|^{-1} v(L)^{-2} |d|^{-n/2} \\ & \times \sum_{r\in L/dL} \sum_{l\in L^*/L} e\left(\frac{-b(h+r,h+r)-2(l,h+r)+c(l,l)}{-2d}\right) e(-(l,k)). \end{aligned}$$

Since $cL^* \subset L$, the map $l \mapsto dl$ induces an automorphism of L^*/L. Taking into account the assumption that $c(x,x) \in 2\mathbb{Z}$ ($\forall x \in L^*$), we have

$$\begin{aligned}&\sum_{l\in L^*/L} e\left(\frac{-b(h+r,h+r)-2(l,h+r)+c(l,l)}{-2d}\right)e(-(l,k))\\ =&e\left(\frac{b(h+r,h+r)}{2d}\right)\sum_{l\in L^*/L} e((l,h-dk))\\ =&e_0 e\left(\frac{b(h+r,h+r)}{2d}\right)\delta_{h,dk}.\end{aligned}$$

On the other hand, the Poisson summation formula implies that $|\det(Q)|^{-1}v(L)^{-2}e_0 = 1$. Furthermore,

$$\begin{aligned}\sum_{r\in L/dL} e\left(\frac{b(h+r,h+r)}{2d}\right) &= \sum_{r\in L/dL} e\left(\frac{b(adh+r,adh+r)}{2d}\right)\\ &= e\left(\frac{ab(h,h)}{2}\right)\sum_{r\in L/dL} e\left(\frac{b(r,r)}{2d}\right).\end{aligned}$$

Thus, we have

$$c(h,k)_\sigma = \delta_{h,dk}\sqrt{\mathrm{i}}^{(p-q)\mathrm{sgn}(cd)} e\left(\frac{ab(h,h)}{2}\right)|d|^{-n/2}\sum_{r\in L/dL} e\left(\frac{b(r,r)}{2d}\right).$$

Now we can use the argument in the proof of Proposition 1.1 and Proposition 1.2 with a slight modification and get

$$|d|^{-n/2}\sum_{r\in L/dL} e\left(\frac{b(r,r)}{2d}\right) = \begin{cases} \varepsilon_d^{-n}(\mathrm{sgn}(c)\mathrm{i})^n\left(\dfrac{2c}{d}\right)^n\left(\dfrac{D}{-d}\right), & \text{if } d<0,\\ \varepsilon_d^{n}\left(\dfrac{-2c}{d}\right)^n\left(\dfrac{D}{d}\right), & \text{if } d>0,\end{cases}$$

which completes the proof. □

Let G be the connected component of the identity of the real orthogonal group of Q. Let Γ be the subgroup of G of all elements which leave the lattice L invariant and leave L^*/L point-wise fixed. Then, as a function on G, $\theta(g\cdot f,h)$ ($\forall\ f\in\phi(V)$, $g\in G$, $g\cdot f$ was defined as in equality (8.6), $h\in L^*/L$) is left Γ-invariant and slowly increasing on G/Γ (For the definitions of slowly increasing functions and rapidly decreasing functions on G/Γ, see R. Godement). Take a rapidly decreasing function Φ on G/Γ and put

$$\theta(f,\Phi;h) = \int_{G/\Gamma}\theta(g\cdot f,h)\Phi(g)\mathrm{d}g,$$

where dg is a Haar measure on G. Now assume that f satisfies (8.11) and set

$$\Theta(z, f, \Phi; h) = v^{-m/4}\theta(r(\sigma_z, Q)f, \Phi; h) \tag{8.14}$$

for $z = u + \mathrm{i}v \in \mathbb{H}$.

If no confusion is likely, we write

$$\Theta(z, h) = \Theta(z, f, \Phi; h).$$

Proposition 8.3 *Assume f satisfies* (8.11). *Then we have*

(1) *If* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *satisfies the condition* (8.13), *then*

$$\sqrt{\mathrm{i}}^{(p-q)\mathrm{sgn}(c)}\sqrt{J(\gamma, z)}^{-m}\Theta(\gamma(z), h) = \sum_{k\in L^*/L} c(h, k)_\gamma \Theta(z, k), \quad c \neq 0.$$

(2) *Assume that Φ satisfies the differential equation $L_Q\Phi = \lambda\Phi$ on G. Then*

$$\begin{aligned}&\left\{4v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) - 2\mathrm{i}mv\left(\frac{\partial}{\partial u} + \mathrm{i}\frac{\partial}{\partial v}\right)\right\}\Theta(z, h)\\ =&\left\{\lambda - m\left(\frac{m}{4} - 1\right) + n\left(\frac{n}{4} - 1\right)\right\}\Theta(z, h)\end{aligned} \tag{8.15}$$

for $z = u + \mathrm{i}v \in \mathbb{H}$.

Proof (1) It follows easily from (8.8) that

$$r(\gamma, Q)r(\sigma_z, Q) = r(\sigma_{\gamma(z)}, Q)r(k_\theta, Q),$$

where $\mathrm{e}^{-\mathrm{i}\theta} = J(\gamma, z)/|J(\gamma, z)|$ and $k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Since f satisfies (8.11),

$$r(k_\theta, Q)f = \sqrt{\mathrm{i}}^{(p-q)\mathrm{sgn}(c)}\sqrt{J(\gamma, z)/|J(\gamma, z)|}^{-m} f$$

(see (8.9)). So, by Proposition 8.2, we have

$$\sqrt{\mathrm{i}}^{(p-q)\mathrm{sgn}(c)}\sqrt{J(\gamma, z)}^{-m}\Theta(\gamma(z), h) = \sum_{k\in L^*/L} c(h, k)_\gamma \Theta(z, k).$$

(2) By Lemma 8.3, we have

$$\begin{aligned}&\left\{4v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) - 2\mathrm{i}mv\left(\frac{\partial}{\partial u} + \mathrm{i}\frac{\partial}{\partial v}\right)\right\}\Theta(z, f, \Phi; h)\\ =&m\left(1 - \frac{m}{4}\right)\Theta(z, h) + \Theta(z, C_Q f, \Phi; h).\end{aligned}$$

By Lemma 8.2, Lemma 8.4 and integration by parts, we have (8.15). This completes the proof. □

Example 8.1 Let $n=1$, $Q=(2/N)$, $L=N\mathbb{Z}$ and $f(x)=\exp(-2\pi x^2/N)$. Then we have $p=1$, $q=0$, $L^*=\mathbb{Z}/2$, $r(k(\theta))f=(\cos\theta-\mathrm{i}\sin\theta)^{-1/2}f$ and $\theta(z,\, f,\, 0)=\theta(Nz)$, where $\theta(z,\, f,\, h)=v^{-1/4}\theta(r(\sigma_z,\, Q)f,\, h)$ and $\theta(z)$ is defined as in Chapter 1. From Proposition 8.3 we have for $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(4N)$ that

$$(\sqrt{\mathrm{i}})^{\mathrm{sgn}(c)}(cz+d)^{-1/2}\theta(N\sigma(z))=c(0,0)_\sigma\theta(Nz),$$
$$c(0,0)_\sigma=(\sqrt{\mathrm{i}})^{\mathrm{sgn}(c)}j(\sigma,z)(cz+d)^{-1/2}\left(\frac{N}{d}\right).$$

Of course these formulas are the same as the transformation formula for Theta-function in Chapter 1.

We note that $c(h,k)_\sigma$ in Proposition 8.2 does not depend on f. We can interpret the Weil representation by the so-called Fock representation. We define a map

$$I: L^2(\mathbb{R})\to H=L^2(\mathbb{C},\exp\{-\pi z\bar{z}\}\mathrm{d}z)$$

by the integral transformation

$$I(f)(z)=\int_{\mathbb{R}} k(x,z)f(x)\mathrm{d}x,$$

where $f\in L^2(\mathbb{R})$ and

$$k(x,z)=\exp\{-\pi mx^2\}e(x\sqrt{m}z)\exp\{\pi z^2/2\}.$$

Then I is bijective and maps the Hermite function $\exp(\pi mx^2)\dfrac{\mathrm{d}^s}{\mathrm{d}x^s}\Big|_{\sqrt{m}x}\exp(-2\pi x^2)$ in $L^2(\mathbb{R})$ to the polynomial z^s in H up to a constant multiple. Moreover, by a direct computation one can easily check that

$$I(r(k(\theta))f)=(\cos\theta-\mathrm{i}\sin\theta)^{-1/2}M(\mathrm{e}^{\mathrm{i}\theta})I(f),$$

where $f\in L^2(\mathbb{R})$, $Q=(m)$ and $M(\mathrm{e}^{\mathrm{i}\theta})$ is the map such that $M(\mathrm{e}^{\mathrm{i}\theta})g(z)=g(\mathrm{e}^{\mathrm{i}\theta}z)$ for $g(z)\in H$. In this way we can find a function $f_{1,s}\in L^2(\mathbb{R})$ satisfying

$$r(k(\theta))f_{1,s}=(\cos\theta-\mathrm{i}\sin\theta)^{-(2s+1)/2}f_{1,s}$$

for a positive integer s. Namely,

$$f_{1,s}(x)=H_s(2\sqrt{\pi m}x^2),$$

where

$$H_s(x)=(-1)^s\exp\{x^2/2\}\frac{\mathrm{d}^s}{\mathrm{d}x^s}\exp\{-x^2/2\}$$

is a Hermite polynomial.

Put again $m = 2/N$ and let L be as above. Then

$$\theta(z, f_{1,s}, 0) = \theta_{1,s}(z) = v^{-1/2} \sum_{x=-\infty}^{\infty} H(2\sqrt{2N\pi v}x) \exp\{2\pi \mathrm{i} N z x^2\}$$

satisfies

$$\theta_{1,s}(\sigma(z)) = \left(\frac{N}{d}\right) j(\sigma, z)(cz + d)^s \theta_{1,s}(z)$$

according to the independence of $c(h,k)_\sigma$ to f. In the same way we can prove

$$\theta_{1,s}(-1/4Nz) = (2N)^{s/2}(\sqrt{-2\mathrm{i}z})^{2s+1}\theta_s(z),$$

where

$$\theta_s(z) = (2v)^{-s/2} \sum_{x=-\infty}^{\infty} \exp\{2\pi \mathrm{i} x^2 z\} H_s(2\sqrt{2\pi v}x).$$

□

Example 8.2 Now we consider the case $n = 2$, $Q = \begin{pmatrix} 0 & -4/N \\ -4/N & 0 \end{pmatrix}$, i.e.,

$$(x, y) = -\frac{4}{N}(x_1 y_2 + x_2 y_1)$$

and

$$L = (4N\mathbb{Z}) \oplus (N\mathbb{Z}/4).$$

Then $p = q = 1, r = r_0, L^* = (\mathbb{Z}) \oplus (\mathbb{Z}/16)$ and $4NL^* = L$ satisfies the assumption of Proposition 8.2. Put $L' = \mathbb{Z} \oplus (N\mathbb{Z}/4)$, $h \in L'$. Then for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$, $c(h,k)_\sigma = \delta_{k,ah}$ and $\theta(r_0(\sigma)f, h) = \theta(f, ah)$ are valid. If $f \in \phi(\mathbb{R}^2)$ satisfies $r(k(\theta))f = \mathrm{e}^{\mathrm{i}s\theta} f$, and if we define $\theta_{2,s}(z, f)$ by

$$\theta_{2,s}(z, f) = \sum_{h \in L'/L^*} \overline{\chi_1}(h)\theta(z, f, h),$$

where $\chi_1 = \chi \left(\frac{-1}{*}\right)^\lambda$ with λ a positive integer and χ a character modulo $4N$. Then we have

$$\theta_{2,s}(\sigma(z), f) = \overline{\chi_1}(d)(cz + d)^s \theta_{2,s}(z, f).$$

We explain how to find f with this property. Put $Q = \begin{pmatrix} 0 & -2m \\ -2m & 0 \end{pmatrix}$, $m > 0$. We define a partial Fourier transformation F by

$$F(f)(x_1, x_2) = \sqrt{2m} \int_{-\infty}^{\infty} f(x_1, t) \exp\{4\pi \mathrm{i} m t x_2\} \mathrm{d}t,$$

$$F^{-1}(f)(x_1, x_2) = \sqrt{2m} \int_{-\infty}^{\infty} f(x_1, t) \exp\{-4\pi \mathrm{i} m t x_2\} \mathrm{d}t.$$

One can easily check that

$$r(\sigma)f = FR(\sigma)F^{-1}(f),$$

where

$$(R(\sigma)f)(x) = f((x_1, x_2)\sigma).$$

And so r is a representation of $SL_2(\mathbb{R})$ although Weil representation is not always a multiplicative representation. Put

$$\begin{aligned} f'(x_1, x_2) &= (x_1 + \mathrm{i}x_2)^s \exp(-2m\pi(x_1^2 + x_2^2)); \\ f_{2,s}(x) &= F(f')(x) = \sqrt{2}(\sqrt{4\pi m})^{-s-1} H(\sqrt{4\pi m}(x_1 - x_2)) \exp(-2m\pi(x_1^2 + x_2^2)). \end{aligned}$$

Then

$$R(k(\theta))f' = \mathrm{e}^{2\mathrm{i}s\theta} f',$$

and $f_{2,s}$ has the required property. Generally, the Weil representation commutes with the action of the orthogonal group of Q on $L^2(\mathbb{R}^n)$. In the present case, the elements of that group are diagonal matrices in $SL_2(\mathbb{R})$. Put $f_\eta(x_1, x_2) = f_{2,s}(\eta^{-1}x_1, \eta x_2)$, and $m = 2/N$. Put $\theta_{2,s}(z, \eta) = \theta_{2,s}(z, f_\eta)$. Then

$$\begin{aligned} \theta_{2,s}(z, \eta) =& v^{(1-s)/2} \sum_{x_1, x_2 \in \mathbb{Z}} \overline{\chi}_1(x_1) \exp\left\{ - 2\pi \mathrm{i}ux_1x_2 - \frac{Nv}{4}\pi x_2^2\eta^2 - \frac{4v}{N}\pi x_1^2\eta^{-2} \right\} \\ & \times H_s\left(2\sqrt{\frac{2}{N}\pi v}\left(x_1\eta^{-1} - \frac{Nx_2}{4}\eta\right)\right). \end{aligned}$$

Observing that $f_{2,s} = F(f')$ and using the Poisson summation formula, we get a different expression for $\theta_{2,s}$:

$$\begin{aligned} \theta_{2,s}(z, \eta) =& \left(\sqrt{\frac{8\pi}{N}}\right)^{s+1} (\sqrt{2\pi})^{-1}\mathrm{i}^s\eta^{-s-1}v^{-s} \\ & \times \sum_{x_1, x_2 \in \mathbb{Z}} \overline{\chi_1}(x_1)(x_1\overline{z} + x_2)^s \exp\left\{ - \frac{4\pi}{N\eta^2 v}|x_1 z + x_2|^2 \right\}. \end{aligned}$$

□

Example 8.3 We denote by $r^{(i)}$ the Weil representation in the vector space V_i, $i = 1, 2, 3$, and by L_i, L_i^*, $r_0^{(i)}$, $h_i \in L_i^*$ and $c_i(h_i, k_i)_\sigma$ corresponding lattices, etc. If V_3 is the orthogonal sum of V_1 and V_2, then $r_0^{(3)} = r_0^{(1)} \otimes r_0^{(2)}, r^{(3)} = r^{(1)} \otimes r^{(2)}$, and $c_3(h_3, k_3)_\sigma = c_1(h_1, k_1)_\sigma c_2(h_2, k_2)_\sigma$ is obvious for $h_3 = (h_1, h_2), k_3 = (k_1, k_2)$. If $n = 3, Q = \frac{2}{N}\begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}$ and $L = 4N\mathbb{Z} \oplus N\mathbb{Z} \oplus (N\mathbb{Z}/4)$, then according to the preceding two examples, we have

$$c(h, k)_\sigma = \delta_{k,ah}(\sqrt{\mathrm{i}})^{\mathrm{sgn}(c)} j(\sigma, z)(cz + d)^{-1/2}\left(\frac{N}{d}\right)$$

for $f \in L^2(\mathbb{R}^3)$, $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$ and $h, k \in L'/L$ with $L' = \mathbb{Z} \oplus N\mathbb{Z} \oplus (N\mathbb{Z}/4)$. Therefore, if $r(k(\theta))(f) = (\cos\theta - \mathrm{i}\sin\theta)^{-\kappa/2} f$ is satisfied, then by Proposition 8.3 we have

$$\theta_\kappa(\sigma(z), f) = \overline{\chi}_1(d)\left(\frac{N}{d}\right) j(\sigma, z)(cz+d)^\lambda \theta_\kappa(z, f)$$

where $\kappa = 2\lambda + 1$, for $h = (h_1, h_2, h_3)$ we define, $\overline{\chi}_1(h) = \overline{\chi}_1(h_1)$ and

$$\theta_\kappa(z, f) = \sum_{h \in L'/L} \overline{\chi}_1(h)\theta(z, f, h).$$

One can take here $f_{1,s}(x_2) f_{2,\lambda-s}(x_1, x_3)$ $(s = 1, 2, \cdots, \lambda)$, or their linear combinations for such $f(x)$. In view of

$$(x - \mathrm{i}y)^\lambda = \sum_{s=0}^{\lambda} \binom{\lambda}{s} H_{\lambda-s}(x) H_s(y)(-\mathrm{i})^s,$$

$f_3(x) = (x_1 - \mathrm{i}x_2 - x_3)^\lambda \exp\{-m\pi(2x_1^2 + x_2^2 + 2x_3^2)\}$ is available, too. On the other hand, the action of $SL_2(\mathbb{R})$ on $\mathbb{R}^3$ is defined as follows: $g \in SL_2(\mathbb{R})$ operates on $\mathbb{R}^3$ through the symmetric tensor representation, i.e., for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $gx = (x_1', x_2', x_3')$ is determined by

$$g \begin{pmatrix} x_1 & x_2/2 \\ x_2/2 & x_3 \end{pmatrix} g^{\mathrm{T}} = \begin{pmatrix} x_1' & x_2'/2 \\ x_2'/2 & x_3' \end{pmatrix},$$

and gives an isomorphism of $SL_2(\mathbb{R})$ with the orthogonal group of Q.

Let N be a positive integer, χ a character modulo $4N$ and $\chi_1 = \chi\left(\frac{-1}{*}\right)^\lambda$ with a positive integer λ. Define a function on $\mathbb{R}^3$ by

$$f(x) = (x_1 - \mathrm{i}x_2 - x_3)^\lambda \exp\{(-2\pi/N)(2x_1^2 + x_2^2 + 2x_3^2)\}.$$

For $\kappa = 2\lambda + 1$, $z = u + \mathrm{i}v \in \mathbb{H}$ and for the lattice $L' = \mathbb{Z} \oplus N\mathbb{Z} \oplus (N\mathbb{Z}/4) \in \mathbb{Q}^3$, we define a theta series $\theta(z, g)$ by

$$\theta(z, g) = \sum_{x \in L'} \overline{\chi_1}(x_1) v^{(3-\kappa)/4}(\exp\{2\pi\mathrm{i}(u/N)(x_2^2 - 4x_1x_3)\}) f(\sqrt{v} g^{-1} x),$$

where $\sqrt{v} \in \mathbb{R}$ is viewed as a scalar of the vector space $\mathbb{R}^3$, and $g \in SL_2(\mathbb{R})$ operates on $\mathbb{R}^3$ as above.

Let $gf \in L^2(\mathbb{R}^3)$ be defined by $(gf)(x) = f(g^{-1}x)$ and take $m = 2/N$ in $f_3(x)$. Then it is clear that $\theta(z, g) = \theta(z, gf_3)$. The action of $r_0(k(\theta))$ commutes with that of g in $L^2(\mathbb{R}^3)$, gf_3 has the same property as f_3, and the required transformation formula of $\theta(z, g)$ is

$$\theta(\sigma(z), g) = \overline{\chi}(d)\left(\frac{N}{d}\right) j(\sigma, z)^\kappa \theta(z, g).$$

We note that f_3 has the property $f_3(k(\alpha)x) = \mathrm{e}^{2\lambda \mathrm{i}\alpha} f_3(x)$, and so $\theta(z, gk(\alpha)) = \mathrm{e}^{-2\mathrm{i}\lambda\alpha}\theta(z, g)$. □

8.2 Shimura Lifting for Cusp Forms

Let $G(z) = \sum_{n=1}^{\infty} a(n)e(nz)$ be an element of $S(4N, k+1/2, \chi)$, t a square-free positive integer, put $\chi_t = \chi\left(\frac{-1}{*}\right)\left(\frac{t}{*}\right)$ and $\Phi_t(w) = \sum_{n=1}^{\infty} A_t(n)e(nw)$ with $A_t(n)$ defined by the following equality

$$\sum_{n=1}^{\infty} A_t(n)e(nw) = \left(\sum_{m=1}^{\infty} \chi_t(m)m^{\lambda-1-s}\right)\left(\sum_{m=1}^{\infty} a(tm^2)m^{-s}\right).$$

Then $\Phi_t(w)$ is called the Shimura t−lifting of $G(z)$. The main theorem of G. Shimura, 1973 asserted that Φ_t belongs to $G(N_t,\ k-1,\ \chi^2)$, and in fact $\Phi_t \in S(N_t,\ k-1,\ \chi^2)$ for $k \geqslant 5$ with a certain positive integer N_t. He proved this result through Weil theorem. He also conjectured the level N_t can be taken as $2N$, and for $k = 1$, $\Phi(w)$ is a cusp form if and only if $G(z)$ is orthogonal to some theta series with respect to the Petersson inner product.

In this section we shall study these problems and prove these results. Our presentation is due to T. Shintani, S. Niwa, 1975, H. Kojima, 1980 and J. Sturm, 1982.

From now on, we always think of $\theta(z, g) = \theta(z, gf_3)$ as the function defined in Section 8.1. Now let $F(z)$ be in $S\left(4N,\ k/2,\ \overline{\chi}\left(\frac{N}{*}\right)\right)$ with $k = 2\lambda+1$ an odd positive integer. Since $F(z)$ is rapidly decreasing at each cusp of $\Gamma_0(4N)$, while $\theta(z, g)$ is at most slowly increasing there, so the following integral, which is the Petersson inner product of $F(z)$ and $\theta(z, g)$, is well-defined:

$$F(g) = \int_{D_0(4N)} F(z)\overline{\theta}(z, g)v^{k/2}\frac{\mathrm{d}u\mathrm{d}v}{v^2},$$

where $D_0(4N)$ is the fundamental domain of $\Gamma_0(4N)$. We have the following

Lemma 8.5 *The function $F(g)$ has the following properties:*

(1) $F(g) \in C^{\infty}(SL_2(\mathbb{R}))$ *is an eigenfunction of the* Casimir *operator D_g*, i.e., $D_g F = \lambda(\lambda-1)F$, *where*

$$D_g = \frac{1}{4}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^2 + 2\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + 2\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right);$$

(2) $F\left(g\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right) = \exp\{2\lambda\theta \mathrm{i}\}F(g);$

(3) *$F(\gamma g) = \chi^2(d)F(g)$ for every* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \Gamma_0(2N) \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$.

Proof The first conclusion is a direct consequence of the Proposition 8.3. In fact, by the proposition, we have

$$D_g\theta(z,g) = \left[4v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) - 2\mathrm{i}kv\left(\frac{\partial}{\partial u} + \mathrm{i}\frac{\partial}{\partial v}\right) + k\left(\frac{k}{4} - 1\right) + \frac{3}{4}\right],$$

where D_g is the Casimir operator on $SL_2(\mathbb{R})$. By Green's formula we have

$$D_g \int\limits_{D_0(4N)} F(z)\overline{\theta}(z,g)v^{k/2}\frac{\mathrm{d}u\mathrm{d}v}{v^2} = \lambda(\lambda-1)\int\limits_{D_0(4N)} F(z)\overline{\theta}(z,g)v^{k/2}\frac{\mathrm{d}u\mathrm{d}v}{v^2},$$

which is just (1).

Since $\theta(z, gk(\alpha)) = \mathrm{e}^{-2\mathrm{i}\lambda\alpha}\theta(z,g)$, so

$$\begin{aligned} F(gk(\alpha)) &= \int\limits_{D_0(4N)} F(z)\overline{\theta}(z,gk(\alpha))v^{k/2}\frac{\mathrm{d}u\mathrm{d}v}{v^2} \\ &= \int\limits_{D_0(4N)} F(z)\overline{\mathrm{e}^{-2i\lambda\alpha}\theta}(z,g)v^{k/2}\frac{\mathrm{d}u\mathrm{d}v}{v^2} \\ &= \exp\{2\mathrm{i}\lambda\alpha\}F(g), \end{aligned}$$

which is (2).

Now we prove that $\theta(z,\gamma g) = \chi^2(d)\theta(z,g)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \Gamma_0(2N)$ $\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$ from which (3) is deduced. Recalling the definition of $\theta(z,g)$:

$$\theta(z,g) = \sum_{x\in L'} \overline{\chi_1}(x_1)v^{(3-\kappa)/4}(\exp\{2\pi\mathrm{i}(u/N)(x_2^2 - 4x_1x_3)\})f(\sqrt{v}g^{-1}x),$$

where $L' = \mathbb{Z} \oplus N\mathbb{Z} \oplus (N\mathbb{Z}/4)$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \Gamma_0(2N) \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}$, it is easy to verify that $a, d \in \mathbb{Z}, c \in N\mathbb{Z}/2, b \in 4\mathbb{Z}$. By the definition of the symmetric tensor representation, for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\gamma x = (x_1', x_2', x_3')$ is determined by

$$\gamma\begin{pmatrix} x_1 & x_2/2 \\ x_2/2 & x_3 \end{pmatrix}\gamma^{\mathrm{T}} = \begin{pmatrix} x_1' & x_2'/2 \\ x_2'/2 & x_3' \end{pmatrix}.$$

That is,

$$\begin{aligned} x_1' &= a^2x_1 + abx_2 + b^2x_3, \\ x_2' &= 2cax_1 + (ad+bc)x_2 + 2bdx_3, \\ x_3' &= c^2x_1 + cdx_2 + d^2x_3. \end{aligned}$$

It is clear that both lattices $L = 4N\mathbb{Z} \oplus N\mathbb{Z} \oplus (N\mathbb{Z}/4)$ and L' are stable by γ and $x_1' \equiv a^2x_1 \pmod{4N}$ for $x = (x_1,\ x_2,\ x_3) \in L'$ which imply that $\theta(z,\ \gamma g) = (\overline{\chi}(a))^2\theta(z,\ g) = \chi^2(d)\theta(z,\ g)$ since $\overline{\chi}^2(a) = \chi^2(d)$. This completes the proof. □

We define two functions $\Psi(w)$ and $\Phi(w)$ $(w = \xi + \mathrm{i}\eta \in \mathbb{H})$ by

$$\Psi(w) = F\left(\begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} \eta^{1/2} & \xi\eta^{-1/2} \\ 0 & \eta^{-1/2} \end{pmatrix}\right)(4\eta)^{-\lambda}$$

and

$$\Phi(w) = \Psi\left(-\frac{1}{2Nw}\right)(2N)^{\lambda}(-2Nw)^{-2\lambda}.$$

Let W be the isomorphism of $S\left(4N,\, k/2,\, \overline{\chi}\left(\frac{N}{*}\right)\right)$ onto $S(4N,\, k/2,\, \chi)$ defined by

$$G(z) = (F|[W(4N)])(z) = F(-1/4Nz)(4N)^{-k/4}(-\mathrm{i}z)^{-k/2}$$

for all $F(z) \in S\left(4N,\, k/2,\, \overline{\chi}\left(\frac{N}{*}\right)\right)$. Then $G(z)$ has the Fourier expansion

$$G(z) = \sum_{n=1}^{\infty} a(n)e(nz)$$

at ∞. Define a sequence $\{A(n)\}_{n=1}^{\infty}$ by the following relation

$$\sum_{n=1}^{\infty} A(n)n^{-s} = L(s-\lambda+1, \chi_1)\sum_{n=1}^{\infty} a(n^2)n^{-s},$$

where $\chi_1 = \chi\left(\frac{-1}{*}\right)^{\lambda}$. Then we define the Shimura lifting I_k $(k \geqslant 3)$ by

$$I_k(G(z)) = \sum_{n=1}^{\infty} A(n)e(nz) \quad \text{for } G(z) \in S(4N, k/2, \chi).$$

Now we can present the main result of this chapter as follows.

Theorem 8.1 *If $k \geqslant 3$, then $\Phi(w)$ belongs to $G(2N,\, k-1,\, \chi^2)$ and $\Phi(w) = cI_k(G(z))$ with*

$$c = \mathrm{i}^{k-1}N^{k/4}2^{(-9k+15)/4}\mathrm{Re}((2-\mathrm{i})^{(k-1)/2}).$$

Moreover, if $k \geqslant 5$, then $\Phi(w)$ belongs to $S(2N,\, k-1,\, \chi^2)$.

Proof By Lemma 8.5, we have

$$\theta(z, \gamma' g) = \chi^2(d')\theta(z, g)$$

for any

$$\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}\Gamma_0(2N)\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix}.$$

And consequently, by the definition of $\Psi(w)$ we have

$$\Psi(\gamma(w)) = \overline{\chi}^2(d)(cw+d)^{2\lambda}\,\Psi(w)$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2N)$. This implies that

$$\Phi(\gamma(w)) = \chi^2(d)(cz+d)^{k-1}\,\Phi(w)$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2N)$. Therefore, if $\Phi(w)$ is holomorphic on $\mathbb{H}$, then we can conclude that $\Phi(w)$ is an integral modular form of weight $2\lambda = k-1$ for the congruence subgroup $\Gamma_0(2N)$. Now we prove that $\Phi(w)$ is holomorphic on $\mathbb{H}$. For the simplicity we assume $k=3$ though the method is applicable in all cases. By virtue of Lemma 8.5 and the invariance of the Casimir operator D_g, we have

$$\left(\eta^2\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right) - 2\mathrm{i}\eta\left(\frac{\partial}{\partial\xi}+\mathrm{i}\frac{\partial}{\partial\eta}\right)\right)\Phi(w) = 0.$$

Now $\Phi(w)$ has the Fourier expansion

$$\Phi(w) = \sum_{m=-\infty}^{\infty} a_m(\eta)\exp\{2\pi\mathrm{i}m\xi\}$$

at ∞. So $a_m(\eta)$ is a solution of the differential equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} + \frac{2}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta} + (-4\pi^2m^2 + 4\pi m/\eta)\right)a_m(\eta) = 0.$$

Therefore we get

$$a_m(\eta) = \begin{cases} b_m\exp\{-2\pi m\eta\} + c_m u_m(\eta), & \text{if } m \neq 0, \\ b_0 + c_0\eta^{-1}, & \text{if } m = 0. \end{cases}$$

where

$$u_m(\eta) = \begin{cases} \exp\{-2\pi m\eta\}\displaystyle\int_1^{\eta} \eta^{-2}\exp\{4\pi m\eta\}\mathrm{d}\eta, & \text{if } m > 0, \\ \exp\{-2\pi m\eta\}\displaystyle\int_{\eta}^{\infty} \eta^{-2}\exp\{4\pi m\eta\}\mathrm{d}\eta, & \text{if } m < 0. \end{cases}$$

By integration by parts, we have the following asymptotic behavior of $u_m(\eta)$:

$$|u_m(\eta)| \geqslant (4\pi m - \pi)^{-1}\exp\{-2\pi m\eta\}|\exp\{(4\pi m-\pi)\eta\} - \exp\{4\pi m - \pi\}| \quad (8.16)$$

for $m > 0$, and

$$u_m(\eta) = -\frac{\exp\{2\pi m\eta\}}{4\pi m\eta^2} + \alpha_m(\eta) \quad (8.17)$$

for $m < 0$, where

$$|\alpha_m(\eta)| \leqslant \exp\{2\pi m\eta\}(1/8\pi^2|m^2|\eta^3 + 15/32\pi^3|m^3|\eta^4).$$

Moreover we have

$$\eta\Phi(w) = O(\eta + \eta^{-1}) \quad \text{for } \eta \to 0 \text{ and } \eta \to \infty\ , \tag{8.18}$$

uniformly in ξ, which will be proved later. Since

$$\int_0^1 \eta^2|\Phi(w)|^2 \mathrm{d}\xi = \sum_{m=-\infty}^{\infty} |a_m(\eta)|^2\eta^2,$$

we get from (8.18)

$$|a_m(\eta)| \leqslant M((\eta + \eta^{-1})\eta^{-1}), \tag{8.19}$$

where M is independent of m and η. Hence by (8.16) and (8.17), we have $c_m = 0$ for all $m > 0$ and $b_m = 0$ for all $m < 0$. Hence we see

$$\begin{aligned}\Phi(w) = &\sum_{m=1}^{\infty} b_m \exp\{-2\pi m\eta\} \exp\{2\pi i m\xi\} \\ &+ \sum_{m=1}^{\infty} c_{-m}u_{-m}(\eta) \exp\{-2\pi\mathrm{i} m\xi\} + a_0(\eta).\end{aligned} \tag{8.20}$$

By (8.19) we have $|a_m(1/|m|)| \leqslant M(1 + m^2)$. Hence we get $b_m = O(m^\nu)(m \to \infty)$ and $c_{-m} = O(m^\nu)(m \to \infty)$ for some $\nu > 0$. We shall prove that $\Phi(\mathrm{i}\eta)$ has the following asymptotic behavior later:

$$\Phi(\mathrm{i}\eta) = \begin{cases} O(\eta^{-\mu}), & \eta \to +\infty \quad \text{for all } \mu > 0, \\ O(\eta^{\mu}), & \eta \to 0 \quad\quad \text{for all } \mu > 0. \end{cases} \tag{8.21}$$

In particular, we see that $a_0(\eta) = 0$. Hence we have

$$\begin{aligned}\Phi(w) = &\sum_{m=1}^{\infty} b_m \exp\{-2\pi m\eta\} \exp\{2\pi\mathrm{i} m\xi\} \\ &+ \sum_{m=1}^{\infty} c_{-m}u_{-m}(\eta) \exp\{-2\pi\mathrm{i} m\xi\}.\end{aligned} \tag{8.22}$$

By virtue of (8.21), $\Phi(\mathrm{i}\eta)\eta^{l-1}$ belongs to $L_1(\mathbb{R}^+)$ for a sufficiently large $l > 0$. Let $\Omega(s)$ be the Mellin transformation of $\Phi(\mathrm{i}\eta)$, i.e.,

$$\Omega(s) = \int_0^\infty \Phi(\mathrm{i}\eta)\eta^{s-1}\mathrm{d}\eta.$$

Here we note that $\Phi(\mathrm{i}\eta)$ is a function with bounded variation on all compact subsets of $\mathbb{R}^+$ and $\Phi(\mathrm{i}\eta) = \dfrac{1}{2}(\Phi(\mathrm{i}(\eta+0)) + \Phi(\mathrm{i}(\eta-0)))$ for all $\eta > 0$. Hence the Mellin inversion formula gives

$$\Phi(\mathrm{i}\eta) = \frac{1}{2\pi\mathrm{i}}\int_{l-\mathrm{i}\infty}^{l+\mathrm{i}\infty} \Omega(s)\eta^{-s}\mathrm{d}s. \tag{8.23}$$

On the other hand, we shall compute that

$$\Omega(s) = c(2\pi)^{-s}\Gamma(s)L(s,\chi_1)\sum_{n=1}^{\infty} a(n^2)n^{-s} = (2\pi)^{-s}\Gamma(s)\sum_{n=1}^{\infty} a'_n n^{-s},$$

where $G(z) = \sum_{n=1}^{\infty} a(n)e(nz)$. Consequently, we get

$$\Phi(\mathrm{i}\eta) = \sum_{n=1}^{\infty} a'_n \exp\{-2\pi n\eta\}. \tag{8.24}$$

Therefore, by (8.20), to prove that $\Phi(w)$ is holomorphic it is sufficient to show that $c_{-m} = 0$ for all $m \geqslant 1$. We assume that $c_{-m_0} \neq 0$ and $c_{-m} = 0$ for all $m < m_0$. Then by (8.20) and (8.24) we have

$$\begin{aligned}&\sum_{m>m_0} c_{-m}u_{-m}(\eta)/H_{m_0}(\eta) + c_{-m_0}u_{-m_0}(\eta)/H_{m_0}(\eta)\\ &= \sum_{n=1}^{\infty}(a'_n - b_n)\exp\{-2\pi n\eta\}/H_{m_0}(\eta),\end{aligned} \tag{8.25}$$

where $H_{m_0}(\eta) = \exp\{-2\pi m_0\eta\}/4\pi m_0\eta^2$.

We note that the series on both sides of (8.25) are uniformly convergent on $[1,\infty)$. Set $t = \exp\{-2\pi\eta\}$ for $\eta > 0$. The right hand side of (8.25) is equal to

$$\frac{m_0}{\pi}(\log t)^2\sum_{n=1}^{\infty}(a'_n - b_n)t^{n-m_0}.$$

By virtue of (8.17), we see that the left hand side of (8.25) converges to c_{-m_0} as $\eta \to +\infty$. Hence we get

$$\lim_{t\to 0, t>0}\left\{\frac{m_0}{\pi}(\log t)^2\sum_{n=1}^{\infty}(a'_n - b_n)t^{n-m_0}\right\} = c_{-m_0} \neq 0,$$

which is a contradiction and we proved that $\Phi(w)$ is holomorphic.

There still remains the investigation of the asymptotic behavior of $\Phi(\mathrm{i}\eta)$ as $\eta \to 0$ and ∞, and the computation of the Mellin transformation of $\Phi(\mathrm{i}\eta)$.

We first compute the Mellin transformation of $\Phi(\mathrm{i}\eta)$ for any $k \geqslant 3$. By the definition of Mellin transformation we have

$$\begin{aligned}\Omega(s) &= \int_0^\infty \Phi(\mathrm{i}\eta)\eta^{s-1}\mathrm{d}\eta = (-1)^\lambda(2N)^{\lambda-s}\int_0^\infty \Psi(\mathrm{i}\eta^{-1})\eta^{s-k}\mathrm{d}\eta\\ &= (-1)^\lambda(2N)^{\lambda-s}4^{-\lambda}\int_0^\infty \eta^{s-\lambda}\int_{D_0(4N)} v^{k/2}\overline{\theta}(z,\sigma_{4\mathrm{i}\eta-1})F(z)\mathrm{d}z\frac{\mathrm{d}\eta}{\eta},\end{aligned}$$

where $\mathrm{d}z = \dfrac{\mathrm{d}u\mathrm{d}v}{v^2}$ and $\sigma_w = \begin{pmatrix}\eta^{1/2} & \xi\eta^{-1/2}\\ 0 & \eta^{-1/2}\end{pmatrix}$ for $w = \xi + \mathrm{i}\eta \in \mathbb{H}$.

From the definition of $\theta(z,g)$ and the relation

$$(x-\mathrm{i}y)^\lambda = \sum_{\varepsilon=1}^{\lambda}\binom{\lambda}{\varepsilon}H_{\lambda-\varepsilon}(x)H_\varepsilon(y)(-\mathrm{i})^\varepsilon,$$

we have a simple expression

$$\theta(z,\sigma_{\mathrm{i}\eta}) = \left(2\sqrt{\frac{2\pi}{N}}\right)^{-\lambda}\sum_{\varepsilon=0}^{\lambda}\binom{\lambda}{\varepsilon}(-\mathrm{i})^\varepsilon\theta_{2,\lambda-\varepsilon}(z,\eta)\theta_{1,\varepsilon}(z),$$

where $\theta_{2,\lambda-\varepsilon}$, $\theta_{1,\varepsilon}$ are defined as in Example 8.1 and Example 8.2. Therefore by changing the order of integration whose justification can be deduced from the asymptotic behaviors (8.21) of $\Phi(\mathrm{i}\eta)$, we get

$$\Omega(s) = c_1(s)\sum_{\varepsilon=0}^{\lambda}\binom{\lambda}{\varepsilon}\mathrm{i}^\varepsilon\int_{D_0(4N)} v^{k/2}F(z)\overline{\theta}_{1,\varepsilon}(z)\left[\int_0^\infty \overline{\theta}_{2,\lambda-\varepsilon}(z,\eta^{-1})\eta^{s-\lambda}\frac{\mathrm{d}\eta}{\eta}\right]\mathrm{d}z$$

with $c_1(s) = (-1)^\lambda(2N)^{\lambda-s}4^{s-2\lambda}(2\sqrt{2\pi/N})^{-\lambda}$. Note that we can exchange the order of the summation and the integration as above. In terms of the different expressions of $\theta_{2,\varepsilon}$ given in Example 8.2, the integral in the bracket becomes an Eisenstein series

$$\begin{aligned}&\left(\sqrt{\frac{8\pi}{N}}\right)^{\lambda-\varepsilon+1}\sqrt{2\pi^{-1}}(-\mathrm{i})^{\lambda-\varepsilon}v^{-\lambda+\varepsilon}\left(\frac{1}{2}\right)\left(\frac{N}{4\pi}\right)^{(s-\varepsilon-1)/2}\\ &\times\Gamma\left(\frac{s-\varepsilon+1}{2}\right)\sum_{(x_1,x_2)\in\mathbb{Z}\times\mathbb{Z}}\chi_1(x_1)(x_1z+x_2)^{\lambda-\varepsilon}|x_1z+x_2|^{-s+\varepsilon-1}.\end{aligned}$$

Changing the variable z to $-1/4Nz$ and using $G(z) = F(-1/4Nz)(4N)^{-k/4}(-\mathrm{i}z)^{-k/2}$ and

$$\theta_{1,\varepsilon}(-1/4Nz) = (2N)^{\varepsilon/2}(\sqrt{-2\mathrm{i}z})^{2\varepsilon+1}\theta_\varepsilon(z),$$

we get

$$\Omega(s) = c_2(s)\sum_{\varepsilon=0}^{\lambda}\binom{\lambda}{\varepsilon}(\sqrt{2\pi})^{\lambda-\varepsilon+1}\mathrm{i}^{\varepsilon-\lambda}J_\varepsilon(s),$$

where $c_2(s)$ is like $c_1(s)$ above and $J_\varepsilon(s)$ is given by

$$J_\varepsilon(s) = \int_{D_0(4N)} G(z)\overline{\theta}_\varepsilon(z)v^{(s+\varepsilon+2)/2}\pi^{-(s-\varepsilon+1)/2}\Gamma\left(\frac{s-\varepsilon+1}{2}\right)$$
$$\times \sum_{x_1,x_2\in\mathbb{Z}} \chi_1(x_1)(4Nx_2z+x_1)^{\lambda-\varepsilon}|4Nx_2z+x_1|^{-s+\varepsilon-1}\mathrm{d}z$$
$$=\pi^{-(s-\varepsilon+1)/2}\Gamma\left(\frac{s-\varepsilon+1}{2}\right)L(s-\lambda+1,\chi_1)\int_0^\infty\int_0^1 G(z)\overline{\theta}_\varepsilon(z)v^{(s+\varepsilon+2)/2}\mathrm{d}z.$$

We note that $\theta_\varepsilon(z) = 0$ if ε is odd. The convolution appearing in $J_\varepsilon(s)$ is easily computed by Fourier expansion $\theta_\varepsilon(z) = \sum_{k=-\infty}^{\infty}(2v)^{-\varepsilon/2}H_\varepsilon(2\sqrt{2\pi v}k)\exp\{2\pi k^2 z\}$ and by partial integration, that is,

$$\int_0^\infty\int_0^1 G(z)\overline{\theta}_\varepsilon(z)\mathrm{d}uv^{(s+\varepsilon)/2}\frac{\mathrm{d}v}{v}=2^{1-\varepsilon}(4\pi)^{-s/2}(s-1)(s-2)\cdots(s-\varepsilon)\Gamma\left(\frac{s-\varepsilon}{2}\right)D(s),$$

where $D(s) = \sum_{k=1}^{\infty} a(k^2)k^{-s}$ with $G(z) = \sum_{k=1}^{\infty} a(k)e(kz)$. Therefore we get

$$J_\varepsilon(s) = 2^{2-2s}\pi^{-s+\varepsilon/2}\Gamma(s)L(s-\lambda+1,\chi_1)D(s).$$

Hence we have

$$\Omega(s) = c(2\pi)^{-s}\Gamma(s)L(s-\lambda+1,\chi_1)D(s).$$

By the definition of the Shimura lifting I_k and the computation of the Mellin transformation of $\Phi(\mathrm{i}\eta)$, we see that $\Phi(w) = cI_k(G(z))$. For $k \geqslant 5$, the function $\Phi(w)$ belongs to $S(2N,\, k-1,\, \chi^2)$ by virtue of the magnitude of the growth of $A_1(n)$.

In order to complete the proof of the theorem we only need to give the proofs for (8.18) and (8.21). Now we first prove (8.18). It is easy to see that we only need to show it for $\Psi(w)$ by the relation between $\Phi(w)$ and $\Psi(w)$. In fact, we shall prove a more general result for any $k \geqslant 3$:

$$\eta^\lambda\,\Psi(w) = O(\eta+\eta^{-1}).$$

Recalling the definition of $\theta(z,g)$:

$$\theta(z,g) = \sum_{x\in L'}\overline{\chi_1}(x_1)v^{(3-k)/4}\exp\{2\pi\mathrm{i}(u/N)(x_2^2-4x_1x_3)\}f(\sqrt{v}g^{-1}x),$$

we get

$$|\theta(z,\sigma_{4w})| \leqslant v^{(3-k)/4}\sum_{x\in L'}|f(\sqrt{v}\sigma_{4w}^{-1}x)|.$$

Put $M = \mathbb{Z}/4\oplus\mathbb{Z}/4\oplus\mathbb{Z}/4$, then

$$\sum_{x\in L'}|f(\sqrt{v}\sigma_{4w}^{-1}x)| \leqslant \sum_{x\in M}|f(\sqrt{v}\sigma_w^{-1}x)| = \sum_{x\in M}|f(\sqrt{v}\sigma_{\gamma(w)}^{-1}x)| \quad \text{for } \gamma\in SL_2(\mathbb{Z}).$$

If $\eta > c_1 > 0$ and $|\xi| < c_2$, then there exist $0 < h_j(x) \in \phi(\mathbb{R})$, $j = 1, 2, 3$ such that

$$\left| \begin{pmatrix} 1 & \xi/\eta \\ 0 & 1 \end{pmatrix} f(x) \right| \leqslant h_1(x_1)h_2(x_2)h_3(x_3)$$

for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Thus

$$\begin{aligned} \sum_{x\in M} |f(\sqrt{v}\sigma_w^{-1}x)| &= \sum_{x\in M} \left| \begin{pmatrix} \sqrt{\eta} & 0 \\ 0 & \sqrt{\eta}^{-1} \end{pmatrix} \begin{pmatrix} 1 & \xi/\eta \\ 0 & 1 \end{pmatrix} f(\sqrt{v}x) \right| \\ &\leqslant \Big(\sum_{x_1} h_1(\sqrt{v}\eta^{-1}x_1) \Big) \Big(\sum_{x_2} h_2(\sqrt{v}x_2) \Big) \Big(\sum_{x_3} h_3(\sqrt{v}\eta x_3) \Big), \end{aligned}$$

where $x_j \in \mathbb{Z}/4$. Therefore

$$\sum_{x\in M} |f(\sqrt{v}\sigma_w^{-1}x)| = O((\sqrt{v}^{-1} + 1)^2(\sqrt{v}^{-1}\eta + 1))$$

for $w = \xi + \mathrm{i}\eta$ with $|\xi| < c_2, \eta > c_1 > 0$. Put $U = \{w = \xi + \mathrm{i}\eta \,|\, |\xi| \leqslant 1/2, \eta > 0, |w| \geqslant 1\}$. Let $c_1 < \sqrt{3}/2, c_2 > 1/2$ and choose $\gamma \in SL_2(\mathbb{Z})$ for $w \in \mathbb{H}$ such that $\gamma(w) \in U$. Then

$$\begin{aligned} \sum_{x\in L'} |f(\sqrt{v}\sigma_{4w}^{-1}x)| &\leqslant \sum_{x\in M} |f(\sqrt{v}\sigma_{\gamma(w)}^{-1}x)| \\ &= O(\sqrt{v}^{-1} + 1)^3(\mathrm{Im}(\gamma(w)) + 1) \\ &= O((v^{-3/2} + 1)(\eta + \eta^{-1})). \end{aligned}$$

Thus $|\theta(z, \sigma_{4w})| = O(v^{(3-k)/4}(v^{-3/2} + 1)(\eta + \eta^{-1}))$ for all $w \in \mathbb{H}, z \in \mathbb{H}$, and hence $\eta^\lambda \Psi(w) = O(\eta + \eta^{-1})$ for all $w \in \mathbb{H}$ by the definition of $\Psi(w)$.

Finally we prove (8.21). By the definition of $\theta_{2,\lambda-\varepsilon}(z, \eta)$, we know that it is majorized by $\eta^{-\lambda+\varepsilon-1}v^{-\lambda+\varepsilon}F_\varepsilon(z, \eta)$, where $F_\varepsilon(z, \eta)$ is defined by

$$F_\varepsilon(z, \eta) = \sum_{x_1,x_2} |x_1z + x_2|^{\lambda-\varepsilon} \exp\Big\{ -\frac{4\pi}{N\eta^2 v}|x_1z + x_2|^2 \Big\},$$

where $(0, 0) \neq (x_1, x_2) \in \mathbb{Z}^2$. Therefore, if β is the smallest integer $\geqslant (\lambda - \varepsilon)/2$, then

$$F_\varepsilon(z, \eta) \leqslant \begin{cases} lv^{\beta+1}\mathrm{e}^{-\pi h/v\eta^2}, & \text{if } \eta < 1, v > c > 0, c < \sqrt{3}/2, \\ l'\eta^{2(\lambda-\varepsilon+1)}v^{\beta+1}\mathrm{e}^{-\pi\eta^2 h/v}, & \text{if } \eta > 1, v > c > 0, c < \sqrt{3}/2, \end{cases}$$

where l, l' and h are positive constants depending only on ε and c. Put $U = \{z = u + \mathrm{i}v \in \mathbb{H} \,|\, |u| \leqslant 1/2, |z| \geqslant 1\}$, choose $\gamma_i \in SL_2(\mathbb{Z})$ such that $D_0(4N) \subset \bigcup_{i=1}^{t} \gamma_i(U)$ and put $T(z) = v^{k/2}\overline{\theta}_{1,\varepsilon}(z)F(z)$, then $T(\gamma_i(z)) = O(g_i(v))$ for $z \in U$ where the g_i's are some rapidly decreasing functions. Put $F'_\varepsilon(z, \eta) = \eta^{-\lambda+\varepsilon-1}v^{-\lambda+\varepsilon}F_\varepsilon(z, \eta)$, then

$$\int_{D_0(4N)} |T(z)\overline{\theta}_{2,\lambda-\varepsilon}(z,\eta^{-1})\mathrm{d}z \leqslant \sum_{i=1}^{t} c_i \int_U T(\gamma_i(z))F'_\varepsilon(\gamma_i(z),\eta^{-1})\mathrm{d}z$$
$$\leqslant \sum_{i=1}^{t} e_i \int_c^\infty v^{\nu_i}\eta^\alpha g_i(v)\exp\{-\pi\eta^2 hv^{-1}\}\mathrm{d}v$$

for all $\eta > 1$ with some constants c_i, e_i, ν_i, α. Since $\eta^{2\mu}v^{-\mu}\exp\{-\pi\eta^2 hv^{-1}\} < C_\mu$ for $\mu > 0$ with some constant C_μ and $\eta^{2\mu-\alpha}\int_c^\infty v^{\nu_i}\eta^\alpha g_i(v)\exp\{-\pi\eta^2 hv^{-1}\}\mathrm{d}v < C_\mu\int_c^\infty v^{\nu_i+\mu}g_i(v)\mathrm{d}v = C'_\mu$ with some constant C'_μ. Therefore

$$\int_{D_0(4N)} |T(z)\overline{\theta}_{2,\lambda-\varepsilon}(z,\eta^{-1})|\mathrm{d}z = O(\eta^{-\mu})$$

for any $\mu > 0$ if $\eta > 1$. In the same way, we get

$$\int_{D_0(4N)} |T(z)\overline{\theta}_{2,\lambda-\varepsilon}(z,\eta^{-1})|\mathrm{d}z = O(\eta^{\mu})$$

for any $\mu > 0$ if $\eta < 1$. Hence we get (8.21) by the above estimations, the definition of $\Phi(w)$ and

$$\theta(z,\sigma_{\mathrm{i}\eta}) = \left(2\sqrt{\frac{2\pi}{N}}\right)^{-\lambda}\sum_{\varepsilon=0}^{\lambda}\binom{\lambda}{\varepsilon}(-\mathrm{i})^\varepsilon\theta_{2,\lambda-\varepsilon}(z,\eta)\theta_{1,\varepsilon}(z).$$

This completes the proof. □

Let $G(z) = \sum_{n=1}^{\infty} a(n)e(nz)$ be an element of $S(4N,\, k/2,\, \chi)$, let t be a square-free positive integer, put $\chi_t = \chi\left(\frac{-1}{*}\right)\left(\frac{t}{*}\right)$ and $\Phi_t(w) = \sum_{n=1}^{\infty} A_t(n)e(nw)$ with $A_t(n)$ defined by

$$\sum_{n=1}^{\infty} A_t(n)n^{-s} = \left(\sum_{m=1}^{\infty}\chi_t(m)m^{\lambda-1-s}\right)\left(\sum_{m=1}^{\infty}a(tm^2)m^{-s}\right).$$

Then we have

Corollary 8.1 *$\Phi_t(w) \in G(2N,\, k-1,\, \chi^2)$ for all $k \geqslant 3$ and $\Phi_t(w) \in S(2N,\, k-1,\, \chi^2)$ if $k \geqslant 5$.*

Proof Since $G(tz) = \sum_{n=1}^{\infty} b(n)e(nz)$ belongs to $S\left(4tN,\, k/2,\, \chi\left(\frac{t}{*}\right)\right)$, Theorem 8.1

implies that $\widetilde{\Phi(w)} = \sum_{n=1}^{\infty} B_1(n)e(nw)$, defined by

$$\sum_{n=1}^{\infty} B_1(n)n^{-s} = \left(\sum_{m=1}^{\infty} \chi_t(m)m^{\lambda-1-s}\right)\left(\sum_{m=1}^{\infty} b(m^2)m^{-s}\right)$$

belongs to $G(2tN,\ k-1,\ \chi^2)$ for all $k \geqslant 3$ and $S(2tN,\ k-1,\ \chi^2)$ if $k \geqslant 5$. Since $b(m^2) = a(tj^2)$ or 0 according as $m = tj$ or t does not divide m, we know that

$$\sum_{n=1}^{\infty} B_1(n)n^{-s} = t^{-s} \sum_{m=1}^{\infty} A_t(m)m^{-s}$$

holds and so $B_1(n) = A_t(n/t)$ or 0 according as $t|n$ or $t \nmid n$. Hence we have

$$\widetilde{\Phi(w)} = \Phi_t(tw),$$

and so

$$\Phi_t(\sigma(w)) = (cw+d)^{2\lambda}\chi^2(d)\,\Phi_t(w)$$

for all $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^t(2N)$ with $\Gamma_0^t(2N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2N) | b \equiv 0 \pmod t\right\}$.

Put $\Gamma_\infty = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{Z}\right\}$. Since $\Gamma_0(2N)$ is generated by Γ_∞ and $\Gamma_0^t(2N)$, $\Phi_t(w)$ belongs to $G(2N,\ k-1,\ \chi^2)$ for all $k \geqslant 3$ and $S(2N,\ k-1,\ \chi^2)$ if $k \geqslant 5$. This completes the proof. □

Now we consider the Shimura lifting for cusp forms with weight 3/2. By Theorem 8.1 and Corollary 8.1 we know that, for any $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in S(4N,\ 3/2,\ \chi)$, t a square-free positive integer, the Shimura lifting $I_{3,t}(f)$ of f belongs to $G(2N,\ 2,\ \chi^2)$. It is clear that the Zeta function of $I_{3,t}(f)$ is

$$L(s, I_{3,t}(f)) = L\left(s, \chi\left(\frac{-t}{*}\right)\right) \sum_{m=1}^{\infty} a(tm^2)m^{-s}. \tag{8.26}$$

We shall prove that $I_{3,t}(f)$ is a cusp form if and only if $\langle f, h\rangle = 0$ for all $h \in T$, where T is the vector space spanned by all theta series of $S(4N,\ 3/2,\ \chi)$ associated with some Dirichlet characters.

Proposition 8.4 *Let ψ be a primitive character modulo r, put*

$$h(z,\psi) = \sum_{n=1}^{\infty} \psi(n)ne(n^2z), \quad \forall\ z \in \mathbb{H}.$$

Then $h \in S\left(4r^2,\ 3/2,\ \psi\left(\frac{-1}{*}\right)\right)$.

Proof This is one of the conclusions in Theorem 7.3. □

By (8.26) we get

$$L(s, I_{3,1}(h(z,\psi))) = L(s,\psi)L(s-1,\psi),$$

which shows that $I_{3,1}(h(z,\psi))$ is an Eisenstein series (not a cusp form).

Proposition 8.5 *Let α be a non-negative integer, A a positive integer, ϕ a primitive character modulo A. Define*

$$H_\alpha(s,z,\phi) = \pi^{-s}\Gamma(s)y^s \sum_{m,n}{}' \phi(n)(mAz+n)^\alpha |mAz+n|^{-2s},$$

where $z \in \mathbb{H}$, $(0,0) \neq (m,n) \in \mathbb{Z}^2$. Suppose that $\alpha > 0$ or $A > 1$, then the series above is absolutely convergent for $\mathrm{Re}(s) > 1 + \alpha/2$, $H_\alpha(s,z,\phi)$ can be continued to a holomorphic function on the whole s-plane and satisfies the following functional equation

$$H_\alpha(\alpha+1-s,z,\phi) = (-1)^\alpha g(\phi)A^{3s-\alpha-2}z^\alpha H_\alpha(s,-1/Az,\overline{\phi}),$$

where $g(\phi) = \sum_{k=1}^{A} \phi(k)e(k/A)$.

Proof We have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-\pi t|uz+v|^2/y\}e(ur+vs)\mathrm{d}u\mathrm{d}v \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-\pi t[(ux+v)^2+u^2y^2]/y\}e(ur+vs)\mathrm{d}u\mathrm{d}v \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-\pi t(v^2+u^2y^2)/y\}e(u(r-xs)+vs)\mathrm{d}u\mathrm{d}v \\
&= (ty)^{-1/2}\int_{-\infty}^{\infty} \exp\{-\pi u^2\}e(u(r-xs)/(ty)^{1/2})\mathrm{d}u(ty^{-1})^{-1/2} \\
&\quad \times \int_{-\infty}^{\infty} \exp\{-\pi v^2\}e(vsy^{1/2}/t^{1/2})\mathrm{d}v \\
&= t^{-1}\mathrm{e}^{-\pi[(r-xs)^2/(ty)+s^2y/t]} = t^{-1}\mathrm{e}^{-\pi|r-sz|^2/(ty)}.
\end{aligned} \tag{8.27}$$

Since

$$\left(z\frac{\partial}{\partial r}+\frac{\partial}{\partial s}\right)e(ur+vs) = 2\pi\mathrm{i}(uz+v),$$

$$\left(z\frac{\partial}{\partial r}+\frac{\partial}{\partial s}\right)\exp\{-\pi|r-sz|^2/(ty)\} = -2\pi\mathrm{i}t^{-1}(r-sz)\exp\{-\pi|r-sz|^2/(ty)\},$$

applying α times the differential operator $\left(z\frac{\partial}{\partial r}+\frac{\partial}{\partial s}\right)$ on both sides of (8.27), we get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(uz+v)^\alpha \exp\{-\pi t|uz+v|^2/y\}e(ur+vs)\mathrm{d}u\mathrm{d}v$$
$$=(-1)^\alpha t^{-\alpha-1}(r-sz)^\alpha \exp\{-\pi|r-sz|^2/(ty)\}. \tag{8.28}$$

Put

$$\zeta(t,z,u,v)=\sum_{m,n}((m+u)z+n+v)^\alpha \exp\{-\pi t|(m+u)z+n+v|^2/y\}$$
$$=\sum_{m,n}c(m,n)e(mu+nv).$$

By (8.28) we get

$$c(-m,-n)=\int_0^1\int_0^1\sum_{m',n'}((m'+u)z+n'+v)^\alpha$$
$$\times\exp\{-\pi t|(m'+u)z+n'+v|^2/y\}e(mu+nv)\mathrm{d}u\mathrm{d}v$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(uz+v)^\alpha\exp\{-\pi t|uz+v|^2/y\}e(mu+nv)\mathrm{d}u\mathrm{d}v$$
$$=(-1)^\alpha t^{-\alpha-1}(m-nz)^\alpha\exp\{-\pi|m-nz|^2/(ty)\}.$$

Hence

$$\zeta(t,z,u,v)=(-1)^\alpha t^{-\alpha-1}\sum_{m,n}(mz+n)^\alpha\exp\{-\pi|mz+n|^2/(ty)\}e(mv-nu). \tag{8.29}$$

Suppose that p,q are integers, define

$$\xi(t,z,p,q)=\sum_{(m,n)\equiv(p,q)(\bmod A)}(mz+n)^\alpha\exp\{-\pi t|mz+n|^2/(A^2y)\}$$

and

$$\eta(t,z,p,q)=\sum_{k=1}^{A}\phi(k)\xi(t,z,kp,kq). \tag{8.30}$$

Suppose that $(p,q)\not\equiv(0,0) \pmod A$ if $A>1$. By (8.29) we have

$$\xi(t,z,p,q)=A^\alpha\zeta(t,z,p/A,q/A)$$
$$=(-A)^\alpha t^{-\alpha-1}\sum_{m,n}e((qm-pn)/A)(mz+n)^\alpha\exp\{-\pi|mz+n|^2/(ty)\}$$
$$=(-A)^\alpha t^{-\alpha-1}\sum_{(a,b)\bmod A}e((qa-pb)/A)\xi(A^2t^{-1},z,a,b)$$

and

$$\begin{aligned}\eta(t^{-1},z,p,q) &= \sum_{k=1}^{A}\phi(k)\xi(t^{-1},z,pk,qk)\\ &= (-A)^{\alpha}t^{\alpha+1}\sum_{k=1}^{A}\phi(k)\sum_{(a,b)\bmod A} e(k(qa-pb)/A)\xi(A^2t,z,a,b)\\ &= (-A)^{\alpha}t^{\alpha+1}g(\phi)\sum_{(a,b)\bmod A}\overline{\phi}(qa-pb)\xi(A^2t,z,p,q).\end{aligned} \tag{8.31}$$

If $\alpha > 0$ or $A > 1$, the terms corresponding to $m = n = 0$ on both sides of (8.31) disappear. Hence by (8.30) and (8.31) we have

$$|\eta(t,z,p,q)| \leqslant \begin{cases} M\mathrm{e}^{-ct}, & \text{if } t > 1,\\ M' t^{-\alpha-1}\mathrm{e}^{-c'/t}, & \text{if } t < 1,\end{cases} \tag{8.32}$$

where M, M', c, c' are positive constants dependent only on z, p, q. We can integrate the following integral term by term

$$\begin{aligned}\int_0^{\infty}\eta(t,z,p,q)t^{s-1}\mathrm{d}t &= \sum_{k=1}^{A}\phi(k)\sum_{(m,n)\equiv k(p,q)(\bmod A)}(mz+n)^{\alpha}\\ &\quad\times\int_0^{\infty}\exp(-\pi t|mz+n|^2/(A^2y))t^{s-1}\mathrm{d}t\\ &= A^{2s}\pi^{-s}y^s\Gamma(s)\sum_{k=1}^{A}\phi(k)\sum_{(m,n)\equiv(p,q)(\bmod A)}(mz+n)^{\alpha}|mz+n|^{-2s}.\end{aligned} \tag{8.33}$$

The series on the right hand side of (8.33) is absolutely convergent for $\mathrm{Re}(s) > 1+\alpha/2$. Divide the integral of the right hand side of (8.33) into two parts: $\int_0^1$ and $\int_1^{\infty}$. Using (8.32), we know that these two integrals are holomorphic functions on the s-plane which continues the series of the right hand side of (8.33) to a holomorphic function on the s-plane. And we have

$$A^{2s}H_{\alpha}(s,z,\phi) = \int_0^{\infty}\eta(t,z,0,1)t^{s-1}\mathrm{d}t. \tag{8.34}$$

Therefore for $\alpha > 0$ or $A > 1$, $H_{\alpha}(s,z,\phi)$ can be continued to a holomorphic function on the s-plane. Substituting s by $\alpha+1-s$ in (8.34), we get

$$\begin{aligned}A^{2(\alpha+1-s)}H_{\alpha}(\alpha+1-s,z,\phi) &= \int_0^{\infty}\eta(t,z,0,1)t^{\alpha-s}\mathrm{d}t = \int_0^{\infty}\eta(t^{-1},z,0,1)t^{s-\alpha-2}\mathrm{d}t\\ &= (-A)^{\alpha}g(\phi)\sum_{(a,b)\bmod A}\overline{\phi}(a)\int_0^{\infty}\xi(A^2t,z,a,b)t^{s-1}\mathrm{d}t\end{aligned}$$

$$= (-A)^{\alpha} g(\phi) y^s \pi^{-s} \Gamma(s) {\sum_{m,n}}' \overline{\phi}(m)(mz+n)^{\alpha}|mz+n|^{-2s}$$
$$= (-1)^{\alpha} g(\phi) A^{\alpha+s} z^{\alpha} H_{\alpha}(s, -1/Az, \overline{\phi}),$$

which completes the proof. □

Proposition 8.6 *Let ω be a character modulo A, put*

$$G(s) = \Gamma(s) {\sum_{m,n}}' \omega(n)|mAz+n|^{-2s}.$$

Then $G(s)$ can be continued to a holomorphic function if ω is non-trivial; $G(s)$ can be continued to a meromorphic function with only two poles $s = 0,\ 1$ of order 1 if $A = 1$, and with the corresponding residues -1 and π/y respectively; and $G(s)$ can be continued to a meromorphic function with only one pole 1 of order 1 if $A > 1$ and ω is trivial and with the corresponding residue $\pi \prod_{p|A}(1-p^{-1})/(Ay)$.

Proof Let B be the conductor of ω and $A = BC$, let ϕ be the primitive character modulo B determined by ω. Then

$$\begin{aligned} G(s) &= \Gamma(s) {\sum_{m,n}}' \phi(n) \sum_{d|(n,C)} \mu(d)|mAz+n|^{-2s} \\ &= \Gamma(s) \sum_{d|C} \mu(d)\phi(d) d^{-2s} {\sum_{m,n}}' \phi(n) \left| m\frac{A}{d}z + n \right|^{-2s}. \end{aligned} \tag{8.35}$$

Hence, by Proposition 8.5, $G(s)$ can be continued to a holomorphic function if $B > 1$ (i.e. if ω is non-trivial).

Now suppose that $A = 1$, put

$$\eta(t, z) = \sum_{m,n} \exp\{-\pi t|mz+n|^2/y\}.$$

By (8.31) we get

$$\eta(t^{-1}, z) = t\eta(t, z).$$

We have, for all $\mathrm{Re}(s) > 1$, that

$$\begin{aligned} \pi^{-s} y^s G(s) &= \int_0^{\infty} (\eta(t,z)-1) t^{s-1} \mathrm{d}t \\ &= \int_1^{\infty} (\eta(t^{-1},z)-1) t^{-s-1} \mathrm{d}t + \int_1^{\infty} (\eta(t,z)-1) t^{s-1} \mathrm{d}t \\ &= \int_1^{\infty} (t(\eta(t,z)-1) + t - 1) t^{-s-1} \mathrm{d}t + \int_1^{\infty} (\eta(t,z)-1) t^{s-1} \mathrm{d}t \\ &= \frac{1}{s-1} - \frac{1}{s} + \int_1^{\infty} (\eta(t,z)-1) t^{-s} \mathrm{d}t + \int_1^{\infty} (\eta(t,z)-1) t^{s-1} \mathrm{d}t. \end{aligned}$$

The two integrals on the right hand side of the above are holomorphic, so $G(s)$ can be continued to a meromorphic function with only two poles $s = 0, 1$ of order 1 and residues -1 and π/y respectively.

Now suppose that $B = 1, A > 1$. By (8.35) we get

$$G(s) = \sum_{d|A} \mu(d) d^{-2s} \Gamma(s) \sum_{m,n}{}' |mAz/d + n|^{-2s}.$$

Substituting $\frac{A}{d}z$ by z and using the above result for $A = 1$, we know that $G(s)$ can be continued to a meromorphic function with pole $s = 1$ and the residue

$$\sum_{d|A} \mu(d) d^{-2} \pi d/(Ay) = \pi \prod_{p|A} (1 - p^{-1})/(Ay).$$

This completes the proof. □

Now put

$$T = \{h(tz, \psi) | \psi \text{ is any odd primitive character, } t \text{ is any positive integer}\}$$

and $\widetilde{T}$ the vector space spanned by T. Also put

$$T_1 = \{h(tz, \psi) | \psi \text{ is any odd character, } t \text{ is any positive integer}\}$$

and

$$T_2 = \{\theta(tz, h, N) | t, h, N \in \mathbb{Z}, t > 0, N > 0\},$$

where

$$\theta(z, h, N) = \sum_{m \equiv h (\bmod N)} m e(m^2 z).$$

Denote by $\widetilde{T_i}$ the vector space spanned by T_i for $i = 1, 2$.

Lemma 8.6 *We have $\widetilde{T} = \widetilde{T_1} = \widetilde{T_2}$.*

Proof It is clear that $\widetilde{T} \subset \widetilde{T_1} \subset \widetilde{T_2}$. Let ψ be any odd character modulo N, $\widetilde{\psi}$ the primitive character determined by ψ. Then $\psi(d) = \widetilde{\psi}(d)$ for all $(d, N) = 1$, and

$$\begin{aligned} \sum_{m=1}^{\infty} \psi(m) m e(tm^2 z) &= \sum_{m=1}^{\infty} \sum_{d|(m,N)} \mu(d) \widetilde{\psi}(m) m e(tm^2 z) \\ &= \sum_{d|N} \mu(d) d \widetilde{\psi}(d) h(td^2 z, \widetilde{\psi}) \in \widetilde{T}, \end{aligned}$$

which shows that $\widetilde{T} = \widetilde{T_1}$. Denote $d = (h, N)$. We have

$$\begin{aligned}\theta(tz,h,N) &= d\sum_{m\equiv hd^{-1}(\bmod Nd^{-1})} me(td^2m^2z)\\ &= d\phi(Nd^{-1})^{-1}\sum_{m=1}^{\infty}\sum_{\psi}\overline{\psi}(hd^{-1})\psi(m)me(td^2m^2z)\\ &= d\phi(Nd^{-1})\sum_{\psi}\overline{\psi}(hd^{-1})h(td^2z,\psi)\in\widetilde{T_1},\end{aligned}$$

where ψ runs over all characters modulo Nd^{-1}, ϕ is the Euler function. Therefore $\widetilde{T_1}=\widetilde{T_2}$, which completes the proof. □

If $f(z)=\sum_{n\in\mathbb{Q}} a(n)e(nz)$ is a formal series, define

$$\xi(f(z))=\sum_{n=0}^{\infty}a(n)e(nz).$$

Put

$$F=\{\theta(zA^{-1})|\theta(z)\in\widetilde{T},\ A \text{ is any positive integer}\}.$$

Lemma 8.7 *Let $G(z)\in F$, $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$ and $H(z)=G(\gamma(z))(cz+d)^{-3/2}$. Then $H(z)\in F$, $\xi(G(z))\in\widetilde{T}$.*

Proof Since $SL_2(\mathbb{Z})$ is generated by $\gamma_1=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ and $\gamma_2=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, we only need prove that $H(z)\in F$ for γ_1, γ_2. Without loss of generality, we can assume that $G(z)=\theta(tA^{-1}z,\,h,\,N)$. It is easy to see

$$G(\gamma_1(z))=\sum_{\substack{g\equiv h(\bmod N),\\ g \bmod AN}} e(tg^2/A)\theta(tz/A,AN,g)\in F.$$

Using Lemma 7.5, we can prove that $H(z)\in F$ for γ_2. Now we prove $\xi(G(z))\in\widetilde{T}$. Assume again $G(z)=\theta(tz/A,h,N)$. Then

$$\xi(G(z))=\sum_{\substack{m\equiv h(\bmod N),\\ m^2\equiv 0(\bmod A)}} me(tm^2z/A).$$

Let $A=p_1^{e_1}\cdots p_j^{e_j}$ be the standard factorization of A. Take $B=p_1^{f_1}\cdots p_j^{f_j}$ such that f_i are the smallest positive integers with property $2f_i\geqslant e_i$ for all $1\leqslant i\leqslant j$. Then

$$\xi(G(z))=\sum_{\substack{m\equiv h(\bmod N),\\ m\equiv 0(\bmod B)}} me(tm^2z/A).$$

Denote $d = (B, N)$. If $d \nmid h$, then $\xi(G(z)) = 0$. If $d|h$, put $h' = h/d$, $N' = N/d$, $t' = tB^2/A$ and take B' such that $Bd^{-1}B' \equiv 1 \pmod{N'}$, then

$$\xi(G(z)) = \sum_{n\equiv h'B' (\bmod N')} nBe(tn^2B^2z/A) = \theta(t'z, h'B', N') \in \widetilde{T}.$$

This completes the proof. □

Theorem 8.2 *Let* $4|N$, $f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S(N,\ 3/2,\ \omega)$. *Then for any square-free positive integer* t, $I_{3,t}(f)$ *is a cusp form if and only if* $f(z)$ *is orthogonal to the subspace* $S(N,\ 3/2,\ \omega) \cap \widetilde{T}$.

Proof Let $I_{3,t}(f) = \sum_{n=0}^{\infty} b(n)e(nz) \in G(N/2,\ 2,\ \omega^2)$. By Theorem 7.13, $I_{3,t}(f)$ is a cusp form if and only if for all primitive character ψ and all positive integer r, $L(s, I_{3,t}(f), \psi, r)$ is holomorphic at $s = 2$. Substituting N by $[N, r]$ =l.c.m. of N, r, without loss of generality, we can assume that $r|N^{\infty}$. We have

$$\sum_{n=1}^{\infty} b(n)n^{-s} = L\left(s, \omega\left(\frac{-t}{*}\right)\right) \sum_{n=1}^{\infty} a(tn^2)n^{-s}.$$

Since ω is a character modulo N,

$$\begin{aligned} L(s, I_{3,t}(f), \psi, r) &= \sum_{n=1}^{\infty} \psi(n)b(rn)n^{-s} \\ &= L\left(s, \omega\psi\left(\frac{-t}{*}\right)\right) \sum_{n=1}^{\infty} \psi(n)a(tr^2n^2)n^{-s}. \end{aligned}$$

Put

$$h(z, \overline{\psi}) = \sum_{n=1}^{\infty} \overline{\psi}(n)n^{\nu}e(n^2z),$$

where $\psi(-1) = (-1)^{\nu}, \nu = 0, 1$. Taking a constant $\sigma > 0$, for $\mathrm{Re}(s) > \sigma$, we have

$$\begin{aligned} \int_0^{\infty}\int_0^1 f(z)\overline{h}(tr^2z, \overline{\psi})y^{s-1}\mathrm{d}x\mathrm{d}y &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a(n)\psi(m)m^{\nu}\int_0^{\infty} e(\mathrm{i}(n+tr^2m^2)y)y^{s-1}\mathrm{d}y \\ &\quad \times \int_0^1 e((n-tr^2m^2)x)\mathrm{d}x \\ &= (4\pi tr^2)^{-s}\Gamma(s)\sum_{m=1}^{\infty} \psi(m)a(tr^2m^2)m^{\nu-2s}. \end{aligned}$$

Denote by g the conductor of ψ. Then $h(tr^2z, \overline{\psi}) \in G\left(4tr^2g^2,\ (1+2\nu)/2,\ \overline{\psi}\left(\frac{(-1)^{\nu}tr^2}{*}\right)\right)$

by Theorem 7.3 and Theorem 5.16. Denote $\widetilde{N} = (4tr^2g^2, N)$, define $B(z,s) = f(z)\overline{h}(tr^2z, \overline{\psi})y^{s+1}$. Then for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = \Gamma_0(\widetilde{N})$, we have

$$B(\gamma(z), s) = \omega\psi(d)\left(\frac{-t}{d}\right)(cz+d)^{1-\nu}|cz+d|^{2\nu-1-2s}B(z,s).$$

Therefore

$$L(2s-\nu, I_{3,t}(f), \psi, r) = (4\pi tr^2)^s\Gamma(s)^{-1}\int_{\Gamma\backslash\mathbb{H}} B(z,s)L\left(2s-\nu, \omega\psi\left(\frac{-t}{*}\right)\right) \times \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma} \omega\psi(d)\left(\frac{-t}{d}\right)(cz+d)^{1-\nu}|cz+d|^{2\nu-1-2s}\frac{\mathrm{d}x\mathrm{d}y}{y^2}. \tag{8.36}$$

It is easy to see

$$L\left(2s-\nu, \omega\psi\left(\frac{-t}{*}\right)\right)\sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma} \omega\psi(d)\left(\frac{-t}{d}\right)(cz+d)^{1-\nu}|cz+d|^{2\nu-1-2s} = \sum_{m,n}{}' \omega\psi(n)\left(\frac{-t}{n}\right)(m\widetilde{N}z+n)^{1-\nu}|m\widetilde{N}z+n|^{2\nu-1-2s}. \tag{8.37}$$

If $\nu = 0$, by Proposition 8.5, $L(s, I_{3,t}(f), \psi, r)$ is holomorphic at $s = 2$. If $\nu = 1$, by Proposition 8.6, we know that the series in (8.36) is holomorphic except the case $\omega = \overline{\psi}\left(\frac{-t}{*}\right)$. In that case, it has a pole $s = 3/2$ of order 1 with residue c/y and $c \neq 0$ a constant. Hence, by (8.36), only for $\omega = \overline{\psi}\left(\frac{-t}{*}\right)$, $L(s, I_{3,t}(f), \psi, r)$ has a possible pole $s = 2$ of order 1 with residue $c' < f, h(tr^2z, \overline{\psi}) >$ and $c' \neq 0$ a constant.

Now suppose that $I_{3,t}(f)$ is a cusp form. By the above argumentation we know that f is orthogonal to $h(tr^2z, \overline{\psi})$ if $\omega = \overline{\psi}\left(\frac{-t}{*}\right)$. If $\omega \neq \overline{\psi}\left(\frac{-t}{*}\right)$, put $\omega' = \overline{\psi}\left(\frac{-t}{*}\right)$. Then $f \in S(\widetilde{N}, 3/2, \omega)$, $h(tr^2z, \overline{\psi}) \in S(\widetilde{N}, 3/2, \omega')$. Therefore for any $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma_0(\widetilde{N})$ we have

$$\begin{aligned}\omega(d_\gamma)\overline{\omega'}(d_\gamma)\langle f, h(tr^2z,\overline{\psi})\rangle_{\Gamma_0(\widetilde{N})} &= \langle f|[\gamma], h(tr^2z,\overline{\psi})|[\gamma]\rangle_{\Gamma_0(\widetilde{N})} \\ &= \langle f, h(tr^2z,\overline{\psi})\rangle_{\Gamma_0(\widetilde{N})}.\end{aligned}$$

Since $\omega \neq \omega'$, we can find a $\gamma \in \Gamma_0(\widetilde{N})$ such that $\omega(d_\gamma) \neq \omega'(d_\gamma)$. Hence we get $\langle f, h(tr^2z,\overline{\psi})\rangle = 0$. But any positive integer u can be written as $u = tr^2$ with t square-free. So f is orthogonal to $\widetilde{T}$ and hence is orthogonal to $S(N, 3/2, \omega) \cap \widetilde{T}$.

Conversely, suppose f is orthogonal to $S(N, 3/2, \omega) \cap \widetilde{T}$. Take any $h(uz, \psi) \in T$. Then $h(uz, \psi) \in S\left(4ug^2, 3/2, \psi\left(\frac{-u}{*}\right)\right)$ where g is the conductor of ψ. Denote $\widetilde{N} = [4ug^2, N]$. Suppose $\omega = \psi\left(\frac{-u}{*}\right)$. Let $\Gamma_0(N) = \bigcup_{i=1}^{r} \Gamma(\widetilde{N})\gamma_i$ be the decomposition of $\Gamma_0(N)$ into right cosets with respect to $\Gamma(\widetilde{N})$. Let

$$\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}.$$

Then

$$g(z) = \sum_{i=1}^{r} \omega(a_i) h(uz, \psi) | [\gamma_i]$$

belongs to $S(N, 3/2, \omega)$. By Lemma 8.7 we know that $g(z) \in F$. Since $g(z+1) = g(z)$, $\xi(g(z)) = g(z)$. By Lemma 8.7 we know that $g(z) \in \widetilde{T}$, i.e., $g \in S(N, 3/2, \omega) \cap \widetilde{T}$. By hypothesis, we get

$$\begin{aligned} 0 &= \langle f(z), g(z) \rangle \\ &= \sum_{i=1}^{r} \overline{\omega}(a_i) \langle f(z), h(uz, \psi) | [\gamma_i] \rangle \\ &= \sum_{i=1}^{r} \overline{\omega}(a_i) \langle f | [\gamma_i^{-1}](z), h(uz, \psi) \rangle \\ &= r \langle f(z), h(uz, \psi) \rangle, \end{aligned}$$

which shows that f is orthogonal to $h(uz, \psi)$. Hence $L(s, I_{3,t}(f), \psi, r)$ is holomorphic at $s = 2$ (since whose residue at $s = 2$ is 0 or $c' \langle f, h(tr^2 z, \overline{\psi}) \rangle = 0$). This shows that $I_{3,t}(f)$ is a cusp form.

This completes the proof. □

8.3 Shimura Lifting of Eisenstein Spaces

In this section we deal with Shimura lifting of Eisenstein spaces.

Let χ be a Dirichlet character modulo N, and denote by $L(s, \chi)$ the associated L-series

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}.$$

For a positive integer k we have that $L(1-k, \chi) = -\frac{B_{k,\chi}}{k}$, where the numbers $B_{k,\chi}$ are defined by

$$\sum_{a=1}^{N} \frac{\chi(a) t e^{at}}{e^{Nt} - 1} = \sum_{k=0}^{\infty} B_{k,\chi} \frac{t^k}{k!}.$$

Fix an integer $k \geqslant 2$, we define rational numbers $H(k,n)$ by

$$H(k,n) := \begin{cases} \zeta(1-2k), & \text{if } n=0, \\ L(1-k,\chi_D)\sum_{d|f}\mu(d)\chi_D(d)d^{k-1}\sigma_{2k-1}(f/d), & \text{if } (-1)^k n = Df^2, \\ 0, & \text{otherwise,} \end{cases}$$

where ζ denotes the Riemann ζ-function, μ the Moebius function, D a fundamental discriminant, χ_D the quadratic character associated with $\mathbb{Q}(\sqrt{D})$ and the arithmetical function σ_r is defined by $\sigma_r(m) = \sum_{d|m} d^r$. H.Cohen introduced the rational numbers $H(k,n)$ and proved that

$$H_k(z) := \sum_{n=0}^{\infty} H(k,n)\exp\{2\pi i n z\} \tag{8.38}$$

is a modular form of half-integral weight $k+1/2$ for $\Gamma_0(4)$ in [C] which is now named Cohen-Eisenstein series. For $k=1$ and group $\Gamma_0(4p)$ with p a prime, Cohen-Eisenstein series are defined by

$$H_{1,p}(z) := \sum_{n=0}^{\infty} H(n)_p \exp\{2\pi i n z\}, \tag{8.39}$$

where $H(n)_p := H(p^2 n) - pH(n)$ with $H(n)$(for $n > 0$) the number of classes of positive definite binary quadratic forms of discriminant $-n$(where forms equivalent to a multiple of x^2+y^2 or x^2+xy+y^2 are counted with multiplicity $\frac{1}{2}$ or $\frac{1}{3}$ respectively) and with $H(0) = -\frac{1}{12}$. $H_{1,p}$ is a modular form of weight $3/2$ on $\Gamma_0(4p)$.

The problem of constructing Shimura lifting of non-cusp forms was first considered by W.Kohnen for the Cohen-Eisenstein series and later by A.G.Van Asch for the space of non-cusp forms of weight $k+1/2(k \geqslant 2)$ on $\Gamma_0(4)$ and $\Gamma_0(4p)$ with p an odd prime.

In this section we shall consider more general cases.

Let the rational numbers $H(k,\ l,\ N,\ N;\ n)$ and $H(k,\ l,\ m,\ N;\ n)$ be defined as in Section 7.4 with $N \neq m|N$.

Note that $H(k,\ 1,\ 1,\ 1;\ n) = H(k,\ n)$ is just the rational numbers defined by H.Cohen.

Theorem 8.3 *Let N be a square-free odd positive integer, l a divisor of N and D a fundamental discriminant with $\varepsilon(-1)^k D > 0$. Then*

(1) *If $k=1$ and $\left(\frac{D}{p}\right) \neq 1$ for all $p|N$, then the* Shimura *lifting defined by*

$$L_D\left(\sum_{n=0}^{\infty} a(n)q^n\right) := \frac{a(0)}{2} L_N\left(0, \left(\frac{D}{\cdot}\right)\right) + \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{D}{d}\right) a\left(|D|\frac{n^2}{d^2}\right)\right) q^n$$

gives an one-to-one correspondence from $E^+_{3/2}(4N, \text{id.})$ *to* $\mathcal{E}(N, 2, \text{id.})$.

(2) *If* $k \geqslant 2$, *then the* Shimura *lifting defined by*

$$L_D\left(\sum_{n=0}^{\infty} a(n)q^n\right) := \frac{a(0)}{2} L_N\left(1-k, \chi_l'\left(\frac{D}{\cdot}\right)\right) + \sum_{n=1}^{\infty}\left(\sum_{d|n} d^{k-1}\left(\frac{D}{d}\right)\chi_l'(d) a\left(|D|\frac{n^2}{d^2}\right)\right) q^n$$

gives a one-to-one correspondence from $E^+_{k+1/2}(4N, \chi_l)$ *to* $\mathcal{E}(N,\ 2k,\ \text{id.})$.

Proof We denote by $U(m)(m|N^\infty)$ the following operator defined by

$$U(m)\left(\sum_{n=0}^{\infty} a(n)q^n\right) = \sum_{n=0}^{\infty} a(mn)q^n$$

for any $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in G(4N,\ k+1/2,\ \chi_l)$ or $G(N,\ 2k,\ \text{id.})$. Then $U(m)$ $(m|N^\infty)$ map $G(N,\ 2k, \text{id.})$ to $G(N,\ 2k, \text{id.})$ and $U(m^2)$ $(m|N^\infty)$ map $G(4N,\ k+1/2,$ $\chi_l)$ to $G(4N,\ k+1/2,\ \chi_l)$. A direct calculation shows that $L_D \circ U(m^2) = U(m) \circ L_D$ for any $m|N^\infty$ and any fundamental discriminant D with $\varepsilon(-1)^k D > 0$.

(1) Since L_D is a linear map on the space consisting of all formal power series $\sum_{n=0}^{\infty} a(n)q^n$ with $a(n) \in \mathbb{C}$, we only need to prove that L_D maps a basis of $E^+_{3/2}(4N, \text{id.})$ to a basis of $\mathcal{E}(N,\ 2,\ \text{id.})$. We first consider the case that $N = p$ is a prime. Then the dimension of $E^+_{3/2}(4p,\ \text{id.})$ equals to one and $H_1(\text{id.},\ p,\ p) \in E^+_{3/2}(4p,\ \text{id.})$. Denote that $H_1(\text{id.},\ p,\ p) := \sum_{n=0}^{\infty} a(n)q^n$ and $L_D(H_1(\text{id.},\ p,\ p)) := \sum_{n=0}^{\infty} b(n)q^n$. Then by the definition of L_D, we see that

$$\begin{aligned}
b(n) &= \sum_{d|n}\left(\frac{D}{d}\right) a\left(|D|\frac{n^2}{d^2}\right)\\
&= \sum_{d|n}\left(\frac{D}{d}\right) L_p\left(0, \left(\frac{D'_{|D|\frac{n^2}{d^2}}}{\cdot}\right)\right) \sum_{d_1|f_{|D|\frac{n^2}{d^2}}} \mu(d_1)\left(\frac{D_{|D|\frac{n^2}{d^2}}}{d_1}\right) \sum_{\substack{e|f_{|D|\frac{n^2}{d^2}}/d_1\\ (e,p)=1}} e\\
&= \sum_{d|n}\left(\frac{D}{d}\right) L_p\left(0, \left(\frac{D}{\cdot}\right)\right) \sum_{d_1|n/d} \mu(d_1)\left(\frac{D}{d_1}\right) \sum_{\substack{e|n/dd_1\\ (e,p)=1}} e\\
&= L_p\left(0, \left(\frac{D}{\cdot}\right)\right) \sum_{s|n}\left(\frac{D}{s}\right) \sum_{e|n/s,(e,p)=1} e \sum_{d|s}\mu(d) = L_p\left(0, \left(\frac{D}{\cdot}\right)\right) \sum_{e|n,(e,p)=1} e,
\end{aligned}$$

$$b(0) = \frac{1}{2}a(0)L_p\left(0, \left(\frac{D}{\cdot}\right)\right) = \frac{1}{2}L_p\left(0, \left(\frac{D}{\cdot}\right)\right)L_p(-1, \text{id.}) = \frac{p-1}{24}L_p\left(0, \left(\frac{D}{\cdot}\right)\right).$$

Hence we obtain that

$$L_D(H_1(\text{id.}, p, p)) = L_p\left(0, \left(\frac{D}{\cdot}\right)\right)E_2^{(p)}(z),$$

where

$$E_2^{(p)}(z) = \frac{p-1}{24} + \sum_{n=1}^{\infty}\left(\sum_{d|n, p|d} d\right)q^n \in \mathcal{E}(p, 2, \text{id.})$$

is the normalized Eisenstein series of weight 2 on $\Gamma_0(p)$. By the hypothesis in Theorem 8.3 we see that

$$L_p\left(0, \left(\frac{D}{\cdot}\right)\right) = \left(1 - \left(\frac{D}{p}\right)\right)L\left(0, \left(\frac{D}{\cdot}\right)\right) = \left(1 - \left(\frac{D}{p}\right)\right)\frac{h(D)}{w_D} \neq 0,$$

where w_D is the half of the number of units in $\mathbb{Q}(\sqrt{D})$.

This shows that L_D is a bijection if $N = p$ is a prime. We now prove that this holds for any square-free positive integer $N > 1$. Suppose that $N = p_1p_2\cdots p_t$. For any prime divisor p_i of N, denote the following Eisenstein series by $E_2^{(p_i)}(z)$:

$$E_2^{(p_i)}(z) := \sum_{n=1}^{\infty} a_i(n)q^n := \frac{p_i - 1}{24} + \sum_{n=1}^{\infty}\left(\sum_{d|n, p_i|d} d\right)q^n,$$

which is the normalized Eisenstein series of weight 2 on $\Gamma_0(p_i)$.

Let

$$S_i = \{U(l)(E_2^{(p_i)}(z)) \mid l|N/(p_1\cdots p_i)\}$$

for $1 \leqslant i \leqslant t$. By the properties of $U(m)$ we know that $S_i \subset \mathcal{E}(N, 2, \text{id.})$ for $1 \leqslant i \leqslant t$. We want to prove that $S := \bigcup_{i=1}^{t} S_i$ is a basis of $\mathcal{E}(N, 2, \text{id.})$. Since $\dim(\mathcal{E}(N, 2, \text{id.})) = 2^t - 1 =$ the number of elements in S, we only need to prove that the elements in S are linearly independent. We denote that $E_{2,l}^{(p_i)}(z) := U(l)(E_2^{(p_i)}(z))$. Suppose that there exist complex numbers $c_i(l)$ such that

$$\sum_{i=1}^{t}\sum_{l|N/(p_1\cdots p_i)} c_i(l)E_{2,l}^{(p_i)} = 0. \tag{8.40}$$

We must prove that $c_i(l) = 0$ for all $1 \leqslant i \leqslant t$ and $l|N/(p_1\cdots p_i)$. We prove this by induction on t. If $t = 1$, it is clear that $S = S_1 = \{E_2^{(p_1)}(z)\}$ is a basis of $\mathcal{E}(N, 2, \text{id.}) = \mathcal{E}(p_1, 2, \text{id.})$.

For any modular form $f(z) = \sum_{n=0}^{\infty} a(n)q^n$, let $L(s,f) := \sum_{n=1}^{\infty} a(n)n^{-s}$ be the corresponding Dirichlet series. Then by a direct calculation we see that

$$L(s, E_2^{(p_i)}) = \zeta(s)L(s-1, 1_{p_i}),$$

where 1_m denotes the trivial character modulo m for any positive integer m.

For $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in G(N, 2, \mathrm{id.})$, $r|N$ and ψ any character modulo N, we define

$$L(s, f, \psi, r) := \sum_{n=1}^{\infty} \psi(n)a(rn)n^{-s}.$$

Then we have that $\psi(n) = 0$ if $(n, N) \neq 1$, and so

$$\begin{aligned} L(s, E_{2,l}^{(p_i)}(z), \psi, r) &= \sum_{n=1}^{\infty} \psi(n) \left(\sum_{d|nlr, (p_i,d)=1} d \right) n^{-s} \\ &= \sum_{(n,N)=1} \psi(n) \left(\sum_{d|nlr, (p_i,d)=1} d \right) n^{-s} \\ &= \left(\sum_{a|lr, (p_i,a)=1} a \right) \sum_{(n,N)=1} \psi(n) \left(\sum_{d|n} d \right) n^{-s} \\ &= \prod_{p_i \neq p|lr} (1 + p + p^2 + \cdots + p^{\nu_p(lr)}) L(s,\psi)L(s-1,\psi). \end{aligned} \tag{8.41}$$

Hence from (8.40) and (8.41) we obtain that

$$\begin{aligned} 0 &= \sum_{i=1}^{t} \sum_{l|N/(p_1 \cdots p_i)} c_i(l) L(s, E_{2,l}^{(p_i)}, \psi, r) \\ &= \sum_{i=1}^{t} \sum_{l|N/(p_1 \cdots p_i)} c_i(l) \prod_{p_i \neq p|lr} (1 + p + p^2 + \cdots + p^{\nu_p(lr)}) L(s,\psi)L(s-1,\psi). \end{aligned}$$

This implies that

$$A_r := \sum_{i=1}^{t} \sum_{l|N/(p_1 \cdots p_i)} c_i(l) \prod_{p_i \neq p|lr} (1 + p + p^2 + \cdots + p^{\nu_p(lr)}) = 0, \quad \forall\, r|N. \tag{8.42}$$

That is, $c_i(l)$ must satisfy the above system of linear equations (8.42). Hence we only need to prove that the system of linear equations (8.42) has only the solution zero. It is clear that this holds for $t = 1$. Suppose that (8.42) has only the solution zero for $t-1$. Write that $N = p_1 N_1$ with $(p_1, N_1) = 1$. Let r_1 be a positive divisor of N_1. Then

$$\begin{aligned}
A_{p_1 r_1} - A_{r_1} =& \sum_{i=1}^{t} \sum_{l|N/(p_1\cdots p_i)} c_i(l) \prod_{p_i \neq p | l p_1 r_1} (1+p+p^2+\cdots+p^{\nu_p(lp_1r_1)}) \\
&-\sum_{i=1}^{t} \sum_{l|N/(p_1\cdots p_i)} c_i(l) \prod_{p_i \neq p | l r_1} (1+p+p^2+\cdots+p^{\nu_p(lr_1)}) \\
=& \sum_{i=1}^{t} \sum_{l|N/(p_1\cdots p_i)} c_i(l) \Bigg(\prod_{p_i \neq p | l p_1 r_1} (1+p+p^2+\cdots+p^{\nu_p(lp_1r_1)}) \\
&- \prod_{p_i \neq p | l r_1} (1+p+p^2+\cdots+p_p^{\nu}(lr_1)) \Bigg) \\
=& p_1 \sum_{i=2}^{t} \sum_{l|N/(p_1\cdots p_i)} c_i(l) \prod_{p_i \neq p | l r_1} (1+p+p^2+\cdots+p^{\nu_p(lr_1)}) \\
=& p_1 \sum_{i=2}^{t} \sum_{l|N_1/(p_2\cdots p_i)} c_i(l) \prod_{p_i \neq p | l r_1} (1+p+p^2+\cdots+p^{\nu_p(lr_1)}) = 0, \quad \forall\, r_1 | N_1.
\end{aligned} \tag{8.43}$$

By the induction assumption, we know that (8.43) has only the solution zero. Therefore $c_i(l) = 0$ for all $2 \leqslant i \leqslant t$ and $l | N/(p_1 \cdots p_i)$. Then (8.42) becomes

$$\sum_{l|N/p_1} c_1(l) \prod_{p_1 \neq p | lr} (1+p+p^2+\cdots+p^{\nu_p(lr)}) = \sum_{l|N/p_1} c_1(l) \sum_{d|lr, p_1|d} d = 0, \quad \forall\, r|N. \tag{8.44}$$

This shows that $c_1(l)$ must satisfy the system of linear equations (8.44). So we only need to prove that (8.44) has only the solution zero. For any positive integer $N > 1$ and any prime p with $(p, N) = 1$, we define the following system of linear equations for $x(l)$ with $l|N$

$$B_{p,N}(r) := \sum_{l|N} x(l) \sum_{d|lr, p|d} d = 0, \quad \forall\, r|N. \tag{8.45}$$

It is clear that (8.44) has only the solution zero if we can prove that (8.45) has only the solution zero. We prove that (8.45) has only the solution zero by induction on the number of prime factors of N. For $t = 0$, it is obvious. Suppose that (8.45) has only the solution zero for $t-1$. We want to prove that our assertion holds also for $N = p_1 p_2 \cdots p_t$. Write $N = p_i N_i$ with $(p_i, N_i) = 1$ for all $1 \leqslant i \leqslant t$. Let $r_i | N_i$ be any positive divisor of N_i. Then

$$0 = B_{p,N}(r_i) - B_{p,N_i}(r_i) = \sum_{l|N} x(l) \sum_{d|lr_i, p|d} d - \sum_{l|N_i} x(l) \sum_{d|lr_i, p|d} d$$

$$
\begin{aligned}
&= \left(\sum_{l|N_i} x(l) + \sum_{p_i|l|N} x(l) \right) \sum_{d|lr_i, p|d} d - \sum_{l|N_i} x(l) \sum_{d|lr_i, p|d} d \\
&= \sum_{l_i|N_i} x(p_i l_i) \sum_{d|p_i l_i r_i, p|d} d \\
&= (p_i + 1) \sum_{l_i|N_i} x(p_i l_i) \sum_{d|l_i r_i, p|d} d, \quad \forall\ r_i|N_i \text{ with } 1 \leqslant i \leqslant t,
\end{aligned}
$$

where we used the fact that $(l_i r_i, p_i) = 1$ to deduce the last equality. Hence

$$
\sum_{l_i|N_i} x(p_i l_i) \sum_{d|l_i r_i, p|d} d = 0, \quad \forall\ r_i|N_i \text{ with } 1 \leqslant i \leqslant t.
$$

By the induction hypothesis, we see that $x(p_i l_i) = 0$ for all $l_i|N_i, 1 \leqslant i \leqslant t$. Therefore $x(l) = 0$ for all $l|N$ with $l \neq 1$. Substituting these into (8.45) we obtain that $x(1) = 0$. This shows that (8.45) and hence (8.44) has only the solution zero. We have proved that S is a basis of $\mathcal{E}(N, 2, \text{id.})$. Now let

$$
\begin{aligned}
S_i' &= \{U(l^2)(H_1(\text{id.}, p_i, p_i)(z)) \mid l \mid N/(p_1 \cdots p_i)\}, \quad \text{ for } 1 \leqslant i \leqslant t, \\
S' &= \bigcup_{i=1}^{t} S_i'.
\end{aligned}
$$

We know that $H_1(\text{id.}, p_i, p_i) \in E_{3/2}^+(4p_i, \text{id.}) \subset E_{3/2}^+(4N, \text{id.})$ and hence $U(l^2)(H_1(\text{id.}, p_i, p_i)(z)) \in E_{3/2}^+(4N, \text{id.})$ for all $l \mid N/(p_1 \cdots p_i)$. This shows that $S' \subset E_{3/2}^+(4N, \text{id.})$. Using the properties of $U(l^2)$ and L_D and the result proved above, we see that

$$
\begin{aligned}
&L_D \circ U(l^2)(H_1(\text{id.}, p_i, p_i)(z)) \\
=&U(l) \circ L_D(H_1(\text{id.}, p_i, p_i)(z)) \\
=&U(l) \left(L_{p_i} \left(0, \left(\frac{D}{\cdot} \right) \right) E_2^{(p_i)}(z) \right) \\
=&L_{p_i} \left(0, \left(\frac{D}{\cdot} \right) \right) U(l)(E_2^{(p_i)}(z)) \in \mathcal{E}(N, 2, \text{id.}), \quad \forall 1 \leqslant i \leqslant t \text{ and } l|N/(p_1 \cdots p_i),
\end{aligned}
$$

where we used the fact that $E_2^{(p_i)}(z) \in E_2(p_i, \text{id.}) \subset \mathcal{E}(N, 2, \text{id.})$ and $U(l)(E_2^{(p_i)}(z)) \in \mathcal{E}(N, 2, \text{id.})$ for all $l \mid N$. Since $\left(\frac{D}{p_i} \right) \neq 1$ for all $p_i|N$, then $L_{p_i} \left(0, \left(\frac{D}{\cdot}\right)\right) \neq 0$. Hence we have proved that L_D maps S' to a basis of $\mathcal{E}(N, 2, \text{id.})$. Because L_D is a linear operator, S' is a basis of $E_{3/2}^+(4N, \text{id.})$. This implies that L_D is a bijection from $E_{3/2}^+(4N, \text{id.})$ to $\mathcal{E}(N, 2, \text{id.})$.

(2) Since L_D is a linear operator, we only need to calculate the image of $H_k(\chi, m, N)$ under the Shimura lifting L_D. Denote that $H_k(\chi, N, N) := \sum_{n=0}^{\infty} a_N(n) q^n$ and $L_D(H_k(\chi,$

$N,N)) := \sum_{n=0}^{\infty} b_N(n)q^n$. Then by the definition of L_D, we see that

$$\begin{aligned}
b_N(n) &= \sum_{d|n} \chi_l'(d)\left(\frac{D}{d}\right)d^{k-1}a_N\left(|D|\frac{n^2}{d^2}\right)\\
&= \sum_{d|n} \chi_l'(d)\left(\frac{D}{d}\right)d^{k-1}\left(1-k,\left(\frac{D'_{|D|\frac{n^2}{d^2}}}{\cdot}\right)\right)\\
&\quad \times \sum_{d_1|f_{|D|\frac{n^2}{d^2}}} \mu(d_1)\chi_l'(d_1)\left(\frac{D_{|D|\frac{n^2}{d^2}}}{d_1}\right)d_1^{k-1}\sum_{\substack{e|f_{|D|\frac{n^2}{d^2}}/d_1\\(e,N)=1}} \mathrm{e}^{2k-1}\\
&= \sum_{d|n} \chi_l'(d)\left(\frac{D}{d}\right)d^{k-1}L_N\left(1-k,\chi_l'\left(\frac{D}{\cdot}\right)\right)\\
&\quad \times \sum_{d_1|n/d} \mu(d_1)\chi_l'(d_1)\left(\frac{D}{d_1}\right)d_1^{k-1}\sum_{\substack{e|n/dd_1\\(e,N)=1}} \mathrm{e}^{2k-1}\\
&= L_N\left(1-k,\chi_l'\left(\frac{D}{\cdot}\right)\right)\sum_{s|n}\chi_l'(s)\left(\frac{D}{s}\right)s^{k-1}\sum_{e|n/s,(e,N)=1}\mathrm{e}^{2k-1}\sum_{d|s}\mu(d)\\
&= L_N\left(1-k,\chi_l'\left(\frac{D}{\cdot}\right)\right)\sum_{e|n,(e,N)=1}\mathrm{e}^{2k-1},\\
b(0) &= \frac{1}{2}a_N(0)L_N\left(1-k,\chi_l'\left(\frac{D}{\cdot}\right)\right)\\
&= \frac{1}{2}L_N\left(1-k,\chi_l'\left(\frac{D}{\cdot}\right)\right)L_N(1-2k,\mathrm{id.}).
\end{aligned}$$

Hence

$$L_D(H_k(\chi,N,N)) = L_N\left(1-k,\chi_l'\left(\frac{D}{\cdot}\right)\right)G_{2k,N}(z),$$

where

$$G_{2k,N}(z) := \frac{L_N(1-2k,\mathrm{id.})}{2} + \sum_{n=1}^{\infty}\left(\sum_{d|n,(d,N)=1} d^{2k-1}\right)q^n.$$

For $m|N$ with $m \neq N$ we can compute similarly and obtain that

$$\begin{aligned}
L_D(H_k(\chi,m,N)) =& L_m\left(1-k,\chi_l'\left(\frac{D}{\cdot}\right)\right)\left(\frac{(l,D)}{(l,D,m)}\right)^{2k-1}\\
&\times \prod_{p|N/m}\frac{1-\chi_l'(p)\left(\frac{D}{p}\right)p^{-k}}{1-p^{-2k}}G_{2k,m}(z),
\end{aligned}$$

where

$$G_{2k,m}(z) := \sum_{n=1}^{\infty} \left(\sum_{\substack{d|n,(d,m)=1,\\ (n/d,N/m)=1}} d^{2k-1} \right) q^n, \quad \forall m|N \text{ with } m \neq N.$$

Hence we only need to prove that $\{G_{2k,m}(z) \mid m \mid N\}$ constitute a basis of $\mathcal{E}(N, 2k,$ id.) which is stated as the following

Lemma 8.8 *Let N be a square-free positive integer and $k \geqslant 4$ an even integer. Then*

$$G_{k,N}(z) : = \frac{L_N(1-k, \text{id.})}{2} + \sum_{n=1}^{\infty} \left(\sum_{d|n,(d,N)=1} d^{k-1} \right) q^n,$$

$$G_{k,m}(z) : = \sum_{n=1}^{\infty} \left(\sum_{\substack{d|n,(d,m)=1,\\ (n/d,N/m)=1}} d^{k-1} \right) q^n, \forall m|N \text{ with } m \neq N$$

constitute a basis of $\mathcal{E}(N, k, \text{id.})$.

Proof Let $E_k(z)$ be the Eisenstein series defined by

$$E_k(z) := \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{k-1} \right) q^n.$$

Then it is well known that $\{E_k(lz) \mid l \mid N\}$ constitute a basis of $\mathcal{E}(N, k, \text{id.})$. We define functions $q_{k,m}(z)$ as follows

$$q_{k,N}(z) : = E_k(Nz),$$
$$q_{k,m}(z) : = \sum_{l|N/m} \mu(l) E_k(mlz), \quad \forall\ m|N, m \neq N.$$

Then it is clear that $\{q_{k,m} \mid m \mid N\}$ constitute a basis of $\mathcal{E}(N, k, \text{id.})$. And

$$\begin{aligned} q_{k,m}(z) &= \sum_{n=1}^{\infty} \sum_{l|N/m} \mu(l) \sum_{l|n} \sum_{d|n/l} d^{k-1} q^{mn} \\ &= \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1} \sum_{l|(n/d,N/m)} \mu(l) q^{mn} \\ &= \sum_{n=1}^{\infty} \sum_{\substack{d|n \\ (n/d,N/m)=1}} d^{k-1} q^{mn}. \end{aligned}$$

For $m|N$, denote by $G'_{k,m}(z)$ the following function

$$G'_{k,m}(z) := \sum_{s|m} \prod_{p|s} (1-p^{k-1}) q_{k,s}(z)$$
$$:= \sum_{n=1}^{\infty} a_m(n) q^n, \quad \forall\, m|N, m \neq N,$$

and

$$G'_{k,N}(z) := \sum_{s|N} \prod_{p|s} (1-p^{k-1}) q_{k,s}(z) := \sum_{n=1}^{\infty} a_N(n) q^n.$$

It is clear that these functions constitute a basis of $\mathcal{E}(N,k,\text{id.})$. □

For any fixed n, let it be that $(n,m) = m_1, m = m_1 m_2, n = n' \prod_{p|M_1} p^{\nu_p(n)}$ with $(n',m) = 1$.

$$a_m(n) = \sum_{s|m_1} \prod_{p|s} (1-p^{k-1}) \sum_{\substack{d|n/s \\ (n/sd, N/s)=1}} d^{k-1}$$
$$= \sum_{s|m_1} \prod_{p|s} (1-p^{k-1}) \prod_{p|s} \left(\sum_{t=0}^{\nu_p(n)-1} p^{(k-1)t} \right) \prod_{p|m_1/s} p^{(k-1)\nu_p(n)} \sum_{\substack{d|n,(d,m)=1, \\ (n/d,N/m)=1}} d^{k-1}$$
$$= \sum_{s|m_1} \prod_{p|s} (1-p^{(k-1)\nu_p(n)}) \prod_{p|m_1/s} p^{(k-1)\nu_p(n)} \sum_{\substack{d|n,(d,m)=1, \\ (n/d,N/m)=1}} d^{k-1}$$
$$= \sum_{\substack{d|n,(d,m)=1, \\ (n/d,N/m)=1}} d^{k-1}.$$

This shows that $G_{k,m}(z) = G'_{k,m}(z)$ for all $m|N$ with $m \neq N$. We can prove similarly that $G_{k,N}(z) = G'_{k,N}(z)$. Therefore $\{G_{k,m}(z) \mid m|N\}$ constitute a basis of $\mathcal{E}(N,k,\text{id.})$.

This completes the proof of Theorem 8.3. □

As a Corollary of the above proof, we have

Corollary 8.2 *Let N be a square-free positive odd integer. Define*

$$G_{2,N}(z) : = -\frac{1}{24} \prod_{p|N} (1-p) + \sum_{n=1}^{\infty} \left(\sum_{d|n,(d,N)=1} d \right) q^n,$$
$$G_{2,m}(z) : = \sum_{n=1}^{\infty} \left(\sum_{\substack{d|n,(d,m)=1, \\ (n/d,N/m)=1}} d \right) q^n, \quad \forall\, m|N \textit{ with } m \neq 1, N.$$

Then $\{G_{2,m} \mid m|N, m \neq 1\}$ *constitute a basis of* $\mathcal{E}(N, 2, \text{id.})$.

Proof Completely similar to the proof of Theorem 8.3 (2), we can calculate the images of $H_1(\text{id.}, m, N)$ under L_D for all $m|N$, $m \neq 1$. In particular, if we choose a negative fundamental discriminant D satisfying $D \equiv 0 \pmod N$, then

$$L_D(H_1(\text{id.}, m, N)) = h(D) \prod_{p|N/m} \frac{1}{1 - p^{-2}} G_{2,m}(z), \quad \forall\, m|N, m \neq 1.$$

We have shown in the proof of Theorem 8.3 (1) that L_D is a bijection from $E^+_{3/2}(4N, \text{id.})$ to $\mathcal{E}(N,2, \text{id.})$. Hence $G_{2,m}(z) = L_D(h(D)^{-1} \prod_{p|N/m} (1 - p^{-2}) H_1(\text{id.}, m, N)) \in \mathcal{E}(N, 2, \text{id.})$ and constitute a basis of $\mathcal{E}(N, 2, \text{id.})$. □

8.4 A Congruence Relation between Some Modular Forms

In this section we will give a congruence relation between some modular forms. A special case of our congruence (which was proved by Kohnen and J.A. Antoniadis, 1986) has important applications on the structure of the Selmer groups of some elliptic curves (Please compare J.A. Antoniadis, 1990).

Theorem 8.4 *Let* $N > 3$ *be a square-free positive odd integer with* $N \equiv 3 \pmod 4$, *and let* $l \geqslant 5$ *be a prime which divides the exact numerator of* $\frac{1}{12}\prod_{p|N}(p-1)$, *but does not divide the class number* $h(-N)$ *and* $\prod_{p|N/m}(p+1)$ *for any* $1 < m|N$. *We let*

$$G_{1,-N}(z) := \frac{1}{2}h(-N) + \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{-N}{d}\right)\right)q^n$$

be the Eisenstein series of weight 1 and Nebentypus $\left(\frac{-N}{\cdot}\right)$ *on* $\Gamma_0(N)$ *for the cusp* $\mathrm{i}\infty$. *Put*

$$\mathbb{C}_N := -\frac{1}{12}\prod_{p|N}(1-p)(G_{1,-N}(z))^2 - \frac{1}{2}h(-N)^2 \sum_{1<m|N}\left(\prod_{p|N/m}\frac{-p}{p+1}\right)G_{2,m}(z),$$

$$\mathbb{C}'_N := -\frac{1}{12}\prod_{p|N}(1-p)G_{1,-N}(4z)\theta(Nz) - \frac{1}{2}h(-N)\sum_{1<m|N}\left(\prod_{p|N/m}\frac{1-p}{p}\right)H_1(\text{id.}, m, N)(z),$$

where

$$G_{2,N}(z) := -\frac{\prod_{p|N}(1-p)}{24} + \sum_{n=1}^{\infty}\left(\sum_{d|n,(d,N)=1} d\right)q^n,$$

$$G_{2,m}(z) := \sum_{n=1}^{\infty}\left(\sum_{\substack{d|n,(d,m)=1,\\(n/d,N/m)=1}} d\right)q^n, \quad \forall m|N \text{ with } m \neq 1, N.$$

Then

(1) *The function* $\mathbb{C}'_N(z) \in S^+_{3/2}(4N, \mathrm{id.})$ *has* l*-integral* Fourier *coefficients, is non-zero modulo* l*, and the congruence*

$$\mathbb{C}'_N(z) \equiv -\frac{1}{2}h(-N)\sum_{1<m|N}\left(\prod_{p|N/m}\frac{1-p}{p}\right)H_1(\mathrm{id.}, m, N)(z) \qquad (\mathrm{mod}\ l)$$

holds.

(2) *The function* $\mathbb{C}_N(z) \in S(N,\ 2,\ \mathrm{id.})$ *has* l*-integral* Fourier *coefficients, is non-zero modulo* l*, and the congruence*

$$\mathbb{C}_N(z) \equiv -\frac{1}{2}h(-N)^2\sum_{1<m|N}\left(\prod_{p|N/m}\frac{-p}{p+1}\right)G_{2,m}(z) \qquad (\mathrm{mod}\ l)$$

holds. And one has $L_{-N}(\mathbb{C}'_N(z)) = \mathbb{C}_N$*.*

(3) *Suppose that* $\mathbb{C}'_N(z)$ *belongs to a subspace* V *of* $S^+_{3/2}(4N, \mathrm{id.})$ *which is isomorphic to a subspace of* $S(N,\ 2,\ \mathrm{id.})$ *as modules over the* Hecke *algebra. And suppose that* V *has a basis* $\{f_i(z)\}_{i=1}^r$ *with* $f_i(z)$ *are all Hecke eigenforms and* $f_i(z) := \sum_{n\geqslant 1} c_i(n)q^n$ *corresponding to* $F_i \in S(N,\ 2,\ \mathrm{id.})$*. Then one has*

$$\mathbb{C}'_N = -\frac{1}{12}\prod_{p|N}(1-p)\cdot\alpha'\cdot\sum_{i=1}^{r}\frac{L(F_i,1)c_i(N)}{||f_i||^2}f_i,$$

where α' *is a non-zero constant not depending on* N*,* $L(F_i, s)$ *is the* L*-function associated with* F_i *and* $||f_i||^2 := \int_{\Gamma_0(4N)\backslash H}|f_i|^2y^{-1/2}\mathrm{d}x\mathrm{d}y$ $(x = \mathrm{Re}(z),\ y = \mathrm{Im}(z))$ *the square of the* Petersson *norm of* f_i*.*

Proof (1) We first prove that $\mathbb{C}'_N(z)$ has l-integral Fourier coefficients. Since $\nu_l\left(\frac{1}{12}\prod_{p|N}(1-p)\right) > 0$ and $G_{1,-N}(4z)\theta(Nz)$ has rational Fourier coefficients, we only need to show that

$$\frac{1}{2}h(-N)\sum_{1<m|N}\left(\prod_{p|N/m}\frac{1-p}{p}\right)H_1(\mathrm{id.}, m, N)(z)$$

has l-integral Fourier coefficients. By the definition of $H_1(\text{id.}, m, N)$ we see that the nth Fourier coefficient of

$$\left(\prod_{p|N/m} \frac{1-p}{p}\right) H_1(\text{id.}, m, N)(z)$$

equals to

$$\begin{aligned}\prod_{p|N/m} \frac{1-p}{p} H(1,1,m,N;n) = &\prod_{p|N/m} \frac{1-p}{p} L_m(0,\chi_{D_n}) \prod_{p|N/m} \frac{1-p^{-1}\left(\frac{D_n}{p}\right)}{1-p^{-2}} \\ &\times \sum_{d|f_n} \mu(d)\chi_{D_n}(d)\sigma_{m,N,1}(f_n/d) \\ =&L_m(0,\chi_{D_n}) \prod_{p|N/m} \frac{\left(\frac{D_n}{p}\right)-p}{1+p} \\ &\times \sum_{d|f_n} \mu(d)\chi_{D_n}(d)\sigma_{m,N,1}(f_n/d),\end{aligned}$$

which is l-integral by hypothesis of Theorem 8.4, and hence $\mathbb{C}'_N(z)$ has l-integral Fourier coefficients.

Now we need only to prove that $\mathbb{C}'_N(z) \in S^+_{3/2}(4N, \text{id.})$, as the other assertions are obvious. We must show that $\mathbb{C}'_N(z) \in M^+_{3/2}(4N, \text{id.})$ and the values of $\mathbb{C}'_N(z)$ are zero at all cusp points. In order to do this we introduce the following Eisenstein series: For any positive integer k, and D_1, D_2 relatively prime fundamental discriminants with $(-1)^k D_1 D_2 > 0$ set

$$G_{k,D_1,D_2}(z) := \gamma_{k,D_1}^{-1} \times \frac{1}{2}\sum_{m,n}{}' \left(\frac{D_1}{n}\right)\left(\frac{D_2}{m}\right)(mD_1 z + n)^{-k},$$

where $\gamma_{k,D_1} := \left(\frac{D_1}{-1}\right)^{1/2} |D_1|^{-k+\frac{1}{2}} \frac{(-2\pi \mathrm{i})^k}{(k-1)!}$ and $\sum{}'$ means that (m,n) run over $\mathbb{Z}\times\mathbb{Z}$ except $(0,0)$. The function G_{k,D_1,D_2} is an Eisenstein series in $M_k\left(\Gamma_0(D), \left(\frac{D}{\cdot}\right)\right)$ $(D = D_1 D_2)$ for the cusp $\frac{1}{D_1}$. The Fourier expansion of $G_{k,D_1,D_2}(z)$ is given by

$$G_{k,D_1,D_2}(z) = \sum_{n=0}^{\infty} \sigma_{k-1,D_1,D_2}(n) q^n,$$

where

$$\sigma_{k-1,D_1,D_2}(n) := \begin{cases} -L(1-k,\chi_{D_1})L(0,\chi_{D_2}), & \text{if } n=0, \\ \displaystyle\sum_{\substack{d_1,d_2>0 \\ d_1 d_2 = n}} \left(\frac{D_1}{d_1}\right)\left(\frac{D_2}{d_2}\right) d_1^{k-1}, & \text{if } n>0. \end{cases}$$

We must note that for $k = 1$ or 2 there is a slight problem of convergence. But we can define $G_{k,D_1,D_2}(z)$ as the holomorphic continuation to $s = 0$ of the corresponding non-holomorphic Eisenstein series of weight 1 or 2. Anyway the above formula for the Fourier expansion of G_{k,D_1,D_2} holds for $k = 1, 2$.

Hence we know that for any $k \geqslant 2$

$$\sigma_{k-1,D_1,D_2}(0) := \begin{cases} \frac{1}{2}L(1-k,\chi_D), & \text{if } D_2 = 1, \\ 0, & \text{if } D_2 \neq 1 \end{cases} \tag{8.46}$$

and

$$\sigma_{0,D_1,D_2}(0) := \begin{cases} \frac{1}{2}h(D), & \text{if } D_2 = 1 \text{ or } D_1 = 1, \\ 0, & \text{if } D_2 \neq 1 \text{ and } D_1 \neq 1. \end{cases} \tag{8.47}$$

Denote $G_{k,D}(z)$, $G_{k,4D}(z)$ the following Eisenstein series

$$G_{k,D}(z) : = \frac{1}{2}L(1-k,\chi_D) + \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{D}{d}\right)d^{k-1}\right)q^n \in G\left(D, k, \left(\frac{D}{\cdot}\right)\right),$$

$$G_{k,4D}(z) : = G_{k,D}(4z) - 2^{-k}\left(\frac{D}{2}\right)G_{k,D}(2z) \in G\left(4D, k, \left(\frac{D}{\cdot}\right)\right).$$

Now one can show that

$$\begin{aligned} &(|D_1|z+1)^{-k}G_{k,D}\left(\frac{z}{|D_1|z+1}\right) \\ &= \left(\frac{D_2}{-1}\right)^{-1/2}\left(\frac{D_2}{|D_1|}\right)|D_2|^{-1/2}G_{k,D_1,D_2}\left(\frac{z+|D_1|^*}{|D_2|}\right), \end{aligned} \tag{8.48}$$

where $|D_1|^*$ is an integer with $|D_1||D_1|^* \equiv 1(\bmod D_2)$.

And

$$\begin{aligned} &(4|D_1|z+1)^{-k-\frac{1}{2}}G_{k,D}\left(\frac{4z}{4|D_1|z+1}\right)\theta\left(\frac{|D|z}{4|D_1|z+1}\right) \\ &= \left(\frac{D_2}{-|D_1|}\right)|D_2|^{-1}G_{k,D_1,D_2}\left(\frac{4z+|D_1|^*}{|D_2|}\right)\theta\left(\frac{|D_1|z+4^*}{|D_2|}\right), \end{aligned} \tag{8.49}$$

where $a^* \in \mathbb{Z}$ with $aa^* \equiv 1 \pmod{D_2}$ (Please compare with W. Kohnen, 1981, 192-197).

From (8.47) we see immediately that

$$\begin{aligned} &V\left(G_{k,D}(4z)\theta(|D|z), \frac{1}{4|D_1|}\right) \\ &= \lim_{z\to i\infty}(4|D_1|z+1)^{-k-\frac{1}{2}}G_{k,D}\left(\frac{4z}{4|D_1|z+1}\right)\theta\left(\frac{|D|z}{4|D_1|z+1}\right) \end{aligned}$$

$$= \lim_{z\to i\infty} \left(\frac{D_2}{-|D_1|}\right) |D_2|^{-1} G_{k,D_1,D_2}\left(\frac{4z+|D_1|^*}{|D_2|}\right)\theta\left(\frac{|D_1|z+4^*}{|D_2|}\right)$$
$$= \left(\frac{D_2}{-|D_1|}\right) |D_2|^{-1} V(G_{k,D_1,D_2}(z), i\infty)V(\theta(z), i\infty)$$
$$= \left(\frac{D_2}{-|D_1|}\right) |D_2|^{-1} \sigma_{k-1,D_1,D_2}(0).$$

Especially, from (8.47), we see that

$$V\left(G_{1,D}(4z)\theta(|D|z), \frac{1}{4|D_1|}\right) = \begin{cases} \frac{1}{2}\left(\frac{D_2}{-|D_1|}\right)|D_2|^{-1}h(D), & \text{if } D_2 = 1 \text{ or } D_1 = 1, \\ 0, & \text{if } D_2 \neq 1 \text{ and } D_1 \neq 1. \end{cases} \tag{8.50}$$

Since $4/|D_1|$ and $1/|D_1|$ are $\Gamma_0(4|D|)$-equivalent, we can also calculate the value of $G_{1,D}\ (4z)\theta(|D|z)$ at the cusp point $1/|D_1|$ by Claim 1 of Theorem 10.9 and (8.48) as follows:

$$\begin{aligned} V\left(G_{1,D}(4z)\theta(|D|z), \frac{1}{|D_1|}\right) &= V\left(G_{1,D}(4z), \frac{1}{|D_1|}\right)V\left(\theta(|D|z), \frac{1}{|D_1|}\right) \\ &= \lim_{z\to i\infty}(-|D_1|z)G_{1,D}\left(4(z+\frac{1}{|D_1|})\right) \\ &\quad \times \lim_{z\to i\infty}(-|D_1|z)^{1/2}\theta\left(|D|(z+\frac{1}{|D_1|})\right) \\ &= \frac{1}{4|D_2|^{1/2}}V\left(G_{1,D}(z), \frac{4}{|D_1|}\right)V\left(\theta(z), \frac{|D|}{|D_1|}\right) \\ &= \frac{1}{4|D_2|^{1/2}}\left(\frac{D}{d}\right)V\left(G_{1,D}(z), \frac{1}{|D_1|}\right)V(\theta(z), |D_2|) \\ &= \frac{1-i}{8|D_2|^{1/2}}\left(\frac{D}{d}\right)\left(\frac{D_2}{-1}\right)^{-1/2}\left(\frac{D_2}{|D_1|}\right)|D_2|^{-1/2}\sigma_{0,D_1,D_2}(0) \\ &= -L(0,\chi_{D_1})L(0,\chi_{D_2})\frac{1-i}{8|D_2|}\left(\frac{D}{d}\right)\left(\frac{D_2}{-1}\right)^{-1/2}\left(\frac{D_2}{|D_1|}\right), \end{aligned}$$

where d is an integer such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(|D|) \text{ and } \begin{pmatrix} a & b \\ c & d \end{pmatrix}(4/|D_1|) = 1/|D_1|.$$

Therefore we get that

$$V\left(G_{1,D}(4z)\theta(|D|z), \frac{1}{|D_1|}\right) = \begin{cases} \frac{-1-i}{16|D|}h(D), & \text{if } D_1 = 1, \\ \frac{1-i}{16}h(D), & \text{if } D_1 = D, \\ 0, & \text{if } D_1 \neq 1 \text{ and } D_1 \neq D. \end{cases} \tag{8.51}$$

Finally since $V(\theta(z),1/2)=0$, we see easily that

$$V\left((G_{1,D}(4z)\theta(|D|z),\frac{1}{2|D_1|})\right)=0. \tag{8.52}$$

By Theorem 7.7 we can calculate the values of $H_1(\mathrm{id.},m,N)$ at cusp points as follows: For any positive divisor d of N,

$$\begin{aligned}V(H_1(\mathrm{id.},m,N),1/d)&=V(L_m(-1,\mathrm{id.})H_1^{'}(\mathrm{id.},m,N),1/d)\\&=-\frac{1}{12}\prod_{p|m}(1-p)(V(g_3(\mathrm{id.},4m,4N),1/d)\\&\quad-\frac{3}{2}V(g_3(\mathrm{id.},m,4N),1/d))\\&=-\frac{1}{12}\prod_{p|m}(1-p)\frac{1+\mathrm{i}}{8}\mu(m/d)dm^{-1}\varepsilon_d^{-1}\end{aligned} \tag{8.53}$$

and

$$V(H_1(\mathrm{id.},m,N),1/2d)=0 \tag{8.54}$$

and

$$\begin{aligned}V(H_1(\mathrm{id.},m,N),1/4d)&=V(L_m(-1,\mathrm{id.})H_1'(\mathrm{id.},m,N),1/4d)\\&=-\frac{1}{12}\prod_{p|m}(1-p)(V(g_1(\mathrm{id.},4m,4N),1/4d)\\&\quad-\frac{3}{2}V(g_1(\mathrm{id.},m,4N),1/4d))\\&=-\frac{1}{12}\prod_{p|m}(1-p)\mu(m/d)dm^{-1}.\end{aligned} \tag{8.55}$$

Using the above results we can compute the values of $\mathbb{C}_N'(z)$ at all cusp points. For example, we have for $D_1\neq 1$ and $D_1\neq -N$ by (8.50) and (8.55),

$$\begin{aligned}V(\mathbb{C}_N'(z),1/4|D_1|)&=V\Bigg(-\frac{1}{12}\prod_{p|N}(1-p)G_{1,-N}(4z)\theta(Nz)\\&\quad-\frac{1}{2}h(-N)\sum_{1\neq m|N}\left(\prod_{p|N/m}\frac{1-p}{p}\right)H_1(\mathrm{id.},m,N)(z),1/4|D_1|)\\&=-\frac{1}{2}h(-N)\sum_{1\neq m|N}\left(\prod_{p|N/m}\frac{1-p}{p}\right)V(H_1(\mathrm{id.},m,N)(z),1/4|D_1|)\\&=-\frac{1}{2}h(-N)\sum_{1\neq m|N}\left(\prod_{p|N/m}\frac{1-p}{p}\right)\end{aligned}$$

$$\times \left(-\frac{1}{12} \prod_{p|m} (1-p) \right) \mu(m/|D_1|)|D_1|m^{-1}$$
$$= \frac{h(-N)}{24N} \prod_{p|N} (1-p)|D_1| \sum_{1<m|N} \mu\left(\frac{m}{|D_1|}\right) = 0.$$

For $D_1 = 1$, we have by (8.46) and (8.55)

$$\begin{aligned} V(\mathbb{C}'_N(z), 1/4) &= -\frac{1}{12} \prod_{p|N} (1-p) V(G_{1,-N}(4z)\theta(Nz), 1/4) \\ &\quad + \frac{h(-N)}{24N} \prod_{p|N} (1-p) \sum_{1<m|N} \mu(m) \\ &= \frac{h(-N)}{24N} \prod_{p|N} (1-p) - \frac{h(-N)}{24N} \prod_{p|N} (1-p) = 0. \end{aligned}$$

For $D_1 = -N$, we have by (8.46) and (8.55)

$$\begin{aligned} V(\mathbb{C}'_N(z), 1/4N) &= -\frac{1}{12} \prod_{p|N} (1-p) V(G_{1,-N}(4z)\theta(Nz), 1/4N) \\ &\quad + \frac{h(-N)}{24N} \prod_{p|N} (1-p) N \sum_{1<m|N} \mu(m/N) \\ &= -\frac{h(-N)}{24} \prod_{p|N} (1-p) + \frac{h(-N)}{24} \prod_{p|N} (1-p) = 0. \end{aligned}$$

This shows that for all positive divisors d of N, we have

$$V(\mathbb{C}'_N(z), 1/4d) = 0.$$

It is clear that $V(\mathbb{C}'_N(z), 1/2d) = 0$ for all positive divisors d of N from (8.52) and (8.54). We now compute the values of $\mathbb{C}'_N(z)$ at the cusp point $1/|D_1|$ by (8.51) and (8.53): for $D_1 \neq 1$ and $D_2 \neq 1$,

$$\begin{aligned} V(\mathbb{C}'_N(z), 1/|D_1|) &= 0 - \frac{1}{2} h(-N) \sum_{1<m|N} \prod_{p|N/m} \left(\frac{1-p}{p} \right) \\ &\quad \times L_m(-1, \mathrm{id.}) \frac{1+\mathrm{i}}{8} \mu\left(\frac{m}{|D_1|} \right) m^{-1} |D_1| \varepsilon_{|D_1|}^{-1} \\ &= \frac{(1+\mathrm{i})h(-N)}{192N} |D_1| \varepsilon_{|D_1|}^{-1} \prod_{p|N} (1-p) \sum_{1<m|N} \mu\left(\frac{m}{|D_1|} \right) = 0 \end{aligned}$$

and

$$\begin{aligned} V(\mathbb{C}'_N(z),1) &= -\frac{1}{12}\prod_{p|N}(1-p)\left(-\frac{(1+\mathrm{i})h(-N)}{16N}\right) \\ &\quad -\frac{1}{2}h(-N)\sum_{1<m|N}\prod_{p|N/m}\left(\frac{1-p}{p}\right)L_m(-1,\mathrm{id.})\frac{1+i}{8}\mu(m)m^{-1}\varepsilon_1^{-1} \\ &= \frac{(1+\mathrm{i})h(-N)}{192N}\prod_{p|N}(1-p)+\frac{(1+\mathrm{i})h(-N)}{192N}\prod_{p|N}(1-p)\sum_{1<m|N}\mu(m) \\ &= \frac{(1+\mathrm{i})h(-N)}{192N}\prod_{p|N}(1-p)-\frac{(1+\mathrm{i})h(-N)}{192N}\prod_{p|N}(1-p)=0. \end{aligned}$$

Since $N \equiv 3 \pmod 4$, we have $\varepsilon_N = \mathrm{i}$ and hence

$$\begin{aligned} V(\mathbb{C}'_N(z),1/N) &= -\frac{1}{12}\prod_{p|N}(1-p)\left(-\frac{(1-\mathrm{i})h(-N)}{16}\right) \\ &\quad -\frac{1}{2}h(-N)\sum_{1<m|N}\prod_{p|N/m}\left(\frac{1-p}{p}\right)L_m(-1,\mathrm{id.})\frac{1+\mathrm{i}}{8}\mu\left(\frac{m}{N}\right)m^{-1}N\varepsilon_N^{-1} \\ &= -\frac{(1-\mathrm{i})h(-N)}{192}\prod_{p|N}(1-p)+\frac{(1+\mathrm{i})h(-N)}{192}\prod_{p|N}(1-p)\mathrm{i}^{-1}\sum_{1<m|N}\mu\left(\frac{m}{N}\right) \\ &= -\frac{(1-\mathrm{i})h(-N)}{192}\prod_{p|N}(1-p)+\frac{(1-\mathrm{i})h(-N)}{192}\prod_{p|N}(1-p)=0. \end{aligned}$$

This shows that $V(\mathbb{C}'_N(z),1/d)=0$ for any positive divisor d of N. Hence $\mathbb{C}'_N(z) \in S(4N,3/2,\mathrm{id.})$ is a cusp form. On the other hand, we can prove that $G_{1,-N}(4z)\theta(Nz) = r \times pr(G_{1,-4N}(z)\theta(z))$ with r a constant by the method as exposed in W. Kohnen, 1981 where pr denotes the projection from the space $G(4N,\ 3/2,\ \mathrm{id.})$ to the space $M^+_{3/2}(4N,\mathrm{id.})$ (W. Kohnen, 1982). This shows that $\mathbb{C}'_N(z) \in M^+_{3/2}(4N,\mathrm{id.})$ and hence $\mathbb{C}'_N(z) \in S^+_{3/2}(4N,\mathrm{id.})$. This completes the proof of (1).

(2) It is clear that $\mathbb{C}_N(z)$ has l-integral Fourier coefficients by the hypothesis in Theorem 8.4. We only need to show that $L_{\text{-}N}(\mathbb{C}'_N(z)) = \mathbb{C}_N$. The proof is similar to the arguments used in W. Kohnen, 1981. For the sake of completeness we give it as follows. Write $c(n)$ resp. $b(n)$ for the nth Fourier coefficient of $G_{1,-N}(4z)\theta(Nz)$ resp. $G_{1,-N}(z)$. Then

$$c(n) = \sum_{\substack{r\in\mathbb{Z},Nr^2\leqslant n\\ n\equiv Nr^2 (\bmod 4)}} b\left(\frac{n-Nr^2}{4}\right).$$

Denote that $L_{-N}(G_{1,-N}(4z)\theta(Nz)) := \sum_{n=0}^{\infty} a(n)q^n$. Then for $n>0$ we have that

$$\begin{aligned} a(n) &= \sum_{d|n} \left(\frac{-N}{d}\right) c\left(N\frac{n^2}{d^2}\right) \\ &= \sum_{d|n} \left(\frac{-N}{d}\right) \sum_{\substack{r\in\mathbb{Z}, |r|\leqslant\sqrt{n/d} \\ r\equiv n/d (\bmod 2)}} b\left(N\frac{n^2-r^2d^2}{4d^2}\right). \end{aligned}$$

Observing that $b(Nm) = b(m)$ for any $m \geqslant 0$ and writing $n_1 = \dfrac{n-rd}{2}, n_2 = \dfrac{n+rd}{2}$, we see that the coefficient

$$a(n) = \sum_{\substack{n_1,n_2\geqslant 0 \\ n_1+n_2=n}} \sum_{d|(n_1,n_2)} \left(\frac{-N}{d}\right) b\left(\frac{n_1 n_2}{d^2}\right).$$

By the multiplicative properties of $b(n)$, the inner sum equals $b(n_1)b(n_2)$, hence

$$a(n) = \sum_{\substack{n_1,n_2\geqslant 0 \\ n_1+n_2=n}} b(n_1)b(n_2),$$

which is the nth Fourier coefficient of $G_{1,-N}(z)^2$. But

$$a(0) = \frac{c(0)}{2} L_N(0, \chi_{-N}) = \frac{1}{4} h(-N)^2,$$

which is the constant term of $G_{1,-N}(z)^2$. This shows that $L_{-N}(G_{1,-N}(4z)\theta(Nz)) = G_{1,-N}(z)^2$. But we know that from Corollary 8.2

$$L_{-N}(H_1(\text{id.}, m, N)) = h(-N) \prod_{p|N/m} \frac{1}{1-p^{-2}} G_{2,m}(z), \quad \forall\, m|N, m \neq 1,$$

which implies that $L_{-N}(\mathbb{C}'_N(z)) = \mathbb{C}_N(z)$ as desired.

(3) It can be proved by Rankin's trick, just as used in W. Kohnen, 1981 and J.A. Antoniadis, 1986. We omit the proof because of the complete similarity with the one in W. Kohnen, 1981 and J.A. Antoniadis, 1986. □

Proposition 8.7 *Let $p > 3$ be a prime with $p \equiv 3 \pmod 4$, and let $l \geqslant 5$ be a prime which divides the exact numerator of $\dfrac{p-1}{12}$, but does not divide the class number $h(-p)$. Then*

(1) *The function $\mathbb{C}'_p(z) \in S^+_{3/2}(4N, \text{id.})$ has l-integral* Fourier *coefficients, is non-zero modulo l, and the congruence*

$$\mathbb{C}'_p(z) \equiv -\frac{1}{2} h(-p) H_{1,p}(z) \pmod l$$

holds.

(2) *The function* $\mathbb{C}_p(z) \in S(N,\ 2,\ \text{id.})$ *has* l*-integral* Fourier *coefficients, is non-zero modulo* l*, and the congruence*

$$\mathbb{C}_p(z) \equiv -\frac{1}{2}h(-p)^2 G_{2,p}(z) \pmod{l}$$

holds. And one has

$$L_{-p}(\mathbb{C}'_p(z)) = \mathbb{C}_p.$$

(3) $\mathbb{C}'_N(z)$ *belongs to a subspace* V *of* $S^+_{3/2}(4N, \text{id.})$ *which is isomorphic to a subspace of* $S(N, 2, \text{id.})$ *as modules over the* Hecke *algebra. Suppose that* V *has a basis* $\{f_i(z)\}_{i=1}^r$ *with all* $f_i(z)$ *are* Hecke *eigenforms and* $f_i(z) := \sum_{n\geqslant 1} c_i(n)q^n$ *corresponding to* $F_i \in S(N,\ 2,\ \text{id.})$*. Then one has*

$$\mathbb{C}'_p = -\frac{1-p}{12}\cdot\alpha'\cdot\sum_{i=1}^{r}\frac{L(F_i,1)c_i(p)}{||f_i||^2}f_i,$$

where α' *is a non-zero constant not depending on* p*,* $L(F_i, s)$ *is the* L*-function associated with* F_i *and* $||f_i||^2 := \int_{\Gamma_0(4p)\backslash H} |f_i|^2 y^{-1/2}\mathrm{d}x\mathrm{d}y$ $(x = \mathrm{Re}(z), y = \mathrm{Im}(z))$ *the square of the* Petersson *norm of* f_i.

Proof This is a special case of Theorem 8.4. □

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Chapter 9

Trace Formula

9.1 Eichler-Selberg Trace Formula on $SL_2(\mathbb{Z})$

Throughout this section we write $\Gamma = SL_2(\mathbb{Z})$. And let F be a fundamental domain of Γ. Let $k \geqslant 2$ be a fixed positive integer. We write $\mathrm{T}(n)$ for the Hecke operator on the space of cusp forms $S_k := S(1, 2k, \mathrm{id.})$.

In this section we want to compute the trace of the Hecke operator $\mathrm{T}(n)$ as a Hermitian operator on the space S_k. The method given in this section is owed to D.Zagier.

Let $h(z, z')$ be a function of two variables z, z' in $\mathbb{H}$, and assume that h is a cusp form of weight $2k$ as a function of each variable. We define $f * h$ for any $f \in S_k$ as a function of z' by

$$(f * h)(z') = \int_F f(z)\overline{h(z, -\overline{z'})}y^{k-2}\mathrm{d}x\mathrm{d}y. \tag{9.1}$$

Let m be a positive integer and $z, z' \in \mathbb{H}$. Put

$$\begin{aligned} h_m(z, z') &= \sum_{\substack{a,b,c,d \in \mathbb{Z}, \\ ad-bc=m}} (czz' + dz' + az + b)^{-2k} \\ &= \sum_{\substack{a,b,c,d \in \mathbb{Z}, \\ ad-bc=m}} (cz + d)^{-2k} \big(z' + \frac{az+b}{cz+d}\big)^{-2k} \\ &= \sum_{\substack{a,b,c,d \in \mathbb{Z}, \\ ad-bc=m}} (cz' + a)^{-2k} \big(z + \frac{dz'+b}{cz'+a}\big)^{-2k}. \end{aligned} \tag{9.2}$$

It is clear that the above series converges absolutely and uniformly on any bounded closed set of $\mathbb{H} \times \mathbb{H}$. Therefore $h_m(z, z')$ is an analytic function in z, z'. It is also obvious from (9.2) that $h_m(z, z')$ is a cusp form in every variable separately.

Lemma 9.1 *Set* $c_k = \dfrac{(-1)^k \pi}{2^{2k-3}(2k-1)}$. *Then*

$$c_k m^{-2k+1}(f|\mathrm{T}(m))(z') = (f * h_m)(z')$$

holds for any $f \in S_k$.

Proof Assume first that $m = 1$. Since for any $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $f \in S_k$ we have

$$(c\overline{z} + d)^{-2k} f(z) y^{2k} = f(Mz)(\mathrm{Im}(Mz))^{2k}.$$

Hence from (9.2) we see that

$$f(z)\overline{h_1(z, z')}y^{2k} = \sum_{M \in \Gamma} (\overline{z'} + M\overline{z})^{-2k} f(Mz)(\mathrm{Im}(Mz))^{2k},$$

and

$$(f * h_1)(z') = \int_F \sum_{M \in \Gamma} (-z' + M\overline{z})^{-2k} f(Mz)(\mathrm{Im}(Mz))^{2k} y^{-2} \mathrm{d}x\mathrm{d}y. \tag{9.3}$$

Since the series on the right hand side in (9.3) is absolutely and uniformly convergent, we can interchange the integration and the sum

$$\begin{aligned}(f * h_1)(z') &= \sum_{M \in \Gamma} \int_F (-z' + M\overline{z})^{-2k} f(Mz)(\mathrm{Im}(Mz))^{2k} y^{-2} \mathrm{d}x\mathrm{d}y \\ &= \sum_{M \in \Gamma} \int_{M(F)} (-z' + \overline{z})^{-2k} f(z)\big(\mathrm{Im}(z)\big)^{2k} y^{-2} \mathrm{d}x\mathrm{d}y \qquad (9.4) \\ &= 2\int_0^\infty \int_{-\infty}^\infty (x - \mathrm{i}y - z')^{-2k} f(x + \mathrm{i}y) y^{2k-2} \mathrm{d}x\mathrm{d}y,\end{aligned}$$

where the last equality comes from the fact that the upper half plane is equal to the union of transformations of the fundamental domain F under Γ, disjoint except for boundary points of measure zero, and the factor 2 comes from the fact that $\pm\gamma \in \Gamma$ give the same transformation. Since $f(z)$ is holomorphic on $\mathbb{H}$ and zero at $i\infty$, we obtain from Cauchy's formula that

$$\int_{-\infty}^\infty (x - \mathrm{i}y - z')^{-2k} f(x + \mathrm{i}y) \mathrm{d}x = \frac{2\pi\mathrm{i}}{(2k-1)!} f^{(2k-1)}(2\mathrm{i}y + z'). \tag{9.5}$$

From (9.4) and (9.5) we get

$$\begin{aligned}(f * h_1)(z') &= \frac{4\pi\mathrm{i}}{(2k-1)!} \int_0^\infty y^{2k-2} f^{(2k-1)}(2y\mathrm{i} + z')\mathrm{d}y \\ &= \frac{4\pi\mathrm{i}}{(2k-1)!} \left(\frac{-1}{2\mathrm{i}}\right)^{2k-2} (2k-2)! \frac{-1}{2\mathrm{i}} f(z') \\ &= c_k f(z'),\end{aligned}$$

where we used repeatedly integration by parts and the fact that $f \in S_k$. This implies that the lemma holds for $m = 1$ since $\mathrm{T}(1) = \mathrm{id}$. Let $\mathrm{T}(m)$ operate on h_1 with respect to the first variable z. Then

$$\begin{aligned} h_1(z,z')|\mathrm{T}(m) &= m^{2k-1} \sum_{a_1d_1=m, d_1>0, b_1 \bmod d_1} d_1^{-2k} h_1\left(\frac{a_1z+b_1}{d_1}, z'\right) \\ &= m^{2k-1} \sum_{a,b,c,d\in\mathbb{Z}, ad-bc=m} \sum_{a_1d_1=m,d_1>0,b_1 \bmod d_1} d_1^{-2k}\left(c\frac{a_1z+b_1}{d_1}+d\right)^{-2k} \\ &\quad \times\left(z'+M\left(\frac{a_1z+b_1}{d_1}\right)\right)^{-2k} \\ &= m^{2k-1} \sum_{\substack{a,b,c,d\in\mathbb{Z}, \\ ad-bc=m}} (cz+d)^{-2k}\left(z'+\frac{az+b}{cz+d}\right)^{-2k}, \end{aligned} \tag{9.6}$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and the last equality comes from the fact that the following set

$$\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix}, \quad a_1d_1 = m, \quad d_1 > 0, \quad b_1 \bmod d_1$$

is a complete set of right cosets of Δ_m with respect to Γ where

$$\Delta_m := \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a,b,c,d \in \mathbb{Z}, ad-bc = m \right\}.$$

Hence from (9.6) and the definition of h_m, we see that

$$(h_1|\mathrm{T}(m))(z,z') = m^{2k-1}h_m(z,z'), \tag{9.7}$$

where $\mathrm{T}(m)$ operates on the first variable z. Hence from (9.7), the fact that the lemma holds for $m = 1$ and the properties of the Petersson inner product we see that

$$\begin{aligned} (f * h_m)(z') &= m^{-2k+1}(f * h_1|\mathrm{T}(m))(z') \\ &= m^{-2k+1}((f|\mathrm{T}(m)) * h_1) \\ &= m^{-2k+1}c_k f|\mathrm{T}(m). \end{aligned}$$

This completes the proof. □

Let now $f_1, \cdots, f_r$ be an orthogonal basis of eigenfunctions for the Hecke operators, and assume that they are normalized, i.e., for $1 \leqslant j \leqslant r$,

$$f_j = \sum_{n=1}^{\infty} a_n^{(j)} e(nz), \quad a_1^{(j)} = 1.$$

Then we have

$$f_j|\mathrm{T}(m) = a_m^{(j)} f_j. \tag{9.8}$$

Lemma 9.2 (1) *We have that*

$$c_k^{-1}m^{2k-1}h_m(z,z') = \sum_{j=1}^{r} \frac{a_m^{(j)}}{\langle f_j, f_j\rangle} f_j(z)f_j(z'), \tag{9.9}$$

where $\langle *, *\rangle$ *is the the* Petersson *inner product.*

(2) *We have that*

$$\mathrm{tr}(\mathrm{T}(m)) = c_k^{-1}m^{2k-1}\int_F h_m(z,-\overline{z})y^{2k-2}\mathrm{d}x\mathrm{d}y. \tag{9.10}$$

Proof (1) Since $\mathrm{T}(m)$ is Hermitian and f_j is the eigenfunction of $\mathrm{T}(m)$ with eigenvalue $a_m^{(j)}$, $a_m^{(j)}$ is a real number for all m, j, and hence we have

$$\overline{f_j(-\overline{z})} = f_j(z), \quad 1 \leqslant j \leqslant r. \tag{9.11}$$

Since $f_1, \cdots, f_r$ consist of a basis, and $h_m(z,z')$ as a function of z or z' is in S_k,

$$h_m(z,z') = \sum_{i,j=1}^{r} a_{ij} f_i(z) f_j(z'), \tag{9.12}$$

where a_{ij} are some constants. By Lemma 9.1, (9.8), (9.12) and (9.11) we see that

$$\begin{aligned} c_k m^{-2k+1} a_m^{(l)} f_l(z') &= c_k m^{-2k+1}(f_l|\mathrm{T}(m))(z') = (f_l * h_m)(z') \\ &= \sum_{i,j=1}^{r} \overline{a_{i,j}} \int_F f_l(z)\overline{f_i(z)f_j(-\overline{z'})}y^{2k-2}\mathrm{d}x\mathrm{d}y \\ &= \sum_{i,j=1}^{r} \overline{a_{ij}}\langle f_l, f_i\rangle f_j(z') = \sum_{j=1}^{r} \overline{a_{lj}}\langle f_l, f_l\rangle f_j(z'), \end{aligned}$$

where the last equality comes from the fact that $f_1, \cdots, f_r$ are orthogonal to each other. Hence we have that $a_{lj} = 0$ if $l \neq j$ and

$$a_{ll} = \frac{c_k m^{-2k+1}}{\langle f_l, f_l\rangle} a_m^{(l)}.$$

This shows (1) of the lemma.

(2) By the definition of the trace of a linear operator, we see that

$$\mathrm{tr}(\mathrm{T}(m)) = \sum_{j=1}^{r} a_m^{(j)}.$$

Hence by (1) we know that

$$\begin{aligned} & c_k^{-1}m^{2k-1}\int_F h_m(z,-\overline{z})y^{2k-2}\mathrm{d}x\mathrm{d}y \\ =& \sum_{j=1}^{r} \frac{a_m^{(j)}}{\langle f_j, f_j\rangle} \int_F f_j(z) f_j(-\overline{z}) y^{2k-2}\mathrm{d}x\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{r} \frac{a_m^{(j)}}{\langle f_j, f_j \rangle} \int_F f_j(z)\overline{f_j(z)} y^{2k-2} \mathrm{d}x\mathrm{d}y \\
&= \sum_{j=1}^{r} a_m^{(j)} = \mathrm{tr}(\mathrm{T}(m)),
\end{aligned}
$$

where the second equality comes from (9.11). This completes the proof. □

We shall give an explicit expression for the trace in terms of the (2) of Lemma 9.2. We need some notations and definitions. We define a function $H(n)$ for integers n as follows:

$$H(n) = 0, \quad \forall\, n < 0 \text{ and } H(0) = -\frac{1}{12}.$$

If $n > 0$, let $H(n)$ be the number of equivalence classes with respect to Γ of positive definite binary quadratic forms $ax^2 + bxy + cy^2$ with discriminant $b^2 - 4ac = -n$, counting forms equivalent to a multiple of $x^2 + y^2$ or $x^2 + xy + y^2$ with multiplicity $\frac{1}{2}$ or $\frac{1}{3}$ respectively. By the definition we see that $H(n) = 0$ if $n \equiv 1, 2 \pmod 4$. We also define a polynomial $P_j(t, m)$ as the coefficient of x^{j-2} in the formal power series expansion of $(1 - tx + mx^2)^{-1}$, i.e.,

$$(1 - tx + mx^2)^{-1} = \sum_{j=2}^{\infty} P_j(t, m) x^{j-2}.$$

It is easy to verify that

$$P_j(t, m) = \frac{\rho^{j-1} - \overline{\rho}^{j-1}}{\rho - \overline{\rho}}, \tag{9.13}$$

where $\rho, \overline{\rho}$ are the roots of the equation $x^2 - tx + m = 0$.

Theorem 9.1 *Let $m \geqslant 1$ and $k \geqslant 2$ be positive integers, then the trace of the* Hecke *operator* $\mathrm{T}(m)$ *on the space* S_k *is given by*

$$\mathrm{tr}(\mathrm{T}(m)) = -\frac{1}{2} \sum_{-\infty}^{\infty} P_{2k}(t, m) H(4m - t^2) - \frac{1}{2} \sum_{dd'=m} \min\{d, d'\}^{2k-1}, \tag{9.14}$$

where d and d' are positive integers such that $dd' = m$.

Proof By Lemma 9.2 we have

$$\mathrm{tr}(\mathrm{T}(m)) = c_k^{-1} m^{2k-1} \int_F \sum_{ad-bc=m} \frac{y^{2k}}{(c|z|^2 + d\overline{z} - az - b)^{2k}} \frac{\mathrm{d}x\mathrm{d}y}{y^2}. \tag{9.15}$$

To show the theorem we only need to compute the integral. The sum on the right hand side is invariant under the action of Γ since the integral is independent of the choice of the fundamental domain F.

Suppose that $M = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma$. Write $M^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} M = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$. Then a direct computation shows that

$$c_2|z|^2 + d_2\overline{z} - a_2 z - b_2 = (c|Mz|^2 + d\overline{Mz} - aMz - b)|c_1 z + d_1|^2.$$

Therefore

$$\frac{\mathrm{Im}(Mz)^{2k}}{(c|Mz|^2 + d\overline{Mz} - aMz - b)^{2k}} = \frac{y^{2k}}{(c_2|z|^2 + d_2\overline{z} - a_2 z - b_2)^{2k}},$$

where we used the fact that $\mathrm{Im}(Mz) = |c_1 z + d_1|^{-2}\mathrm{Im}(z)$. This shows that replacing z by Mz amounts to replacing the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by $M^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} M$ in terms of the sum (9.15). These two matrices have the same determinant and the same trace. Therefore we may decompose the sum into pieces which are invariant under Γ, characterized by the condition $a + d =$constant, so that

$$\mathrm{tr}(\mathrm{T}(m)) = \sum_{t=-\infty}^{\infty} I(m,t),$$

where

$$I(m,t) = c_k^{-1} m^{2k-1} \int_F \sum_{\substack{ad-bc=m,\\ a+d=t}} \frac{y^{2k}}{(c|z|^2 + d\overline{z} - az - b)^{2k}} \frac{\mathrm{d}x\mathrm{d}y}{y^2}. \tag{9.16}$$

We shall prove

$$\frac{1}{2}(I(m,t) + I(m,-t)) = \begin{cases} -\dfrac{1}{2}P_{2k}(t,m)H(4m-t^2), & \text{if } t^2 - 4m < 0, \\ \dfrac{2k-1}{24}m^{k-1} - \dfrac{1}{4}m^{k-1/2}, & \text{if } t^2 - 4m = 0, \\ -\dfrac{1}{2}\left(\dfrac{|t|-u}{2}\right)^{2k-1}, & \text{if } t^2 - 4m = u^2 \\ & \text{with } u \text{ a positive integer}, \\ 0, & \text{if } t^2 - 4m > 0 \text{ is non-square}, \end{cases}$$

which implies the theorem. To calculate the integral (9.16), we first note that there is a bijection between the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant m and trace t, and the set of binary quadratic forms g with discriminant $|g| = t^2 - 4m$. In fact, the bijection is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto g(u,v) = cu^2 + (d-a)uv - bv^2,$$

$$g(u,v)=\alpha u^2+\beta uv+\gamma v^2 \mapsto \begin{pmatrix} \dfrac{t-\beta}{2} & -\gamma \\ \alpha & \dfrac{t+\beta}{2} \end{pmatrix}.$$

For any form $g(u,v)=\alpha u^2+\beta uv+\gamma v^2$ and real t, $z=x+\mathrm{i}y\in\mathbb{H}$, put

$$R_g(z,t)=\frac{y^{2k}}{(\alpha(x^2+y^2)+\beta x+\gamma-\mathrm{i}ty)^{2k}}, \tag{9.17}$$

then we see that

$$I(m,t)=c_k^{-1}m^{2k-1}\int_F\sum_{|g|=t^2-4m}R_g(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2}, \tag{9.18}$$

where the sum is taken over all forms with discriminant t^2-4m. Any $M\in\Gamma$ transforms a quadratic form g into a form Mg. A direct computation shows that

$$R_g(Mz,t)=R_{Mg}(z,t). \tag{9.19}$$

Therefore for each discriminant D, i.e., for each integer $D\equiv 0$ or $1 \pmod 4$, we have

$$\begin{aligned}\sum_{|g|=D}R_g(z,t)&=\sum_{|g|=D \bmod \Gamma}\ \sum_{M\in\Gamma/\Gamma_g}R_{Mg}(z,t)\\ &=\sum_{|g|=D \bmod \Gamma}\ \sum_{M\in\Gamma/\Gamma_g}R_g(Mz,t),\end{aligned}$$

where the first sum is taken over a set of representatives for classes of quadratic forms with discriminant D, and the second sum is taken over right cosets of Γ with respect to the isotropy group Γ_g of elements leaving g fixed. For $D\neq 0$, the class number $h(D)$ is finite, and hence the first sum is finite and

$$\int_F\sum_{|g|=D}R_g(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2}=\sum_{|g|=D \bmod \Gamma}\int_{F_g}R_g(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2}, \tag{9.20}$$

where $F_g=\bigcup_{M\in\Gamma/\Gamma_g}M(F)$ is a fundamental domain of Γ_g on $\mathbb{H}$. For $D=0$ we can take $g_r(u,v)=rv^2$ $(r\in\mathbb{Z})$ as a complete set of representatives for the forms with discriminant 0. The isotropy group Γ_{g_r} of g_r is equal to Γ for $r=0$, and is equal to

$$\Gamma_\infty:=\left\{\pm\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\,\middle|\, n\in\mathbb{Z}\right\}$$

for $r\neq 0$. Hence we have

$$\int_F\sum_{|g|=0}R_g(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2}=\int_F R_{g_0}(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2}+\int_{F_\infty}\sum_{r\in\mathbb{Z}^*}R_{g_r}(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2}, \tag{9.21}$$

where $F_\infty = \{z \in \mathbb{H} | 0 \leqslant \mathrm{Re}(z) < 1\}$ is a fundamental domain of Γ_∞ on $\mathbb{H}$. The integrals at the right hand side of (9.20) and (9.21) for $D = t^2 - 4m$ remain to be computed. We distinguish four cases.

Case (1). $D < 0$. In this case Γ_g is finite (and one can prove that its order is 1, 2 or 3). Let

$$g(u, v) = \alpha u^2 + \beta uv + \gamma v^2$$

be a quadratic form with discriminant $|g| = D$, then

$$\begin{aligned}\int_{F_g} R_g(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2} &= \frac{1}{|\Gamma_g|}\int_{\mathbb{H}} R_g(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2} \\ &= \frac{1}{|\Gamma_g|}\int_{\mathbb{H}} \frac{y^{2k}}{(\alpha|z|^2+\beta x+\gamma - ity)^{2k}}\frac{\mathrm{d}x\mathrm{d}y}{y^2}.\end{aligned}$$

For $\alpha > 0$, replacing z by $\dfrac{z}{\alpha} - \dfrac{\beta}{2\alpha}$ we have

$$\int_{F_g} R_g(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2} = \frac{1}{|\Gamma_g|}\int_{\mathbb{H}} \frac{y^{2k}}{(|z|^2 - \mathrm{i}ty - D/4)^{2k}}\frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

For $\alpha < 0$, using $(\alpha, \beta, \gamma) \mapsto (-\alpha, -\beta, -\gamma)$ we can obtain that

$$\int_{F_g} R_g(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2} = \frac{1}{|\Gamma_g|}\int_{\mathbb{H}} \frac{y^{2k}}{(|z|^2 + \mathrm{i}ty - D/4)^{2k}}\frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

Set

$$I(t) = \int_{\mathbb{H}} \frac{y^{2k}}{(|z|^2 - \mathrm{i}ty - D/4)^{2k}}\frac{\mathrm{d}x\mathrm{d}y}{y^2}, \tag{9.22}$$

which is only dependent on D, t and k. Therefore

$$\begin{aligned}\int_F \sum_{|g|=D} R_g(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2} &= (I(t)+I(-t))\sum_{|g|=D \bmod \Gamma}\frac{1}{|\Gamma_g|} \\ &= H(-D)(I(t)+I(-t)).\end{aligned} \tag{9.23}$$

A direct computation shows that

$$\int_{-\infty}^{\infty}(x^2+s)^{-2}\mathrm{d}x = \frac{\pi s^{-3/2}}{2}$$

holds for any $\mathrm{Re}(s) \neq 0$. Taking derivatives with respect to s we obtain that

$$\int_{-\infty}^{\infty}(x^2+s)^{-l}\mathrm{d}x = \frac{\pi}{(l-1)!}\frac{1}{2}\cdot\frac{3}{2}\cdots\left(l-\frac{3}{2}\right)s^{-l+1/2}$$

for any $l \geqslant 2$ and $\mathrm{Re}(s) \neq 0$. Therefore we see that

$$\begin{aligned} I(t) &= \int_0^\infty y^{2k-2} \int_{-\infty}^\infty (x^2+y^2-\mathrm{i}ty-D/4)^{-2k} \mathrm{d}x\mathrm{d}y \\ &= \frac{\pi}{(2k-1)!} \frac{1}{2}\cdot\frac{3}{2}\cdots(2k-3/2) \int_0^\infty (y^2-\mathrm{i}yt-D/4)^{-2k+1/2} y^{2k-2}\mathrm{d}y \\ &= \frac{(-1)^{k-1}\pi}{2(2k-1)!} \frac{\mathrm{d}^{2k-2}}{\mathrm{d}t^{2k-2}} \int_0^\infty (y^2-\mathrm{i}ty-D/4)^{-3/2}\mathrm{d}y \\ &= \frac{(-1)^{k-1}\pi}{2(2k-1)!} \frac{\mathrm{d}^{2k-2}}{\mathrm{d}t^{2k-2}} \left(\frac{4}{t^2-D} \frac{y-\mathrm{i}t/2}{\sqrt{y^2-\mathrm{i}yt-D/4}} \Big|_{y=0}^{y=\infty} \right) \\ &= \frac{(-1)^{k-1}\pi}{2(2k-1)!} \frac{\mathrm{d}^{2k-2}}{\mathrm{d}t^{2k-2}} \left(\frac{4}{\sqrt{|D|}} \frac{1}{\sqrt{|D|}-it} \right) g \\ &= \frac{2\pi}{2k-1} \frac{1}{\sqrt{|D|}} \frac{1}{(\sqrt{|D|}-\mathrm{i}t)^{2k-1}}. \end{aligned}$$

Hence from the above equality and (9.23) we obtain for $t^2-4m<0$ that

$$\begin{aligned} I(m,t) &= c_k^{-1} m^{2k-1} H(4m-t^2) \cdot \frac{2\pi}{2k-1} \cdot \frac{1}{\sqrt{4m-t^2}} \\ &\quad \times \left(\frac{1}{(\sqrt{4m-t^2}-\mathrm{i}t)^{2k-1}} + \frac{1}{(\sqrt{4m-t^2}+\mathrm{i}t)^{2k-1}} \right) \\ &= \frac{H(4m-t^2)}{2} \frac{\overline{\rho}^{2k-1}-\rho^{2k-1}}{\rho-\overline{\rho}}, \end{aligned}$$

where $\rho = (t+\mathrm{i}\sqrt{4m-t^2})/2$. Therefore for $t^2-4m<0$ we have

$$\begin{aligned} \frac{1}{2}(I(m,t)+I(m,-t)) &= \frac{H(4m-t^2)}{2} \frac{\overline{\rho}^{2k-1}-\rho^{2k-1}}{\rho-\overline{\rho}} \\ &= -\frac{1}{2} H(4m-t^2) P_{2k}(t,m) \end{aligned}$$

from the definition of $P_j(t,m)$.

Case (2). $D=0$. In this case we have $t=\pm 2\sqrt{m}$ with m a square. The first term of the right hand side of (9.21) is equal to

$$\int_F R_{g_0}(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2} = \frac{(-1)^k}{t^{2k}} \int_F \frac{\mathrm{d}x\mathrm{d}y}{y^2} = \frac{(-1)^k \pi}{3t^{2k}}. \tag{9.24}$$

And the second term of the right hand side of (9.21) is equal to

$$\begin{aligned} \int_{D_\infty} \sum_{0\neq r\in\mathbb{Z}} R_{g_r}(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2} &= \int_0^\infty \int_0^1 \sum_{0\neq r\in\mathbb{Z}} (r-\mathrm{i}ty)^{-2k} y^{2k-2}\mathrm{d}x\mathrm{d}y \\ &= \int_0^\infty y^{2k-2} \sum_{0\neq r\in\mathbb{Z}} (r-\mathrm{i}ty)^{-2k}\mathrm{d}y \end{aligned}$$

$$\begin{aligned}
&= \frac{(-1)^{k-1}}{(2k-1)!} \frac{\mathrm{d}^{2k-2}}{\mathrm{d}t^{2k-2}} \int_0^\infty \sum_{0\neq r\in\mathbb{Z}} (r - \mathrm{i}ty)^{-2} \mathrm{d}y \\
&= \frac{(-1)^{k-1}}{(2k-1)!} \frac{\mathrm{d}^{2k-2}}{\mathrm{d}t^{2k-2}} \int_0^\infty \left(\frac{1}{t^2y^2} - \frac{\pi^2}{\sinh^2 \pi ty} \right) \mathrm{d}y \\
&= \frac{(-1)^{k-1}}{(2k-1)!} \frac{\mathrm{d}^{2k-2}}{\mathrm{d}t^{2k-2}} \left(-\frac{1}{t^2y} + \frac{\pi \mathrm{cth}(\pi ty)}{t} \right) \Bigg|_{y=0}^{y=\infty} \\
&= \frac{(-1)^{k-1}}{(2k-1)!} \frac{\mathrm{d}^{2k-2}}{\mathrm{d}t^{2k-2}} \left(\frac{\pi}{|t|} \right) = \frac{(-1)^{k-1}}{2k-1} \pi |t|^{-2k+1},
\end{aligned} \tag{9.25}$$

where we used the fact that

$$\mathrm{cth}(x) = \begin{cases} \dfrac{1}{x} + \dfrac{x}{3} + O(x^3), & \text{if } x \to 0, \\ 1, & \text{if } x \to \infty. \end{cases}$$

Therefore we obtain for $t^2 = 4m$ that

$$\begin{aligned}
I(m,t) &= c_k^{-1} m^{2k-1} \left(\frac{(-1)^k \pi}{3(4m)^k} + \frac{(-1)^{k-1}\pi}{2k-1} \frac{1}{(2\sqrt{m})^{2k-1}} \right) \\
&= \frac{2k-1}{24} m^{k-1} - \frac{1}{4} m^{k-1/2}.
\end{aligned}$$

Case (3). $D = l^2$, $l > 0$ a positive integer. Then every form with discriminant D is similar to one of the following standard forms:

$$g_\alpha(u,v) = \alpha u^2 + luv, \quad 1 \leqslant \alpha \leqslant l.$$

It is clear that $|\Gamma_{g_\alpha}| = 1$, and hence

$$\int_F \sum_{|g|=D} R_g(z,t) \frac{\mathrm{d}x\mathrm{d}y}{y^2} = \sum_{\alpha=1}^{l} \int_F \sum_{M\in\Gamma} \frac{y_M^{2k}}{(\alpha|z_M|^2 + lx_M - \mathrm{i}ty_M)^{2k}} \frac{\mathrm{d}x_M \mathrm{d}y_M}{y_M^2}, \tag{9.26}$$

where $z_M = Mz = x_M + \mathrm{i}y_M$. We write the integral on the right hand side of (9.26) as

$$I_\alpha = \lim_{\varepsilon\to 0} \int_{F_\varepsilon} \sum_{M\in\Gamma} \frac{y_M^{2k}}{(\alpha|z_M|^2 + lx_M - \mathrm{i}ty_M)^{2k}} \frac{\mathrm{d}x_M \mathrm{d}y_M}{y_M^2}, \tag{9.27}$$

where

$$F_\varepsilon := \{ z \in F | \mathrm{Im}(z) \leqslant 1/\varepsilon \}$$

is a compact set for any $\varepsilon > 0$. So we can interchange the integral and the summation, but we must be careful by taking limit because there are probably some problems at the points which are the roots of $\alpha|z_M|^2 + lx_M - \mathrm{i}ty_M = 0$. So we have

$$I_\alpha = \int_{\mathbb{H}} \frac{y^{2k}}{(\alpha|z|^2 + lx - \mathrm{i}ty)^{2k}} \frac{\mathrm{d}x\mathrm{d}y}{y^2} - \lim_{\varepsilon\to 0} I_\varepsilon - \lim_{\varepsilon\to 0} J_\varepsilon, \tag{9.28}$$

where

$$I_\varepsilon = \int\limits_{|z-\varepsilon \mathrm{i}/\alpha|\leqslant\varepsilon/\alpha} \frac{y^{2k-2}}{(\alpha|z|^2+lx-\mathrm{i}ty)^{2k}}\mathrm{d}x\mathrm{d}y,$$

$$J_\varepsilon = \int\limits_{|z+l/\alpha-\varepsilon \mathrm{i}/\alpha|\leqslant\varepsilon/\alpha} \frac{y^{2k-2}}{(\alpha|z|^2+lx-\mathrm{i}ty)^{2k}}\mathrm{d}x\mathrm{d}y$$

for a sufficiently small $\varepsilon > 0$. Replacing z by $\dfrac{z}{\alpha} - \dfrac{l}{2\alpha}$, then we see from (9.28) that

$$I_\alpha = \int_0^\infty \int_{-\infty}^\infty \frac{y^{2k-2}}{(x^2+y^2-\mathrm{i}ty-l^2/4)^{2k}}\mathrm{d}x\mathrm{d}y - \lim_{\varepsilon\to 0} I_\varepsilon^+ - \lim_{\varepsilon\to 0} I_\varepsilon^-,$$

where

$$I_\varepsilon^+ = \int\limits_{|z/\alpha-l/(2\alpha)-\varepsilon \mathrm{i}/\alpha|\leqslant\varepsilon/\alpha} \frac{y^{2k-2}}{(x^2+y^2-\mathrm{i}ty-l^2/4)^{2k}}\mathrm{d}x\mathrm{d}y, \tag{9.29}$$

$$I_\varepsilon^- = \int\limits_{|z+l/(2\alpha)-\varepsilon \mathrm{i}/\alpha|\leqslant\varepsilon/\alpha} \frac{y^{2k-2}}{(x^2+y^2-\mathrm{i}ty-l^2/4)^{2k}}\mathrm{d}x\mathrm{d}y. \tag{9.30}$$

Similar to the case $D < 0$, we have that

$$\int_0^\infty \int_{-\infty}^\infty \frac{y^{2k-2}}{(x^2+y^2-\mathrm{i}ty-l^2/4)^{2k}}\mathrm{d}x\mathrm{d}y = \frac{(-1)^k 2\pi/l}{2k-1}(l+|t|)^{1-2k}.$$

Substituting x and y by $\pm l/2+\varepsilon a$ and $\varepsilon+\varepsilon b$ resp. in (9.29) and (9.30), we see that

$$I_\varepsilon^\pm = \int\limits_{a^2+b^2\leqslant 1} \frac{(1+b)^{2k-2}\mathrm{d}a\mathrm{d}b}{(\pm al - \mathrm{i}t(1+b)+\varepsilon(a^2+1+b^2))^{2k}}.$$

Hence

$$\lim_{\varepsilon\to 0} I_\varepsilon^\pm = \int\limits_{a^2+b^2\leqslant 1} \frac{(1+b)^{2k-2}\mathrm{d}a\mathrm{d}b}{(al \mp it(1+b))^{2k}} = I_{\pm t}$$

and

$$\begin{aligned} I_t &= \int\limits_{a^2+b^2\leqslant 1} \frac{(1+b)^{2k-2}\mathrm{d}a\mathrm{d}b}{(al-\mathrm{i}t(1+b))^{2k}} \\ &= \int_{-1}^1 (1+b)^{2k-2} \int_{-\sqrt{1-b^2}}^{\sqrt{1-b^2}} \frac{\mathrm{d}a}{(la-\mathrm{i}t(1+b))^{2k}}\mathrm{d}b \\ &= \int_{-1}^1 \frac{-(1+b)^{2k-2}}{l(2k-1)}((l\sqrt{1-b^2}-\mathrm{i}t(1+b))^{1-2k} \\ &\quad + (l\sqrt{1-b^2}+it(1+b))^{1-2k})\mathrm{d}b, \end{aligned}$$

replacing b by $\dfrac{1-v^2}{1+v^2}$, then

$$\begin{aligned}
I_t &= -\frac{2}{(2k-1)l}\int_0^\infty ((lv+\mathrm{i}t)^{1-2k}+(lv-\mathrm{i}t)^{1-2k})\frac{v\mathrm{d}v}{1+v^2} \\
&= -\frac{2}{(2k-1)l}\int_{-\infty}^\infty (lv+\mathrm{i}t)^{1-2k}\frac{v\mathrm{d}v}{1+v^2} \\
&= -\frac{2}{(2k-1)l}\int_{-\infty}^\infty \frac{(-1)^{k-1}}{(2k-2)!}\frac{\mathrm{d}^{2k-2}}{\mathrm{d}t^{2k-2}}(lv+\mathrm{i}t)^{-1}\frac{v\mathrm{d}v}{1+v^2} \\
&= \frac{2(-1)^k}{l(2k-1)!}\frac{\mathrm{d}^{2k-2}}{\mathrm{d}t^{2k-2}}\int_{-\infty}^\infty \frac{v\mathrm{d}v}{(lv+it)(1+v^2)}.
\end{aligned}$$

But

$$\begin{aligned}
\int_{-\infty}^\infty \frac{v\mathrm{d}v}{(lv+it)(1+v^2)} &= \int_{-\infty}^\infty \frac{(lv-\mathrm{i}t)v\mathrm{d}v}{(l^2v^2+t^2)(1+v^2)} \\
&= \int_{-\infty}^\infty \frac{lv^2\mathrm{d}v}{(l^2v^2+t^2)(1+v^2)} - \mathrm{i}t\int_{-\infty}^\infty \frac{v\mathrm{d}v}{(l^2v^2+t^2)(1+v^2)},
\end{aligned}$$

the second integral in the above line is zero since the function is odd, so that

$$\begin{aligned}
\int_{-\infty}^\infty \frac{v\mathrm{d}v}{(lv+\mathrm{i}t)(1+v^2)} &= \frac{l}{t^2-l^2}\int_{-\infty}^\infty\left(\frac{t^2}{l^2v^2+t^2}-\frac{1}{1+v^2}\right)\mathrm{d}v \\
&= \frac{t^2}{t^2-l^2}\frac{1}{t}\arctan\left(\frac{v}{t}\right)\Big|_{-\infty}^\infty - \frac{l}{t^2-l^2}\arctan(v)\Big|_{-\infty}^\infty \\
&= \frac{\pi}{|t|+l}
\end{aligned}$$

and hence

$$\begin{aligned}
I_t &= \frac{(-1)^k 2\pi}{l(2k-1)}(|t|+l)^{1-2k}, \\
I_\alpha &= \frac{(-1)^{k-1}2\pi}{l(2k-1)}(|t|+l)^{1-2k}.
\end{aligned}$$

Therefore we obtain that

$$\int_F \sum_{|g|=l^2} R_g(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2} = \sum_{\alpha=1}^{l} I_\alpha = \frac{(-1)^{k-1}2\pi}{2k-1}(|t|+l)^{1-2k},$$

so that

$$I(m,t) = c_k^{-1}m^{2k-1}\frac{(-1)^{k-1}2\pi}{2k-1}(|t|+l)^{1-2k} - \frac{1}{2}\left(\frac{|t|-l}{2}\right)^{2k-1}.$$

Therefore for $t^2-4m=l^2$ with l a positive integer we have

$$\frac{1}{2}(I(m,t)+I(m,-t)) = -\frac{1}{2}\left(\frac{|t|-l}{2}\right)^{2k-1}.$$

Case (4). $D > 0$ and D is non-square. Then the isotropy group of the form

$$g(u,v) = \alpha u^2 + \beta uv + \gamma v^2$$

is an infinitely cyclic group, where the discriminant of g is equal to D. Let $w > w'$ be two real roots of the equation $\alpha x^2 + \beta x + \gamma = 0$, then $w + w' = -\dfrac{\alpha}{\beta}$, $ww' = \dfrac{\gamma}{\alpha}$. Put

$$J = \frac{1}{\sqrt{w-w'}} \begin{pmatrix} w & w' \\ 1 & 1 \end{pmatrix} \in SL_2(\mathbb{R})$$

and

$$J^{\mathrm{T}} \begin{pmatrix} \alpha & \beta/2 \\ \beta/2 & \gamma \end{pmatrix} = \frac{\sqrt{D}}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where J^{T} is the transpose of J. If $\pm I \neq T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ such that

$$T^{\mathrm{T}} \begin{pmatrix} \alpha & \beta/2 \\ \beta/2 & \gamma \end{pmatrix} T = \begin{pmatrix} \alpha & \beta/2 \\ \beta/2 & \gamma \end{pmatrix},$$

i.e., $\pm T \in \Gamma_g$, then $\pm I \neq S := J^{-1}TJ \in SL_2(\mathbb{R})$ and

$$S^{\mathrm{T}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

From this we know that $S = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$ with a real $\varepsilon \neq \pm 1$. Then

$$T = J \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} J^{-1}. \tag{9.31}$$

This shows that w, w' are fixed points of T, i.e., w, w' are roots of the equation $cx^2 + (d-a)x - b = 0$, so that

$$c = m\alpha, \quad d - a = m\beta, \quad -b = m\gamma. \tag{9.32}$$

Set $t = a + d$, then

$$\begin{aligned} &a = \frac{t - m\beta}{2}, \quad b = -m\gamma, \quad c = m\alpha, \quad d = \frac{t + m\beta}{2}, \\ &t^2 - Dm^2 = (a+d)^2 - m^2(\beta^2 - 4\alpha\gamma) = 4. \end{aligned} \tag{9.33}$$

From (9.31) we know that $\varepsilon + \varepsilon^{-1} = a + d = t$, so that

$$\varepsilon^{\pm 1} = \frac{t \pm \sqrt{t^2 - 4}}{2} = \frac{t \pm m\sqrt{D}}{2}. \tag{9.34}$$

Set $m = s/q$ with $q \geqslant 1, s$ integers and $(s,q) = 1$. Then from (9.32) we see that $q | (\alpha, \beta, \gamma)$. It is also clear that $(\alpha, \beta, \gamma)^2 | D$. Put

$$D = \varDelta(\alpha,\beta,\gamma)^2, \tag{9.35}$$

with $\varDelta \in \mathbb{N}$ a non-square. So we can write

$$m = \frac{p}{(\alpha,\beta,\gamma)} \tag{9.36}$$

with $p \in \mathbb{Z}$. From (9.34), (9.35) and (9.36) we get

$$\varepsilon^{\pm 1} = \frac{t \pm p\sqrt{\varDelta}}{2}$$

so that from (9.33), (9.35) and (9.36)

$$t^2 - \varDelta p^2 = 4.$$

We know that Pell's equation has the solutions

$$\varepsilon = \pm\varepsilon_0^n, \quad 0 \neq n \in \mathbb{Z}, \tag{9.37}$$

where $\varepsilon_0 = \dfrac{t_0 + p_0\sqrt{\varDelta}}{2} > 1$ is the fundamental unit. By (9.31) and (9.37) we have

$$T = \pm J \begin{pmatrix} \varepsilon_0^n & 0 \\ 0 & \varepsilon_0^{-n} \end{pmatrix} J^{-1} = \pm\left(J \begin{pmatrix} \varepsilon_0 & 0 \\ 0 & \varepsilon_0^{-1} \end{pmatrix} J^{-1} \right)^n,$$

which implies that the isotropy group $\varGamma_g$ is the infinitely cyclic group generated by

$$J \begin{pmatrix} \varepsilon_0 & 0 \\ 0 & \varepsilon_0^{-1} \end{pmatrix} J^{-1}.$$

Therefore

$$\int\limits_{F_g} R_g(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2} = \int\limits_{F_g} R_{Jg}(J^{-1}z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2} = \int\limits_{J^{-1}F_g} R_{Jg}(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2}. \tag{9.38}$$

Since $\varGamma_g$ is generated by $J \begin{pmatrix} \varepsilon_0 & 0 \\ 0 & \varepsilon_0^{-1} \end{pmatrix} J^{-1}$, we may assume that F_g can be chosen such that $J^{-1}F_g = \{z = x + \mathrm{i}y | y > 0, 1 < |z| \leqslant \varepsilon_0^2\}$. Hence from (9.38) and the fact that

$$J^{\mathrm{T}} \begin{pmatrix} \alpha & \beta/2 \\ \beta/2 & \gamma \end{pmatrix} = \frac{\sqrt{D}}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

we see that

$$\int\limits_{F_g} R_g(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2} = \int\limits_{y>0,1<|z|<\varepsilon_0^2} (\sqrt{D}d - \mathrm{i}ty)^{-2k} y^{2k-2}\mathrm{d}x\mathrm{d}y,$$

replacing x and y by $\rho\cos\theta$ and $\rho\sin\theta$ resp., we see that

$$\int_{F_g} R_g(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2} = \int_0^{\pi}\int_1^{\varepsilon_0^2}(\sqrt{D}\cos\theta - \mathrm{i}t\sin\theta)^{-2k}(\sin\theta)^{2k-2}\rho^{-1}\mathrm{d}\rho\mathrm{d}\theta$$
$$= (2\log(\varepsilon_0))\int_0^{\pi}(\sqrt{D}\cos\theta - \mathrm{i}t\sin\theta)^{-2k}(\sin\theta)^{2k-2}\mathrm{d}\theta,$$

so that

$$I := \int_{F_g} R_g(z,t)\frac{\mathrm{d}x\mathrm{d}y}{y^2} + \int_{F_g} R_g(z,-t)\frac{\mathrm{d}x\mathrm{d}y}{y^2}$$
$$= (2\log(\varepsilon_0))\int_{-\pi}^{\pi}(\sqrt{D}\cos\theta - \mathrm{i}t\sin\theta)^{-2k}(\sin\theta)^{2k-2}\mathrm{d}\theta$$
$$= 8(-1)^k\mathrm{i}\log(\varepsilon_0)\int_{|\zeta|=1}\frac{(\zeta^2-1)^{2k-2}\zeta}{((\sqrt{D}-t)\zeta^2+(\sqrt{D}+t))^{2k}}\mathrm{d}\zeta,$$

where $\zeta = \mathrm{e}^{\mathrm{i}\theta}$. Since

$$f(z) = \frac{(z^2-1)^{2k-2}z}{((\sqrt{D}-t)z^2+(\sqrt{D}+t))^{2k}}$$

is holomorphic in $|z| \leqslant 1$, we see that $I = 0$ by residue theorem. This shows that $I(m,t) + I(m,-t) = 0$ if $D = t^2 - 4m > 0$ is non-square.

This completes the proof of the Eichler-Selberg trace formula on $SL_2(\mathbb{Z})$. □

9.2 Eichler-Selberg Trace Formula on Fuchsian Groups

In this section we shall discuss the Eichler-Selberg trace formula of the Hecke operator on a Fuchsian group.

Let $\Gamma \subset G = SL_2(\mathbb{R})$ be a Fuchsian group of the first kind. For any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and any real α, put

$$J_g(z) = cz + d, \quad J_g^{\alpha}(z) = \exp\{\alpha\log(J_g(z))\},$$

where

$$\log(w) = \log(|w|) + \mathrm{i}\arg(w), \quad -\pi < \arg(w) \leqslant \pi.$$

It is clear that the function on $G \times G$

$$C_{\alpha}(g_1,g_2) = \frac{J_{g_1}^{\alpha}(g_2(z))J_{g_2}^{\alpha}(z)}{J_{g_1g_2}^{\alpha}(z)}, \quad g_1,g_2 \in G$$

is independent of the choice of z, its value is equal to 1 or $\exp\{\pm 2\pi\mathrm{i}\alpha\}$, and $|C_{\alpha}(g_1, g_2)| = 1$.

Let V be a complex Hilbert space with dimension n, inner product $(*,*)$ and norm $\| * \|$.

Definition 9.1 *A map ε from the group Γ to the group of unitary operators on V is called a multiplier of weight α if it satisfies the following conditions:*

$$\varepsilon(\gamma_1\gamma_2) = C_\alpha(\gamma_1, \gamma_2)\varepsilon(\gamma_1)\varepsilon(\gamma_2), \quad \forall \gamma_1, \gamma_2 \in \Gamma$$

and

$$\varepsilon(-I) = \exp\{-\pi i\alpha\}E, \quad \text{if} -I \in \Gamma,$$

where E is the identity on V and I is the identity of Γ.

Let $A_\alpha^\infty(\Gamma, V, \varepsilon)$ be the vector space on $\mathbb{C}$ of analytical functions Φ from $\mathbb{H}$ to V satisfying the following conditions:

(1) $\|\Phi\|_\infty := \sup\limits_{z\in\mathbb{H}}(\|\Phi\|y^{\alpha/2}) < \infty$;

(2) $\Phi(\gamma z) = J_\gamma^\alpha(z)\varepsilon(\gamma)\Phi(z)$ for any $\gamma \in \Gamma$.

Let Δ be a Γ double coset such that every element of Δ belongs to a subgroup of G which is commensurable with Γ. For any $\gamma \in \Gamma$, set

$$\psi_\gamma(z) = J_\gamma^{-\alpha}(z)\varepsilon(\gamma)^{-1}.$$

For any $\xi \in \Delta$, take an operator $\eta(\xi)$ on V such that $\psi_\xi(z) = J_\xi^{-\alpha}(z)\eta(\xi)$ satisfies

$$\psi_{\gamma_1\xi\gamma_2}(z) = \psi_{\gamma_2}(z)\psi_\xi(\gamma_2 z)\psi_{\gamma_1}(\xi\gamma_2 z), \quad \forall \gamma_1, \quad \gamma_2 \in \Gamma, \quad \xi \in \Delta. \tag{9.39}$$

Note that (9.39) is equivalent to

$$\eta(\gamma_1\xi\gamma_2) = C_{-\alpha}(\gamma_1, \xi\gamma_2)C_{-\alpha}(\xi, \gamma_2)\varepsilon(\gamma_2)^{-1}\eta(\xi)\varepsilon(\gamma_1)^{-1} \tag{9.40}$$

for any γ_1, $\gamma_2 \in \Gamma$, $\xi \in \Delta$

Now let $\Delta = \bigcup\limits_v \Gamma\delta_v$ be a disjoint union of right cosets of Γ. We define the Hecke operator $\mathrm{T}(\Delta)$ on the space $A_\alpha^\infty(\Gamma, V, \varepsilon)$ as follows:

$$(\Phi|\mathrm{T}(\Delta))(z) = \sum_v \psi_{\delta_v}\Phi(\delta_v z).$$

It is clear that $\mathrm{T}(\Delta)$ is independent of the choice of δ_v and

$$\mathrm{T}(\Delta)A_\alpha^\infty(\Gamma, V, \varepsilon) \subset A_\alpha^\infty(\Gamma, V, \varepsilon).$$

For ξ, $\xi' \in \Delta$ we define an equivalent relation between ξ and ξ' as follows:

(1) If ξ, ξ' are scalars, then $\xi = \xi'$;

(2) If ξ, ξ' are all elliptic (or hyperbolic), there exists a $\gamma \in \Gamma$ such that $\gamma^{-1}\xi\gamma = \xi'$;

(3) If ξ, ξ' are all parabolic, there exists a $\gamma \in \Gamma$ such that $\gamma^{-1}\xi'\gamma \in \Gamma_\xi\xi$ with $\Gamma_\xi = \{\gamma \in \Gamma | \gamma^{-1}\xi\gamma = \xi\}$.

Now let $C = C(\xi)$ be the equivalent class of ξ. We define $I(C)$ as follow:

(1) If ξ is a scalar, then

$$I(C) = \frac{\alpha - 1}{4\pi[\pm\Gamma : \Gamma]}\mathrm{Vol}(F)\mathrm{tr}(\psi_\xi),$$

where $\mathrm{Vol}(F) = \int_F y^{-2}\mathrm{d}x\mathrm{d}y$ with F a fundamental domain of Γ.

(2) If $\xi \in \Delta$ is an elliptic element with fixed points z_0, $\overline{z_0}$ and $z_0 \in \mathbb{H}$, set $\rho = \begin{pmatrix} \overline{z_0} & z_0 \\ 1 & 1 \end{pmatrix}$, then $\rho^{-1}\xi\rho = \begin{pmatrix} \overline{\lambda} & 0 \\ 0 & \lambda \end{pmatrix}$ with $|\lambda| = 1$, define

$$I(C) = \frac{\mathrm{tr}(\psi_\xi(z_0))}{[\Gamma_\xi : 1](1 - \lambda^{-2})}.$$

(3) If $\xi \in \Delta$ is hyperbolic whose fixed points are not cusp points of Γ, then $I(C) = 0$;

(4) If $\xi \in \Delta$ is hyperbolic whose fixed points u, v are cusp points of Γ, we take $\rho \in G$ such that $\rho(0) = u$, $\rho(\infty) = v$, then $\rho^{-1}\xi\rho = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ with λ a real and $|\lambda| > 1$. Set

$$\psi = C_\alpha(\rho, \rho^{-1}\xi)C_\alpha(\rho^{-1}\xi\rho, \rho^{-1})C_{-\alpha}(\rho^{-1}, \rho)\lambda^{-\alpha}\mathrm{tr}(\eta(\xi)),$$

then

$$I(C) = -\frac{\psi}{[\pm\Gamma : \Gamma](1 - \lambda^{-2})}.$$

(5) If $\xi \in \Delta$ is parabolic whose fixed point s is a cusp point of Γ, then $\Gamma_\xi/\Gamma_\xi \cap \{\pm I\}$ is an infinitely cyclic group with a generator δ, there exists $\rho \in G$ such that $\rho^{-1}\delta\rho = t\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \delta_1$, $t = \pm 1$ and $\rho^{-1}\xi\rho = c\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} = \xi_1$, $c = \pm 1$ with r a real. Set

$$\psi = C_\alpha(\rho, \rho^{-1}\xi)C_\alpha(\rho^{-1}\xi\rho, \rho^{-1})C_{-\alpha}(\rho^{-1}, \rho)c^{-\alpha}\eta(\xi),$$
$$\varepsilon_\delta = C_{-\alpha}(\rho, \rho^{-1}\delta)C_{-\alpha}(\rho^{-1}\delta\rho, \rho^{-1})C_\alpha(\rho^{-1}, \rho)t^\alpha\varepsilon(\delta)$$

and denote by $\mathrm{e}^{2\pi\mathrm{i}n_j} (0 < n_j \leqslant 1, j = 1, \cdots, n)$ the eigenvalues of ε_δ, (ψ_{jl}) the matrix of ψ under the basis consisting of the eigenvectors of ε_δ. Then

$$I(C) = \sum_{j=1}^n \psi_{jj}\mathrm{e}^{2\pi\mathrm{i}rn_j} \times \begin{cases} \dfrac{1}{2} - n_j, & \text{if } C = C(\xi) \subset \pm\Gamma, \text{ i.e., } r \text{ is an integer,} \\ \dfrac{1}{1 - \mathrm{e}^{2\pi\mathrm{i}r}}, & \text{if } C = C(\xi) \not\subset \pm\Gamma, \text{ i.e., } r \text{ is not an integer.} \end{cases}$$

Theorem 9.2 *Let $\alpha \geqslant 2$ be a real, then the space $A_\alpha^\infty(\Gamma, V, \varepsilon)$ is finite dimensional and*

$$\mathrm{tr}(\mathrm{T}(\Delta)) = \sum_C I(C),$$

where C runs over the set of equivalent classes of Δ.

Our destination in this section is to prove this theorem. To do this we shall first express the Hecke operator as an integral operator on the space $A_\alpha^\infty(\Gamma, V, \varepsilon)$. Let $A(\mathbb{H}, V)$ be the set of analytical functions on $\mathbb{H}$ with values in V and

$$A_\alpha(\Gamma, V, \varepsilon) = \{\Phi \in A(\mathbb{H}, V) | \Phi(\gamma z) = J_\gamma^\alpha(z)\varepsilon(\gamma)\Phi(z) \text{ for any } \gamma \in \Gamma\}.$$

Denote by μ the G-invariant measure of $\mathbb{H}$, i.e., for any $z = x + \mathrm{i}y \in \mathbb{H}$, we have that $d\mu(z) = y^{-2}\mathrm{d}x\mathrm{d}y$. We also denote by μ the measure on the Riemann surface $\Gamma \setminus \mathbb{H}$. It is clear that $\|\Phi(z)\|y^{\alpha/2}$ is Γ-invariant for any $\Phi \in A_\alpha(\Gamma, V, \varepsilon)$. Let $l \geqslant 1$ be a real, define a l-Norm of $\Phi \in A_\alpha(\Gamma, V, \varepsilon)$:

$$\|\Phi\|_l = \left(\int_F \|\Phi(z)\|^l y^{l\alpha/2}\mathrm{d}\mu(z)\right)^{1/l}$$

and set

$$A_\alpha^l(\Gamma, V, \varepsilon) = \{\Phi \in A_\alpha(\Gamma, V, \varepsilon) | \|\Phi\|_l < \infty\}.$$

The following fact is clear.

Lemma 9.3 *$A_\alpha^l(\Gamma, V, \varepsilon)$ is a* Banach *space for any $1 \leqslant l \leqslant \infty$.*

Since $\mu(F) < \infty$, we see for any $1 < l < \infty$ that

$$A_\alpha^\infty(\Gamma, V, \varepsilon) \subset A_\alpha^l(\Gamma, V, \varepsilon) \subset A_\alpha^1(\Gamma, V, \varepsilon)$$

and the imbeddings are continuous. Put

$$A_\alpha^1(\mathbb{H}, V) = \left\{\Phi \in A(\mathbb{H}, V) | \int_{\mathbb{H}} \|\Phi\| y^{\alpha/2}\mathrm{d}\mu(z) < \infty\right\}.$$

For any $\Phi \in A_\alpha^1(\mathbb{H}, V), \alpha > 2$, we define the Poincare series as follows

$$\Theta_\alpha\Phi(z) = \sum_{\gamma\in\Gamma} \psi_\gamma(z)\Phi(\gamma z),$$

where

$$\psi_\gamma(z) = J_\gamma^{-\alpha}(z)\varepsilon(\gamma)^{-1}.$$

Lemma 9.4 *$\Theta_\alpha\Phi \in A_\alpha^1(\mathbb{H}, V, \varepsilon)$ for any $\Phi \in A_\alpha^1(\mathbb{H}, V)$.*

Proof It follows from the fact that

$$\|\Theta_\alpha\Phi\|_1 \leqslant \int_{\mathbb{H}} \|\Phi(z)\| y^{\alpha/2}\mathrm{d}\mu(z).$$

□

In particular $A^2_\alpha(\Gamma, V, \varepsilon)$ is a Hilbert space with inner product

$$\langle \Phi, \Psi \rangle = \int_F (\Phi(z), \Psi(z)) y^\alpha \mathrm{d}\mu(z), \quad \Phi, \Psi \in A^2_\alpha(\Gamma, V, \varepsilon),$$

where (Φ, Ψ) is the inner product on the space V.

For any real α put

$$k_\alpha(z, w) = \left(\frac{z - \overline{w}}{2i}\right)^{-\alpha}, \quad z, w \in \mathbb{H}. \tag{9.41}$$

Then we have the following

Lemma 9.5 *For any $g \in G$ we have that*

$$k_\alpha(gz, gw) = J_g^\alpha(z)\overline{J_g^\alpha(w)} k_\alpha(z, w), \quad z, w \in \mathbb{H}. \tag{9.42}$$

For any given $z \in \mathbb{H}$, we consider the map from $D = \{w \in \mathbb{C} \| w| < 1\}$ to $\mathbb{H}$:

$$\rho(w) = \frac{\overline{z}w + z}{w + 1} \tag{9.43}$$

and set

$$J_\rho^\alpha(w) = 2y^{-\alpha/2} \mathrm{e}^{\pi \mathrm{i} \alpha/4} (1 + w)^\alpha, \quad z = x + \mathrm{i}y. \tag{9.44}$$

Then

$$k_\alpha(\rho w, \rho \tau) = 2^\alpha J_\rho^\alpha(w) \overline{J_\rho^\alpha(\tau)} (1 - w\overline{\tau})^{-\alpha}, \tag{9.45}$$

in particular, we have

$$k_\alpha(z, \rho\tau) = 2^\alpha J_\rho^\alpha(0) \overline{J_\rho^\alpha(\tau)}, \tag{9.46}$$

$$(\mathrm{Im}(\rho\tau))^\alpha = k_{-\alpha}(\rho\tau, \rho\tau) = 2^{-\alpha} |J_\rho(\tau)|^{-2\alpha} (1 - |\tau|^2)^\alpha, \tag{9.47}$$

and

$$\frac{\mathrm{d}(\rho\tau)}{\mathrm{d}\tau} = J_\rho^{-2}(\tau). \tag{9.48}$$

Lemma 9.6 *Let $\alpha > 2$ be real and $f(z)$ a holomorphic function on $\mathbb{H}$ such that for any $z \in \mathbb{H}$ the integral*

$$I(z) = \int_{\mathbb{H}} k_\alpha(z, \xi) f(\xi) (\mathrm{Im}(\xi))^\alpha \mathrm{d}\mu(\xi)$$

converges absolutely. Then for any $z \in \mathbb{H}$ we have

$$f(z) = \frac{\alpha - 1}{4\pi} \int_{\mathbb{H}} k_\alpha(z, \xi) f(\xi) (\mathrm{Im}(\xi))^\alpha \mathrm{d}\mu(\xi).$$

Proof By (9.41)-(9.48) we can obtain that

$$I(z) = 2J_\rho^\alpha(0)\int\limits_D J_\rho^{-\alpha}(\tau)f(\rho\tau)(1-|\tau|^2)^{\alpha-2}|\mathrm{d}\tau\wedge\mathrm{d}\overline{\tau}|.$$

Put $\tau = r\mathrm{e}^{\mathrm{i}\phi}$. Then

$$I(z) = 4J_\rho^\alpha(0)\int_0^1(1-r^2)^{\alpha-2}r\left(\int_0^{2\pi}J_\rho^{-\alpha}(r\mathrm{e}^{\mathrm{i}\phi})f(\rho(r\mathrm{e}^{\mathrm{i}\phi}))\mathrm{d}\phi\right)\mathrm{d}r$$

and applying Cauchy's integral theorem we see that

$$\int_0^{2\pi}J_\rho^{-\alpha}(r\mathrm{e}^{\mathrm{i}\phi})f(\rho(r\mathrm{e}^{\mathrm{i}\phi}))\mathrm{d}\phi = 2\pi J_\rho^{-\alpha}(0)f(\rho(0)) = 2\pi J_\rho^{-\alpha}(0)f(z),$$

thus,

$$I(z) = 8\pi f(z)\int_0^1(1-r^2)^{\alpha-2}r\mathrm{d}r = \frac{4\pi}{\alpha-1}f(z).$$

This completes the proof. □

Lemma 9.7 *Let $\alpha > 2$ be a real, then*

(1) $k_\alpha(\cdot,\xi)\in A_\alpha^1(\mathbb{H},\mathbb{C})$ *for any* $\xi\in\mathbb{H}$;

(2) $\overline{k_\alpha(z,\cdot)}\in A_\alpha^1(\mathbb{H},\mathbb{C})$ *for any* $z\in\mathbb{H}$.

Proof Obvious. □

Denote by V' and ε the space of all linear operators of V and the multiplier in V' induced from ε respectively. Put

$$K_\alpha(z,\xi) = \sum_{\gamma\in\Gamma}k_\alpha(\gamma z,\xi)\psi_\gamma(z). \tag{9.49}$$

Then it is easy to show the following:

Lemma 9.8 *Let $\alpha > 2$, then for any ξ*

(1) $K_\alpha(\cdot,\xi)\in A_\alpha^1(\Gamma,V',\varepsilon)$;

(2) $K_\alpha(\cdot,\xi)\in A_\alpha^\infty(\Gamma,V',\varepsilon)$ *and the right hand side of* (9.49) *is convergent for* (z,ξ) *with respect to the norm.*

(3) *If we consider $K_\alpha(z,\xi)$ with respect to the variable ξ, then $(K_\alpha(z,\xi))^* = K_\alpha(\xi,z)$ for $z,\xi\in\mathbb{H}$ where $(K_\alpha(z,\xi))^*$ is the conjugate operator of $K_\alpha(z,\xi)$ in V. In particular, for any given $z\in\mathbb{H}$, $K_\alpha(z,\xi)$ is anti-analytical with respect to ξ (i.e., is analytical with respect to $\overline{\xi}$) and*

$$K_\alpha(z,\gamma\xi) = K_\alpha(z,\xi)\overline{J_\gamma^\alpha(\xi)}\varepsilon(\gamma)^{-1},\quad \gamma\in\Gamma.$$

By Lemma 9.7 and Lemma 9.8 we have immediately the following

Lemma 9.9 *Let $\alpha > 2$. Then for any $\Phi \in A^l_\alpha(\Gamma, V, \varepsilon)$ with $1 \leqslant l \leqslant \infty$ we have*

$$\Phi(z) = \frac{\alpha - 1}{4\pi[\pm\Gamma : \Gamma]} \int_F K_\alpha(z, \xi)(\mathrm{Im}(\xi))^\alpha \Phi(\xi) \mathrm{d}\mu(\xi), \quad z \in \mathbb{H}.$$

From the above lemma we see that $A^\infty_\alpha(\Gamma, V, \varepsilon)$ is dense everywhere in $A^2_\alpha(\Gamma, V, \varepsilon)$. Put

$$\mathcal{K}_\alpha(z, \xi) = \frac{\alpha - 1}{4\pi[\pm\Gamma : \Gamma]} K_\alpha(z, \xi).$$

Then from Lemma 9.9 and Lemma 9.8 we have the following:

Theorem 9.3 *Let $\{\Phi_m(z)\}$ be a standard orthogonal basis of $A^2_\alpha(\Gamma, V, \varepsilon)$. Then for any vector $v \in V$ we have that*

$$\mathcal{K}_\alpha(z, \xi)v = \sum_m (v, \Phi_m(\xi)) \Phi_m(z), \quad z, \xi \in \mathbb{H}$$

and the series on the right hand side above is absolutely and uniformly convergent on any compact subset of $\mathbb{H} \times \mathbb{H}$.

Corollary 9.1 *Let $v_1, \cdots, v_n$ be a standard orthogonal basis of V. Then for any $z, \xi \in \mathbb{H}$ we have*

$$\mathrm{tr}(\mathcal{K}_\alpha(z, \xi)) = \sum_m \sum_{i=1}^n (v_i, \Phi_m(\xi))(\Phi_m(z), v_i)$$

and the series on the right hand side above is absolutely and uniformly convergent on any compact subset of $\mathbb{H} \times \mathbb{H}$. In particular we have

$$\mathrm{tr}(\mathcal{K}_\alpha(z, z)) = \sum_m \|\Phi_m(z)\|^2,$$

$$\dim(A^2_\alpha(\Gamma, V, \varepsilon)) = \int_F \mathrm{tr}(\mathcal{K}_\alpha(z, z) y^\alpha \mathrm{d}\mu(z).$$

Remark 9.1 We will know that the dimension of $A^2_\alpha(\Gamma, V, \varepsilon)$ is finite. Hence $A^\infty_\alpha(\Gamma, V, \varepsilon) = A^2_\alpha(\Gamma, V, \varepsilon)$. So we can consider the Hecke operator in the space $A^2_\alpha(\Gamma, V, \varepsilon)$.

By the definition of the Hecke operator $\mathrm{T}(\Delta)$ we obtain immediately the following:

Lemma 9.10 *We have*

$$\mathcal{K}_\alpha(z, w)|\mathrm{T}(\Delta) = \frac{\alpha - 1}{4\pi[\pm\Gamma : \Gamma]} \sum_{\gamma \in \Delta} k_\alpha(\gamma z, \xi)\psi_\gamma(z)$$

and the series on the right hand side above is absolutely and uniformly convergent on any compact subset of $\mathbb{H} \times \mathbb{H}$, where the Hecke *operator $\mathrm{T}(\Delta)$ is supposed to operate on the variable z. Write $\mathcal{T}_\alpha(\Delta)(z, \xi) := \mathcal{K}_\alpha(z, w)|\mathrm{T}(\Delta)$.*

Lemma 9.11 (1) *For any $z \in \mathbb{H}$, the function $\mathcal{T}_\alpha(\varDelta)(z,\xi)$ is anti-analytical with respect to ξ and*

$$\sup_{\xi\in\mathbb{H}}(\|\mathcal{T}_\alpha(\varDelta)(z,\xi)\|(\mathrm{Im}(\xi))^{\alpha/2} < \infty.$$

(2) *For any $\gamma \in \varGamma$ we have*

$$\mathcal{T}_\alpha(\varDelta)(z,\gamma\xi) = \mathcal{T}_\alpha(\varDelta)(z,\xi)\overline{J_\gamma^\alpha(\xi)}\varepsilon(\gamma)^{-1}.$$

Proof Obvious. □

By Lemma 9.9, Theorem 9.3, Lemma 9.10 and Lemma 9.11 we have the following:

Theorem 9.4 *Let $\alpha > 2$ be a real, $\varPhi \in A_\alpha^\infty(\varGamma, V, \varepsilon) = A_\alpha^2(\varGamma, V, \varepsilon)$. Then*

$$(\varPhi|\mathrm{T}(\Delta))(z) = \int_F \mathcal{T}_\alpha(\varDelta)(z,\xi)\varPhi(\xi)(\mathrm{Im}(\xi))^\alpha \mathrm{d}\mu(\xi)$$

and

$$\mathrm{tr}(\mathrm{T}(\varDelta)) = \int_F \mathrm{tr}(\mathcal{T}_\alpha(\varDelta)(z,z))y^\alpha \mathrm{d}\mu(z).$$

The Proof of Theorem 9.2 We know that

$$\mathrm{tr}(\mathrm{T}(\varDelta)) = \frac{\alpha-1}{4\pi[\pm\varGamma:\varGamma]}\int_F \sum_{\xi\in\varDelta} k_\alpha(\xi z,z)\mathrm{tr}(\psi_\xi(z))y^\alpha \mathrm{d}\mu(z). \tag{9.50}$$

We take a fundamental domain F of $\varGamma$ as follows:

$$F = F_0 \bigcup_{k=1}^h F_k, \qquad \text{disjoint union,} \tag{9.51}$$

where F_0 is compact, and for every $k = 1, \cdots, h$ there exists a $g_k \in G$ such that $g_k F_k = \varPi_a$ with $a > 0$ and

$$\varPi_a = \{z \in \mathbb{H} \,|\, \mathrm{Im}(z) > a, 0 \leqslant \mathrm{Re}(z) \leqslant 1\}. \tag{9.52}$$

Suppose $A > a$, set $F_{k,A} = F_k - g_k^{-1}\varPi_A$ and let x_k be the cusp point corresponding to F_k. Put

$$\varDelta_k = \{\zeta \in \varDelta | \zeta x_k = x_k\}. \tag{9.53}$$

By (9.50)-(9.53), we see that

$$\begin{aligned}\mathrm{tr}(\mathrm{T}(\varDelta)) = \frac{\alpha-1}{4\pi[\pm\varGamma:\varGamma]}\lim_{A\to\infty}\Bigg(&\sum_{\xi\in\varDelta}\int_{F_0} k_\alpha(\xi z,z)\mathrm{tr}(\psi_\xi(z))y^\alpha \mathrm{d}\mu(z)\\ &+\sum_{k=1}^h \sum_{\xi\in\varDelta-\varDelta_k}\int_{F_k} k_\alpha(\xi z,z)\mathrm{tr}(\psi_\xi(z))y^\alpha \mathrm{d}\mu(z)\\ &+\sum_{k=1}^h \sum_{\xi\in\varDelta_k}\int_{F_{k,A}} k_\alpha(\xi z,z)\mathrm{tr}(\psi_\xi(z))y^\alpha \mathrm{d}\mu(z)\Bigg).\end{aligned} \tag{9.54}$$

Classifying them according to the equivalence classes $C \subset \varDelta$, we have from (9.54) that

$$\mathrm{tr}(\mathrm{T}(\varDelta)) = \frac{\alpha - 1}{4\pi[\pm\varGamma : \varGamma]} \sum_{C \subset \varDelta} I'(C). \tag{9.55}$$

We now compute $I'(C)$ as follows.

Case (1): $C = C(\xi), \xi = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\psi_\xi(z) = \psi_\xi$ is independent of z, and

$$I'(C) = \int_F \left(\frac{z - \overline{z}}{2\mathrm{i}}\right)^{-\alpha} \mathrm{tr}(\psi_\xi(z)) y^{\alpha-2} \mathrm{d}x\mathrm{d}y = \mu(F)\mathrm{tr}(\psi_\xi). \tag{9.56}$$

Case (2): $C = C(\xi)$ with $\xi \in \varDelta$ an elliptic element. Then we have

$$I'(C) = 2\mathrm{tr}(\eta(\xi)) \frac{[\pm\varGamma : \varGamma]}{[\varGamma_\xi : 1]} \iint_D \frac{J_\rho^\alpha(\rho^{-1}\xi\rho(w))}{J_\xi^\alpha(\rho(w))J_\rho^\alpha(w)}$$
$$\times (1 - \overline{\lambda}^2 |w|^2)^{-\alpha} (1 - |w|^2)^{\alpha-2} |\mathrm{d}w \wedge \mathrm{d}\overline{w}|.$$

It is easy to verify that

$$\frac{J_\rho^\alpha(\rho^{-1}\xi\rho(w))}{J_\xi^\alpha(\rho(w))J_\rho^\alpha(w)} = J_\xi^{-\alpha}(z_0), \quad \mathrm{d}\mu(z) = \frac{2|\mathrm{d}w \wedge \mathrm{d}\overline{w}|}{(1 - |w|^2)^2}, \quad z = \frac{\overline{z_0}w + z_0}{w + 1}$$

and

$$\int_D (1 - \overline{\lambda}^2 |w|^2)^{-\alpha} (1 - |w|^2)^{\alpha-2} |\mathrm{d}w \wedge \mathrm{d}\overline{w}| = \frac{2\pi}{(\alpha - 1)(1 - \overline{\lambda}^2)},$$

hence

$$I'(C) = \frac{4\pi[\pm\varGamma : \varGamma]}{(\alpha - 1)(1 - \overline{\lambda}^2)[\varGamma_\xi : 1]} \mathrm{tr}(\psi_\xi(z_0)). \tag{9.57}$$

Case (3): $C = C(\xi)$ with $\xi \in \varDelta$ a hyperbolic element with non-cuspidal fixed points. Then

$$I'(C) = \int_{F_\xi} k_\alpha(\xi z, z) \mathrm{tr}(\psi_\xi(z)) y^\alpha \mathrm{d}\mu(z),$$

where F_ξ is a fundamental domain of $\varGamma_\xi$. In this case the group $\varGamma_\xi / \varGamma_\xi \cap (\pm I)$ is infinitely cyclic. Taking $\rho \in G$ with $\rho^{-1}\xi\rho = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ and $\lambda > 1$ a real, we have

$$I'(C) = c \int_1^b r^{-1} \mathrm{d}r \int_0^\pi \left(\frac{(\lambda^{-2} - 1)\cot\phi + \mathrm{i}(\lambda^{-2} + 1)}{2\mathrm{i}}\right)^{-\alpha} \frac{\mathrm{d}\phi}{\sin^2\phi} = 0, \tag{9.58}$$

where b, c are constants.

Case (4): $C = C(\xi)$ with $\xi \in \varDelta$ a hyperbolic element with two cuspidal fixed points

u, v. Taking $\rho \in G$ such that $\rho(0) = u$, $\rho(\infty) = v$, we have that $\rho^{-1}\xi\rho = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}$ with λ a real and $|\lambda| > 1$. Set

$$\psi = C_\alpha(\rho, \rho^{-1}\xi)C_\alpha(\rho^{-1}\xi\rho, \rho^{-1})C_{-\alpha}(\rho^{-1}, \rho)\lambda^{-\alpha}\mathrm{tr}(\eta(\xi)),$$

then

$$I'(C) = \lim_{A\to\infty, B\to\infty} \mathrm{tr}(\eta(\xi)) \int_{\mathbb{H}-C_0(A)-C_\infty(B)} \frac{J_\rho^\alpha(\rho^{-1}\xi\rho z)}{J_\xi^\alpha(\rho(z))J_\rho^\alpha(z)} k_\alpha(\rho^{-1}\xi\rho z, z) y^\alpha \mathrm{d}\mu(z),$$

where

$$C_0(A) = \{z \in \mathbb{H} | |z - \mathrm{i}/(2A)| < 1/(2A)\}, \quad C_\infty(B) = \{z \in \mathbb{H} | \mathrm{Im}(z) > B\}.$$

Since

$$\frac{J_\rho^\alpha(\rho^{-1}\xi\rho z)}{J_\xi^\alpha(\rho(z))J_\rho^\alpha(z)} = C_\alpha(\rho, \rho^{-1}\xi)C_\alpha(\rho^{-1}\xi\rho, \rho^{-1})C_{-\alpha}(\rho^{-1}, \rho)\lambda^{-\alpha},$$

we see that

$$\begin{aligned} I'(C) =& \psi \lim_{\substack{A\to\infty,\\ B\to\infty}} \int_0^\pi \left(\frac{(\lambda^{-2}-1)\cos\phi + \mathrm{i}(\lambda^{-2}+1)\sin\phi}{2\mathrm{i}}\right)^{-\alpha} \sin^{\alpha-2}\phi \int_{\sin\phi/A}^{B/\sin\phi} r^{-1}\mathrm{d}r\mathrm{d}\phi \\ =& -2\psi \int_0^\pi \left(\frac{(\lambda^{-2}-1)\cot\phi + \mathrm{i}(\lambda^{-2}+1)}{2\mathrm{i}}\right)^{-\alpha} \frac{\log(\sin\phi)}{\sin^2\phi}\mathrm{d}\phi \\ =& \frac{4\mathrm{i}\psi}{(1-\lambda^{-2})(\alpha-1)} \int_0^\pi \left(\frac{(\lambda^{-2}-1)\cot\phi + \mathrm{i}(\lambda^{-2}+1)}{2\mathrm{i}}\right)^{1-\alpha} \cot\phi\mathrm{d}\phi, \end{aligned}$$

where we used integration by parts to obtain the last equality. Set $u = \cot\phi$, we see that

$$\begin{aligned} I'(C) =& \frac{4\mathrm{i}\psi}{(1-\lambda^{-2})(\alpha-1)} \int_\infty^{-\infty} \left(\frac{(\lambda^{-2}-1)u + \mathrm{i}(\lambda^{-2}+1)}{2\mathrm{i}}\right)^{1-\alpha} \frac{u\mathrm{d}u}{1+u^2} \\ =& \frac{4\mathrm{i}\psi}{(1-\lambda^{-2})(\alpha-1)} \operatorname*{Res}_{u=-\mathrm{i}} \left(\frac{(\lambda^{-2}-1)u + \mathrm{i}(\lambda^{-2}+1)}{2\mathrm{i}}\right)^{1-\alpha} \frac{u}{u-\mathrm{i}} \\ =& -\frac{4\pi\psi}{(\alpha-1)(1-\lambda^{-2})}. \end{aligned} \tag{9.59}$$

Case (5): $C = C(\xi)$ with ξ a parabolic element. Then $\Gamma/\Gamma_\xi \cap (\pm I)$ is an infinitely cyclic group and

$$\begin{aligned} I'(C) = [\pm\Gamma : \Gamma] \lim_{A\to\infty} {\sum_m}' C_\alpha(\rho, \rho^{-1}\delta^m\xi)C_\alpha(\rho^{-1}\delta^m\xi\rho, \rho^{-1}) \\ \times C_{-\alpha}(\rho^{-1}, \rho)\mathrm{tr}(\eta(\delta^m\xi)) \int_0^A y^{\alpha-2} \int_0^1 k_\alpha(\rho^{-1}\delta^m\xi\rho z, z) J_{\rho^{-1}\delta^m\xi\rho}^{-\alpha}(z)\mathrm{d}x\mathrm{d}y \end{aligned}$$

$$= [\pm\Gamma : \Gamma] \lim_{A\to\infty} \sum_m{}' C_\alpha(\delta_1^m \xi_1, \rho^{-1}) C_{-\alpha}(\rho^{-1}, \rho) \mathrm{tr}(\eta(\rho\delta_1^m \xi \rho^{-1}))$$

$$\times \int_0^A y^{\alpha-2} \int_0^1 k_\alpha(\delta_1^m \xi_1 z, z) J^{-\alpha}_{\delta_1^m \xi_1}(z) \mathrm{d}x\mathrm{d}y,$$

where $\sum_m{}'$ means that m runs over all integers with $\delta^m \xi \neq 1 \neq \delta_1^m \xi_1$. It is easy to see that

$$\widetilde{\psi}_{\delta_1^m \xi_1}(z) = C_\alpha(\rho, \delta_1^m \xi_1 \rho^{-1}) C_\alpha(\delta_1^m \xi, \rho^{-1}) C_{-\alpha}(\rho^{-1}, \rho) J^{-\alpha}_{\delta_1^m \xi_1}(z) \eta(\rho \delta_1^m \xi \rho^{-1})$$

is independent of z and satisfies

$$\widetilde{\psi}_{\delta_1^m \xi_1}(z) = \psi \cdot \varepsilon_\delta^{-m}.$$

Therefore we have

$$\widetilde{\psi}_{\delta_1^m \xi_1}(z) = \sum_{j=1}^n \psi_{jj} \mathrm{e}^{-2\pi \mathrm{i} m n_j},$$

so

$$I'(C) = [\pm\Gamma : \Gamma] \sum_{j=1}^n \psi_{jj} \lim_{A\to\infty} \sum_m{}' \mathrm{e}^{-2\pi \mathrm{i} m n_j} \int_0^A \left(y + \frac{m+r}{2i}\right)^{-\alpha} y^{\alpha-2} \mathrm{d}y$$

$$= \frac{2\mathrm{i}[\pm\Gamma : \Gamma]}{\alpha - 1} \sum_{j=1}^n \psi_{jj} \lim_{A\to\infty} \sum_m{}' \frac{\mathrm{e}^{-2\pi \mathrm{i} m n_j}}{m+r} \left(1 + \frac{m+r}{2\mathrm{i}A}\right)^{1-\alpha}, \tag{9.60}$$

where $\sum_m{}'$ means that m runs over all integers with $m + r \neq 0$.

For any real n with $0 < n \leqslant 1$, put

$$S(n,r) = \lim_{A\to\infty} \sum_{\substack{m\in\mathbb{Z},\\ m+r\neq 0}} \frac{\mathrm{e}^{-2\pi \mathrm{i} m n}}{m+r} \left(1 + \frac{m+r}{2\mathrm{i}A}\right)^{1-\alpha}, \tag{9.61}$$

then

$$S(n,r) = \begin{cases} 2\pi\mathrm{i}(n - 1/2), & \text{if } r \text{ is an integer,} \\ -2\pi\mathrm{i}\dfrac{\mathrm{e}^{2\pi\mathrm{i} r n}}{1 - \mathrm{e}^{2\pi\mathrm{i} r}}, & \text{if } r \text{ is not an integer} \end{cases} \tag{9.62}$$

In fact, since $S(n,r) = \mathrm{e}^{2\pi\mathrm{i}[r]n} S(n, \{r\})$ where $\{r\}$ is the fractional part of r, we may assume that $0 \leqslant r < 1$.

If $0 < n < 1$, since

$$\sum_m{}' \frac{\mathrm{e}^{-2\pi\mathrm{i} m n}}{m+r} = \begin{cases} 2\pi\mathrm{i}(n - 1/2), & \text{if } r = 0, \\ -2\pi\mathrm{i}\dfrac{\mathrm{e}^{2\pi\mathrm{i} r n}}{1 - \mathrm{e}^{2\pi\mathrm{i} r}}, & \text{if } 0 < r < 1 \end{cases}$$

is boundedly convergent, we may interchange the summation and the limitation in (9.61) and hence obtain (9.62). So we may now assume that $n = 1$. Then

$$S(n,r) = S(1,r) = \lim_{A\to\infty} \sum_{\substack{m\in\mathbb{Z},\\ m+r\neq 0}} \frac{1}{m+r}\left(1+\frac{m+r}{2\mathrm{i}A}\right)^{1-\alpha}.$$

If $r = 0$, then

$$\begin{aligned} S(n,0) &= \lim_{A\to\infty}\sum_{m=1}^{\infty}\left(\frac{1}{m}\left(1+\frac{m}{2\mathrm{i}A}\right)^{1-\alpha} - \frac{1}{m}\left(1-\frac{m}{2\mathrm{i}A}\right)^{1-\alpha}\right)\\ &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{x}\left(\left(1+\frac{x}{2\mathrm{i}}\right)^{1-\alpha} - \left(1-\frac{x}{2\mathrm{i}}\right)^{1-\alpha}\right)\mathrm{d}x\\ &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{x}\left(\left(1+\frac{x}{2\mathrm{i}}\right)^{1-\alpha} - \frac{1}{x^2+1}\right)\mathrm{d}x\\ &\quad + \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{x}\left(\frac{1}{x^2+1} - \left(1-\frac{x}{2\mathrm{i}}\right)^{1-\alpha}\right)\mathrm{d}x\\ &= \pi\mathrm{i}, \end{aligned}$$

where we used the residue theorem to get the last equality.

If $r \neq 0$, then

$$\begin{aligned} S(n,r) =& \frac{1}{r} + \frac{1}{r+1} - \frac{1}{1-r} + \int_1^{\infty}\frac{\{x\}}{(x-r)^2}\mathrm{d}x - \int_1^{\infty}\frac{\{x\}}{(x+r)^2}\mathrm{d}x\\ &+\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{x}\left(\left(1+\frac{x}{2\mathrm{i}}\right)^{1-\alpha} - \left(1-\frac{x}{2\mathrm{i}}\right)^{1-\alpha}\right)\mathrm{d}x - \log\frac{1+r}{1-r}, \end{aligned} \tag{9.63}$$

where the last term came from

$$\begin{aligned} &-\lim_{A\to\infty,\varepsilon\to 0}\int_{\varepsilon}^{(1+r)/A}\frac{1}{x}\left(1+\frac{x}{2\mathrm{i}}\right)^{1-\alpha}\mathrm{d}x - \int_{\varepsilon}^{(1-r)/A}\frac{1}{x}\left(1-\frac{x}{2\mathrm{i}}\right)^{1-\alpha}\mathrm{d}x\\ =& -\lim_{A\to\infty,\varepsilon\to 0}\left(\int_{\varepsilon}^{(1+r)/A}\frac{1}{x}\mathrm{d}x - \int_{\varepsilon}^{(1-r)/A}\frac{1}{x}\mathrm{d}x\right)\\ =& -\log\frac{1+r}{1-r}. \end{aligned}$$

On the other hand we have

$$\begin{aligned} &\int_1^{\infty}\frac{\{x\}}{(x-r)^2}\mathrm{d}x - \int_1^{\infty}\frac{\{x\}}{(x+r)^2}\mathrm{d}x\\ =& \sum_{m=1}^{\infty}\left(-\int_m^{m+1}\frac{x-m}{(x+r)^2}\mathrm{d}x + \int_m^{m+1}\frac{x-m}{(x-r)^2}\mathrm{d}x\right)\\ =& \sum_{m=1}^{\infty}\left(\frac{1}{r+m+1} + \frac{1}{r-m-1} + \int_m^{m+1}\left(\frac{1}{x-r} - \frac{1}{x+r}\right)\mathrm{d}x\right) \end{aligned}$$

$$=\sum_{m=2}^{\infty}\left(\frac{1}{r+m}-\frac{1}{m}\right)+\sum_{m=-2}^{-\infty}\left(\frac{1}{r+m}-\frac{1}{m}\right)+\int_1^{\infty}\left(\frac{1}{x-r}-\frac{1}{x+r}\right)\mathrm{d}x$$
$$=\sum_{m\in\mathbb{Z},m\neq 0,\pm 1}^{\infty}\left(\frac{1}{r+m}-\frac{1}{m}\right)+\log\frac{1+r}{1-r}.$$

By the above equality and (9.63) we obtain that

$$S(n,r)=\pi\cot(\pi r)+\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{x}\left(\left(1+\frac{x}{2\mathrm{i}}\right)^{1-\alpha}-\left(1-\frac{x}{2\mathrm{i}}\right)^{1-\alpha}\right)\mathrm{d}x$$
$$=\pi\mathrm{i}+\pi\cot(\pi r), \tag{9.64}$$

where to obtain the last equality we used the following fact:

$$\int_{-\infty}^{\infty}\frac{1}{x}\left(\left(1+\frac{x}{2\mathrm{i}}\right)^{1-\alpha}-\left(1-\frac{x}{2\mathrm{i}}\right)^{1-\alpha}\right)\mathrm{d}x$$
$$=\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{x}\left(\left(1+\frac{x}{2\mathrm{i}}\right)^{1-\alpha}-\frac{1}{x^2+1}\right)\mathrm{d}x$$
$$+\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{x}\left(\frac{1}{x^2+1}-\left(1-\frac{x}{2\mathrm{i}}\right)^{1-\alpha}\right)\mathrm{d}x$$
$$=\pi\mathrm{i}$$

by the residue theorem.

This completes the proof of Theorem 9.2 because of (9.56)-(9.62). □

Let $k\geqslant 1$ and N be positive integers. Applying Theorem 9.2, we can prove the following trace formula (please compare J. Oesterlé, 1977).

Theorem 9.5 *We have the following trace formula:*

$$\mathrm{tr}(T_{2k,N}(n),S(N,2k,\mathrm{id.}))=B_1+B_2+B_3+B_4+B_5,\quad (n,N)=1$$

with

$$B_1=\delta\left(\frac{1}{n}\right)\frac{(2k-1)N}{12}\prod_{p|N}\left(1+\frac{1}{p}\right),$$

$$B_2=-\frac{1}{2}\sum_{|s|<2\sqrt{n}}P_{2k}(s,n)\sum_{\substack{f^2|s^2-4n,\\(s^2-4n)/f^2\equiv 0,1 \bmod 4}}h'\left(\frac{s^2-4n}{f^2}\right)\mu(s,f,n,N),$$

$$B_3=-\sigma_0(N)\sum_{\substack{0<\lambda'<\lambda,\\\lambda\lambda'=n}}\frac{\lambda'^{2k-1}\phi((\lambda-\lambda'),n)}{(\lambda-\lambda',n)},$$

$$B_4=-\frac{1}{2}n^{k-1}\phi(\sqrt{n})\sigma_0(N),$$

$$B_5=\begin{cases}0, & \text{if } k>1,\\ \sigma_1(n), & \text{if } k=1,\end{cases}$$

where $P_k(t,j)$ is defined in equality (9.13), $\mu(t,f,n,M)$ is defined in Lemma 5.23, $h'(m)$ is defined as the number of $SL_2(\mathbb{Z})$-equivalence classes of positive definite primitive integral binary quadratic forms of discriminant m, and a form equivalent to X^2+Y^2 or X^2+XY+Y^2 is counted with multiplicity $1/2$ or $1/3$ respectively, $\sigma_0(N)$ is defined as the number of positive divisors of N. $\delta(x)$ is 1 or 0 according as $x \in \mathbb{Z}$ or not, and $\phi(k)$ is Euler *function.*

Also, applying Theorem 9.2, S. Niwa proved the similar result for Hecke operators half-integral weight (please compare S. Niwa, 1977).

Theorem 9.6 *Let $k \geqslant 1$ be a positive integer and N an odd positive integer. Then for any $(n,2N)=1$ we have*

$$\operatorname{tr}(T_{k+1/2,4N}(n^2), S(4N,k+1/2,\mathrm{id.})) = \operatorname{tr}(T_{2k,2N}(n), S(2N,2k,\mathrm{id.})).$$

We shall not give the proofs of Theorem 9.5 and Theorem 9.6, since they are similar to the proof of Kohnen's trace formula for Kohnen's + space which will be given in next section.

9.3 Trace Formula on the Space $S_{k+1/2}(N,\chi)$

In this section we compute the traces of Hecke operators on the space $S_{k+1/2}(N,\chi)$ discussed in Section 6.2. We will use the notations of Section 6.2. Our presentation is due to W. Kohnen, 1982.

Denote by $\mathscr{H}_N$ the subalgebra of the Hecke algebra with respect to $\Gamma_0(N)$ and

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod{N}, (a,N)=1, ad-bc>0 \right\},$$

which is generated by the double cosets $\Gamma_0(N)\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\Gamma_0(N)$ with $n \in \mathbb{N}$ and $(n,2N)=1$. Then the elements $\Gamma_0(N)\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\Gamma_0(N)$ form a $\mathbb{C}$-basis of $\mathscr{H}_N$, where a, $d>0$, $a|d$ and $(d,2N)=1$.

Define a linear map R from $\mathscr{H}_N$ to $\operatorname{End}_{\mathbb{C}}(S_{k+1/2}(n,\chi))$ by requiring that

$$R\left(\Gamma_0(N)\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\Gamma_0(N)\right) = a(ad)^{k-3/2}\left[\Delta_0(4N,\chi_1)\left\{\begin{pmatrix} a^2 & 0 \\ 0 & d^2 \end{pmatrix}, (d/a)^{1/2}\right\}\Delta_0(4N,\chi)\right]$$

restricted to $S_{k+1/2}(N,\chi)$, where $\chi_1 = \left(\dfrac{4\chi(-1)}{\cdot}\right)\chi$. Since

$$\left[\Delta_0(4N,\chi_1)\left\{\begin{pmatrix} a^2 & 0 \\ 0 & d^2 \end{pmatrix}, (d/a)^{1/2}\right\}\Delta_0(4N,\chi)\right]$$

is a polynomial in $\mathrm{T}_{k+1/2,4N,\chi_1}(p^2)$ (p prime, $p \nmid 2N$), it preserves $S_{k+1/2}(N,\chi)$. Then R is a representation of $\mathscr{H}_N$. On the other hand, we have a representation $\widetilde{R} : \mathscr{H}_N \to \mathrm{End}_{\mathbb{C}}(S(N,2k,\mathrm{id.}))$ defined by

$$\widetilde{R}\left(\Gamma_0(N)\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\Gamma_0(N)\right) = (ad)^{2k-1}\left[\Gamma_0(N)\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\Gamma_0(N)\right]_{2k}.$$

The aim of this section is to prove that the representations R and $\widetilde{R}$ are equivalent which can be deduced from the following:

Theorem 9.7 *Let notations be as above. Then*

$$\mathrm{tr}(\mathrm{T}_{k+1/2,N,\chi}(n), S_{k+1/2}(N,\chi)) = \mathrm{tr}(\mathrm{T}_{2k,N}(n), S(N,2k,\mathrm{id.})), \quad (n,2N)=1, \quad (9.65)$$

where $\mathrm{T}_{k+1/2,N,\chi}(n)$ resp. $\mathrm{T}_{2k,N}(n)$ *are the images of* $\Gamma_0(N)\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}\Gamma_0(N)$ *under* R resp. $\widetilde{R}$.

An explicit expression of the trace of $\mathrm{T}_{2k,N}(n)$ on $S_{2k}(N)$ was obtained in Theorem 9.5. We will show that the left-hand side of (9.65) is also given by this expression. We know that $U(t)$ is an isomorphism from $S_{k+1/2}(N)$ onto $S_{k+1/2}(N,\chi)$ by the results of Section 6.2. Since $U(t)$ commutes with Hecke operators, it is sufficient to compute the left-hand side of (9.65) for trivial χ. In the following we write $\mathrm{T}_{k+1/2,N}(n)$ resp. $\mathrm{T}_{k+1/2,4N}(n^2)$ for $\mathrm{T}_{k+1/2,N,\mathrm{id.}}(n)$ resp. $\mathrm{T}_{k+1/2,4N,\left(\frac{4}{\cdot}\right)}(n^2)$ and abbreviate $\Delta_0\left(4N,\left(\frac{4}{\cdot}\right)\right)$ as $\Delta_0(4N)$. By the definition of $\mathrm{T}_{k+1/2,N}(n)$ we have

$$\mathrm{tr}(\mathrm{T}_{k+1/2,N}(n), S_{k+1/2}(N)) = \mathrm{tr}\left(\mathrm{T}_{k+1/2,4N}(n^2)\mathrm{pr}, S\left(4N, k+1/2, \left(\frac{4}{\cdot}\right)\right)\right).$$

Substituting the definition of pr we see that

$$\begin{aligned} &\mathrm{tr}(\mathrm{T}_{k+1/2,N}(n), S_{k+1/2}(N)) \\ =&\frac{1}{6}(-1)^{[(k+1)/2]}\sqrt{2}\,\mathrm{tr}\left(\mathrm{T}_{k+1/2,4N}(n^2)Q, S\left(4N,k+1/2,\left(\frac{4}{\cdot}\right)\right)\right) \\ &+\frac{1}{3}\mathrm{tr}\left(\mathrm{T}_{k+1/2,4N}(n^2), S(4N,k+1/2,\left(\frac{4}{\cdot}\right)\right)\right) \end{aligned} \quad (9.66)$$

The second trace on the right-hand side was computed in Theorem 9.6 for $k \geqslant 2$, in which case the term is equal to the trace of the usual Hecke operator of degree n on the space of weight $2k$ for $\Gamma_0(2N)$. So we only need to compute the first summand on the right of (9.66).

Lemma 9.12 *We have the following equality:*

$$\begin{aligned}&\Delta_0(4N)\left\{\begin{pmatrix}1&0\\0&n^2\end{pmatrix},n^{1/2}\right\}\Delta_0(4N)\cdot\Delta_0(4N)\xi\Delta_0(4N)\\=&\Delta_0(4N)\left\{\begin{pmatrix}4&1\\0&4n^2\end{pmatrix},\mathrm{e}^{\pi\mathrm{i}/4}n^{1/2}\right\}\Delta_0(4N),\end{aligned}\tag{9.67}$$

where $\xi:=\xi_{k+1/2,\epsilon}:=\left\{\begin{pmatrix}4&1\\0&4\end{pmatrix},\epsilon^{1/2}\mathrm{e}^{\pi\mathrm{i}/4}\right\}$.

Proof For any double coset $\mathscr{D}$ denote by $\deg(\mathscr{D})$ its degree, i.e. the number of right cosets contained in $\mathscr{D}$. Using $(n,2N)=1$ we can check that

$$\deg\left(\Delta_0(4N)\left\{\begin{pmatrix}1&0\\0&n^2\end{pmatrix},n^{1/2}\right\}\Delta_0(4N)\right)=[\Gamma_0(4N):\Gamma_0(4Nn^2)].$$

Furthermore, by (6.27), we see that

$$\deg(\Delta_0(4N)\xi\Delta_0(4N))=4,$$

so the degree of the left-hand side of (9.67) is equal to $4[\Gamma_0(4N):\Gamma_0(4Nn^2)]$. On the other hand we can verify that

$$\Delta_0(4N)\cap\alpha\Delta_0(4N)\alpha^{-1}=\Delta_0(16Nn^2),$$

where $\alpha=\left\{\begin{pmatrix}4&1\\0&4n^2\end{pmatrix},\mathrm{e}^{\pi\mathrm{i}/4}n^{1/2}\right\}$. So the degree of the right-hand side of (9.67) is equal to

$$\begin{aligned}[\Gamma_0(4N):\Gamma_0(16Nn^2)]&=[\Gamma_0(4Nn^2):\Gamma_0(16Nn^2)][\Gamma_0(4N):\Gamma_0(4Nn^2)]\\&=4[\Gamma_0(4N):\Gamma_0(4Nn^2)].\end{aligned}$$

Therefore the degree of the expressions on both sides of (9.67) are equal. Since

$$\left\{\begin{pmatrix}1&0\\0&n^2\end{pmatrix},n^{1/2}\right\}\xi=\left\{\begin{pmatrix}4&1\\0&4n^2\end{pmatrix},\mathrm{e}^{\pi\mathrm{i}/4}n^{1/2}\right\},$$

the set on the right of (9.67) is contained in the set on the left, the desired equality holds. □

From Lemma 9.12 we see that

$$\begin{aligned}&\frac{1}{6}(-1)^{[(k+1)/2]}\sqrt{2}\mathrm{tr}\left(\mathrm{T}_{k+1/2,4N}(n^2)Q,S\left(4N,k+1/2,\left(\frac{4}{\cdot}\right)\right)\right)\\=&c_{n,k}\mathrm{tr}\left([\Delta_0(4N)(C_n,n^{1/2})\Delta_0(4N)],S\left(4N,k+1/2,\left(\frac{4}{\cdot}\right)\right)\right),\end{aligned}$$

where $c_{n,k}=\dfrac{1}{6}n^{k-3/2}(1-(-1)^k\mathrm{i})$ and $C_n=\begin{pmatrix}4&1\\0&4n^2\end{pmatrix}$.

We shall now apply the Eichler-Selberg trace formula proved in Section 9.2. We suppose $k \geqslant 1$. If $C = D_1 C_n D_2$ with $D_1, D_2 \in \Gamma_0(4N)$ we put $C^* = D_1^*(C_n, n^{1/2})D_2^*$. Then it is easy to verify that the map $C \mapsto C^*$ is a well-defined bijection between $\Gamma_0(4N)C_n\Gamma_0(4N)$ and $\Delta_0(4N)(C_n,\, n^{1/2})\Delta_0(4N)$. For $C \in GL_2^+(\mathbb{R})$ we denote by $\Gamma_0(4N)_C = \{D \in \Gamma_0(4N) | D^{-1}CD = C\}$ the stabilizer of C in $\Gamma_0(4N)$, and for a cusp $x \in \mathbb{Q} \cup \{i\infty\}$ we write $\Delta_0(4N)_{(x)} = \{D^* \in \Delta_0(4N) | Dx = x\}$.

Two elements C and C' in $\Gamma_0(4N)C_n\Gamma_0(4N)$ are equivalent if one of the following conditions is satisfied:

(1) There exists $D \in \Gamma_0(4N)$ with $C' = D^{-1}CD$;

(2) C, C' are parabolic and there exists $D \in \Gamma_0(4N)$ and $D' \in \Gamma_0(4N)_C$ with $C' = D^{-1}D'CD$.

Then according to S. Niwa, 1977 we have

$$\frac{1}{6}(-1)^{[(k+1)/2]}\sqrt{2}\mathrm{tr}\left(\mathrm{T}_{k+1/2,4N}(n^2), S\left(4N, k+1/2, \left(\frac{4}{\cdot}\right)\right)\right) = c_{n,k}\left(\sum_{\widetilde{C}} I(C) + r\right),$$

where the summation extends over all classes $\widetilde{C}$ in $\Gamma_0(4N)C_n\Gamma_0(4N)$ modulo the equivalence relation defined above and the complex number $I(C)$ is dependent only on the class of C and is given as follows:

(1) If C is scalar, $C^* = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \eta \right\}$, one has

$$I(C) = (k - 1/2)\frac{1}{4\pi\eta} \int\limits_{\Gamma_0(4N)\backslash\mathbb{H}} y^{-2}\mathrm{d}x\mathrm{d}y.$$

(2) If C is elliptic, $C^* = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, t_C(cz+d)^{1/2} \right\}$, one has

$$I(C) = (\sigma_C t_C \rho^{k-1/2}(\rho - \overline{\rho}))^{-1},$$

where ρ and $\overline{\rho}$ are the eigenvalues of C with $\mathrm{sgn}(\mathrm{Im}(\rho)) = \mathrm{sgn}(c)$ and σ_C is the order of $\Gamma_0(4N)_C$.

(3) If C is hyperbolic and its fixed points are not cusps of $\Gamma_0(4N)$, $I(C) = 0$.

(4) If C is hyperbolic and its fixed points are cusps of $\Gamma_0(4N)$, let $\widehat{G}$ be the group consisting of pairs $(A, \phi(z))$, where $A \in GL_2^+(\mathbb{R})$ and $\phi(z)$ is a complex-valued holomorphic function on the upper half-plane satisfying $|\phi(z)| = (\det(A))^{-1/4}|cz + d|^{1/2}$ (please see Section 4.1), choose $\delta \in S\widehat{G}$ (the subset of $\widehat{G}$ of elements whose first components have determinant 1) such that $\delta^{-1}C^*\delta = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda' \end{pmatrix}, \eta \right\}$ with $|\lambda'| < |\lambda|$. Then

$$I(C) = \frac{1}{2}\left(\eta\left(\frac{\lambda'}{\lambda} - 1\right)\right)^{-1}.$$

(5) If C is parabolic with fixed point $x \in \mathbb{Q} \cup \{\mathrm{i}\infty\}$, there exists $\delta \in S\widehat{G}$, $X \in \mathbb{R}$, $\mu \in \mathbb{C}$ and $\rho \in \{\pm 1\}$ such that $\delta\left\{\rho\begin{pmatrix}1 & X\\ 0 & 1\end{pmatrix}, \mu\right\}\delta^{-1}$ generates $\Delta_0(4N)_{(x)}/\{(\pm 1, 1)\}$. If $\delta^{-1}C^*\delta = \left\{\begin{pmatrix}a & auX\\ 0 & 1a\end{pmatrix}, \eta\right\}$ and $\mu = \mathrm{e}^{-2\pi\mathrm{i}\alpha}$ $(0 \leqslant \alpha < 1)$, one has

$$I(C) = \begin{cases} -\dfrac{1}{2\eta}\mathrm{e}^{-2\pi\mathrm{i}u\alpha}(1-2\alpha), & \text{if } u \in \mathbb{Z}\ , \\ -\dfrac{1}{2\eta}\mathrm{e}^{-2\pi\mathrm{i}u\alpha}(1-i\cot(\pi u)), & \text{if } u \notin \mathbb{Z}\ . \end{cases}$$

Finally

$$r = \begin{cases} 0, & \text{if } k > 1, \\ \operatorname{tr}\left([\Delta_0(4N)(C_n^{-1}, n^{-1/2})\Delta_0(4N)], G(4N, 1/2, \mathrm{id.})\right), & \text{if } k = 1, \end{cases}$$

where $G(4N, 1/2, \mathrm{id.})$ is the space of modular forms of weight $1/2$ on $\Gamma_0(4N)$.

Therefore we can write

$$c_{n,k}\left(\sum_{\widetilde{C}} I(C) + r\right) = A_1 + A_2 + A_3 + A_4 + A_5,$$

where A_1, A_2, A_3 and A_4 is $c_{n,k}$ times the contribution from the scalar, elliptic, hyperbolic and parabolic elements, respectively, and $A_5 = c_{n,k}r$.

Since the upper right entry of a matrix in $\Gamma_0(4N)C_n\Gamma_0(4N)$ is always odd, $\Gamma_0(4N) \cdot C_n\Gamma_0(4N)$ contains no scalar matrices and so $A_1 = 0$.

Computation of A_2 For an elliptic matrix $C = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma_0(4N)C_n\Gamma_0(4N)$ put $C' = \begin{pmatrix}a & -b\\ -c & d\end{pmatrix}$ and $J(C) = I(C) + I(C')$. Then

$$A_2 = c_{n,k}\sum_{\widetilde{C}} J(C),$$

where the summation extends over those elliptic $\Gamma_0(4N)-$conjugacy classes $\widetilde{C}$ for which the lower left entry of C is positive. Now the fact that $C \in \Gamma_0(4N)C_n\Gamma_0(4N)$ implies $t \equiv 0(\mathrm{mod}\ 4)$ and f odd. Since C is primitive, we have $(f, n) = 1$. Therefore we may write

$$A_2 = c_{n,k}\sum_t{}^{(1)}\sum_f{}^{(2)}\sum_A{}^{(3)}\sum_B{}^{(4)} J(B^{-1}AB), \tag{9.68}$$

where $\sum^{(1)}$ extends over $t \in \mathbb{Z}$ with $|t| < 8n$, $t \equiv 0 \pmod 4$ and where $\sum^{(2)}$ extends over $f \in \mathbb{N}$ with $f^2|(t^2 - 64n^2)$ and $(f, 2n) = 1$; for t and f fixed A in $\sum^{(3)}$ runs over a set of representatives of $\Gamma(1)$-conjugacy classes of primitive matrices in $GL_2(\mathbb{Z})$ with determinant $16n^2$, invariants t and f and lower left entry positive, and B in $\sum^{(4)}$ runs over those elements in $\Gamma(1)/\Gamma_0(4N)$ for which $B^{-1}AB \in \Gamma_0(4N)C_n\Gamma_0(4N)$.

Now we want to compute $J(C)$ for a matrix $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c > 0$ representing a class in $\Gamma_0(4N)C_n\Gamma_0(4N)$. By Lemma 5.21 resp. Lemma 5.22 we may suppose that

$$d > 0, \quad (b,d) = 1, \quad \left(\frac{b}{f}, \frac{t^2 - 64n^2}{f^2}\right) = 1 \tag{9.69}$$

and that

$$C^* = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (4n)^{-1/2}\left(\frac{d}{b}\right)\left(\frac{-4}{b}\right)^{-1/2}(cz+d)^{1/2} \right\}.$$

Put

$$\rho = \frac{t + \mathrm{i}\sqrt{64n^2 - t^2}}{2}.$$

Then it is easily checked that

$$J(C) = c_{n,k}^{-1}\frac{1}{3}n^{2k-1}2^{2k}\sigma_C^{-1}\left(\frac{d}{b}\right)\left(1 - \left(\frac{-4}{b}\right)(-1)^k\mathrm{i}\right)\frac{\rho^{1/2-k} - \left(\frac{-4}{b}\right)(-1)^k\mathrm{i}\overline{\rho}^{1/2-k}}{\rho - \overline{\rho}}. \tag{9.70}$$

Since $(f, t+8n, t-8n) = 1$ we can decompose f as $f = f_1 f_2$ with $f_1^2|(t+8n)$, $f_2^2|(t-8n)$ and $(t+8n, f_2) = (t-8n, f_1) = 1$. Then noticing $d > 0, b$ odd and $(d, b/f) = ((t+8n)/f_1^2, b/f) = 1$ we find

$$\left(\frac{d}{b/f}\right)\left(\frac{(t+8n)/f_1^2}{b/f}\right) = \left(\frac{(4n+d)/f_1^2}{b/f}\right) = 1,$$

i.e.,

$$\left(\frac{d}{b/f}\right) = \left(\frac{(t+8n)/f_1^2}{b/f}\right).$$

Similarly

$$\left(\frac{d}{b/f}\right) = \left(\frac{(t-8n)/f_2^2}{b/f}\right)$$

and

$$\left(\frac{d}{f_1}\right) = \left(\frac{t-8n}{f_1}\right), \quad \left(\frac{d}{f_2}\right) = \left(\frac{t+8n}{f_2}\right).$$

Thus we see that

$$\left(\frac{d}{b}\right) = \left(\frac{t-8n}{f_1}\right)\left(\frac{t+8n}{f_2}\right)\left(\frac{(t+8n)/f_1^2}{b/f}\right)$$
$$= \left(\frac{t-8n}{f_1}\right)\left(\frac{t+8n}{f_2}\right)\left(\frac{(t-8n)/f_2^2}{b/f}\right).$$

Finally put

$$\lambda_{t,n} = \frac{\sqrt{8n-t}+\mathrm{i}\sqrt{8n+t}}{4}, \quad p_{k+1/2}(t,n) = \frac{\lambda_{t,n}^{2k-1} - \overline{\lambda}_{t,n}^{2k-1}}{\lambda_{t,n} - \overline{\lambda}_{t,n}}.$$

Then $\lambda_{t,n}^2 = -\overline{\rho}/4$, $\lambda_{-t,n}^2 = \rho/4$ and

$$\left(1-\left(\frac{-4}{b}\right)(-1)^k\mathrm{i}\right)\frac{\rho^{1/2-k} - \left(\frac{-4}{b}\right)(-1)^k\mathrm{i}\overline{\rho}^{1/2-k}}{\rho-\overline{\rho}}$$
$$= -n^{2k-1}2^{-2k}\left(\left(\frac{-4}{b}\right)\frac{1}{\sqrt{8b-t}}p_{k+1/2}(t,n) + \frac{1}{\sqrt{8n+t}}p_{k+1/2}(-t,n)\right). \quad (9.71)$$

Substituting (9.70) and (9.71) into (9.69) and setting $\eta(C) = \left(\frac{(8n-t)/f_2^2}{b/f}\right)$, we obtain

$$J(C) = -\frac{c_{n,k}^{-1}}{3}\left(\left(\frac{8n-t}{f_1}\right)\left(\frac{-8n-t}{f_2}\right)\frac{1}{\sqrt{8n-t}}p_{k+1/2}(t,n)\sigma_C^{-1}\eta(C)\right.$$
$$\left.+\left(\frac{t-8n}{f_1}\right)\left(\frac{t+8n}{f_2}\right)\frac{1}{\sqrt{8n+t}}p_{k+1/2}(-t,n)\sigma_C^{-1}\eta(-C)\right).$$

If we assume that the matrices $B^{-1}AB$ in (9.68) satisfy the conditions (9.69), substitute the expression in terms of N, n, t and f found for $J(B^{-1}AB)$ into (9.68) and observe that $\sigma_{B^{-1}AB}$ equals the order ω_A of the $\Gamma(1)$-stabilizer of A, we see that

$$A_2 = -\frac{2}{3}\left(\sum_t{}^{(1)}\sum_f{}^{(2)}\left(\frac{8n-t}{f_1}\right)\left(\frac{-8n-t}{f_2}\right)\frac{1}{\sqrt{8n-t}}p_{k+1/2}(t,n)\right.$$
$$\left.\times\sum_A{}^{(3)}\omega_A^{-1}\sum_B{}^{(4)}\eta(B^{-1}AB)\right).$$

Now recall that the association

$$A \longmapsto F_A(X,Y) = \frac{1}{f}(-bX^2+(a-d)XY+cY^2), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a bijective correspondence between the set $\mathfrak{B}_{t,f,n}$ of integral elliptic primitive matrices with determinant $16n^2$, invariants t and f and lower left entry positive and the set of positive definitive primitive integral binary quadratic forms of discriminant $\dfrac{t^2-64n^2}{f^2}$ with $(f,2n)=1$, and that conjugation by $\Gamma(1)$ corresponds to the usual action of $\Gamma(1)$ on quadratic forms.

For $A \in \mathfrak{B}_{t,f,n}$ denote by $\mathfrak{a}_A$ the ideal $bf^{-1}\mathbb{Z} \oplus \dfrac{(a-d)f^{-1}+\sqrt{(t^2-64n^2)f^{-2}}}{2}\mathbb{Z}$ corresponding to F_A and contained in the order $\mathfrak{O}$ of discriminant $(t^2-64n^2)/f^2$ of the imaginary quadratic field $\mathbb{Q}(\sqrt{(t^2-64n^2)f^{-2}})$. Then the norm $\mathfrak{N}(\mathfrak{a}_A)$ of $\mathfrak{a}_A$ equals $-b/f$, hence

$$\eta(B^{-1}AB)=\left(\frac{(8n-t)/f_2^2}{b/f}\right)=\left(\frac{(8n-t)/f_2^2}{\mathfrak{N}(\mathfrak{a}_A)}\right)=\psi(\mathfrak{a}_A),$$

where $\psi=\psi_{(8n-t)/f_2^2}$ is the genus character of the ideal class group of $\mathfrak{O}$ corresponding to the decomposition $\dfrac{t^2-64n^2}{f^2}=\dfrac{8n-t}{f_2^2}\dfrac{-8n-t}{f_1^2}$ of $\dfrac{t^2-64n^2}{f^2}$ into a product of two discriminants.

If $(8n-t)/f_2^2$ is not a perfect square, then ψ is a non-trivial character, and for any set of representatives $\{A_\nu\}$ for $\Gamma(1)\backslash\mathfrak{B}_{t,f,n}$ we have

$$\sum_{A_\nu}{}^{(3)}\omega_{A_\nu}^{-1}\psi(\mathfrak{a}_{A_\nu})=0,$$

hence also

$$\sum_{A}{}^{(3)}\omega_A^{-1}\sum_{B}{}^{(4)}\eta(B^{-1}AB)=0,$$

since the number of $B\in\Gamma(1)/\Gamma_0(4N)$ such that $B^{-1}AB\in\Gamma_0(4N)C_n\Gamma_0(4N)$ depends only on N, n, t and f (cf. Lemma 5.23).

On the other hand, if $(8n-t)/f_2^2$ is a perfect square, then $\eta(B^{-1}AB)=1$, so we obtain

$$\sum_{A}{}^{(3)}\omega_A^{-1}\sum_{B}{}^{(4)}\eta(B^{-1}AB)=\frac{1}{2}\mu\left(\frac{t}{4},f,n^2,N\right)h'\left(\frac{t^2-64n^2}{f^2}\right),$$

where μ is defined as in Lemma 5.23 and $h'(m)$ is the number of $\Gamma(1)$-equivalence classes of positive definite primitive integral binary quadratic forms of discriminant m and a form equivalent to X^2+Y^2 (resp.X^2+XY+Y^2) is counted with multiplicity 1/2 (resp. 1/3).

Now if $8n-t=4s^2$ with $s\in\mathbb{N}$, then

$$8n+t=4(4n-s^2),\quad t^2-64n^2=16s^2(s^2-4n)$$

and the condition $|t|<8n$ is equivalent to $|s|<2\sqrt{n}$, and $f^2|(t^2-64n^2)$ means $f_1^2|(s^2-4n)$, $f_2|s$. Put

$$p_{2k}(s,n)=\frac{\lambda^{2k-1}-\overline{\lambda}^{2k-1}}{\lambda-\overline{\lambda}}$$

with λ, $\overline{\lambda}$ the solutions of $X^2-sX+n=0$ we may therefore rewrite A_2 as

$$A_2=-\frac{1}{3}\cdot\frac{1}{2s}\sum_{\substack{|s|<2\sqrt{n},\\ s>0}}p_{2k}(s,n)\sum_{\substack{f_1^2|s^2-4n,\\ (f_1,2n)=1}}\sum_{f_2|s}\mu(2n-s^2,f_1f_2,n^2,N)$$
$$\times\left(\frac{s^2-4n}{f_2}\right)h'\left(16\frac{s^2}{f_2^2}\frac{s^2-4n}{f_1^2}\right). \tag{9.72}$$

Now we want to compute the sum over f_2 in (9.72). We claim that

$$\frac{1}{2s}\sum_{f_2|s}\mu(2n-s^2,f_1f_2,n^2,N)\left(\frac{s^2-4n}{f_2}\right)h'\left(16\frac{s^2}{f_2^2}\frac{s^2-4n}{f_1^2}\right)$$
$$=\left(2+\left(\frac{4}{s}\right)\right)\mu(s,f_1,n,N)h'((s^2-4n)/f_1^2). \tag{9.73}$$

We shall give the proof of (9.73) only for the case $N=1$ and $N=l$ an odd prime. (The general case is of course similar.)

Using the formula

$$h'(Dm^2)=m\prod_{p|m}\left(1-\left(\frac{D}{p}\right)p^{-1}\right)h'(D), \tag{9.74}$$

where D is a fundamental discriminant and $m\in\mathbb{N}$, one first checks that (please compare with S. Niwa, 1977)

$$\frac{1}{2s}\sum_{f_2|s}\left(\frac{s^2-4n}{f_2}\right)h'\left(16\frac{s^2}{f_2^2}\frac{s^2-4n}{f_1^2}\right)=\left(2+\left(\frac{4}{s}\right)\right)h'((s^2-4n)/f_1^2), \tag{9.75}$$

which proves (9.73) for $N=1$.

Next consider $N=l$. Write $s=l^\gamma S$ with $l\nmid S$ and $\gamma\geqslant 0$. If $\gamma=0$, then (9.73) again follows immediately from (9.75). If $\gamma\geqslant 1$, then $l\nmid(s^2-4n)$, hence $l\nmid f_1$, and by definition of μ, the left-hand side of (9.73) equals

$$\frac{1}{2l^\gamma S}\left(\sum_{f_2|S}\left(\frac{S^2-4n}{f_2}\right)h'\left(16l^{2\gamma}\frac{S^2}{f_2^2}\frac{s^2-4n}{f_1^2}\right)\right.$$
$$\left.+(l+1)\sum_{1\leqslant\nu\leqslant\gamma}\left(\frac{s^2-4n}{l^\nu}\right)\sum_{f_2|S}\left(\frac{s^2-4n}{f_2}\right)h'\left(16l^{2\gamma-2\nu}\frac{S^2}{f_2^2}\frac{s^2-4n}{f_1^2}\right)\right).$$

By (9.74) we have for $0<\nu\leqslant\gamma$

$$h'\left(16l^{2\gamma-2\nu}\frac{S^2}{f_2^2}\frac{s^2-4n}{f_1^2}\right)=l^{\gamma-\nu}\left(1-\left(\frac{s^2-4n}{l}\right)l^{-1}\right)h'\left(16\frac{S^2}{f_2^2}\frac{s^2-4n}{f_1^2}\right).$$

Since

$$l^{-\gamma}\Bigg(l^{\gamma}\left(1-\left(\frac{s^2-4n}{l}\right)l^{-1}\right)+(l+1)\left(1-\left(\frac{s^2-4n}{l}\right)l^{-1}\right)$$
$$\times\sum_{1\leqslant\nu\leqslant\gamma-1}\left(\frac{s^2-4n}{l^{\nu}}\right)l^{\gamma-\nu}+(l+1)\left(\frac{s^2-4n}{l^{\gamma}}\right)\Bigg)$$
$$=1+\left(\frac{s^2-4n}{l}\right)$$

and

$$\frac{1}{2S}\sum_{f_2|S}\left(\frac{s^2-4n}{f_2}\right)h'\left(16\frac{S^2}{f_2^2}\frac{s^2-4n}{f_1^2}\right)=\left(2+\left(\frac{4}{s}\right)\right)h'((s^2-4n)/f_1^2),$$

we see that the left-hand side of (9.73) is

$$\left(2+\left(\frac{4}{s}\right)\right)\left(1+\left(\frac{(s^2-4n)/f_1^2}{l}\right)\right)h'((s^2-4n)/f_1^2)$$

as was to be proved.

Substituting now (9.73) into (9.72) we find

$$A_2=\frac{1}{3}\left(2+\left(\frac{4}{s}\right)\right)\sum_{|s|<2\sqrt{n},s>0}p_{2k}(s,n)$$
$$\times\sum_{\substack{f_1^2|s^2-4n,\\(f_1,2n)=1}}h'((s^2-4n)/f_1^2)\mu(s,f_1,n,N). \tag{9.76}$$

Computation of A_3 For a hyperbolic matrix $C=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(4N)C_n\Gamma_0(4N)$, put $C'=\begin{pmatrix}a&-b\\-c&d\end{pmatrix}$ and $J(C)=I(C)+I(C')$. If C runs through a set of representatives of hyperbolic $\Gamma_0(4N)$-conjugacy classes in $\Gamma_0(4N)C_n\Gamma_0(4N)$, whose fixed points are cusps of $\Gamma_0(4N)$, then so does C'. Noticing $I(C)=I(-C)$ we therefore have

$$A_3=c_{n,k}\sum_{\widetilde{C}}J(C),$$

where the sum is over all hyperbolic $\Gamma_0(4N)$-conjugacy classes $\widetilde{C}$ such that C fixes a cusp of $\Gamma_0(4N)$ and $t>0$.

Now

$$\left\{\begin{pmatrix}\nu'&\tau\\0&\nu\end{pmatrix}\in M_2(\mathbb{Z})\,\middle|\,\nu\nu'=16n^2,0<\nu'<\nu,0\leqslant\tau<\nu-\nu'\right\}$$

is a set of representatives of $\Gamma(1)$-conjugacy classes of integral hyperbolic matrices with determinant $16n^2$ and positive trace, whose fixed points are cusps of $\Gamma(1)$. Hence

$$A_3 = c_{n,k} \sum_{\nu,\nu',\tau}{}^{(1)} \sum_B J\left(B^{-1}\begin{pmatrix}\nu' & \tau \\ 0 & \nu\end{pmatrix} B\right),$$

where ν, ν', τ in $\sum^{(1)}$ run over integers satisfying $\nu\nu' = 16n^2$, $0 < \nu' < \nu$, $0 \leqslant \tau < \nu - \nu'$, $(\nu - \nu', \tau, 2n) = 1$ and for ν, ν' and τ fixed, B runs over a set of representatives of $\Gamma(1)/\mathit{\Gamma}_0(4N)$ such that $B^{-1}\begin{pmatrix}\nu' & \tau \\ 0 & \nu\end{pmatrix} B$ is in $\mathit{\Gamma}_0(4N)C_n\mathit{\Gamma}_0(4N)$.

Let $C = \begin{pmatrix}a & b \\ c & d\end{pmatrix} \in \mathit{\Gamma}_0(4N)C_n\mathit{\Gamma}_0(4N)$ be a hyperbolic matrix with $d > 0$, $(b,d) = 1$ and $(b/f, (t^2 - 64n^2)/f^2) = 1$ (cf. Lemma 5.22). Suppose that $C = B^{-1}\begin{pmatrix}\nu' & \tau \\ 0 & \nu\end{pmatrix} B$ with $B \in \Gamma(1)$. For $B = \begin{pmatrix}* & * \\ v & w\end{pmatrix}$ put

$$\delta = \left\{\begin{pmatrix}1 & -\tau/(\nu-\nu') \\ 0 & 1\end{pmatrix}, 1\right\}\{B, (vz+w)^{1/2}\}.$$

Then by Lemma 5.21

$$\delta C^* \delta^{-1} = \left\{\begin{pmatrix}\nu' & 0 \\ 0 & \nu\end{pmatrix}, (4n)^{-1/2}\nu^{1/2}\left(\frac{-4}{b}\right)^{-1/2}\left(\frac{d}{b}\right)\right\}.$$

From this one can easily see that

$$J(C) = -c_{n,k}^{-1}\frac{2^{-2k+2}}{3}\frac{\nu'^{k-1/2}}{\nu - \nu'}\left(\frac{d}{b}\right).$$

Arguing now as in the elliptic case we find that

$$\sum_{\nu,\nu',\tau}{}^{(1)} \sum_B J\left(B^{-1}\begin{pmatrix}\nu' & \tau \\ 0 & \nu\end{pmatrix} B\right) = 0$$

unless $\nu + \nu' + 8n$ is a perfect square, and that in the latter case

$$\sum_B J\left(B^{-1}\begin{pmatrix}\nu' & \tau \\ 0 & \nu\end{pmatrix} B\right) = \mu((\nu+\nu')/4, (\nu-\nu',\tau), n^2, N). \tag{9.77}$$

Since for a divisor b of $\nu - \nu'$ there are $\phi((\nu-\nu')/b)$ different values of τ satisfying $0 \leqslant \tau < \nu - \nu'$ and $b = (\nu - \nu', \tau)$, and since $(\nu - \nu', \tau, 2n) = 1$, we can rewrite (9.77) as

$$\sum_B J\left(B^{-1}\begin{pmatrix}\nu' & \tau \\ 0 & \nu\end{pmatrix} B\right) = \sum_{\substack{b\mid\nu-\nu',\\ (b,2n)=1}} \phi((\nu-\nu')/b)\mu((\nu+\nu')/4, b, n^2, N).$$

Now if $\nu+\nu'+8n=4s^2$ and $\nu-\nu'-8n=4r^2$ with $r,\, s\in\mathbb{N}$, define $\lambda,\lambda'\in\mathbb{Z}$ by $s=\lambda+\lambda'$, $r=\lambda-\lambda'$. Then $\nu'=4\lambda'^2$, $\nu=4\lambda^2$ and the conditions $\nu\nu'=16n^2$ resp. $0<\nu'<\nu$ are equivalent to $\lambda\lambda'=n$ resp. $0<\lambda'<\lambda$. Thus we have

$$A_3=-\frac{1}{6}\sum_{\substack{0<\lambda'<\lambda,\\ \lambda'\lambda=n}}\frac{\lambda'^{2k-1}}{\lambda^2-\lambda'^2}\sum_{\substack{b|\lambda^2-\lambda'^2,\\ (b,2n)=1}}\phi(4(\lambda^2-\lambda'^2)/b)\mu(\lambda^2+\lambda'^2,b,n^2,N). \tag{9.78}$$

Now one can easily check the formula for $0<\lambda'<\lambda, \lambda\lambda'=n$

$$\frac{1}{2(\lambda^2-\lambda'^2)}\sum_{\substack{b|(\lambda^2-\lambda'^2),\\ (b,2n)=1}}\phi(4(\lambda^2-\lambda'^2)/b)\mu(\lambda^2+\lambda'^2,b,n^2,N)=\frac{\phi((\lambda-\lambda',n))}{(\lambda-\lambda',n)}\sigma_0(N)$$

here $\sigma_0(N)$ denotes the number of positive divisors of N. Substituting the above identity into (9.78) we obtain

$$A_3=-\frac{\sigma_0(N)}{3}\sum_{\substack{0<\lambda'<\lambda,\\ \lambda\lambda'=n}}\frac{\lambda'^{2k-1}\phi((\lambda-\lambda',n))}{(\lambda-\lambda',n)}. \tag{9.79}$$

Computation of A_4 Since N is odd and square-free, the cusp of $\Gamma_0(4N)$ are represented by the numbers $1/t$, where t runs over all positive divisors of $4N$. For such a t put $A_t=\begin{pmatrix}1+t & -1\\ -t & 1\end{pmatrix}$ and $\delta_t=\{A_t,(-tz+1)^{1/2}\}$. Then the stabilizer of $1/t$ in $\Delta_0(4N)/\{\pm1,1\}$ is generated by

$$\begin{aligned}&\delta_t^{-1}\left\{\begin{pmatrix}1 & T\\ 0 & 1\end{pmatrix},\left(\frac{-t^2T}{1+tT}\right)\left(\frac{-4}{1+tT}\right)^{-1/2}\right\}\delta_t\\ &=\left\{\begin{pmatrix}1-tT & T\\ -t^2T & 1+tT\end{pmatrix},\left(\frac{-t^2T}{1+tT}\right)\left(\frac{-4}{1+tT}\right)^{-1/2}(-t^2Tz+1+tT)^{1/2}\right\},\end{aligned}$$

where T is the least natural number such that $t^2T\equiv 0 \pmod{4N}$.

Let $C\in GL_2(\mathbb{Z})$ be a parabolic matrix with $\det C=16n^2$ and with fixed point $1/t$. Then

$$C=A_t^{-1}\begin{pmatrix}4n & \tau\\ 0 & 4n\end{pmatrix}A_t=\begin{pmatrix}4n-t\tau & \tau\\ -t^2\tau & 4n+t\tau\end{pmatrix}$$

with some $\tau\in\mathbb{Z}$. Using Lemma 5.20 one can easily see that C is in $\Gamma_0(4N)C_n\Gamma_0(4N)$ if and only if $t\equiv 0(\bmod 4), T=4N/t$ and $\tau=4Nv/t$ with some $v\in\mathbb{Z},(v,4N)=1$. Thus a set of representatives of parabolic matrices in $\Gamma_0(4N)C_n\Gamma_0(4N)$ for the equivalence relation defined above is formed by the matrices

$$C_{v,t}=\begin{pmatrix}4n-4Nv & Nv/t\\ -16Nvt & 4n+4Nv\end{pmatrix},$$

where t runs over all positive divisors of N and v runs over a reduced residue system mod $4n$, and we have $u = u_{v,t} = v/4n$ and $\alpha = \alpha_{v,t} = 0$.

Now note that $(Nv/t, 4n + 4Nv) = 1$. Hence if we assume $v > 0$ one sees from Lemma 5.21 that

$$\eta = \eta_{v,t} = \left(\frac{4n}{Nv/t}\right)\left(\frac{-4}{Nv/t}\right)^{-k-1/2}$$

and if we suppose $4n - 4Nv < 0$ we have

$$\eta_{-v,-t} = \eta_{v,t}.$$

Since $u_{v,t} = u_{-v,-t}$ and cot is an odd function it follows

$$I(C_{v,t}) + I(C_{-v,-t}) = -\frac{1}{\eta_{v,t}},$$

hence

$$A_4 = \frac{c_{n,k}}{2}\sum_{v,t}(I(C_{v,t}) + I(C_{-v,-t})) = -\frac{1}{2}\sum_{v,t}\frac{1}{\eta_{v,t}}.$$

If we replace t with N/t and substitute the value of $\eta_{v,t}$ we find

$$A_4 = -\frac{1}{12}n^{k-3/2}(1-(-1)^k\mathrm{i})\sum_{t|N}\left(\frac{4n}{t}\right)\sum_{v \bmod 4n}\left(\frac{4n}{v}\right)\left(\frac{-4}{vt}\right)^{k+1/2}.$$

But the sum over v equals

$$\delta(\sqrt{n})(1+(-1)^k\mathrm{i})\phi(n),$$

where $\delta(x)$ is 1 or 0 according as $x \in \mathbb{Z}$ or not. Thus

$$A_4 = -\frac{1}{6}n^{k-3/2}\delta(\sqrt{n})\phi(n)\sigma_0(N).$$

Using $\phi(m^2) = m\phi(m)($ for $m \in \mathbb{Z})$ we obtain the final formula

$$A_4 = -\frac{1}{6}n^{k-1}\phi(\sqrt{n})\sigma_0(N). \tag{9.80}$$

Computation of A_5 Let $k = 1$. Then

$$A_5 = c_{n,1}\mathrm{tr}\left([\Delta_0(4N)\{C_n^{-1}, n^{-1/2}\}\Delta_0(4N)], G(4N, 1/2, \mathrm{id.})\right).$$

We have by Lemma 9.12

$$[\Delta_0(4N)\{C_n^{-1}, n^{-1/2}\}\Delta_0(4N)] = \left[\Delta_0(4N)\left\{\begin{pmatrix} n^2 & 0 \\ 0 & 1\end{pmatrix}, n^{-1/2}\right\}\Delta_0(4N)\right]$$
$$\times\left[\Delta_0(4N)\left\{\begin{pmatrix} 4 & -1 \\ 0 & 4\end{pmatrix}, 1\right\}\Delta_0(4N)\right],$$

and the first factor on the right equals

$$\left[\Delta_0(4N)\left\{\begin{pmatrix}1&0\\0&n^2\end{pmatrix},n^{1/2}\right\}\Delta_0(4N)\right]=n^{3/2}\mathrm{T}_{1/2,4N}(n^2).$$

Consequently

$$A_5=\frac{n(1+\mathrm{i})}{6}\mathrm{tr}\left(\mathrm{T}_{1/2,4N}(n^2)\left[\Delta_0(4N)\left\{\begin{pmatrix}4&-1\\0&4\end{pmatrix},1\right\}\Delta_0(4N)\right],G(4N,1/2,\mathrm{id.})\right).$$

But by Section 7.2,

$$G(4N,1/2,\mathrm{id.})=\mathbb{C}\theta,$$

where $\theta(z)=\sum_{n\in\mathbb{Z}}q^{n^2}$ is the standard theta function. Noticing

$$n\theta|\mathrm{T}_{1/2,4N}(n^2)=\sigma_1(n)\theta\left|\left[\Delta_0(4N)\left\{\begin{pmatrix}4&-1\\0&4\end{pmatrix},1\right\}\Delta_0(4N)\right]=\theta\left(z-\frac{1}{4}\right)\right|\mathrm{Tr}$$

and

$$\theta\left(z-\frac{1}{4}\right)\Big|\mathrm{Tr}=2(1-\mathrm{i})\theta(z).$$

We conclude that

$$A_5=\frac{2}{3}\sigma_1(n).$$

Thus

$$A_5=\begin{cases}0, & \text{if } k>1,\\ \frac{2}{3}\sigma_1(n), & \text{if } k=1.\end{cases}\tag{9.81}$$

Summarizing we have proved up to now that

$$\frac{(-1)^{[(k+1)/2]}}{6}\sqrt{2}\mathrm{tr}(\mathrm{T}_{k+1/2,4N}(n^2)Q,S(4N,k+1/2,\mathrm{id.}))=A_2+A_3+A_4+A_5$$

with A_2,A_3,A_4 and A_5 given by (9.76), (9.79), (9.80) and (9.81), respectively. Now let us consider the second summand of (9.66). As Niwa proved that for $k\geqslant 2$

$$\mathrm{tr}(\mathrm{T}_{k+1/2,4N}(n^2),S(4N,k+1/2,\mathrm{id.}))=\mathrm{tr}(T_{2k,2N}(n),S(2N,2k,\mathrm{id.})).\tag{9.82}$$

Identity (9.82) is also correct for $k=1$, as one sees as above using $G(4N,1/2,\mathrm{id.})=\mathbb{C}\theta$.

Now we have from Theorem 9.5

$$\mathrm{tr}(T_{2k,2N}(n),S(2N,2k,\mathrm{id.}))=A_1'+A_2'+A_3'+A_4'+A_5'$$

with

$$A_1'=\delta\left(\frac{1}{n}\right)\frac{(2k-1)2N}{12}\prod_{p|2N}\left(1+\frac{1}{p}\right),\tag{9.83}$$

$$A_2' = -\frac{1}{2}\sum_{|s|<2\sqrt{n}} p_{2k}(s,n) \sum_{\substack{f^2|s^2-4n,\\ (s^2-4n)/f^2\equiv 0,1 \bmod 4,\\ (f,n)=1}} h'\left(\frac{s^2-4n}{f^2}\right)\mu(s,f,n,2N), \quad (9.84)$$

$$A_3' = -\sigma_0(2N)\sum_{0<\lambda'<\lambda,\lambda\lambda'=n} \frac{\lambda'^{2k-1}\phi((\lambda-\lambda',n))}{(\lambda-\lambda',n)}, \quad (9.85)$$

$$A_4' = -\frac{1}{2}n^{k-1}\phi(\sqrt{n})\sigma_0(2N) \quad (9.86)$$

and

$$A_5' = \begin{cases} 0, & \text{if } k>1, \\ \sigma_1(n), & \text{if } k=1. \end{cases} \quad (9.87)$$

Now $\frac{1}{3}A_1'$, the sum $A_2 + \frac{1}{3}A_2', A_3 + \frac{1}{3}A_3'$ and $A_4 + \frac{1}{3}A_4'$ are also given by the right hand side of (9.83)-(9.86), respectively, except that $2N$ has to be replaced by N. Furthermore

$$A_5 + \frac{1}{3}A_5' = \begin{cases} 0, & \text{if } k>1, \\ \sigma_1(n), & \text{if } k=1. \end{cases}$$

So we see again that

$$\sum_{2\leqslant\nu\leqslant 5} A_\nu + \frac{1}{3}\sum_{1\leqslant\nu\leqslant 5} A_\nu'$$

is exactly the trace of $T_{2k,N}(n)$ on $S(2N, 2k, \text{id.})$. Thus we have proved the theorem.

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Chapter 10

Integers Represented by Positive Definite Quadratic Forms

10.1 Theta Function of a Positive Definite Quadratic Form and Its Values at Cusp Points

In the first chapter we introduced the theta function of a positive definite quadratic form and discussed its transformation formula under the action of the modular group. We want now to show that the theta function is a modular form.

Let $f(x_1, \cdots, x_k)$ be a positive definite quadratic form with integral coefficients. Define the matrix A of $f(x_1, \cdots, x_k)$ as follows:

$$A = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right).$$

It is clear that A is a symmetric matrix with even diagonal entries. Put

$$\theta_f(z) = \sum_{m \in \mathbb{Z}^k} e(zmAm^{\mathrm{T}}/2), \quad z \in \mathbb{H}.$$

It is clear that $\theta_f(z)$ is a holomorphic function on $\mathbb{H}$. Let N be the level of $f(x_1, \cdots, x_k)$, i.e., the minimal positive integer N such that NA^{-1} is an integral matrix with even diagonal entries. Set

$$\chi = \begin{cases} \left(\dfrac{2 \det A}{\cdot} \right), & \text{if } k \text{ is odd}, \\ \left(\dfrac{(-1)^{k/2} \det A}{\cdot} \right), & \text{if } k \text{ is even}. \end{cases}$$

Theorem 10.1 *$\theta_f(z)$ is in $G(N, k/2, \chi)$.*

Proof By the results in Chapter 1 we need only to consider the behavior of $\theta_f(z)$ at the cusp points of $\Gamma_0(N)$. It is clear that

$$\lim_{z \to i\infty} \theta_f(z) = 1,$$

i.e., $\theta_f(z)$ is holomorphic at i∞. Let a/c be any cusp point with $c > 0$. Take

$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, then $\rho(\infty) = a/c$. We have that

$$\theta_f(z)\left(\frac{az+b}{cz+d}\right) = \sum_{x \bmod c} e(axAx^{\mathrm{T}}/2c) \sum_{m\in\mathbb{Z}^k} e(-(m+x/c)A(m+x/c)^{\mathrm{T}}/2(z+d/c)), \tag{10.1}$$

where $x \in \mathbb{Z}^k$. By the proof of Proposition 1.2 we see that

$$\sum_{m\in\mathbb{Z}^k} e(-(x+m)A(x+m)^{\mathrm{T}}/2z) = (-\mathrm{i}z)^{k/2}(\det A)^{-1/2} \sum_{m\in\mathbb{Z}^k} e(zmA^{-1}m^{\mathrm{T}}/2 + x\cdot m^{\mathrm{T}}),$$

where $x \in \mathbb{R}^k$. Replacing x by x/c in the above equality we get

$$\begin{aligned}\theta_f\left(\frac{az+b}{cz+d}\right) = (-\mathrm{i}(z+d/c))^{k/2}(\det A)^{-1/2} \sum_{m\in\mathbb{Z}^k} e(zmAm^{\mathrm{T}}/2)\\ \times \sum_{x \bmod c} e(axAx^{\mathrm{T}}/2c + x\cdot m^{\mathrm{T}}/c + dmA^{-1}m^t/2c),\end{aligned}$$

hence

$$\lim_{z\to i\infty} (z+d/c)^{-k/2}\theta_f\left(\frac{az+b}{cz+d}\right) = (-\mathrm{i})^{k/2}(\det A)^{-1/2} \sum_{x \bmod c} e(axAx^{\mathrm{T}}/2c), \tag{10.2}$$

i.e., $\theta_f(z)$ is holomorphic at the cusp point a/c. This completes the proof. □

Let $f_1 = f_1(x_1, \cdots, x_k)$ and $f_2 = f_2(x_1, \cdots, x_k)$ be two positive definite quadratic forms with integral coefficients, A_1 and A_2 the corresponding matrices of f_1 and f_2 respectively. f_1 and f_2 are called equivalent if there exists an integral matrix S with determinant ± 1 such that $SA_1S^{\mathrm{T}} = A_2$. f_1 and f_2 are called equivalent over the real field $\mathbb{R}$ if there exists a real invertible matrix S_r such that $S_rA_1S_r^{\mathrm{T}} = A_2$. Let p be a prime and take A_1, A_2 as matrices over the finite field $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. f_1 and f_2 are called equivalent over $\mathbb{F}_p$ if there exists an invertible matrix S_p on $\mathbb{F}_p$ such that $S_pA_1S_p^{\mathrm{T}} = A_2$. f_1 and f_2 are called in the same genus if f_1 and f_2 are equivalent over $\mathbb{R}$ and over $\mathbb{F}_p$ for any prime p. It is clear that f_1 and f_2 are in the same genus if they are equivalent. It can be proved that there are only finite equivalence classes in a genus.

Let $f = f(x_1, x_2, \cdots, x_k)$ be a positive definite quadratic form, and $f_1, f_2, \cdots, f_h$ be a full system of representations of all different classes in the genus of f. Let n be an arbitrary non-negative integer, and $r(f_i, n)$ denote the number of integral solutions of the equation $f_i(x) = n$. It is difficult to find an analytical expression for the number $r(f_i, n)$ in general cases.

Denote by $M_k(\mathbb{Z})$ the set of all $k \times k$ integral matrices. Put $O(f) = \#\{S \in M_k(\mathbb{Z}) | SAS^{\mathrm{T}} = A\}$, define the theta function θ of the genus of f:

$$\theta(\text{gen}.f, z) = \left(\sum_{i=1}^{h} \frac{1}{O(f_i)}\right)^{-1} \sum_{i=1}^{h} \frac{\theta_{f_i}(z)}{O(f_i)},$$

Then

$$\begin{aligned}\theta(\text{gen}.f, z) &= \sum_{n=0}^{\infty} r(\text{gen}.f, n)\exp\{2\pi i n z\} \\ &= \sum_{i=1}^{h} \left(\frac{1}{O(f_i)}\right)^{-1} \sum_{i=1}^{h} \sum_{n=0}^{\infty} \frac{r(f_i, n)}{O(f_i)} \exp\{2\pi i n z\},\end{aligned}$$

it follows that

$$r(\text{gen}.f, n) = \sum_{i=1}^{h} \left(\frac{1}{O(f_i)}\right)^{-1} \sum_{i=1}^{h} \frac{r(f_i, n)}{O(f_i)},$$

i.e., the number $r(\text{gen}.f, n)$ is a mean of the numbers $r(f_i, n)$, $(n \geqslant 0)$ when $k \geqslant 5$. This result is called Siegel theorem C.L.Siegel, 1966, which is equivalent to the fact that the function is an Eisenstein series of the weight $k/2$. A.N. Andrianov, 1980 obtained the same conclusion of Siegel theorem in the case of $k = 4$. Finally R. Schulze, 1984 reduced the same result of Siegel theorem in the case of $k = 3$. He proved that the function $\theta(\text{gen}.f, z)$ is an Eisenstein series of the weight $3/2$ when $k = 3$. Under certain conditions, if the function $\theta(\text{gen}.f, z)$ belongs to the space $\mathcal{E}(4D, 3/2, \chi_l)$ or $\mathcal{E}(8D, 3/2, \chi_l)$ then it can be represented as a linear combination of the basis functions for these spaces given in the Theorem 7.7 and Theorem 7.8 respectively. The coefficients of the linear combination can be determined using the values of the function $\theta(\text{gen}.f, z)$, thus an analytic expression for the number $r(\text{gen}.f, n)$ can be reduced in this way.

The Scholze-Pillot's Proof for Siegel theorem will be described below.

Let f_1 and f_2 be in the same genus. Then the corresponding matrices of f_1 and f_2 have the same determinant. If a/c is a cusp point with $c > 0$, then there exists an integral matrix S such that $(\det S, 2c) = 1$ and $SA_1S^{\mathrm{T}} \equiv A_2 \pmod{2c}$ by the above definitions and the Chinese remainder theorem. This shows that $\theta_{f_1}(z)$ and $\theta_{f_2}(z)$ have the same value at the cusp point a/c by (10.2). Hence $\theta_{f_1}(z) - \theta_{f_2}(z)$ is a cusp from.

Theorem 10.2 *Let p be a prime, $p \nmid N$. Set*

$$\lambda_p = \begin{cases} p^{k-2} + 1, & \textit{if } 2 \nmid k, \\ p^{k-2} + 2p^{k/2-1} \left(\dfrac{(-1)^{k/2} \det A}{p}\right) + 1, & \textit{if } 2 | k. \end{cases}$$

Then

$$\theta(\text{gen}.f, z)|\mathrm{T}(p^2) = \lambda_p \theta(\text{gen}.f, z),$$

where $\mathrm{T}(p^2)$ *is the* Hecke *operator on the space* $G(N, k/2, \chi)$.

Proof Please see R. Schulze, 1984 and P. Ponomarev, 1981. □

Theorem 10.3 *The function* $\theta(\text{gen}.f, z)$ *is in the space* $\mathcal{E}(N, k/2, \chi)$.

Proof We assume first that $k \geqslant 4$ is an even. Since

$$G(N, k/2, \chi) = \mathcal{E}(N, k/2, \chi) \oplus S(N, k/2, \chi),$$

there exist two functions $g_1(z)$ and $g_2(z)$ such that

$$\theta(\text{gen}.f, z) = g_1(z) + g_2(z), \quad g_1(z) \in S(N, k/2, \chi), g_2(z) \in \mathcal{E}(N, k/2, \chi).$$

Let $g_1(z) = \sum_{n=n_0}^{\infty} c(n)e(nz), c(n_0) \neq 0$. For any $p \nmid N$, by Theorem 10.2, we see that $g_1(z)|\mathrm{T}(p^2) = \lambda_p g_1(z)$, and hence

$$\lambda_p c(n_0) = c(n_0 p^2) + \chi(p)\left(\frac{-n_0}{p}\right)a(n_0).$$

By Lemma 7.24 we have that $c(n) = O(n^{k/4})$, so $\lambda_p = O(p^{k/2})$. If $k \geqslant 6$, we see that $\lambda_p \sim p^{k-2}$ $(p \to \infty)$ which contradicts $\lambda_p = O(p^{k/2})$. Hence we have $g_1(z) = 0$, which shows the theorem. If $k = 4$, we can prove the theorem similarly in terms of a more precise estimation $c(n) = O(n^{k/4-1/5})$ proved by R.A. Rankin, 1939. This shows the theorem for $k \geqslant 4$ even.

Now assume that k is an odd. For $k \geqslant 5$ we can prove the theorem by a similar method as for the case $k \geqslant 6$ an even. Now let $k = 3$ and $V := S(N, 3/2, \chi) \bigcap \widetilde{T}$ be as in Theorem 8.2. Denote by $V^{\perp}$ the orthogonal complement of V in $S(N, 3/2, \chi)$. Then we have

$$\theta(\text{gen}.f, z) = g_1 + g_2 + g_3, \quad g_1 \in V, \quad g_2 \in V^{\perp}, \quad g_3 \in \mathcal{E}(N, 3/2, \chi).$$

By Theorem 10.2 we see that $g_i|\mathrm{T}(p^2) = (p+1)g_i$ for any $p \nmid N$ and $i = 1, 2, 3$. But by the definition of $\widetilde{T}$ we know that g_1 is a finite linear combination of functions $h(tz; \psi)$ with $\chi = \psi\left(\frac{-t}{\cdot}\right)$. Hence we have

$$h(tz; \psi)|\mathrm{T}(p^2) = \chi(p)\left(\frac{-t}{p}\right)(p+1)h(tz; \psi).$$

There must be a prime p such that

$$h(tz; \psi)|\mathrm{T}(p^2) = -(p+1)h(tz; \psi)$$

holds for all finite functions $h(tz;\psi)$, so that $g_1(z)|\mathrm{T}(p^2) = -(p+1)g_1$, from which we get $g_1 = 0$ since we have also $g_1(z)|\mathrm{T}(p^2) = (p+1)g_1$. $g_2(z)$ is mapped in $S(N/2, 2, \mathrm{id.})$ under the Shimura lifting S and the image $S(g_2)$ of g_2 is also an eigenfunction of $\mathrm{T}(p)$ with eigenvalue $p+1$. In terms of Rankin's estimation $c(n) = O(n^{4/5})$ we can show that $g_2 = 0$. Therefore $\theta(\mathrm{gen}.f, z) \in \mathcal{E}(N, 3/2, \chi)$. This completes the proof. □

Let $f(x_1, x_2, \cdots, x_k)$ be a positive definite quadratic form with integral coefficients. Put

$$\theta_f(z) = \sum_{m \in \mathbb{Z}^k} e(zmAm^{\mathrm{T}}/2), \quad z \in \mathbb{H},$$

$$O(f) = \#\{S \in M_k(\mathbb{Z}) | SAS^{\mathrm{T}} = A\},$$

$$\theta(\mathrm{gen}.f, z) = \left(\sum_{f_i} \frac{1}{O(f_i)} \right)^{-1} \sum_{f_i} \frac{\theta_{f_i}(z)}{O(f_i)},$$

where the f_i run over a complete set of representatives of the equivalence classes in the genus of f.

Suppose that N is the level of f, i.e.,

$$N = \min\{N | NA^{-1} \text{ is integral and the diagonal entries are even, } N \text{ positive integer}\}.$$

Let now $S(N)$ denote a complete set of representatives of equivalence classes of cusp points for the group $\varGamma_0(N)$. In fact we can choose $S(N) = \{d/c \,|\, c|N, d \in (\mathbb{Z}/(c, N/c)\mathbb{Z})^*$ and $(d, c) = 1\}$.

We want to compute the values of $\theta_f(z)$ at cusp points for $\varGamma_0(N)$. It is clear that

$$\lim_{z \to i\infty} \theta_f(z) = 1.$$

Now suppose that a/c is a cusp point, where $(a, c) = 1, c|N, a \in (\mathbb{Z}/(c, N/c)\mathbb{Z})^*$. Choose a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, then $\gamma(\mathrm{i}\infty) = a/c$. So in terms of the equality (10.2) we obtain

$$\begin{aligned} V(\theta_f, a/c) &= \lim_{z \to \mathrm{i}\infty} (cz+d)^{-k/2} \theta_f \left(\frac{az+b}{cz+d} \right) \\ &= (-\mathrm{i})^{k/2} (\det A)^{-1/2} c^{-k/2} \sum_{x \bmod c} e(axAx^{\mathrm{T}}/2c) \end{aligned}$$

This shows that in order to get the values of $\theta_f(z)$ at cusp points we only need to evaluate the Gauss sum

$$\sum_{x \bmod c} e(axAx^{\mathrm{T}}/2c)$$

where c, a are co-prime positive integers.

Now we will calculate the Gauss sum

$$G(a,c) := \sum_{x \bmod c} e(axAx^{\mathrm{T}}/2c), \quad (c,a)=1.$$

Lemma 10.1 *If* $(c,c')=1$*, then*

$$G(a,cc') = G(ac,c')G(ac',c).$$

Proof Let $x = cy + c'z$, then

$$\begin{aligned} G(a,cc') &= \sum_{x \bmod cc'} e(axAx^{\mathrm{T}}/2cc') \\ &= \sum_{y \bmod c'} \sum_{z \bmod c} e(a(cy+c'z)A(cy+c'z)^{\mathrm{T}}/2cc') \\ &= \sum_{y \bmod c'} e(acyAy^{\mathrm{T}}/2c') \sum_{z \bmod c} e(ac'zAz^{\mathrm{T}}/2c) \\ &= G(ac,c')G(ac',c). \end{aligned}$$

This completes the proof. □

By Lemma 10.1, we only need to evaluate the Gauss sum $G(a,p^m)$ where $p \nmid a$ a prime and m is a positive integer.

We first assume that p is an odd prime. Then there exists an invertible matrix S over the ring $\mathbb{Z}_p$ of p-adic integers such that

$$SAS^{\mathrm{T}} = \mathrm{diag}\{\alpha_1 p^{\beta_1}, \alpha_2 p^{\beta_2}, \cdots, \alpha_k p^{\beta_k}\},$$

where $\alpha_i, \det S \in \mathbb{Z}_p^*$, $0 \leqslant \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_k$ are rational integers. Let $l_m = \#\{\beta_i | \beta_i \geqslant m\}$.

Hence

$$\begin{aligned} G(a,p^m) &= \sum_{x \bmod p^m} e(axAx^{\mathrm{T}}/2p^m) \\ &= \sum_{x \bmod p^m} e\left(ax\left(\bigoplus_{i=1}^{k} \alpha_i p^{\beta_i}\right)x^{\mathrm{T}}/2p^m\right) \\ &= \sum_{x=(x_1,\cdots,x_k) \bmod p^m} \prod_{i=1}^{k} e(a\alpha_i p^{\beta_i} x_i^2/2p^m) \\ &= p^{ml_m} \prod_{\beta_i<m} \left(\sum_{x \bmod p^m} e(a\alpha_i' x^2/p^{m-\beta_i})\right) \quad (\text{where } \alpha_i' \equiv 2^{-1}\alpha_i \bmod p^{m-\beta_i}) \\ &= p^{ml_m} \prod_{\beta_i<m} \left(\sum_{z \bmod p^{\beta_i}} \sum_{y \bmod p^{m-\beta_i}} e(a\alpha_i'(y+p^{m-\beta_i}z)^2/p^{m-\beta_i})\right) \end{aligned}$$

$$
\begin{aligned}
&= p^{ml_m} \prod_{\beta_i < m} \left(\sum_{z \bmod p^{\beta_i}} \sum_{y \bmod p^{m-\beta_i}} e(a\alpha_i' y^2/p^{m-\beta_i}) \right) \\
&= p^{ml_m} \prod_{\beta_i < m} p^{\beta_i} S(a\alpha_i', p^{m-\beta_i}) \\
&= p^{ml_m} \prod_{\beta_i < m} p^{\beta_i} \left(\frac{a\alpha_i'}{p^{m-\beta_i}} \right) \varepsilon_{p^{m-\beta_i}} p^{\frac{m-\beta_i}{2}} \\
&= p^{ml_m} \prod_{\beta_i < m} \left(\frac{a\alpha_i'}{p^{m-\beta_i}} \right) \varepsilon_{p^{m-\beta_i}} p^{\frac{m+\beta_i}{2}},
\end{aligned}
$$

where $S(\alpha, p^\beta) = \sum_{x \bmod p^\beta} e(\alpha x^2/p^\beta)$ is the classical Gauss sum, and $\varepsilon_d = 1$ or i according to $d \equiv 1$ or $3 \pmod 4$ respectively.

Now consider the case $p = 2$. In this case, there exists an invertible matrix S over the ring $\mathbb{Z}_2$ of 2-adic integers such that

$$
SAS^{\mathrm{T}} = \bigoplus_{i=1}^{l} \alpha_i 2^{s_i} \bigoplus_{j=1}^{l_1} \beta_j 2^{t_j} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bigoplus_{s=1}^{l_2} \gamma_s 2^{u_s} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},
$$

where α_i, β_j, $\gamma_s \in \mathbb{Z}_2^*$, $s_i \geqslant 1$, t_j, $u_s \geqslant 0$ are rational integers.

Hence we have

$$
\begin{aligned}
G(a, 2^m) &= \sum_{x \bmod 2^m} e(axAx^{\mathrm{T}}/2^{k+1}) \\
&= \sum_{x \bmod 2^m} e\Bigg(ax \Bigg(\bigoplus_{i=1}^{l} \alpha_i 2^{s_i} \bigoplus_{j=1}^{l_1} \beta_j 2^{t_j} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&\qquad \bigoplus_{s=1}^{l_2} \gamma_s 2^{u_s} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \Bigg) x^{\mathrm{T}}/2^{k+1} \Bigg),
\end{aligned}
$$

which implies that we only need to evaluate the following kinds of Gauss sums:

$$
\begin{aligned}
G_{1,t}(a\alpha_i, 2^m) &:= \sum_{x \bmod 2^m} e(a\alpha_i x^2/2^t), \\
G_{2,t}(a\beta_j, 2^m) &:= \sum_{(x,y) \bmod 2^m} e(a\beta_j xy/2^t), \\
G_{3,t}(a\gamma_s, 2^m) &:= \sum_{(x,y) \bmod 2^m} e(a\gamma_s(x^2 + xy + y^2)/2^t),
\end{aligned}
$$

where t is a positive integer and $t \leqslant m$.

Now we compute the above Gauss sums:

$$\begin{aligned}
G_{1,t}(a\alpha_i, 2^m) &= \sum_{x \bmod 2^m} e(a\alpha_i x^2/2^t) \\
&= \sum_{y \bmod 2^t} \sum_{z \bmod 2^{m-t}} e(a\alpha_i (y+2^t z)^2/2^t) \\
&= \sum_{z \bmod 2^{m-t}} \sum_{y \bmod 2^t} e(a\alpha_i y^2/2^t) = 2^{k-t} S(a\alpha_i, 2^t) \\
&= \begin{cases} 0, & \text{if } t = 1, \\ (1+i^{a\alpha_i})2^{m-\frac{t}{2}}, & \text{if } t \text{ is even}, \\ 2^{m-\frac{t-1}{2}} e^{\frac{\pi i a\alpha_i}{4}}, & \text{if } t > 1 \text{ and odd.} \end{cases}
\end{aligned}$$

$$\begin{aligned}
G_{2,t}(a\beta_j, 2^m) &= \sum_{(x,y) \bmod 2^m} e(a\beta_j xy/2^t) = \sum_{x \bmod 2^m} \sum_{y \bmod 2^m} e(a\beta_j xy/2^t) \\
&= \sum_{x \bmod 2^m} 2^{m-t} \sum_{y \bmod 2^t} e(a\beta_j xy/2^t) = 2^{m-t} \sum_{\substack{x \bmod 2^m, \\ 2^t | x}} 2^t = 2^{2m-t},
\end{aligned}$$

$$\begin{aligned}
G_{3,t}(a\gamma_s, 2^m) &= \sum_{(x,y) \bmod 2^m} e(a\gamma_s(x^2+xy+y^2)/2^t) \\
&= \sum_{x \bmod 2^m} \sum_{y \bmod 2^m} e(a\gamma_s(x^2+xy+y^2)/2^t) \\
&= \sum_{x \bmod 2^m} e(a\gamma_s x^2/2^t) \sum_{y \bmod 2^m} e(a\gamma_s(xy+y^2)/2^t) \\
&= \sum_{x \bmod 2^m} e(a\gamma_s x^2/2^t) \sum_{z \bmod 2^{m-t}} \sum_{y \bmod 2^t} \\
&\quad e(a\gamma_s(x(y+2^t z)+(y+2^t z)^2)/2^t) \\
&= \sum_{x \bmod 2^m} e(a\gamma_s x^2/2^t) \sum_{z \bmod 2^{m-t}} \sum_{y \bmod 2^t} e(a\gamma_s(xy+y^2)/2^t) \\
&= 2^{m-t} \sum_{x \bmod 2^m} e(a\gamma_s x^2/2^t) \sum_{y \bmod 2^t} e(a\gamma_s(xy+y^2)/2^t) \\
&= 2^{2(m-t)} \sum_{x \bmod 2^t} e(a\gamma_s x^2/2^t) \sum_{y \bmod 2^t} e(a\gamma_s(xy+y^2)/2^t).
\end{aligned}$$

Now let $w = \left[\frac{t+1}{2}\right]$, then

$$\begin{aligned}
&\sum_{y \bmod 2^t} e(a\gamma_s(xy+y^2)/2^t) \\
&= \sum_{u \bmod 2^w} \sum_{v \bmod 2^{t-w}} e(a\gamma_s(x(u+2^w v)+(u+2^w v)^2)/2^t) \\
&= \sum_{u \bmod 2^w} e(a\gamma_s(xu+u^2)/2^t) \sum_{v \bmod 2^{t-w}} e(a\gamma_s(x+2u)v/2^{t-w})
\end{aligned}$$

$$= \sum_{\substack{u \bmod 2^w, \\ 2^{t-w}|(x+2u)}} 2^{t-w} e(a\gamma_s(xu+u^2)/2^t).$$

Therefore, we obtain

$$\begin{aligned}
&G_{3,t}(a\gamma_s, 2^m) \\
&= 2^{2(m-t)} \sum_{x \bmod 2^t} e(a\gamma_s x^2/2^t) \sum_{\substack{u \bmod 2^w, \\ 2^{t-w}|(x+2u)}} 2^{t-w} e(a\gamma_s(xu+u^2)/2^t) \\
&= 2^{2m-t-w} \sum_{u \bmod 2^w} e(a\gamma_s u^2/2^t) \sum_{\substack{x \bmod 2^t, \\ x+2u\equiv 0(2^{t-w})}} e(a\gamma_s(xu+x^2)/2^t) \\
&= 2^{2m-t-w} \sum_{u \bmod 2^w} e(a\gamma_s u^2/2^t) \sum_{y \bmod 2^w} \\
&\quad e(a\gamma_s((-2u+2^{t-w}y)u+(-2u+2^{t-w}y)^2)/2^t) \\
&= 2^{2m-t-w} \sum_{u \bmod 2^w} e(3a\gamma_s u^2/2^t) \sum_{y \bmod 2^w} e(-3a\gamma_s yu/2^w) e(a\gamma_s 2^{2(t-w)} y^2/2^t).
\end{aligned}$$

Now, if $t = 2g$ is even, then $w = \left[\dfrac{t+1}{2}\right] = g$, and $t - w = g$, $2^{2(t-w)}y^2/2^t = y^2$. Therefore we get

$$\begin{aligned}
G_{3,t}(a\gamma_s, 2^m) &= 2^{2m-t-w} \sum_{u \bmod 2^w} e(3a\gamma_s u^2/2^t) \sum_{y \bmod 2^w} e(-3a\gamma_s yu/2^w) \\
&= 2^{2m-t-w} \sum_{\substack{u \bmod 2^w, \\ 2^w|u}} 2^w e(3a\gamma_s u^2/2^t) \\
&= 2^{2m-t}.
\end{aligned}$$

If $t = 2g+1$ is odd, then $w = \left[\dfrac{t+1}{2}\right] = g+1$, and $t-w = g$, $2^{2(t-w)}y^2/2^t = y^2/2$. Therefore we get

$$\begin{aligned}
&G_{3,t}(a\gamma_s, 2^m) \\
&= 2^{2m-t-w} \sum_{u \bmod 2^w} e(3a\gamma_s u^2/2^t) \sum_{y \bmod 2^w} e(-3a\gamma_s yu/2^w) e(a\gamma_s y^2/2) \\
&= 2^{2m-t-w} \sum_{u \bmod 2^w} e(3a\gamma_s u^2/2^t) \Bigg(- \sum_{\substack{y \bmod 2^w, \\ y \text{ is odd}}} e(-3a\gamma_s yu/2^w) \\
&\quad + \sum_{\substack{y \bmod 2^w, \\ y \text{ is even}}} e(-3a\gamma_s yu/2^w) \Bigg)
\end{aligned}$$

$$= 2^{2m-t-w} \sum_{u \bmod 2^w} e(3a\gamma_s u^2/2^t)\Bigg(- \sum_{y \bmod 2^w} e(-3a\gamma_s yu/2^w)$$
$$+ 2 \sum_{\substack{y \bmod 2^w,\\ y \text{ is even}}} e(-3a\gamma_s yu/2^w)\Bigg)$$
$$= -2^{2m-t-w} \sum_{u \bmod 2^w} e(3a\gamma_s u^2/2^t) \sum_{y \bmod 2^w} e(-3a\gamma_s yu/2^w)$$
$$+ 2^{2m-t-w+1} \sum_{u \bmod 2^w} e(3a\gamma_s u^2/2^t) \sum_{\substack{y \bmod 2^w,\\ y \text{ is even}}} e(-3a\gamma_s yu/2^w)$$
$$= -2^{2m-t-w} \sum_{u \bmod 2^w, 2^w|u} 2^w e(3a\gamma_s u^2/2^t)$$
$$+ 2^{2m-t-w+1} \sum_{u \bmod 2^w} e(3a\gamma_s u^2/2^t) \sum_{y \bmod 2^{w-1}} e(-3a\gamma_s yu/2^{w-1})$$
$$= -2^{2m-t} + 2^{2m-t-w+1} \sum_{\substack{u \bmod 2^w\\ 2^{w-1}|u}} 2^{w-1} e(3a\gamma_s u^2/2^t)$$
$$= -2^{2m-t} + 2^{2m-t}(1 + e(3a\gamma_s(2^{w-1})^2/2^t))$$
$$= -2^{2m-t} + 2^{2m-t}(1 + e(3a\gamma_s/2))$$
$$= -2^{2m-t},$$

where $e(3a\gamma_s/2) = -1$ since $3a\gamma_s \equiv 1 \pmod 2$.

Therefore we have proved

$$G_{3,t}(a\gamma_s, 2^m) = (-1)^t 2^{2m-t}.$$

Now let $l_m = \#\{s_i | s_i \geqslant m+1\} + 2\#\{t_j | t_j \geqslant m\} + 2\#\{u_s | u_s \geqslant m\}$. Finally we have

$$G(a, 2^m)$$
$$= 2^{ml_m} \prod_{s_i < m+1} G_{1,m+1-s_i}(a\alpha_i, 2^m) \prod_{t_j < m} G_{2,m-t_j}(a\beta_j, 2^m) \prod_{u_s < m} G_{3,m-u_s}(a\gamma_s, 2^m)$$
$$= 2^{ml_m} \prod_{s_i < m+1} G_{1,m+1-s_i}(a\alpha_i, 2^m) \prod_{t_j < m} 2^{2m-(m-t_j)} \prod_{u_s < m} (-1)^{m-u_s} 2^{2m-(m-u_s)}$$
$$= 2^{ml_m} \prod_{s_i < m+1} G_{1,m+1-s_i}(a\alpha_i, 2^m) \prod_{t_j < m} 2^{m+t_j} \prod_{u_s < m} (-1)^{m-u_s} 2^{m+u_s}.$$

So we can compute the values of $\theta_f(z)$ at each cusp point.

Example 10.1 Let $f(x, y) = ax^2 + bxy + cy^2$ be an integral primitive, positive definite, binary quadratic form with fundamental discriminant D. We want to evaluate $\theta_f(z)$ at cusp point $1/\alpha$ where $\alpha | D$. Since D is a fundamental discriminant, the odd part of D is square free. If $p|D$ is an odd prime, then $p \nmid a$ or $p \nmid c$ since f is primitive. Hence we have

(1) If $p \nmid a$, then

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \sim \begin{pmatrix} 2a & 0 \\ 0 & (2a)^{-1}\det A \end{pmatrix}$$

over $\mathbb{Z}_p$.

(2) If $p \nmid c$, then

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \sim \begin{pmatrix} 2c & 0 \\ 0 & (2c)^{-1}\det A \end{pmatrix}$$

over $\mathbb{Z}_p$.

Therefore

$$G(n,p) = \begin{cases} pS(an,p) = \left(\dfrac{an}{p}\right)\varepsilon_p p^{3/2}, & \text{if } p \nmid a, \\ pS(cn,p) = \left(\frac{cn}{p}\right)\varepsilon_p p^{3/2}, & \text{if } p \nmid c. \end{cases}$$

So for $\alpha = p_1 p_2 \cdots p_s | D$, p_i odd, we have

$$G(1,\alpha) = \prod_{i=1}^{s} G(\alpha/p_i, p_i) = \prod_{i=1}^{s}\left(\frac{\delta_i\alpha/p_i}{p_i}\right)\varepsilon_{p_i} p_i^{3/2} = \alpha^{3/2}\prod_{i=1}^{s}\left(\frac{\delta_i\alpha/p_i}{p_i}\right)\varepsilon_{p_i},$$

where $\delta_i = a$ or c according to $p_i \nmid a$ or $p_i \nmid c$. Hence

$$\begin{aligned} V(\theta_f, 1/\alpha) &= -i(\det A)^{-1/2}\alpha^{-1}G(1,\alpha) \\ &= -i\left(\frac{\alpha}{\det A}\right)^{1/2}\prod_{i=1}^{s}\left(\frac{\delta_i\alpha/p_i}{p_i}\right)\varepsilon_{p_i} = -\left(\frac{\alpha}{D}\right)^{1/2}\prod_{i=1}^{s}\left(\frac{\delta_i\alpha/p_i}{p_i}\right)\varepsilon_{p_i}. \end{aligned}$$

We now compute the Gauss sum for $p = 2$.

(3) If $D = b^2 - 4ac \equiv 1 \pmod 4$, and $a \equiv c \equiv 1 \pmod 2$, then

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \sim \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

over $\mathbb{Z}_2$. Therefore

$$G(n, 2^m) = (-1)^m 2^m \quad \text{for any odd positive integer } n.$$

(4) If $D \equiv 1 \pmod 4$, $ac \equiv 0 \pmod 2$, then

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

over $\mathbb{Z}_2$. Therefore

$$G(n, 2^m) = 2^m, \quad \text{for any odd positive integer } n.$$

(5) If $D \equiv 0 \pmod 4$, then $2|b$. Denote $b = 2b'$. It is clear that $2 \nmid a$ or $2 \nmid c$ since $(a, b, c) = 1$. We assume that $2 \nmid a$. Hence

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} = 2\begin{pmatrix} a & b' \\ b' & c \end{pmatrix} \sim 2\begin{pmatrix} a & 0 \\ 0 & a^{-1}\frac{D}{4} \end{pmatrix}$$

over $\mathbb{Z}_2$. Therefore we have

$$G(n, 2^m) = G_{1,m}(na, 2^m)G_{1,m-t}(n\beta, 2^m),$$

where $t = \nu_2(c - a^{-1}b'^2) = \nu_2(D/4)$, $\beta = (c - a^{-1}b'^2)2^{-t} = a^{-1}\dfrac{D}{2^{2+t}}$, and we think

$$G_{1,m-t}(n\beta, 2^m) = 2^m$$

for any $m \leqslant t$. In particular, we know that

$$G(n, 2) = G(n, 2^{t+1}) = 0.$$

Since D is a fundamental discriminant, $t = \nu_2(D/4) = 0$ or 1 according to $D \equiv 12$ or $8 \pmod{16}$ respectively.

So for $\alpha = 2^m|D$, we have

$$\begin{aligned} V(\theta_f, 1/2^m) &= -\mathrm{i}(\det A)^{-1/2}2^{-m}G(1, 2^m) \\ &= -(D)^{-1/2}2^{-m}G_{1,m}(a, 2^m)G_{1,m-t}(\beta, 2^m). \end{aligned}$$

In particular

$$V(\theta_f, 1/\alpha) = 0$$

for any $\alpha = 2^m\alpha_1|D$ where $m = 1$ or $t+1$, $2 \nmid \alpha_1$. For $\alpha = 2^m\alpha_1 = 2^m\prod_{i=1}^{s} p_i|D$ with $m \neq 1, t+1$, we have

$$\begin{aligned} V(\theta_f, 1/\alpha) &= -\mathrm{i}(\det A)^{-1/2}\alpha^{-1}G(1, \alpha) \\ &= -(D)^{-1/2}\alpha^{-1}G(2^m, \alpha_1)G(\alpha_1, 2^m) \\ &= -(D)^{-1/2}\alpha^{-1}\alpha_1^{3/2}G_{1,m}(a\alpha_1, 2^m)G_{1,m-t}(\alpha_1\beta, 2^m)\prod_{i=1}^{s}\left(\frac{\delta_i\alpha/p_i}{p_i}\right)\varepsilon_{p_i} \\ &= -(\alpha/D)^{1/2}2^{-3m/2}G_{1,m}(a\alpha_1, 2^m)G_{1,m-t}(\alpha_1\beta, 2^m)\prod_{i=1}^{s}\left(\frac{\delta_i\alpha/p_i}{p_i}\right)\varepsilon_{p_i}. \end{aligned}$$

□

Remark 10.1 If D is an odd fundamental discriminant, our result is just Lemma IV(2.3) in B.H.Gross, D.B.Zagier 1986. If D is even, our result is just Proposition 2 in I. Kiming, 1995.

Example 10.2 Let $f = f(x_1, \cdots, x_k)$ be a positive definite quadratic form with k odd. Suppose that the level of f is $4D$ with D square free odd integer. Let

$D = p_1 p_2 \cdots p_t$. Since D is square free, there exists an invertible matrix S_i over $\mathbb{Z}_{p_i}$ such that

$$S_i A S_i^{\mathrm{T}} = \mathrm{diag}\{\alpha_{i,1}, \alpha_{i,2}, \cdots, \alpha_{i,s_i}, \alpha_{i,s_i+1} p_i, \cdots, \alpha_{i,k} p_i\}$$

with $\alpha_{i,j} \in \mathbb{Z}_{p_i}^*$. Hence

$$G(n, p_i) = p_i^{k-s_i} \prod_{g=1}^{s_i} \left(\frac{n\alpha'_{i,g}}{p_i} \right) \varepsilon_{p_i} p_i^{1/2} = p_i^{k-\frac{s_i}{2}} \varepsilon_{p_i}^{s_i} \left(\frac{n^{s_i} A_i}{p_i} \right),$$

where $A_i = \prod_{g=1}^{s_i} \alpha'_{i,g}$ and $\alpha'_{i,g} \equiv 2^{-1}\alpha_{i,g} \pmod{p_i}$. Therefore for any $\alpha = \prod_{i=1}^{t} p_i^{\delta_i} | D, \delta_i = 0$ or 1, we can evaluate

$$G(1, \alpha) = \prod_{i=1}^{t} G(\alpha/p_i^{\delta_i}, p_i^{\delta_i}) = \prod_{i=1}^{t} \left(p_i^{k-\frac{s_i}{2}} \varepsilon_{p_i}^{s_i} \left(\frac{n^{s_i} A_i}{p_i} \right) \right)^{\delta_i}.$$

Since $4D$ is the level of f and D square free, there exists an invertible matrix S over $\mathbb{Z}_2$ such that

$$SAS^{\mathrm{T}} = \bigoplus_{i=1}^{l} \alpha_i 2^{a_i} \bigoplus_{j=1}^{l_1} \beta_j 2^{t_j} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bigoplus_{s=1}^{l_2} \gamma_s 2^{u_s} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Since k is odd, α_i appears at least one time and $s_i = 1, t_j, u_s \leqslant 2$. Hence we have

$$\begin{aligned} G(n,2) &= 0, \\ G(n,4) &= 2^{4a} \prod_{i=1}^{l} G_{1,2}(n\alpha_i, 4) \prod_{t_j<2} 2^{2+t_j} \prod_{u_s<2} (-1)^{u_s} 2^{2+u_s} \\ &= (-1)^e 2^{2a+l+2b+2c+d+e} \prod_{i=1}^{l} (1 + i^{n\alpha_i}), \end{aligned}$$

where $a = \#\{t_j, u_s | t_j = u_s = 2\}, b = \#\{t_j | t_j < 2\}$, $c = \#\{u_s | u_s < 2\}$, $d = \sum_{t_j<2} t_j$, $e = \sum_{u_s<2} u_s$. From the above calculation we obtain the value

$$V(\theta_f, 1/\alpha) = (-\mathrm{i})^{k/2} (\det A)^{-1/2} \alpha^{-k/2} G(1, \alpha)$$

for any $\alpha | 4D$. In particular we know that $V(\theta_f, 1/2\beta) = 0$ for any $\beta | D$ since $G(n, 2) = 0$ for any odd integer n. $\square$

10.2 The Minimal Integer Represented by a Positive Definite Quadratic Form

We consider the following problem: for a given positive definite quadratic form f, find an upper bound on the size for the minimal positive integer represented by f.

We first consider the case that the level of f is equal to 1. Let

$$E_k(z) = \frac{1}{2} \sum_{l,m}{}' \frac{1}{(lz+m)^k}, \quad k = 4, 6, 8, \cdots, \tag{10.3}$$

where (l, m) run over all pairs of integers except $(0, 0)$. By Section 7.5 we know that

$$E_k(z) = \zeta(k) + \frac{(2\pi \mathrm{i})^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \tag{10.4}$$

where

$$\sigma_g(n) = \sum_{d|n} d^g.$$

In view of

$$\zeta(k) = -\frac{(2\pi \mathrm{i})^k B_k}{2(k)!}, \tag{10.5}$$

$E_k(z)$ can be expressed by the formulae

$$E_k(z) = \zeta(k) G_k(z), G_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \quad k = 4, 6, 8, \cdots. \tag{10.6}$$

In particular, we have the Bernoulli numbers:

$$B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{14} = \frac{7}{6},$$

and hence

$$\begin{cases} G_4(z) = 1 + 240 \sum \sigma_3(n) q^n, \\ G_6(z) = 1 - 504 \sum \sigma_5(n) q^n, \\ G_8(n) = 1 + 480 \sum \sigma_7(n) q^n, \\ G_{10}(z) = 1 - 264 \sum \sigma_9(n) q^n, \\ G_{14}(z) = 1 - 24 \sum \sigma_{13}(n) q^n \end{cases} \tag{10.7}$$

with integral coefficients and constant 1. By the dimension formula we see that the dimension r_h of the linear space of modular forms of weight h is equal to $\left[\frac{h}{12}\right] + 1$ or $\left[\frac{h}{12}\right]$ according to $h \not\equiv 2 \pmod{12}$ or $h \equiv 2 \pmod{12}$ respectively. In particular we have

$$G_4^2 = G_8, \quad G_4G_6 = G_{10}, \quad G_4^2G_6 = G_{14}, \quad G_lG_{14-l} = G_{14}, \quad l = h - 12r_h + 12 = 0, 4, 6, 8, 10, 14 \tag{10.8}$$

and for the modular form

$$\Delta = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

of weight 12,

$$1728\Delta = G_4^3 - G_6^2.$$

Let

$$j(z) = G_4^3/\Delta(z) = q^{-1} + \cdots \tag{10.9}$$

be the absolute invariant, then

$$\begin{aligned}\Delta^2\frac{\mathrm{d}j}{\mathrm{d}z} &= 3G_4^2\frac{\mathrm{d}G_4}{\mathrm{d}z}\Delta - G_4^3\frac{\mathrm{d}\Delta}{\mathrm{d}z}\\ &= \frac{1}{1728}G_4^2G_6\left(2G_4\frac{\mathrm{d}G_6}{\mathrm{d}z} - 3G_6\frac{\mathrm{d}G_4}{\mathrm{d}z}\right)\end{aligned}$$

and the expression in the brackets is a modular form of weight 12 and indeed a cusp form which can therefore differ from Δ at most by a constant factor. Comparing the coefficients of q in the Fourier expansions, we get

$$\frac{\mathrm{d}j}{\mathrm{d}\log q} = -G_{14}\Delta^{-1}. \tag{10.10}$$

Let hereafter, $h > 2$, and hence $r_h > 0$. The power-products $G_4^aG_6^b$, where the exponents a, b run over all non-negative rational integer solutions of

$$4a + 6b = h$$

form a basis of the space $G(1, h, \mathrm{id.}) := G(h)$. It follows from this that, for every function $M \in G(h)$, $MG_{h-12r+12}^{-1}$ always belongs to $G(12r - 12)$. Since Δ^{r-1} is a modular form of weight $12r - 12$, not vanishing anywhere in the interior of the upper half-plane,

$$MG_{h-12r+12}^{-1}\Delta^{1-r} := w(f) := w, \tag{10.11}$$

is an entire modular function and hence a polynomial in j with constant coefficients.

Let

$$T_h = G_{12r-h+2}\Delta^{-r} \tag{10.12}$$

with Fourier expansion

$$T_h = c_{hr}q^{-r} + \cdots + c_{h1}q^{-1} + c_{h0} + \cdots \tag{10.13}$$

and first coefficient $c_{hr} = 1$. Since

$$\varDelta^{-1} = q^{-1} \prod_{n=1}^{\infty} (1 + q^n + q^{2n} + \cdots)^{24}, \tag{10.14}$$

all the Fourier coefficients of T_h turn out to be rational integers.

Theorem 10.4 *Let*

$$M = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots \tag{10.15}$$

be the Fourier *series of a modular form M of weight h. Then*

$$c_{h0}a_0 + c_{h1}a_1 + \cdots + c_{hr}a_r = 0.$$

Proof For $l = 0, 1, 2, \cdots$, we have

$$j^l \frac{\mathrm{d}j}{\mathrm{d}z} = \frac{1}{l+1} \frac{\mathrm{d}j^{l+1}}{\mathrm{d}z},$$

and hence, by (10.9), it has a Fourier series without constant term. Since the function w defined by (10.11) is a polynomial in j, the product $w\dfrac{\mathrm{d}j}{\mathrm{d}z}$ has also a Fourier series without constant term. Because of (10.8) and (10.10), we have

$$-\frac{1}{2\pi i} w \frac{\mathrm{d}j}{\mathrm{d}z} = MG^{-1}_{h-12r+12}\varDelta^{1-r}G_{14}\varDelta^{-1} = MG_{12r-h+2}\varDelta^{-r} = MT_h$$

from which the theorem follows on substituting the series (10.13) and (10.15) for T_h and M respectively. □

Put $c_{h0} := c_h$ for brevity. We have the following:

Theorem 10.5 *We have $c_h \neq 0$.*

Proof First, consider the case $h \equiv 2 \pmod 4$. So that $h \equiv 2t \pmod{12}$ with $t = 1, 3, 5$. Then correspondingly $12r = h - 2$, $h + 6$, $h + 2$, hence $12r - h + 2 = 0$, 8, 4 and

$$G_{12r-h+2} = G_0, G_4^2, G_4.$$

Since by (10.7), G_4 has all its Fourier coefficients positive and the same holds for $\varDelta^{-r}$ as a consequence of (10.14). We conclude from (10.12) that all the coefficients in the expansion (10.13) are positive. Therefore the integers $c_{h0}, c_{h1}, \cdots, c_{hr}$ are all positive and in particular, $c_h = c_{h0} > 0$, i.e., $c_h \neq 0$.

Let now $h \equiv 0 \pmod 4$, so that $h \equiv 4t \pmod{12}$ with $t = 0, 1, 2$ whence $12r = h - 4t + 12$, $h - 12r + 12 = 4t$ and

$$g_{h-12r+12} = G_{4t} = G_4^t.$$

Furthermore we have now

$$\begin{aligned} T_h &= -G_{12r-h+2}\Delta^{1-r}G_{14}^{-1}\frac{\mathrm{d}j}{\mathrm{d}\log q} \\ &= -G_4^{-t}\Delta^{1-r}\frac{\mathrm{d}j}{\mathrm{d}\log q} = \frac{3}{t-3}\Delta^{1-r-t/3}\frac{\mathrm{d}j^{1-t/3}}{\mathrm{d}\log q} \\ &= \frac{3}{t-3}\frac{\mathrm{d}(G_4^{3-t}\Delta^{-r})}{\mathrm{d}\log q} + \frac{3r+t-3}{(3-t)r}G_4^{3-t}\frac{\mathrm{d}\Delta^{-r}}{\mathrm{d}\log q}; \end{aligned}$$

hence c_{h0} is also the constant term in the Fourier expansion of the function

$$V_h = \frac{3r+t-3}{(3-t)r}G_4^{3-t}\frac{\mathrm{d}\Delta^{-r}}{\mathrm{d}\log q}.$$

Because of the assumption $h > 2$, we see that $3r + t - 3 > 0$. The series for G_4^{3-t} begins with 1 and has again all its coefficients positive. Furthermore, by (10.14), the coefficients of the negative powers $q^{-1}, \cdots, q^{-r}$ of q in the derivative of Δ^{-r} with respect to $\log q$ are all negative while the constant term is absent. Hence the constant term in V_h is negative and $c_h = c_{h0} < 0$, i.e., $c_h \neq 0$. This completes the proof. □

A most important consequence of Theorem 10.4 and Theorem 10.5 is the fact that, for every modular form M of weight h and level 1, the constant term a_0 in its Fourier expansion is determined by the r Fourier coefficients $a_1, \cdots, a_r$, which comes out of the formula

$$a_0 = c_h^{-1}(c_{h1}a_1 + \cdots + c_{hr}a_r). \tag{10.16}$$

If, in particular, $a_0 \neq 0$, then there must be some i $(1 \leqslant i \leqslant r)$ such that $a_i \neq 0$. In particular, if taking the theta function of a positive definite even unimodular quadratic form Q in $2h$ variables as our M, we have that $a_0 = 1 \neq 0$, and hence conclude that Q represents a positive integer $n \leqslant r_h$(Please compare [?]).

We now want to extend Siegel's results above to the case with level 2.

Let $G(2, h)$ be the vector space of holomorphic modular forms of weight h for $\Gamma_0(2)$, $r = r(2, h) := \dim(G(2, h))$. Then by the dimension formula we see that $r(2,h) = 1 + \left[\frac{h}{4}\right]$ for any even nonnegative number h.

We introduce some analogues of the above function T_h. In order to do this, we need some more Eisenstein series.

Put

$$\sigma_k^{\mathrm{odd}}(n) := \sum_{\substack{0<d\mid n \\ 2\nmid d}} d^k, \quad \sigma_k^{\mathrm{alt}}(n) := \sum_{0<d\mid n} (-1)^d d^k, \quad \sigma_{N,k}^*(n) := \sum_{\substack{0<d\mid n \\ N\nmid (n/d)}} d^k.$$

Since $r(2,2) = \left[\frac{2}{4}\right] + 1 = 1$, let $E_{\infty,2}$ be the unique normalized modular form (in fact, the Eisenstein series) in $G(2,2)$ defined by

$$E_{\infty,2}(z) := 1 + 24\sum_{n=1}^{\infty}\sigma_1^{\text{odd}}(n)q^n.$$

Since $r(2,4) = \left[\frac{4}{4}\right] + 1 = 2$, the vector space $G(2,4)$ is spanned by two Eisenstein series $E_{0,4}(z)$ and $E_{\infty,4}(z)$ with respect to the cusp points 0 and ∞ respectively. They have Fourier expansions:

$$E_{0,4} = 1 + 16\sum_{n=1}^{\infty}\sigma_3^{\text{alt}}(n)q^n, \quad E_{\infty,4} = \sum_{n=1}^{\infty}\sigma_{2,3}^{*}(n)q^n.$$

In fact, in terms of the results in Section 7.5, we can easily see that all the functions $E_{\infty,2}(z)$ and $E_{0,4}(z), E_{\infty,4}(z)$ are in $\mathcal{E}(2,2,\text{id.})$ and $\mathcal{E}(2,4,\text{id.})$ respectively.

We also denote by $j_2 = j_2(z)$ the following modular function for $\Gamma_0(2)$:

$$j_2(z) := E_{\infty,2}^2 E_{\infty,4}^{-1},$$

which is a level two analogue of $j(z)$ for $\Gamma_0(1)$. Finally, we introduce analogues of the T_h:

$$T_{2,h} := E_{\infty,2}E_{0,4}E_{\infty,4}^{-r} \text{ if } r = r(2,h) \equiv 0 (\text{mod } 4),$$
$$T_{2,h} := E_{\infty,2}^2 E_{0,4}E_{\infty,4}^{-1-r} \text{ if } r = r(2,h) \equiv 2 (\text{mod } 4).$$

We need the following:

Lemma 10.2 *The function j_2 is a modular function for $\Gamma_0(2)$. It is holomorphic on $\mathbb{H}$ with a simple pole at infinity and defines a bijection of $\mathbb{H}/\Gamma_0(2)$ onto $\mathbb{C}$ by passage to the quotient.*

Proof The first two conclusions are clear. Let $S : z \to -1/z$ and $T : z \to z+1$ be two linear fractional transformations. Let

$$F = \{z \in \mathbb{H} \mid |z| > 1, |\text{Re}(z)| < 1/2\}$$

be the fundamental domain of $\Gamma_0(1)$. Denote by V the closure of $F \bigcup S(F) \bigcup ST(F)$, and put $F_2 = V \bigcup \{\mathrm{i}\infty\}$. Then F_2 is a fundamental domain for $\Gamma_0(2)$ which has two $\Gamma_0(2)$-inequivalent cusp points: zero and $\mathrm{i}\infty$. The only non-cusp in F_2 fixed by a map in $\Gamma_0(2)$ is $\gamma = -\frac{1}{2} + \frac{1}{2}\mathrm{i}$. The number of zeros in a fundamental domain of a non zero function in $G(2,h)$ is $h/4$. Now let $f_\lambda = E_{\infty,2}^2 - \lambda E_{\infty,4}$ for any $\lambda \in \mathbb{C}$. Then $f_\lambda \in G(2,4)$. The sum of its zero orders in a fundamental domain is 1. If f_λ has multiple zeros in a fundamental domain, there must be exactly two of them in the equivalence class of γ, or exactly three in the one of $\rho = \mathrm{e}^{2\pi\mathrm{i}/3}$. This completes the proof. □

Lemma 10.3 *Let f be a meromorphic function on $\mathbb{H}^*$. Then the following statements are equivalent:*

(1) *f is a modular function for $\Gamma_0(2)$;*

(2) *f is a quotient of two modular forms for $\Gamma_0(2)$ of equal weight;*

(3) *f is a rational function of j_2.*

Proof It is clear that (3) $\Rightarrow$ (2) $\Rightarrow$ (1). for $z \in \mathbb{H}^*$, denote by $[z]$ the equivalence class of z in $\mathbb{H}/\Gamma_0(2)$. By an abuse of the notation we may take f as in (1) as a function from $\mathbb{H}^*/\Gamma_0(2)$ to $\widehat{\mathbb{C}}$. The function j_2, also regarded in this fashion, is invertible. Let $\widetilde{f} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ satisfy $\widetilde{f} = f \circ j_2^{-1}$. Then $\widetilde{f}$ is meromorphic on $\widehat{\mathbb{C}}$, so that it is rational. If $z \in \widehat{\mathbb{C}}$, let $u = j_2^{-1}(z) \in \mathbb{H}^*/\Gamma_0(2)$. Then $f(u) = f(j_2^{-1}(z)) = \widetilde{f}(z) = \widetilde{f}(j_2(z))$. Thus f is a rational function in j_2. □

Lemma 10.4 *For $z \in \mathbb{H}$, we have that*

$$\frac{\mathrm{d}}{\mathrm{d}z} j_2(z) = -2\pi \mathrm{i} E_{\infty,2}(z) E_{0,4}(z) E_{\infty,4}^{-1}(z).$$

Proof It is clear from the definition of a modular function that the derivative of a modular function has weight two. Therefore both sides of the equality in the lemma are meromorphic modular forms of weight 2 for $\Gamma_0(2)$. The only poles of either functions lie at infinity. On both sides, the principal parts of the Fourier expansions at infinity consist only of the term $-2\pi \mathrm{i} q^{-1}$. Hence the modular form

$$\alpha := \frac{\mathrm{d}}{\mathrm{d}z} j_2(z) + 2\pi \mathrm{i} E_{\infty,2}(z) E_{0,4}(z) E_{\infty,4}^{-1}(z)$$

is holomorphic with weight two. For a non zero modular form in $G(2,h)$, the number of zeros in a fundamental domain is $h/4$, we can easily check that the exponent of the first nonzero Fourier coefficient in the expansion of α exceeds $h/4 = 1/2$. This exponent counts the number of zeros at $\mathrm{i}\infty$. Hence $\alpha = 0$ and the lemma holds. □

We now introduce an analogue of the map w in (10.11).

For $h \equiv 0 \pmod 4$ and $f \in G(2,h)$, let

$$W_2(f) = f E_{\infty,4}^{-h/4}.$$

For $h \equiv 2 \pmod 4$ and $f \in G(2,h)$, let

$$W_2(f) = f E_{\infty,2} E_{\infty,4}^{-(h+2)/4}.$$

Lemma 10.5 *Let h be an even positive integer. Then*

(1) *the restriction of W_2 to $G(2,h)$ is an isomorphism from the vector space $G(2,h)$ to the vector space of polynomials in j_2 of degree less than $r = r(2,h)$ or of degree between 1 and r inclusive according to $r \equiv 0 \pmod 4$ or $h \equiv 2 \pmod 4$ respectively.*

(2) *for any $f \in G(2,h)$, the constant term in the Fourier expansion at infinity of $fT_{2,h}$ is zero.*

Proof (1) Suppose $h \equiv 0 \pmod 4$ and $f \in G(2,h)$, then

$$W_2(f) = fE_{\infty,4}^{-h/4} = fE_{\infty,4}^{1-r}.$$

For $d = 0, 1, 2, \cdots, r-1$, the products $j_2^d E_{\infty,4}^{r-1}$ belong to $G(2,h)$. We have $W_2(j_2^d E_{\infty,4}^{r-1}) = j_2^d$. Let V be the subspace of $G(2,h)$ generated by the modular forms $j_2^d E_{\infty,4}^{r-1}$ for $d = 0, 1, 2, \cdots, r-1$. And denote by V_1 the space of polynomials in j_2 of degree at most $r-1$. W_2 carries V isomorphically onto V_1. Hence $\dim(V) = r$ which implies that $V = G(2,h)$. Now let $h \equiv 2 \pmod 4$. Then

$$W_2(f) = fE_{\infty,2}E_{\infty,4}^{-r}.$$

For $d = 0, 1, 2, \cdots, r-1$, the products $j_2^d E_{\infty,2} E_{\infty,4}^{r-1}$ belong to $G(2,h)$ and

$$W_2(j_2^d E_{\infty,2} E_{\infty,4}^{r-1}) = j_2^{d+1}.$$

W_2 carries $E_{\infty,2}V$ isomorphically onto $j_2 V_1$. Therefore $\dim(E_{\infty,2}V) = r$. Hence $E_{\infty,2}V = G(2,h)$.

(2) Suppose $h \equiv 0 \pmod 4$. Then

$$W_2(f)\frac{\mathrm{d}j_2}{\mathrm{d}z} = -fE_{\infty,4}^{1-r} 2\pi \mathrm{i} E_{\infty,2}E_{0,4}E_{\infty,4}^{-1} = -2\pi \mathrm{i} f T_{2,h}.$$

We can obtain the same result for $h \equiv 2 \pmod 4$ by a similar computation. Thus $fT_{2,h}$ is the derivative of a polynomial in j_2, so it can be expressed in a neighborhood of infinity as the derivative with respect to z of a power series in the variable $q = \mathrm{e}^{2\pi \mathrm{i} z}$. This derivative is a power series in q with vanishing constant term. This completes the proof. □

Lemma 10.6 (1)

$$E_{\infty,4}(z) = q \prod_{0<n\in 2\mathbb{Z}} (1-q^n)^8 \prod_{0<n\in \mathbb{Z}\backslash 2\mathbb{Z}} (1-q^n)^{-8};$$

(2) *For a given set* A *and a given arithmetical function* f, *the number* $p_{A,f}(n)$ *defined by the equation*

$$\prod_{n\in A} (1-x^n)^{-f(n)/n} = 1 + \sum_{n=1}^{\infty} p_{A,f}(n)x^n$$

satisfies the recursion formula

$$np_{A,f}(n) = \sum_{k=1}^{n} f_A(k)p_{A,f}(n-k),$$

where $p_{A,f}(0) = 1$ *and* $f_A(k) = \sum_{d|k, d\in A} f(d)$.

Proof (1) This is equivalent to show that

$$E_{\infty,4}(z) = \eta(2z)^{16}\eta(z)^{-8}.$$

Denote by $f(z)$ the right hand side of the above. The function f is holomorphic on $\mathbb{H}$ because η is non-vanishing on $\mathbb{H}$. We see that f has the product expansion

$$f(z) = q \prod_{0<n\in 2\mathbb{Z}} (1-q^n)^8 \prod_{0<n\in \mathbb{Z}\setminus 2\mathbb{Z}} (1-q^n)^{-8}$$

from the product expansion of η. It follows that f has a simple zero at infinity. The number of zeros in a $\Gamma_0(2)$ fundamental domain for a modular form in $G(2,4)$ is one. But from the transformation formula of the η function we know easily that f is in $G(2,4)$. This shows that f and $E_{\infty,4}$ are monic modular forms with the same weight, level and divisor (both equal to $1\cdot \mathrm{i}\infty$), hence identical.

(2) By induction. □

Theorem 10.6 *For any even positive integer h, the constant term in the Fourier expansion at infinity of $T_{2,h}$ is non zero.*

Proof Let $h \equiv 0 \pmod 4$. Put $u = 2\pi \mathrm{i} z = \log q$. Write D for the operator $\dfrac{\mathrm{d}}{\mathrm{d}u}$. It is clear that $D(q^n) = nq^n$. Put $m_2 = j_2 - 64$. It is easy to see that $E_{\infty,2}^2 = E_{0,4} + 64E_{\infty,4}$. So that $m_2 = E_{0,4}E_{\infty,4}^{-1}$. Thus

$$\frac{\mathrm{d}m_2}{\mathrm{d}z} = \frac{\mathrm{d}j_2}{\mathrm{d}z} = -2\pi\mathrm{i}E_{\infty,2}E_{0,4}E_{\infty,4}^{-1}$$

and $D(m_2) = -E_{\infty,2}E_{0,4}E_{\infty,4}^{-1}$. It follows that

$$T_{2,h} = -E_{\infty,4}^{1-r}D(m_2).$$

Therefore

$$\begin{aligned} E_{\infty,4}^{1-r}D(m_2) &= D(E_{\infty,4}^{1-r}m_2) - m_2 D(E_{\infty,4}^{1-r}) \\ &= D(E_{\infty,4}^{1-r}m_2) - m_2(1-r)E_{\infty,4}^{-r}D(E_{\infty,4}) \\ &= D(E_{\infty,4}^{1-r}m_2) + (r-1)m_2E_{\infty,4}^{-r}\left(-\frac{1}{r}E_{\infty,4}^{1+r}D(E_{\infty,4}^{-r})\right) \\ &= D(E_{\infty,4}^{1-r}m_2) + \frac{1-r}{r}m_2E_{\infty,4}D(E_{\infty,4}^{-r}) \\ &= D(E_{\infty,4}^{1-r}m_2) + \frac{1-r}{r}E_{0,4}D(E_{\infty,4}^{-r}). \end{aligned}$$

The term $D(E_{\infty,4}^{1-r}m_2)$ makes no contribution to the constant term. Hence the constant term of $T_{2,h}$ is equal to that of $\dfrac{r-1}{r}E_{0,4}D(E_{\infty,4}^{-r})$. We now compute the principal part of $D(E_{\infty,4}^{-r})$.

By Lemma 10.6, for fixed s, if we write

$$E_{\infty,4}^{-s} = q^{-s} \sum_{n=0}^{\infty} R(n) q^n,$$

then $R(0) = 1$ and

$$R(n) = \frac{8s}{n} \sum_{a=1}^{n} \sigma_1^{\text{alt}}(a) R(n-a), \quad \forall\, n > 0. \tag{10.17}$$

Because $\sigma_1^{\text{alt}}(a)$ alternates sign, the alternation of the sign of $R(n)$ follows by an induction from (10.16). So we can write $R(n) = U_n(-1)^n$ with some $U_n > 0$. Therefore we have

$$E_{\infty,4}^{-r} = U_0(-1)^0 q^{-r} + U_1(-1)^1 q^{-r+1} + \cdots + U_{r-1}(-1)^{r-1} q^{-1} + 0 + \cdots,$$

hence

$$\begin{aligned} D(E_{\infty,4}^{-r}) &= -rU_0(-1)^0 q^{-r} + (1-r)U_1(-1)^1 q^{1-r} \\ &\quad + \cdots + (-1)U_{r-1}(-1)^{r-1} q^{-1} + 0 + \cdots \\ &= V_r(-1)^1 q^{-r} + V_{r-1}(-1)^2 q^{1-r} + \cdots + V_1(-1)^r q^{-1} + 0 + \cdots, \end{aligned}$$

where $V_i = iU_{r-i} > 0$ for $1 \leqslant i \leqslant r$. On the other hand, the Fourier coefficient of $q^n (n \geqslant 0)$ in the expansion of $E_{0,4}$ is $W_n(-1)^n$ for positive W_n, by the definition of $E_{0,4}$. Therefore the constant term of $E_{0,4}D(E_{\infty,4}^{-r})$ is equal to

$$\sum_{n=1}^{r} V_n(-1)^{r+1-n} W_n(-1)^n = (-1)^{r+1} \sum_{n=1}^{r} V_n W_n,$$

so that the constant term of $T_{2,h}$ is equal to

$$\frac{r-1}{r}(-1)^{r+1} \sum_{n=1}^{r} V_n W_n \neq 0$$

for $h \geqslant 4, h \equiv 0(4)$ (since $r > 1$ in this case).

Now we assume that $h \equiv 2 \pmod 4$. We have proved the following equality above

$$\frac{\mathrm{d}}{\mathrm{d}z} m_2(z) = \frac{\mathrm{d}}{\mathrm{d}z} j_2(z) = -2\pi \mathrm{i} E_{\infty,2}(z) E_{0,4}(z) E_{\infty,4}^{-1}(z).$$

So $D(m_2(z)) = -E_{\infty,2}(z)E_{0,4}(z)E_{\infty,4}^{-1}(z)$. This implies that

$$T_{2,h} = E_{\infty,2}^2 E_{0,4} E_{\infty,4}^{-1-r} = -E_{\infty,2} E_{\infty,4}^{-r} D(m_2).$$

Therefore

$$
\begin{aligned}
& E_{\infty,2}E_{\infty,4}^{-r}D(m_2) \\
&= D(E_{\infty,2}E_{\infty,4}^{-r}m_2) - E_{\infty,2}m_2 D(E_{\infty,4}^{-r}) - E_{\infty,4}^{-r}m_2 D(E_{\infty,2}) \\
&= D(E_{\infty,2}E_{\infty,4}^{-r}m_2) - E_{\infty,2}m_2(-r)E_{\infty,4}^{-r-1}D(E_{\infty,4}) - E_{0,4}E_{\infty,4}^{-r-1}D(E_{\infty,2}) \\
&= D(E_{\infty,2}E_{\infty,4}^{-r}m_2) - E_{\infty,2}m_2(-r)E_{\infty,4}^{-r-1}\left(\frac{1}{-r-1}\right)E_{\infty,4}^{r+2}D(E_{\infty,4}^{-r-1}) \\
&\quad -E_{0,4}E_{\infty,4}^{-r-1}D(E_{\infty,2}) \\
&= D(E_{\infty,2}E_{\infty,4}^{-r}m_2) - \frac{r}{r+1}E_{\infty,2}m_2E_{\infty,4}^{-r-1}E_{\infty,4}^{r+2}D(E_{\infty,4}^{-r-1}) - E_{0,4}E_{\infty,4}^{-r-1}D(E_{\infty,2}) \\
&= D(E_{\infty,2}E_{\infty,4}^{-r}m_2) - \frac{r}{r+1}E_{0,4}E_{\infty,2}D(E_{\infty,4}^{-r-1}) - E_{0,4}E_{\infty,4}^{-r-1}D(E_{\infty,2}) \\
&= D(E_{\infty,2}E_{\infty,4}^{-r}m_2) - \frac{r}{r+1}E_{0,4}\left(D(E_{\infty,2}E_{\infty,4}^{-r-1}) - E_{\infty,4}^{-r-1}D(E_{\infty,2})\right) \\
&\quad -E_{0,4}E_{\infty,4}^{-r-1}D(E_{\infty,2}) \\
&= D(E_{\infty,2}E_{\infty,4}^{-r}m_2) - \frac{r}{r+1}E_{0,4}D(E_{\infty,2}E_{\infty,4}^{-r-1}) - \frac{1}{r+1}E_{0,4}E_{\infty,4}^{-r-1}D(E_{\infty,2}).
\end{aligned}
$$

The term $D(E_{\infty,2}E_{\infty,4}^{-r}m_2)$ makes no contribution to the constant term of $T_{2,h}$ because for any formal series $\sum_{n=0}^{\infty} b_n q^n$ we have that $D\left(\sum_{n=0}^{\infty} b_n q^n\right) = \sum_{n=0}^{\infty} n b_n q^n$ which has no constant term. Hence we only need to compute the constant terms of $\frac{r}{r+1}E_{0,4}$ $D(E_{\infty,2}E_{\infty,4}^{-r-1})$ and $\frac{1}{r+1}E_{0,4}E_{\infty,4}^{-r-1}D(E_{\infty,2})$.

For any positive integer s, we write

$$E_{\infty,4}^{-s} := q^{-s}\sum_{n=0}^{\infty} R_s(n)q^n.$$

Then by Lemma 10.6 and by an easy induction we can prove that $R_s(n) = (-1)^n U_s(n)$ with $U_s(n) > 0$.

But we know

$$E_{\infty,2}(z) := 1 + 24\sum_{n=1}^{\infty} \sigma_1^{\text{odd}}(n)q^n.$$

Hence we have

$$
\begin{aligned}
E_{\infty,2}E_{\infty,4}^{-r-1} &:= q^{-r-1}\sum_{i=0}^{\infty} a_i q^i \\
&= \left(1 + 24\sum_{n=1}^{\infty} \sigma_1^{\text{odd}}(n)q^n\right)\left(\sum_{n=0}^{\infty}(-1)^n U_{r+1}(n)q^n\right),
\end{aligned}
$$

where

$$a_i = 24 \sum_{j=0}^{i} \sigma_1^{\mathrm{odd}}(j) U_{r+1}(i-j)(-1)^{i-j}, \quad \sigma_1^{\mathrm{odd}}(0) := \frac{1}{24}. \tag{10.18}$$

Hence

$$D(E_{\infty,2} E_{\infty,4}^{-r-1}) = q^{-r-1} \sum_{i=0}^{\infty} (i-r-1) a_i q^i.$$

Noting that the nth Fourier coefficient of

$$E_{0,4} = 1 + 16 \sum_{n=1}^{\infty} \sigma_3^{\mathrm{alt}}(n) q^n$$

has the form $(-1)^n W_n$ with $W_n = (-1)^n 16 \sigma_3^{\mathrm{alt}}(n)$ a positive integer, we see that

$$\begin{aligned} E_{0,4} D(E_{\infty,2} E_{\infty,4}^{-r-1}) &:= \sum_{n=-r-1}^{\infty} a'_n q^n \\ &= \left(\sum_{n=0}^{\infty} (-1)^n W_n q^n \right) \left(\sum_{i=0}^{\infty} (i-r-1) a_i q^{i-r-1} \right). \end{aligned}$$

In particular, we have

$$a'_0 = \sum_{i=0}^{r} (i-r-1) a_i (-1)^{r+1-i} W_{r+1-i}. \tag{10.19}$$

On the other hand, we have

$$E_{0,4} E_{\infty,4}^{-r-1} := \sum_{i=0}^{\infty} b_i q^{i-r-1} = \left(\sum_{n=0}^{\infty} (-1)^n W_n q^n \right) \left(\sum_{n=0}^{\infty} (-1)^n U_{r+1}(n) q^n \right),$$

where

$$b_i := \sum_{j=0}^{i} (-1)^i U_{r+1}(i-j) W_j \tag{10.20}$$

and

$$D(E_{\infty,2}) = 24 \sum_{n=1}^{\infty} n \sigma_1^{\mathrm{odd}}(n) q^n$$

Hence

$$E_{0,4} E_{\infty,4}^{-r-1} D(E_{\infty,2}) := \sum_{n=-r}^{\infty} b'_n q^n = \left(\sum_{i=0}^{\infty} b_i q^{i-r-1} \right) \left(24 \sum_{n=1}^{\infty} n \sigma_1^{\mathrm{odd}}(n) q^n \right).$$

In particular, we have

$$b'_0 = 24 \sum_{i=0}^{r} b_i (r+1-i) \sigma_1^{\mathrm{odd}}(r+1-i) \tag{10.21}$$

From (10.17)–(10.20) we see

$$a_0' = \sum_{i=0}^{r}\sum_{j=0}^{i} 24\sigma_1^{\text{odd}}(j)U_{r+1}(i-j)(-1)^{r+1-j}(i-r-1)W_{r+1-i}$$
$$= 24\sum_{i=0}^{r}(i-r-1)W_{r+1-i}\sum_{j=0}^{i}(-1)^{r+1-j}U_{r+1}(i-j)\sigma_1^{\text{odd}}(j),$$
$$b_0' = 24\sum_{i=0}^{r}\sum_{j=0}^{i} U_{r+1}(i-j)W_j(-1)^i(r+1-i)\sigma_1^{\text{odd}}(r+1-i)$$
$$= 24\sum_{i=0}^{r}(-1)^i(r+1-i)\sigma_1^{\text{odd}}(r+1-i)\sum_{j=0}^{i}U_{r+1}(i-j)W_j.$$

Therefore the constant term of $T_{2,h}$ is equal to

$$-\frac{r}{r+1}a_0' - \frac{1}{r+1}b_0'$$
$$= -\frac{24r}{r+1}\sum_{i=0}^{r}(i-r-1)W_{r+1-i}\sum_{j=0}^{i}(-1)^{r+1-j}U_{r+1}(i-j)\sigma_1^{\text{odd}}(j)$$
$$-\frac{24}{r+1}\sum_{i=0}^{r}(-1)^i(r+1-i)\sigma_1^{\text{odd}}(r+1-i)\sum_{j=0}^{i}U_{r+1}(i-j)W_j$$
$$= -\frac{24}{r+1}\left((-1)^r r\sum_{i=0}^{r}(r+1-i)W_{r+1-i}\sum_{j=0}^{i}(-1)^jU_{r+1}(i-j)\sigma_1^{\text{odd}}(j)\right.$$
$$\left.+\sum_{i=0}^{r}(-1)^i(r+1-i)\sigma_1^{\text{odd}}(r+1-i)\sum_{j=0}^{i}U_{r+1}(i-j)W_j\right)$$
$$= -\frac{24}{r+1}\sum_{i=0}^{r}(r+1-i)((-1)^r rW_{r+1-i} + (-1)^i\sigma_1^{\text{odd}}(r+1-i))$$
$$\times\sum_{j=0}^{i}((-1)^j\sigma_1^{\text{odd}}(j) + W_j).$$

For any nonnegative even integer n, it is clear that $(-1)^n\sigma_1^{\text{odd}}(n)+W_n > 0$ because $\sigma_1^{\text{odd}}(n) > 0$ and $W_n > 0$ for any nonnegative integer n. For any odd integer n we have

$$(-1)^n\sigma_1^{\text{odd}}(n) + W_n = -\sum_{\substack{0<d|n\\ 2\nmid d}} d - 16\sum_{0<d|n}(-1)^d d^3$$
$$= \sum_{\substack{0<d|n\\ 2\nmid d}}(16d^3-d) - \sum_{\substack{0<d|n\\ 2|d}}(16d^3) = \sum_{\substack{0<d|n\\ 2\nmid d}}(16d^3-d) > 0.$$

And it is clear that $rW_{r+1-i}+(-1)^{r-i}\sigma_1^{\mathrm{odd}}(r+1-i)>0$ if $r-i$ is even. If $r-i$ is odd, then $r+1-i$ is even, so we see that

$$\begin{aligned}
&rW_{r+1-i}+(-1)^{r-i}\sigma_1^{\mathrm{odd}}(r+1-i)\\
=&16r\sum_{0<d\mid r+1-i}(-1)^d d^3-\sum_{\substack{0<d\mid n\\ 2\nmid d}} d\\
=&16r\sum_{\substack{0<d\mid r+1-i\\ 2\mid d}} d^3-\sum_{\substack{0<d\mid n\\ 2\nmid d}}(16d^3+d)\geqslant r\sum_{i=1}^{t}16(2d_i)^3-\sum_{i=1}^{t}(16d_i^3+d_i)\\
=&\sum_{i=1}^{t}(128rd_i^2-16d_i^2-1)d_i>0 \quad \text{for all } r\geqslant 1,
\end{aligned}$$

where d_i with $1\leqslant i\leqslant t$ are all distinct odd divisors of $r+1-i$.

This shows that

$$\begin{aligned}
-\frac{r}{r+1}a_0'-\frac{1}{r+1}b_0'=&-\frac{24}{r+1}\sum_{i=0}^{r}(r+1-i)((-1)^r rW_{r+1-i}+(-1)^i\sigma_1^{\mathrm{odd}}(r+1-i))\\
&\times\sum_{j=0}^{i}U_{r+1}(i-j)((-1)^j\sigma_1^{\mathrm{odd}}(j)+W_j)\\
=&(-1)^{r+1}\frac{24}{r+1}\sum_{i=0}^{r}X_i\sum_{j=0}^{i}Y_j,
\end{aligned}$$

where

$$X_i:=(r+1-i)(rW_{r+1-i}+(-1)^{r-i}\sigma_1^{\mathrm{odd}}(r+1-i))>0$$

and

$$Y_j:=U_{r+1}(i-j)((-1)^j\sigma_1^{\mathrm{odd}}(j)+W_j)>0,$$

This proves that the constant term of $T_{2,h}$ is

$$\frac{r}{r+1}a_0'+\frac{1}{r+1}b_0'\neq 0$$

for any positive integer r. This completes the proof. □

Theorem 10.7 *Suppose $f\in G(2,h)$ with* Fourier *expansion at infinity*

$$f(z)=\sum_{n=0}^{\infty}A_nq^n \quad \textit{with } A_0\neq 0.$$

If $h\equiv 0 \pmod 4$, then there is some $A_n\neq 0$ for $1\leqslant n\leqslant r(2,h)$. If $h\equiv 2 \pmod 4$, then there is some $A_n\neq 0$ for $1\leqslant n\leqslant 1+r(2,h)$.

Proof First suppose that $h \equiv 0 \pmod 4$. We denote the coefficient of q^n in the Fourier coefficient of any modular form g at infinity as $c_n(g)$. The meromorphic modular form $T_{2,h}$ has a Fourier expansion

$$T_{2,h} = \sum_{n=-r}^{\infty} c_n(T_{2,h})q^n$$

with $c_{-r}(T_{2,h}) = 1$. By the part (2) of Lemma 10.5, we see that

$$0 = c_0(T_{2,h}f) = \sum_{i=0}^{r} c_{-i}(T_{2,h})A_i.$$

By hypothesis, $A_0 \neq 0$. By Theorem 10.6, $c_0(T_{2,h}) \neq 0$, so

$$A_0 = -(c_0(T_{2,h}))^{-1} \sum_{i=1}^{r} c_{-i}(T_{2,h})A_i,$$

which implies that there exists an n with $1 \leqslant n \leqslant r$ such that $A_n \neq 0$.

If $h \equiv 2 \pmod 4$, then

$$T_{2,h} = \sum_{n=-r-1}^{\infty} c_n(T_{2,h})q^n$$

with $c_{-r-1}(T_{2,h}) = 1$. By the part (2) of Lemma 10.5, we see that

$$0 = c_0(T_{2,h}f) = \sum_{i=0}^{r+1} c_{-i}(T_{2,h})A_i.$$

By hypothesis, $A_0 \neq 0$. By Theorem 10.6, $c_0(T_{2,h}) \neq 0$, so that

$$A_0 = -(c_0(T_{2,h}))^{-1} \sum_{i=1}^{r+1} c_{-i}(T_{2,h})A_i,$$

which implies that there exists an n with $1 \leqslant n \leqslant r+1$ such that $A_n \neq 0$. This completes the proof. □

Theorem 10.8 *Let Q be an even positive definite quadratic form of level two in v variables. Then Q represents a positive integer $2n \leqslant 2 + v/4$ or a positive integer $2n \leqslant 3 + v/4$ according to $v \equiv 0 \pmod 8$ or $v \equiv 4 \pmod 8$ respectively.*

Proof Suppose that Q is an even positive definite quadratic form of level two in v variables with $v \equiv 4 \pmod 8$. Put $v = 8k + 4$. Then by the well-known facts on θ-functions we know that the function defined by

$$\theta_Q(z) := \sum_{n=0}^{\infty} \#Q^{-1}(2n)q^n \in G(2, v/2)$$

is a holomorphic modular form where

$$\#Q^{-1}(2n) := \#\{(x_1, x_2, \cdots, x_v) \in \mathbb{Z}^v | Q(x_1, x_2, \cdots, x_v) = 2n\}.$$

It is clear that $\#Q^{-1}(0) = 1$. Hence by Theorem 10.7 we know that there exists an n_0 with $1 \leqslant n_0 \leqslant 1 + r(2, v/2)$ such that $\#Q^{-1}(2n_0) > 0$. That means Q represents the integer $2n_0$ with $n_0 \leqslant 1 + r(2, v/2) = 1 + r(2, 4k + 2) = 2 + \left[\dfrac{4k+2}{4}\right] = 2 + k$. Hence Q represents the integer $2n_0 \leqslant 2(2 + k) = 4 + 2k = 3 + v/4$. We can prove the case $h \equiv 0 \pmod 8$ similarly. This completes the proof. □

10.3 The Eligible Numbers of a Positive Definite Ternary Quadratic Form

In this section we study the problem of how to find the integers represented by a positive definite ternary quadratic form. It is a classical result that, taken together, the forms of a genus represent all numbers not ruled out by some corresponding congruences B.W. Jones, 1931; B.W. Jones, 1950. Following Kaplansky, we call these the eligible numbers of the genus I. Kaplansky, 1995. But it is very difficult to determine which of these eligible numbers can be represented by a form in the genus. In general we have the following results:

(R1) A positive definite ternary quadratic form f represents all of sufficiently large numbers which are represented by the spinor genus of f. (cf. W. Duke, 1990.)

(R2) Let n_0 be a square-free positive integer represented primitively by the genus of a positive definite ternary quadratic form f with discriminant d, then f primitively represents all of sufficiently large integers n_0t^2 if $(t, 2d) = 1$ and n_0t^2 are primitively represented by the spinor genus of f. (cf. J. Hsia, 1997.)

But there are no effective algorithm to determine all exceptions because (R1) and (R2) are dependent on Siegel's ineffective lower bound for the class numbers and the Iwaniec's estimation for the coefficients of cusp forms (cf. Remark 10.3). Even for the simplest cases, we can not do this. For example, let $f_1 = x^2 + y^2 + 7z^2$, $f_2 = x^2 + 7y^2 + 7z^2$. Then f_1 and $g_1 = x^2 + 2y^2 + 4z^2 + 2yz$ belong to the same genus, f_2 and $g_2 = 2x^2 + 4y^2 + 7z^2 - 2xy$ belong to another genus. The eligible numbers of f_1 and g_1 (f_2 and g_2 respectively)are numbers which are not the product of an odd (even respectively) power of 7 and a number congruent to 3, 5 or 6 mod 7 (see Example 10.1 and Example 10.2). We also can not determine which of them are represented by f_1 and f_2 respectively.

In I. Kaplansky, 1995 Kaplansky proved the following result and pointed out the following tables:

Theorem The form f_1 represents all eligible numbers which are multiples of 9;

it also represents all eligible numbers congruent to 2 mod 3 which are not of the form $14t^2$.

List I: Up to 100, 000 there are 27 eligible numbers prime to 7 not represented by f_1: 3, 6, 19, 22, 31, 51, 55, 66, 94, 139, 142, 159, 166, 214, 235, 283, 439, 534, 559, 670, 874, 946, 1726, 2131, 2419, 3559, 4759.

List II: Up to $100,000$ there are 26 eligible numbers congruent to $1, 2$ or 4 mod 7 which are not represented by f_2: 2, 22, 46, 58, 85, 93, 102, 205, 298, 310, 330, 358, 466, 478, 697, 862, 949, 1222, 1402, 1513, 1957, 1978, 2962, 3502, 7165, 10558.

List III: Up to $100,000$ there are 3 eligible numbers prime to 7 not represented by $f_3 = x^2 + 2y^2 + 7y^2$: 5, 20, 158.

List IV: Up to $100,000$ there are 3 eligible numbers congruent to $1, 2$ or $4 \mod 7$ which are not represented by $f_4 = x^2 + 7y^2 + 14z^2$: 2, 74, 506.

It is clear that $14 \cdot 7^{2k} \equiv 2 \pmod 3$ and f_1 does not represent $14 \cdot 7^{2k}$ for any non-negative integer k by a simple induction. We call the numbers of $14 \cdot 7^{2k}$ to be of trivial type. Hence there are indeed eligible numbers of the form $14t^2$ which are missed by f_1. But as Kaplansky pointed out, **List II** shows, that up to $700,000$ there are no further eligible numbers of form $14t^2$ that are missed by f_1 and which are not of trivial type. This motivated Kaplansky to make the following:

Conjecture f_1 represents all eligible numbers congruent to 2 mod 3 which are not of trivial type.

Kaplansky also conjectured that these four lists describe all exceptions, and so our knowledge of the integers represented by f_1 and g_1 would be complete.

In this section we want to show some general results about the eligible numbers of positive definite ternary forms by using modular forms of weight $3/2$, and give an algorithm for the number of representations of a positive integer n by a genus of positive definite ternary quadratic forms which is of an independent interest because it is a generalization of the classical theorem of Gauss concerning the number of representations of a natural number as a sum of three squares. By this algorithm, we can more precisely deal with eligible numbers and prove that the above **Conjecture** holds. We will also show how to use the algorithm to compute the number of representations and eligible numbers of a positive integer n by a genus of a positive definite ternary quadratic forms. We will study the relationships between the numbers of representations of ternary positive definite quadratic forms and elliptic curves.

Now let α, β, γ be square-free positive odd integers with $(\alpha, \beta, \gamma) = 1$, $D = [\alpha, \beta, \gamma]$, and λ_{4m} $(m|D)$ and λ_m $(1 \neq m|D)$ be the unique solution of the following system of linear equations:

$$(\star)\begin{cases}\displaystyle\sum_{m|D}(C_{4m}\cdot\mu(m/d)m^{-1})+\sum_{1\neq m|D}(C_m\cdot\mu(m/d)m^{-1}\\ \displaystyle\quad=\frac{1}{D}\left(\frac{-1}{d}\right)\left(\frac{\alpha\beta/(\alpha,\beta)^2}{(d,\alpha,\beta)(d,l,\gamma)}\right)\left(\frac{\beta\gamma/(\beta,\gamma)^2}{d,\beta,\gamma)(d,l,\alpha)}\right)\left(\frac{\gamma\alpha/(\gamma,\alpha)^2}{(d,\gamma,\alpha)(d,l,\beta)}\right),\\ \displaystyle\sum_{m|D}C_{4m}\cdot\mu(m/d)m^{-1}=\frac{1}{D}\frac{-1}{(D/d)}\left(\frac{\alpha\beta/(\alpha,\beta)^2}{\gamma(\alpha,\beta)(\alpha,\beta,d)^{-1}(\gamma,\alpha\beta d)^{-1}}\right)\\ \displaystyle\quad\times\left(\frac{\beta\gamma/(\beta,\gamma)^2}{\alpha(\beta,\gamma)(\alpha,\beta\gamma d)^{-1}(\beta,\gamma,d)^{-1}}\right)\left(\frac{\gamma\alpha/(\gamma,\alpha)^2}{\beta(\gamma,\alpha)(\beta,\alpha\gamma d)^{-1}(\gamma,\alpha,d)^{-1}}\right)\quad d|D,\end{cases}$$

which will be proved to have a unique solution later (cf. The proof of Theorem 10.9). It is clear that λ_{4m} $(m|D)$ and λ_m $(1\neq m|D)$ are only dependent on α,β,γ.

For positive integers n, D, l we define:

$$\alpha(n)=\begin{cases}3\times 2^{-(1+\nu_2(n))/2}, & \text{if } 2\nmid\nu_2(n),\\ 3\times 2^{-(1+\nu_2(n)/2)}, & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)}\equiv 1(\text{mod } 4),\\ 2^{-\nu_2(n)/2}, & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)}\equiv 3(\text{mod } 8),\\ 0, & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)}\equiv 7(\text{mod } 8)\end{cases}$$

and

$$\beta_{l,p}(n)=\begin{cases}(1+p)p^{(1-\nu_p(ln))/2}, & \text{if } 2\nmid\nu_p(ln),\\ 2p^{1-\nu_p(ln)/2}, & \text{if } 2|\nu_p(ln), \left(\dfrac{-ln/p^{\nu_p(ln)}}{p}\right)=-1,\\ 0, & \text{if } 2|\nu_p(ln), \left(\dfrac{-ln/p^{\nu_p(ln)}}{p}\right)=1.\end{cases}$$

and

$$\beta_3(n,\chi_D,4D)=\sum_{\substack{(ab)^2|n,(ab,2D)=1\\ a,b \text{ positive integers}}}\mu(a)\left(\frac{-n}{a}\right)(ab)^{-1}.$$

Note that $\beta_3(n,\chi_D,4D)=1$ if n is square-free.

Let f be a positive definite ternary quadratic form, $\{f_1=f,\ f_2,\ \cdots,\ f_t\}$ a set of representatives of equivalence class in the genus of f. Denote by $r_i(n)=r(f_i,n)$ the number of representations of n by f_i. Put $G(n)=\displaystyle\sum_{i=1}^{t}\frac{r_i(n)}{O(f_i)}$. With these notations we get the following

Theorem 10.9 *Let α, β, γ be square-free odd positive integers such that $(\alpha,\beta,\gamma)=1$, $f=\alpha x^2+\beta y^2+\gamma z^2$. Let $\mathbb{A}=\{f_1=f,\ f_2,\ \cdots,\ f_t\}$ be a set of representatives for the equivalence classes in the genus of f. Then for any positive integer n we have that*

$$G(n)=r(\alpha,\beta,\gamma;n)\cdot h(-ln),$$

where $l = \alpha\beta\gamma/((\alpha,\beta)^2(\alpha,\gamma)^2(\beta,\gamma)^2)$ *and* $r(\alpha,\beta,\gamma;n)$ *is given by the following formula:*

$$\begin{aligned}
&r(\alpha,\beta,\gamma;n)\\
&= \frac{32}{\omega_{ln}}\alpha(ln)(1-2^{-1}\chi_{-ln}(2))\left(\frac{ln}{\delta_{ln}}\right)^{\frac{1}{2}}\beta_3(ln,\chi_D,4D)\left(\sum_{i=1}^{t}\frac{1}{0(f_i)}\right)\\
&\quad\times\Bigg(\sum_{m|D}(-1)^{t(m)}\lambda_{4m}\prod_{p|D/m}\frac{(1-\chi_{-ln}(p)p^{-1})p^2}{p^2-1}\prod_{p|m}\frac{(1-\chi_{-ln}(p)p^{-1})}{p^2-1}\beta_{l,p}(n)\\
&\quad+\sum_{1\neq m|D}(-1)^{t(m)}\lambda_m\prod_{p|D/m}\frac{(1-\chi_{-ln}(p)p^{-1})p^2}{p^2-1}\prod_{p|m}\frac{(1-\chi_{-ln}(p)p^{-1})}{p^2-1}\beta_{l,p}(n)\Bigg).
\end{aligned}$$

Proof We recall the following notations introduced in Section 7.3

$$\lambda_3(n,4D) = L_{4D}(2,\mathrm{id.})^{-1}L_{4D}(1,\chi_{-n})\beta_3(n,\chi_D,4D)$$

$$A_3(2,n) = \begin{cases} 4^{-1}(1-\mathrm{i})(1-3\cdot 2^{-(1+\nu_2(n))/2}), & \text{if } 2\nmid \nu_2(n),\\ 4^{-1}(1-\mathrm{i})(1-3\cdot 2^{-(1+\nu_2(n)/2)}), & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)}\equiv 1 \pmod 4,\\ 4^{-1}(1-\mathrm{i})(1-2^{-\nu_2(n)/2}), & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)}\equiv 3 \pmod 8,\\ 4^{-1}(1-\mathrm{i}), & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)}\equiv 7 \pmod 8.\end{cases}$$

$$A_3(p,n) = \begin{cases} p^{-1}-(1+p)p^{-(3+\nu_p(n))/2}, & \text{if } 2\nmid \nu_p(n),\\ p^{-1}-2p^{-1-\nu_p(n)/2}, & \text{if } 2|\nu_p(n), \left(\dfrac{-n/p^{\nu_p(n)}}{p}\right)=-1,\\ p^{-1}, & \text{if } 2|\nu_p(n), \left(\dfrac{-n/p^{\nu_p(n)}}{p}\right)=1,\end{cases}$$

$$L_N(s,\chi) = \sum_{(n,N)=1}^{\infty}\chi(n)n^{-s} = \prod_{p\nmid N}(1-\chi(p)p^{-S})^{-1},$$

$$\beta_3(n,\chi_D,4D) = \sum_{\substack{(ab)^2|n,(ab,2D)=1\\ a,b \text{ positive integers}}}\mu(a)\left(\frac{-n}{a}\right)(ab)^{-1},$$

where $\nu_2(n)$ is the maximal non–negative integer such $p^{\nu_2(n)}|n$.

We define functions $g(\chi_l,4m,4D)(z)$ $(m|D)$ and $g(\chi_l,m,4D)(z)$ $(m\neq 1, m|D)$, where D is a square–free odd positive integer and $l|D$ as follows:

$$\begin{aligned}
g(\chi_l,4D,4D)(z) &= 1-4\pi(1+\mathrm{i})l^{\frac{1}{2}}\sum_{n=1}^{\infty}\lambda_3(ln,4D)(A(2,ln)-4^{-1}(1-\mathrm{i}))\\
&\quad\times\prod_{p|D}(A(p,ln)-p^{-1})n^{\frac{1}{2}}\exp\{2\pi\mathrm{i}nz\},
\end{aligned}$$

$$g(\chi_l,4m,4D)(z) = -4\pi(1+\mathrm{i})l^{\frac{1}{2}}\sum_{n=1}^{\infty}\lambda_3(ln,4D)(A(2,ln)-4^{-1}(1-\mathrm{i}))$$

$$\times \prod_{p|m}(A(p,ln)-p^{-1})n^{\frac{1}{2}}\exp\{2\pi i nz\}, \forall D \neq m|D,$$

$$g(\chi_l,m,4D)(z)=2\pi l^{\frac{1}{2}}\sum_{n=1}^{\infty}\lambda(ln,4D)\prod_{p|m}(A(p,ln)-p^{-1})n^{\frac{1}{2}}\exp\{2\pi i nz\}.$$

By the results of Section 7.3, the set of functions

$$g(\chi_l,4m,4D)(m|D), \quad g(\chi_l,m,4D), \quad 1\neq m|D$$

is a basis of $\mathcal{E}(4D,\ 3/2,\ \chi_l)$, and we have

$$V(g(\chi_l,4m,4D),1/\alpha) = -4^{-1}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1}l^{\frac{1}{2}}(l,\alpha)^{-\frac{1}{2}}\varepsilon^{-1}_{\alpha/(l,\alpha)}\left(\frac{l/(l,\alpha)}{d/(l,\alpha)}\right),$$
$$V(g(\chi_l,4m,4D),1/(4\alpha)) = \mu(m/\alpha)\alpha m^{-1}l^{\frac{1}{2}}(l,\alpha)^{-\frac{1}{2}}\varepsilon_{l/(l,\alpha)}\left(\frac{\alpha/(l,\alpha)}{l/(l,\alpha)}\right),$$
$$V(g(\chi_l,4m,4D),1/(2\alpha)) = 0,$$
$$V(g(\chi_l,m,4D),1/\alpha) = -4^{-1}(1+\mathrm{i})\mu(m/\alpha)\alpha m^{-1}l^{\frac{1}{2}}(l,\alpha)^{-\frac{1}{2}}\varepsilon^{-1}_{\alpha/(l,\alpha)}\left(\frac{l/(l,\alpha)}{\alpha/(l,\alpha)}\right),$$
$$V(g(\chi_l,m,4D),1/(2\alpha)) = 0,$$
$$V(g(\chi_l,m,4D),1/(4\alpha)) = 0,$$

where α is any positive divisor odd D and $V(f,p)$ represents the value of f at the cusp point p.

For $f=\alpha x^2+\beta y^2+\gamma z^2$, we see that $\theta_f(z)\in G(4D,3/2,\chi_l)$ and $\theta(\text{gen}.f,z)\in \mathcal{E}(4D,3/2,\chi_l)$ by the results in Section 10.1, where $D=[\alpha,\beta,\gamma]$ and $l=\alpha\beta\gamma/((\alpha,\beta)^2 \cdot(\alpha,\gamma)^2(\beta,\gamma)^2)$. Therefore there exist complex numbers $c_{4m}(m|D)$ and $c_m(m|D,m\neq 1)$ such that

$$\theta(\text{gen}.f,z)=\sum_{m|D}c_{4m}g(\chi_l,4m,4D)+\sum_{1\neq m|D}c_m g(\chi_l,m,4D).$$

If we can compute explicitly these complex numbers, then we can obtain the explicit expression of $G(n):=\sum_{i=1}^{t}\frac{r_i(n)}{O(f_i)}$ by comparing the Fourier coefficients of the two sides of the above equality. In order to do this, we only need to calculate the values of $\theta(\text{gen}.f,z)$ at cusp points.

Claim 1 Let d/c be a cusp point $(c>0,(c,d)=1)$. Then

$$V(\theta,d/c)=\begin{cases}\varepsilon_d^{-1}\left(\dfrac{d}{c}\right), & \text{if } 4|c,\\ \dfrac{1-i}{2}\varepsilon_c\left(\dfrac{d}{c}\right), & \text{if } 2\nmid c,\\ 0, & \text{if } 2\parallel c,\end{cases}$$

where $\theta(z) = \sum_{m=-\infty}^{\infty} \exp\{m^2 z\}$.

Claim 2 Let d be a square-free odd positive integer, then

$$\varepsilon_d = \prod_{p|d} \varepsilon_p \left(\frac{dp^{-1}}{p}\right).$$

The proofs of these two claims are just simple calculations, and hence they are omitted.

It is easy to see that for square-free positive odd D, $S(4D) := \{1/d,\, 1/2d,\, 1/4d \mid d|D\}$ is a representative system of all equivalent classes of cusp points of $\Gamma_0(4D)$.

Claim 3 Let be $f = \alpha x^2 + \beta y^2 + \gamma z^2$, where α, β, γ are square–free positive odd integers such that $(\alpha, \beta, \gamma) = 1$. Then

$$\begin{aligned}
V(\theta_f, 1/d) =& -\frac{(1+\mathrm{i})dl^{1/2}}{4D(l,d)^{1/2}} \varepsilon_{d/(d,l)}^{-1} \left(\frac{-1}{d}\right) \left(\frac{l/(l,d)}{d/(l,d)}\right) \\
& \cdot \left(\frac{\alpha\beta/(\alpha,\beta)^2}{(d,\alpha,\beta)(d,l,\gamma)}\right) \left(\frac{\beta\gamma/(\beta,\gamma)^2}{(d,\beta,\gamma)(d,l,\alpha)}\right) \times \left(\frac{\gamma\alpha/(\gamma,\alpha)^2}{(d,\gamma,\alpha)(d,l,\beta)}\right), \\
V(\theta_f, 1/4d) =& dD^{-1} l^{1/2} (l,d)^{-1/2} \varepsilon_{l/(l,d)} \left(\frac{-1}{D/d}\right) \left(\frac{d/(l,d)}{l/(l,d)}\right) \\
& \times \left(\frac{\alpha\beta/(\alpha,\beta)^2}{\gamma(\alpha,\beta)(\alpha,\beta,d)^{-1}(\gamma,\alpha\beta d)^{-1}}\right) \\
& \times \left(\frac{\beta\gamma/(\beta,\gamma)^2}{\alpha(\beta,\gamma)(\alpha,\beta\gamma d)^{-1}(\beta,\gamma,d)^{-1}}\right) \left(\frac{\gamma\alpha/(\gamma,\alpha)^2}{\beta(\gamma,\alpha)(\beta,\alpha\gamma d)^{-1}(\gamma,\alpha,d)^{-1}}\right), \\
V(\theta_f, 1/2d) =& 0,
\end{aligned}$$

where $d|D$.

This is a special case of our general result in Section 10.1. But now we can give a new proof for this fact. We have that

$$\begin{aligned}
V(\theta_f, 1/d) =& \lim_{z\to 0} (-dz)^{3/2} \theta_f \left(z + \frac{1}{d}\right) \\
=& \lim_{z\to 0} (-dz)^{3/2} \theta(\alpha(z+1/d))\theta(\beta(z+1/d))\theta(\gamma(z+1/d)) \\
=& \lim_{z\to 0} (-dz)^{3/2} \theta\left(\alpha z + \frac{\alpha/(\alpha,d)}{d/(\alpha,d)}\right) \theta\left(\beta z + \frac{\beta/(\beta,d)}{\alpha/(\beta,d)}\right) \theta\left(\gamma z + \frac{\gamma/(\gamma,d)}{d/(\gamma,d)}\right) \\
=& \left(\frac{(\alpha,d)(\beta,d)(\gamma,d)}{\alpha\beta\gamma}\right)^{\frac{1}{2}} V\left(\theta, \frac{\alpha/(\alpha,d)}{d/(\alpha,d)}\right) \\
& \cdot V\left(\theta, \frac{\beta/(\beta,d)}{d/(\beta,d)}\right) \cdot V\left(\theta, \frac{\gamma/(\gamma,d)}{d/(\gamma,d)}\right).
\end{aligned}$$

We express d as $d = (d,l) \times \dfrac{d}{(d,l)}$. Suppose that p is a prime factor of d. Then $p|(d,l)$

if and only if only one of α,β,γ is divisible by p, $p|d/(d,l)$ if and only if only two of α,β,γ are divisible by p. This shows that $\alpha\beta\gamma = D^2/l$, $(\alpha,d)(\beta,d)(\gamma,d) = d^2/(d,l)$. Hence by the above claims we obtain that

$$V(\theta_f, 1/d) = -4^{-1}(1+\mathrm{i})dD^{-1}l^{1/2}(d,l)^{-1/2}V_1,$$

where

$$\begin{aligned}
V_1 = &\varepsilon_{d/(\alpha,d)}\varepsilon_{d/(\beta,d)}\varepsilon_{d/(\gamma,d)}\left(\frac{\alpha/(\alpha,d)}{d/(\alpha,d)}\right)\left(\frac{\beta/(\beta,d)}{d/(\beta,d)}\right)\left(\frac{\gamma/(\gamma,d)}{d/(\gamma,d)}\right)\\
= &\prod_{p|d}\varepsilon_p^2\prod_{p|d/(d,l)}\varepsilon_p^{-1}\prod_{p|d/(\alpha,d)}\left(\frac{\alpha d/p}{p}\right)\\
&\prod_{p|d/(\beta,d)}\left(\frac{\beta d/p}{p}\right)\prod_{p|d/(\gamma,d)}\left(\frac{\gamma d/p}{p}\right)\\
= &\left(\frac{-1}{d}\right)\varepsilon_{d/(d,l)}^{-1}\prod_{p|d/(d,l)}\left(\frac{d(p(d,l))^{-1}}{p}\right)\\
&\prod_{p|d/(\alpha,d)}\left(\frac{\alpha d/p}{p}\right)\prod_{p|d/(\beta,d)}\left(\frac{\beta d/p}{p}\right)\prod_{p|d/(\gamma,d)}\left(\frac{\gamma d/p}{p}\right)\\
= &\left(\frac{-1}{d}\right)\varepsilon_{d/(d,l)}^{-1}\left(\frac{\alpha(d,l)}{(d,\beta,\gamma)}\right)\left(\frac{\beta(d,l)}{(d,\gamma,\alpha)}\right)\\
&\left(\frac{\gamma(d,l)}{(d,\alpha,\beta)}\right)\left(\frac{\alpha\beta}{(d,l,\gamma)}\right)\left(\frac{\beta\gamma}{(d,l,\alpha)}\right)\left(\frac{\gamma\alpha}{(d,l,\beta)}\right)\\
= &\left(\frac{-1}{d}\right)\varepsilon_{d/(d,l)}^{-1}\left(\frac{l/(d,l)}{d/(d,l)}\right)\left(\frac{\alpha\beta/(\alpha,\beta)^2}{(d,l,\gamma)(d,\alpha,\beta)}\right)\\
&\left(\frac{\beta\gamma/(\beta,\gamma)^2}{(d,l,\alpha)(d,\beta,\gamma)}\right)\left(\frac{\gamma\alpha/(\gamma,\alpha)^2}{(d,l,\beta)(d,\gamma,\alpha)}\right),
\end{aligned}$$

which implies the expression of $V(\theta_f, 1/d)$.

Similarly we have that

$$\begin{aligned}
V(\theta_f, 1/4d) = &\lim_{z\to 0}(-4dz)^{\frac{3}{2}}\theta_f(z+1/4d)\\
= &\lim_{z\to 0}(-4dz)^{\frac{3}{2}}\theta(\alpha(z+1/4d))\theta(\beta(z+1/4d))\theta(\gamma(z+1/4d))\\
= &\left(\frac{(\alpha,d)(\beta,d)(\gamma,d)}{\alpha\beta\gamma}\right)^{\frac{1}{2}}V\left(\theta, \frac{\alpha/(\alpha,d)}{4d/(\alpha,d)}\right)
\end{aligned}$$

$$\times V\left(\theta, \frac{\beta/(\beta,d)}{4d/(\beta,d)}\right) V\left(\theta, \frac{\gamma/(\gamma,d)}{4d/(\gamma,d)}\right)$$
$$= dD^{-\frac{1}{2}} l^{\frac{1}{2}} (l,d)^{-\frac{1}{2}} V_2,$$

where

$$\begin{aligned} V_2 &= \varepsilon^{-1}_{\alpha/(\alpha,d)} \varepsilon^{-1}_{\beta/(\beta,d)} \varepsilon^{-1}_{\gamma/(\gamma,d)} \left(\frac{d/(\alpha,d)}{\alpha/(\alpha,d)}\right) \left(\frac{d/(\beta,d)}{\beta/(\beta,d)}\right) \left(\frac{d/(\gamma,d)}{\gamma/(\gamma,d)}\right) \\ &= \prod_{p|D/p} \varepsilon_p^{-2} \prod_{p|l/(l,\alpha)} \varepsilon_p \prod_{p|\alpha/(\alpha,d)} \left(\frac{\alpha d/p}{p}\right) \prod_{p|\beta/(\beta,d)} \left(\frac{\beta d/p}{p}\right) \prod_{p|\gamma/(\gamma,d)} \left(\frac{\gamma d/p}{p}\right) \\ &= \varepsilon_{l/(l,d)} \left(\frac{-1}{D/d}\right) \prod_{p|l/(l,d)} \left(\frac{l(p(l,d))^{-1}}{p}\right) \prod_{p|\alpha/(\alpha,d)} \left(\frac{\alpha d/p}{p}\right) \\ &\quad \times \prod_{p|\beta/(\beta,d)} \left(\frac{\beta d/p}{p}\right) \prod_{p|\gamma/(\gamma,d)} \left(\frac{\gamma d/p}{p}\right), \end{aligned}$$

since

$$l/(l,d) = \alpha/(\alpha,\beta\gamma d) \times \beta/(\beta,\gamma\alpha d) \times \gamma/(\gamma,\alpha\beta d).$$

Hence,

$$\begin{aligned} V_2 &= \varepsilon_{l/(l,d)} \left(\frac{-1}{D/d}\right) \left(\frac{\alpha\beta/(\alpha,\beta)^2}{(\alpha,\beta)/(\alpha,\beta,d)}\right) \left(\frac{\beta\gamma/(\beta,\gamma)^2}{(\beta,\gamma)/(\beta,\gamma,d)}\right) \left(\frac{\gamma,\alpha/(\gamma,\alpha)^2}{(\gamma,\alpha)/(\gamma,\alpha,d)}\right) \\ &\quad \times \left(\frac{\alpha dl(d,l)^{-1}(\alpha,l)^{-2}}{\alpha/(\alpha,\beta\gamma d)}\right) \left(\frac{\beta dl(d,l)^{-1}(\beta,l)^{-2}}{\beta/(\beta,\gamma\alpha d)}\right) \left(\frac{\gamma dl(d,l)^{-1}(\gamma,l)^{-2}}{\gamma/(\gamma,\alpha\beta,d)}\right) \\ &= \varepsilon_{l/(l,d)} \left(\frac{-1}{D/d}\right) \left(\frac{d/(d,l)}{l/(d,l)}\right) \left(\frac{\alpha\beta/(\alpha,\beta)^2}{(\alpha,\beta)/(\alpha,\beta,d) \times \gamma/(\gamma,\alpha\beta d)}\right) \\ &\quad \times \left(\frac{\beta\gamma/(\beta,\gamma)^2}{(\beta,\gamma)/(\beta,\gamma,d) \times \alpha/(\alpha,\beta\gamma d)}\right) \left(\frac{\gamma\alpha/(\gamma,\alpha)^2}{(\gamma,\alpha)/(\gamma,\alpha,d) \times \beta/(\beta,\gamma\alpha d)}\right), \end{aligned}$$

which implies the expressions for $V(\theta_f, 1/4d)$. Finally we can show that $V(\theta_f, 1/2d) = 0$ by the fact that $V(\theta, 1/2) = 0$. This completes the proof of Claim 3.

Since $\theta_f(z)$ and $\theta(\text{gen}.f, z)$ have the same values at each cusp point, we see that

$$\begin{aligned} &V(\theta(\text{gen}.f,z),p) = V(\theta_f(z),p) \\ &= \sum_{m|D} C_{4m} V(g(\chi_l, 4m, 4D), p) + \sum_{1\neq m|D} C_m V(g(\chi_l, m, 4D), p) \end{aligned}$$

for each cusp point p. Hence we obtain a system of equations:

$$\left\{\begin{array}{l} \displaystyle\sum_{m|D} C_{4m}V(g(\chi_l,4m,4D),1/\alpha)+\sum_{1\neq m|D} C_mV(g(\chi_l,m,4D),1/\alpha) \\ =V(\theta_f,1/\alpha),(\alpha|D), \\ \displaystyle\sum_{m|D} C_{4m}V(g(\chi_l,4m,4D),1/(2\alpha))+\sum_{1\neq m|D} C_mV(g(\chi_l,m,4D),1/(2\alpha)) \\ =V(\theta_f,1/(2\alpha))=0,(\alpha|D), \\ \displaystyle\sum_{m|D} C_{4m}V(g(\chi_l,4m,4D),1/(4\alpha))+\sum_{1\neq m|D} C_mV(g(\chi_l,m,4D),1/(4\alpha)) \\ =V(\theta_f,1/(4\alpha)),(\alpha|D). \end{array}\right. \tag{10.22}$$

Inserting the values of the functions at cusp points into equality (10.22), we have that

$$\left\{\begin{array}{l} \displaystyle\sum_{m|D}(C_{4m}\cdot\mu(m/d)m^{-1})+\sum_{1\neq m|D}(C_m\cdot\mu(m/d)m^{-1} \\ \displaystyle=\frac{1}{D}\left(\frac{-1}{d}\right)\left(\frac{\alpha\beta/(\alpha,\beta)^2}{(d,\alpha,\beta)(d,l,\gamma)}\right)\left(\frac{\beta\gamma/(\beta,\gamma)^2}{d,\beta,\gamma)(d,l,\alpha)}\right)\left(\frac{\gamma\alpha/(\gamma,\alpha)^2}{(d,\gamma,\alpha)(d,l,\beta)}\right), \\ \displaystyle\sum_{m|D}C_{4m}\cdot\mu(m/d)m^{-1}=\frac{1}{D}\frac{-1}{(D/d)}\left(\frac{\alpha\beta/(\alpha,\beta)^2}{\gamma(\alpha,\beta)(\alpha,\beta,d)^{-1}(\gamma,\alpha\beta d)^{-1}}\right) \\ \displaystyle\qquad\times\left(\frac{\beta\gamma/(\beta,\gamma)^2}{\alpha(\beta,\gamma)(\alpha,\beta\gamma d)^{-1}}(\beta,\gamma,d)^{-1}\right) \\ \displaystyle\qquad\times\left(\frac{\gamma\alpha/(\gamma,\alpha)^2}{\beta(\gamma,\alpha)(\beta,\alpha\gamma d)^{-1}(\gamma,\alpha,d)^{-1}}\right),\quad (d|D). \end{array}\right. \tag{10.23}$$

We must prove that the system (10.23) has a unique solution for C_{4m} $(m|D)$ and C_m $(1\neq m|D)$. This is equivalent to proving that the corresponding homogeneous system has only zero as a solution. Otherwise, suppose that $C_{4m}=\lambda_{4m}$ $(m|D)$ and $C_m=\lambda_m$ $(1\neq m|D)$ is a non-zero solution of (10.23), i.e.,

$$\left\{\begin{array}{l} \displaystyle\sum_{m|D}(\lambda_{4m}\cdot\mu(m/d)m^{-1})+\sum_{1\neq m|D}(\lambda_m\cdot\mu(m/d)m^{-1}=0, \\ \displaystyle\sum_{m|D}\lambda_{4m}\cdot\mu(m/d)m^{-1}=0,\quad d|D. \end{array}\right. \tag{10.24}$$

Consider the following function:

$$h(z)=\sum_{m|D}\lambda_{4m}g(\chi_l,4m,4D)+\sum_{1\neq m|D}\lambda_m g(\chi_l,m,4D),$$

which belongs to the space $\mathcal{E}(4D,3/2,\chi_l)$. We now compute the values of $h(z)$ at all cusp points. For any $d|D$, we see that:

$$\begin{aligned}
V(h(z),1/d) &= \sum_{m|D} \lambda_{4m} V(g(\chi_l,4m,4D),1/d) + \sum_{1\neq m|D} \lambda_m V(g(\chi_l,m,4D),1/d) \\
&= -4^{-1}(1+i)dl^{\frac{1}{2}}(l,d)^{-\frac{1}{2}}\varepsilon_{d/(l,d)}^{-1} \\
&\quad \times \left(\frac{l/(l,d)}{d/(l,d)}\right)\left(\sum_{m|D}\lambda_{4m}\mu(m/d)m^{-1} + \sum_{1\neq m|D}\lambda_m\mu(m/d)m^{-1}\right) \\
&= 0, \\
V(h(z),1/(2d)) &= \sum_{m|D} \lambda_{4m} V(g(\chi_l,4m,4D),1/(2d)) + \sum_{1\neq m|D} \lambda_m V(g(\chi_l,m,4D),1/(2d)) \\
&= \sum_{m|D} \lambda_{4m}\cdot 0 + \sum_{1\neq m|D} \lambda_m \cdot 0 = 0, \\
V(h(z),1/(4d)) &= \sum_{m|D} \lambda_{4m} V(g(\chi_l,4m,4D),1/(4d)) + \sum_{1\neq m|D} \lambda_m V(g(\chi_l,m,4D),1/(4d)) \\
&= dl^{\frac{1}{2}}(l,d)^{-\frac{1}{2}}\varepsilon_{l/(l,d)}\left(\frac{d/(l,d)}{l/(l,d)}\right)\left(\sum_{m|D}\lambda_{4m}\mu(m/d)m^{-1}\right) = 0.
\end{aligned}$$

These imply that the values of modular form $h(z)$ are equal to zero at all cusp points of $\Gamma_0(4D)$. Hence $h(z) \in S(4D,3/2,\chi_l)$ which shows that $h(z) \in S(4D, 3/2,\chi_l)\bigcap \mathcal{E}(4D,3/2,\chi_l) = \{0\}$, i.e.,

$$\sum_{m|D} \lambda_{4m} g(\chi_l,4m,4D) + \sum_{1\neq m|D} \lambda_m g(\chi_l,m,4D) = 0.$$

But $g(\chi_l,4m,4D)$ $(m|D)$ and $g(\chi_l,m,4D)$ $(1\neq m|D)$ are linearly independent. Therefore $\lambda_{4m} = 0$ $(m|D)$ and $\lambda_m = 0$ $(1 \neq m|D)$ which contradicts the assumption for λ_{4m} and λ_m and hence show that the system (10.23) has only zero as a solution.

From (10.23) we can easily calculate explicitly all the $C_m(1\neq m|D)$ and $C_{4m}(m|D)$, it is clear that all these are rational numbers and only dependent on α,β,γ.

That is, we obtain explicitly rational numbers C_m and C_{4m} such that

$$\theta(\text{gen}.f,z) = \sum_{m|D} C_{4m} g(\chi_l,4m,4D) + \sum_{1\neq m|D} C_m g(\chi_l,m,4D). \tag{10.25}$$

On the other hand, let

$$\begin{aligned}
\alpha(n) &= 2(1+i)(4^{-1}(1-i) - A_3(2,n)) \\
&= \begin{cases} 3\times 2^{-(1+\nu_2(n))/2}, & \text{if } 2\nmid \nu_2(n), \\ 3\times 2^{-(1+\nu_2(n)/2)}, & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)} \equiv 1 (\text{mod } 4), \\ 2^{-\nu_2(n)/2}, & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)} \equiv 3 (\text{mod } 8), \\ 0, & \text{if } 2|\nu_2(n), n/2^{\nu_2(n)} \equiv 7 (\text{mod } 8) \end{cases}
\end{aligned} \tag{10.26}$$

and

$$\beta_{l,p}(n) = p^2(p^{-1} - A_3(p, ln))$$
$$= \begin{cases} (1+p)p^{(1-\nu_p(ln))/2}, & \text{if } 2 \nmid \nu_p(ln), \\ 2p^{1-\nu_p(ln)/2}, & \text{if } 2|\nu_p(ln), \left(\dfrac{-ln/p^{\nu_p(ln)}}{p}\right) = -1, \\ 0, & \text{if } 2|\nu_p(ln), \left(\dfrac{-ln/p^{\nu_p(ln)}}{p}\right) = 1. \end{cases} \quad (10.27)$$

Let δ_{ln} be the conductor of the character χ_{-ln} and $h(-ln)$ be the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-ln})$. Then the class number formula shows that

$$h(-ln) = (2\pi)^{-1}\delta_{ln}^{\frac{1}{2}}\omega_{ln}L(1, \chi_{-ln}),$$

where

$$\omega_{ln} = \begin{cases} 6, & \text{if } \delta_{ln} = 3, \\ 4, & \text{if } \delta_{ln} = 4, \\ 2, & \text{if otherwise.} \end{cases}$$

Hence

$$\begin{aligned} \lambda_3(ln, 4D) &= L_{4D}(2, \mathrm{id})^{-1}L_{4D}(1, \chi_{-ln})\beta_3(ln, \chi_D, 4D) \\ &= L(2, \mathrm{id})^{-1}\prod_{p|4D}(1-p^{-2})^{-1}L(1, \chi_{-ln}) \\ &\quad \cdot \prod_{p|4D}(1-\chi_{-ln}(p)p^{-1}) \cdot \beta_3(ln, \chi_D, 4D) \\ &= \frac{6}{\pi^2} \cdot \prod_{p|4D}(1-p^{-2})^{-1}(1-\chi_{-ln}(p)p^{-1}) \\ &\quad \cdot h(-ln) \cdot 2\pi \cdot \omega_{ln}^{-1}\delta_{ln}^{-\frac{1}{2}}\beta_3(ln, \chi_D, 4D) \\ &= \frac{12}{\pi}\prod_{p|4D}\frac{(1-\chi_{-ln}(p)p^{-1})p^2}{p^2-1} \cdot \frac{h(-ln)}{\omega_{ln}\sqrt{\delta_{ln}}} \cdot \beta_3(ln, \chi_D, 4D). \end{aligned}$$

This implies that

$$\begin{aligned} g(\chi_l, 4D, 4D) &= 1 + (-1)^{t(D)}32\sum_{n=1}^{\infty}h(-ln)\omega_{ln}^{-1}\alpha(ln)(1-2^{-1}\chi_{-ln}(2)) \\ &\quad \times \prod_{p|D}\left[\frac{(1-\chi_{ln}(p)p^{-1})}{p^2-1}\beta_{l,p}(n)\right] \\ &\quad \cdot \left(\frac{ln}{\delta_{ln}}\right)^{1/2}\beta_3(ln, \chi_D, 4D)\exp\{2\pi\mathrm{i}nz\}, \end{aligned}$$

$$g(\chi_l, 4m, 4D) = (-1)^{t(m)}32\sum_{n=1}^{\infty}h(-ln)\omega_{ln}^{-1}\alpha(ln)(1-2^{-1}\chi_{-ln}(2))$$

$$\times \prod_{p|D/m} \frac{(1-\chi_{-ln}(p)p^{-1})p^2}{p^2-1} \prod_{p|m} \frac{(1-\chi_{-ln}(p)p^{-1})}{p^2-1} \beta_{l,p}(n)$$

$$\times \left(\frac{ln}{\delta_{ln}}\right)^{\frac{1}{2}} \beta_3(ln, \chi_D, 4D) \exp\{2\pi i n z\},$$

$$g(\chi_l, m, 4D) = (-1)^{t(m)} 32 \sum_{n=1}^{\infty} h(-ln)\omega_{ln}^{-1}(1-2^{-1}\chi_{ln}(2)) \tag{10.28}$$

$$\times \prod_{p|D/m} \frac{(1-\chi_{-ln}(p)p^{-1})p^2}{p^2-1} \prod_{p|m} \frac{(1-\chi_{-ln}(p)p^{-1})}{p^2-1} \beta_{l,p}(n)$$

$$\times \left(\frac{ln}{\delta_{ln}}\right)^{\frac{1}{2}} \beta_3(ln, \chi_D, 4D) \exp\{2\pi i n z\},$$

where $t(m)$ is the number of distinct prime factors of m. Let be $ln = ds^2$ with d square-free, then $\delta_{ln} = d$ or $4d$ according to $d \equiv 1 \pmod 4$ or $d \equiv 2, 3 \pmod 4$ which implies that $\left(\frac{ln}{\delta_{ln}}\right)^{1/2} = \left(\frac{ds^2}{d}\right)^{1/2} = s$ or $\left(\frac{ln}{\delta_{ln}}\right)^{1/2} = \left(\frac{ds^2}{4d}\right)^{1/2} = \frac{s}{2}$ according to $d \equiv 1 \pmod 4$ or $d \equiv 2, 3 \pmod 4$. Anyway, $\left(\frac{ln}{\delta_{ln}}\right)^{1/2}$ is an explicitly determined rational number. Now we compare the Fourier coefficients of the two sides of (10.24), and use (10.27) to obtain that

$$G(n) = r(\alpha, \beta, \gamma; n)h(-ln),$$

where $r(\alpha, \beta, \gamma; n)$ is defined as in 10.9. This completes the proof of the theorem. □

By Theorem 10.9 we obtain the following:

An Algorithm for $G(n)$ and eligible numbers of f:

Input: A positive definite ternary quadratic form f;

Output: $G(n)$ and the set $\mathbb{E}$ of eligible numbers of f;

Step 1: Solve the system $(\star)$;

Step 2: Use Theorem 10.9 to compute $G(n)$;

Step 3: Put $\mathbb{E} = \{n \in \mathbb{N} | r(\alpha, \beta, \gamma; n) = 0\}$.

We will compute some examples with this algorithm.

It is clear that Theorem 10.9 holds indeed for any positive definite ternary quadratic form f with level $4D$ (D a square-free odd positive integer). Hence by Theorem 10.9 we can always give the precise major part for the number $r(f, n)$ of representations for n by f. Especially if the space $S(N, 3/2, \chi_l)$ is the null space, we can obtain the precise formula for $r(f, n)$ by Theorem 10.9. For example, by the dimension formulae for the space of modular forms, we can find that the following spaces are all null spaces:

$$\begin{array}{lll} S(4,3/2,\chi_1), & S(8,3/2,\chi_1), & S(8,3/2,\chi_2), \\ S(12,3/2,\chi_1), & S(12,3/2,\chi_3), & S(16,3/2,\chi_1), \\ S(20,3/2,\chi_1), & S(20,3/2,\chi_5), & S(24,3/2,\chi_1), \\ S(24,3/2,\chi_2), & S(24,3/2,\chi_3), & S(24,3/2,\chi_6), \\ S(32,3/2,\chi_1), & S(32,3/2,\chi_2), & S(64,3/2,\chi_2). \end{array}$$

Hence we can obtain the following formulae: Let be $N(a,b,c;n) = r(ax^2+by^2+cz^2,n)$, $\delta(x) = 1$ or 0 according to x is an integer or not, then

$$\begin{aligned} N(1,1,1;n) &= 2\pi n^{\frac{1}{2}}\lambda(n,4)\alpha(n), \quad \text{(Gauss formula)} \\ N(1,2,2;n) &= 2\pi n^{\frac{1}{2}}\lambda(n,4)\left(\alpha(n) - \delta\left(\frac{n-1}{4}\right) - \delta\left(\frac{n-2}{4}\right)\right), \\ N(1,3,3;n) &= 2\pi n^{\frac{1}{2}}\lambda(n,12)(1/3 - A(3,n))(2-\alpha(n)), \\ N(1,5,5;n) &= 2\pi n^{\frac{1}{2}}\lambda(n,20)\alpha(n)(A(5,n)+1/5), \\ N(2,3,6;n) &= 2\pi n^{\frac{1}{2}}\lambda(n,12)(1/3 + A(3,n))\left(\alpha(n) - \delta\left(\frac{n-1}{4}\right) - \delta\left(\frac{n-2}{4}\right)\right), \text{etc.} \end{aligned}$$

From this point of view we see that Theorem 10.9 is a generalization of the classical result of Gauss concerning the number of representations of a natural number as a sum of three squares.

Corollary 10.1 *Let $f = x^2 + y^2 + pz^2$, p an odd prime, then*

$$G(n) := \sum_{i=1}^{t} \frac{r_i(n)}{O(f_i)} = \begin{cases} \dfrac{32}{\omega_{pn}(p^2-1)} h(-pn)\alpha(pn)(2p - \beta_{p,p}(n))\gamma_p(n) \cdot \left(\displaystyle\sum_{i=1}^{t} \frac{1}{O(f_i)}\right), \\ \qquad\qquad \text{if} p \equiv 1 \pmod 4, \\ \dfrac{32}{\omega_{pn}(p^2-1)} h(-pn)(2 - \alpha(pn))\beta_{p,p}(n)\gamma_p(n) \cdot \left(\displaystyle\sum_{i=1}^{t} \frac{1}{O(f_i)}\right), \\ \qquad\qquad \text{if} p \equiv 3 \pmod 4, \end{cases}$$

where $\gamma_p(n) = (1 - 2^{-1}\chi_{-pn}(2))(pn/\delta_{pn})^{1/2} \displaystyle\sum_{\substack{(ab)^2 | n \\ (ab,2p)=1}} \mu(a)\chi_{-pn}(a)(ab)^{-1}$.

Proof Just as in the proof of Theorem 10.9, we have that $D = l = p$. So by (10.28) we see that $\mathcal{E}(4p, 3/2, \chi_p)$ has a basis as follows:

$$g(\chi_p, 4p, 4p) = 1 - \frac{32}{p^2-1}\sum_{n=1}^{\infty} h(-pn)\omega_{pn}^{-1}\alpha(pn)\beta_{p,p}(n)\gamma_p(n)\exp\{2\pi i nz\},$$
$$g(\chi_p, 4, 4p) = \frac{32p^2}{p^2-1}\sum_{n=1}^{\infty} h(-pn)\omega_{pn}^{-1}\alpha(pn)\gamma_p(n)\exp\{2\pi i nz\},$$
$$g(\chi_p, p, 4p) = -\frac{32}{p^2-1}\sum_{n=1}^{\infty} h(-pn)\omega_{pn}^{-1}\beta_{p,p}(n)\gamma_p(n)\exp\{2\pi i nz\}.$$

We can easily calculate the solution of the system of equations (10.23):

$$\begin{pmatrix} c_4 \\ c_{4p} \\ c_p \end{pmatrix} = \begin{pmatrix} \frac{2}{p} \\ 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

according to $p \equiv 1$ or $3 \pmod 4$. Hence we see that

$$\theta(\text{gen}.f, z) = \begin{cases} g(\chi_p, 4p, 4p) + 2p^{-1}g(\chi_p, 4, 4p), & \text{if } p \equiv 1 (\text{mod } 4), \\ g(\chi_p, 4p, 4p) - 2g(\chi_p, p, 4p), & \text{if } p \equiv 3 (\text{mod } 4). \end{cases}$$

Hence we see that

$$G(n) = \begin{cases} \dfrac{32}{\omega_{pn}(p^2-1)} h(-pn)\alpha(pn)(2p - \beta_{p,p}(n))\gamma_p(n) \cdot \left(\sum_{i=1}^{t} \dfrac{1}{O(f_i)}\right), \\ \qquad\qquad \text{if } p \equiv 1 \pmod 4 \\ \dfrac{32}{\omega_{pn}(p^2-1)} h(-pn)(2 - \alpha(pn))\beta_{p,p}(n)\gamma_p(n) \cdot \left(\sum_{i=1}^{t} \dfrac{1}{O(f_i)}\right), \\ \qquad\qquad \text{if } p \equiv 3 \pmod 4 \end{cases}$$

as stated in the corollary. □

Example 10.3 Let $p = 7$, then $f = f_1 = x^2 + y^2 + 7z^2$ and $g_1 = x^2 + 2y^2 + 4z^2 + 2yz$ belong to the same genus, $O(f_1) = 8$, $O(g_1) = 4$. Hence

$$G_1(n) = \frac{r_1(n)}{8} + \frac{r_1'(n)}{4} = \frac{1}{4}\omega_{7n}^{-1} \cdot (2 - \alpha(7n))\beta_{7,7}(n)\gamma_7(n)h(-7n).$$

□

Corollary 10.2 *Let $f = x^2 + py^2 + pz^2$, p an odd prime, then*

$$G(n) := \sum_{i=1}^{t} \frac{r_i(n)}{O(f_i)}$$

$$= \begin{cases} \dfrac{32}{\omega_n(p^2-1)} h(-n)\alpha(n)(2p - \beta_{1,p}(n))\gamma_p'(n) \left(\displaystyle\sum_{i=1}^{t} \frac{1}{O(f_i)}\right), & \text{if } p \equiv 1 (\text{mod } 4), \\ \dfrac{32}{\omega_n(p^2-1)} h(-n)(2 - \alpha(n))\beta_{1,p}(n)\gamma_p'(n) \left(\displaystyle\sum_{i=1}^{t} \frac{1}{O(f_i)}\right), & \text{if } p \equiv 3 (\text{mod } 4), \end{cases}$$

where $\gamma_p'(n) = (1 - 2^{-1}\chi_{-n}(2))(1 - \chi_{-n}(p) \cdot p^{-1})(n/\delta_n)^{1/2} \sum \mu(a)\chi_{-n}(a)(ab)^{-1}$.

Proof Just as in the proof of Theorem 10.9, we have that $D = p$, $l = 1$. So by (10.28) we see that $\mathcal{E}(4P, 3/2, \chi_1)$ has a basis as follows:

$$g(\chi_1, 4p, 4p) = 1 - \frac{32}{p^2 - 1} \sum_{n=1}^{\infty} h(-n)\omega_n^{-1}\alpha(n)\beta_{1,p}(n)\gamma_p'(n) \exp\{nz\},$$

$$g(\chi_1, 4, 4p) = \frac{32p^2}{p^2 - 1} \sum_{n=1}^{\infty} h(-n)\omega_n^{-1}\alpha(n)\gamma_p'(n) \exp\{nz\},$$

$$g(\chi_1, p, 4p) = -\frac{32}{p^2 - 1} \sum_{n=1}^{\infty} h(-n)\omega_n^{-1}\beta_{1,p}(n)\gamma_p'(n) \exp\{nz\}.$$

We can also calculate the solution of the system of equations (10.23):

$$\begin{pmatrix} c_4 \\ c_{4p} \\ c_p \end{pmatrix} = \begin{pmatrix} \frac{2}{p} \\ 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

according to $p \equiv 1$ or $3 \pmod 4$. Hence we see that

$$\theta(\text{gen.} f, z) = \begin{cases} g(\chi_1, 4p, 4p) + 2p^{-1}g(\chi_1, 4, 4p), & \text{if } p \equiv 1 \,(\text{mod } 4), \\ g(\chi_1, 4p, 4p) - 2g(\chi_1, p, 4p), & \text{if } p \equiv 3 \,(\text{mod } 4). \end{cases}$$

Therefore we see that

$$G(n) = \begin{cases} \dfrac{32}{\omega_n(p^2-1)} h(-n)\alpha(n)(2p - \beta_{1,p}(n))\gamma_p'(n) \cdot \left(\displaystyle\sum_{i=1}^{t} \frac{1}{O(f_i)}\right), \\ \hfill \text{if } p \equiv 1 \pmod 4, \\ \dfrac{32}{\omega_n(p^2-1)} h(-pn)(2 - \alpha(n))\beta_{1,p}(n)\gamma_p'(n) \cdot \left(\displaystyle\sum_{i=1}^{t} \frac{1}{O(f_i)}\right), \\ \hfill \text{if } p \equiv 3 \pmod 4. \end{cases}$$

This completes the proof. □

Example 10.4 Let $f = f_2 = x^2+7y^2+7z^2$, then f_2 and $g_2 = 2x^2+4y^2+7z^2-2xy$ belong to the same genus, $O(f_2) = 8$, $O(g_2) = 4$. Hence

$$G_2(n) := \frac{r_2(n)}{8} + \frac{r_2'(n)}{4} = \frac{1}{4}\omega_n^{-1} \cdot (2-\alpha(n))\beta_{1,7}(n)\gamma_7'(n)h(-n).$$

□

By Corollary 10.1 and Corollary 10.2, we can prove the following

Corollary 10.3 *Let $f_{(p)} = x^2 + y^2 + pz^2$, p an odd prime, then*

(1) *if $p \equiv 3 \pmod 4$, the eligible numbers of the genus of $f_{(p)}$ are numbers which are not the product of an odd power of p and a number n satisfying* $\left(\frac{-n}{p}\right) = 1$;

(2) *if $p \equiv 1 \pmod 8$, the eligible numbers of the genus of $f_{(p)}$ are numbers which are not the product of an even power of 2 and a number congruent to 7 mod 8;*

(3) *if $p \equiv 5 \pmod 8$, the eligible numbers of the genus of $f_{(p)}$ are numbers which are not the product of an even power of 2 and a number congruent to 3 mod 8.*

Corollary 10.4 *Let $g_{(p)} = x^2 + py^2 + pz^2$, p an odd prime, then*

(1) *if $p \equiv 3 \pmod 4$, the eligible numbers of the genus of $g_{(p)}$ are numbers which are not the product of an even power of p and a number n satisfying* $\left(\frac{-n}{p}\right) = 1$;

(2) *if $p \equiv 1 \pmod 4$, the eligible numbers of the genus of $g_{(p)}$ are numbers which are not the numbers n satisfying* $\left(\frac{n}{p}\right) = -1$ *or the product of an even power of 2 and a number congruent to 7 mod 8.*

Proof By definition, a positive integer n is eligible if and only if $G(n) > 0$, i.e., n is not an eligible integer if and only if $G(n) = 0$. If $p \equiv 3 \pmod 4$, then

$$G(n) = \frac{32}{\omega_{pn}} h(-pn)(2-\alpha(pn))\beta_{p,p}(n)\gamma_p(n) \cdot \left(\sum_{i=1}^{t} \frac{1}{O(f_i)}\right),$$

which implies that $G(n) = 0$ if and only if one of the factors at the right end of the above equality equals zero. But it is clear that $\frac{32}{\omega_{pn}} h(-pn)\left(\sum_{i=1}^{t} \frac{1}{O(f_i)}\right) > 0$. So we only need to consider the other three factors. By (10.26) we see that $2-\alpha(pn) \geqslant 2-3/2 = 1/2$. So the only possibilities are that $\beta_{p,p}(n) = 0$ or $\gamma_p(n) = 0$. By (10.27) we know that $\beta_{p,p}(n) = 0$ if and only if $\nu_p(n) \equiv 1 \pmod 2$ and $\left(\frac{-n/p^{\nu_p(n)}}{p}\right) = 1$. Hence if we can prove that $\gamma_p(n) \neq 0$, then this completes the proof of (1). In fact, we can prove the following claim which completes the proof of (1). The proofs of (2) and Corollary 10.4 are similar.

Claim Let D be a square-free positive integer, then

$$\beta_3(n,\chi_D,4D) = \sum_{\substack{(ab)^2|n,(ab,2D)=1\\ a,b \text{ positive integers}}} \mu(a)\left(\frac{-n}{a}\right)(ab)^{-1} \neq 0$$

for any positive integer n.

In fact, by definition, we see that

$$\begin{aligned}\beta_3(n,\chi_D,4D) &= \sum_{\substack{(ab)^2|n,(ab,2D)=1\\ a,b \text{ positive integers}}} \mu(a)\left(\frac{-n}{a}\right)(ab)^{-1}\\ &= \prod_{p\nmid 2D, p|D_n} \sum_{t=0}^{h(p,f_n)} p^{-t} \cdot \prod_{p\nmid 2DD_n}\left(\sum_{t=0}^{\nu_p(f_n)} p^{-t} - p^{-1}\left(\frac{D_n}{p}\right)\sum_{t=0}^{\nu_p(f_n)-1} p^{-t}\right),\end{aligned}$$

where $-n = D_n f_n^2$ such that D_n is a fundamental discriminant and f_n is a positive integer. The above equality implies that $\beta_3(n,\chi_D,4D) \neq 0$. This completes the proofs. □

Example 10.5 The eligible numbers of $f_1 = f_{(7)} = x^2 + y^2 + 7z^2$ are numbers which are not the product of an odd power of 7 and a number congruent to $3, 5$ or 6 mod 7 since $\left(\frac{-n}{7}\right) = 1$ if and only if n congruent to $3, 5$ or 6 mod 7. □

Example 10.6 The eligible numbers of $f_2 = g_{(7)} = x^2 + 7y^2 + 7z^2$ are numbers which are not the product of an even power of 7 and a number congruent to $3, 5$ or 6 mod 7 since $\left(\frac{-n}{7}\right) = 1$ if and only if n is congruent to 3, 5 or 6 mod 7. □

Theorem 10.10 *Let f be a positive definite quadratic form with matrix A. Then there are only finitely many square-free eligible integers which are prime to $2|A|$ and not represented by f.*

Proof The proof of this theorem is similar to the one in W. Duke, 1990. For the sake of completeness we include it here. In order to prove the theorem, we need some of the results in B.W. Jones, 1950, esp. Theorem 86 in B.W. Jones, 1950 which can be described as the following claim:

Claim: Let f be a positive definite ternary quadratic form with matrix A, $d = |A|$, Ω the g.c.d. of the 2-rowed minor determinants of A and $\Delta = qd/\Omega^2$ with q prime to $2d$, then for any eligible number q of the genus of f with $(q, 2d) = 1$ we have that

$$G(A,q) = 2^{-t(d/\Omega^2)} H(\Delta)\rho_\Delta$$

where $t(w)$ is the number of odd prime factors of w, $H(\Delta)$ is the number of properly primitive classes of positive binary forms $ax^2 + 2bxy + cy^2$ of determinant $\Delta = ac - b^2$,

ρ_Δ is a rational number equal to 1/8, 1/6, 1/4, 1/3, 1/2, 2/3, 1, 2, 4 according to the different cases of the values of Δ, and $G(A, q)$ is the number of essentially distinct primitive representations of q by the genus of f. Please compare Theorem 86 in B.W. Jones, 1950 for details.

Now let $\mathbb{G} = \{f = f_1, f_2, \cdots, f_t\}$ be a set of representatives of the genus of f. Define

$$\theta_f(z) = \sum_{m\in\mathbb{Z}^3} e(zmAm^{\mathrm{T}}/2), \quad z \in \mathbb{H},$$
$$O(f) = \#\{S \in M_3(\mathbb{Z}) | SAS^{\mathrm{T}} = A\},$$
$$\theta(\text{gen}.f, z) = \left(\sum_{f_i} \frac{1}{O(f_i)}\right)^{-1} \sum_{f_i} \frac{\theta_{f_i}(z)}{O(f_i)},$$

then we have that

$$\theta_f(z) - \theta(\text{gen}.f, z) \in S(N, 3/2, \chi)$$

by the results in Section 10.1. Now let $r_i(n)$ be the number of representations of n by f_i, then

$$\begin{aligned}\theta_f(z) - \theta(\text{gen}.f, z) &:= \sum_{n=1}^{\infty} a(n)q^n \\ &= \sum_{n=1}^{\infty} r_1(n)q^n - \left(\sum_{f_i} \frac{1}{O(f_i)}\right)^{-1} \sum_{n=1}^{\infty} \left(\sum_{f_i} \frac{r_i(n)}{O(f_i)}\right) q^n.\end{aligned}$$

Now suppose that n_0 is a square-free eligible number of $\mathbb{G}$ which can not be represented by $f = f_1$, i.e., $r_1(n_0) = 0$. Then by Iwaniec's H. Iwaniec, 1987 and Duke's W. Duke, 1988 we have that

$$|a(n_0)| = \left(\sum_{f_i} \frac{1}{O(f_i)}\right)^{-1} \left(\sum_{f_i} \frac{r_i(n)}{O(f_i)}\right) \ll \tau(n_0) n_0^{\frac{3}{7}} (\log 2n_0)^2.$$

On the other hand, let $G_i(n)$ be the essentially distinct primitive representations of n by f_i, it is clear that $2G_i(n) \leqslant r_i(n)$ because every positive definite ternary quadratic form has at least two automorphs. So we see that

$$\begin{aligned}G(A, n) &= \sum_{f_i} G_i(n) \leqslant \frac{1}{2} \sum_{f_i} r_i(n) \\ &\leqslant \frac{O(\mathbb{G})}{2} \sum_{f_i} \frac{r_i(n)}{O(f_i)} = \frac{O(\mathbb{G}) \left(\sum_{f_i} \frac{1}{O(f_i)}\right)}{2} |a(n)|,\end{aligned}$$

where $O(\mathbb{G}) = \max\{O(f_i)\}$. So by the above **Claim** and Siegel's lower bounds for the class numbers we see that

$$|a(n_0)| \gg G(A, n_0) \gg H(\varDelta) = H(n_0 d/\varOmega^2) \gg n_0^{1/2-\epsilon}.$$

Comparing these two estimations we see that there are only finitely many square-free eligible integers prime to $2|A|$ which can not be represented by f. This completes the proof. □

Remark 10.2 Notice that there are some similarities between our Theorem 10.9 and Theorem 86 in B.W. Jones, 1950, but they differ from one another in the following aspects:

(1) In general $G(n) \neq G(A, n)$ and there is no simple equality between them. Of course we have the inequality $G(n) \leqslant G(A, n) \leqslant \dfrac{O(\mathbb{G})}{2} G(n)$ just as we saw in the proof of Theorem 10.10;

(2) In Jones' Theorem 86, it is assumed that $(n, N) = 1$ where N is the level of the quadratic form f. But we need not this assumption in our Theorem 10.9;

(3) Jones' Theorem 86 can not tell us which are the eligible numbers for the genus but our Theorem 10.9 can do this (cf. Example 10.5 and Example 10.6). Anyway neither does our Theorem 10.9 contain Jones' Theorem 86, nor is the converse the case.

Since we employed Theorem 86 (i.e., our **Claim**) in B.W. Jones, 1950 in our proof of Theorem 10.10, we have to limit ourselves to the case with n_0 prime to $2d$. For the case with n_0 not prime to $2|A|$, we may employ our Theorem 10.9. For a concrete positive definite ternary quadratic form f, we can always investigate any square-free natural number n (prime or not prime to $2|A|$) by Theorem 10.9. For example we take the forms in Corollary 10.1 and Corollary 10.2. Suppose that $p \equiv 3 \pmod 4$, N a square-free eligible number not represented by $f_p = x^2 + x^2 + pz^2$ or $f_p = x^2 + py^2 + pz^2$, by (10.26), (10.26):

$$\begin{aligned}
\alpha(pN) &= \alpha(N) = \frac{3}{2},\ 1 \text{ or } 0,\\
\beta_{p,p}(N) &= p + 1 \text{ or } 2,\\
\gamma_p(N) &= \left(1 - \frac{\chi_{-pN}(2)}{2}\right)\left(\frac{pN}{\delta_{pN}}\right)^{\frac{1}{2}} \geqslant \frac{1}{4},\\
\beta_{1,p}(N) &= p + 1 \text{ or } 2p,\\
\gamma'_p(N) &= \left(1 - \frac{\chi_{-N}(2)}{2}\right)\left(1 - \frac{\chi_{-N}(p)}{p}\right)\left(\frac{N}{\delta_N}\right)^{\frac{1}{2}} \geqslant \frac{p-1}{4p}.
\end{aligned}$$

Then Corollary 10.1 and Corollary 10.2 imply that

$$|a(N)| = \left(\sum_{f_i} \frac{1}{O(f_i)}\right)^{-1} \left(\sum_{f_i} \frac{r_i(n)}{O(f_i)}\right) \gg h(-pN) \gg N^{1/2-\epsilon},$$

$$|a(N)| = \left(\sum_{f_i} \frac{1}{O(f_i)}\right)^{-1} \left(\sum_{f_i} \frac{r_i(n)}{O(f_i)}\right) \gg h(-N) \gg N^{1/2-\epsilon}$$

because of Siegel's lower bounds for class numbers. Together with the estimations in H. Iwaniec, 1987 and W. Duke, 1988 as above, we obtain that there exist at most finitely many square-free eligible integers which are not represented by $f_p = x^2 + y^2 + pz^2$ or $f_p = x^2 + py^2 + pz^2$ for $p \equiv 3 \pmod 4$. We can similarly discuss this phenomenon for $p \equiv 1 \pmod 4$.

Remark 10.3 Even though there exist only finitely many square-free eligible numbers prime to $2|A|$ which can not be represented by a positive definite ternary quadratic forms, it is not implementable to find all of these eligible numbers through computation for two reasons: ① Siegel's lower bounds for class numbers are not effective; ② it is impossible to obtain a contradiction through computation even if we assume that the lower bounds are effective since we have to compute all of n with $n^{1/2} \leqslant \tau(n)n^{3/7}(\log(2n))^2$ which requires that n is about 10^{75}. Even if we replace Iwaniec's bound by a sharper bound, cf. V.A. Bykovskii, 1998, we also can not implement the algorithm to find all of these exceptional eligible integers by calculation.

Theorem 10.11 *Let $\mathbb{A} = \{f_1, f_2, \cdots, f_t\}$ be a set of representatives of the genus of a positive definite ternary quadratic form of level N. Assume that there are the following linear combinations of Theta- functions:*

$$\widetilde{f}_i(z) := \sum_{n=1}^{\infty} b_i(n)q^n = \sum_{j=1}^{i+1} \alpha_{i,j}\theta(f_j)$$

with $\alpha_{i,1}\alpha_{i,i+1} \neq 0$ for $1 \leqslant i \leqslant t-1$, such that $\widetilde{f}_i(z)$ is an eigenfunction for all Hecke *operators whose Shimura lifting is a cusp form corresponding to an elliptic curve E_i. Then we can find an effectively determinable finite set $P_{\mathbb{A}} = \{p_0, p_1, \cdots, p_s\}$ of primes such that for every square-free eligible number n_0 of $\mathbb{A}$ with $(n_0, N) = 1$* (i.e., *$(n_0, N) = 1$ and n_0 can be represented by one of the forms in $\mathbb{A}$) and for every prime p not in $P_{\mathbb{A}}$, we have that $p^2 n_0$ can be represented by f_1.*

Proof We only consider the case $t = 3$ because the general case is similar. Let N be the level of f_1, P_N the set of all distinct prime factors of N, and $F_i(z) := \sum_{n=1}^{\infty} B_i(n)q^n$ the Shimura lifting of $\widetilde{f}_i(z)$. Since $\widetilde{f}_i(z)$ is an eigenfunction for all Hecke operators, there exist complex numbers α_{ip} such that $T_{p^2}(\widetilde{f}_i(z)) = \alpha_{ip}\widetilde{f}_i(z)$. But Hecke operators commute with Shimura liftings. Therefore

$$T_p(F_i(z)) = T_p(S(\widetilde{f_i}(z))) = S(T_{p^2}(\widetilde{f_i}(z))) = S(\alpha_{ip}\widetilde{f_i}(z)) = \alpha_{ip}F_i(z).$$

But because $F_i(z)$ is a new form corresponding to the elliptic curve E_i, it shows that $T_p(F_i(z)) = B_i(p)F_i(z)$. Hence we see that $\alpha_{ip} = B_i(p)$ for any $p \notin P_N$. This implies that

$$B_i(p)b_i(n) = b_i(p^2n) + \chi(p)\left(\frac{-n}{p}\right)b_i(n) + pb_i(n/p^2)$$

for any prime p with $(p, N) = 1$ and any positive integer n. Especially for any square-free positive integer n we have that

$$B_i(p)b_i(n) = b_i(p^2n) + \chi(p)\left(\frac{-n}{p}\right)b_i(n).$$

Hence we see that

$$\alpha_{11}r_1(p^2n) + \alpha_{12}r_2(p^2n) = (\alpha_{11}r_1(n) + \alpha_{12}r_2(n))\left(B_1(p) - \chi(p)\left(\frac{-n}{p}\right)\right) \quad (10.29)$$

$$\begin{aligned}&\alpha_{21}r_1(p^2n) + \alpha_{22}r_2(p^2n) + \alpha_{23}r_3(p^2n)\\ =&(\alpha_{21}r_1(n) + \alpha_{22}r_2(n) + \alpha_{23}r_3(n))(B_2(p) - \chi(p)\left(\frac{-n}{p}\right),\end{aligned} \quad (10.30)$$

where $r_i(n)$ is the number of representations of n by f_i. We want to prove that for any square-free eligible number n_0 of $\mathbb{A}$ which is prime to N and not represented by f_1, p^2n_0 can be represented by f_1 where $p \notin P_{\mathbb{A}}$ and $P_{\mathbb{A}}$ containing P_N is an effectively determinable finite set of primes. Otherwise, suppose that $p \notin P_N$ is a prime such that p^2n_0 can not be represented by f_1. Let be $n = n_0$ in (10.29) and (10.30), then

$$r_2(p^2n_0) = r_2(n_0)\left(B_1(p) - \chi(p)\left(\frac{-n_0}{p}\right)\right), \quad (10.31)$$

$$\alpha_{22}r_2(p^2n_0) + \alpha_{23}r_3(p^2n_0) = \alpha_{22}r_2(n_0) + \alpha_{23}r_3(n_0))\left(B_2(p) - \chi(p)\left(\frac{-n_0}{p}\right)\right), \quad (10.32)$$

since $r_1(n_0) = r_1(p^2n_0) = 0$. By (10.31) and (10.32) it is clear that

$$\begin{aligned}\alpha_{23}r_3(p^2n_0) :=& \alpha r_2(n_0) + \beta r_3(n_0)\\ =& \alpha_{22}(B_2(p) - B_1(p))r_2(n_0) + \alpha_{23}\left(B_2(p) - \chi(p)\left(\frac{-n_0}{p}\right)\right)r_3(n_0).\end{aligned}$$

Now let $G(n)$ and $G_i(n)$ be the essentially distinct primitive representations of n by $\mathbb{A}$ and f_i respectively. Then we have that $2G_i(n) \leqslant r_i(n)$ and $G_i(n) \geqslant \dfrac{r_i(n)}{O(f_i)}$. So

$$G(n) = \sum_{i=1}^{t} G_i(n) \leqslant \frac{1}{2}\sum_{i=1}^{t} r_i(n) \leqslant \frac{O(\mathbb{A})}{2}\sum_{i=1}^{t}\frac{r_i(n)}{O(f_i)},$$

$$G(n) = \sum_{i=1}^{t} G_i(n) \geqslant \sum_{i=1}^{t}\frac{r_i(n)}{O(f_i)},$$

where $O(\mathbb{A}) = \max\{O(f_i)\}$. From these and the **Claim** in the proof of Theorem 10.10 we see that

$$\frac{H(p^2\Delta)\rho_{p^2\Delta}}{H(\Delta)\rho_\Delta} = \frac{G(p^2n_0)}{G(n_0)} \leqslant \frac{O(\mathbb{A})}{2} \frac{\sum_{i=1}^{t} \frac{r_i(p^2n_0)}{O(f_i)}}{\sum_{i=1}^{t} \frac{r_i(n_0)}{O(f_i)}}$$
$$= \frac{O(\mathbb{A})}{2} \frac{\delta_2 r_2(p^2n_0) + \delta_3 r_3(p^2n_0)}{\delta_2 r_2(n_0) + \delta_3 r_3(n_0)}$$
$$= \frac{O(\mathbb{A})}{2} \frac{\delta_2 r_2(p^2n_0) + \delta_3\alpha_{23}^{-1}\alpha r_2(n_0) + \delta_2\alpha_{23}^{-1}\beta r_3(n_0)}{\delta_2 r_2(n_0) + \delta_3 r_3(n_0)}, \quad (10.33)$$

where $\delta_i = \dfrac{1}{O(f_i)}$ and $\Delta = n_0 d/\Omega^2$ as in the proof of Theorem 10.10. Now consider two cases:

Case (1) Suppose that $r_3(n_0) \leqslant r_2(n_0)$, then (10.31)–(10.33) show that

$$\frac{1}{3}(p-1) \leqslant \frac{O(\mathbb{A})}{2} \frac{\left|\delta_2 \frac{r_2(p^2n_0)}{r_2(n_0)} + \delta_3\alpha\alpha_{23}^{-1} + \delta_3\beta\alpha_{23}^{-1}\frac{r_3(n_0)}{r_2(n_0)}\right|}{\delta_2 + \delta_3\frac{r_3(n_0)}{r_2(n_0)}}$$
$$\leqslant \frac{O(\mathbb{A})}{2} \frac{\delta_2 \left|B_1(p) - \chi(p)\left(\frac{-n_0}{p}\right)\right| + |\ \delta_3\alpha\alpha_{23}^{-1}\ | + |\ \delta_3\beta\alpha_{23}^{-1}\ |}{\delta_2}.$$

Case (2) Suppose that $r_2(n_0) \leqslant r_3(n_0)$, a similar computation shows that

$$\frac{1}{3}(p-1) \leqslant \frac{O(\mathbb{A})}{2} \frac{\left|\delta_2 \frac{r_2(p^2n_0)}{r_3(n_0)} + \delta_3\beta\alpha_{23}^{-1} + \delta_3\alpha\alpha_{23}^{-1}\frac{r_2(n_0)}{r_3(n_0)}\right|}{\delta_3 + \delta_2\frac{r_2(n_0)}{r_3(n_0)}}$$
$$\leqslant \frac{O(\mathbb{A})}{2} \frac{\delta_2 \left|\frac{r_2(p^2n_0)}{r_2(n_0)}\right| + |\ \delta_3\beta\alpha_{23}^{-1}\ | + |\ \delta_3\alpha\alpha_{23}^{-1}\ |}{\delta_3}$$
$$\leqslant \frac{O(\mathbb{A})}{2} \frac{\delta_2 \left|B_1(p) - \chi(p)\left(\frac{-n_0}{p}\right)\right| + |\ \delta_3\alpha\alpha_{23}^{-1}\ | + |\ \delta_3\beta\alpha_{23}^{-1}\ |}{\delta_3},$$

where we used the facts that $H(p^2\Delta)/H(\Delta) = p - \left(\dfrac{\Delta}{p}\right)$ and $\rho_{p^2\Delta}/\rho_\Delta \geqslant 1/3$ (cf. Theorem 86 in B.W. Jones, 1950). Anyway we have obtained the following inequality:

$$p - 1 \leqslant C_1 \mid B_1(p) \mid + C_2 \mid B_2(p) \mid + C_3,$$

where C_1, C_2, C_3 are positive constants only dependent on α_{ij} and $O(f_i)$. On the other hand we have that $| B_i(p) | \leqslant 2p^{1/2}$ which implies that

$$p - 1 \leqslant 2(C_1 + C_2)\sqrt{p} + C_3.$$

It is clear that this inequality only holds for finitely many primes. Denote it by P. Then for any $p \notin P_{\mathbb{A}} = P \bigcup P_N$ we have that $p^2 n_0$ can be represented by f_1 which completes the proof. □

The argumentation in the above proof implies the following

Corollary 10.5 *Let $\mathbb{A} = \{f, g\}$ be a genus consisting of two equivalence classes such that $\widetilde{f}(z) = \alpha\theta(f) + \beta\theta(g)$ is an eigenfunction for all* Hecke *operators and its* Shimura *lifting is a cusp form corresponding to an elliptic curve E. Then for any eligible integer n_0 which is prime to $2|A|$ and not represented by f and any prime $p \notin P_{\mathbb{A}}$, $p^2 n_0$ can be represented by f where $P_{\mathbb{A}} = \left\{ p \text{ prime} \mid p|N \text{ or } \frac{1}{3}(p-1) \leqslant \frac{O(\mathbb{A})}{2}(2\sqrt{p}+1) \right\}$ and N is the level of f.*

Remark 10.4 Just as pointed out in Remark 10.2, to investigate the case n not prime to the level or to obtain more precise result about the set $P_{\mathbb{A}}$, we may employ our Theorem 10.9. The following proof of Theorem 10.12 is an example together with the ideas in Theorem 10.11 and Theorem 10.9.

Theorem 10.12 *Let be $f_2 = x^2 + 7y^2 + 7z^2$. If n is a positive integer with $\left(\frac{n}{7}\right) = 1$* (i.e., *$n$ is an eligible integer prime to* 7) *which can not be represented by f_2, then n is square-free.*

Proof By Example 10.4 and the fact that n is an eligible integer, we know that

$$0 < G(n) := \frac{r_2(n)}{8} + \frac{r_2'(n)}{4} = \frac{1}{4}\omega_n^{-1}(2-\alpha(n))\beta_{1,7}(n)\gamma_7'(n)h(-n), \tag{10.34}$$

where $r_2(n)$ and $r_2'(n)$ denote the numbers of representations of n by f_2 and $g_2 = 2x^2 + 4y^2 + 7z^2 - 2xy$ respectively. We also easily know that

$$\widetilde{f}_2(z) := \sum_{n=1}^{\infty} b(n)\mathrm{e}^{2\pi \mathrm{i} nz} = \frac{1}{2}\sum_{n=1}^{\infty}(r_2(n) - r_2'(n))\exp\{2\pi \mathrm{i} nz\},$$

is an eigenfunction of all Hecke operators T_{n^2} in the space $S(28, 3/2, \chi_1)$ by a direct computation. And the Shimura lifting $F_2(z) = S(\widetilde{f}_2(z))$ of $\widetilde{f}_2(z)$ is a new form with weight 2, character χ_1 and level 14, i.e., $F_2(z) \in S^{\text{new}}(14, 3/2, \chi_1)$. So there exist complex numbers α_n such that $T_{n^2}(\widetilde{f}_2(z)) = \alpha_n \widetilde{f}_2(z)$. But Hecke operators commute with the Shimura lifting. So we see that

$$T_n(F_2(z)) = T_n(S(\widetilde{f}_2(z))) = S(T_{n^2}(\widetilde{f}_2(z))) = S(\alpha_n \widetilde{f}_2(z)) = \alpha_n S(\widetilde{f}_2(z)) = \alpha_n F_2(z)$$

which implies that α_n are also the eigenvalues of T_n for $F_2(z)$. But because $F_2(z)$ is a new form with weight 2 shows that for any positive integer m with $(m, 14) = 1$, $\alpha_m = B(m)$ where $F_2(z) = \sum_{n=1}^{\infty} B(n)\mathrm{e}^{2\pi i n z}$ is the Fourier expansion of $F_2(z)$. These facts show that

$$B(p)b(n) = \alpha_p b(n) = b(p^2 n) + \left(\frac{-n}{p}\right) b(n) + pb(n/p^2) \tag{10.35}$$

for any prime p with $(p, 14) = 1$ and any positive integer n. We obtain by $\dfrac{r_2(n) - r_2'(n)}{2}$ instead of $b(n)$ that

$$r_2(p^2n) - r_2'(p^2n) = \left(B(p) - \left(\frac{-n}{p}\right)\right)(r_2(n) - r_2'(n)) + p(r_2(n/p^2) - r_2'(n/p^2)).$$

In particular, if n is a square-free positive integer, then for any prime p with $(p, 14) = 1$, we see that

$$r_2(p^2n) - r_2'(p^2n) = \left(B(p) - \left(\frac{-n}{p}\right)\right)(r_2(n) - r_2'(n)). \tag{10.36}$$

For a prime p such that $p|14$, by the definition of Hecke operators, we see that $T_{p^2}(\widetilde{f}_2(z)) = \sum_{n=1}^{\infty} b(p^2n)\mathrm{e}^{2\pi i n z}$ which implies that

$$\alpha_p b(n) = b(p^2 n),$$

i.e.

$$r_2(p^2n) - r_2'(p^2n) = \alpha_p(r_2(n) - r_2'(n)). \tag{10.37}$$

An easy calculation shows that $\alpha_2 = -1$ and $\alpha_7 = 1$. We now want to prove that if n_0 is square-free eligible number such that $r_2(n_0) = 0$ (i.e., n_0 is not represented by f_2) then $r_2(p^2n_0) \neq 0$ (i.e., p^2n_0 can be represented by f_2) for any prime p with $(p, 7) = 1$. Otherwise, we have by (10.36), (10.37) that

$$\begin{aligned} \frac{r_2'(p^2n_0)}{r_2'(n_0)} &= B(p) - \left(\frac{-n_0}{p}\right) \leqslant B(p) + 1, \\ \frac{r_2'(2^2n_0)}{r_2'(n_0)} &= \alpha_2 = -1. \end{aligned} \tag{10.38}$$

On the other hand, we have that by (10.34)

$$\begin{aligned} \frac{r_2'(p^2n_0)}{r_2'(n_0)} = \frac{G_2(p^2n_0)}{G_2(n_0)} \\ &= \frac{\omega_{p^2n_0}^{-1}(2 - \alpha(p^2n_0))\beta_{1,7}(p^2n_0)\gamma_7'(p^2n_0)h(-p^2n_0)}{\omega_{n_0}^{-1}(2 - \alpha(n_0))\beta_{1,7}(n_0)\gamma_7'(n_0)h(-n_0)} \\ &= \frac{(2 - \alpha(p^2n_0))\beta_{1,7}(p^2n_0)\gamma_7'(p^2n_0)}{(2 - \alpha(n_0))\beta_{1,7}(n_0)\gamma_7'(n_0)}. \end{aligned} \tag{10.39}$$

We now suppose that $p \neq 7$ and 2, then by the definitions of $\alpha(n), \beta_{1,7}(n), \gamma_7'(n)$ and n_0 a square-free integer, we easily obtain that

$$\begin{aligned}
\alpha(p^2 n_0) &= \alpha(n_0), \\
\beta_{1,7}(p^2 n_0) &= \beta_{1,7}(n_0), \\
\gamma_7'(p^2 n_0) &= (p+1)\gamma_7'(n_0), \\
\alpha(2^2 n_0) &= \frac{1}{2}\alpha(n_0), \\
\beta_{1,7}(2^2 n_0) &= \beta_{1,7}(n_0), \\
\gamma_7'(2^2 n_0) &= \frac{3}{1 - 2^{-1}\chi_{-n_0}(2)}\gamma_7'(n_0), \\
\alpha(7^2 n_0) &= \alpha(n_0), \\
\beta_{1,7}(7^2 n_0) &= \frac{1}{7}\beta_{1,7}(n_0), \\
\gamma_7'(7^2 n_0) &= \frac{8}{1 - \chi_{-n_0}(7)7^{-1}}\gamma_7'(n_0).
\end{aligned}$$

Hence we see that

$$\frac{r_2'(p^2 n_0)}{r_2'(n_0)} = \begin{cases} p+1, & \text{if } p \neq 2, 7, \\ 5, & \text{if } p = 2, \nu_2(n_0) = 1, \\ 15, & \text{if } p = 2, n_0 \equiv 1 \pmod 4, \\ 9, & \text{if } p = 2, n \equiv 3 \pmod 8, \\ 6, & \text{if } p = 2, n \equiv 7 \pmod 8, \\ \dfrac{8}{7 - \chi_{-n_0}(7)}, & \text{if } p = 7. \end{cases} \tag{10.40}$$

For any prime $p \neq 2, 7$, by equalities (10.38) and (10.40) we have that

$$B(p) \geqslant p$$

and

$$0 < \frac{r_2'(2^2 n_0)}{r_2'(n_0)} = -1 < 0,$$

which is impossible, since n_0 is an eligible integer. On the other hand, it is well known that $B(p) \leqslant 2p^{\frac{1}{2}}$ by Deligne's estimation for coefficients of modular forms. This implies that $2p^{\frac{1}{2}} \geqslant p$ for any prime $p \neq 2$ and 7 which is impossible.

What we have proved is that if n is any square-free eligible number of the genus of f_2 which is not represented by f_2, then $p^2 n$ can be represented by f_2 for any prime p with $p \neq 7$. This, of course, is equivalent to saying that if an eligible number n prime to 7 can not be represented by f_2 then n is square-free. This completes the proof. □

As a conclusion of Theorem 10.12 we have that

Theorem 10.13 *The form $f_1 = x^2 + y^2 + 7z^2$ represents all eligible numbers which are multiples of 9; it also represents all eligible numbers congruent to 2 mod 3 except those of the trivial type. In other words, the* Kaplansky's Conjecture *holds.*

Proof We first show the following fact: $f_1 = x^2 + y^2 + 7z^2$ does not represent $7A$ if and only if $f_2 = x^2 + 7y^2 + 7z^2$ does not represent A.

In fact, it is obvious that if f_2 represents A, i.e., there are integers a, b, c such that $a^2 + 7b^2 + 7c^2 = A$, then $7A = (7b)^2 + (7c)^2 + 7a^2$. Conversely, if $7A = x^2 + y^2 + 7z^2$, then $x^2 + y^2 \equiv 0 \pmod 7$ which implies that $x \equiv 0 \pmod 7$ and $y \equiv 0 \pmod 7$. Let be $x = 7x'$, $y = 7y'$, we see that $A = z^2 + 7(x')^2 + 7(y')^2$ which shows that f_2 represents A.

By Example 10.6, we know that the eligible numbers of f_2 are precisely all integers which are not the product of an even power of 7 and a number congruent to 3, 5, 6 mod 7. Hence, to prove Theorem 10.13 we only need to show that f_2 represents all eligible numbers which are congruent to 1, 2, 4 mod 7 and of form $2t^2$ with $t \neq 1$ and $7 \nmid t$. If $2|t$, it is clear that f_2 represents $2t^2$ because f_2 represents 8. Hence we can assume that t is an odd integer. This shows that Theorem 10.12 implies Theorem 10.13. □

Remark 10.5 If n is not prime to 7, the result in Theorem 10.12 does not hold. For example $n = 98 = 2 \cdot 7^2$ can not be represented by f_2. In fact, for $p = 7$, the above proof is not suitable because we can not obtain a contradiction as above for $p \neq 7$. For if we assume that n_0 is an eligible number such that $r_2(n_0) = r_2(7^2 n_0) = 0$, then the calculations above show that $\dfrac{8}{7 - \chi_{-n_0}(7)} = \dfrac{r_2'(7^2 n_0)}{r_2'(n_0)} = \alpha_7 = 1$, which possibly holds, e.g., $n_0 = 2$ makes it hold. In this proof we need not introduce the concept of essentially distinct primitive representations. And for the formula giving the number of representations for a genus of positive definite ternary quadratic forms, we also need not assume that our discussion is limited to the integers prime to the level of the quadratic form because we do not employ Theorem 86 in B.W. Jones, 1950. In fact, the argumentation of the above proof can also be applied to other genera consisting of two equivalent classes. For example, we can prove the following result:

Corollary 10.6 *Let $f_{(p)} = x^2 + py^2 + pz^2$ with an odd prime p and assume that the genus of $f_{(p)}$ consists of two equivalence classes which we denote by $f_{(p)}$ and $g_{(p)}$. Denote*

$$\widetilde{f}_{(p)}(z) := \sum_{n=1}^{\infty} b(n) e^{2\pi i n z} = \frac{1}{2} \sum_{n=1}^{\infty} (r(n) - r'(n)) e^{2\pi i n z},$$

where $r(n)$ and $r'(n)$ are the numbers of representations of n by $f_{(p)}$ and $g_{(p)}$ respectively. And assume that the Shimura *lifting $F_{(p)}(z) = S(\widetilde{f}_{(p)}(z))$ of $\widetilde{f}_{(p)}(z)$ is a new form of weight 2 corresponding to a modular elliptic curve, then every eligible number prime to $2p$ of the genus of $f_{(p)}$ not represented by $f_{(p)}$ is square-free.*

Proof It is completely similar to the proof of Theorem 10.12. □

Example 10.7 Every eligible integer prime to 34 not represented by $f_{(17)} = x^2 + 17y^2 + 17z^2$ is square free. This is because that the genus for $f_{(17)}$ consists of $f_{(17)}$ and $g_{(17)} = 2x^2 + 9y^2 + 17z^2 + 2xy$ and the Shimura lifting of $\widetilde{f}_{(17)}(z)$ is the new form corresponding to the modular elliptic curve (34A). □

Combining Theorem 10.10, Theroem 10.12, Remark 10.2 and the result of Corollary 10.6 we indeed obtain:

Corollary 10.7 *Let* $f_{(p)} = x^2 + py^2 + pz^2$ *be as in* Corollary 10.6. *Then there are only finitely many eligible numbers which are prime to* $2p$ *and not represented by the quadratic form* $f_{(p)}$ *. In particular, there are only finitely many eligible numbers prime to* 7 *and* 34 *not represented by the forms* $f_{(7)}$ *and* $f_{(17)}$ *respectively.*

We now consider the following problem: Let n be a square free positive integer, f and g be two ternary positive definite quadratic forms in the same genus, then when do we have that $r(f,n) \neq r(g,n)$ where $r(f,n)$ and $r(g,n)$ are the numbers of representation of n by f and g respectively. For example, if $f_{(7)} = x^2 + 7y^2 + 7z^2$, $g_{(7)} = 2x^2 + 4y^2 + 7z^2 - 2xy$, then $f_{(7)}$ and $g_{(7)}$ are in the same genus, and we want to know when do we have that $r(f_{(7)}, n) \neq r(g_{(7)}, n)$ for a positive integer. It is clear that we only need to consider eligible numbers n because $r(f,n) = r(g,n) = 0$ if n is not eligible.

We now assume always that f and g are in the same genus and $r(f,1) \neq r(g,1)$. Let

$$\widetilde{f}(z) =: \sum_{n=1}^{\infty} b_n \mathrm{e}^{2\pi \mathrm{i} nz} = \frac{1}{r}\sum_{n=1}^{\infty}(r(f,n) - r(g,n))\exp\{2\pi \mathrm{i} nz\},$$

where $r = r(f,1) - r(g,1) \neq 0$. Then $\widetilde{f}(z) \in S(N,\ 3/2,\ \chi_l)$. For example, we have that

$$\begin{aligned}\widetilde{f}_{(7)}(z) &= \frac{1}{2}\sum_{n=1}^{\infty}(r(f_{(7)},n) - r(g_{(7)},n))\exp\{2\pi \mathrm{i} nz\} \\ &= q + \cdots \in S(28, 3/2, \chi_1), \quad q = \exp\{2\pi \mathrm{i} z\}.\end{aligned}$$

We assume further that the Shimura lifting $F(z)$ of $\widetilde{f}(z)$ is a new form corresponding to a modular elliptic curve $E/\mathbb{Q}$. For example, we see that $F_{(7)}(z) = S(\widetilde{f}_{(7)}(z))$ is the new form corresponding to the modular elliptic curve (14C):

$$(14\text{C}): \qquad y^2 = x^3 + x^2 + 72x - 368.$$

$F_{(11)}(z) = S(\widetilde{f}_{(11)}(z))$ is the new form corresponding to the modular elliptic curve (11B) where $f_{(11)} = x^2 + 11y^2 + 11z^2$:

$$(11\text{B}): \qquad y^2 + y = x^3 - x^2 - 10x - 20.$$

By the definition of $\widetilde{f}(z)$, what we want to know is that when are the coefficients of $\widetilde{f}(z)$ not equal to zero. In order to do this we need the following result of Waldspurger:

Lemma 10.7 *Assume that $E/\mathbb{Q}$ is a modular elliptic curve with corresponding cusp form f_E, and that*

$$F \in S(N, 3/2, \chi_t) \bigcap S_0(N, \chi_t)^{\perp}$$

with

$$S(F) = f_E, \quad F = \sum_{n=1}^{\infty} a_n \mathrm{e}^{2\pi i n z},$$

where $S_0(N, \psi)$ is the subspace of $S(N, 3/2, \psi)$ generated by the form F of the following type: There is a $t \in \mathbb{N}$ and a quadratic character χ with conductor r such that $F = \sum_{m=1}^{\infty} \chi(m) m q^{tm^2}$ and $N = 4r^2 t$, $\psi = \chi \cdot \chi_t \cdot \chi_{-1}$. Assume that d and d_0 are natural square free numbers with

$$d \equiv d_0 \mod \left(\prod_{p|N} \mathbb{Q}_p^{*2} \right), \qquad \text{and } (dd_0, N) = 1.$$

Then

$$L_{E_{-td}}(1) \sqrt{d} a_{d_0}^2 = L_{E_{-td_0}}(1) \sqrt{d_0} a_d^2.$$

So especially: if

$$L_{E_{-td_0}} a_{d_0} \neq 0,$$

then

$$L_{E_{-td}}(1) = 0 \text{ if and only if } a_d = 0,$$

where $L_{E_D}(s)$ is the Hasse-Weil Zeta function of the D-th twist of elliptic curves E.

Now denote the set of representatives of all inequivalent integers mod $\prod_{p|N} \mathbb{Q}_p^{*2}$ which are eligible numbers for the genus of f and prime to N by D_N, then D_N is finite. Let be $D_N = \{d_1, d_2, \cdots, d_l\}$.

We have that for any square free eligible natural integer d such $(d, N) = 1$, there exist unique $d_i \in D_N$ such that

$$\frac{L_{E_{-td}}(1)\sqrt{d}}{a_d^2} = \frac{L_{E_{-td_i}}(1)\sqrt{d_i}}{a_{d_i}^2}.$$

Using this equality, we can deduce when the coefficients a_d are different from zero.

Example 10.8 Let $f = f_{(7)}$, $g = g_{(7)}$, $E = (14\text{C})$, then

$$\widetilde{f}_{(7)}(z) = \frac{1}{2} \left(\theta(f_{(7)}) - \theta(g_{(7)}) \right) \in S_{3/2}(28, \chi_1)$$

and

$$F_{(7)}(z) = S(\widetilde{f}_{(7)}(z)) \in S_2^{\text{new}}(14)$$

corresponding to the modular elliptic curve (14C). And we can calculate that

$$\begin{aligned}
D_{28} &= \{1, 11, 15, 29\},\\
L_{E_{-d_i}} &\neq 0, \text{for all} d_i \in D_{28},\\
b_1 &= \frac{1}{2}(r(f_{(7)}, 1) - r(g_{(7)}, 1)) = 1,\\
b_{11} &= \frac{1}{2}(r(f_{(7)}, 11) - r(g_{(7)}, 11)) = \frac{1}{2}(8-8) = 0,\\
b_{15} &= \frac{1}{2}(r(f_{(7)}, 15) - r(g_{(7)}, 15)) = \frac{1}{2}(8-8) = 0,\\
b_{29} &= \frac{1}{2}(r(f_{(7)}, 29) - r(g_{(7)}, 29)) = \frac{1}{2}(8-4) = 2.
\end{aligned}$$

These calculations and Waldspurger's Theorem show that for square free eligible numbers d such that $(d, 14) = 1$:

$$r(f_{(7)}, d) = r(g_{(7)}, d), \quad \text{if } d \equiv 11, 15 \mod \left(\prod_{p|28} \mathbb{Q}_p^{*2}\right),$$

$$r(f_{(7)}, d) \neq r(g_{(7)}, d) \text{ if and only if } L_{E_{-d}}(1) \neq 0 \text{ for } d \equiv 1, 29 \mod \left(\prod_{p|28} \mathbb{Q}_p^{*2}\right).$$

□

Hence we have the following:

Theorem 10.14 *Let be* $f_{(7)} = x^2 + 7y^2 + 7z^2, g_{(7)} = 2x^2 + 4y^2 + 7z^2 - 2xy$*,* E *the corresponding modular elliptic curve of the cusp form* $\frac{1}{2}(\theta(f_{(7)}) - \theta(g_{(7)}))$ *and* E_{-d} *the* $-d$*-twist of* E*. Then for any square free eligible numbers* d *such that* $(d, 14) = 1$*, we have that*

(1) $r(f_{(7)}, d) = r(g_{(7)}, d)$, *if* $d \equiv 11, 15 \mod \left(\prod_{p|28} \mathbb{Q}_p^{*2}\right)$;

(2) $r(f_{(7)}, d) \neq r(g_{(7)}, d)$ *if and only if* $L_{E_{-d}}(1) \neq 0$ *for* $d \equiv 1, 29 \mod \left(\prod_{p|28} \mathbb{Q}_p^{*2}\right)$,

where $L_{E_{-d}}(s)$ *is the* Hasse-Weil *L-function of the elliptic curve* E_{-d}*. Especially, if* n *is a square free natural number such that*

$$n \equiv 3 \pmod 8 \text{ and } \left(\frac{n}{7}\right) = 1$$

or

$$n \equiv 7 \,(\mathrm{mod}\ 8)\ \textit{and}\ \left(\frac{n}{7}\right) = 1,$$

then

$$r(f_{(7)}, n) = r(g_{(7)}, n).$$

Proof Above all proved except for the last assertion. But

$$n \equiv 3 \,(\mathrm{mod}\ 8) \quad \text{and} \quad \left(\frac{n}{7}\right) = 1$$

implies that

$$n \equiv 11 \,\mathrm{mod} \left(\prod_{p|28} \mathbb{Q}_p^{*2} \right).$$

And

$$n \equiv 7 \,(\mathrm{mod}\ 8) \quad \text{and} \quad \left(\frac{n}{7}\right) = 1$$

implies that

$$n \equiv 15 \,\mathrm{mod} \left(\prod_{p|28} \mathbb{Q}_p^{*2} \right),$$

which shows this theorem. □

From this theorem, we see that for the cases of $d \equiv 11, 15 \mod \left(\prod_{p|28} \mathbb{Q}_p^{*2} \right)$, the result (1) is completely pleasant. And for the cases of $d \equiv 1, 29 \mod \left(\prod_{p|28} \mathbb{Q}_p^{*2} \right)$, the result (2) is not so pleasant because it is not an easy task to determine if $L_{E_{-d}}(1) = 0$. But we have the following:

Theorem 10.15 *Let* $p \equiv 1 \,\mathrm{mod} \left(\prod_{p|28} \mathbb{Q}_p^{*2} \right)$ *be a prime not dividing* 14, *then* $r(f_{(7)}, p) \neq r(g_{(7)}, p)$ *if* p *is represented by* $2X^2 + 7Y^2$.

Proof As in J.A. Antoniadis, 1990, we denote

$$F_0 = \left(\theta(X^2 + 14Y^2) - \theta(2X^2 + 7Y^2)\right) \cdot \theta_{\mathrm{id},14} := \sum_{n=1}^{\infty} a_n \mathrm{e}^{2\pi i n z} \in S_{3/2}(56, \chi_1),$$

where

$$\theta_{\mathrm{id},14} := \sum_{n=-\infty}^{\infty} q^{14n^2} \in M_{1/2}(56, \chi_{14}), \quad \theta(X^2 + 14Y^2) - \theta(2X^2 + 7Y^2) \in S_1(56, \chi_{-14}).$$

Then by the results in J.A. Antoniadis, 1990, we know that F_0 is mapped to the cusp form corresponding to the modular elliptic curve (14C) under Shimura lifting and $a_p \neq 0$ if p is a prime not dividing 14 and represented by $2X^2+7Y^2$. Since $p \equiv 1 \mod \left(\prod_{p|28} \mathbb{Q}_p^{*2}\right)$, by Waldspurger's Theorem, we see that

$$L_{E_{-p}}(1)\sqrt{p}a_1^2 = L_{E_{-1}}(1)a_p^2.$$

A direct computation shows that $a_1 \cdot L_{E_{-1}}(1) \neq 0$ which implies that

$$L_{E_{-p}}(1) = 0 \text{ if and only if } a_p = 0.$$

Therefore by Lemma 10.7, we have proved that

$$r(f_{(7)}, p) \neq r(g_{(7)}, p) \text{ if and only if } a_p \neq 0,$$

which completes the proof since $a_p \neq 0$ if $(p, 14) = 1$ and represented by $2X^2 + 7Y^2$. □

Our method can be used for other ternary positive definite quadratic forms. For example, we can similarly study the forms $f_{(11)}, g_{(11)}$. In this case, we calculate:

$$\begin{aligned} &D_{44} = \{1, 3, 5, 15\}, \\ &L_{E_{-d_i}} \neq 0, \quad \text{for all } d_i \in D_{44}, \\ &b_1 = 1, \quad b_3 = -1m \quad b_5 = -1, \quad b_{15} = 1. \end{aligned}$$

Hence we conclude that

Theorem 10.16 *Let be $f_{(11)} = x^2+11y^2+11z^2$, $g_{(11)} = 3x^2+4y^2+11z^2+2xy$, E the corresponding modular elliptic curve of the cusp form $\frac{1}{2}(\theta(f_{(11)}) - \theta(g_{(11)}))$ and E_{-d} the $-d$-twist of E. Then for square free eligible numbers d such that $(d, 22) = 1$, we have that*

$$r(f_{(11)}, d) \neq r(g_{(11)}, d) \quad \textit{if and only if } L_{E_{-d}}(1) \neq 0,$$

where $L_{E_{-d}}(s)$ is the Hasse-Weil *L-function of the elliptic curve E_{-d}. Especially, we have that $r(f_{(11)}, d) \neq r(g_{(11)}, d)$ if d satisfies one of the following conditions*:

(1) $d = p$ is a prime not splitting in $\mathbb{Q}(\sqrt{-11})_{(2)}/\mathbb{Q}(\sqrt{-11})$, where $\mathbb{Q}(\sqrt{-11})_{(2)}$ is the class field of $\mathbb{Q}(\sqrt{-11})$ with conductor 2;

(2) $d = p$ is a prime with $(p, 22) = 1$ such that p is represented by $3X^2+2XY+4Y^2$;

(3) $5 \nmid h(-d)$.

Proof Since $L_{E_{-d_i}}(1) \cdot b_{d_i} \neq 0$ for all $d_i \in D_{44}$, we know that $r(f_{(11)}, d) \neq r(g_{(11)}, d)$ if and only if $L_{E_{-d}}(1) \neq 0$ by Waldspurger's Theorem. All other assertions are immediate conclusions of Proposition 4.2 and Proposition 4.8 in J.A. Antoniadis, 1990. □

Remark 10.6 Our method in this section can be used to any other positive definite quadratic forms satisfying our assumptions in the paragraph before Lemma 10.7. For example, we can study similarly the forms $f_{(17)}$ and $g_{(17)}$, etc.

Finally we consider the following problem: for a given positive definite quadratic form with integral coefficients, find an exact formula for the number of representations of integers by this form. In general it is a difficult classical problem. Even for the simplest cases, i.e., binary forms and ternary forms, the problem is still open. For the general case, what we know is that the sum of the numbers of representations of an integer by all classes in a fixed genus is in relation to the coefficients of some modular forms in an Eisenstein subspace. But even for the sum, it is non-trivial to give an exact formula for a form given generally. In any case, the number of representations of an integer by one form in the genus has never been formulated if the class number of the genus is larger than one.

We shall consider some ternary quadratic forms with class number two of their genus, and give exact formulae for the numbers of representations of an integer by these forms. The main idea is as follows. For a positive definite ternary form f, let f and g be the representatives of classes in the genus of f. On the one hand, some linear combination of the numbers of representations of an integer by f and g can be related to the class number of a certain quadratic field; on the other hand, sometimes, we can find another linear combination of these numbers which is related to the L-function of an elliptic curve. By these two linear combinations, in terms of class number of a quadratic field and the special value of the L-function of an elliptic curve, we can get exact formulae for the number of representations of an integer by f and g respectively. This also shows the difficulty of the classical problem mentioned above because of the mysterious properties of the special values of L-functions and class numbers.

Theorem 10.17 *Let $f = \alpha x^2 + \beta y^2 + \gamma z^2$ be a positive definite ternary quadratic form with level N. Suppose the genus of f consists of two classes, f and g are the representatives of the classes. We assume further that $\mu O(f) - \nu O(g) \neq 0$, and*

$$\mu\theta_f + \nu\theta_g = \sum_{n=1}^{\infty} a_n \mathrm{e}^{2\pi i n z} \in S(N, 3/2, \chi_t) \bigcap S_0(N, \chi_t)^{\perp}$$

and the Shimura *lifting $F(z)$ of $\mu\theta_f + \nu\theta_g$ is a new form corresponding to an elliptic curve $E/\mathbb{Q}$. Let n with $(n, N) = 1$ be any square-free eligible number of the genus* (i.e., *d can be represented by the genus of f) with $n \equiv d_i \bmod \prod_{p|N} \mathbb{Q}_p^{*2}$ and $L_{E_{-ld_i}}(1) \neq 0$, then*

$$r(f,n)=\frac{O(f)a_{d_i}\sqrt{\dfrac{L_{E_{-ln}}(1)}{L_{E_{-ld_i}}(1)}}-\nu O(f)O(g)r(a,b,c;n)h(-ln)}{\mu O(f)-\nu O(g)},$$

$$r(g,n)=\frac{\mu O(f)O(g)r(a,b,c;n)h(-ln)-O(g)a_{d_i}\sqrt{\dfrac{L_{E_{-ln}}(1)}{L_{E_{-ld_i}}(1)}}}{\mu O(f)-\nu O(g)},$$

where $d_i \in D_N = \{d_1, d_2, \cdots, d_l\}$, $L_{E_D}(s)$ *is the* Hasse-Weil Zeta *function of the* D*-th twist of the elliptic curve* E.

Proof In Lemma 10.7, we take $F(z) = \mu\theta_f(z) + \nu\theta_g(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$. Then by Theorem 10.9 we obtain the following system of equations:

$$\begin{cases} \mu r(f,n) + \nu r(g,n) = a_n, \\ \dfrac{r(f,n)}{O(f)} + \dfrac{r(g,n)}{O(g)} = r(a,b,c;n)h(-ln). \end{cases} \tag{10.41}$$

For the positive integer n, there is a unique $d_i \in D_N$ with $n \equiv d_i \mod \left(\prod_{p|N} \mathbb{Q}_p^{*2}\right)$. By the above Lemma 10.7, under the assumptions of the theorem, we have that

$$a_n = a_{d_i}\sqrt{\frac{L_{E_{-ln}}(1)}{L_{E_{-ld_i}}(1)}},$$

solving the system (10.41) for $r(f,n), r(g,n)$, and inserting above the expression for a_n, we get the results desired, which completes the proof. □

Remark 10.7 Because the set $D_N = \{d_1, d_2, \cdots, d_l\}$ is finite, we see that $r(f,n)$ and $r(g,n)$ can be represented explicitly in terms of the classnumber $h(-ln)$ and the special value $L_{E_{-ln}}(1)$ of L-function of the twist of the elliptic curve E.

Example 10.9 Let be $f_1 = x^2 + 7y^2 + 7z^2$, $g_1 = 2x^2 + 4y^2 + 7z^2 - 2xy$.Then $O(f_1) = 8$, $O(g_1) = 4$, and

$$\begin{aligned} \widetilde{f}(z) &= \sum_{n=1}^{\infty} a_n \exp\{2\pi i n z\} \\ &:= \frac{1}{2}\theta_{f_1}(z) - \frac{1}{2}\theta_{g_1}(z) = \frac{1}{2}\sum_{n=1}^{\infty}(r(f_1,n) - r(g_1,n)) \exp\{2\pi i n z\} \\ &= q + \cdots \in S(28, 3/2, \chi_1), \quad q = \exp\{2\pi i z\} \end{aligned}$$

and $F(z) = S(\widetilde{f}(z))$ is the new form corresponding to the elliptic curve (14C):

$$(14\text{C}): \qquad y^2 = x^3 + x^2 + 72x - 368.$$

We can easily calculate that

$$\begin{aligned}
D_{28} &= \{1, 11, 15, 29\}, \\
L_{E_{-d_i}} &\neq 0 \quad \text{for all } d_i \in D_{28}, \\
a_1 &= \frac{1}{2}(r(f,1) - r(g,1)) = 1, \\
a_{11} &= \frac{1}{2}(r(f,11) - r(g,11)) = \frac{1}{2}(8-8) = 0, \\
a_{15} &= \frac{1}{2}(r(f,15) - r(g,15)) = \frac{1}{2}(8-8) = 0, \\
a_{29} &= \frac{1}{2}(r(f,29) - r(g,29)) = \frac{1}{2}(8-4) = 2.
\end{aligned}$$

Hence by Theorem 10.17, for any square-free eligible integer n , we have that

$$\begin{aligned}
r(f_1,n) &= \frac{4}{3}\sqrt{\frac{L_{E_{-n}}(1)}{L_{E_{-1}}(1)}} + \frac{8}{3}r(1,7,7;n)h(-n), \quad \text{if } n \equiv 1 \bmod \prod_{p|28} \mathbb{Q}_p^{*2}, \\
r(g_1,n) &= \frac{8}{3}r(1,7,7;n)h(-n) + \frac{1}{3}\sqrt{\frac{L_{E_{-n}}(1)}{L_{E_{-1}}(1)}}, \quad \text{if } n \equiv 1 \bmod \prod_{p|28} \mathbb{Q}_p^{*2}, \\
r(f_1,n) &= r(g_1,n) = \frac{8}{3}r(1,7,7;n)h(-n), \quad \text{if } n \equiv 11 \bmod \prod_{p|28} \mathbb{Q}_p^{*2}, \\
r(f_1,n) &= r(g_1,n) = \frac{8}{3}r(1,7,7;n)h(-n), \quad \text{if } n \equiv 15 \bmod \prod_{p|28} \mathbb{Q}_p^{*2}, \\
r(f_1,n) &= \frac{8}{3}\sqrt{\frac{L_{E_{-n}}(1)}{L_{E_{-29}}(1)}} + \frac{8}{3}r(1,7,7;n)h(-n), \quad \text{if } n \equiv 29 \bmod \prod_{p|28} \mathbb{Q}_p^{*2}, \\
r(g_1,n) &= \frac{8}{3}r(1,7,7;n)h(-n) + \frac{2}{3}\sqrt{\frac{L_{E_{-n}}(1)}{L_{E_{-29}}(1)}}, \quad \text{if } n \equiv 29 \bmod \prod_{p|28} \mathbb{Q}_p^{*2},
\end{aligned}$$

where

$$\begin{aligned}
r(1,7,7;n) &= \frac{1}{4}\omega_n^{-1} \cdot (2 - \alpha(n))\beta_{1,7}(n)\gamma_7'(n); \\
\gamma_p'(n) &= (1 - 2^{-1}\chi_{-n}(2))(1 - \chi_{-n}(p) \cdot p^{-1})(n/\delta_n)^{\frac{1}{2}} \\
&\quad \times \sum_{\substack{(ab)^2|n \\ (ab,2p)=1}} \mu(a)\chi_{-n}(a)(ab)^{-1}
\end{aligned}$$

for any prime p; In particular we know that $\gamma_p'(n) = (1 - 2^{-1}\chi_{-n}(2))(1 - \chi_{-n}(p) \cdot p^{-1})(n/\delta_n)^{1/2}$ for any square-free positive integer n.

From these results, we can get very explicit formulae for the number of representations of the square-free positive eligible number n with $(n, 28) = 1$ by f and g. E.g., for any square-free positive integer $n > 3$ with $n \equiv 3 \mod 8$ and $\left(\frac{n}{7}\right) = 1$, then $n \equiv 11 \mod \left(\prod_{p|28} \mathbb{Q}_p^{*2}\right)$. By the definitions of $\alpha(n), \beta_{1,7}(n)$ and $\gamma_7'(n)$, we have that

$$\alpha(n) = 1, \quad \beta_{1,7}(n) = 14, \quad \gamma_7'(n) = \frac{12}{7}.$$

So

$$r(f_1, n) = r(g_1, n) = 8h(-n).$$

Of course, we can discuss also other square-free positive integers n in a similar way. □

Example 10.10 Let be $f_2 = x^2 + 11y^2 + 11z^2$, $g_2 = 3x^2 + 4y^2 + 11z^2 + 2xy$. Then we have that

$$D_{44} = \{1, 3, 5, 15\}, \quad L_{E_{-d_i}} \neq 0, \quad \text{for all } d_i \in D_{44},$$
$$a_1 = 1, \quad a_3 = -1, \quad a_5 = -1, \quad a_{15} = 1.$$

And $O(f_2) = 8, O(g_2) = 4$,

$$\begin{aligned}\widetilde{f}(z) &= \sum_{n=1}^{\infty} a_n \exp(2\pi i n z) := \frac{1}{2}\theta_{f_2}(z) - \frac{1}{2}\theta_{g_2}(z) \\ &= \frac{1}{2}\sum_{n=1}^{\infty}(r(f_2, n) - r(g_2, n)) \exp\{2\pi i n z\} \\ &= q + \cdots \in S(28, 3/2, \chi_1), \quad q = \exp\{2\pi i z\}\end{aligned}$$

and $F(z) = S(\widetilde{f}(z))$ is the new form corresponding to the elliptic curve (11B):

$$(11\mathrm{B}): \qquad y^2 + y = x^3 - x^2 - 10x - 20,$$

So by Theorem 10.17, we can get the exact formulae for the number of representations of any square-free eligible integer n with $(n, 22) = 1$ by f and g in terms of $h(-n)$ and $L_{E_{-n}}(1)$. We omit the calculations. □

Theorem 10.18 *Suppose that n is an odd square-free positive integer congruent to 1 or 3 modulo 8. $f_3 = x^2 + 2y^2 + 32z^2, g_3 = 2x^2 + 4y^2 + 9z^2 - 4yz$. Then*

$$r(f_3, n) = c(n)h(-n) + 2\sqrt{\frac{L_{E_{n^2}}(1)}{\omega\sqrt{n}}},$$

$$r(g_3, n) = c(n)h(-n) - 2\sqrt{\frac{L_{E_{n^2}}(1)}{\omega\sqrt{n}}},$$

where $c(n) = 2$ *or* 6 *according to* $n \equiv 1$ *or* $3 \pmod 8$, ω *is the real period of the elliptic curve* $E : y^2 = 4x^3 - 4x$ *and* $L_{E_{n^2}}(s)$ *is the L-function of the congruent elliptic curve defined by* $y^2 = x^3 - n^2x$.

Proof Let $f_3 = x^2 + 2y^2 + 32z^2$, $g_3 = 2x^2 + 4y^2 + 9z^2 - 4yz$. We want to give the formula for the number of representations of n by f_3 and g_3. It is clear that $r(f_3, n) = r(g_3, n) = 0$ for any $n \equiv 5$ or $7 \pmod 8$. So we only need to consider positive integers congruent to 1 or 3 modulo 8. Now let $f_3' = 2x^2 + y^2 + 32z^2$, $g_3' = 2x^2 + y^2 + 8z^2$, then by Tunnell's paper J.B. Tunnell, 1983, for any odd positive integer n, we have

$$\frac{L_{E_{n^2}}(1)}{\omega\sqrt{n}} = \frac{1}{4}a(n)^2,$$

where E_{n^2} is the congruent elliptic curve defined by $y^2 = x^3 - n^2x$, ω is the real period of the elliptic curve $y^2 = 4x^3 - 4x$ and $a(n) = r(f_3', n) - \frac{1}{2}r(g_3', n)$. It is not difficult to see that $a(n) = \frac{1}{2}(r(f_3, n) - r(g_3, n))$ for any odd n. So we have

$$\frac{L_{E_{n^2}}(1)}{\omega\sqrt{n}} = \frac{1}{4}a(n)^2, \tag{10.42}$$

where $a(n) = \frac{1}{2}(r(f_3, n) - r(g_3, n))$.

In order to get the formulae for the number of representations of n by f_3 and g_3, we only need to find the number $r(f_3, n) + r(g_3, n)$ by (10.42). But by the definitions of $r(f_3, n)$ and $r(g_3, n)$, we see that $r(f_3, n) + r(g_3, n) = r(x^2 + 2y^2 + 8z^2, n)$. So we only need to calculate the number $r(x^2 + 2y^2 + 8z^2, n)$. We shall prove that for $n > 3$ square-free,

$$r(x^2 + 2y^2 + 8z^2, n) = \begin{cases} 4h(-n) \text{ if } n \equiv 1 \pmod 8, \\ 12h(-n) \text{ if } n \equiv 3 \pmod 8. \end{cases}$$

In fact, if $n \equiv 1 \pmod 8$, then for any triple $(x, y, z) \in \mathbb{Z}^3$ such that $x^2+2y^2+2z^2 = n$, the x must be odd and y, z are both even. So we have a one-to-one correspondence:

$$\{(x, y, z) \in \mathbb{Z}^3 | x^2 + 2y^2 + 2z^2 = n\} \leftrightarrow \{(x, y, z) \in \mathbb{Z}^3 | x^2 + 2y^2 + 8z^2 = n\},$$
$$(x, y, z) \leftrightarrow (x, y, z/2).$$

If $n \equiv 3 \pmod 8$, then for any triple $(x, y, z) \in \mathbb{Z}^3$ such that $x^2 + 2y^2 + 2z^2 = n$, the x must be odd and there is exactly one of y, z that is odd. We let z be the even one. Then we have a two-to-one correspondence:

$$\{(x, y, z) \in \mathbb{Z}^3 | x^2 + 2y^2 + 2z^2 = n\} \leftrightarrow \{(x, y, z) \in \mathbb{Z}^3 | x^2 + 2y^2 + 8z^2 = n\}$$
$$\begin{cases} (x, y, z) \\ (x, z, y) \end{cases} \leftrightarrow (x, y, z/2).$$

So we have

$$r(x^2+2y^2+8z^2,n)=\begin{cases} r(x^2+2y^2+2z^2,n) \text{ if } n\equiv 1 \,(\text{mod } 8),\\ \dfrac{1}{2}r(x^2+2y^2+2z^2,n) \text{ if } n\equiv 3 \,(\text{mod } 8).\end{cases}$$

Now we can compute the number $r(x^2+2y^2+2z^2,n)$ in terms of our Theorem 10.9. By Theorem 10.9 it can be proved that for any positive integer n

$$\begin{aligned} r(x^2+2y^2+2z^2,n) = &\frac{32h(-n)\sqrt{n}}{\omega_n\sqrt{\delta_n}}\left(1-\frac{1}{2}\chi_{-n}(2)\right)\\ &\times\left(\alpha(n)-\delta\left(\frac{n-1}{4}\right)-\left(\frac{n-2}{n}\right)\right)\\ &\sum_{\substack{(ab)^2|n,(ab,2)=1\\ a,b \text{ positive integers}}}\mu(a)\left(\frac{-n}{a}\right)(ab)^{-1},\end{aligned}$$

where $\delta(x)=1$ or 0 according to x an integer or not.

In particular, for any square-free odd positive integer n, the sum is equal to 1, and since the conductor δ_n of χ_{-n} is equal to $4n$ or n according to $n\equiv 1$ or 3 (mod 4), we have

$$r(x^2+2y^2+2z^2,n)=\begin{cases} 2, & \text{if } n=1,\\ 8, & \text{if } n=3,\\ 4h(-n), & \text{if } n\equiv 1\,(\text{mod } 8), n\neq 1,\\ 24h(-n), & \text{if } n\equiv 3\,(\text{mod } 8), n\neq 3.\end{cases}$$

Therefore we have for any square-free odd positive integer $n>3$

$$r(f_3,n)+r(g_3,n)=r(x^2+2y^2+8z^2,n)=\begin{cases} 4h(-n), & \text{if } n\equiv 1\,(\text{mod } 8),\\ 12h(-n), & \text{if } n\equiv 3\,(\text{mod } 8),\end{cases} \tag{10.43}$$

By the above (10.40) and (10.42) we have proved the theorem. □

Let $N=p_1p_2\cdots p_m$ with $p_1,p_2,\cdots,p_m$ distinct odd primes, at most two of them congruent to 3 modulo 8 and others congruent to 1 modulo 8. If there is at most one of p_i congruent to 3 modulo 8, then we define a simple graph $G_N=(V(G_N),E(G_N))$ with vertices $V(G_N)=\{p_1,p_2,\cdots,p_m\}$ and edges $E(G_N)=\left\{(p_i,p_j)\,\middle|\,\left(\dfrac{p_j}{p_i}\right)=-1\right\}$ where $(-)$ is the Legendre symbol as usual. Otherwise, without loss of generality, we may assume $p_1\equiv p_2\equiv 3$ (mod8) and $p_i\equiv 1$ (mod8) for $i\geqslant 3$. We define a simple graph $G_N=(V(G_N),E(G_N))$ with vertices $V(G_N)=\{p_1,p_2,\cdots,p_m\}$ and edges $E(G_N)=\left\{(p_1,p_2)\bigcup(p_i,p_j)\,\middle|\,\left(\dfrac{p_j}{p_i}\right)=-1,\{i,j\}\neq\{1,2\}\right\}$. By the quadratic reciprocity law, the graph G_N is a non-directed graph. We denote the number of spanning trees of G_N by $\tau(G_N)$ if N has at most one prime factor congruent to 3 modulo 8, otherwise τ is the number of spanning trees containing the special edge (p_1, p_2)(a subgraph of a non-directed simple graph is called a spanning tree if it is a tree and its vertices coincide with that of the original graph). Let $\nu_2(n)$ be the 2-adic additive valuation normalized by $\nu_2(2) = 1$.

Theorem 10.19 *Let $N = p_1p_2\cdots p_m > 3$ congruent to 1 or 3 modulo 8, with $p_1, p_2, \cdots, p_m$ distinct odd primes, at most two of them congruent to 3 modulo 8 and all others congruent to 1 modulo 8. Let f_3, g_3 be as in* Theorem 10.18. *Then*

(1) $\nu_2(r(f_3, N)) \geqslant m, \nu_2(r(g_3, N)) \geqslant m$;

(2) *if all $p_i (i = 1, 2, \cdots, m)$ are congruent to 1 modulo 8, then the equality in (1) holds if and only if $\nu_2(h(-N)) = m - 1$;*

(3) *if there is only one or two p_i $(i = 1, 2, \cdots, m)$ congruent to 3 modulo 8, then the equality in (1) holds if and only if one of the following conditions is satisfied*: i) $\nu_2(h(-N)) = m - 1$ *and $\tau(G_N)$ is even*; ii) $\nu_2(h(-N)) > m - 1$ *and $\tau(G_N)$ is odd.*

Proof In order to prove the theorem, we need the following facts(for the proofs of these facts please see C. Zhao, 1991, C. Zhao, 2001, C. Zhao, 2003):

Claim Let the notations be as in the theorem. Then

(1) $\nu_2\left(\dfrac{L_{E_{N^2}}(1)}{\omega\sqrt{N}}\right) \geqslant 2m$ if all p_i $(i = 1, 2, \cdots, m)$ are congruent to 1 modulo 8;

(2) $\nu_2\left(\dfrac{L_{E_{N^2}}(1)}{\omega\sqrt{N}}\right) \geqslant 2m - 2$ if one or two of $p_i (i = 1, 2, \cdots, m)$ are congruent to 3 modulo 8 and others are congruent to 1 modulo 8. Moreover, the equality holds if and only if $\tau(G_N)$ is odd.

We consider the 2-adic valuation of the terms on the right side of the conclusion of Theorem 10.18. It is clear that $\nu_2(c(N)) = 1$. From the Gauss genus theory we know that

$$\nu_2(h(-N)) \geqslant m - 1, \tag{10.44}$$

where m is the number of prime factors of N. By the claim we see that $\nu_2\left(4\dfrac{L_{E_{N^2}}(1)}{\omega\sqrt{N}}\right) \geqslant 2m$. So the first conclusion (1) of the theorem is valid.

Now suppose that $N = p_1p_2\cdots p_m$ with all $p_i \equiv 1 \pmod 8$. Then we have that

$$\nu_2\left(4\frac{L_{E_{N^2}}(1)}{\omega\sqrt{N}}\right) \geqslant 2m + 2.$$

Therefore, by Theorem 10.18, (10.43) and (10.44), we see that $\nu_2(r(f_3, N)) = \nu_2(r(g_3, N)) = m$ if and only if $\nu_2(c(N)h(-N)) = m$, which is equivalent to $\nu_2(h(-N)) = m - 1$. This is the second assertion (2) of the theorem.

Finally, suppose that $N = p_1p_2\cdots p_m$ as in (3) of the theorem. By the claim we have

$$\nu_2\left(4\frac{L_{E_{N^2}}(1)}{\omega\sqrt{N}}\right) \geqslant 2m. \tag{10.45}$$

And the equality holds if and only if $\tau(G_N)$ is odd. By (10.43), we have

$$\nu_2(c(N)h(-N)) \geqslant m \tag{10.46}$$

and the equality holds if and only if $\nu_2(h(-N)) = m-1$. Therefore by Theorem 10.18, $\nu_2(r(f_3,N)) = \nu_2(r(g_3,N)) = m$ if and only if one of the inequalities in (10.45) and (10.46) holds while the other one does not hold. This is the assertion (3) of the theorem which completes the proof. □

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