8

Intuitionistic Fuzzy Relations (IFRs)

8.1 Cartesian Products over IFSs

First, we define six versions of another operation over IFSs – namely, Cartesian products of two IFSs. To introduce the concept of intuitionistic fuzzy relation, we use these operations.

Let E_1 and E_2 be two universes and let

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E_1\},$$
$$B = \{\langle y, \mu_B(y), \nu_B(y)\rangle | y \in E_2\},$$

be two IFSs over E_1 and over E_2, respectively.

Now, following [39], define,

$$A \times_1 B = \{\langle\langle x, y\rangle, \mu_A(x).\mu_B(y), \nu_A(x).\nu_B(y)\rangle | x \in E_1 \& y \in E_2\}, \tag{8.1}$$

$$A \times_2 B = \{\langle\langle x, y\rangle, \mu_A(x) + \mu_B(y) - \mu_A(x).\mu_B(y), \nu_A(x).\nu_B(y)\rangle | x \in E_1 \& y \in E_2\}, \tag{8.2}$$

$$A \times_3 B = \{\langle\langle x, y\rangle, \mu_A(x).\mu_B(y), \nu_A(x) + \nu_B(y) - \nu_A(x).\nu_B(y)\rangle | x \in E_1 \& y \in E_2\}, \tag{8.3}$$

$$A \times_4 B = \{\langle\langle x, y\rangle, \min(\mu_A(x), \mu_B(y)), \max(\nu_A(x), \nu_B(y))\rangle | x \in E_1 \& y \in E_2\}, \tag{8.4}$$

$$A \times_5 B = \{\langle\langle x, y\rangle, \max(\mu_A(x), \mu_B(y)), \min(\nu_A(x), \nu_B(y))\rangle | x \in E_1 \& y \in E_2\}, \tag{8.5}$$

$$A \times_6 B = \{\langle\langle x, y\rangle, \frac{\mu_A(x)+\mu_B(y)}{2}, \frac{\nu_A(x)+\nu_B(y)}{2})\rangle | x \in E_1 \& y \in E_2\}. \tag{8.6}$$

Operation $\times_6$ was introduced by my student Velin Andonov in [8], who studied its properties.

From,

$$0 \leq \mu_A(x).\mu_B(y) + \nu_A(x).\nu_B(y) \leq \mu_A(x) + \nu_A(x) \leq 1,$$

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it follows that $A \times_1 B$ is an IFS over the universe $E_1 \times E_2$, where "$\times$" is the classical Cartesian product on ordinary sets (E_1 and E_2). For the five other products the computations are analogous.

For every three universes E_1, E_2 and E_3 and four IFSs A, B (over E_1), C (over E_2) and D (over E_3):

(a) $(A \times C) \times D = A \times (C \times D)$, where $\times \in \{\times_1, \times_2, \times_3, \times_4, \times_5\}$.
(b) $(A \cup B) \times C = (A \times C) \cup (B \times C)$,
(c) $(A \cap B) \times C = (A \times C) \cap (B \times C)$,
(d) $C \times (A \cup B) = (C \times A) \cup (C \times B)$,
(e) $C \times (A \cap B) = (C \times A) \cap (C \times B)$,

where $\times \in \{\times_1, \times_2, \times_3, \times_4, \times_5, \times_6\}$.

For every two universes E_1 and E_2 and three IFSs A, B (over E_1) and C (over E_2):

(a) $(A + B) \times C \subseteq (A \times C) + (B \times C)$,
(b) $(A.B) \times C \supseteq (A \times C).(B \times C)$,
(c) $(A@B) \times C = (A \times C)@(B \times C)$,
(d) $C \times (A + B) \subseteq (C \times A) + (C \times B)$,
(e) $C \times (A.B) \supseteq (C \times A).(C \times B)$,
(f) $C \times (A@B) = (C \times A)@(C \times B)$,

where $\times \in \{\times_1, \times_2, \times_3, \times_6\}$.

If A is an IFS over E_1 and B is an IFS over E_2, then,

(a) $\square(A \times_1 B) \subseteq \square A \times_1 \square B$,
(b) $\lozenge(A \times_1 B) \supseteq \lozenge A \times_1 \lozenge B$,

(c) $\square(A \times_2 B) = \square A \times_2 \square B$,
(d) $\lozenge(A \times_2 B) = \lozenge A \times_2 \lozenge B$,

(e) $\square(A \times_3 B) = \square A \times_3 \square B$,
(f) $\lozenge(A \times_3 B) = \lozenge A \times_3 \lozenge B$,

(g) $\square(A \times_4 B) = \square A \times_4 \square B$,
(h) $\lozenge(A \times_4 B) = \lozenge A \times_4 \lozenge B$,

(i) $\square(A \times_5 B) = \square A \times_5 \square B$,
(j) $\lozenge(A \times_5 B) = \lozenge A \times_5 \lozenge B$.

(k) $\square(A \times_6 B) = \square A \times_6 \square B$,
(l) $\lozenge(A \times_6 B) = \lozenge A \times_6 \lozenge B$.

Formulae similar to De Morgan's laws hold for the above Cartesian products. If A is an IFS over E_1 and B is an IFS over E_2, then,

(a) $\overline{\overline{A} \times_1 \overline{B}} = A \times_1 B$,

(b) $\overline{\overline{A} \times_2 \overline{B}} = A \times_3 B$,

(c) $\overline{\overline{A} \times_3 \overline{B}} = A \times_2 B$,

(d) $\overline{\overline{A} \times_4 \overline{B}} = A \times_5 B$,

(e) $\overline{\overline{A} \times_5 \overline{B}} = A \times_4 B$,

(f) $\overline{\overline{A} \times_6 \overline{B}} = A \times_6 B$.

Therefore, operations $\times_2$ and $\times_3$; $\times_4$ and $\times_5$ are dual and the operations $\times_1$ and $\times_6$ are autodual.

The following inclusions hold for every two universes E_1 and E_2, and two IFSs A (over E_1) and B (over E_2):

(a) $A \times_3 B \subseteq A \times_1 B \subseteq A \times_2 B$,

(b) $A \times_3 B \subseteq A \times_4 B \subseteq A \times_6 B \subseteq A \times_5 B \subseteq A \times_2 B$.

For every two universes E_1 and E_2, and two IFSs A (over E_1) and B (over E_2) and for every $\alpha, \beta \in [0, 1]$,

$$G_{\alpha,\beta}(A \times_1 B) = G_{\alpha/\gamma,\beta/\delta}(A) \times_1 G_{\gamma,\delta}(B),$$

for every $0 < \gamma \leq 1, 0 < \delta \leq 1$ for which $\frac{\alpha}{\gamma}, \frac{\beta}{\delta} \in [0, 1]$.

For every two IFSs A and B, and for every $\alpha, \beta \in [0, 1]$, the following relations hold:

(a) $D_\alpha(A \times_4 B) \subseteq D_\alpha(A) \times_4 D_\alpha(B)$,

(b) $F_{\alpha,\beta}(A \times_4 B) \subseteq F_{\alpha,\beta}(A) \times_4 F_{\alpha,\beta}(B)$, where $\alpha + \beta \leq 1$,

(c) $G_{\alpha,\beta}(A \times_4 B) = G_{\alpha,\beta}(A) \times_4 G_{\alpha,\beta}(B)$,

(d) $H_{\alpha,\beta}(A \times_4 B) \subseteq H_{\alpha,\beta}(A) \times_4 H_{\alpha,\beta}(B)$,

(e) $H^*_{\alpha,\beta}(A \times_4 B) \subseteq H^*_{\alpha,\beta}(A) \times_4 H^*_{\alpha,\beta}(B)$,

(f) $J_{\alpha,\beta}(A \times_4 B) \subseteq J_{\alpha,\beta}(A) \times_4 J_{\alpha,\beta}(B)$,

(g) $J^*_{\alpha,\beta}(A \times_4 B) \subseteq J^*_{\alpha,\beta}(A) \times_4 J^*_{\alpha,\beta}(B)$,

(h) $P_{\alpha,\beta}(A \times_4 B) = P_{\alpha,\beta}(A) \times_4 P_{\alpha,\beta}(B)$,

(i) $Q_{\alpha,\beta}(A \times_4 B) = Q_{\alpha,\beta}(A) \times_4 Q_{\alpha,\beta}(B)$,

(j) $\boxdot_{\alpha,\beta,\gamma,\delta}(A \times_4 B) = \boxdot_{\alpha,\beta,\gamma,\delta}(A) \times_4 \boxdot_{\alpha,\beta,\gamma,\delta}(B)$,

(k) $D_\alpha(A \times_5 B) \supseteq D_\alpha(A) \times_5 D_\alpha(B)$,

(l) $F_{\alpha,\beta}(A \times_5 B) \supseteq F_{\alpha,\beta}(A) \times_5 F_{\alpha,\beta}(B)$, where $\alpha + \beta \leq 1$,

(m) $G_{\alpha,\beta}(A \times_5 B) = G_{\alpha,\beta}(A) \times_5 G_{\alpha,\beta}(B)$,

(n) $H_{\alpha,\beta}(A \times_5 B) \supseteq H_{\alpha,\beta}(A) \times_5 H_{\alpha,\beta}(B)$,

(o) $H^*_{\alpha,\beta}(A \times_5 B) \supseteq H^*_{\alpha,\beta}(A) \times_5 H^*_{\alpha,\beta}(B)$,

(p) $J_{\alpha,\beta}(A \times_5 B) \supseteq J_{\alpha,\beta}(A) \times_5 J_{\alpha,\beta}(B)$,

(q) $J^*_{\alpha,\beta}(A \times_5 B) \supseteq J^*_{\alpha,\beta}(A) \times_5 J^*_{\alpha,\beta}(B)$.
(r) $P_{\alpha,\beta}(A \times_5 B) = P_{\alpha,\beta}(A) \times_5 P_{\alpha,\beta}(B)$,
(s) $Q_{\alpha,\beta}(A \times_5 B) = Q_{\alpha,\beta}(A) \times_5 Q_{\alpha,\beta}(B)$,
(t) $\boxdot_{\alpha,\beta,\gamma,\delta}(A \times_5 B) = \boxdot_{\alpha,\beta,\gamma,\delta}(A) \times_5 \boxdot_{\alpha,\beta,\gamma,\delta}(B)$,
(u) $D_\alpha(A \times_6 B) = D_\alpha(A) \times_6 D_\alpha(B)$,
(v) $F_{\alpha,\beta}(A \times_6 B) = F_{\alpha,\beta}(A) \times_6 F_{\alpha,\beta}(B)$, where $\alpha + \beta \leq 1$,
(w) $G_{\alpha,\beta}(A \times_6 B) = G_{\alpha,\beta}(A) \times_6 G_{\alpha,\beta}(B)$,
(x) $H_{\alpha,\beta}(A \times_6 B) = H_{\alpha,\beta}(A) \times_6 H_{\alpha,\beta}(B)$,
(y) $H^*_{\alpha,\beta}(A \times_6 B) = H^*_{\alpha,\beta}(A) \times_6 H^*_{\alpha,\beta}(B)$,
(z) $J_{\alpha,\beta}(A \times_6 B) = J_{\alpha,\beta}(A) \times_6 J_{\alpha,\beta}(B)$,
(α) $J^*_{\alpha,\beta}(A \times_6 B) = J^*_{\alpha,\beta}(A) \times_6 J^*_{\alpha,\beta}(B)$,
(β) $P_{\alpha,\beta}(A \times_6 B) \subseteq P_{\alpha,\beta}(A) \times_6 P_{\alpha,\beta}(B)$,
(γ) $Q_{\alpha,\beta}(A \times_6 B) \supseteq Q_{\alpha,\beta}(A) \times_5 Q_{\alpha,\beta}(B)$,
(δ) $\boxcircle_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}(A \times_6 B) = \boxcircle_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}(A) \times_6 \boxcircle_{\alpha,\beta,\gamma,\delta,\varepsilon,\zeta}(B)$.

8.2 Index Matrix

The concept of Index Matrix (IM) was introduced in 1984 in [12, 15]. During the last 25 years, some of its properties were studied (see [75, 76]), but in general it was only used as an auxiliary tool for describing the transitions of the generalized nets (see [23, 65]), the intuitionistic fuzzy relations with finite universes, the intuitionistic fuzzy graphs with finite set of vertrices, as well as in some decision making algorithms based on intuitionistic fuzzy estimations. Some authors found an application of the IMs in the area of number theory, [337].

Following [75, 76], in Subsection **8.2.1**, we give the basic definition of an IM and the operations over two IMs, as well as some properties of IMs. In Subsection **8.2.2**, operations extending those from Subsection **8.2.1** and essentially new ones, are introduced for the first time and their properties has been discussed. Since the proofs of the formulated theorems are based on the respective definitions, only one proof is given as an illustration.

8.2.1 Basic Definitions and Properties

Following [15], the basic definitions and properties related to IMs are given.

Let I be a fixed set of indices and $\mathcal{R}$ be the set of all real numbers. By IM with index sets K and L $(K, L \subset I)$, we mean the object,

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_1 & a_{k_1,l_1} & a_{k_1,l_2} & \dots & a_{k_1,l_n} \\ k_2 & a_{k_2,l_1} & a_{k_2,l_2} & \dots & a_{k_2,l_n} \\ \vdots & & & & \\ k_m & a_{k_m,l_1} & a_{k_m,l_2} & \dots & a_{k_m,l_n} \end{array},$$

where $K = \{k_1, k_2, ..., k_m\},\ L = \{l_1, l_2, ..., l_n\},$ for $1 \le i \le m$, and $1 \le j \le n : a_{k_i,l_j} \in \mathcal{R}$.

For the IMs $A = [K, L, \{a_{k_i,l_j}\}], B = [P, Q, \{b_{p_r,q_s}\}]$, operations that are analogous of the usual matrix operations of addition and multiplication are defined, as well as other, specific ones.

(a) addition $A \oplus B = [K \cup P, L \cup Q, \{c_{t_u,v_w}\}]$, where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ a_{k_i,l_j} + b_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ 0, & \text{otherwise} \end{cases}$$

(b) termwise multiplication $A \otimes B = [K \cap P, L \cap Q, \{c_{t_u,v_w}\}]$, where

$$c_{t_u,v_w} = a_{k_i,l_j}.b_{p_r,q_s}, \text{ for } \begin{array}{l} t_u = k_i = p_r \in K \cap P \text{ and} \\ v_w = l_j = q_s \in L \cap Q; \end{array}$$

(c) multiplication $A \odot B = [K \cup (P - L), Q \cup (L - P), \{c_{t_u,v_w}\}]$, where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - P \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P - L \text{ and } v_w = q_s \in Q \\ \sum\limits_{l_j = p_r \in L \cap P} a_{k_i,l_j}.b_{p_r,q_s}, & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\ 0, & \text{otherwise} \end{cases}$$

(d) structural subtraction $A \ominus B = [K - P, L - Q, \{c_{t_u,v_w}\}]$, where "–" is the set–theoretic difference operation and

$$c_{t_u,v_w} = a_{k_i,l_j}, \text{ for } t_u = k_i \in K - P \text{ and } v_w = l_j \in L - Q.$$

(e) multiplication with a constant $\alpha.A = [K, L, \{\alpha.a_{k_i,l_j}\}]$, where α is a constant.

(f) termwise subtraction $A - B = A \oplus (-1).B$.
For example, if we have the IMs X and Y

$$X = \begin{array}{c|ccc} & c & d & e \\ \hline a & 1 & 2 & 3 \\ b & 4 & 5 & 6 \end{array}, \quad Y = \begin{array}{c|cc} & c & r \\ \hline a & 10 & 11 \\ p & 12 & 13 \\ q & 14 & 15 \end{array},$$

then

$$X \oplus Y = \begin{array}{c|cccc} & c & d & e & r \\ \hline a & 11 & 2 & 3 & 11 \\ b & 4 & 5 & 6 & 0 \\ p & 12 & 0 & 0 & 13 \\ q & 14 & 0 & 0 & 15 \end{array}$$

and

$$X \otimes Y = \begin{array}{c|c} & c \\ \hline a & 10 \end{array}$$

If IM Z has the form

$$Z = \begin{array}{c|c} & u \\ \hline c & 10 \\ d & 11 \\ s & 12 \\ t & 13 \end{array},$$

then

$$X \odot Z = \begin{array}{c|cc} & e & u \\ \hline a & 3 & 1 \times 10 + 2 \times 11 \\ b & 6 & 4 \times 10 + 5 \times 11 \\ s & 0 & 12 \\ t & 0 & 13 \end{array} = \begin{array}{c|cc} & e & u \\ \hline a & 3 & 32 \\ b & 6 & 95 \\ s & 0 & 12 \\ t & 0 & 13 \end{array}.$$

Now, it is seen that when

$$K = P = \{1, 2, ..., m\},$$

$$L = Q = \{1, 2, ..., n\},$$

we obtain the definitions for standard matrix operations. In IMs, we use different symbols as indices of the rows and columns and they, as we have seen above, give us additional information and possibilities for description.

Let $\mathcal{IM}_{\mathcal{R}}$ be the set of all IMs with their elements being real numbers, $\mathcal{IM}_{\{0,1\}}$ be the set of all $(0, 1)$-IMs. i.e., IMs with elements only 0 or 1, and $\mathcal{IM}_{\mathcal{P}}$ be the class of all IMs with elements – predicates[1]. The problem with the "zero"-IM is more complex than in the standard matrix case. We introduce "zero"-IM for $\mathcal{IM}_{\mathcal{R}}$ as

[1] All IMs over the class of the predicates also generate a class in Neuman-Bernaus-Gödel set theoretical sense.

$$I_0 = [K, L, \{0\}],$$

an IM whose elements are equal to 0 and $K, L \subset I$ are arbitrary index sets, as well as the IM

$$I_\emptyset = [\emptyset, \emptyset, \{a_{k_i,l_j}\}].$$

In the second case there are no matrix cells where the elements a_{k_i,l_j} may be inserted. In both cases, for each IM $A = [K, L, \{b_{k_i,l_j}\}]$ and for $I_0 = [K, L, \{0\}]$ with the same index sets, we have,

$$A \oplus I_0 = A = I_0 \oplus A.$$

The situation with $\mathcal{IM}_{\{0,1\}}$ is similar, while in the case of $\mathcal{IM}_{\mathcal{P}}$ the "zero"-IM can be either IM

$$I_f = [K, L, \{\text{“}false\text{”}\}],$$

or the IM

$$I_\emptyset = [\emptyset, \emptyset, \{a_{k_i,l_j}\}]$$

where the elements a_{k_i,l_j} are arbitrary predicates.

Let

$$I_1 = [K, L, \{1\}]$$

denote the IM, whose elements are equal to 1, and where $K, L \subset I$ are arbitrary index sets.

The operations defined above are oriented to IMs, whose elements are real or complex numbers. Let us denote these operations, respectively, by $\oplus_+$, $\otimes_\times$, $\odot_{+,.}$, $\ominus_-$.

The following properties of the IM are discussed in [15].

Theorem 8.1: (a) $\langle \mathcal{IM}_{\mathcal{R}}, \oplus_+ \rangle$ is a commutative semigroup,
(b) $\langle \mathcal{IM}_{\mathcal{R}}, \otimes_\times \rangle$ is a commutative semigroup,
(c) $\langle \mathcal{IM}_{\mathcal{R}}, \odot_{+,\times} \rangle$ is a semigroup,
(d) $\langle \mathcal{IM}_{\mathcal{R}}, \oplus_+, I_\emptyset \rangle$ is a commutative monoid.

8.2.2 Other Definitions and Properties

Now, a series of new operations and relations over IMs are introduced. They have been collected by the author during the last 10-15 years, but now they are published for the first time.

8.2.2.1. Modifications of the IM-operations

It is well-known that the $(0, 1)$-matrices have applications in the areas of discrete mathematics and combinatorial analysis. When we choose to work with this kind of matrices, the above operations have the following forms.

(a′) $A \oplus_{\max} B = [K \cup P, L \cup Q, \{c_{t_u,v_w}\}]$, where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{and } v_w = q_s \in Q; \\ \max(a_{k_i,l_j}, b_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ 0, & \text{otherwise} \end{cases}$$

(b′) $A \otimes_{\min} B = [K \cap P, L \cap Q, \{c_{t_u,v_w}\}]$, where

$$c_{t_u,v_w} = \min(a_{k_i,l_j}, b_{p_r,q_s}), \text{ for } \begin{array}{l} t_u = k_i = p_r \in K \cap P \text{ and} \\ v_w = l_j = q_s \in L \cap Q; \end{array}$$

(c′) $A \odot_{\max,\min} B = [K \cup (P - L), Q \cup (L - P), \{c_{t_u,v_w}\}]$, where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \\ & \text{and } v_w = l_j \in L - P \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P - L \\ & \text{and } v_w = q_s \in Q \\ \max\limits_{l_j = p_r \in L \cap P} \min(a_{k_i,l_j}, b_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \\ & \text{and } v_w = q_s \in Q \\ 0, & \text{otherwise} \end{cases}$$

Operation (d) from Subsection **8.2.1** preserves its form, operation (e) is possible only in the case when $\alpha \in \{0, 1\}$, while operation (f) is impossible.

The three operations are applicable also to the IMs, whose elements are real numbers.

Theorem 8.2: (a) $\langle \mathcal{IM}_{\{0,1\}}, \oplus_{\max} \rangle$ is a commutative semigroup,
(b) $\langle \mathcal{IM}_{\{0,1\}}, \otimes_{\min} \rangle$ is a commutative semigroup,
(c) $\langle \mathcal{IM}_{\{0,1\}}, \odot_{\max,\min} \rangle$ is a semigroup,
(d) $\langle \mathcal{IM}_{\{0,1\}}, \oplus_{\max}, I_\emptyset \rangle$ is a commutative monoid.

When working with matrices, whose elements are sentences or predicates, the forms of the above operations become

(a″) $A \oplus_\vee B = [K \cup P, L \cup Q, \{c_{t_u,v_w}\}]$, where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ & \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ & \\ a_{k_i,l_j} \vee b_{p_r,q_s}, & \text{if } t_u = k_i = p_r \in K \cap P \\ & \text{and } v_w = l_j = q_s \in L \cap Q \\ & \\ false, & \text{otherwise} \end{cases}$$

(b″) $A \otimes_{\wedge} B = [K \cap P, L \cap Q, \{c_{t_u,v_w}\}]$, where

$$c_{t_u,v_w} = a_{k_i,l_j} \wedge b_{p_r,q_s}, \text{ for } \begin{array}{l} t_u = k_i = p_r \in K \cap P \text{ and} \\ v_w = l_j = q_s \in L \cap Q; \end{array}$$

(c″) $A \odot_{\vee,\wedge} B = [K \cup (P - L), Q \cup (L - P), \{c_{t_u,v_w}\}]$, where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - P \\ & \\ b_{p_r,q_s}, & \text{if } t_u = p_r \in P - L \text{ and } v_w = q_s \in Q \\ & \\ \bigvee\limits_{l_j = p_r \in L \cap P} (a_{k_i,l_j} \wedge b_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \text{ and } v_w = q_s \in Q \\ & \\ false, & \text{otherwise} \end{cases}$$

Operation (d) from Subsection **8.2.1** preserves its form, while operations (e) and (f) are impossible.

Theorem 8.3: (a) $\langle \mathcal{IM}_{\mathcal{P}}, \oplus_{\vee} \rangle$ is a commutative semigroup,
(b) $\langle \mathcal{IM}_{\mathcal{P}}, \otimes_{\wedge} \rangle$ is a commutative semigroup,
(c) $\langle \mathcal{IM}_{\mathcal{P}}, \odot_{\vee,\wedge} \rangle$ is a semigroup,
(d) $\langle \mathcal{IM}_{\mathcal{P}}, \oplus_{\vee}, I_{\emptyset} \rangle$ is a commutative monoid.

8.2.2.2. Relations over IMs

Let two IMs $A = [K, L, \{a_{k,l}\}]$ and $B = [P, Q, \{b_{p,q}\}]$ be given. We introduce the following (new) definitions where $\subset$ and $\subseteq$ denote the relations *"strong inclusion"* and *"weak inclusion"*.

The strict relation *"inclusion about dimension"* is

$$A \subset_d B \text{ iff } ((K \subset P)\&(L \subset Q)) \vee (K \subseteq P)\&(L \subset Q) \vee (K \subset P)\&(L \subseteq Q))$$

$$\&(\forall k \in K)(\forall l \in L)(a_{k,l} = b_{k,l}).$$

The non-strict relation *"inclusion about dimension"* is

$$A \subseteq_d B \text{ iff } (K \subseteq P)\&(L \subseteq Q)\&(\forall k \in K)(\forall l \in L)(a_{k,l} = b_{k,l}).$$

The strict relation *"inclusion about value"* is

$$A \subset_v B \text{ iff } (K = P)\&(L = Q)\&(\forall k \in K)(\forall l \in L)(a_{k,l} < b_{k,l}).$$

The non-strict relation *"inclusion about value"* is

$$A \subseteq_v B \text{ iff } (K = P)\&(L = Q)\&(\forall k \in K)(\forall l \in L)(a_{k,l} \leq b_{k,l}).$$

The strict relation *"inclusion"* is

$$A \subset B \text{ iff } (((K \subset P)\&(L \subset Q)) \vee ((K \subseteq P)\&(L \subset Q)) \vee ((K \subset P)$$

$$\&(L \subseteq Q))) \;\&\; (\forall k \in K)(\forall l \in L)(a_{k,l} < b_{k,l}).$$

The non-strict relation *"inclusion"* is

$$A \subseteq B \text{ iff } (K \subseteq P)\&(L \subseteq Q)\&(\forall k \in K)(\forall l \in L)(a_{k,l} \leq b_{k,l}).$$

It is obvious that for every two IMs A and B,

- if $A \subset_d B$, then $A \subseteq_d B$;
- if $A \subset_v B$, then $A \subseteq_v B$;
- if $A \subset B$, $A \subseteq_d B$, or $A \subseteq_v B$, then $A \subseteq B$;
- if $A \subset_d B$ or $A \subset_v B$, then $A \subseteq B$.

8.2.2.3. Operations "Reduction" over an IM

First, we introduce operations $(k, *)$-reduction and $(*, l)$-reduction of a given IM $A = [K, L, \{a_{k_i,l_j}\}]$,

$$A_{(k,*)} = [K - \{k\}, L, \{c_{t_u,v_w}\}]$$

where

$$c_{t_u,v_w} = a_{k_i,l_j} \quad \text{for } t_u = k_i \in K - \{k\} \text{ and } v_w = l_j \in L$$

and

$$A_{(*,l)} = [K, L - \{l\}, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = a_{k_i,l_j} \quad \text{for } t_u = k_i \in K \text{ and } v_w = l_j \in L - \{l\}.$$

Second, we define (k, l)-reduction

$$A_{(k,l)} = (A_{(k,*)})_{(*,l)} = (A_{(*,l)})_{(k,*)},$$

i.e.,

$$A_{(k,l)} = [K - \{k\}, L - \{l\}, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = a_{k_i,l_j} \quad \text{for } t_u = k_i \in K - \{k\} \text{ and } v_w = l_j \in L - \{l\}.$$

For every IM A and for every $k_1, k_2 \in K$, $l_1, l_2 \in L$,

$$(A_{(k_1,l_1)})_{(k_2,l_2)} = (A_{(k_2,l_2)})_{(k_1,l_1)}.$$

Third, let $P = \{k_1, k_2, ..., k_s\} \subseteq K$ and $Q = \{q_1, q_2, ..., q_t\} \subseteq L$. Now, we define the following three operations:

$$A_{(P,*)} = (...((A_{(k_1,*)})_{(k_2,*)})...)_{(k_s,*)},$$

$$A_{(*,Q)} = (...((A_{(*,l_1)})_{(*,l_2)})...)_{(*,l_t)},$$

$$A_{(P,Q)} = (A_{(P,*)})_{(*,Q)} = (A_{(*,Q)})_{(P,*)}.$$

Obviously,

$$A_{(K,L)} = I_\emptyset,$$

$$A_{(\emptyset,\emptyset)} = A.$$

For every two IMs $A = [K, L, \{a_{k_i,l_j}\}]$ and $B = [P, Q, \{b_{p_r,q_s}\}]$:

$$A \subseteq_d B \text{ iff } A = B_{(P-K,Q-L)}.$$

Let $A \subseteq_d B$. Therefore, $K \subseteq P$ and $L \subseteq Q$, and for every $k \in K, l \in L$: $a_{k,l} = b_{k,l}$. From the definition,

$$B_{(P-K,Q-L)} = (...((B_{(p_1,q_1)})_{(p_1,q_2)})...)_{(p_r,q_s)},$$

where $p_1, p_2, ..., p_r \in P - K$, i.e., $p_1, p_2, ..., p_r \in P$, and $p_1, p_2, ..., p_r \notin K$, and $q_1, q_2, ..., q_s \in Q - L$, i.e., $q_1, q_2, ..., q_s \in Q$, and $q_1, q_2, ..., q_s \notin L$. Therefore,

$$B_{(P-K,Q-L)} = [P - (P - K), Q - (Q - L), \{b_{k,l}\}]$$

$$= [K, L, \{b_{k,l}\}] = [K, L, \{a_{k,l}\}] = A,$$

because, by definition the elements of the two IMs which are indexed by equal symbols coincide.

Conversely, if $A = B_{(P-K,Q-L)}$, then

$$A = B_{(P-K,Q-L)} \subseteq_d B_{\emptyset,\emptyset} = B.$$

8.2.2.4. Operation "Projection" over an IM

Let $M \subseteq K$ and $N \subseteq L$. Then,

$$pr_{M,N}A = [M, N, \{b_{k_i,l_j}\}],$$

where

$$(\forall k_i \in M)(\forall l_j \in N)(b_{k_i,l_j} = a_{k_i,l_j}).$$

For every IM A, and sets $M_1 \subseteq M_2 \subseteq K$ and $N_1 \subseteq N_2 \subseteq L$, the equality

$$pr_{M_1,N_1}(pr_{M_2,N_2}A) = pr_{M_1,N_1}A$$

holds.

8.2.2.5. Hierarhical Operations over IMs

Let A be an ordinary IM, and let its element a_{k_f,e_g} be an IM by itself,

$$a_{k_f,l_g} = [P, Q, \{b_{p_r,q_s}\}],$$

where

$$K \cap P = L \cap Q = \emptyset.$$

Here, introduce the following hierarchical operation

$$A|(a_{k_f,l_g}) = [(K - \{k_f\}) \cup P, (L - \{l_g\}) \cup Q, \{c_{t_u,v_w}\}],$$

where

$$c_{t_u,v_w} = \begin{cases} a_{k_i,l_j}, \text{ if } t_u = k_i \in K - \{k_f\} \text{ and } v_w = l_j \in L - \{l_g\} \\ b_{p_r,q_s}, \text{ if } t_u = p_r \in P \text{ and } v_w = q_s \in Q \\ 0, \quad \text{otherwise} \end{cases}$$

Assume that, if a_{k_f,l_g} is not an element of IM A, then

$$A|(a_{k_f,l_g}) = A.$$

Therefore,

$$A|(a_{k_f,l_g})$$

$$= \begin{array}{c|ccccccccc} & l_1 & \ldots & l_{g-1} & q_1 & \ldots & q_u & l_{g+1} & \ldots & l_n \\ \hline k_1 & a_{k_1,l_1} & \ldots & a_{k_1,l_{g-1}} & 0 & \ldots & 0 & a_{k_1,l_{g+1}} & \ldots & a_{k_1,l_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{f-1} & a_{k_{f-1},l_1} & \ldots & a_{k_{f-1},l_{g-1}} & 0 & \ldots & 0 & a_{k_{f-1},l_{g+1}} & \ldots & a_{k_{f-1},l_n} \\ p_1 & 0 & \ldots & 0 & b_{p_1,q_1} & \ldots & b_{p_1,q_v} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_u & 0 & \ldots & 0 & b_{p_u,q_1} & \ldots & b_{p_u,q_v} & 0 & \ldots & 0 \\ k_{f+1} & a_{k_{f+1},l_1} & \ldots & a_{k_{f+1},l_{g-1}} & 0 & \ldots & 0 & a_{k_{f+1},l_{g+1}} & \ldots & a_{k_{f+1},l_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ k_m & a_{k_m,l_1} & \ldots & a_{k_m,l_{g-1}} & 0 & \ldots & 0 & a_{k_m,l_{g+1}} & \ldots & a_{k_m,l_n} \end{array} .$$

From this form of the IM $A|(a_{k_f,l_g})$ we see that for the hierarchical operation the following equality holds.

$$A|(a_{k_f,l_g}) = (A \ominus [\{k_f\}, \{l_g\}, \{0\}]) \oplus a_{k_f,l_g}.$$

We see that the elements $a_{k_f,l_1}, a_{k_f,l_2}, ..., a_{k_f,l_{g-1}}, a_{k_f,l_{g+1}}, ..., a_{k_f,l_n}$ in the IM A now are changed with 0. Therefore, in a result of this operation information is lost.

Below, we modify the hierarchical operation, so, all information from the IMs, participating in it, to be keeped. The new form of this operation, for the above defined IM A and its fixed element a_{k_f,l_g}, is

$$A|^*(a_{k_f,l_g})$$

$$= \begin{array}{c|ccccccccc} & l_1 & \ldots & l_{g-1} & q_1 & \ldots & q_u & l_{g+1} & \ldots & l_n \\ \hline k_1 & a_{k_1,l_1} & \ldots & a_{k_1,l_{g-1}} & a_{k_1,l_g} & \ldots & a_{k_1,l_g} & a_{k_1,l_{g+1}} & \ldots & a_{k_1,l_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ k_{f-1} & a_{k_{f-1},l_1} & \ldots & a_{k_{f-1},l_{g-1}} & a_{k_{f-1},l_g} & \ldots & a_{k_{f-1},l_g} & a_{k_{f-1},l_{g+1}} & \ldots & a_{k_{f-1},l_n} \\ p_1 & a_{k_f,l_1} & \ldots & a_{k_f,l_{g-1}} & b_{p_1,q_1} & \ldots & b_{p_1,q_v} & a_{k_f,l_{g+1}} & \ldots & a_{k_f,l_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_u & a_{k_f,l_1} & \ldots & a_{k_f,l_{g-1}} & b_{p_u,q_1} & \ldots & b_{p_u,q_v} & a_{k_f,l_{g+1}} & \ldots & a_{k_f,l_n} \\ k_{f+1} & a_{k_{f+1},l_1} & \ldots & a_{k_{f+1},l_{g-1}} & a_{k_{f+1},l_g} & \ldots & a_{k_{f+1},l_g} & a_{k_{f+1},l_{g+1}} & \ldots & a_{k_{f+1},l_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ k_m & a_{k_m,l_1} & \ldots & a_{k_m,l_{g-1}} & a_{k_m,l_g} & \ldots & a_{k_m,l_g} & a_{k_m,l_{g+1}} & \ldots & a_{k_m,l_n} \end{array} .$$

Now, the following equality is valid.

$$A|^*(a_{k_f,l_g}) = (A \ominus [\{k_f\},\{l_g\},\{0\}]) \oplus a_{k_f,l_g} \oplus [P, L-\{l_g\},\{c_{x,l_j}\}]$$

$$\oplus [K-\{k_f\}Q,\{d_{k_i,y}\}],$$

where for each $t \in P$ and for each $l_j \in L - \{l_g\}$,

$$c_{x,l_j} = a_{k_f,l_j}$$

and for each $k_i \in K - \{k_f\}$ and for each $y \in Q$,

$$d_{k_i,y} = a_{k_i,l_g}.$$

Let for $i = 1, 2, ..., s$,

$$a^i_{k_{i,f},l_{i,g}} = [P_i, Q_i, \{b^i_{p_{i,r},q_{i,s}}\}],$$

where for every i, j $(1 \le i < j \le s)$,

$$P_i \cap P_j = Q_i \cap Q_j = \emptyset,$$

$$P_i \cap K = Q_i \cap L = \emptyset.$$

Then, for $k_{1,f}, k_{2,f}, ..., k_{s,f} \in K$ and $l_{1,g}, l_{2,g}, ..., l_{s,g} \in L$,

$$A|(a^1_{k_{1,f},l_{1,g}}, a^2_{k_{2,f},l_{2,g}}, ..., a^s_{k_{s,f},l_{s,g}})$$

$$= (...((A|(a^1_{k_{1,f},l_{1,g}}))|(a^2_{k_{2,f},l_{2,g}}))...)|(a^s_{k_{s,f},l_{s,g}})$$

and

$$A|^*(a^1_{k_{1,f},l_{1,g}}, a^2_{k_{2,f},l_{2,g}}, ..., a^s_{k_{s,f},l_{s,g}})$$
$$= (...((A|^*(a^1_{k_{1,f},l_{1,g}}))|^*(a^2_{k_{2,f},l_{2,g}}))...)|^*(a^s_{k_{s,f},l_{s,g}}).$$

Let the IM A be given and let for $i = 1, 2$: $k_{1,f} \neq k_{2,f}$ and $l_{1,g} \neq l_{2,g}$ and

$$a^i_{k_{i,f},l_{i,g}} = [P_i, Q_i, \{b^i_{p_{i,r},q_{i,s}}\}],$$

where

$$P_1 \cap P_2 = Q_1 \cap Q_2 = \emptyset,$$
$$P_i \cap K = Q_i \cap L = \emptyset.$$

Then,

$$A|(a^1_{k_{1,f},l_{1,g}}, a^2_{k_{2,f},l_{2,g}}) = A|(a^2_{k_{2,f},l_{2,g}}, a^1_{k_{1,f},l_{1,g}})$$

and

$$A|^*(a^1_{k_{1,f},l_{1,g}}, a^2_{k_{2,f},l_{2,g}}) = A|^*(a^2_{k_{2,f},l_{2,g}}, a^1_{k_{1,f},l_{1,g}}).$$

Let A and a_{k_f,l_g} be as above, let b_{m_d,n_e} be the element of the IM a_{k_f,l_g}, and let

$$b_{m_d,n_e} = [R, S, \{c_{t_u,v_w}\}],$$

where

$$K \cap R = L \cap S = P \cap R = Q \cap S = K \cap P = L \cap Q = \emptyset.$$

Then,

$$(A|(a_{k_f,l_g}))|(b_{m_d,n_e})$$
$$= [(K - \{k_f\}) \cup (P - \{m_d\}) \cup R, (L - \{l_g\}) \cup (Q - \{n_e\} \cup S\,\{\alpha_{\beta_\gamma,\delta_\varepsilon}\}],$$

where

$$\alpha_{\beta_\gamma,\delta_\varepsilon} = \begin{cases} a_{k_i,l_j}, & \text{if } \beta_\gamma = k_i \in K - \{k_f\} \text{ and } \delta_\varepsilon = l_j \in L - \{l_g\} \\ b_{p_r,q_s}, & \text{if } \beta_\gamma = p_r \in P - \{m_d\} \text{ and } \delta_\varepsilon = q_s \in Q - \{n_e\} \\ c_{t_u,v_w}, & \text{if } \beta_\gamma = t_u \in R \text{ and } \delta_\varepsilon = v_w \in S \\ 0, & \text{otherwise} \end{cases}$$

For the above IMs A, a_{k_f,l_g} and b_{m_d,n_e}

$$(A|(a_{k_f,l_g}))|(b_{m_d,n_e}) = A|((a_{k_f,l_g})|(b_{m_d,n_e})).$$

8.2.2.6. Operation "Substitution" over an IM

Let IM $A = [K, L, \{a_{k,l}\}]$ be given.

First, local substitution over the IM is defined for the couples of indices (p, k) and/or (q, l), respectively, by

$$[\frac{p}{k}]A = [(K - \{k\}) \cup \{p\}, L, \{a_{k,l}\}],$$

$$[\frac{q}{l}]A = [K, (L - \{l\}) \cup \{q\}, \{a_{k,l}\}],$$

Secondly,

$$[\frac{p}{k}\frac{q}{l}]A = [\frac{p}{k}][\frac{q}{l}]A,$$

i.e.

$$[\frac{p}{k}\frac{q}{l}]A = [(K - \{k\}) \cup \{p\}, (L - \{l\}) \cup \{q\}, \{a_{k,l}\}].$$

Obviously, for the above indices k, l, p, q,

$$[\frac{k}{p}]([\frac{p}{k}]A) = [\frac{l}{q}]([\frac{q}{l}]A) = [\frac{k}{p}\frac{l}{q}]([\frac{p}{k}\frac{q}{l}]A) = A,$$

Let the sets of indices $P = \{p_1, p_2, ..., p_m\}$, $Q = \{q_1, q_2, ..., q_n\}$ be given. Third, for them define sequentially,

$$[\frac{P}{K}]A = [\frac{p_1}{k_1}\frac{p_2}{k_2}...\frac{p_n}{k_n}]A,$$

$$[\frac{Q}{L}]A = ([\frac{q_1}{l_1}\frac{q_2}{l_2}...\frac{q_n}{l_n}]A),$$

$$[\frac{P}{K}\frac{Q}{L}]A) = [\frac{P}{K}][\frac{Q}{L}]A,$$

i.e.,

$$[\frac{P}{K}\frac{Q}{L}]A = [\frac{p_1}{k_1}\frac{p_2}{k_2}...\frac{p_m}{k_m}\frac{q_1}{l_1}\frac{q_2}{l_2}...\frac{q_n}{l_n}]A = [P, Q, \{a_{k,l}\}]$$

Obviously, for the sets K, L, P, Q:

$$[\frac{K}{P}]([\frac{P}{K}]A) = [\frac{L}{Q}]([\frac{Q}{L}]A) = [\frac{K}{P}\frac{L}{Q}]([\frac{P}{K}\frac{Q}{L}]A) = A.$$

For every four sets of indices P_1, P_2, Q_1, Q_2

$$[\frac{P_2}{P_1}\frac{Q_2}{Q_1}][\frac{P_1}{K}\frac{Q_1}{L}]A = [\frac{P_2}{K}\frac{Q_2}{L}]A.$$

8.2.3 Intuitionistic Fuzzy IMs (IFIMs)

In this Section, basic definitions and properties related to IFIMs are given, by extending the results from the previous Section.

8.2.3.1. Basic Definitions and Properties
Now, the new object – the IFIM – has the form

$$[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$$

$$\equiv \begin{array}{c|cccc} & l_1 & l_2 & \ldots & l_n \\ \hline k_1 & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \langle \mu_{k_1,l_2}, \nu_{k_1,l_2} \rangle & \ldots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ k_2 & \langle \mu_{k_2,l_1}, \nu_{k_2,l_1} \rangle & \langle \mu_{k_2,l_2}, \nu_{k_2,l_2} \rangle & \ldots & \langle \mu_{k_2,l_n}, \nu_{k_2,l_n} \rangle \\ \vdots & & & & \\ k_m & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \langle \mu_{k_m,l_2}, \nu_{k_m,l_2} \rangle & \ldots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array},$$

where for every $1 \leq i \leq m, 1 \leq j \leq n$: $0 \leq \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \leq 1$.

For the IFIMs $A = [K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}], B = [P, Q, \{\langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle\}]$, operations that are analogous of the usual matrix operations of addition and multiplication are defined, as well as other specific ones.

(a) addition $A \oplus B = [K \cup P, L \cup Q, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}]$, where

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle =$$

$$= \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \text{ and } v_w = l_j \in L; \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = p_r \in P \text{ and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \text{ and } v_w = q_s \in Q; \\ \langle \max(\mu_{k_i,l_j}, \rho_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \cap P \\ \quad \min(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle, & \text{and } v_w = l_j = q_s \in L \cap Q \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

(b) termwise multiplication $A \otimes B = [K \cap P, L \cap Q, \langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}]$, where

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \langle \min(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \max(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle,$$

if $t_u = k_i = p_r \in K \cap P$ and $v_w = l_j = q_s \in L \cap Q$.

(c) multiplication $A \odot B = [K \cup (P - L), Q \cup (L - P), \langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}]$, where

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle =$$

$$
= \begin{cases}
\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \text{ and } v_w = l_j \in L - P \\
\langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = p_r \in P - L \text{ and } v_w = q_s \in Q \\
\langle \max\limits_{l_j = p_r \in L \cap P} (\min(\mu_{k_i,l_j}, \rho_{p_r,q_s})), & \text{if } t_u = k_i \in K \text{ and } v_w = q_s \in Q \\
\quad \min\limits_{l_j = p_r \in L \cap P} (\max(\nu_{k_i,l_j}, \sigma_{p_r,q_s})) \rangle, & \\
\langle 0, 1 \rangle, & \text{otherwise}
\end{cases}
$$

(d) structural subtraction $A \ominus B = [K - P, L - Q, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}]$, where "–" is the set–theoretic difference operation and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, \text{ for } t_u = k_i \in K - P \text{ and } v_w = l_j \in L - Q.$$

(e) negation of an IFIM $\neg A = [K, L, \{\neg \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$, where $\neg$ is one of the negations, defined in Subsection **9.2.1**.

(f) termwise subtraction $A - B = A \oplus \neg B$.

For example, consider two IFIMs X and Y

$$
X = \begin{array}{c|cc} & c & d \\ \hline a & \langle 0.5, 0.3 \rangle & \langle 0.4, 0.2 \rangle \\ b & \langle 0.1, 0.8 \rangle & \langle 0.7, 0.1 \rangle \end{array}, \quad
Y = \begin{array}{c|cc} & c & g \\ \hline a & \langle 0.3, 0.1 \rangle & \langle 0.6, 0.2 \rangle \\ e & \langle 0.3, 0.6 \rangle & \langle 0.3, 0.6 \rangle \\ f & \langle 0.5, 0.2 \rangle & \langle 0.6, 0.1 \rangle \end{array},
$$

then

$$
X \oplus Y = \begin{array}{c|ccc} & c & d & g \\ \hline a & \langle 0.5, 0.1 \rangle & \langle 0.4, 0.2 \rangle & \langle 0.6, 0.2 \rangle \\ b & \langle 0.1, 0.8 \rangle & \langle 0.7, 0.1 \rangle & \langle 0.0, 1.0 \rangle \\ e & \langle 0.3, 0.6 \rangle & \langle 0.0, 1.0 \rangle & \langle 0.3, 0.6 \rangle \\ f & \langle 0.5, 0.2 \rangle & \langle 0.0, 1.0 \rangle & \langle 0.6, 0.1 \rangle \end{array}.
$$

Obviously when

$$K = P = \{1, 2, ..., m\},$$

$$L = Q = \{1, 2, ..., n\}$$

we obtain the definitions for standard matrix operations with intuitionistic fuzzy pairs. In the IFIM case, we use different symbols as indices of the rows and columns and they, as we have seen above, give us additional information and possibilities for description.

Let $\mathcal{IM}_{\mathcal{IF}}$ be the set of all IFIMs with their elements being intuitionistic fuzzy pairs. The problem with the "zero"-IFIM is more complex than in the standard matrix case. We introduce "zero"-IFIM for $\mathcal{IM}_{\mathcal{IF}}$ as the IFIM

$$I_0 = [K, L, \{\langle 0.0, 1.0 \rangle\}]$$

whose elements are equal to $\langle 0.0, 1.0 \rangle$ and $K, L \subset I$ are arbitrary index sets, as well as the IFIM

$$I_\emptyset = [\emptyset, \emptyset, \{a_{k_i,l_j}\}].$$

In the second case, there are no matrix cells where the elements a_{k_i,l_j} may be inserted. In both cases, for each IFIM $A = [K, L, \{b_{k_i,l_j}\}]$ and for I_0 with the same index sets, we obtain

$$A \oplus I_0 = A = I_0 \oplus A.$$

Let $I_1 = [K, L, \{\langle 1.0, 0.0\rangle\}]$ denote the IFIM, whose elements are equal to $\langle 1.0,\ 0.0\rangle$, and where $K, L \subset I$ are arbitrary index sets.

The following properties of the IFIM are valid, similar from Section **8.2.1**

Theorem 8.4: (a) $\langle \mathcal{IM}_{\mathcal{IF}}, \oplus\rangle$ is a commutative semigroup,
(b) $\langle \mathcal{IM}_{\mathcal{IF}}, \otimes\rangle$ is a commutative semigroup,
(c) $\langle \mathcal{IM}_{\mathcal{IF}}, \odot\rangle$ is a semigroup,
(d) $\langle \mathcal{IM}_{\mathcal{IF}}, \oplus, I_\emptyset\rangle$ is a commutative monoid.

8.2.3.2. Relations over IFIMs

Let the two IFIMs $A = [K, L, \{\langle a_{k,l}, b_{k,l}\rangle\}]$ and $B = [P, Q, \{\langle c_{p,q}, d_{p,q}\rangle\}]$ be given. We introduce the following (new) definitions where $\subset$ and $\subseteq$ denote the relations *"strong inclusion"* and *"weak inclusion", respectively.*

The strict relation *"inclusion about dimension"* is

$$A \subset_d B \text{ iff } ((K \subset P)\&(L \subset Q)) \vee (K \subseteq P)\&(L \subset Q) \vee (K \subset P)\&(L \subseteq Q))$$

$$\&(\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle = \langle c_{k,l}, d_{k,l}\rangle).$$

The non-strict relation *"inclusion about dimension"* is

$$A \subseteq_d B \text{ iff } (K \subseteq P)\&(L \subseteq Q)\&(\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle = \langle c_{k,l}, d_{k,l}\rangle).$$

The strict relation *"inclusion about value"* is

$$A \subset_v B \text{ iff } (K = P)\&(L = Q)\&(\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle < \langle c_{k,l}, d_{k,l}\rangle).$$

The non-strict relation *"inclusion about value"* is

$$A \subseteq_v B \text{ iff } (K = P)\&(L = Q)\&(\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle \leq \langle c_{k,l}, d_{k,l}\rangle).$$

The strict relation *"inclusion"* is

$$A \subset B \text{ iff } ((K \subset P)\&(L \subset Q)) \vee (K \subseteq P)\&(L \subset Q) \vee (K \subset P)\&(L \subseteq Q))$$

$$\&(\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle < \langle c_{k,l}, d_{k,l}\rangle).$$

The non-strict relation *"inclusion"* is

$$A \subseteq B \text{ iff } (K \subseteq P)\&(L \subseteq Q)\&(\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle \leq \langle c_{k,l}, d_{k,l}\rangle).$$

Obviously, for every two IFIMs A and B,

- if $A \subset_d B$, then $A \subseteq_d B$;
- if $A \subset_v B$, then $A \subseteq_v B$;
- if $A \subset B$, $A \subseteq_d B$, or $A \subseteq_v B$, then $A \subseteq B$;
- if $A \subset_d B$ or $A \subset_v B$, then $A \subseteq B$.

Operations "reduction", "projection" and "substitution" coincide with the respective operations defined over IMs, while hierarhical operations over IMs are not applied here.

8.3 Intuitionistic Fuzzy Relations (IFRs)

The concept of Intuitionistic Fuzzy Relation (IFR) is based on the definition of the IFSs. It was introduced in different forms and approached from different starting points, and independently, in 1984 and 1989 by the author (in two partial cases; see [13, 39]), in 1989 in [148] by Buhaescu and in 1992-1995 in [156, 154, 155, 157] by Bustince and Burillo. We must note that the approaches in the various IFR definitions differ in the different authors' researches. On the other hand, the author's results were not widely known; first he got acquainted with Stoyanova's results and after this he learned about Buhaescu's (obtained earlier); then he sent parts of the above works to Burillo and Bustince after they had obtained their own results.

Thus the idea of IFR was generated independently in four different places (Sofia and Varna in Bulgaria, Romania and Spain). The Spanish authors' approach is in some sense the most general. In the present form it includes Buhaescu's results.

First, we introduce Burillo and Bustince's definition of the concept of IFR, following [154, 155].

Let X and Y be arbitrary finite non-empty sets.

An $\circ$-IFR (or briefly, IFR, for a fixed operation $\circ \in \{\times_1, \times_2, \dots, \times_6\}$) will mean an IFS $R \subseteq X \times Y$ of the form:

$$R = \{\langle\langle x, y\rangle, \mu_R(x, y), \nu_R(x, y)\rangle | x \in X \& y \in Y\},$$

where $\mu_R : X \times Y \to [0, 1], \nu_R : X \times Y \to [0, 1]$ are degrees of membership and non-membership as in the ordinary IFSs (or degrees of truth and falsity) of the relation R, and for all $\langle x, y\rangle \in X \times Y$,

$$0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1,$$

where the "$\times$" operation is the standard Cartesian product and the form of μ_R and ν_R is related to the form of the Cartesian product $\circ$.

Now, we introduce an index matrix approach of IFR.

Let $IFR_\circ(X, Y)$ be the set of all IFRs over the set $X \times Y$, where $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$ are fixed finite sets (universes), the $\times$ operation between them is the standard Cartesian product, and $\circ \in \{\times_1, \times_2, \dots, \times_6\}$. Therefore, the set $R \in IFR_\circ(X, Y)$ can be represented in the form [32],

$$\begin{array}{c|ccc} & y_1 & \dots & y_n \\ \hline x_1 & \langle \mu_R(x_1, y_1), \nu_R(x_1, y_1)\rangle & \dots & \langle \mu_R(x_1, y_n), \nu_R(x_1, y_n)\rangle \\ x_2 & \langle \mu_R(x_2, y_1), \nu_R(x_2, y_1)\rangle & \dots & \langle \mu_R(x_2, y_n), \nu_R(x_2, y_n)\rangle \\ \vdots & & \dots & \\ x_m & \langle \mu_R(x_m, y_1), \nu_R(x_m, y_1)\rangle & \dots & \langle \mu_R(x_m, y_n), \nu_R(x_m, y_n)\rangle \end{array}$$

This IM-representation allows for a more pictorial description of the elements of R and their degrees of membership and non-membership. Let $R \in IFR_\circ(X_1, Y_1)$ and $S \in IFR_\circ(X_2, Y_2)$, where X_1, Y_1, X_2 and Y_2 are fixed finite sets and $X_1 \cap X_2 \cap Y_1 = X_1 \cap X_2 \cap Y_2 = X_1 \cap Y_1 \cap Y_2 = X_2 \cap Y_1 \cap Y_2 = \emptyset$.

Using the definitions of the operations over IMs, we shall define three operations over IFRs:

$$1.\ R \cup S \in IFR_\circ(X_1 \cup X_2, Y_1 \cup Y_2)$$

and has the form

$$\begin{array}{c|ccc} & y_1 & \dots & y_N \\ \hline x_1 & \langle \mu_{R\cup S}(x_1, y_1), \nu_{R\cup S}(x_1, y_1)\rangle & \dots & \langle \mu_{R\cup S}(x_1, y_N), \nu_{R\cup S}(x_1, y_N)\rangle \\ x_2 & \langle \mu_{R\cup S}(x_2, y_1), \nu_{R\cup S}(x_2, y_1)\rangle & \dots & \langle \mu_{R\cup S}(x_2, y_N), \nu_{R\cup S}(x_2, y_N)\rangle \\ \vdots & & & \\ x_M & \langle \mu_{R\cup S}(x_M, y_1), \nu_{R\cup S}(x_M, y_1)\rangle & \dots & \langle \mu_{R\cup S}(x_M, y_N), \nu_{R\cup S}(x_M, y_N)\rangle \end{array}$$

where

$$X_1 \cup X_2 = \{x_1, x_2, \dots, x_M\} \text{ and } Y_1 \cup Y_2 = \{y_1, y_2, \dots, y_N\}, \text{ and}$$

$$\langle \mu_{R\cup S}(x_i, y_j), \nu_{R\cup S}(x_i, y_j)\rangle = \begin{cases} \langle \mu_R(x'_a, y'_b), \nu_R(x'_a, y'_b)\rangle, \\ \quad \text{if } x_i = x'_a \in X_1 \text{ and } y_j = y'_b \in Y_1 - Y_2 \\ \quad \text{or } x_i = x'_a \in X_1 - X_2 \text{ and } y_j = y'_b \in Y_1 \\ \langle \mu_S(x''_c, y''_d), \nu_S(x''_c, y''_d)\rangle, \\ \quad \text{if } x_i = x''_c \in X_2 \text{ and } y_j = y''_d \in Y_2 - Y_1 \\ \quad \text{or } x_i = x''_c \in X_2 - X_1 \text{ and } y_j = y''_d \in Y_2 \\ \langle \max(\mu_R(x', y'), \mu_S(x'', y'')), \\ \ \min(\nu_R(x', y'), \nu_S(x'', y''))\rangle, \\ \quad \text{if } x_i = x'_a = x''_c \in X_1 \cap X_2 \text{ and} \\ \quad y_j = y'_b = y''_d \in Y_1 \cap Y_2 x \\ \langle 0, 1\rangle, \ \text{otherwise} \end{cases}$$

$$2.\ R \cap S \in IFR_\circ(X_1 \cap X_2, Y_1 \cap Y_2)$$

and has the form of the above IM, but with elements

$$\langle \mu_{R\cap S}(x_i, y_j), \nu_{R\cap S}(x_i, y_j)\rangle$$

$$= \langle \min(\mu_R(x', y'), \mu_S(x'', y'')), \max(\nu_R(x', y'), \nu_S(x'', y''))\rangle,$$

where $x_i = x'_a = x"_c \in X_1 \cap X_2$ and $y_j = y'_b = y"_d \in Y_1 \cap Y_2$ (therefore $X_1 \cap X_2 = \{x_1, x_2, ..., x_M\}$ and $Y_1 \cap Y_2 = \{y_1, y_2, ..., y_N\}$).

$$3.\ R \bullet S \in IFR_\circ(X_1 \cup (X_2 - Y_1), Y_2 \cup (Y_1 - X_2))$$

and has the form of the above IM, but with elements

$$\langle \mu_{R\bullet S}(x_i, y_j), \nu_{R\bullet S}(x_i, y_j)\rangle = \begin{cases} \langle \mu_R(x'_a, y'_b), \nu_R(x'_a, y'_b)\rangle, \\ \text{ if } x_i = x'_a \in X_1 \text{ and } y_j = y'_b \in Y_1 - X_2 \\ \langle \mu_S(x''_c, y''_d), \nu_S(x''_c, y''_d)\rangle, \\ \text{ or } x_i = x''_c \in X_2 - Y_1 \text{ and } y_j = y''_d \in Y_2 \\ \langle \max\limits_{y'_b = x''_c \in Y_1 \cap X_2} \min(\mu_R(x'_a, y'_b), \mu_S(x''_c, y''_d)), \\ \min\limits_{y'_b = x''_c \in Y_1 \cap X_2} \max(\nu_R(x'_a, y'_b), \nu_S(x''_c, y''_d))\rangle, \\ \text{ if } x_i = x'_a \in X_1 \text{ and } y_j = y''_d \in Y_2 \\ \langle 0, 1\rangle, \text{ otherwise} \end{cases}$$

Therefore,

$$X_1 \cup (X_2 - Y_1) = \{x_1, x_2, \ldots, x_M\}$$

and

$$Y_2 \cup (Y_1 - X_2) = \{y_1, y_2, \ldots, y_N\}.$$

8.4 Intuitionistic Fuzzy Graphs (IFGs)

Now, we consider the applications of IFSs, IFRs and IMs to graph theory. Following [31, 33, 45, 424, 425] the concept of an *Intuitionistic Fuzzy Graph (IFG)* are introduced.

Let E_1 and E_2 be two sets. In this Section we assume that $x \in E_1$ and $y \in E_2$ and operation $\times$ denotes the standard Cartesian product operation. Therefore $\langle x, y\rangle \in E_1 \times E_2$. Let the operation $o \in \{\times_1, \times_2, \ldots, \times_6\}$.

The set

$$G^* = \{\langle\langle x, y\rangle, \mu_G(x, y), \nu_G(x, y)\rangle\ \langle x, y\rangle \in E_1 \times E_2\}$$

is called an o-IFG (or briefly, an IFG) if the functions $\mu_G : E_1 \times E_2 \to [0, 1]$ and $\nu_G : E_1 \times E_2 \to [0, 1]$ define the degree of membership and the degree of non-membership, respectively, of the element $\langle x, y\rangle \in E_1 \times E_2$ to the set

$G \subseteq E_1 \times E_2$; these functions have the forms of the corresponding components of the o-Cartesian product over IFSs; and for all $\langle x, y\rangle \in E_1 \times E_2$,

$$0 \leq \mu_G(x, y) + \nu_G(x, y) \leq 1.$$

For simplicity, we write G instead of G^*.

As in [301], we illustrate the above definition by an example of a Berge's graph (see Fig. 8.1; the labels of the arcs show the corresponding degrees). Let the following two tables giving μ- and ν-values be defined for it (for example, the data can be obtained as a result of some observations).

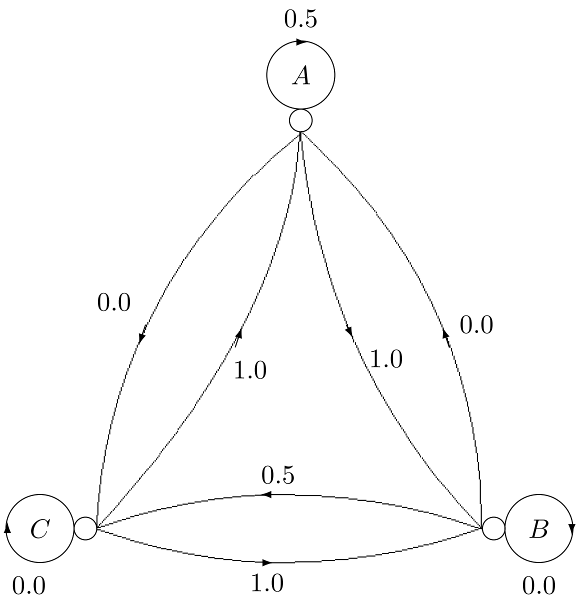

Fig. 8.1

μ_G	A	B	C
A	0.5	1	0
B	0	0	0.5
C	1	1	0

ν_G	A	B	C
A	0.3	0	1
B	1	0.4	0.2
C	0	0	0.7

The data for $\mu_G(x, y)$ are taken from [301]. On the other hand, the IFG G has the form shown in Fig. 8.2.

Let the oriented graph $G = (V, A)$ be given, where V is a set of vertices and A is a set of arcs. Every graph arc connects two graph vertices. Therefore, $A \subseteq V \times V$ and hence A can be described as a $(1, 0)$-IM. If the graph is fuzzy, the IM has elements from the set $[0, 1]$; if the graph is an IFG, the IM has elements from the set $[0, 1] \times [0, 1]$.

The IM of the graph G is given by

$$A = \begin{array}{c|cccc} & v_1 & v_2 & \dots & v_n \\ \hline v_1 & a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ v_2 & a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \dots & \dots & \dots & \dots \\ v_n & a_{n,1} & a_{n,2} & \dots & a_{n,n} \end{array}$$

where

$$a_{i,j} = \langle \mu_{i,j}, \nu_{i,j} \rangle \in [0,1] \times [0,1](1 \leq i, j \leq n),$$
$$0 \leq \mu_G(x,y) + \nu_G(x,y) \leq 1,$$
$$V = \{v_1, v_2, \dots, v_n\}.$$

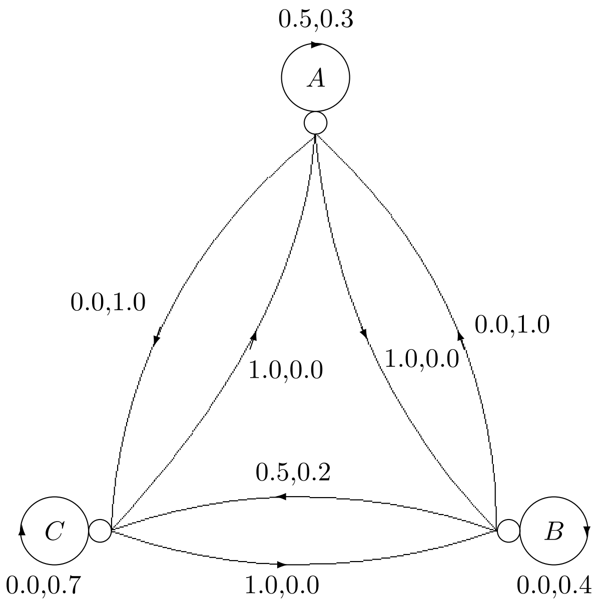

Fig. 8.2

We write briefly,

$$G = [V, V, \{a_{i,j}\}].$$

It can be easily seen that the above IM can be modified to the following form:

$$G = [V_I \cup \overline{V}, \overline{V} \cup V_O, \{a_{i,j}\}],$$

where V_I, V_O and $\overline{V}$ are respectively the sets of the input, output and internal vertices of the graph. At least one arc leaves every vertex of the first type, but none enters; at least one arc enters each vertex of the second type but none leaves it; every vertex of the third type has at least one arc ending in it and at least one arc starting from it.

Obviously, the graph matrix (in the sense of IM) now will be of a smaller dimension than the ordinary graph matrix. Moreover, it can be nonsquare, unlike the ordinary graph, matrices.

As in the ordinary case, the vertex $v_p \in \overline{V}$ has a loop iff $a_{p,p} = \langle \mu_{p,p}, \nu_{p,p} \rangle$ for the vertex v_p and $\mu_{p,p} > 0$ and $\nu_{p,p} < 1$.

Let the graphs G_1 and G_2 be given and let $G_s = [V'_s, V''_s, \{a^s_{i,j}\}]$, where $s = 1, 2$ and V'_s and V''_s are the sets of the graph vertices (input and internal, and output and internal, respectively).

Then, using the apparatus of the IMs, we construct the graph which is a union of the graphs G_1 and G_2. The new graph has the description

$$G = G_1 \cup G_2 = [V'_1 \cup V'_2, V''_1 \cup V''_2, \{\overline{a}_{i,j}\}],$$

where $\overline{a}_{i,j}$ is determined by the above IM-formulae, using min-max operations between its elements, for the case of operation "+" between IMs.

Analogously, we can construct a graph which is the intersection of the two given graphs G_1 and G_2. It would have the form

$$G = G_1 \cap G_2 = [V'_1 \cap V'_2, V''_1 \cap V''_2, \{\overline{\overline{a}}_{i,j}\}],$$

where $\overline{\overline{a}}_{i,j}$ is determined by the above IM-formulae, using min-max operations between its elements, for the case of operation "." between IMs.

Following the definitions from Section **6.1**, for some given $\alpha, \beta \in [0, 1]$ and for a given IFG $G = [V, V, A]$, we define the following three IFGs:

$$G_1 = N_\alpha(G) = [V', V'', A_1]$$

$$G_2 = N^\beta(G) = [V', V'', A_2]$$

$$G_3 = N_{\alpha,\beta}(G) = [V', V'', A_3]$$

For the first graph the arc between the vertices $v_i \in V'$ and $v_j \in V''$ is indexed by $\langle a_{i,j}, b_{i,j} \rangle$, where,

$$a_{i,j} = \begin{cases} \mu(v_i, v_j), & \text{if } \mu(v_i, v_j) \geq \alpha \\ 0, & \text{otherwise} \end{cases}$$

$$b_{i,j} = \begin{cases} \nu(v_i, v_j), & \text{if } \mu(v_i, v_j) \geq \alpha \\ 1, & \text{otherwise} \end{cases}$$

for the second graph - the same pair of numbers, but now having the values:

$$a_{i,j} = \begin{cases} \mu(v_i, v_j), & \text{if } \nu(v_i, v_j) \leq \beta \\ 0, & \text{otherwise} \end{cases}$$

$$b_{i,j} = \begin{cases} \nu(v_i, v_j), & \text{if } \nu(v_i, v_j) \leq \beta \\ 1, & \text{otherwise} \end{cases}$$

for the third graph - the same pair of numbers, but having the values:

$$a_{i,j} = \begin{cases} \mu(v_i, v_j), & \text{if } \mu(v_i, v_j) \geq \alpha \text{ and } \nu(v_i, v_j) \leq \beta \\ 0, & \text{otherwise} \end{cases}$$

$$b_{i,j} = \begin{cases} \nu(v_i, v_j), & \text{if } \mu(v_i, v_j) \geq \alpha \text{ and } \nu(v_i, v_j) \leq \beta \\ 1, & \text{otherwise} \end{cases}.$$

We must note that $v_i \in V'$ and $v_j \in V''$, iff $v_i, v_j \in V$ and in the first and in the third cases $a_{i,j} \geq \alpha$; in the second and in the third cases $b_{i,j} \leq \beta$.

Therefore, in this way we transform a given IFG to a new one whose arcs have high enough degrees of truth and low enough degrees of falsity.

For example, if we apply the operator $N_{\alpha,\beta}$ for $\alpha = 0.5, \beta = 0.25$ to the IFG in Fig. 8.1, we obtain the IFG as in Fig. 8.3.

The following statements hold for every two IFGs A and B and every two numbers $\alpha, \beta \in [0, 1]$:

(a) $N_{\alpha,\beta}(A) = N_\alpha(A) \cap N^\beta(A)$,
(b) $N_{\alpha,\beta}(A \cap B) = N_{\alpha,\beta}(A) \cap N_{\alpha,\beta}(B)$,
(c) $N_{\alpha,\beta}(A \cup B) = N_{\alpha,\beta}(A) \cup N_{\alpha,\beta}(B)$.

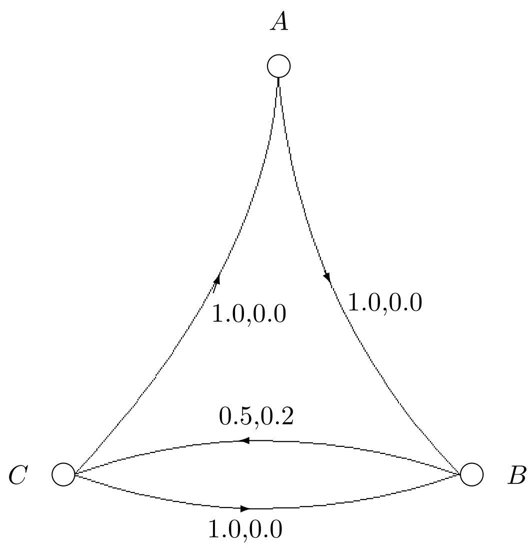

Fig. 8.3

Let us have a (fixed) set of vertices $\mathcal{V}$. An IFTree T (over $\mathcal{V}$) will be the ordered pair $T = (V^*, A^*)$ (see [169, 171, 432]), where

$$V \subset \mathcal{V},$$

$$V^* = \{\langle v, \mu_V(v), \nu_V(v)\rangle | v \in V\},$$

$$A \subset V \times V,$$

$$A^* = \{\langle g, \mu_A(g), \nu_A(g)\rangle | (\exists v, w \in V)(g = \langle v, w\rangle \in A)\},$$

where $\mu_V(v)$ and $\nu_V(v)$ are degrees of membership and non-membership of the element $v \in \mathcal{V}$ to V and

$$0 \leq \mu_V(v) + \nu_V(v) \leq 1.$$

The IFTree $T = (V^*, A^*)$ is:

a) *weak well constructed (WWC-IFTree)* if

$$(\forall v, w \in V)((\exists g \in A)(g = \langle v, w \rangle) \rightarrow (\mu_V(v) \geq \mu_V(w) \ \& \ \nu_V(v) \leq \nu_V(w));$$

b) *strong well constructed (SWC-IFTree)* if

$$(\forall v, w \in V)((\exists g \in A)(g = \langle v, w \rangle)$$

$$\rightarrow (\mu_V(v) \geq \max(\mu_V(w), \mu_A(g)) \ \& \ \nu_V(v) \leq \min(\nu_V(w), \nu_A(g)));$$

c) *average well constructed (AWC-IFTree)* if

$$(\forall v, w \in V)((\exists g \in A)$$

$$(g = \langle v, w \rangle) \rightarrow (\mu_V(v) \geq \frac{\mu_V(w) + \mu_A(g)}{2} \ \& \ \nu_V(v) \leq \frac{\nu_V(w) + \nu_A(g)}{2}).$$

Let two IFTrees $T_1 = (V_1^*, G_1^*)$ and $T_2 = (V_2^*, G_2^*)$ be given. We define:

$$T_1 \cup T_2 = (V_1^*, A_1^*) \cup (V_2^*, A_2^*) = (V_1^* \cup V_2^*, A_1^* \cup A_2^*),$$

$$T_1 \cap T_2 = (V_1^*, A_1^*) \cap (V_2^*, A_2^*) = (V_1^* \cap V_2^*, A_1^* \cap A_2^*).$$

Let

$$\mathcal{P}(X) = \{Y | Y \subset X\},$$

and let for $T = (V^*, A^*)$

$$T_{full} = (E(V), E(A)),$$

$$T_{empty} = (O(V), O(A)),$$

where

$$E(V) = \{\langle v, 1, 0 \rangle | v \in \mathcal{V}\},$$

$$O(V) = \{\langle v, 0, 1 \rangle | v \in \mathcal{V}\},$$

$$E(A) = \{\langle g, 1, 0 \rangle | (\exists v, w \in V)(g = \langle v, w \rangle \in \mathcal{V} \times \mathcal{V})\},$$

$$O(A) = \{\langle g, 0, 1 \rangle | (\exists v, w \in V)(g = \langle v, w \rangle \in \mathcal{V} \times \mathcal{V})\}.$$

Theorem 8.5: $(\mathcal{P}(\mathcal{V}), \cup, T_{empty})$ and $(\mathcal{P}(\mathcal{V}), \cap, T_{full})$ are commutative monoids.

Let $G = (V, A)$ be a given IFTree. We construct its standard incidence matrix. After this, we change the elements of the matrix with their degrees of membership and non-membership. Finally, numbering the rows and columns of the matrix with the identifiers of the IFTree vertices, will result an IM.

For example, if we have the IFTree as in Fig. 8.4, we can construct the IM that corresponds to its incidence matrix:

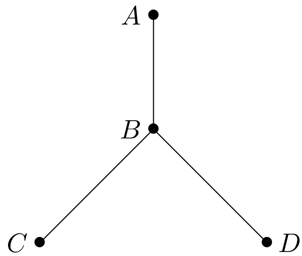

Fig. 8.4

$$[\{A,B,C,D\},\{A,B,C,D\},$$

	A	B
A	$\langle\mu(A,A),\nu(A,A)\rangle$	$\langle\mu(A,B),\nu(A,B)\rangle$
B	$\langle\mu(B,A),\nu(B,A)\rangle$	$\langle\mu(B,B),\nu(B,B)\rangle$
C	$\langle\mu(C,A),\nu(C,A)\rangle$	$\langle\mu(C,B),\nu(C,B)\rangle$
D	$\langle\mu(D,A),\nu(D,A)\rangle$	$\langle\mu(D,B),\nu(D,B)\rangle$

...

...

	C	D
A	$\langle\mu(A,C),\nu(A,C)\rangle$	$\langle\mu(A,D),\nu(A,D)\rangle$
B	$\langle\mu(B,C),\nu(B,C)\rangle$	$\langle\mu(B,D),\nu(B,D)\rangle$
C	$\langle\mu(C,C),\nu(C,C)\rangle$	$\langle\mu(C,D),\nu(C,D)\rangle$
D	$\langle\mu(D,C),\nu(D,C)\rangle$	$\langle\mu(D,D),\nu(D,D)\rangle$

],

where here and below by "..." we note the fact that the IM from the first row continues on the second row.

Having in mind that arcs AA, AC, AD, BB, CC, CD and DD do not exist, we can modify the above IM to the form:

$$[\{A,B,C,D\},\{A,B,C,D\},$$

	A	B	C	D
A	$\langle 0,1\rangle$	$\langle\mu(A,B),\nu(A,B)\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
B	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle\mu(B,C),\nu(B,C)\rangle$	$\langle\mu(B,D),\nu(B,D)\rangle$
C	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
D	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$

].

Now, it is seen, that all elements of the column indexed with A and all elements of the rows indexed with C and D are $\langle 0,1\rangle$. Therefore, we can omit these two rows and the column and we obtain the simpler IM as

$$[\{A,B,C,D\},\{A,B,C,D\},$$

$$\begin{array}{c|ccc} & B & C & D \\ \hline A & \langle\mu(A,B),\nu(A,B)\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ B & \langle 0,1\rangle & \langle\mu(B,C),\nu(B,C)\rangle & \langle\mu(B,D),\nu(B,D)\rangle \end{array}].$$

Finally, having in mind that there is no more a column indexed with A and rows indexed with C and D, we obtain a final form of the IM as

$$[\{A,B\},\{B,C,D\},$$

$$\begin{array}{c|ccc} & B & C & D \\ \hline A & \langle\mu(A,B),\nu(A,B)\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ B & \langle 0,1\rangle & \langle\mu(B,C),\nu(B,C)\rangle & \langle\mu(B,D),\nu(B,D)\rangle \end{array}].$$

Let us have an IFTree $G = (V, A)$ and let L be one of its leaves. Let $F = (W, B)$ be another IFTree so that

$$V \cap W = \{L\},$$

$$A \cup B = \emptyset.$$

Now, we describe the result of operation "substitution of an IFTree's leaf L with the IFTree F. The result will have the form of the IFTree $(V \cup W, A \cup B)$.

For example, if G is the IFTree as in Fig. 8.4 and if we substitute its leaf D with the IFTree F as in Fig. 8.5 that has the shorter IM-representation as

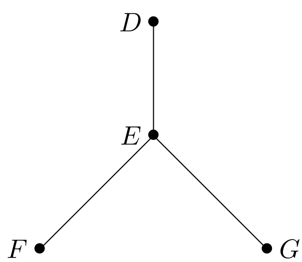

Fig. 8.5

$$[\{D,E\},\{E,F,G\},$$

$$\begin{array}{c|ccc} & E & F & G \\ \hline D & \langle\mu(D,E),\nu(D,E)\rangle & \langle 0,1\rangle & \langle 0,1\rangle \\ E & \langle 0,1\rangle & \langle\mu(E,F),\nu(E,F)\rangle & \langle\mu(E,G),\nu(E,G)\rangle \end{array}$$

then, the result will be the IFTree as in Fig. 8.6 and it has the IM-representation as

$$[\{A,B,D,E\},\{B,C,D,E,F,G\},$$

$$
\begin{array}{c|ccc}
 & B & C & D \\
\hline
A & \langle \mu(A,B), \nu(A,B) \rangle & \langle 0,1 \rangle & \langle 0,1 \rangle \\
B & \langle 0,1 \rangle & \langle \mu(B,C), \nu(B,C) \rangle & \langle \mu(B,D), \nu(B,D) \rangle \\
D & \langle 0,1 \rangle & \langle 0,1 \rangle & \langle 0,1 \rangle \\
E & \langle 0,1 \rangle & \langle 0,1 \rangle & \langle 0,1 \rangle
\end{array}
\quad \ldots
$$

$$
\ldots \quad
\begin{array}{c|ccc}
 & E & F & G \\
\hline
A & \langle 0,1 \rangle & \langle 0,1 \rangle & \langle 0,1 \rangle \\
B & \langle 0,1 \rangle & \langle 0,1 \rangle & \langle 0,1 \rangle \\
D & \langle \mu(D,E), \nu(D,E) \rangle & \langle 0,1 \rangle & \langle 0,1 \rangle \\
E & \langle 0,1 \rangle & \langle \mu(E,F), \nu(E,F) \rangle & \langle \mu(E,G), \nu(E,G) \rangle
\end{array}
$$

$$
= [\{A, B, D, E\}, \{B, C, D, E, F, G\},
$$

$$
\begin{array}{c|ccc}
 & B & C & D \\
\hline
A & \langle \mu(A,B), \nu(A,B) \rangle & \langle 0,1 \rangle & \langle 0,1 \rangle \\
B & \langle 0,1 \rangle & \langle \mu(B,C), \nu(B,C) \rangle & \langle \mu(B,D), \nu(B,D) \rangle
\end{array}
$$

$$
\oplus \quad
\begin{array}{c|ccc}
 & E & F & G \\
\hline
D & \langle \mu(D,E), \nu(D,E) \rangle & \langle 0,1 \rangle & \langle 0,1 \rangle \\
E & \langle 0,1 \rangle & \langle \mu(E,F), \nu(E,F) \rangle & \langle \mu(E,G), \nu(E,G) \rangle
\end{array}
$$

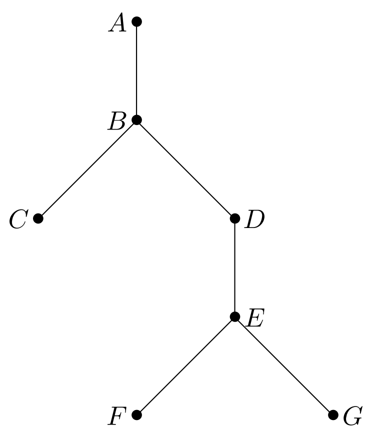

Fig. 8.6

Let the IFTree $T = [V, A]$ be given, where V is the set of its vertices and A is the set of its arcs, and let it has the following IM-form

$$
T = [V, V, \{a_{k_i, l_j}\}].
$$

Let its vertex w be fixed and let the subtree with source vertex w be

$$U = [W, W, \{b_{k_i,l_j}\}],$$

where

$$W = \{w, w_1, w_2, ..., w_s\} \subseteq V.$$

Let P be the new IFTree to be inserted at the vertex w of the IFTree T and has the IM-form

$$P = [Q, Q, \{c_{k_i,l_j}\}],$$

for which $w \in Q$ and $\{q_1, q_2, ..., q_r\} \subset Q$ are destination vertices.

Then the IM-form of the new IFTree T^* is

$$T^* = ([V, V, \{a_{k_i,l_j}\}] \ominus [W, W, \{\overline{a}_{k_i,l_j}\}]) \oplus [Q, Q, \{c_{k_i,l_j}\}]$$

$$\oplus \sum_{i=1}^{r}\!\!\oplus \left[\frac{q_i}{w}\right] \left[\frac{w_{i,1}}{w_1}\right] ... \left[\frac{w_{i,s}}{w_s}\right] [W, W, \{b_{k_i,l_j}\}].$$

We illustrate these definitions by two examples.

Let the ordered IFTree T_1, in Fig. 8.7, be given, and let P (see Fig. 8.8) be the new ordered IFTree to be inserted at vertex w of T_1. The resultant IFTree T_1^* is given in Fig. 8.9.

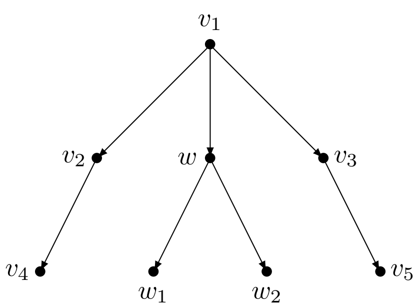

Fig. 8.7

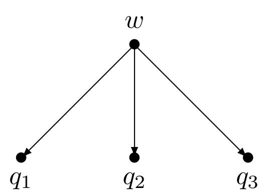

Fig. 8.8

Let the two IFTrees have representations, respectively

$$T_1 = \begin{array}{c|cccccccc}
 & v_1 & v_2 & w & v_3 & v_4 & w_1 & w_2 & v_5 \\ \hline
v_1 & \alpha_{v_1,v_1} & \alpha_{v_1,v_2} & \alpha_{v_1,w} & \alpha_{v_1,v_3} & \alpha_{v_1,v_4} & \alpha_{v_1,w_1} & \alpha_{v_1,w_2} & \alpha_{v_1,v_5} \\
v_2 & \alpha_{v_2,v_1} & \alpha_{v_2,v_2} & \alpha_{v_2,w} & \alpha_{v_2,v_3} & \alpha_{v_2,v_4} & \alpha_{v_2,w_1} & \alpha_{v_2,w_2} & \alpha_{v_2,v_5} \\
w & \alpha_{w,v_1} & \alpha_{w,v_2} & \alpha_{w,w} & \alpha_{w,v_3} & \alpha_{w,v_4} & \alpha_{w,w_1} & \alpha_{w,w_2} & \alpha_{w,v_5} \\
v_3 & \alpha_{v_3,v_1} & \alpha_{v_3,v_2} & \alpha_{v_3,w} & \alpha_{v_3,v_3} & \alpha_{v_3,v_4} & \alpha_{v_3,w_1} & \alpha_{v_3,w_2} & \alpha_{v_3,v_5} \\
v_4 & \alpha_{v_4,v_1} & \alpha_{v_4,v_2} & \alpha_{v_4,w} & \alpha_{v_4,v_3} & \alpha_{v_4,v_4} & \alpha_{v_4,w_1} & \alpha_{v_4,w_2} & \alpha_{v_4,v_5} \\
w_1 & \alpha_{w_1,v_1} & \alpha_{w_1,v_2} & \alpha_{w_1,w} & \alpha_{w_1,v_3} & \alpha_{w_1,v_4} & \alpha_{w_1,w_1} & \alpha_{w_1,w_2} & \alpha_{w_1,v_5} \\
w_2 & \alpha_{w_2,v_1} & \alpha_{w_2,v_2} & \alpha_{w_2,w} & \alpha_{w_2,v_3} & \alpha_{w_2,v_4} & \alpha_{w_2,w_1} & \alpha_{w_2,w_2} & \alpha_{w_2,v_5} \\
v_5 & \alpha_{v_5,v_1} & \alpha_{v_5,v_2} & \alpha_{v_5,w} & \alpha_{v_5,v_3} & \alpha_{v_5,v_4} & \alpha_{v_5,w_1} & \alpha_{v_5,w_2} & \alpha_{v_5,v_5}
\end{array},$$

where $\alpha_{a,b} = \langle \mu_{a,b}, \nu_{a,b} \rangle$ for $a, b \in \{v_1, v_2, w, v_3, v_4, w_1, w_2, v_5\}$ and

$$P = \begin{array}{c|cccc} & w & q_1 & q_2 & q_3 \\ \hline w & \beta_{w,w} & \beta_{w,q_1} & \beta_{w,q_2} & \beta_{w,q_3} \\ q_1 & \beta_{q_1,w} & \beta_{q_1,q_1} & \beta_{q_1,q_2} & \beta_{q_1,q_3} \\ q_2 & \beta_{q_2,w} & \beta_{q_2,q_1} & \beta_{q_2,q_2} & \beta_{q_2,q_3} \\ q_3 & \beta_{q_3,w} & \beta_{q_3,q_1} & \beta_{q_3,q_2} & \beta_{q_3,q_3} \end{array},$$

where $\beta_{a,b} = \langle \mu_{a,b}, \nu_{a,b} \rangle$ for $a, b \in \{w, q_1, q_2, q_3\}$.

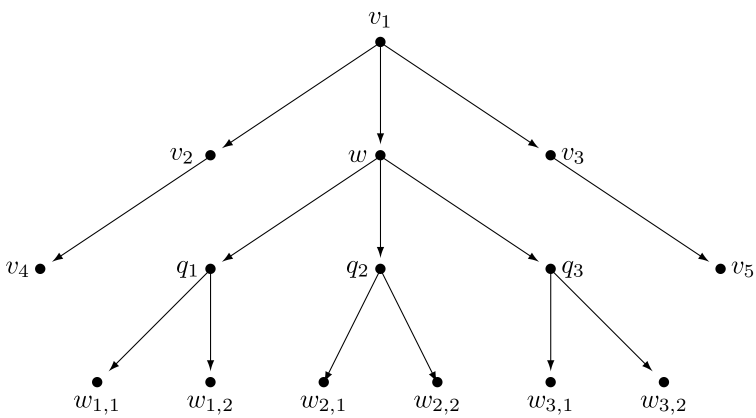

Fig. 8.9

Hawing in mind the above remark for reduction of the IM-representation of a graph and the fact that the IFTree is ordered, we can rewrite the IMs T_1 and P to the (equivalent) forms

$$T_1 = \begin{array}{c|ccccccc} & v_2 & w & v_3 & v_4 & w_1 & w_2 & v_5 \\ \hline v_1 & \alpha_{v_1,v_2} & \alpha_{v_1,w} & \alpha_{v_1,v_3} & \overline{O} & \overline{O} & \overline{O} & \overline{O} \\ v_2 & \overline{O} & \overline{O} & \overline{O} & \alpha_{v_2,v_4} & \overline{O} & \overline{O} & \overline{O} \\ w & \overline{O} & \overline{O} & \overline{O} & \overline{O} & \alpha_{w,w_1} & \alpha_{w,w_2} & \overline{O} \\ v_3 & \overline{O} & \overline{O} & \overline{O} & \overline{O} & \overline{O} & \overline{O} & \alpha_{v_3,v_5} \end{array}$$

and

$$P = \begin{array}{c|ccc} & q_1 & q_2 & q_3 \\ \hline w & \beta_{w,q_1} & \beta_{w,q_2} & \beta_{w,q_3} \end{array},$$

where $\overline{O} = \langle 0, 1 \rangle$.

The IM-form of the IFTree T_1^* is

$$T_1^* = ([\{v_1, v_2, w, v_3, v_4, w_1, w_2, v_5\}, \{v_1, v_2, w, v_3, v_4, w_1, w_2, v_5\}, \{a_{k_i,l_j}\}]$$

$$\ominus[\{w, w_1, w_2\}, \{w, w_1, w_2\}, \{\overline{a}_{k_i,l_j}\}])$$

$$\oplus[\{w, q_1, q_2, q_3\}, \{w, q_1, q_2, q_3\}, \{c_{k_i,l_j}\}]$$

$$\oplus \sum_{i=1}^{3}\!\!\!\!\!\!\oplus \left[\frac{q_i}{w}\right]\left[\frac{w_{i,1}}{w_1}\right]\left[\frac{w_{i,2}}{w_2}\right][\{w, w_1, w_2\}, \{w, w_1, w_2\}, \{b_{k_i,l_j}\}]$$

$$= [\{v_1, v_2, w, v_3, v_4, q_1, q_2, q_3, v_5, w_{1,1}, w_{1,2}, w_{2,1}, w_{2,2}, w_{3,1}, w_{3,2}\},$$

$$\{v_1, v_2, w, v_3, v_4, q_1, q_2, q_3, v_5, w_{1,1}, w_{1,2}, w_{2,1}, w_{2,2}, w_{3,1}, w_{3,2}\}, \{d_{a,b}\}]$$

$$= [\{v_1, v_2, w, v_3, q_1, q_2, q_3\},$$

$$\{v_2, w, v_3, v_4, q_1, q_2, q_3, v_5, w_{1,1}, w_{1,2}, w_{2,1}, w_{2,2}, w_{3,1}, w_{3,2},\}, \{d_{a,b}\}],$$

where the values of the elements $d_{a,b}$ are determined as above.

Let the ordered IFTree T_2, in Fig. 8.10, be given, and let P (see Fig. 8.8) be the new ordered IFTree to be inserted at vertex w of T_2. The resultant IFTree T_2^* is given in Fig. 8.11.

The IFTree T_2 has representation

$$T_2 = \begin{array}{c|cccc} & w & w_1 & w_2 & w_3 \\ \hline w & \beta_{w,w} & \beta_{w,w_1} & \beta_{w,w_2} & \beta_{w,w_3} \\ w_1 & \beta_{w_1,w} & \beta_{w_1,w_1} & \beta_{w_1,w_2} & \beta_{w_1,w_3} \\ w_2 & \beta_{w_2,w} & \beta_{w_2,w_1} & \beta_{w_2,w_2} & \beta_{w_2,w_3} \\ w_3 & \beta_{w_3,w} & \beta_{w_3,w_1} & \beta_{w_3,w_2} & \beta_{w_3,w_3} \end{array} = \begin{array}{c|ccc} & w_1 & w_2 & w_3 \\ \hline w & \beta_{w,w_1} & \beta_{w,w_2} & \beta_{w,w_3} \end{array}.$$

Then

$$T_2^* = ([\{w, w_1, w_2, w_3\}, \{w, w_1, w_2, w_3\}, \{a_{k_i,l_j}\}]$$

$$\ominus[\{w, w_1, w_2, w_3\}, \{w, w_1, w_2, w_3\}, \{a_{k_i,l_j}\}])$$

$$\oplus[\{w, q_1, q_2, q_3\}, \{w, q_1, q_2, q_3\}, \{c_{k_i,l_j}\}]$$

$$\oplus \sum_{i=1}^{3}\!\!\!\!\!\!\oplus \left[\frac{q_i}{w}\right]\left[\frac{w_{i,1}}{w_1}\right]\left[\frac{w_{i,2}}{w_2}\right]\left[\frac{w_{i,3}}{w_3}\right][\{w, w_1, w_2, w_3\}, \{w, w_1, w_2, w_3\}, \{b_{k_i,l_j}\}]$$

$$= [\{w, q_1, q_2, q_3\}, \{w, q_1, q_2, q_3\}, \{c_{k_i,l_j}\}]$$

$$\oplus \sum_{i=1}^{3}\!\!\!\!\!\!\oplus \left[\frac{q_i}{w}\right]\left[\frac{w_{i,1}}{w_1}\right]\left[\frac{w_{i,2}}{w_2}\right]\left[\frac{w_{i,3}}{w_3}\right][\{w, w_1, w_2, w_3\}, \{w, w_1, w_2, w_3\}, \{b_{k_i,l_j}\}]$$

$$= [\{w, q_1, q_2, q_3, w_{1,1}, w_{1,2}, w_{1,3}, w_{2,1}, w_{2,2}, w_{2,3}, w_{3,1}, w_{3,2}, w_{3,3}\},$$

$$\{w, q_1, q_2, q_3, w_{1,1}, w_{1,2}, w_{1,3}, w_{2,1}, w_{2,2}, w_{2,3}, w_{3,1}, w_{3,2}, w_{3,3}\}, \{c_{k_i,l_j}\}].$$

$$= [\{w, q_1, q_2, q_3\}, \{q_1, q_2, q_3, w_{1,1}, w_{1,2}, w_{1,3}, w_{2,1}, w_{2,2}, w_{2,3}, w_{3,1}, w_{3,2}, w_{3,3}\},$$

$$\{c_{k_i,l_j}\}].$$

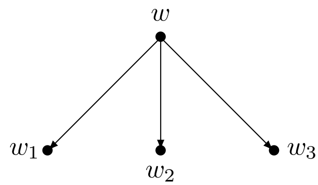

Fig. 8.10

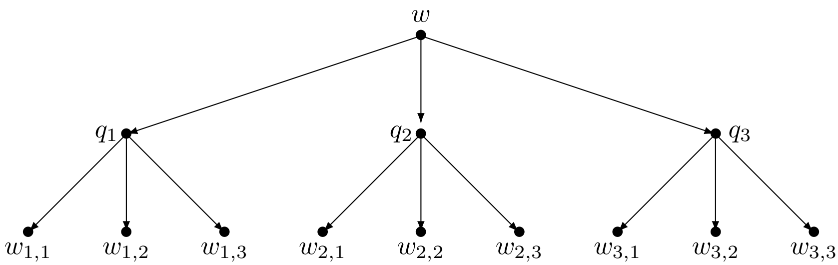

Fig. 8.11

Finally, let the ordered IFTree T_3, in Fig. 8.12, be given, and let P (see Fig. 8.8) be the new ordered IFTree to be inserted at vertex w of T_3. The resultant IFTree T_2^* is given in Fig. 8.13.

The IFTree T_3 has representation

$$T_3 = \begin{array}{c|cccc} & v_1 & v_2 & w & v_3 \\ \hline v_1 & \beta_{v_1,v_1} & \beta_{v_1,v_2} & \beta_{v_1,w} & \beta_{v_1,v_3} \\ v_2 & \beta_{v_2,v_1} & \beta_{v_2,v_2} & \beta_{v_2,w} & \beta_{v_2,v_3} \\ w & \beta_{w,v_1} & \beta_{w,v_2} & \beta_{w,w} & \beta_{w,v_3} \\ v_3 & \beta_{v_3,v_1} & \beta_{v_3,v_2} & \beta_{v_3,w} & \beta_{v_3,v_3} \end{array} = \begin{array}{c|ccc} & v_2 & w & v_3 \\ \hline v_1 & \beta_{v,v_2} & \beta_{v_1,w} & \beta_{v_1,v_3} \end{array}.$$

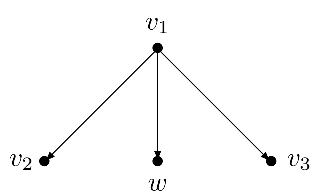

Fig. 8.12

The IM-form of the IFTree T_3^* is

$$\begin{aligned} T_3^* &= ([\{v_1, v_2, w, v_3\}, \{v_1, v_2, w, v_3\}, \{a_{k_i,l_j}\}] \ominus [\{w\}, \{w\}, \{0\}]) \\ &\quad \oplus [\{w, q_1, q_2, q_3\}, \{w, q_1, q_2, q_3\}, \{c_{k_i,l_j}\}] \\ &= [\{v_1, v_2, w, v_3, q_1, q_2, q_3\}, \{v_1, v_2, w, v_3, q_1, q_2, q_3\}, \{d_{a,b}\}] \\ &= [\{v_1, w, \}, \{v_2, w, v_3, q_1, q_2, q_3\}, \{d_{a,b}\}]. \end{aligned}$$

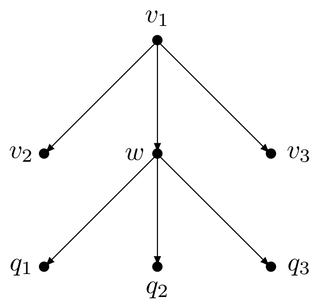

Fig. 8.13

8.5 Example: Intuitionistic Fuzzy Interpretations of Multi-criteria Multi-person and Multi-measurement Tool Decision Making

In Group Decision Making (GDM) a set of experts in a given field is involved in a decision process concerning the selection of the best alternative(s) among a set of predefined ones. An evaluation of the alternatives is performed independently by each decision maker: the experts express their evaluations on the basis of some decision scheme, which can be either implicitly assumed or explicitly specified in the form of a set of predefined criteria [228, 284]. In both cases, the aim is to obtain an evaluation (performance judgment or rating) of the alternatives by each expert. In the case in which a set of predefined criteria is specified, a performance judgment is expressed by each expert for each criterion; this kind of decision problem is called multi-person multi-criteria decision making [284]. Its aim is to compute a consensual judgment and a consensus degree for a majority of the experts on each alternative. As the main actors in a multi-person multi-criteria decision making activity are individuals with their inherent subjectivity, it often happens that performance judgments cannot be precisely assessed; the imprecision may arise from the nature of the characteristics of the alternatives, which can be either unquantifiable or unavailable. It may also be derived from the inability of the experts to formulate a precise evaluation [229, 230, 582]. Several works in the literature have approached the problem of simplifying the experts' formulation of evaluations. To this aim some fuzzy models of GDM have been proposed which relieve experts from quantifying qualitative concepts [140, 165, 230, 258, 294]. This objective has been pursued by dealing directly with performance or preference judgments expressed linguistically.

The second phase of a group decision process is the definition of a collective evaluation for each alternative: once the alternatives have been evaluated, the main problem is to aggregate the experts' performance judgments to obtain an overall rating for each alternative. A consequent problem is to compare the experts' judgments to verify the consensus among them. In the case of unanimous consensus, the evaluation process ends with the selection of the best alternative(s). As in real situation humans rarely come to an unanimous agreement, in the literature some fuzzy approaches to evaluate a “soft” degree of consensus have been proposed. It is important to notice that full consensus (degree = 1) is not necessarily the result of an unanimous agreement, but it can be obtained [140, 295, 294]. Each expert is asked to evaluate at least a part of the alternatives in terms of their performance with respect to each predefined criterion: the experts evaluations are expressed as a pair of numeric values, interpreted in the intuitionistic fuzzy framework: these numbers express a “positive” and a “negative” evaluations, respectively. With each expert a pair of values is associated, which express the expert’s reliability (confidence in her/his evaluation with respect to each criterion). Distinct reliability values are associated with distinct criteria. The *proposed formulation is based on the assumption of alternatives’ independence.*

The contents of this Section is based on paper [98] written by Gabriella Pasi, Ronald Yager and the author.

Here, the described procedure gives the possibility to use partial orders, i.e., orders represented by oriented graphs.

The following basic notation is adopted below:
$E = \{E_1, E_2, ..., E_e\}$ is the set of experts involved in the decision process;
$M = \{M_1, M_2, ..., M_m\}$ is the set of measurement tools employed in the decision process;
$A = \{A_1, A_2, ..., A_p\}$ is the set of considered alternatives;
$C = \{C_1, C_2, ..., C_q\}$ is the set of criteria used for evaluating the alternatives.

Using the apparatus of the IFSs, we discuss the possibility of constructing an overall performance judgment, related to the following distinct, although similar, problems.

Problem 1. *Let alternatives $A_1, ...A_p$ be given and let experts $E_1, ..., E_e$ have to order the alternatives with respect to criteria $C_1, C_2, ..., C_q$. Produce an aggregated order of the objects based on experts’ opinions.*

Problem 2. *Let alternatives $A_1, ...A_p$ be given and let us have the measurement tools $M_1, ..., M_m$, which estimate the alternatives with respect to the criteria $C_1, C_2, ..., C_q$. The problem consists in producing an aggregated estimation of the objects on the basis of the measurement tool estimations.*

Each measurement tool can work using (at a given time) exactly one criterion, but in distinct times it can be tuned to use different criteria. The quality of the estimation of each measurement tool with respect to the other criteria is subjective. The following basic assumptions are considered:

- at each moment the tools use only one criterion;
- we determine the order of criteria;
- we determine for each moment which tool and which criteria will be used.

First, the proposed method of multi-person multi-criterion decision making will be described and then, the proposed method of multi-measurement tools multi-criteria decision making is given. Finally, some examples of the proposed method in the context of public relation and mass communication are discussed.

8.5.1 Experts Who Order Alternatives

Let there be m experts, $E_1, E_2, \ldots, E_m$, p alternatives which have to be evaluated by the experts A_1, $A_2, \ldots, A_p$ and q evaluation criteria C_1, $C_2, \ldots, C_q$. Let i-th expert have his/her own (current) reliability score $\langle \delta_i, \varepsilon_i \rangle \in [0,1]^2$ and his/her own (current) number of participations in expert investigations γ_i (these two values correspond to her/his last evaluation). Expert's reliability scores can be interpreted, as

$$\begin{cases} \delta_i = \dfrac{\sum\limits_{j=1}^{q} \delta_{i,j}}{q} \\ \varepsilon_i = \dfrac{\sum\limits_{j=1}^{q} \varepsilon_{i,j}}{q} \end{cases},$$

where $\langle \delta_{i,j}, \varepsilon_{i,j} \rangle$ are elements of the IM

$$T = \begin{array}{c|c} & C_1 \quad C_2 \quad \ldots \quad C_q \\ \hline E_1 & \\ & \langle \delta_{i,j}, \varepsilon_{i,j} \rangle \\ E_2 & \\ & (1 \leq i \leq m, \\ \vdots & \\ & 1 \leq j \leq q) \\ E_m & \end{array}$$

and $\langle \delta_{i,j}, \varepsilon_{i,j} \rangle$ is the rating of the i-th expert with respect to the j-th criterion (assume that the i-th expert's knowledge reliability may differ over different criteria).

To illustrate the expert's reliability score, we give the following example: a sport journalist gave 10 prognoses for the results of 10 football matches. In 5 of the cases he/she guessed the winner, in 3 of the cases he failed and in the rest two cases he did not engage with a final opinion about the result. That is why, we determine his reliability score as $\langle 0.5, 0.3 \rangle$.

Let each of the experts show which criteria they shall use for a concrete evaluation. We use the set of all criteria provided by the experts. For example, each expert will obtain cards with the different criteria written on them. Each expert ranks these criteria (or a part of them, if he/she deems some of them unnecessary), on the vertices of a graph. The highest vertices of this graph corresponds to the most relevant criteria according to the respective expert. The second top-down vertices interpret the criteria that are "an idea" weaker than the first ones. There are no arcs between vertices which are incomparable due to some criterion. Therefore, each of the experts not only ranks the criteria that he/she uses (it is possible, omitting some of them), but his/her order is not linear one. As a result, we obtain m different graphs. Now, transform these graphs to IFGs, labelling each arc of the i-th expert's graph with a pair of values, corresponding to his/her expert's reliability score.

Using operation "$\cup$" over the IFGs, we obtain a new IFG, say G. It represents all expert opinions about the criteria ordering. Now, its arcs have intuitionistic fuzzy weights being the disjunctions of the weights, of the same arcs in the separate IFGs. Of course, the new graph may not be well ordered, while the expert graphs are well orderd. Now, we reconfigure IFG G as follows. If there is a loop between two vertices V_1 and V_2, i.e., there are vertices $U_1, U_2, ..., U_u$ and vertices $W_1, W_2, ..., W_w$, such that $V_1, U_1, U_2, ..., U_u, V_2$ and $V_2, W_1, W_2, ..., W_w, V_1$ are simple paths in the graph, then we calculate the weights of both paths as conjunctions of the weights of the arcs which take part in the respective paths. The path that has smaller weight must be cut into two, removing its arc with smallest weight. If both arcs have equal weights, these arcs will be removed. Therefore, the new graph is already loop-free. Now, determine the priorities of the vertices of the IFG, i.e., the priorities of the criteria. Let them be $\varphi_1, \varphi_2, ..., \varphi_q$. For example, they have values $\frac{s-1}{t}$ for the vertices from the s-th level bottom-top of the IFG with $t+1$ levels. We use these values below.

This procedure will be used in a next authors' research, but with another form of the algorithm for decision making, using the so constructed IFGs more actively. Here, we use the above construction only to propose the experts' possibility to work with non-linearly ordered criteria and to obtain priorities of these criteria.

Having in mind that the i-th expert can use only a part of the criteria and can estimate only a part of the alternatives, we can construct the IM of his/her estimations in the form

$$S_i = \begin{array}{c|ccc} & A_{l_1} & A_{l_2} \quad \dots & A_{l_{p_i}} \\ \hline C_{i_1} & & & \\ & & \langle \alpha^i_{i_j,l_k} \beta^i_{i_j,l_k} \rangle & \\ C_{i_2} & & & \\ & & (1 \le i_j \le q_i \le q, & \\ \vdots & & & \\ & & 1 \le l_k \le p_i \le p) & \\ C_{i_{q_i}} & & & \end{array}$$

where $\alpha^i_{i_j,k}, \beta^i_{i_j,k} \in [0,1]$, $\alpha^i_{i_j,k} + \beta^i_{i_j,k} \le 1$ and $\langle \alpha^i_{j,k}, \beta^i_{j,k} \rangle$ is the i-th expert estimation for the k-th alternative about the j-th criterion; $C_{i_1}, ..., C_{i_{q_i}}$ and $A_{l_1}, ..., A_{l_{p_i}}$ are only those of the criteria and alternatives which the i-th expert prefers. Let us assume that in cases when pair $\langle \alpha^i_{j,k}, \beta^i_{j,k} \rangle$ does not exist, we work with pair $\langle 0, 1 \rangle$.

Now, construct an IM containing the aggregated estimations of the form

$$S = \begin{array}{c|cccc} & A_1 & A_2 & \dots & A_p \\ \hline C_1 & & & & \\ & & \langle \alpha_{j,k} \beta_{j,k} \rangle & & \\ C_2 & & & & \\ & & (1 \le j \le q, & & \\ \vdots & & & & \\ & & 1 \le k \le p) & & \\ C_q & & & & \end{array}$$

where $\alpha_{j,k}$ and $\beta_{j,k}$ can be calculated by different formulae, with respect to some specific aims. For example, the formulae are

$$\begin{cases} \alpha_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \delta_i . \alpha^i_{j,k}}{m} \\ \beta_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \varepsilon_i . \beta^i_{j,k}}{m} \end{cases}$$

(here, only the average degrees of experts' reliability participate),

$$\begin{cases} \alpha_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \delta_{i,j} . \alpha^i_{j,k}}{m} \\ \beta_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \varepsilon_{i,j} . \beta^i_{j,k}}{m} \end{cases}$$

(here estimated by the corresponding criteria, only the experts' degrees of reliability participate),

$$\begin{cases} \alpha_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \overline{\alpha}^i_{j,k}}{m} \\ \beta_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \overline{\beta}^i_{j,k}}{m} \end{cases}$$

(here only the experts' degrees of reliability estimated by the corresponding criteria participate), where $\overline{\alpha}^i_{j,k}$ and $\overline{\beta}^i_{j,k}$ can be calculated by various formulae, according to the particular goals and the experts' knowledge. For example, the formulae are

$$\begin{cases} \overline{\alpha}^i_{j,k} = \gamma_i . \dfrac{\alpha^i_{j,k}.\delta_{i,j} + \beta^i_{j,k}.\varepsilon_{i,j}}{\gamma_i + 1} \\ \\ \overline{\beta}^i_{j,k} = \gamma_i . \dfrac{\alpha^i_{j,k}.\varepsilon_{i,j} + \beta^i_{j,k}.\delta_{i,j}}{\gamma_i + 1} \end{cases}$$

or

$$\begin{cases} \overline{\alpha}^i_{j,k} = \alpha^i_{j,k} . \dfrac{\delta_{i,j} + 1 - \varepsilon_{i,j}}{2} \\ \\ \overline{\beta}^i_{j,k} = \beta^i_{j,k} . \dfrac{\varepsilon_{i,j} + 1 - \delta_{i,j}}{2} \end{cases} .$$

The first formula takes into account not only the rating of each expert by the different criteria, but also the number of times he has made a prognosis (his first time is neglected, for the lack of previous experience). Obviously, the so constructed elements of the IM satisfy the inequality: $\alpha_{j,k} + \beta_{j,k} \leq 1$. This IM contains the average experts' estimations taking into account the experts' ratings. As we noted above, each of the criteria $C_j (1 \leq j \leq q)$ has itself a priority, denoted by $\varphi_j \in [0, 1]$. For every alternative A_k, we determine the global estimation $\langle \alpha_k, \beta_k \rangle$, where

$$\begin{cases} \alpha_k = \dfrac{\sum\limits_{j=1}^{q} \varphi_j . \alpha_{j,k}}{q} \\ \\ \beta_k = \dfrac{\sum\limits_{j=1}^{q} \varphi_j . \beta_{j,k}}{q} \end{cases} .$$

Let alternatives (processes) have the following (objective) values with regard to the different criteria after the end of the expert estimations:

$$\begin{array}{c|cccc} & A_1 & A_2 & \ldots & A_p \\ \hline C_1 & & & & \\ & & \langle a_{j,k} b_{j,k} \rangle & & \\ C_2 & & & & \\ & & (1 \leq j \leq q, & & \\ \vdots & & & & \\ & & 1 \leq k \leq p) & & \\ C_q & & & & \end{array}$$

where $a_{j,k}, b_{j,k} \in [0, 1]$ and $a_{j,k} + b_{j,k} \leq 1$. Then the i-th expert's new rating, $\langle \delta_i, \varepsilon_i \rangle$, and new number of participations in expert investigations, γ'_i will be:

$$\gamma'_i = \gamma_i + 1,$$

and

$$\begin{cases} \delta'_i = \dfrac{\gamma_i.\delta_i + \frac{c_M - c_i}{2}}{\gamma'_i}, \\ \\ \varepsilon'_i = \dfrac{\gamma_i.\varepsilon_i - \frac{c_M - c_i}{2}}{\gamma'_i}, \end{cases},$$

where

$$c_i = \frac{\sum\limits_{j=1}^{q}\sum\limits_{k=1}^{p}((\alpha_{j,k} - a_{j,k})^2 + (\beta_{j,k} - b_{j,k})^2)^{1/2}}{p.q},$$

and

$$c_M = \frac{\sum\limits_{i=1}^{n} c_i}{n}.$$

8.5.2 Measurement Tools That Evaluate Alternatives

Assume that we have m measurement tools $M_1, M_2, \ldots, M_m$, p alternatives A_1,A_2,..., A_p that have to be evaluated by the m measurement tools, and q evaluation criteria C_1, C_2,..., C_q. Also, assume that the j-th criterion has a given preliminary score $\varphi_j \in [0, 1]$, which denotes the importance of the criterion in the evaluation strategy. This score can be determined, by some experts.

Let us assume that each measurement tool has its own (current) reliability score $\langle \delta_i, \varepsilon_i \rangle \in [0, 1]^2$, and its own (current) number of use in the measurement investigations γ_i. These two values correspond to the measurement tool's last using. The measurement tool reliability scores can be obtained, e.g., by the elements of the IM

$$T = \begin{array}{c|ccc} & C_1 & C_2 & \ldots & C_q \\ \hline M_1 & & & & \\ M_2 & & \langle \delta_{i,j}, \varepsilon_{i,j} \rangle & & \\ \vdots & & (1 \leq i \leq m, & & \\ M_m & & 1 \leq j \leq q) & & \end{array}$$

where $\langle \delta_{i,j}, \varepsilon_{i,j} \rangle$ is the rating of the i-th measurement tool with respect to the j-th criterion. Here, we assume that each measurement tool can be used to evaluate only one criterion. Hence, it is important that we must find the most suitable measurement tool for each criterion.

The following is the procedure aimed at determining the different couples $\langle \delta_{i,j}, \varepsilon_{i,j} \rangle$ with the highest values with respect to formulae:

$$\langle a, b\rangle \leq \langle c, d\rangle \text{ iff } a \leq c \text{ and } b \geq d, \tag{8.7}$$

where $a, b, c, d \in [0, 1]$ and $a + b \leq 1, c + d \leq 1$.

1 Define the empty IM $U = [\emptyset, \emptyset, *]$, where "$*$" denotes the lack of elements in a matrix (i.e., a matrix with a dimension 0×0).

2 Choose $\langle \delta_{i,j}, \varepsilon_{i,j}\rangle$ as the maximal element with respect to the order $\geq$ by using (8.7).

3 We construct the reduced IM $T_{(M_i,C_j)}$ and construct the IM

$$U := U + [\{M_i\}, \{C_j\}, A_{i,j}],$$

where $A_{i,j}$ is an ordinary matrix of dimension 1×1 and with a unique element $\langle \delta_{i,j}, \varepsilon_{i,j}\rangle$.

4 We check whether some of the index sets of IM $T_{(M_i,C_j)}$ are not already empty. If yes - end; if not - go to **1**.

As a result of the above procedure, we obtain a list of measurement tool scores. For brevity, write

$$\langle \delta_{i,j}, \varepsilon_{i,j}\rangle = \langle \delta_i, \varepsilon_i\rangle,$$

because for the concrete sitting the i-th measurement tool uses only these values of its score. The procedure shows that we determine the most suitable measurement tools for the criteria with the highest sense for the estimated alternatives. Perhaps, a part of the criteria or a part of the measurement tools may not be used for the current sitting. Of course, the procedure shows that for the i-th measurement tool (if it is in the list with the most suitable measurement tools), there exists a j-th criterion and therefore, the present value of j is function of i, i.e., $j = j(i)$.

Briefly, let the i-th measurement tool M_i $(1 \leq i \leq m)$, that uses $j = j(i)$-th criterion, have given the following estimations, which are described by the IM

$$S_i = \begin{array}{c|cccc} & A_1 & A_2 & \dots & A_p \\ \hline C_j & \langle \alpha^i_{j,1}\beta^i_{j,1}\rangle & \langle \alpha^i_{j,2}\beta^i_{j,2}\rangle & \dots & \langle \alpha^i_{j,k}\beta^i_{j,k}\rangle \end{array},$$

where: $\alpha^i_{j,k}, \beta^i_{j,k} \in [0, 1]$ and $\alpha^i_{j,k} + \beta^i_{j,k} \leq 1$ for $1 \leq k \leq p$.

Then, we construct the IM

$$S = \begin{array}{c|cccc} & A_1 & A_2 & \dots & A_p \\ \hline C_1 & & & & \\ & & \langle \alpha_{j,k}\beta_{j,k}\rangle & & \\ C_2 & & & & \\ & & (1 \leq j \leq q, & & \\ \vdots & & & & \\ & & 1 \leq k \leq p) & & \\ C_q & & & & \end{array}$$

where $\alpha_{j,k}$ and $\beta_{j,k}$ can be calculated by using

$$\begin{cases} \alpha_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \delta_i.\alpha^i_{j,k}}{m} \\ \beta_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \varepsilon_i.\beta^i_{j,k}}{m} \end{cases}$$

(here only the average degrees of measurement tool reliability is taken),

$$\begin{cases} \alpha_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \delta_{i,j}.\alpha^i_{j,k}}{m} \\ \beta_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \varepsilon_{i,j}.\beta^i_{j,k}}{m} \end{cases}$$

(what participates here is only the measurement tool degrees of reliability, estimated according to the corresponding criteria),

$$\begin{cases} \alpha_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \overline{\alpha}^i_{j,k}}{m} \\ \beta_{j,k} = \dfrac{\sum\limits_{i=1}^{m} \overline{\beta}^i_{j,k}}{m} \end{cases},$$

where $\overline{\alpha}^i_{j,k}$ and $\overline{\beta}^i_{j,k}$ can also be calculated by various formulae, according to particular goals and measurement tool estimations, by using the formulae

$$\begin{cases} \overline{\alpha}^i_{j,k} = \gamma_i.\dfrac{\alpha^i_{j,k}.\delta_{i,j} + \beta^i_{j,k}.\varepsilon_{i,j}}{\gamma_i + 1} \\ \overline{\beta}^i_{j,k} = \gamma_i.\dfrac{\alpha^i_{j,k}.\varepsilon_{i,j} + \beta^i_{j,k}.\delta_{i,j}}{\gamma_i + 1} \end{cases}$$

or

$$\begin{cases} \overline{\alpha}^i_{j,k} = \alpha^i_{j,k}.\dfrac{\delta_{i,j} + 1 - \varepsilon_{i,j}}{2} \\ \overline{\beta}^i_{j,k} = \beta^i_{j,k}.\dfrac{\varepsilon_{i,j} + 1 - \delta_{i,j}}{2} \end{cases}.$$

The first formula takes into account not only the score of each measurement tool by the different criteria, but also the number of times it has been used so far. Obviously, the so constructed elements of the IM satisfy the inequality $\alpha_{j,k} + \beta_{j,k} \leq 1$.

Now, we discuss another possibility to use measurement tool estimations, accounting our opinion about the separate tools. On the basis of the measurement tool estimations and the measurement tool scores, we deform the measurement tool estimations, as follows:

• optimistic estimation:

$$\langle \alpha^i_{j,1} \beta^i_{j,1} \rangle = J_{\delta_i,\varepsilon_i}(\langle \alpha^i_{j,1} \beta^i_{j,1} \rangle)$$

or

$$\langle \alpha^i_{j,1} \beta^i_{j,1} \rangle = J^*_{\delta_i,\varepsilon_i}(\langle \alpha^i_{j,1} \beta^i_{j,1} \rangle);$$

• optimistic estimation with restrictions:

$$\langle \alpha^i_{j,1} \beta^i_{j,1} \rangle = P_{\alpha,\beta}(\langle \alpha^i_{j,1} \beta^i_{j,1} \rangle),$$

where $\alpha, \beta \in [0,1]$ are fixed levels and $\alpha + \beta \leq 1$;

• pessimistic estimation:

$$\langle \alpha^i_{j,1} \beta^i_{j,1} \rangle = H_{\delta_i,\varepsilon_i}(\langle \alpha^i_{j,1} \beta^i_{j,1} \rangle)$$

or

$$\langle \alpha^i_{j,1} \beta^i_{j,1} \rangle = H^*_{\delta_i,\varepsilon_i}(\langle \alpha^i_{j,1} \beta^i_{j,1} \rangle);$$

• pessimistic estimation with restrictions:

$$\langle \alpha^i_{j,1} \beta^i_{j,1} \rangle = Q_{\alpha,\beta}(\langle \alpha^i_{j,1} \beta^i_{j,1} \rangle),$$

where $\alpha, \beta \in [0,1]$ are fixed levels and $\alpha + \beta \leq 1$;

• estimation with decreasing uncertainty:

$$\langle \alpha^i_{j,1} \beta^i_{j,1} \rangle = F_{\delta_i,\varepsilon_i}(\langle \alpha^i_{j,1} \beta^i_{j,1} \rangle)$$

(the condition $\delta_i + \varepsilon_i \leq 1$ is obviously valid);

• estimation with increasing uncertainty:

$$\langle \alpha^i_{j,1} \beta^i_{j,1} \rangle = G_{\delta_i,\varepsilon_i}(\langle \alpha^i_{j,1} \beta^i_{j,1} \rangle).$$

After calculating new values of $\langle \alpha^i_{j,1} \beta^i_{j,1} \rangle$, they are used in the above formulae.

We determine, for every alternative A_k, the global estimation $\langle \alpha_k, \beta_k \rangle$, where

$$\begin{cases} \alpha_k = \dfrac{\sum\limits_{j=1}^{q} \varphi_j . \alpha_{j,k}}{q} \\ \beta_k = \dfrac{\sum\limits_{j=1}^{q} \varphi_j . \beta_{j,k}}{q} \end{cases}.$$

Let alternatives (processes) have the following (objective) values with regard to the different criteria after the end of the evaluations measurement tools:

$$\begin{array}{c|cccc} & A_1 & A_2 & \dots & A_p \\ \hline C_1 & & & & \\ & & \langle a_{j,k} b_{j,k} \rangle & & \\ C_2 & & & & \\ & & (1 \leq j \leq q, & & \\ \vdots & & & & \\ & & 1 \leq k \leq p) & & \\ C_q & & & & \end{array}$$

where: $a_{j,k}, b_{j,k} \in [0,1]$ and $a_{j,k} + b_{j,k} \leq 1$. Then, the measurement tool's new score, $\langle \delta_i, \varepsilon_i \rangle$, and the new number of measurement tools usage, γ_i', is calculated similar to Section **8.5.1**.

The present algorithm should be quite useful when searching for an objective answer on the basis of subjective initial data. Having in mind the experts' reliability scores with respect to their successful prognoses hitherto (objective data), and their present evaluations (subjective data), we try to derive an objective estimation about the current event, so that it would cover the estimations of the widest possible circle of people involved. This formulation of the problem implies that the areas, which will find the proposed algorithm a suitable tool for analysis and representation of the data, are the areas involving evaluation of the public opinion about currently flowing processes and tendencies in the society, or evaluating the ratings of the politicians, the media and other similar phenomena. If there exists some causal relation between two of the chosen parameters in our evaluation, it seems natural to grade them from top to bottom in the graph representation of the problem. If both factors are indepedent but equal in weight, their place is next to one another on the same hierarchy level in the graph. The experts themselves must definitely be specialists in the area they are giving estimations about.

8.6 Some Ways of Determining Membership and Non-membership Functions

In the theory of fuzzy sets, various methods are discussed for the generation of values of the membership function (for instance, see [144, 145, 146, 301, 147, 593, 592]).

Here, we discuss a way of generation of the two degrees – of membership and of non-membership that exist in the intuitionistic fuzzy sets (IFSs). For other approaches of assigning membership and non-membership functions of IFSs see [456].

Let us have k different generators $G_1, G_2, ..., G_k$ of fuzzy estimations for n different objects $O_1, O_2, ..., O_n$. In [144, 145, 146] these generators are called "estimators".

Let the estimations be collected in the IM

	O_1	O_2	...	O_j	...	O_n
G_1	$\alpha_{1,1}$	$\alpha_{1,2}$	...	$\alpha_{1,j}$	...	$\alpha_{1,n}$
G_2	$\alpha_{2,1}$	$\alpha_{2,2}$	...	$\alpha_{2,j}$	...	$\alpha_{2,n}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
G_i	$\alpha_{i,1}$	$\alpha_{i,2}$	...	$\alpha_{i,j}$	...	$\alpha_{i,n}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
G_k	$\alpha_{k,1}$	$\alpha_{k,2}$	...	$\alpha_{k,j}$	...	$\alpha_{k,n}$

On the basis of the values of the IM, we can construct the following two types of fuzzy sets:

$$O_1^* = \{\langle G_i, \alpha_{i,1}\rangle | 1 \le i \le k\},$$

$$O_2^* = \{\langle G_i, \alpha_{i,2}\rangle | 1 \le i \le k\},$$

$$\dots$$

$$O_n^* = \{\langle G_i, \alpha_{i,n}\rangle | 1 \le i \le k\},$$

and

$$G_1^* = \{\langle O_j \alpha_{1,j}\rangle | 1 \le j \le n\},$$

$$G_2^* = \{\langle O_j \alpha_{2,j}\rangle | 1 \le j \le n\},$$

$$\dots$$

$$G_k^* = \{\langle O_j \alpha_{k,j}\rangle | 1 \le j \le n\}.$$

Now, using these sets we construct new different sets, already IFSs.

First, we construct the IFSs:

$$O_1^I = \{\langle G_i, \alpha_{i,1}, \sum_{2 \le s \le n} \alpha_{i,s}\rangle | 1 \le i \le k\},$$

$$O_2^I = \{\langle G_i, \alpha_{i,2}, \sum_{1 \le s \le n;\ s \ne 2} \alpha_{i,s}\rangle | 1 \le i \le k\},$$

$$\dots$$

$$O_n^I = \{\langle G_i, \alpha_{i,n}, \sum_{1 \le s \le n-1} \alpha_{i,s}\rangle | 1 \le i \le k\},$$

or

$$O_j^I = \{\langle G_i, \alpha_{i,j}, \sum_{1 \le s \le n;\ s \ne j} \alpha_{i,s}\rangle | 1 \le i \le k\},\ \text{for } j = 1, 2, ..., n;$$

and

$$G_1^I = \{\langle O_j, \alpha_{1,j}, \sum_{2 \le s \le n} \alpha_{s,j}\rangle | 1 \le j \le n\},$$

$$G_2^I = \{\langle O_j, \alpha_{2,j}, \sum_{1 \le s \le n;\ s \ne 2} \alpha_{s,j}\rangle | 1 \le j \le n\},$$

$$\dots$$

$$G^I_k = \{\langle O_j, \alpha_{k,j}, \sum_{1\le s\le n-1} \alpha_{j,s}\rangle | 1 \le j \le n\},$$

or

$$G^I_i = \{\langle O_j, \alpha_{i,j}, \sum_{1\le s\le n;\ s\ne i} \alpha_{j,s}\rangle | 1 \le j \le n\}, \text{ for } j = 1, 2, ..., k;$$

Second, we construct the IFSs:

$$G^I_{\text{max,min}} = \{\langle O_j, \max_{1\le i\le n} \alpha_{i,j}, \min_{1\le i\le n} \alpha_{i,j}\rangle | 1 \le j \le n\},$$

$$G^I_{\text{av}} = \{\langle O_j, \frac{1}{k}\sum_{i=1}^{k} \alpha_{i,j}, \frac{1}{k}\sum_{1\le s\le n; s\ne j}\ \sum_{i=1}^{k} \alpha_{i,s}\rangle | 1 \le j \le n\},$$

$$G^I_{\text{min,max}} = \{\langle O_j, \min_{1\le i\le n} \alpha_{i,j}, \max_{1\le i\le n} \alpha_{i,j}\rangle | 1 \le j \le n\}.$$

Now, we illustrate the above constructions.

Let five experts E_1, E_2, E_3, E_4 and E_5 offer their evaluations of the percentage of votes, obtained by the political parties P_1, P_2 and P_3:

	P_1	P_2	P_3
E_1	32%	9%	37%
E_2	27%	7%	39%
E_3	26%	11%	35%
E_4	31%	8%	39%
E_5	29%	9%	41%

Now, we are able to generate the fuzzy sets

$$P^*_1 = \{\langle E_1, 0.32\rangle, \langle E_2, 0.27\rangle, \langle E_3, 0.26\rangle, \langle E_4, 0.31\rangle, \langle E_5, 0.29\rangle\},$$

$$P^*_2 = \{\langle E_1, 0.09\rangle, \langle E_2, 0.07\rangle, \langle E_3, 0.11\rangle, \langle E_4, 0.08\rangle, \langle E_5, 0.09\rangle\},$$

$$P^*_3 = \{\langle E_1, 0.37\rangle, \langle E_2, 0.39\rangle, \langle E_3, 0.35\rangle, \langle E_4, 0.39\rangle, \langle E_5, 0.41\rangle\},$$

$$E^*_1 = \{\langle P_1, 0.32\rangle, \langle P_2, 0.09\rangle, \langle P_3, 0.37\rangle\},$$

$$E^*_2 = \{\langle P_1, 0.27\rangle, \langle P_2, 0.07\rangle, \langle P_3, 0.39\rangle\},$$

$$E^*_3 = \{\langle P_1, 0.26\rangle, \langle P_2, 0.11\rangle, \langle P_3, 0.35\rangle\},$$

$$E^*_4 = \{\langle P_1, 0.31\rangle, \langle P_2, 0.08\rangle, \langle P_3, 0.39\rangle\},$$

$$E^*_5 = \{\langle P_1, 0.29\rangle, \langle P_2, 0.09\rangle, \langle P_3, 0.41\rangle\}.$$

We can aggregate the last five sets, e.g., by operation @ and obtain the fuzzy set

$$E_{FS} = \{\langle P_1, 0.29\rangle, \langle P_2, 0.088\rangle, \langle P_3, 0.382\rangle\}.$$

Below, we show why we use the above information for constructing IFSs.

It is easy to figure out that if expert E_1 believes that party P_1 would obtain 32% of the election votes, then he deems that 68% of the voters are against this party. If we take for granted that all the five experts are equally competent, i.e. their opinions are of equal worth, then we may conclude that party P_1 will receive between 26% and 32% of the votes, therefore, the opposers of this party will count between 68% and 74% of the voters. Now, an IFS can be constructed for the universe $\{P_1, P_2, P_3\}$ that would have the form:

$$E_{IFS,1} = \{\langle P_1, 0.26, 0.68\rangle, \langle P_2, 0.07, 0.89\rangle, \langle P_3, 0.35, 0.59\rangle\}.$$

This shows that at least 26% of the voters would support party P_1 and at least 68% would oppose it.

Another possible IFS that we can construct on the basis of the above data, is

$$E_{IFS,2} = \{\langle P_1, 0.29, 0.47\rangle, \langle P_2, 0.088, 0.672\rangle, \langle P_3, 0.382, 0.378\rangle\}.$$

The μ-components of this IFS are obtained directly from E_{FS}, while the ν-components are sums of the μ-components of the other two parties.

Following the above formulae, we can construct the next IFSs:

$$P_1^* = \{\langle E_1, 0.32, 0.46\rangle, \langle E_2, 0.27, 0.46\rangle, \langle E_3, 0.26, 0.46\rangle, \langle E_4, 0.31, 0.47\rangle,$$
$$\langle E_5, 0.29, 0.50\rangle\},$$
$$P_2^* = \{\langle E_1, 0.09, 0.59\rangle, \langle E_2, 0.07, 0.66\rangle, \langle E_3, 0.11, 0.61\rangle, \langle E_4, 0.08, 0.70\rangle,$$
$$\langle E_5, 0.09, 0.70\rangle\},$$
$$P_3^* = \{\langle E_1, 0.37, 0.41\rangle, \langle E_2, 0.39, 0.34\rangle, \langle E_3, 0.35, 0.37\rangle, \langle E_4, 0.39, 0.39\rangle,$$
$$\langle E_5, 0.41, 0.38\rangle\},$$
$$E_1^* = \{\langle P_1, 0.32, 0.46\rangle, \langle P_2, 0.09, 0.59\rangle, \langle P_3, 0.37, 0.41\rangle\},$$
$$E_2^* = \{\langle P_1, 0.27, 0.46\rangle, \langle P_2, 0.07, 0.66\rangle, \langle P_3, 0.39, 0.34\rangle\},$$
$$E_3^* = \{\langle P_1, 0.26, 0.46\rangle, \langle P_2, 0.11, 0.61\rangle, \langle P_3, 0.35, 0.37\rangle\},$$
$$E_4^* = \{\langle P_1, 0.31, 0.47\rangle, \langle P_2, 0.08, 0.70\rangle, \langle P_3, 0.39, 0.39\rangle\},$$
$$E_5^* = \{\langle P_1, 0.29, 0.50\rangle, \langle P_2, 0.09, 0.70\rangle, \langle P_3, 0.41, 0.38\rangle\}.$$

When some of the estimators are incorrect, we can use the algorithms from Section **1.7** for correction of their estimations.