Bifurcations of Solutions of the 2-Dimensional Navier–Stokes System

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Dedicated to the 80th Anniversary of Professor Stephen Smale

Abstract For the 2-dimensional Navier–Stokes System written for the stream functions we construct a set of initial data for which initial critical points bifurcate into three critical points. This can be interpreted as the birth of new viscous vortices from a single one. In another class of solutions vortices merge, i.e. the number of critical points decrease.

1 Introduction

We are very glad to dedicate this paper to Professor S. Smale. The works of Smale in the theory of dynamical systems played a great role in the development of this important field and led to the appearance of new concepts and methods. We wish Professor Smale a very good health and many new important results.

The usual bifurcation theory deals with one-parameter families of smooth maps or vector fields. In this situation fixed points or periodic orbits become functions of this parameter. Bifurcations appear when their linearized spectrum changes its structure. The main role in the theory is played by the so-called versal deformations, i.e. by special families such that arbitrary families can be represented as some

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projections of versal deformations (see, for example [1]). In this approach the positions of bifurcating orbits and their dependence on the parameter are known.

In this paper we consider a dynamical system generated by the 2-dimensional Navier–Stokes System and deformations are produced by solutions of this system. Certainly, this is a very special case of a much more general problem in which Navier–Stokes System is replaced by linear or non-linear PDE for which strong existence and uniqueness results are known. The next step is to choose fixed points or periodic orbits and sometimes this can be a difficult problem. In our case this is done under the assumption of an additional symmetry of the problem.

We write Navier–Stokes System for the stream function $\psi = \psi(\tilde{x}_1, \tilde{x}_2, t)$ on the 2-dimensional square $0 \le \tilde{x}_1,\ \tilde{x}_2 \le \pi$:

$$\frac{\partial \psi}{\partial t} + \Delta^{-1}\left(\frac{\partial \psi}{\partial \tilde{x}_1} \cdot \frac{\partial \Delta \psi}{\partial_{\tilde{x}_2}} - \frac{\partial \psi}{\partial \tilde{x}_2} \cdot \frac{\partial \Delta \psi}{\partial \tilde{x}_1}\right) = \Delta \psi. \tag{1}$$

In (1) the viscosity is taken to be 1 and the external forcing terms are absent. The velocity of the fluid $u = (u_1, u_2)$ is expressed from ψ through the relations

$$u_1 = -\frac{\partial \psi}{\partial \tilde{x}_2}, \quad u_2 = \frac{\partial \psi}{\partial \tilde{x}_1} \tag{2}$$

which show that u is a local function of ψ. This is one of the advantages of ψ. Moreover, the velocity u given by (2) always satisfies incompressibility condition

$$\operatorname{div}(u) = \frac{\partial u_1}{\partial \tilde{x}_1} + \frac{\partial u_2}{\partial x_2} = 0.$$

We consider the space of functions ψ written as a series

$$\psi(\tilde{x}_1, \tilde{x}_2, t) = \sum_{m^2+n^2 \neq 0} f_{mn} \sin m\tilde{x}_1 \sin n\tilde{x}_2. \tag{3}$$

The coefficients f_{mn} are odd functions of its arguments and decay fast enough so that all appearing series converge. In Sect. 2 we reproduce the proof of the theorem from [4] in which we show that the space of such ψ is invariant under the dynamics generated by (1).

In (1) the operator Δ^{-1} has the form

$$\Delta^{-1}\psi = -\sum_{m^2+n^2 \neq 0} \frac{1}{m^2 + n^2} \sin m\tilde{x}_1 \sin n\tilde{x}_2.$$

The formulas (2) and (3) show that on the boundary the velocity vector u is directed along the boundary. This situation is called the slip boundary condition. From the physical point of view it is not so natural but it is quite satisfactory as a mathematical model.

Let us write down an infinite-dimensional system of ODE for the coefficients f_{mn} which follows from (1) and actually is equivalent to (1)

$$\frac{df_{mn}}{dt} - \frac{1}{m^2+n^2}\sum_{\substack{m'+m''=m\\ n'+n''=n}} f_{m'n'}f_{m''n''}\cdot((m'')^2+(n'')^2)\cdot(m'n''-m''n')$$
$$= -(m^2+n^2)f_{mn}. \tag{4}$$

Introduce the vorticity

$$\omega = \Delta\psi = \sum_{m,n}\omega_{mn}\sin m\tilde{x}_1 \sin n\tilde{x}_2$$

which shows that $\omega_{mn} = -(m^2+n^2)f_{mn}$. For the coefficients ω_{mn} we have even a simpler system of ODE equivalent to (4)

$$\frac{d\omega_{mn}}{dt} + \sum_{\substack{m'+m''=m\\ n'+n''=n}} \omega_{m'n'}\omega_{m''n''}\cdot\frac{m'n''-m''n'}{(m')^2+(n')^2}$$
$$= -(m^2+n^2)\omega_{mn}. \tag{5}$$

In [2–4], the following theorem was proven

Theorem 1 (Global wellposedness and decay). *Let* $\gamma > 1$, $A > 0$ *and*

$$|\omega_{mn}(0)| \le \frac{A}{(m^2+n^2)^{\frac{\gamma}{2}}}, \tag{6}$$

for all m, n, $m^2+n^2 \neq 0$. *Then for some absolute constant* $K_1 > 0$ *and all* $t > 0$,

$$|\omega_{mn}(t)| \le \frac{AK_1}{(m^2+n^2)^{\frac{\gamma}{2}}}. \tag{7}$$

The proof of Theorem 1 is given in Sect. 2. In the periodic case it was given in [3] and [4] and extended to other boundary conditions in [2]. The inequality (7) implies that for the stream function

$$|f_{mn}(t)| \le \frac{AK_1}{(m^2+n^2)^{\frac{\gamma}{2}+1}}, \quad \forall\, m,n.$$

We shall take γ to be so large that the decay of f_{mn} will be sufficient for our purposes. Actually the decay of f_{mn} is much faster but we do not dwell on this here.

Remark 1. Our flow (1)–(3) is closely connected with a special class of 2π-periodic flows on the whole plane. Namely suppose $\tilde{\psi} = \tilde{\psi}(\tilde{x}_1, \tilde{x}_2, t)$ is a solution to the Navier–Stokes equation with 2π*-periodic boundary condition*, and satisfy

$$\tilde{\psi}(-\tilde{x}_1, \tilde{x}_2, t) = -\tilde{\psi}(\tilde{x}_1, \tilde{x}_2, t) = \tilde{\psi}(\tilde{x}_1, -\tilde{x}_2, t), \quad \forall\, \tilde{x}_1, \tilde{x}_2. \tag{8}$$

It is not difficult to check that the special symmetry (8) is preserved under the dynamics of the Navier–Stokes flow. Furthermore if we write

$$\tilde{\psi}(\tilde{x}_1, \tilde{x}_2, t) = \sum_{m,n} \tilde{f}_{mn} e^{i(m\tilde{x}_1 + n\tilde{x}_2)},$$

then

$$-\tilde{f}_{mn} = \tilde{f}_{-m,n} = \tilde{f}_{m,-n}, \quad \forall\, m, n.$$

Therefore from a simple computation

$$\tilde{\psi}(\tilde{x}_1, \tilde{x}_2, t) = -\sum_{m,n} \tilde{f}_{mn} \sin m\tilde{x}_1 \sin n\tilde{x}_2 \tag{9}$$

which corresponds exactly to (3) up to a minus sign. This shows that $\tilde{\psi}$ is also a solution to our problem (1)–(3).

We shall call extremal points of the stream function ψ the points of local minima or maxima of ψ. Near these points the velocity u is tangent to the level sets of ψ (or $\tilde{\psi}$) which are closed curves. It is natural to call extremal points of ψ viscous vortices. The main purpose of this paper is to show that these vortices can split or merge.

Now we can formulate our main results of this paper.

Theorem 2 (Existence of bifurcations). *There exists an open set $\mathcal{A}$ in the space of stream functions such that the following holds true:*

For each stream function $\psi_0 \in \mathcal{A}$, there is an open neighborhood U of the point $(\tilde{x}_1, \tilde{x}_2) = (\frac{\pi}{2}, \frac{\pi}{2})$, two moments of times $0 < t_1 < t_2$ such that the corresponding stream function $\psi = \psi(\tilde{x}_1, \tilde{x}_2, t)$ solves (1) with initial data ψ_0 and satisfy

1. *At $t = 0$, $(\frac{\pi}{2}, \frac{\pi}{2})$ is a non-degenerate minimum of ψ in the neighborhood U.*
2. *For any $0 < t \le t_1$, ψ has only one critical point in U given by $(\tilde{x}_1, \tilde{x}_2) = (\frac{\pi}{2}, \frac{\pi}{2})$.*
3. *At $t = t_1$, $(\tilde{x}_1, \tilde{x}_2) = (\frac{\pi}{2}, \frac{\pi}{2})$ is a degenerate local minimum of ψ in U.*
4. *For $t_1 < t \le t_2$, ψ has exactly three critical points in U. The point $(\frac{\pi}{2}, \frac{\pi}{2})$ becomes a saddle. Two other critical points are of the form $(\frac{\pi}{2} + x^*, \frac{\pi}{2} + y^*)$, $(\frac{\pi}{2} - x^*, \frac{\pi}{2} - y^*)$ where $x^* \ne 0$, $y^* \ne 0$ and are local minima.*

Remark 2. Under our conditions the point $(\frac{\pi}{2}, \frac{\pi}{2})$ is the extremal point of the stream function for all time. This property plays the same role as the knowledge of fixed points or periodic orbits in the usual theory of bifurcations.

Remark 3. The fact that the extra critical points emerge in the form $(\frac{\pi}{2} + x^*, \frac{\pi}{2} + y^*)$, $(\frac{\pi}{2} - x^*, \frac{\pi}{2} - y^*)$ is not surprising. As we shall see later in Sect. 3, by the inversion symmetry (18), our stream function ψ is invariant under the reflection about the point $(\frac{\pi}{2}, \frac{\pi}{2})$.

Our next result is in some sense the reversal of the process described in Theorem 2. For a class of initial data having three critical points near the special point $(\frac{\pi}{2}, \frac{\pi}{2})$, we show that they merge into one critical point in finite time.

Theorem 3 (Merging of critical points). *There exists an open set $\mathcal{A}$ in the space of stream functions such that the following holds true:*

For each stream function $\psi_0 \in \mathcal{A}$, there is an open neighborhood U of the point $(\tilde{x}_1, \tilde{x}_2) = (\frac{\pi}{2}, \frac{\pi}{2})$, two moments of times $0 < t_1 < t_2$ such that the corresponding stream function $\psi = \psi(\tilde{x}_1, \tilde{x}_2, t)$ solves (1) with initial data ψ_0 and satisfy

1. *For $0 \leq t < t_1$, ψ has exactly three critical points in U. The point $(\frac{\pi}{2}, \frac{\pi}{2})$ is a saddle. Two other critical points are of the form $(\frac{\pi}{2} + x^*, \frac{\pi}{2} + y^*)$, $(\frac{\pi}{2} - x^*, \frac{\pi}{2} - y^*)$ where $x^* \neq 0$, $y^* \neq 0$ and are local minima.*
2. *At $t = t_1$, $(\frac{\pi}{2}, \frac{\pi}{2})$ is a degenerate minimum of ψ in the neighborhood U.*
3. *For any $t_1 < t \leq t_2$, ψ has only one critical point in U given by $(\tilde{x}_1, \tilde{x}_2) = (\frac{\pi}{2}, \frac{\pi}{2})$.*

This paper is organized as follows. In Sect. 2 we give the proof of Theorem 1. In Sect. 3 we derive the equation for extremal points and formulate sufficient conditions for bifurcations needed in Theorem 2. Section 4 is devoted to the construction of bifurcations in the degenerate case. In Sect. 5 we prove the existence of bifurcation for non-degenerate initial data by using a perturbation argument. In Sect. 6 we give the construction of stream functions satisfying the needed conditions. In Sects. 7 and 8 we describe the proof of Theorem 3 and construction of initial conditions.

2 Proof of Theorem 1

In this section we give the proof of Theorem 1 using the trapping argument from [4].

We shall use the letter C with or without indices to denote different absolute constants whose values may vary from line to line. The actual value of C does not play any role in our arguments.

To simplify notations, denote $Z_*^2 = \{(m,n) \in \mathbb{Z}^2, m \neq 0, n \neq 0\}$ and $r = (m,n) \in \mathbb{Z}_*^2$, $r' = (m',n') \in \mathbb{Z}_*^2$, $r'' = (m'',n'') \in \mathbb{Z}_*^2$, and also denote $\omega_r = \omega_{mn}$, $\omega_{r'} = \omega_{m'n'}$ and so on.

By standard enstrophy inequality, we have

$$\|\omega(t)\|_{L^2_{\tilde{x}_1\tilde{x}_2}([0,\pi]\times[0,\pi])} \leq \mathcal{E}_0, \quad \forall\, t > 0,$$

where $\mathcal{E}_0 > 0$ is the enstrophy at $t = 0$.

By Fourier transform, this implies

$$\left(\sum_{r\in\mathbb{Z}^2_*} |\omega_r(t)|^2 \right)^{\frac{1}{2}} \leq C_1\mathcal{E}_0, \forall\, t > 0. \tag{10}$$

Let $K_1 > 0$ be a constant depending on A which will be taken sufficiently large. By (10), we get

$$|\omega_r(t)| \leq \frac{C_1K_1\mathcal{E}_0}{|r|^{\frac{\gamma}{2}}}, \quad \forall\, |r| \leq K_1^{\frac{2}{\gamma}}, \quad \forall\, t > 0.$$

Define the trapping set

$$\Omega(K_1) = \left\{ (\tilde{\omega}_r) : |\tilde{\omega}_r| \leq \frac{C_1K_1\mathcal{E}_0}{|r|^{\frac{\gamma}{2}}}, \quad \forall\, |r| \geq K_1^{\frac{2}{\gamma}} \right\}. \tag{11}$$

Now we show that for all $t > 0$ the trajectories of our system remain inside the set $\Omega(K_1)$. Indeed at $t = 0$, by choosing $K_1 > 2A$ (see (6)), we get that our system lies strictly inside $\Omega(K_1)$. Assume $t_1 > 0$ is the first moment of time when our system reaches the boundary $\partial\Omega(K_1)$.[1]

Then for some $|r^*| \geq K_1^{\frac{2}{\gamma}}$,

$$|\omega_{r^*}(t_1)| = \frac{C_1K_1\mathcal{E}_0}{|r^*|^{\gamma}}.$$

WLOG assume

$$\omega_{r^*}(t_1) = \frac{C_1K_1\mathcal{E}_0}{|r^*|^{\gamma}}.$$

The case $\omega_{r^*}(t_1) = -\frac{C_1K_1\mathcal{E}_0}{|r^*|^{\gamma}}$ is similar and therefore its discussion is omitted. We then aim to show that

$$\partial_t\omega_{r^*}(t)\Big|_{t=t_1} < 0.$$

[1] Strictly speaking, we should consider the Galerkin approximations of our system to avoid issues connected with the infinite dimensionality of our system.

This will guarantee that the trajectory of our system cannot exit the trapping set $\Omega(K_1)$ and will remain inside $\Omega(K_1)$.

Recall the vorticity equation

$$\partial_t \omega + \Delta^{-1}\nabla^{\perp}\omega \cdot \nabla\omega = \Delta\omega. \tag{12}$$

By using (12), we have

$$\Delta^{-1}\nabla^{\perp}\omega \cdot \nabla\omega = \sum_{(m,n)\in\mathbb{Z}_*^2} N_{mn} \sin m\tilde{x}_1 \sin n\tilde{x}_2,$$

where

$$|N_{mn}| \leq \sum_{r'+r''=r} \frac{|\omega_{r'}|}{|r'|} \cdot |r''| \cdot |\omega_{r''}|. \tag{13}$$

There are two cases.

Case 1. $|r'| > \frac{1}{3}|r|$. Then

$$\frac{|r''|}{|r'|} \leq \frac{|r| + |r'|}{|r'|} \leq C.$$

Hence

$$\begin{aligned}
\text{RHS of (13)} &\leq C \sum_{\substack{r'+r''=r \\ |r'|>\frac{1}{3}|r|}} |\omega_{r'}| \cdot |\omega_{r''}| \\
&\leq C \left(\sum_{|r'|>\frac{1}{3}|r|} |\omega_{r'}|^2 \right)^{\frac{1}{2}} \cdot \left(\sum_{r''} |\omega_{r''}|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{CK_1}{|r|^{\gamma-1}} \mathcal{E}_1.
\end{aligned}$$

Case 2. $|r''| > \frac{1}{3}|r|$ and $|r'| \leq \frac{1}{3}|z|$. Then

$$\begin{aligned}
\text{RHS of (12)} \ &\leq \frac{CK_1}{|r|^{\gamma-1}} \sum_{|r'|\leq\frac{1}{3}|r|} \frac{|\omega_{r'}|}{|r'|} \\
&\leq \frac{CK_1}{|r|^{\gamma-1}} \log|r| \cdot \mathcal{E}_1.
\end{aligned}$$

Concluding from the above cases, we get

$$|N_{r^*}(t_1)| \leq \frac{CK_1\mathcal{E}_1}{|r^*|^{\gamma-1}} \log |r^*|$$

and hence by (12)

$$\begin{aligned} \partial_t \omega_{r^*}(t)\Big|_{t=t_1} &\leq \frac{CK_1\mathcal{E}_1}{|r^*|^{\gamma-1}} \log |r^*| - \frac{C_1K_1\mathcal{E}_1}{|r^*|^{\gamma-2}} \\ &< 0, \end{aligned}$$

if K_1 is sufficiently large (recall that by (11), $|r^*| \geq K_1^{\frac{2}{\gamma}}$). This finishes the trapping argument and Theorem 1 is proved.

3 The Equation for Extremal Points

We consider a special class of flows

$$\psi(\tilde{x}_1, \tilde{x}_2, t) = \sum_{m+n \text{ is even}} f_{mn} \sin(m\tilde{x}_1) \sin(n\tilde{x}_2). \tag{14}$$

It is also invariant under the Navier–Stokes dynamics. If this condition is valid, then on the vertical boundaries, for any $0 \leq \tilde{x}_2 \leq \pi$, the velocity vector at the point $(0, \tilde{x}_2)$ has the same magnitude but opposite direction to the velocity at the point $(0, \pi - \tilde{x}_2)$. In some sense they form a dipole with center at $(\frac{\pi}{2}, \frac{\pi}{2})$. Similar statements also hold for the horizontal boundaries.

In this paper we study bifurcations of the stream function near the point $(\frac{\pi}{2}, \frac{\pi}{2})$.

After the change of variables,

$$\tilde{x}_1 = \frac{\pi}{2} + x, \quad \tilde{x}_2 = \frac{\pi}{2} + y, \tag{15}$$

we shift our coordinate system and define

$$\phi(x, y, t) = \psi(\frac{\pi}{2} + x, \frac{\pi}{2} + y, t). \tag{16}$$

By (16), (14) and (9), we get

$$\begin{aligned} \phi(x, y, t) &= -\sum_{m+n \text{ is even}} f_{mn} e^{i(\frac{m+n}{2}\pi + mx + ny)} \\ &= -\sum_{m+n \text{ is even}} f_{mn} (-1)^{\frac{m+n}{2}} e^{i(mx+ny)}. \end{aligned}$$

Since ϕ and f_{mn} are both real-valued, we get

$$\phi(x, y, t) = -\sum_{m+n \text{ is even}} f_{mn}(-1)^{\frac{m+n}{2}} \cos(mx + ny) \tag{17}$$

$$= \phi(-x, -y, t), \tag{18}$$

i.e. ϕ satisfies the inversion symmetry. It implies that at the point $(x, y) = (0, 0)$ the gradient of ϕ vanishes.

Introduce a neighborhood $U_\delta = \{(x, y) :\ x^2 + y^2 \leq \delta^2\}$. Later we shall choose δ to be sufficiently small.

For sufficiently small $t_2 > 0$ consider the time interval $[0, t_2]$ and write the following expansion of ϕ in the neighborhood U_δ:

$$\begin{aligned}\phi(x, y, t) = {}& \phi(0, 0, t) + a_1(t)x^2 + a_2(t)y^2 + a_3(t)xy \\ & + b_1(t)x^4 + b_2(t)y^4 + b_3(t)x^3y + b_4(t)x^2y^2 + b_5(t)xy^3 \\ & + \epsilon(x, y, t),\end{aligned} \tag{19}$$

where the remainder term satisfies the inequalities

$$\begin{aligned}\epsilon(x, y, t) &= O(x^6 + y^6), \\ \frac{\partial\epsilon}{\partial x}(x, y, t) &= O(|x|^5 + |y|^5), \\ \frac{\partial\epsilon}{\partial y}(x, y, t) &= O(|x|^5 + |y|^5).\end{aligned} \tag{20}$$

In the expansion (19), terms of odd degree are not present because of the symmetry (18).

The first equation for the critical point takes the form

$$\partial_x \phi = 0.$$

By (19), we get

$$\begin{aligned}& 2a_1x + a_3y + 4b_1x^3 + 3b_3x^2y + 2b_4xy^2 \\ & \quad + b_5y^3 + \frac{\partial\epsilon}{\partial x} = 0.\end{aligned} \tag{21}$$

Here and later we occasionally suppress the time dependence and write $a_i(t)$, $b_i(t)$ simply as a_i, b_i when the context is clear.

Assume for $0 \leq t \leq t_2$

$$a_3(t) \sim O(1). \tag{22}$$

More precisely

$$const \le a_3(t) \le const.$$

The values of constants play some role later. This will be clarified below (see (34)).

Equation 21 takes the form

$$y = -\frac{2a_1}{a_3}x - \frac{4b_1}{a_3}x^3 - \frac{3b_3}{a_3}x^2y - \frac{2b_4}{a_3}xy^2 - \frac{b_5}{a_3}y^3 - \frac{1}{a_3}\frac{\partial\epsilon}{\partial x}. \tag{23}$$

Assume also that in formula (19)

$$b_2(0) = b_3(0) = b_4(0) = b_5(0) = 0. \tag{24}$$

For sufficiently small t_2 it implies that for $0 \le t \le t_2$,

$$b_i(t) \sim O(t), \quad i = 2, \cdots, 5. \tag{25}$$

Write (23) in the form

$$y = -\frac{2a_1}{a_3}x - \frac{4b_1}{a_3}x^3 + O(t) \cdot O(|x|^3 + |y|^3) + O(|x|^5 + |y|^5). \tag{26}$$

Since $(x, y) \in U_\delta$, we have the rough estimate

$$y = O(x). \tag{27}$$

Consider the other critical point equation

$$\frac{\partial\phi}{\partial y} = 0.$$

By (19), we get

$$2a_2y + a_3x + 4b_2y^3 + b_3x^3 + 2b_4x^2y + 3b_5xy^2 + \frac{\partial\epsilon}{\partial y} = 0.$$

In view of the assumptions (25) and (20), we obtain

$$2a_2y + a_3x + O(t) \cdot O(|x|^3 + |y|^3) + O(|x|^5 + |y|^5) = 0.$$

Using (27), we get

$$2a_2 y + a_3 x + O(t) \cdot O(|x|^3) + O(|x|^5) = 0. \tag{28}$$

Substituting (26) into (28) and using again (27), we have

$$2a_2 \left(-\frac{2a_1}{a_3} x - \frac{4b_1}{a_3} x^3 \right) + a_3 x + O(t) \cdot O(|x|^3) + O(|x|^5) = 0.$$

Or simply,

$$\frac{a_3^2 - 4a_1 a_2}{a_3} x - \frac{8a_2 b_1}{a_3} x^3 + O(t) \cdot O(|x|^3) + O(|x|^5) = 0. \tag{29}$$

It is obvious that (29) has a solution $x = 0$. We now look for other possible solutions in U_δ. Dividing both sides of (29) by $\frac{x}{a_3}$, we obtain

$$(a_3^2 - 4a_1 a_2) - 8a_2 b_1 x^2 + O(t) \cdot O(x^2) + O(x^4) = 0. \tag{30}$$

We shall choose initial data very carefully so that the needed bifurcation happens on the time interval $[0, t_2]$. This will be done in two stages. At the first stage we consider the degenerate case in which the bifurcation happens immediately for $t > 0$. In the second stage we perturb the degenerate data so that the bifurcation is "delayed" to a later time $0 < t_1 < t_2$. In other words, we show that for sufficiently small (and special) perturbations, the desired bifurcation happens at $t = t_1$.

4 Stage 1: The Bifurcation in the Degenerate Case

Rewrite (30) as

$$-(a_3^2 - 4a_1 a_2) + 8a_2 b_1 x^2 + O(t) \cdot O(x^2) + O(x^4) = 0. \tag{31}$$

Choose $\phi_0 = \phi_0(x, y)$ so that

$$\begin{gathered} a_3(0)^2 - 4a_1(0)a_2(0) = 0, \\ \frac{d}{dt}\left(a_3^2(t) - 4a_1(t)a_2(t)\right)|_{t=0} > 0, \\ a_2(0) > 0,\ a_3(0) > 0,\ b_1(0) > 0. \end{gathered} \tag{32}$$

In addition, we also need

$$b_2(0) = b_3(0) = b_4(0) = b_5(0) = 0. \tag{33}$$

The possibility of choosing ϕ_0 with properties (32)–(33) will be shown later (see Sect. 6). Assume for the moment that these conditions are met, then for sufficiently small $t_2 > 0$, we have for $0 < t \leq t_2$,

$$\begin{aligned}
&A_3'' \geq a_3(t) \geq A_3' > 0,\\
&A_2'' \geq a_2(t) \geq A_2' > 0,\\
&B_1'' \geq \frac{d}{dt}\Big(a_3^2(t) - 4a_1(t)a_2(t)\Big) \geq B_1' > 0,\\
&B_2'' \geq b_1(t) \geq B_2' > 0,
\end{aligned} \tag{34}$$

where A_i', A_i'', B_i', B_i'' are constants.

By (32)–(34), we have for $0 < t \leq t_2$

$$(a_3^2(t) - 4a_1(t)a_2(t)) \sim t,$$

which means that

$$const \cdot t \leq a_3^2(t) - 4a_1(t)a_2(t) \leq const \cdot t.$$

Also we have

$$8a_2(t)b_1(t) \sim const.$$

It follows that for $0 < t \leq t_2$, the equation (31) is of the form

$$-O(t) + O(1) \cdot x^2 + O(t) \cdot O(x^2) + O(x^4) = 0. \tag{35}$$

For sufficiently small δ and sufficiently small t_2, the equation (35) has two and only two solutions

$$x = \pm O(\sqrt{t})$$

because $O(t)$ is of order of t, $O(1) > 0$ and other terms do not play any essential role. In this sense solutions to (31) bifurcates into two solutions for $0 < t \leq t_2$.

Remark that at $t = 0$, the only solution to (31) is $x = 0$ due to the conditions $a_3(0)^2 - 4a_1(0)a_2(0) = 0$ and $a_2(0)b_1(0) \sim const$.

5 Stage 2: Bifurcation from Non-degenerate Initial Data, a Perturbation Argument

In stage 2 we finish our construction of bifurcation from non-degenerate initial data. The main idea is to perturb the initial data considered in Stage 1. The perturbation will be chosen so that initially we will have only one local non-degenerate minimum located at $(x, y) = (0, 0)$.

To this end, consider $\tilde{\phi}_0 = \tilde{\phi}_0(x, y) \in C^\infty$ with the following properties:

$$\tilde{\phi}_0(x, y) = \tilde{\phi}_0(-x, -y), \quad \forall\, x, y,$$

$$\frac{\partial^4 \tilde{\phi}_0}{\partial x^m \partial y^n}\Big|_{(x,y)=(0,0)} = 0, \quad \forall\, m + n = 4, 0 \le m \le 4,$$

$$\frac{\partial^2 \tilde{\phi}_0}{\partial x \partial y}\Big|_{(x,y)=(0,0)} = 0, \quad \frac{\partial^2 \tilde{\phi}_0}{\partial x^2}\Big|_{(x,y)=(0,0)} > 0, \quad \frac{\partial^2 \tilde{\phi}_0}{\partial y^2}\Big|_{(x,y)=(0,0)} > 0. \tag{36}$$

Fix $\phi_0 = \phi_0(x, y)$ taken from Stage 1 which has the properties (32)–(33). We shall consider the perturbation by $\tilde{\phi}_0$ having the form

$$\tilde{\phi}_0^\epsilon(x, y) = \phi_0(x, y) + \epsilon\tilde{\phi}_0(x, y),$$

where $\epsilon > 0$ is sufficiently small.

Denote the corresponding solution of the main equation (1) (in the shifted coordinates) by $\phi^\epsilon = \phi^\epsilon(x, y, t)$. To simplify the notations, we expand $\phi^\epsilon(x, y, t)$ in the form corresponding to (19), i.e. we write

$$\begin{aligned}\phi^\epsilon(x, y, t) = \phi^\epsilon(0, 0, t) + a_1^\epsilon(t)x^2 + a_2^\epsilon y^2 + a_3^\epsilon xy \\ + b_1^\epsilon(t)x^4 + b_2^\epsilon(t)y^4 + b_3^\epsilon(t)x^3 y + b_4^\epsilon(t)x^2 y^2 + b_5^\epsilon(t)xy^3 \\ + \tilde{\epsilon}(x, y, t)\end{aligned} \tag{37}$$

where $\tilde{\epsilon}$ satisfies an estimate similar to (20).

We now check the properties of $\phi^\epsilon(x, y, t)$.

(a) At $t = 0$, the point $(x, y) = (0, 0)$ is the unique extremum of $\phi^\epsilon(x, y, 0)$ in the neighborhood U_δ. Also $(0, 0)$ is a non-degenerate local minimum.

To prove this, we note that due to (32), (33) and (36), the critical point equation (30) still holds for $\phi^\epsilon(x, y, t)$ for sufficiently small $\epsilon > 0$ with corresponding coefficients a_1, a_2, a_3, b_1 now replaced by a_1^ϵ, a_2^ϵ, a_3^ϵ, b_1^ϵ. In particular this gives us

$$(a_3^\epsilon(0))^2 - 4a_1^\epsilon(0)a_2^\epsilon(0) - 8a_2^\epsilon(0)b_1^\epsilon(0)x^2 + O(x^4) = 0. \tag{38}$$

Denote

$$\tilde{a}_1 = \frac{\partial^2 \tilde{\phi}_0}{\partial x^2}\Big|_{(x,y)=(0,0)} > 0,$$

$$\tilde{a}_2 = \frac{\partial^2 \tilde{\phi}_0}{\partial y^2}\Big|_{(x,y)=(0,0)} > 0.$$

By (32) and (36), we have

$$\begin{aligned}(a_3^\epsilon(0))^2 &- 4a_1^\epsilon(0)a_2^\epsilon(0)\\ &= a_3(0)^2 + O(\epsilon^2) - 4(a_1(0) + \epsilon\tilde{a}_1)(a_2(0) + \epsilon\tilde{a}_2)\\ &= -4(a_1(0)\tilde{a}_2 + a_2(0)\tilde{a}_1)\epsilon + O(\epsilon^2). \end{aligned} \tag{39}$$

On the other hand, for sufficiently small $\epsilon > 0$, by using (38) and (36), we have

$$\begin{aligned} a_2^\epsilon(0)b_1^\epsilon(0) &= (a_2(0) + O(\epsilon)) \cdot (b_1(0) + O(\epsilon^2))\\ &= a_2(0)b_1(0) + O(\epsilon)\\ &\sim const. \end{aligned} \tag{40}$$

Therefore by (39) and (40), the equation (38) takes the form

$$-O(1)\epsilon - O(1) \cdot x^2 + O(x^4) = 0,$$

or simply

$$O(1) \cdot \epsilon + O(1) \cdot O(x^2) = 0.$$

It is clear that for $\epsilon > 0$ this equation does not have any real-valued solution in U_δ.

To show that $(0, 0)$ is a non-degenerate local minimum at $t = 0$, we observe that by (39), for sufficiently small $\epsilon > 0$,

$$(a_3^\epsilon(0))^2 - 4a_1^\epsilon(0)a_2^\epsilon(0) < 0. \tag{41}$$

Also we have by (32)

$$a_1^\epsilon(0) > 0, \quad a_2^\epsilon(0) > 0. \tag{42}$$

Equations 41 and 42 show that the Hessian matrix

$$\begin{pmatrix} a_1^\epsilon(0) & \frac{1}{2}a_3^\epsilon(0)\\ \frac{1}{2}a_3^\epsilon(0) & a_2^\epsilon(0)\end{pmatrix}$$

is strictly positive definite. Hence $(0, 0)$ is a non-degenerate local minimum.

(b) Consider the function

$$D^\epsilon(t) = (a_3^\epsilon(t))^2 - 4a_1^\epsilon(t)a_2^\epsilon(t).$$

It will be proven below that for sufficiently small $\epsilon > 0$, the following holds: There exists unique $t_1 = t_1(\epsilon) > 0$ such that

$$\begin{aligned} D^{\epsilon}(t) < 0, &\quad \text{for } 0 \le t < t_1, \\ D^{\epsilon}(t) = 0, &\quad \text{for } t = t_1, \\ D^{\epsilon}(t) > 0, &\quad \text{for } t_1 < t \le t_2. \end{aligned} \tag{43}$$

Furthermore, the reduced critical-point equation (see (30))

$$D^{\epsilon}(t) - 8a_2^{\epsilon}(t)b_1^{\epsilon}(t)x^2 + O(t) \cdot O(x^2) + O(x^4) = 0 \tag{44}$$

has

- No solution for $0 \le t < t_1$,
- Exactly one solution given by $x = 0$ for $t = t_1$,
- Two nonzero solutions for $t_1 < t \le t_2$.

To prove (43), we recall the bound (34) , where for $0 \le t \le t_2$

$$B_1'' \ge \frac{d}{dt}\left(a_3^2(t) - 4a_1(t)a_2(t)\right) \ge B_1' > 0. \tag{45}$$

Since our initial data are given by

$$\phi_0^{\epsilon}(x, y) = \phi_0(x, y) + \epsilon\tilde{\phi}_0(x, y),$$

it follows from simple perturbation theory that for sufficiently small $\epsilon > 0$, we have

$$\|\phi^{\epsilon}(x, y, t) - \phi(x, y, t)\|_{H^m_{t,x,y}} \le \eta(\epsilon, m), \tag{46}$$

where $\eta(\epsilon, m) \to 0$ as $\epsilon \to 0$ and m is fixed.

The notation $H^m_{t,x,y}$ denotes m^{th} Sobolev norms of ψ:

$$\|\psi\|_{H^m_{t,x,y}} = \sum_{0\le\alpha+\beta+\gamma\le m} \left\|\partial_t^{\alpha}\partial_x^{\beta}\partial_y^{\gamma}\psi\right\|_{L^2}.$$

Take m to be sufficiently large and then ϵ sufficiently small. It follows from (45) and (46) that

$$2B_1'' \ge \frac{d}{dt}\left((a_3^{\epsilon}(t))^2 - 4a_1^{\epsilon}(t)a_2^{\epsilon}(t)\right) \ge \frac{B_1'}{2} > 0, \tag{47}$$

for any $0 \le t \le t_2$.

This means in particular that $D^{\epsilon}(t)$ is strictly increasing for $0 \le t \le t_2$.

By (39), we have for $t = 0$ and ϵ sufficiently small,

$$D^{\epsilon}(0) < 0. \tag{48}$$

On the other hand for $t = t_2$, by using the analysis from Stage 1, we have

$$(a_3(t_2))^2 - 4a_1(t_2)a_2(t_2) > 0.$$

Since

$$D^{\epsilon}(t_2) = (a_3(t_2))^2 - 4a_1(t_2)a_2(t_2) + O(\epsilon),$$

it follows easily that for ϵ sufficiently small

$$D^{\epsilon}(t_2) > 0. \tag{49}$$

Now (47)–(49) easily yield (43).

Finally the conclusion after (44) is a simple corollary of the properties of $D^{\epsilon}(t)$ and perturbation theory. We omit the details.

In summary, we have proved the following:

For sufficiently small $\epsilon > 0$, the function $\phi^{\epsilon}(x, y, t)$ has the following properties in the neighborhood U_{δ}:

There exists $0 < t_1 < t_2$, such that

- For $0 \le t < t_1$, $(x, y) = (0, 0)$ is the only critical point in U_{δ}. Furthermore it is a non-degenerate local minimum.
- For $t = t_1$, $(x, y) = (0, 0)$ is the only critical point in U_{δ}.
- For $t_1 < t \le t_2$, there are three critical points in U_{δ}. The point $(x, y) = (0, 0)$ is a saddle. Two other critical points are of the form (x_*, y_*), $(-x_*, -y_*)$, where $x_* > 0$, $y_* > 0$.

Remark that due to our inversion symmetry (18), if (x_*, y_*) is a critical point with $x_* \neq 0$, then $(-x_*, -y_*)$ is also a critical point.

6 Construction of ϕ_0 Satisfying (32)–(33)

We now demonstrate the existence of $\phi_0 = \phi_0(x, y)$ which satisfies conditions (32)–(33) and also has inversion symmetry (18).

By (17), we choose

$$\phi_0(x, y) = - \sum_{\substack{m+n \text{ is even} \\ |m| \le N, |n| \le N}} \tilde{f}_{mn}(-1)^{\frac{m+n}{2}} \cos(mx + ny). \tag{50}$$

To simplify matters, we impose the following conditions on $\tilde{f}_{mn}$:

- $\tilde{f}_{mn}$ is real-valued;
- $\tilde{f}_{mn} = 0$ if $m = 0$ or $n = 0$;
- $\tilde{f}_{mn}$ are odd in each of its variables m and n.

The above conditions imply that

$$\begin{aligned}\phi_0(x, y) = &- \sum_{\substack{1 \le m,n \le N \\ m+n \text{ is even}}} \tilde{f}_{mn} \cdot \Big(2(-1)^{\frac{m+n}{2}} \cos(mx + ny) \\ &\qquad - (-1)^{\frac{-m+n}{2}} \cos(-mx + ny) - (-1)^{\frac{m-n}{2}} \cos(mx - ny) \Big) \\ = &- \sum_{\substack{1 \le m,n \le N \\ m+n \text{ is even}}} 2\tilde{f}_{mn}(-1)^{\frac{m+n}{2}} \Big(\cos(mx + ny) - (-1)^n \cos(mx - ny) \Big).\end{aligned} \tag{51}$$

Define

$$f_{mn} = -2\tilde{f}_{mn}(-1)^{\frac{m+n}{2}}.$$

Then we have

$$\phi_0(x, y) = \sum_{\substack{1 \le m,n \le N \\ m+n \text{ is even}}} f_{mn} \Big(\cos(mx + ny) - (-1)^n \cos(mx - ny) \Big), \tag{52}$$

where f_{mn} are the coefficients to be determined.

Now recall the conditions (32) and (33) and choose

$$\begin{aligned} &a_3(0) = 2, \\ &a_1(0) = a_2(0) = 1, \\ &b_1(0) = \frac{r_1}{24} > 0, \\ &b_2(0) = b_3(0) = b_4(0) = b_5(0) = 0, \end{aligned} \tag{53}$$

where r_1 is a parameter whose value will be specified later.

We still have to check the second condition in (32). This condition can be simplified a little bit. By (53),

$$\begin{aligned} &\frac{d}{dt} \left(a_3^2(t) - 4a_1(t)a_2(t) \right) \Big|_{t=0} \\ &= 4(\dot{a}_3(0) - \dot{a}_1(0) - \dot{a}_2(0)) \\ &= 2 \left(\frac{\partial^3 \phi}{\partial t \partial x \partial y}(0, 0, 0) - \left(\frac{\partial}{\partial t} \Delta \phi \right)(0, 0, 0) \right). \end{aligned}$$

By (1), (19), (16) and (53), we have

$$\left(\frac{\partial}{\partial t}\Delta\phi\right)(0,0,0) = \frac{\partial}{\partial t}\Delta\psi\left(\frac{\pi}{2},\frac{\pi}{2},0\right)$$
$$= \Delta^2\psi_0\left(\frac{\pi}{2},\frac{\pi}{2}\right) + (\nabla^\perp\psi_0)\left(\frac{\pi}{2},\frac{\pi}{2}\right)\cdot(\nabla\Delta\psi_0)\left(\frac{\pi}{2},\frac{\pi}{2}\right)$$
$$= r_1.$$

Similarly

$$\frac{\partial^3\phi}{\partial t\partial x\partial y}(0,0,0) = \left(\partial_{xy}\Delta^{-1}(\nabla^\perp\phi_0\cdot\nabla\Delta\phi_0)\right)(0,0).$$

Therefore the condition

$$\frac{d}{dt}\Big(a_3^2(t) - 4a_1(t)a_2(t)\Big)\Big|_{t=0} > 0$$

is equivalent to

$$\partial_{xy}\Delta^{-1}(\nabla^\perp\phi_0\cdot\nabla\Delta\phi_0)(0,0) > r_1. \tag{54}$$

Our goal is to find (f_{mn}) in (52) such that both (53) and (54) hold. In our formulae below, the summation is understood to be in the region $\{(m,n): 1\le m,n\le N$ and $m+n$ is even$\}$. In terms of f_{mn}, the conditions (53) now take the form

$$\sum f_{mn}\cdot mn\cdot(1+(-1)^n) = -2,$$
$$\sum f_{mn}\cdot m^2\cdot(1-(-1)^n) = -1,$$
$$\sum f_{mn}\cdot n^2\cdot(1-(-1)^n) = -1,$$
$$\sum f_{mn}\cdot m^4\cdot(1-(-1)^n) = r_1,$$
$$\sum f_{mn}\cdot m^3 n\cdot(1+(-1)^n) = 0,$$
$$\sum f_{mn}\cdot m^2n^2\cdot(1-(-1)^n) = 0,$$
$$\sum f_{mn}\cdot mn^3\cdot(1+(-1)^n) = 0,$$
$$\sum f_{mn}\cdot n^4\cdot(1-(-1)^n) = 0. \tag{55}$$

Due to the factors $(1\pm(-1)^n)$ which can vanish depending on the parity of n in the summation, we distinguish two types of coefficients. We shall say f_{mn} is even if both m and n are even. Otherwise f_{mn} is called odd. Notice that due to the constraint that $m+n$ is even we shall only have either odd or even coefficients.

We consider first the equations for even coefficients. From (55) we only need

$$\sum_{\substack{m,n\geq 2\\ m,n \text{ are even}}} f_{mn}\cdot mn=-1,$$

$$\sum_{\substack{m,n\geq 2\\ m,n \text{ are even}}} f_{mn}\cdot m^3n=0,$$

$$\sum_{\substack{m,n\geq 2\\ m,n \text{ are even}}} f_{mn}\cdot mn^3=0, \tag{56}$$

Now we assume that we only have two nonzero even coefficients f_{22} and f_{44}. Then from (56) we get

$$f_{22}\cdot 2^2+f_{44}\cdot 4^2=-1,$$

$$f_{22}\cdot 2^4+f_{44}\cdot 4^4=0.$$

A simple computation gives that

$$f_{22}=-1/3,\quad f_{44}=1/48; \tag{57}$$

Next we turn to odd coefficients.

From (55), we get

$$\sum_{\substack{1\leq m,n\leq N\\ m,n \text{ are odd}}} f_{mn}\cdot m^2=-\frac{1}{2},$$

$$\sum_{\substack{1\leq m,n\leq N\\ m,n \text{ are odd}}} f_{mn}\cdot n^2=-\frac{1}{2},$$

$$\sum_{\substack{1\leq m,n\leq N\\ m,n \text{ are odd}}} f_{mn}\cdot m^4=\frac{r_1}{2},$$

$$\sum_{\substack{1\leq m,n\leq N\\ m,n \text{ are odd}}} f_{mn}\cdot m^2n^2=0,$$

$$\sum_{\substack{1\leq m,n\leq N\\ m,n \text{ are odd}}} f_{mn}\cdot n^4=0, \tag{58}$$

To simplify matters, we assume that the only nonzero odd coefficients are f_{11}, f_{31}, f_{33}, f_{15}, f_{51}.

Let r_2 be another parameter whose value will be specified later. We shall choose $f_{51} = r_2$ and add this condition to (58). For the coefficients f_{11}, f_{31}, f_{33}, f_{15}, f_{51} we then have the matrix equation

$$\begin{pmatrix} 1 & 1 & 3^2 & 3^2 & 1 & 5^2 \\ 1 & 3^2 & 1 & 3^2 & 5^2 & 1 \\ 1 & 1 & 3^4 & 3^4 & 1 & 5^4 \\ 1 & 3^2 & 3^2 & 9^2 & 5^2 & 5^2 \\ 1 & 3^4 & 1 & 3^4 & 5^4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_{11} \\ f_{13} \\ f_{31} \\ f_{33} \\ f_{15} \\ f_{51} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{r_1}{2} \\ 0 \\ 0 \\ r_2 \end{pmatrix} \tag{59}$$

Choose $r_1 = \frac{1}{10}$ and $r_2 = -10$. From (59), we obtain

$$\begin{pmatrix} f_{11} \\ f_{13} \\ f_{31} \\ f_{33} \\ f_{15} \\ f_{51} \end{pmatrix} = \begin{pmatrix} -580.5698 \\ 90.0012 \\ 90.0008 \\ -6.6598 \\ -10.0001 \\ -10.0000 \end{pmatrix} \tag{60}$$

We have completely solved (53). It remains to check the condition (54).
To simplify the computation, we rewrite (52) as

$$\begin{aligned} \phi_0(x, y) &= \sum_{\substack{1 \le m,n \le N \\ m+n \text{ is even}}} f_{mn} \cdot \frac{e^{i(mx+ny)} + e^{-i(mx+ny)}}{2} \\ &\quad + \sum_{\substack{1 \le m,n \le N \\ m+n \text{ is even}}} f_{mn} \cdot (-1)^{n+1} \cdot \frac{e^{i(mx-ny)} + e^{-i(mx-ny)}}{2}. \\ &= \sum_{|m| \le N, |n| \le N} g_{mn} e^{i(mx+ny)}, \end{aligned} \tag{61}$$

where the coefficients g_{mn} satisfy

- $g_{mn} = 0$ if $(m + n)$ is not even or $m = 0$ or $n = 0$.
- $g_{mn} = \frac{1}{2} f_{|m|,|n|}$ if $mn > 0$.
- $g_{mn} = \frac{1}{2} f_{|m|,|n|} \cdot (-1)^{n+1}$ if $mn < 0$.

To find the LHS of (54), we use the coefficients g_{mn} and calculate

$$\Delta^{-1}\nabla\phi_0(x,y) = \sum_{|m|\le N, |n|\le N} g_{mn}\cdot i\cdot \frac{\binom{m}{n}}{(-1)(m^2+n^2)} e^{i(mx+ny)},$$

$$\nabla^{\perp}\phi_0(x,y) = \sum_{|\tilde m|\le N, |\tilde n|\le N} g_{\tilde m\tilde n}\cdot i\cdot \begin{pmatrix}-\tilde n\\ \tilde m\end{pmatrix}\cdot e^{i(\tilde m x+\tilde n y)}.$$

Hence

$$(\nabla^{-1}\nabla\phi_0\cdot\nabla^{\perp}\phi_0)(x,y) = \sum_{\substack{|m|\le N, |n|\le N\\ |\tilde m|\le N, |\tilde n|\le N}} g_{mn}g_{\tilde m\tilde n}\cdot \frac{\tilde m n - m\tilde n}{m^2+n^2}\cdot e^{i\left((m+\tilde m)x+(n+\tilde n)y\right)}. \tag{62}$$

Note that in the summation of the RHS of (62), the zero-th mode is not present since if $m=-\tilde m$, $n=-\tilde n$ then $\tilde m n - m\tilde n = 0$.

We then apply the operator $\partial_{xy}\Delta^{-1}$ to both sides of (62) to obtain

$$\begin{aligned}&\partial_{xy}\Delta^{-1}\left(\Delta^{-1}\nabla\phi_0\cdot\nabla^{\perp}\phi_0\right)\Big|_{(x,y)=(0,0)}\\ &\quad= \sum_{\substack{|m|\le N, |n|\le N\\ |\tilde m|\le N, |\tilde n|\le N}} g_{mn}g_{\tilde m\tilde n}\cdot\frac{\tilde m n - m\tilde n}{m^2+n^2}\cdot\frac{(m+\tilde m)(n+\tilde n)}{(m+\tilde m)^2+(n+\tilde n)^2}.\end{aligned} \tag{63}$$

By using (57), (60), (61), and (63) and a tedious calculation, we obtain

$$\text{LHS of (54)} = 0.1436 > 0.1 = r_1.$$

Clearly this gives us all the needed estimates.

We have finished the construction of the desired initial data ϕ_0 needed in Stage 1. The proof of Theorem 2 is now completed.

7 Proof of Theorem 3

In this section we give the proof of Theorem 3. The argument is similar to the proof of Theorem 2 and is again done in two stages. We sketch the details as follows.

- Stage 1: degenerate case. Recall the reduced critical point equation,

$$-(a_3^2-4a_1a_2)+8a_2b_1x^2+O(t)\cdot O(x^2)+O(x^4)=0. \tag{64}$$

Choose $\phi_0 = \phi_0(x, y)$ so that

$$a_3(0)^2 - 4a_1(0)a_2(0) = 0,$$
$$\frac{d}{dt}\left(a_3^2(t) - 4a_1(t)a_2(t)\right)\Big|_{t=0} < 0,$$
$$a_2(0) > 0,\ a_3(0) > 0,\ b_1(0) > 0, \tag{65}$$

and also

$$b_2(0) = b_3(0) = b_4(0) = b_5(0) = 0. \tag{66}$$

The possibility of choosing ϕ_0 with properties (65)–(66) will be shown in Sect. 8. Assume for the moment that these conditions are met, then for sufficiently small $t_2 > 0$, we have for $0 < t \le t_2$,

$$A_3'' \ge a_3(t) \ge A_3' > 0,$$
$$A_2'' \ge a_2(t) \ge A_2' > 0,$$
$$B_1'' \ge \frac{d}{dt}\Big(4a_1(t)a_2(t) - a_3^2(t)\Big) \ge B_1' > 0,$$
$$B_2'' \ge b_1(t) \ge B_2' > 0, \tag{67}$$

where A_i', A_i'', B_i', B_i'' are constants.

By (65)–(67), we have for $0 < t \le t_2$

$$const \cdot t \le 4a_1(t)a_2(t) - a_3^2(t) \le const \cdot t,$$

and also

$$8a_2(t)b_1(t) \sim const.$$

It follows that for $0 < t \le t_2$, the equation (64) is of the form

$$O(t) + O(1) \cdot x^2 + O(t) \cdot O(x^2) + O(x^4) = 0 \tag{68}$$

which clearly has no real-valued solution for $0 < t \le t_2$.

- Stage 2: a perturbation argument. In stage 2 we perturb the initial data considered in Stage 1 so that initially we will have three critical points.

To this end, consider $\tilde{\phi}_0 = \tilde{\phi}_0(x, y) \in C^\infty$ with the following properties:

$$\tilde{\phi}_0(x, y) = \tilde{\phi}_0(-x, -y), \quad \forall\, x, y,$$

$$\frac{\partial^4 \tilde{\phi}_0}{\partial x^m \partial y^n}\Big|_{(x,y)=(0,0)} = 0, \quad \forall\, m + n = 4, 0 \le m \le 4,$$

$$\frac{\partial^2 \tilde{\phi}_0}{\partial x \partial y}\Big|_{(x,y)=(0,0)} = 0, \quad \frac{\partial^2 \tilde{\phi}_0}{\partial x^2}\Big|_{(x,y)=(0,0)} > 0, \quad \frac{\partial^2 \tilde{\phi}_0}{\partial y^2}\Big|_{(x,y)=(0,0)} > 0. \tag{69}$$

Fix $\phi_0 = \phi_0(x, y)$ taken from Stage 1 which has the properties (65)–(66) and consider the perturbation by $\tilde{\phi}_0$ having the form

$$\tilde{\phi}_0^\epsilon(x, y) = \phi_0(x, y) - \epsilon \tilde{\phi}_0(x, y), \tag{70}$$

where $\epsilon > 0$ is sufficiently small.

Denote the corresponding solution of the main equation (1) (in the shifted coordinates) by $\phi^\epsilon = \phi^\epsilon(x, y, t)$. Expand $\phi^\epsilon(x, y, t)$ in the form

$$\begin{aligned}\phi^\epsilon(x, y, t) = \phi^\epsilon(0, 0, t) &+ a_1^\epsilon(t)x^2 + a_2^\epsilon y^2 + a_3^\epsilon xy \\ &+ b_1^\epsilon(t)x^4 + b_2^\epsilon(t)y^4 + b_3^\epsilon(t)x^3 y + b_4^\epsilon(t)x^2 y^2 + b_5^\epsilon(t)xy^3 \\ &+ \tilde{\epsilon}(x, y, t)\end{aligned} \tag{71}$$

where $\tilde{\epsilon}$ satisfies an estimate similar to (20).

We now check that $\phi^\epsilon(x, y, t)$ has the desired properties needed in Theorem 3.

(a) At $t = 0$, $\phi^\epsilon(x, y, 0)$ has three critical points in the neighborhood U_δ. Also $(0, 0)$ is a saddle point.

To prove this, we note that due to (65), (66) and (69), the reduced critical point equation for $\phi^\epsilon(x, y, t)$ takes the form

$$(a_3^\epsilon(0))^2 - 4a_1^\epsilon(0)a_2^\epsilon(0) - 8a_2^\epsilon(0)b_1^\epsilon(0)x^2 + O(x^4) = 0. \tag{72}$$

Denote

$$\tilde{a}_1 = \frac{\partial^2 \tilde{\phi}_0}{\partial x^2}\Big|_{(x,y)=(0,0)} > 0,$$

$$\tilde{a}_2 = \frac{\partial^2 \tilde{\phi}_0}{\partial y^2}\Big|_{(x,y)=(0,0)} > 0.$$

By (65), (69), and (70), we have

$$
\begin{aligned}
&(a_3^\epsilon(0))^2 - 4a_1^\epsilon(0)a_2^\epsilon(0) \\
&\quad = a_3(0)^2 + O(\epsilon^2) - 4(a_1(0) - \epsilon\tilde{a}_1)(a_2(0) - \epsilon\tilde{a}_2) \\
&\quad = 4(a_1(0)\tilde{a}_2 + a_2(0)\tilde{a}_1)\epsilon + O(\epsilon^2). \qquad (73)
\end{aligned}
$$

On the other hand, for sufficiently small $\epsilon > 0$, by using (72), (69), and (70), we have

$$
\begin{aligned}
a_2^\epsilon(0)b_1^\epsilon(0) &= (a_2(0) - O(\epsilon)) \cdot (b_1(0) + O(\epsilon^2)) \\
&= a_2(0)b_1(0) - O(\epsilon) \\
&\sim const. \qquad (74)
\end{aligned}
$$

Therefore by (73) and (74), the equation (72) takes the form

$$O(1)\epsilon - O(1) \cdot x^2 + O(x^4) = 0,$$

or simply

$$O(1) \cdot \epsilon - O(1) \cdot O(x^2) = 0.$$

It is clear that for $\epsilon > 0$ sufficiently small this equation has two real-valued solutions in U_δ.

To verify that $(0, 0)$ is a saddle point at $t = 0$, we observe that by (73), for sufficiently small $\epsilon > 0$,

$$(a_3^\epsilon(0))^2 - 4a_1^\epsilon(0)a_2^\epsilon(0) > 0. \qquad (75)$$

Also we have by (65)

$$a_1^\epsilon(0) > 0, \quad a_2^\epsilon(0) > 0. \qquad (76)$$

Equations 75 and 76 show that the Hessian matrix

$$\begin{pmatrix} a_1^\epsilon(0) & \frac{1}{2}a_3^\epsilon(0) \\ \frac{1}{2}a_3^\epsilon(0) & a_2^\epsilon(0) \end{pmatrix}$$

has one positive eigen-value and one negative eigen-value. Hence $(0, 0)$ is a saddle.

(b) Consider the function

$$D^\epsilon(t) = (a_3^\epsilon(t))^2 - 4a_1^\epsilon(t)a_2^\epsilon(t).$$

It will be proven below that for sufficiently small $\epsilon > 0$, the following holds: There exists unique $t_1 = t_1(\epsilon) > 0$ such that

$$\begin{aligned} D^\epsilon(t) &> 0, \quad \text{for } 0 \le t < t_1, \\ D^\epsilon(t) &= 0, \quad \text{for } t = t_1, \\ D^\epsilon(t) &< 0, \quad \text{for } t_1 < t \le t_2. \end{aligned} \tag{77}$$

Furthermore, the reduced critical-point equation

$$D^\epsilon(t) - 8a_2^\epsilon(t)b_1^\epsilon(t)x^2 + O(t)\cdot O(x^2) + O(x^4) = 0 \tag{78}$$

has

- Two nonzero solutions for $0 \le t < t_1$,
- Exactly one solution given by $x = 0$ for $t = t_1$,
- No solutions for $t_1 < t \le t_2$.

To prove (77), we recall the bound (67), where for $0 \le t \le t_2$

$$B_1'' \ge \frac{d}{dt}\left(a_3^2(t) - 4a_1(t)a_2(t)\right) \ge B_1' > 0. \tag{79}$$

Since our initial data are given by

$$\phi_0^\epsilon(x,y) = \phi_0(x,y) + \epsilon\tilde{\phi}_0(x,y),$$

it follows from simple perturbation theory that

$$2B_1'' \ge \frac{d}{dt}\left(4a_1^\epsilon(t)a_2^\epsilon(t) - (a_3^\epsilon(t))^2\right) \ge \frac{B_1'}{2} > 0, \tag{80}$$

for any $0 \le t \le t_2$.

This means in particular that $D^\epsilon(t)$ is strictly decreasing for $0 \le t \le t_2$.

By (73), we have for $t = 0$ and ϵ sufficiently small,

$$D^\epsilon(0) > 0. \tag{81}$$

On the other hand for $t = t_2$, by using the analysis from Stage 1, we have

$$(a_3(t_2))^2 - 4a_1(t_2)a_2(t_2) < 0.$$

Since

$$D^\epsilon(t_2) = (a_3(t_2))^2 - 4a_1(t_2)a_2(t_2) + O(\epsilon),$$

it follows easily that for ϵ sufficiently small

$$D^{\epsilon}(t_2) < 0. \tag{82}$$

Now (80)–(82) easily yield (77).

8 Construction of ϕ_0 Satisfying (65)–(66)

We now demonstrate the existence of $\phi_0 = \phi_0(x, y)$ which satisfies conditions (65)–(66). The construction is similar to the one in Sect. 6 and therefore we shall only sketch the details.

Choose ϕ_0 in the form

$$\phi_0(x, y) = \sum_{\substack{1 \le m,n \le N \\ m+n \text{ is even}}} f_{mn} \left(\cos(mx + ny) - (-1)^n \cos(mx - ny)\right), \tag{83}$$

where f_{mn} are the coefficients to be determined.

Now recall the conditions (65) and (66) and set

$$\begin{aligned} a_3(0) &= 2, \\ a_1(0) &= a_2(0) = 1, \\ b_1(0) &= \frac{r_1}{24} > 0, \\ b_2(0) &= b_3(0) = b_4(0) = b_5(0) = 0, \end{aligned} \tag{84}$$

where r_1 is a parameter whose value will be specified later.

The second condition in (65) simplifies to

$$\partial_{xy} \Delta^{-1} (\nabla^{\perp} \phi_0 \cdot \nabla \Delta \phi_0)(0, 0) < r_1. \tag{85}$$

Our goal is to find (f_{mn}) in (83) such that both (84) and (85) hold. In our formulae below, the summation is understood to be in the region $\{(m, n) : 1 \le m, n \le N$ and $m + n$ is even$\}$. In terms of f_{mn}, the conditions (84) now take the form

$$\begin{aligned} \sum f_{mn} \cdot mn \cdot (1 + (-1)^n) &= -2, \\ \sum f_{mn} \cdot m^2 \cdot (1 - (-1)^n) &= -1, \\ \sum f_{mn} \cdot n^2 \cdot (1 - (-1)^n) &= -1, \\ \sum f_{mn} \cdot m^4 \cdot (1 - (-1)^n) &= r_1, \end{aligned}$$

$$\sum f_{mn} \cdot m^3 n \cdot (1 + (-1)^n) = 0,$$
$$\sum f_{mn} \cdot m^2 n^2 \cdot (1 - (-1)^n) = 0,$$
$$\sum f_{mn} \cdot m n^3 \cdot (1 + (-1)^n) = 0,$$
$$\sum f_{mn} \cdot n^4 \cdot (1 - (-1)^n) = 0. \tag{86}$$

Due to the factors $(1 \pm (-1)^n)$ which can vanish depending on the parity of n in the summation, we distinguish two types of coefficients. We shall say f_{mn} is even if both m and n are even. Otherwise f_{mn} is called odd. Notice that due to the constraint that $m + n$ is even we shall only have either odd or even coefficients.

Consider first the equations for even coefficients. From (86) we only need

$$\sum_{\substack{m,n \geq 2 \\ m,n \text{ are even}}} f_{mn} \cdot mn = -1,$$
$$\sum_{\substack{m,n \geq 2 \\ m,n \text{ are even}}} f_{mn} \cdot m^3 n = 0,$$
$$\sum_{\substack{m,n \geq 2 \\ m,n \text{ are even}}} f_{mn} \cdot m n^3 = 0, \tag{87}$$

Now we assume that we only have two nonzero even coefficients f_{22} and f_{44}. Then from (87) we get

$$f_{22} \cdot 2^2 + f_{44} \cdot 4^2 = -1,$$
$$f_{22} \cdot 2^4 + f_{44} \cdot 4^4 = 0.$$

A simple computation gives that

$$f_{22} = -1/3, \quad f_{44} = 1/48; \tag{88}$$

Next we turn to odd coefficients.

From (86), we get

$$\sum_{\substack{1 \leq m,n \leq N \\ m,n \text{ are odd}}} f_{mn} \cdot m^2 = -\frac{1}{2},$$
$$\sum_{\substack{1 \leq m,n \leq N \\ m,n \text{ are odd}}} f_{mn} \cdot n^2 = -\frac{1}{2},$$

$$\sum_{\substack{1 \le m,n \le N \\ m,n \text{ are odd}}} f_{mn} \cdot m^4 = \frac{r_1}{2},$$

$$\sum_{\substack{1 \le m,n \le N \\ m,n \text{ are odd}}} f_{mn} \cdot m^2 n^2 = 0,$$

$$\sum_{\substack{1 \le m,n \le N \\ m,n \text{ are odd}}} f_{mn} \cdot n^4 = 0, \tag{89}$$

To simplify matters, we assume that the only nonzero odd coefficients are f_{11}, f_{31}, f_{33}, f_{15}, f_{51}.

Let r_2 be another parameter whose value will be specified later. We shall choose $f_{51} = r_2$ and add this condition to (89). For the coefficients f_{11}, f_{31}, f_{33}, f_{15}, f_{51} we then have the matrix equation

$$\begin{pmatrix} 1 & 1 & 3^2 & 3^2 & 1 & 5^2 \\ 1 & 3^2 & 1 & 3^2 & 5^2 & 1 \\ 1 & 1 & 3^4 & 3^4 & 1 & 5^4 \\ 1 & 3^2 & 3^2 & 9^2 & 5^2 & 5^2 \\ 1 & 3^4 & 1 & 3^4 & 5^4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_{11} \\ f_{13} \\ f_{31} \\ f_{33} \\ f_{15} \\ f_{51} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{r_1}{2} \\ 0 \\ 0 \\ r_2 \end{pmatrix} \tag{90}$$

Choose $r_1 = r_2 = 1$. From (90), we obtain

$$\begin{pmatrix} f_{11} \\ f_{13} \\ f_{31} \\ f_{33} \\ f_{15} \\ f_{51} \end{pmatrix} = \begin{pmatrix} 57.3646 \\ -8.9883 \\ -8.9922 \\ 0.6727 \\ 0.9987 \\ 1.0000 \end{pmatrix} \tag{91}$$

We have completely solved (84). It remains to check the condition (85). For this purpose, we rewrite (83) as

$$\begin{aligned} \phi_0(x,y) = & \sum_{\substack{1 \le m,n \le N \\ m+n \text{ is even}}} f_{mn} \cdot \frac{e^{i(mx+ny)} + e^{-i(mx+ny)}}{2} \\ & + \sum_{\substack{1 \le m,n \le N \\ m+n \text{ is even}}} f_{mn} \cdot (-1)^{n+1} \cdot \frac{e^{i(mx-ny)} + e^{-i(mx-ny)}}{2}. \\ = & \sum_{|m| \le N, |n| \le N} g_{mn} e^{i(mx+ny)}, \end{aligned} \tag{92}$$

where the coefficients g_{mn} satisfy

- $g_{mn} = 0$ if $(m + n)$ is not even or $m = 0$ or $n = 0$.
- $g_{mn} = \frac{1}{2} f_{|m|,|n|}$ if $mn > 0$.
- $g_{mn} = \frac{1}{2} f_{|m|,|n|} \cdot (-1)^{n+1}$ if $mn < 0$.

In terms of the coefficients g_{mn}, the LHS of (85) takes the form

$$(\nabla^{-1}\nabla\phi_0 \cdot \nabla^{\perp}\phi_0)(x, y) = \sum_{\substack{|m|\le N,|n|\le N \\ |\tilde{m}|\le N,|\tilde{n}|\le N}} g_{mn} g_{\tilde{m}\tilde{n}} \cdot \frac{\tilde{m}n - m\tilde{n}}{m^2 + n^2} \cdot e^{i\left((m+\tilde{m})x+(n+\tilde{n})y\right)}.$$

By a tedious calculation, we obtain

$$\text{LHS of (85)} = -0.1420 < 1 = r_1.$$

Clearly this gives us all the needed estimates.

We have finished the construction of the desired initial data ϕ_0 needed in Stage 1 of Sect. 7. The proof of Theorem 3 is now completed.

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