Chapter 9
Semi-explicit Distributed NMPC

Abstract. In this chapter, a suboptimal approach to *distributed* NMPC for systems consisting of nonlinear subsystems with linearly *coupled* dynamics, subject to both state and input constraints, is considered. The approach applies the dynamic dual decomposition method and reformulates the original *centralized* NMPC problem into a *distributed* quasi-NMPC problem by linearization of the nonlinear system dynamics. The approach is based on *distributed on-line* optimization (by gradient iterations) and can be applied to large-scale nonlinear systems. Further, a *semi-explicit* NMPC approach to efficiently solve the *distributed* NMPC problem for small- and medium-scale systems is described. It combines the explicit approximate solution with the on-line optimization and the result is a decrease of the on-line computational complexity. Both the *on-line* optimization based *distributed* NMPC and the *semi-explicit distributed* NMPC are illustrated in a problem to solve a NMPC problem for a nonlinear system consisting of two subsystems.

9.1 Introduction

Recall that NMPC involves the solution at each sampling instant of a finite horizon optimal control problem subject to the system dynamics, and state and input constraints. However, solving in a *centralized* way NMPC problems for medium- and large-scale systems may be impractical due to the complexity of the Nonlinear Programming (NLP) problem, the topology of the plant and data communication, and the large number of decision variables. Therefore, there is a strong motivation for development of methods for *distributed* solution of NMPC problems. At the same time, the multi-core computer architectures available nowadays would encourage *parallel* and *distributed* NMPC computations [5]. Recently, several approaches for *decentralized* implementation of MPC algorithms have been developed, [19]. As it is pointed out in [15], the possibility to use MPC in a *decentralized* fashion has the advantage to reduce the original, large size, optimization problem into a number of smaller and more tractable ones. In [20], [13], [22], [9], approaches for *distributed* MPC for systems consisting of *linear* interconnected subsystems have been developed. The approach in [9] is based on the dual decomposition methods [1, 6, 4],

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where large-scale optimization problems are handled by using Lagrange multipliers to relax the couplings between the sub-problems. In [17, 18], the dual decomposition is used for analysis and synthesis of distributed feedback controllers.

Further, approaches for *distributed* MPC for systems composed of several *nonlinear* subsystems have been proposed in [15, 7, 12]. In [15], a stabilizing *decentralized* MPC algorithm for nonlinear systems consisting of several interconnected local subsystems is developed. It is derived under the main assumptions that no information can be exchanged between local control laws, i.e. the coupling between the subsystems is ignored, and only input constraints are imposed on the system. In [7], it is supposed that the dynamics and constraints of the nonlinear subsystems are *decoupled*, but their state vectors are *coupled* in a single cost function of a finite horizon optimal control problem. In [12], an optimal control problem for a set of dynamically *decoupled* nonlinear systems, where the cost function and constraints couple the dynamical behavior of the systems, is solved.

In this chapter, the suboptimal approach [11] to *distributed* NMPC for a more general class of systems consisting of *nonlinear* subsystems with *coupled* dynamics subject to both state and input constraints is considered. Like in [15], it is supposed that the couplings between the subsystems are linear. However in difference to [15], the *distributed* NMPC method proposed here takes into account these couplings, as well as state constraints. The approach [11] applies the dynamic dual decomposition method [4, 18, 9] and reformulates the original *centralized* NMPC problem into a *distributed* quasi-NMPC problem by linearization of the nonlinear system dynamics. The approach is based entirely on *distributed* on-line optimization (by gradient iterations) and can be applied to large-scale nonlinear systems. Further, a *semi-explicit* NMPC approach to efficiently solve the *distributed* NMPC problem for small- and medium-scale systems is proposed.

9.2 Formulation of NMPC Problem for Interconnected Systems

Consider a system composed by the interconnection of M subsystems (shown in Fig. 9.1), which is described by the following nonlinear discrete-time models [15]:

$$x_i(t+1) = f_i(x_i(t), u_i(t)) + g_i(x(t)) + d_i(t)\,,\ i = 1, 2, \ldots, M \tag{9.1}$$

where $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$, and $d_i(t) \in \mathbb{R}^{n_i}$ are the state, control input, and disturbance vectors, related to the i-th subsystem, and $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$ and $g_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ are nonlinear functions.

In (9.1), the mutual influence of the M subsystems is described by the functions g_i, which depend on the overall state:

$$x(t) = [x_1(t), x_2(t), \ldots, x_M(t)] \in \mathbb{R}^n\,,\ n = \sum_{i=1}^{M} n_i \tag{9.2}$$

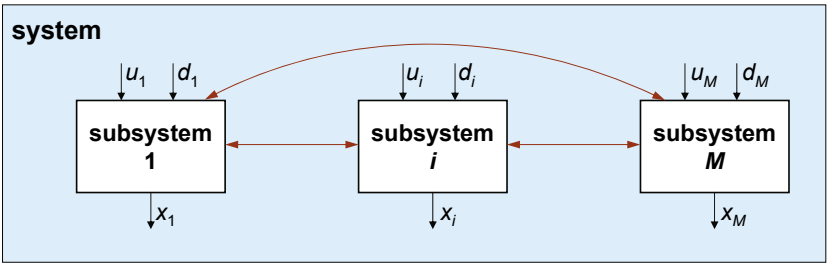

Fig. 9.1 System composed by the interconnection of M subsystems.

Similarly, the overall control input is denoted:

$$u(t) = [u_1(t), u_2(t), \ldots, u_M(t)] \in \mathbb{R}^m, \; m = \sum_{i=1}^{M} m_i \tag{9.3}$$

The following control input and state constraints are imposed on the subsystems:

$$u_{\min,i} \leq u_i(t) \leq u_{\max,i}, \; x_{\min,i} \leq x_i(t) \leq x_{\max,i}, \; i = 1, 2, \ldots, M \tag{9.4}$$

and the following assumptions are made [11]:

Assumption 9.1. *The functions f_i and g_i, $i=1, \ldots, M$ are C^1 functions with $f_i(0,0) = 0$, $g_i(0) = 0$.*

Assumption 9.2. $x_{\min,i} < 0 < x_{\max,i}$, $u_{\min,i} < 0 < u_{\max,i}$, $i = 1, \ldots, M$.

Assumption 9.3. *The disturbances d_i, $i = 1, \ldots, M$ are bounded by:*

$$|d_i(t)| \leq d_{\max,i}, \; i = 1, 2, \ldots, M \tag{9.5}$$

with $d_{\max,i} \in \mathbb{R}^{n_i}_{>0}$, $d_{\max,i} < |x_{\min,i}|$ and $d_{\max,i} < x_{\max,i}$, $i = 1, 2, \ldots, M$, where the operation $|\cdot|$ is taken element-wise.

It is supposed that a full measurement $x = [x_1, x_2, \ldots, x_M]$ of the overall state is available at the current time t. The optimal regulation problem is considered where the goal is to steer the overall state of the system (9.1) to the origin. For the current $x = [x_1, x_2, \ldots, x_M]$, the regulation NMPC solves the optimization problem [11]:

Problem 9.1 (Centralized NMPC):

$$V^{opt}(x) = \min_{U} J(U, x) \tag{9.6}$$

subject to $x_{t|t} = x$ and:

$$x_{i,t+k|t} \in \mathscr{X}_i\,,\ i=1,\ldots,M\,,\ k=1,\ldots,N-1 \tag{9.7}$$
$$u_{i,t+k} \in \mathscr{U}_i\,,\ i=1,\ldots,M\,,\ k=0,1,\ldots,N-1 \tag{9.8}$$
$$x_{i,t+k+1|t} = f_i(x_{i,t+k|t},u_{i,t+k}) + g_i(x_{t+k|t})\,,\ i=1,\ldots,M\,,\ k=0,1,\ldots,N-2 \tag{9.9}$$
$$x_{t+k|t} = [x_{1,t+k|t}, x_{2,t+k|t}, \ldots, x_{M,t+k|t}]\,,\ k=0,1,\ldots,N-1 \tag{9.10}$$
$$u_{t+k} = [u_{1,t+k}, u_{2,t+k}, \ldots, u_{M,t+k}]\,,\ k=0,1,\ldots,N-1 \tag{9.11}$$

with $U=[u_t, u_{t+1}, \ldots, u_{t+N-1}]$ and the cost function given by:

$$J(U,x) = \sum_{k=0}^{N-1}\sum_{i=1}^{M} l_i(x_{i,t+k|t},u_{i,t+k}) \tag{9.12}$$

Here, $l_i(x_{i,t+k|t},u_{i,t+k}) = \|x_{i,t+k|t}\|^2_{Q_i} + \|u_{i,t+k}\|^2_{R_i}$ is the stage cost for the i-th subsystem with weighting matrices $Q_i, R_i \succ 0$, and N is a finite horizon. The sets $\mathscr{X}_i$ and $\mathscr{U}_i$ are defined by:

$$\mathscr{X}_i = \{\lambda_i \in \mathbb{R}^{n_i} \,|\, x_{\min,i} + d_{\max,i} \le \lambda_i \le x_{\max,i} - d_{\max,i}\} \tag{9.13}$$
$$\mathscr{U}_i = \{\eta_i \in \mathbb{R}^{m_i} \,|\, u_{\min,i} \le \eta_i \le u_{\max,i}\} \tag{9.14}$$

It follows from (9.13)–(9.14) that $\mathscr{X}_i$ and $\mathscr{U}_i$ are convex (polyhedral) sets, which include the origin in their interior (due to Assumptions 9.2 and 9.3).

It should be noted that the state constraints (9.7) with the admissible set $\mathscr{X}_i$ defined by (9.13) guarantee the robust feasibility of the solution in sense that the original state constraints (9.4) will be satisfied for the worst-case disturbances.

9.3 Distributed NMPC for Interconnected Nonlinear Systems with Linear Couplings

9.3.1 Distributed NMPC by Dual Decomposition

Problem 9.1 can be decomposed by using the dynamic dual decomposition approach [4, 18]. First, the following assumption is made [11]:

Assumption 9.4. *The functions $g_i(x(t))$ have the form:*

$$g_i(x(t)) = \sum_{j=1,\,j\neq i}^{M} A_{ij}x_j(t)\,,\ i=1,\ldots,M \tag{9.15}$$

where $A_{ij} \in \mathbb{R}^{n_i\times n_j}$ are constant matrices.

The following *decoupled* state equations can be formulated:

$$x_i(t+1) = f_i(x_i(t),u_i(t)) + v_i(t) + d_i(t)\,,\ i=1,\ldots,M \tag{9.16}$$

with the additional constraints, [18]:

$$v_i(t) = \sum_{j=1,\, j\neq i}^{M} A_{ij} x_j(t)\,,\ i = 1, \ldots, M \ \text{ for all } t \tag{9.17}$$

The variable $v_i \in \mathbb{R}^{n_i}$ can be interpreted as the influence of the other subsystems in the update of x_i.

Then, similar to [18], the constraints (9.17) are relaxed by introducing the corresponding vectors of Lagrange multipliers $p_i \in \mathbb{R}^{n_i}$ in the cost function (9.12) and the Problem 9.1 is reformulated as a *distributed* NMPC problem [11]:

Problem 9.2 (Distributed NMPC):

$$\begin{aligned}
&V^{opt}(x) = \\
&\max_{P} \min_{U,X,V} \sum_{k=0}^{N-1} \sum_{i=1}^{M} [l_i(x_{i,t+k|t}, u_{i,t+k}) + p_{i,t+k}^T (v_{i,t+k} - \sum_{\substack{j=1 \\ j\neq i}}^{M} A_{ij} x_{j,t+k|t})] = \\
&\max_{P} \sum_{i=1}^{M} (\min_{U_i,X_i,V_i} \sum_{k=0}^{N-1} \underbrace{[l_i(x_{i,t+k|t}, u_{i,t+k}) + p_{i,t+k}^T v_{i,t+k} - x_{i,t+k|t}^T \sum_{\substack{j=1 \\ j\neq i}}^{M} A_{ji}^T p_{j,t+k}]}_{l_i^P(x_{i,t+k|t}, u_{i,t+k}, v_{i,t+k}, P)})
\end{aligned} \tag{9.18}$$

subject to $x_{t|t} = x$, constraints (9.7)–(9.8) and:

$$x_{i,t+k+1|t} = f_i(x_{i,t+k|t}, u_{i,t+k}) + v_{i,t+k}\,,\ i = 1, \ldots, M\,,\ k = 0, 1, \ldots, N-2 \tag{9.19}$$

$$p_{t+N-1} = 0 \tag{9.20}$$

Here:

$$\begin{aligned}
&P = [p_t, p_{t+1}, \ldots, p_{t+N-1}] \text{ with } p_{t+k} = [p_{1,t+k}, p_{2,t+k}, \ldots, p_{M,t+k}]\,, \\
&\quad k = 0, 1, \ldots, N-1 \\
&U = [u_t, u_{t+1}, \ldots, u_{t+N-1}]\,,\ U_i = [u_{i,t}, u_{i,t+1}, \ldots, u_{i,t+N-1}] \\
&X = [x_{t|t}, x_{t+1|t}, \ldots, x_{t+N-1|t}]\,,\ X_i = [x_{i,t|t}, x_{i,t+1|t}, \ldots, x_{i,t+N-1|t}] \\
&V = [v_t, v_{t+1}, \ldots, v_{t+N-1}]\,,\ V_i = [v_{i,t}, v_{i,t+1}, \ldots, v_{i,t+N-1}] \\
&\quad \text{with } v_{t+k} = [v_{1,t+k}, v_{2,t+k}, \ldots, v_{M,t+k}]\,,\ k = 0, 1, \ldots, N-1
\end{aligned} \tag{9.21}$$

The Lagrange multipliers P are also referred to as prices [18] and the Problem 9.2 can be interpreted as a game with two players for each subsystem. Given the prices, the objective of the first player for the i-th subsystem is to select the inputs $U_i = [u_{i,t}, u_{i,t+1}, \ldots, u_{i,t+N-1}]$ to minimize the local cost $\sum_{k=0}^{N-1} l_i^P(x_{i,t+k|t}, u_{i,t+k}, v_{i,t+k}, P)$. The other player for the i-th subsystem chooses $P_i = [p_{i,t}, p_{i,t+1}, \ldots, p_{i,t+N-1}]$ with the objective to maximize $\sum_{k=0}^{N-1} p_{i,t+k}^T (v_{i,t+k} - \sum_{j=1,\, j\neq i}^{M} A_{ij} x_{j,t+k|t})$.

The inner *decoupled* optimization problems in Problem 9.2 represent Nonlinear Programming (NLP) sub-problems corresponding to the NMPC of the i-th

subsystem, since the constraints (9.19) are nonlinear in the optimization variables. Each NLP sub-problem is presented as follows [11]:

Problem 9.3^i (i-th NLP sub-problem):

$$V_i^{opt}(P,x_i) = \min_{U_i,X_i,V_i} \sum_{k=0}^{N-1} l_i^P(x_{i,t+k|t}, u_{i,t+k}, v_{i,t+k}, P) \tag{9.22}$$

subject to $x_{i,t|t} = x_i$ and:

$$x_{i,t+k|t} \in \mathscr{X}_i\,,\ k = 1, \ldots, N-1 \tag{9.23}$$
$$u_{i,t+k} \in \mathscr{U}_i\,,\ k = 0, 1, \ldots, N-1 \tag{9.24}$$
$$x_{i,t+k+1|t} = f_i(x_{i,t+k|t}, u_{i,t+k}) + v_{i,t+k}\,,\ k = 0, 1, \ldots, N-2 \tag{9.25}$$

Denote with $U_i^{opt} = [u_{i,t}^{opt}, u_{i,t+1}^{opt}, \ldots, u_{i,t+N-1}^{opt}]$, $X_i^{opt} = [x_{i,t|t}^{opt}, x_{i,t+1|t}^{opt}, \ldots, x_{i,t+N-1|t}^{opt}]$ and $V_i^{opt} = [v_{i,t}^{opt}, v_{i,t+1}^{opt}, \ldots, v_{i,t+N-1}^{opt}]$ the optimal solution of Problem 9.3^i.

9.3.2 Local QP Approximations of the NLP Sub-problems

The cost functions $\sum_{k=0}^{N-1} l_i^P(x_{i,t+k|t}, u_{i,t+k}, v_{i,t+k}, P)$ in the Problems 9.3^i, $i = 1, 2, \ldots, M$ are convex, however the constraints (9.25) may be non-convex in the general case. In [11], the constraints (9.25) are locally approximated by linear constraints, leading to a *quasi-nonlinear* approach. Let $x_{i,t|t} = x_i^0 \in \mathscr{X}_i$ be arbitrary and denote the corresponding optimal solution to the sub-problem 9.3^i with:

$$\begin{aligned} U_i^0 &= U_i^{opt}(x_i^0) = [u_{i,t}^0, u_{i,t+1}^0, \ldots, u_{i,t+N-1}^0] \\ X_i^0 &= X_i^{opt}(x_i^0) = [x_{i,t|t}^0, x_{i,t+1|t}^0, \ldots, x_{i,t+N-1|t}^0] \\ V_i^0 &= V_i^{opt}(x_i^0) = [v_{i,t}^0, v_{i,t+1}^0, \ldots, v_{i,t+N-1}^0] \end{aligned} \tag{9.26}$$

The optimal solution (9.26) depends on the values of the prices P. In Section 9.3.3, it is described how P and the solution (9.26) are updated iteratively. Further, a first order truncated Taylor series expansions of the right-hand side of constraints (9.25) around the point $(U_i^0, X_i^0, V_i^0, x_i^0)$ lead to the locally linear constraints [11]:

$$\tilde{X}_i^+ = S_{X_i}(\tilde{X}_i - \tilde{X}_i^0) + S_{U_i}(\tilde{U}_i - \tilde{U}_i^0) + (\tilde{V}_i - \tilde{V}_i^0) + E_i^0 \tag{9.27}$$

where:

$$\begin{aligned} &\tilde{X}_i = [x_{i,t|t}, \ldots, x_{i,t+N-2|t}]\,,\ \tilde{X}_i^0 = [x_{i,t|t}^0, \ldots, x_{i,t+N-2|t}^0] \\ &\tilde{U}_i = [u_{i,t}, \ldots, u_{i,t+N-2}]\,,\ \tilde{U}_i^0 = [u_{i,t}^0, \ldots, u_{i,t+N-2}^0] \\ &\tilde{V}_i = [v_{i,t}, \ldots, v_{i,t+N-2}]\,,\ \tilde{V}_i^0 = [v_{i,t}^0, \ldots, v_{i,t+N-2}^0] \\ &\tilde{X}_i^+ = [x_{i,t+1|t}, \ldots, x_{i,t+N-1|t}] \end{aligned}$$

Here, the matrices S_{X_i}, S_{U_i}, and E_i^0 are given by:

$$S_{X_i} = \begin{bmatrix} \nabla_{x_i} f_i(x_{i,t|t}^0, u_{i,t}^0) & 0_{n_i} & \dots & 0_{n_i} \\ \vdots & & & \\ 0_{n_i} & 0_{n_i} & \dots & \nabla_{x_i} f_i(x_{i,t+N-2|t}^0, u_{i,t+N-2}^0) \end{bmatrix} \tag{9.28}$$

$$S_{U_i} = \begin{bmatrix} \nabla_{u_i} f_i(x_{i,t|t}^0, u_{i,t}^0) & 0_{n_i,m_i} & \dots & 0_{n_i,m_i} \\ \vdots & & & \\ 0_{n_i,m_i} & 0_{n_i,m_i} & \dots & \nabla_{u_i} f_i(x_{i,t+N-2|t}^0, u_{i,t+N-2}^0) \end{bmatrix} \tag{9.29}$$

$$E_i^0 = \begin{bmatrix} f_i(x_{i,t|t}^0, u_{i,t}^0) + v_{i,t}^0 \\ \vdots \\ f_i(x_{i,t+N-2|t}^0, u_{i,t+N-2}^0) + v_{i,t+N-2}^0 \end{bmatrix} \tag{9.30}$$

where 0_{n_i} is the n_i-dimensional square zero matrix and $0_{n_i,m_i}$ is the $n_i \times m_i$-dimensional zero matrix. It can be observed that (9.27) is a linear time-varying approximation of the constraints (9.25). Then, the NLP sub-problems 9.3^i for the subsystems are approximated with the QP sub-problems [11]:

Problem 9.4^i (i-th QP sub-problem):

$$V_i^*(P, x_i) = \min_{U_i, X_i, V_i} \sum_{k=0}^{N-1} l_i^P(x_{i,t+k|t}, u_{i,t+k}, v_{i,t+k}, P) \tag{9.31}$$

subject to $x_{i,t|t} = x_i$, (9.23), (9.24), and (9.27).

Denote with $U_i^* = [u_{i,t}^*, \dots, u_{i,t+N-1}^*]$, $X_i^* = [x_{i,t|t}^*, \dots, x_{i,t+N-1|t}^*]$ and $V_i^* = [v_{i,t}^*, \dots, v_{i,t+N-1}^*]$ the optimal solution of Problem 9.4^i. Then, the following *centralized* NMPC problem with linearized constraints is formulated [11]:

Problem 9.5 (Centralized NMPC with linearized constraints):

$$V^*(x) = \min_U J(U, x) \tag{9.32}$$

subject to $x_{t|t} = x$, constraints (9.7), (9.8), and:

$$\begin{aligned} x_{i,t+k+1|t} = {} & \nabla_{x_i} f_i(x_{i,t+k|t}^0, u_{i,t+k}^0)(x_{i,t+k|t} - x_{i,t+k|t}^0) + \\ & \nabla_{u_i} f_i(x_{i,t+k|t}^0, u_{i,t+k}^0)(u_{i,t+k} - u_{i,t+k}^0) + \sum_{j=1, j\neq i}^{M} A_{ij}(x_{j,t+k|t} - x_{j,t+k|t}^0) \\ & + f_i(x_{i,t+k|t}^0, u_{i,t+k}^0) + \sum_{j=1, j\neq i}^{M} A_{ij} x_{j,t+k|t}^0 \\ & i = 1, \dots, M,\ k = 0, 1, \dots, N-2 \end{aligned} \tag{9.33}$$

where the cost function $J(U,x)$ is given by (9.12). Here, (9.33) includes the constraints due to the linearized dynamics for all subsystems.

Then, the *distributed* NMPC problem with linearized dynamic constraints is as follows [11]:

Problem 9.6 (Distributed NMPC with linearized constraints):

$$\max_{P} \sum_{i=1}^{M} V_i^*(P,x_i) = \max_{P} \sum_{i=1}^{M} \min_{U_i,X_i,V_i} \sum_{k=0}^{N-1} l_i^P(x_{i,t+k|t}, u_{i,t+k}, v_{i,t+k}, P) \tag{9.34}$$

subject to $x_{t|t} = x$, constraints (9.7), (9.8), and:

$$\tilde{X}_i^+ = S_{X_i}(\tilde{X}_i - \tilde{X}_i^0) + S_{U_i}(\tilde{U}_i - \tilde{U}_i^0) + (\tilde{V}_i - \tilde{V}_i^0) + E_i^0,\ i = 1, \ldots, M \tag{9.35}$$

$$p_{t+N-1} = 0 \tag{9.36}$$

Then, the decomposition of the optimization Problem 9.5 is given by the following proposition [11]:

Proposition 9.1. *Suppose that $x = [x_1, x_2, \ldots, x_M]$ is a feasible point for Problem 9.5. Then:*

$$V^*(x) = \max_{P} \sum_{i=1}^{M} V_i^*(P,x_i) \tag{9.37}$$

where maximization is subject to $p_{t+N-1} = 0$.

Proof. [11] The proof follows similar arguments as in [9]. Since the stage cost functions $l_i(x_{i,t+k|t}, u_{i,t+k})$, $i = 1, 2, \ldots, M$ are convex, from the duality theory [2] it follows that there is no duality gap between the dual Problem 9.6 and the Problem 9.5. The requirement $p_{t+N-1} = 0$ follows from the optimality conditions of Pontryagin's maximum-principle for discrete-time systems [3] and the fact that the state is not specified at the terminal time $t+N-1$. Therefore, (9.37) holds. Further, the maximum in (9.37) is attained when all elements of the gradient of $\sum_{i=1}^{M} V_i^*(P,x_i)$ with respect to P are zero, i.e. $v_{i,t+k}^* - \sum_{j=1,\, j\neq i}^{M} A_{ij} x_{j,t+k|t}^* = 0$, $i = 1, \ldots, M$, $k = 0, 1, \ldots, N-1$. This means that the constraints (9.17) are satisfied at the optimum.

Proposition 9.1 shows that the computation of U_i^*, X_i^* and V_i^* for given prices P is completely *decentralized*. However, as described in [9], finding the optimal prices requires coordination. According to the duality theory [2], $V_i^*(P,x_i)$, $i = 1, \ldots, M$ are concave functions of P. Therefore, the optimal price sequence $P^* = [p_t^*, p_{t+1}^*, \ldots, p_{t+N-1}^*]$ can be found as the limits of a gradient iteration [9]. Given a price prediction sequence $P_i^r = [p_{i,t}^r, \ldots, p_{i,t+N-1}^r]$ for the r-th iteration, the corresponding sequences $U_i^{*r} = [u_{i,t}^{*r}, \ldots, u_{i,t+N-1}^{*r}]$, $X_i^{*r} = [x_{i,t|t}^{*r}, \ldots, x_{i,t+N-1|t}^{*r}]$ and $V_i^{*r} = [v_{i,t}^{*r}, \ldots, v_{i,t+N-1}^{*r}]$ are computed locally by solving Problem 9.4^i. Then, the prices can be updated *distributedly* by a gradient step:

$$p_{i,t+k}^{r+1} = p_{i,t+k}^{r} + \gamma_i^r (v_{i,t+k}^{*r} - \sum_{\substack{j=1 \\ j\neq i}}^{M} A_{ij} x_{j,t+k|t}^{*r}),\ k = 0, 1, \ldots, N-2$$

$$\text{with } p_{i,t+N-1}^{r+1} = p_{i,t+N-1}^{r} = 0 \tag{9.38}$$

It should be noted that Proposition 9.1 holds only locally due to linearization, in a neighborhood of the optimal solution U_i^0, X_i^0, V_i^0 to sub-problems 9.3^i, $i = 1, 2, \ldots, M$, where the linear constraints (9.33) can sufficiently accurately approximate the nonlinear constraints (9.19). Therefore, it would be necessary to periodically update the linear constraints (9.33) and then to apply formula (9.38) for a number of steps.

9.3.3 A Suboptimal Approach to Distributed NMPC Based on On-Line Optimization

In [9], an approach to *distributed* MPC for *linear* systems in the absence of disturbances has been suggested, where the prices are updated according to (9.38). In [11], a suboptimal algorithm to *distributed* quasi-NMPC is proposed that considers a more general class of systems, since it refers to *nonlinear* systems with linear couplings in the presence of bounded disturbances (see Section 9.2). The suggested algorithm includes two loops. In the outer loop, the NLP sub-problems 9.3^i, $i = 1, 2, \ldots, M$, are solved and the matrices of the linear constraints of the approximating QP sub-problems 9.4^i, $i = 1, 2, \ldots, M$ are computed. Then, in the inner loop, the price sequences and solution are updated based on Proposition 9.1 and applying formula (9.38) for a given number of steps. The algorithm is described by [11]:

Algorithm 9.1. Distributed quasi-NMPC by on-line optimization.

---

1. Given numbers Q and L, step sizes γ_i, $i = 1, 2, \ldots, M$ and arbitrary guesses P_i^0, $i = 1, 2, \ldots, M$ for the price sequences. Let $t = 0$.
2. Let the state at time t be $x(t) = x = [x_1, \ldots, x_M]$.
3. for $q = 1, 2, \ldots, Q$ **do**
4. For $x_{i,t|t} = x_i$ compute *distributedly* the optimal solutions $U_i^0 = U_i^{opt}(x_i)$, $X_i^0 = X_i^{opt}(x_i)$, $V_i^0 = V_i^{opt}(x_i)$ to the NLP sub-problems 9.3^i, $i = 1, 2, \ldots, M$, corresponding to the price sequences $P_i^0 = [p_{i,t}^0, \ldots, p_{i,t+N-1}^0]$. Compute the matrices S_{X_i}, S_{U_i}, and E_{0i} associated to the approximating QP sub-problems 9.4^i, $i = 1, 2, \ldots, M$.
5. **for** $r = 0, 1, \ldots, L-1$ **do**
6. For i-th subsystem, $i = 1, 2, \ldots, M$, communicate the price sequences $P_j^r = [p_{j,t}^r, \ldots, p_{j,t+N-1}^r]$, $j = 1, \ldots, M$, $j \neq i$ of the interconnected subsystems.
7. Compute the sequences $U_i^{*r} = [u_{i,t}^{*r}, \ldots, u_{i,t+N-1}^{*r}]$, $X_i^{*r} = [x_{i,t|t}^{*r}, \ldots, x_{i,t+N-1|t}^{*r}]$

and $V_i^{*r} = [v_{i,t}^{*r}, \ldots, v_{i,t+N-1}^{*r}]$ corresponding to the price sequence $P^r = [p_t^r, \ldots, p_{t+N-1}^r]$ by solving *distributedly* the QP sub-problems 9.4^i, $i = 1, 2, \ldots, M$.

8. For i-th subsystem, $i = 1, 2, \ldots, M$, communicate the state trajectories $X_j^{*r} = [x_{j,t|t}^{*r}, x_{j,t+1|t}^{*r}, \ldots, x_{j,t+N-1|t}^{*r}]$, $j = 1, \ldots, M$, $j \neq i$ of the interconnected subsystems.

9. Compute *distributedly* the updates $P_i^{r+1} = [p_{i,t}^{r+1}, \ldots, p_{i,t+N-1}^{r+1}]$, $i = 1, 2, \ldots, M$ of the price sequences by applying (9.38) for $\gamma_i^r = \gamma_i$, $i = 1, 2, \ldots, M$.

10. end

11. Let $P_i^0 = P_i^L$, $i = 1, 2, \ldots, M$.

12. end

13. Apply to the overall system the control inputs $u_i(t) = u_{i,t}^{*L-1}$, $i = 1, 2, \ldots, M$.

14. Let $t = t + 1$ and go to step 2.

The steps 4 to 11 in Algorithm 9.1 include an iterative solution of the NLP sub-problems 9.3^i, approximating them with the QP sub-problems 9.4^i, and then updating the prices by utilizing Proposition 9.1.

It should be noted that alternatively, an approach similar to [14, 16] can be applied, where the idea would be to avoid solving the NLP sub-problems 9.3^i in step 4 and to formulate the approximating QP sub-problems 9.4^i by using the optimal sequences U_i^*, X_i^* and V_i^*, computed in the previous time instant.

9.4 A Semi-explicit Approach to Efficient Distributed NMPC for Interconnected Systems with Linear Couplings

Although the original *centralized* NMPC problem (Problem 9.1) has been represented as a *distributed* quasi-NMPC problem (Problem 9.6), its approximate solution with Algorithm 9.1 may still require significant computational efforts. This is mainly due to step 4, where the NLP sub-problems 9.3^i, $i = 1, 2, \ldots, M$ are solved and approximated with QP sub-problems. Therefore, there is a motivation to precompute off-line the optimal price sequence $P^* = [p_t^*, p_{t+1}^*, \ldots, p_{t+N-1}^*]$ as an *explicit* function of the overall state x by applying a parametric programming approach. However, it is known that the off-line computational complexity with the *explicit* approach tends to increase rapidly with the number of states and thus they can be applied only to small-scale processes. Therefore, the use of a *semi-explicit* approach to efficiently solve *distributed* NMPC problems for interconnected nonlinear systems (which are supposed to be of medium-scale) is more appropriate. The idea of the *semi-explicit* approaches has been used in [21] to solve *centralized linear* MPC problems and later applied to *nonlinear* MPC formulations [10]. The *semi-explicit* approaches [21, 10] combine the two paradigms of *explicit* and *on-line* MPC in order to overcome their individual limitations. They consist in using a piecewise linear (PWL) approximation of the optimal control law (which is computed off-line) to warm-start the on-line optimization.

9.4.1 Formulation of the Distributed NMPC Problem as an mp-NLP Problem

Here, a *semi-explicit* NMPC approach to efficiently solve the *distributed* NMPC problem (Problem 9.2) is proposed. With off-line computations, an approximate explicit solution $\hat{P}(x)$ for the price sequences is obtained. Then, on-line, this solution is used as an initial guess (warm start) for a gradient iterations algorithm, similar to Algorithm 9.1. The expected result would be a decrease of the number of iterations in the gradient algorithm and thus of the on-line computational efforts. The following multi-parametric Nonlinear Programming (mp-NLP) problem is solved:

Problem 9.7 (mp-NLP):

$$V^{opt}(x) = \max_{P} \sum_{i=1}^{M} V_i^{opt}(P, x_i) \text{ subject to } p_{t+N-1} = 0 \tag{9.39}$$

In general, $V_i^{opt}(P, x_i)$ are nonlinear functions of the initial state x_i because they are obtained by solving the NLP sub-problems 9.3^i, $i = 1, 2, \ldots, M$ where the dynamic equality constraints are eliminated using direct single shooting. Thus, the Problem 9.7 is a mp-NLP problem since it is a NLP problem in P parameterized by x [8]. Define the set of N-step feasible initial states as follows:

$$X_f = \{x \in \mathbb{R}^n \,|\, \text{Problems } 9.3^i,\ i = 1, \ldots, M \text{ are feasible for some } P \in \mathbb{R}^{Nn}\} \tag{9.40}$$

If Problem 9.2 is feasible, then X_f is a non-empty set. The purpose of the *semi-explicit* approach is to obtain first an approximate explicit solution $\hat{P}(x)$ to Problem 9.7 in some set $X \subseteq X_f \subseteq \mathbb{R}^n$ by applying an approximate mp-NLP method.

9.4.2 Approximate mp-NLP Approach to Semi-explicit Distributed NMPC

Let $X \subset \mathbb{R}^n$ be a hyper-rectangle where we seek to approximate the optimal solution $P^{opt}(x)$ to the Problem 9.7. The approximate solution $\hat{P}(x)$ to $P^{opt}(x)$ is found by applying the approximate mp-NLP approach, described in Chapter 1.Thus, it is required that the state space partition is orthogonal and can be represented as a $k-d$ tree. The idea is to construct a piecewise linear (PWL) approximation $\hat{P}(x)$ to $P^{opt}(x)$ on X, where the constituent affine functions are defined on hyper-rectangles covering X. The computation of an affine approximation $\hat{P}_0(x) = K_0 x + g_0$, associated to a given region X_0, includes the following steps. First, the optimal solution of Problem 9.7 is computed at the vertices and the center point of X_0. Then, based on the solutions at these points, a local linear approximation $\hat{P}_0(x)$ to the optimal solution $P^{opt}(x)$, valid in the whole hyper-rectangle X_0, is determined. By taking into

account that the constraint $p_{t+N-1} = 0$ should be satisfied, K_0 and g_0 should have the form:

$$K_0 = \begin{bmatrix} \tilde{K}_0 \\ 0_{1,n} \end{bmatrix}, \; g_0 = \begin{bmatrix} \tilde{g}_0 \\ 0 \end{bmatrix} \tag{9.41}$$

where $0_{1,n}$ is a zero vector with dimension $1 \times n$. Then, $\tilde{K}_0$ and $\tilde{g}_0$ are determined by applying the following procedure:

Procedure 9.1 (Computation of explicit approximate solution). *Consider any hyper-rectangle $X_0 \subseteq X$ with center point w_0 and vertices $\{w_1, w_2, \ldots, w_{2^n}\}$. Compute K_0 and g_0 by solving the following NLP:*

$$\min_{\tilde{K}_0, \tilde{g}_0} \sum_{q=0}^{2^n} \Big(\sum_{i=1}^{M} V_i^{opt}(K_0 w_q + g_0, w_q^i) - V^{opt}(w_q) + \mu \| K_0 w_q + g_0 - P^{opt}(w_q) \|_2^2 \Big) \tag{9.42}$$

subject to (9.41).

In (9.42), w_q is the value of the overall state, i.e. $w_q = [w_q^1, w_q^2, \ldots, w_q^M]$, while w_q^i is the value of the state associated to the i-th subsystem. Also in (9.42), $V_i^{opt}(K_0 w_q + g_0, w_q^i)$ is the i-th optimal local cost obtained by solving the local NLP Problem 9.3^i for $P = K_0 w_q + g_0$, $V^{opt}(w_q)$ denotes the optimal cost associated to the whole system, corresponding to the optimal solution $P^{opt}(w_q)$, i.e. $V^{opt}(w_q) = \sum_{i=1}^{M} V_i^{opt}(P^{opt}(w_q), w_q^i)$, and the parameter $\mu > 0$ is a weighting coefficient.

After a linear approximation $\hat{P}_0(x) = K_0 x + g_0$ has been determined, an estimate $\hat{\varepsilon}_0$ of the maximal cost function approximation error ε_0 in X_0 is computed as follows:

$$\hat{\varepsilon}_0 = \max_{q \in \{0,1,2,\ldots,2^n\}} \Big(\sum_{i=1}^{M} V_i^{opt}(K_0 w_q + g_0, w_q^i) - V^{opt}(w_q) \Big) \tag{9.43}$$

If the maximal cost function approximation error ε_0 in X_0 is greater than a specified tolerance $\bar{\varepsilon} > 0$, the region X_0 is split and the above procedure is repeated for the new regions, as described in Chapter 1.

After an approximate PWL solution $\hat{P}(x)$ for the price sequences has been found, the control inputs are determined on-line by applying a modification of Algorithm 9.1 (where the approximate solution $\hat{P}(x)$ is used as a warm start). It should be noted that with the *semi-explicit* approach the number of gradient iterations at which the NLP Problems 9.3^i, $i = 1, 2, \ldots, M$ are solved can be smaller than that with the approach based entirely on on-line optimization.

Although the *semi-explicit* approach is characterized with less off-line computational complexity compared to the purely explicit approach, the complexity would increase with the number of the parameters in the mp-NLP problem [10]. Therefore, the application of the proposed *semi-explicit* approach is restricted to small- and medium-scale systems.

9.5 Application: Distributed NMPC of a Nonlinear System Consisting of Two Sub-systems

Consider the following second order system composed of two subsystems S_1 and S_2 [15]:

$$S_1 : x_1(t+1) = \sqrt{x_1(t)^2+1} + u_1(t) - 1 + \eta_1 x_2(t) + d_1(t) \tag{9.44}$$

$$S_2 : x_2(t+1) = e^{-\sin(x_2(t))} + u_2(t) - 1 + \eta_2 x_1(t) + d_2(t) \tag{9.45}$$

Thus, the functions f_i, g_i, $i = 1, 2$ in the formulation (9.1) are:

$$f_1(x_1(t), u_1(t)) = \sqrt{x_1(t)^2+1} + u_1(t) - 1 \tag{9.46}$$

$$f_2(x_2(t), u_2(t)) = e^{-\sin(x_2(t))} + u_2(t) - 1 \tag{9.47}$$

$$g_1(x(t)) = \eta_1 x_2(t)\,,\ g_2(x(t)) = \eta_2 x_1(t) \tag{9.48}$$

The functions g_i satisfy Assumption 9.4 and they describe the mutual influence of the two subsystems. The disturbances are assumed to be the states of the following asymptotically stable first order systems [15]:

$$d_i(t+1) = 0.9 d_i(t)\,,\ i = 1, 2 \tag{9.49}$$

and they are bounded by:

$$|d_i(t)| \leq 1\,,\ i = 1, 2 \tag{9.50}$$

The following constraints are imposed on the system (9.44)–(9.45):

$$-0.3 \leq u_i(t) \leq 0.5\,,\ i = 1, 2 \tag{9.51}$$

$$-2 \leq x_1(t) \leq 2\,,\ -1.5 \leq x_2(t) \leq 1.5 \tag{9.52}$$

It should be noted that here the input constraints are different from those in [15]. Also in addition, state constraints are imposed on the system (while in [15] only input constraints are considered). The coefficients related to the couplings between the two subsystems are $\eta_1 = \eta_2 = 0.4$. The prediction horizon in the *centralized* NMPC problem (Problem 9.1) is $N = 5$ and the weighting matrices are $Q_i = R_i = 1$, $i = 1, 2$.

9.5.1 Results with the Distributed NMPC Based on On-Line Optimization

The *centralized* NMPC problem is represented as a *distributed* NMPC problem (Problem 9.6) by applying the dual decomposition approach. Then, Algorithm 9.1 with parameters $Q = 5$, $L = 3$, $\gamma_i = 0.3$, $i = 1, 2$ is used to generate the two control inputs for an initial state $x(0) = [0.3\ 0.3]$ and initial disturbances $d(0) = [1\ 1]$. The corresponding trajectories of the prices p_1, p_2, the control inputs u_1, u_2, the states

x_1, x_2 and the disturbances d_1, d_2 associated to the two subsystems are depicted in Fig. 9.2 to Fig. 9.5. The trajectories obtained with the following approaches are compared:

- The *suboptimal* distributed NMPC approach *with* linearization of the nonlinear constraints (9.25) (described in Section 9.3);
- A *suboptimal* distributed NMPC approach *without* linearization of the nonlinear constraints (9.25). In this case, a modification of Algorithm 9.1 is used for the on-line computation of the control inputs. It has only one loop, where the optimal solutions of the NLP sub-problems 9.3^i, $i = 1, 2, \ldots, M$ are computed *distributedly*, and then the price sequences are updated by applying (9.38) by using the computed optimal solutions. The loop is repeated $Q = 5$ times and the step size in (9.38) is $\gamma_i = 0.3$, $i = 1, 2$.
- The *exact* distributed NMPC approach, which solves Problem 9.6 at each time instant.
- The *centralized* NMPC approach, which solves Problem 9.1 at each time instant.

The computational complexity of both *suboptimal distributed* NMPC approaches is compared to that of the *exact distributed* NMPC approach and the *centralized* NMPC approach. For this aim, the respective trajectories are determined for 100 initial states, obtained by gridding the state space $[-1,\ 1] \times [-1,\ 1]$. The results are presented in Table 9.1, where also the possibility of these approaches to find a feasible solution is compared and expressed in terms of percentage from the total number of initial states. The computations are performed on a 3 GHz Intel Core 2 Duo processor.

Table 9.1 Comparison of different NMPC approaches

Method	Percentage of feasible solutions	Average CPU time [s]	Maximal CPU time [s]
Suboptimal distributed NMPC *without* linearization	79 %	0.66	0.74
Suboptimal distributed NMPC *with* linearization	79 %	1.81	1.86
Exact distributed NMPC	79 %	10.28	14.86
Centralized NMPC	28 %	0.66	2.30

It can be seen from Table 9.1 that the *suboptimal distributed* NMPC approach *without* linearization of the nonlinear constraints is the most computationally efficient approach. Both *suboptimal* approaches are more efficient in comparison to the *exact distributed* NMPC approach. A disadvantage of the *centralized* approach is that it fails to find a feasible solution for a significant number of initial states (in this sense it is about three times less efficient than the *distributed* NMPC approaches). A possible reason for this is the fact that in the presence of disturbances, the worst-case state constraints related to both subsystems (constraints (9.7) in the Problem 9.1) can be difficult to be satisfied at the same time for some of the initial states.

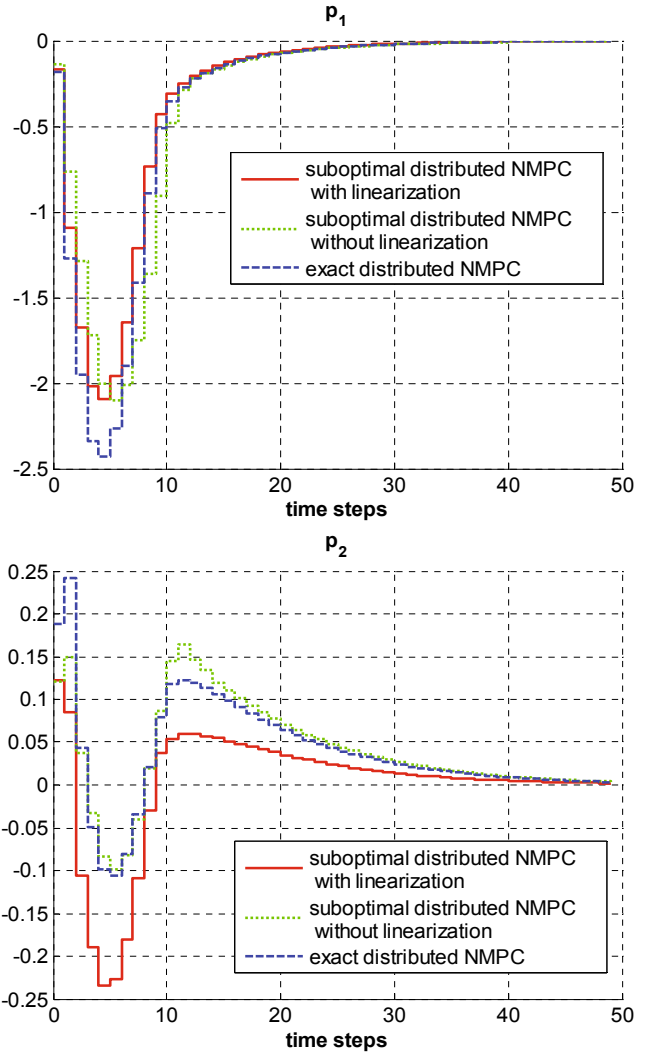

Fig. 9.2 The prices p_1 and p_2.

In contrast, the *distributed* NMPC approaches lead to two completely *decentralized* NMPC problems, where worst-case disturbance assumptions of only the individual subsystems are made (not simultaneously).

9.5.2 Results with the Semi-explicit Distributed NMPC

The *semi-explicit* approach from Section 9.4 is used to obtain an approximate PWL solution $\hat{P}(x)$ for the price sequences. The state space partition of this solution is

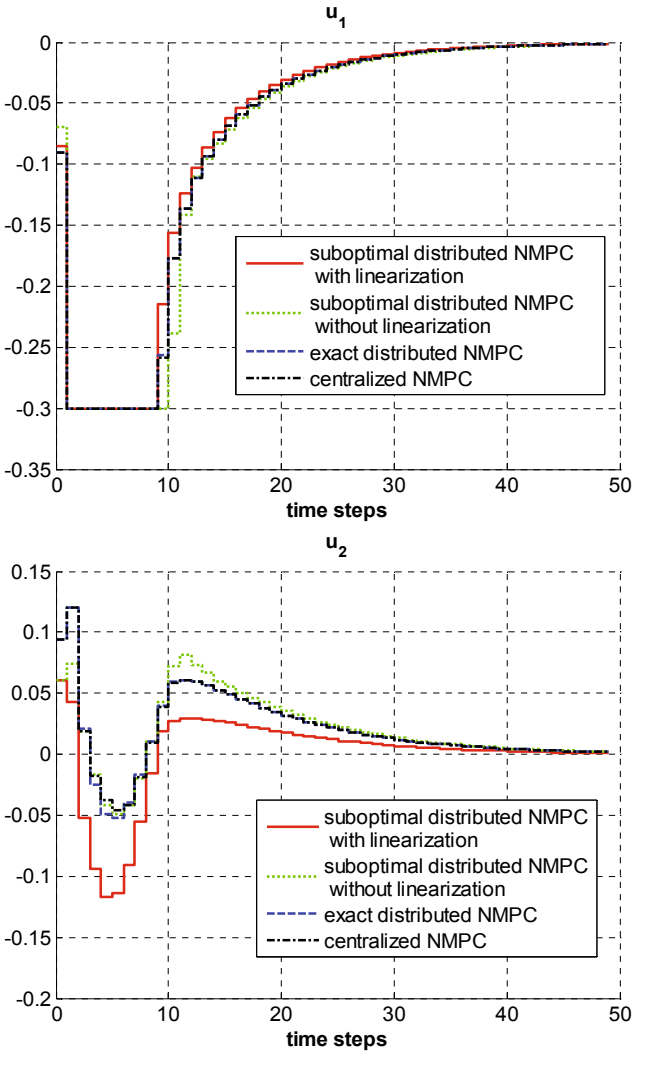

Fig. 9.3 The control inputs u_1 and u_2 for subsystems S_1 and S_2.

shown in Fig. 9.6. Then on-line, a modified version of Algorithm 9.1 is used to generate the two control inputs for the initial state $x(0) = [0.3\ 0.3]$ and initial disturbances $d(0) = [1\ 1]$. In the modified Algorithm 9.1, the number of gradient iterations, performed on-line is $Q = 3$ and the step sizes are $\gamma_i = 0.3, i = 1, 2$. The corresponding trajectories of the prices p_1, p_2, the control inputs u_1, u_2, and the states x_1, x_2, associated to the two subsystems, are depicted in Fig. 9.7 to Fig. 9.9. For comparison, in Fig. 9.7 to Fig. 9.9, the trajectories, obtained with the distributed NMPC approach, based entirely on *on-line* optimization (with number of gradient

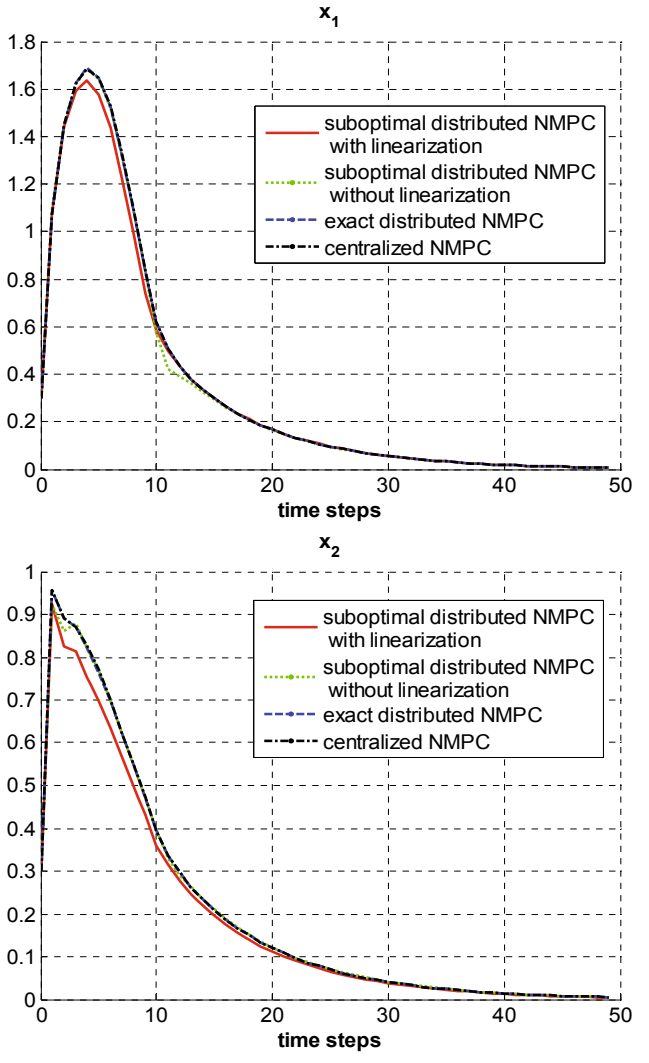

Fig. 9.4 The states x_1 and x_2 of subsystems S_1 and S_2.

iterations $Q = 5$ and step sizes $\gamma_i = 0.3$, $i = 1, 2$) and those, obtained with the exact distributed NMPC are also shown.

In Table 9.2, the computational complexity of the *semi-explicit* suboptimal distributed NMPC and the *on-line* optimization-based distributed NMPC is compared, as well as their possibility to find a feasible solution and the average accumulated cost function value. For this aim, the respective trajectories are determined for 100 initial states, obtained by gridding the state space $[-1, 1] \times [-1, 1]$. The computations are performed on a 3 GHz Intel Core 2 Duo processor.

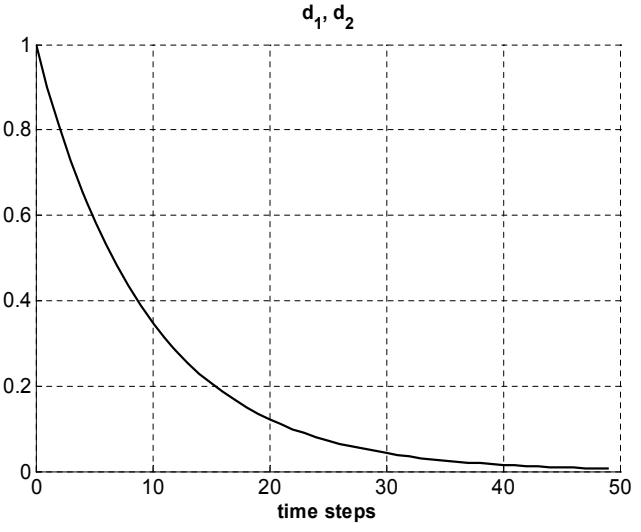

Fig. 9.5 The disturbances d_1, d_2.

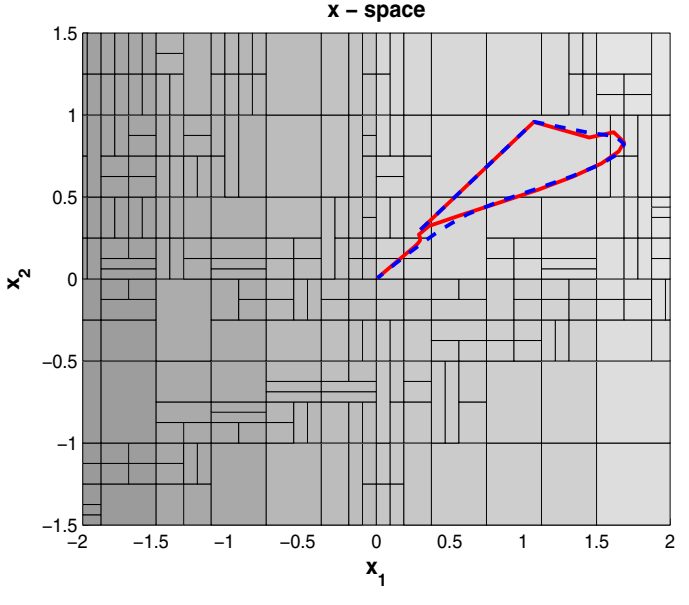

Fig. 9.6 State space partition associated to $\hat{P}(x)$ and state trajectories obtained with the semi-explicit distributed NMPC with number of gradient iterations $Q = 3$ (the solid curve) and with the exact distributed NMPC (the dashed curve).

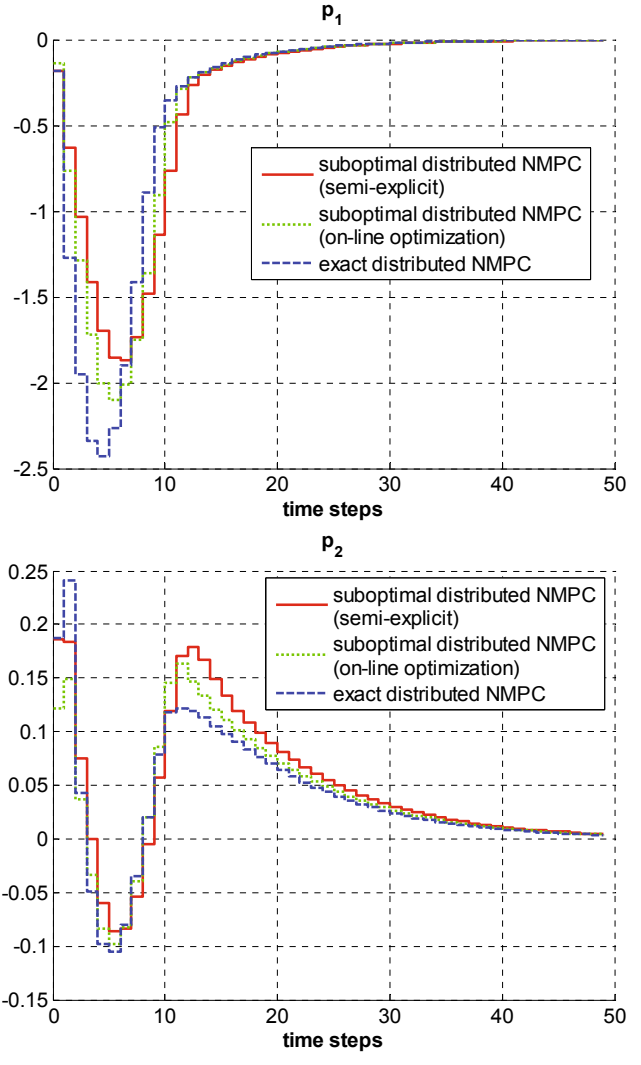

Fig. 9.7 The prices p_1 and p_2.

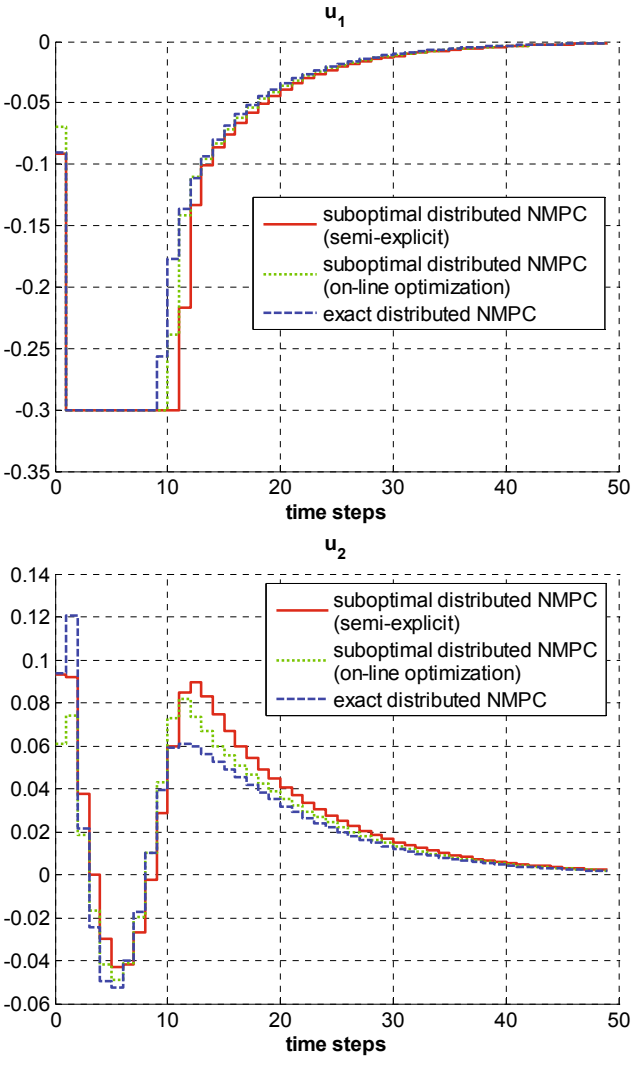

Fig. 9.8 The control inputs u_1 and u_2 for subsystems S_1 and S_2.

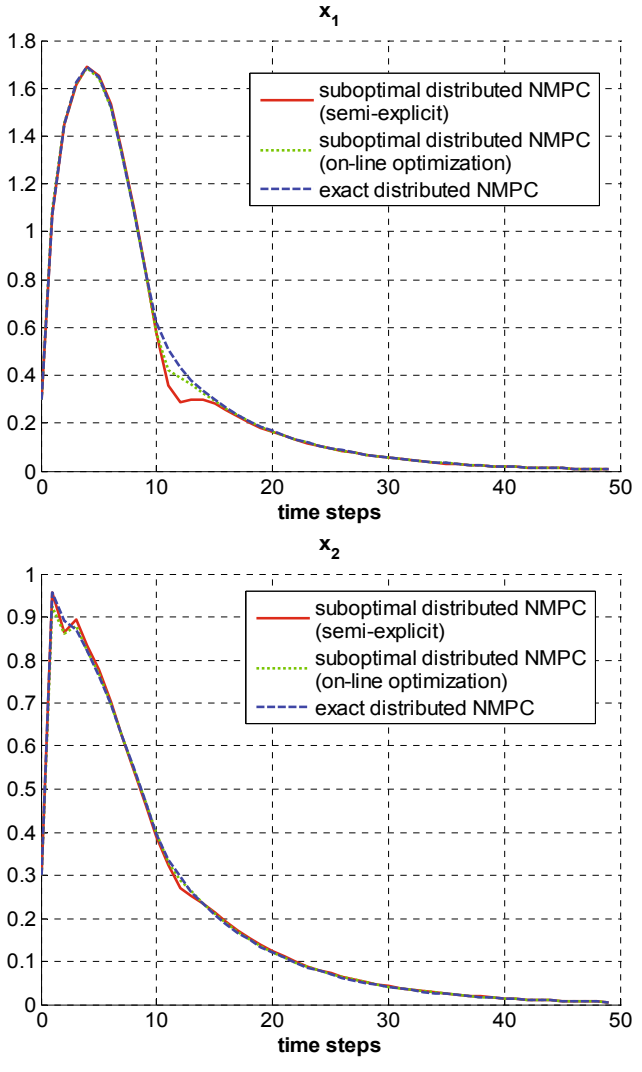

Fig. 9.9 The states x_1 and x_2 of subsystems S_1 and S_2.

Table 9.2 Comparison of the semi-explicit distributed NMPC approach (with number of gradient iterations $Q = 3$) and the on-line optimization-based distributed NMPC approach (with number of gradient iterations $Q = 5$)

Method	Percentage of feasible solutions	Average accumulated cost function value	Average CPU time [s]	Maximal CPU time [s]
Semi-explicit distributed NMPC with $Q = 3$	74 %	25.31	0.39	0.46
On-line optimization -based distributed NMPC with $Q = 5$	79 %	25.35	0.66	0.74

It can be seen from Table 9.2 that the *semi-explicit* suboptimal distributed NMPC approach allows to use less number of gradient iterations in comparison to the distributed NMPC approach based entirely on *on-line* optimization. Thus, it leads to a decrease in the on-line computational complexity, while keeping nearly the same ability to find a feasible solution and control quality (in terms of the average accumulated cost function value).

References

1. Arrow, K.J., Hurwicz, L., Uzawa, H.: Studies in linear and nonlinear programming. Stanford University Press (1958)
2. Boyd, S., Vandenberghe, L.: Convex optimization. Cambridge University Press (2004)
3. Bryson, A., Ho, Y.: Applied optimal control: Optimization, estimation and control. Blaisdell Publ. Co., Waltham (1969)
4. Cohen, G., Miara, B.: Optimization with an auxiliary constraint and decomposition. SIAM Journal on Control and Optimization 28, 137–157 (1990)
5. Constantinides, G.A.: Parallel architectures for model predictive control. In: Proceedings of the European Control Conference, Budapest, Hungary, pp. 138–143 (2009)
6. Dantzig, G.B., Wolfe, P.: The decomposition algorithm for linear programs. Econometrica 29, 767–778 (1961)
7. Dunbar, W.B., Murray, R.M.: Distributed receding horizon control for multi-vehicle formation stabilization. Automatica 42, 549–558 (2006)
8. Fiacco, A.V.: Introduction to sensitivity and stability analysis in nonlinear programming. Academic Press, Orlando (1983)
9. Giselsson, P., Rantzer, A.: Distributed model predictive control with suboptimality and stability guarantees. In: Proceedings of the Conference on Decision and Control, Atlanta, GA, pp. 7272–7277 (2010)
10. Grancharova, A., Johansen, T.A.: Approaches to explicit nonlinear model predictive control with reduced partition complexity. In: Proceedings of the European Control Conference, Budapest, Hungary, pp. 2414–2419 (2009)
11. Grancharova, A., Johansen, T.A.: Distributed quasi-nonlinear model predictive control by dual decomposition. In: Proceedings of the 18th IFAC World Congress, Milano, Italy, pp. 1429–1434 (2011), `http://www.IFAC-PapersOnLine.net`

12. Keviczky, T., Borrelli, F., Balas, G.J.: Decentralized receding horizon control for large scale dynamically decoupled systems. Automatica 42, 2105–2115 (2006)
13. Kim, T.H., Sugie, T.: Robust decentralized MPC algorithm for a class of dynamically interconnected constrained systems. In: Proceedings of the Conference on Decision and Control, Seville, Spain, pp. 290–295 (2005)
14. Li, W.C., Biegler, L.T.: Multistep, Newton-type control strategies for constrained nonlinear processes. Chem. Eng. R & Des. 67, 562–577 (1989)
15. Magni, L., Scattolini, R.: Stabilizing decentralized model predictive control of nonlinear systems. Automatica 42, 1231–1236 (2006)
16. Oliveira, N.M.C., Biegler, L.T.: An extension of Newton-type algorithms for nonlinear process control. Automatica 31, 281–286 (1995)
17. Rantzer, A.: On prize mechanisms in linear quadratic team theory. In: Proceedings of the Conference on Decision and Control, New Orleans, pp. 1112–1116 (2007)
18. Rantzer, A.: Dynamic dual decomposition for distributed control. In: Proceedings of the American Control Conference, St. Louis, MO, USA, pp. 884–888 (2009)
19. Scattolini, R.: Architectures for distributed and hierarchical Model Predictive Control – A review. Journal of Process Control 19, 723–731 (2009)
20. Venkat, A.N., Rawlings, J.B., Wright, S.J.: Plant-wide optimal control with decentralized MPC. In: Proceedings of the IFAC Conference on Dynamics and Control of Process Systems, Boston, MA (2004)
21. Zeilinger, M., Jones, C.N., Morari, M.: Real-time suboptimal model predictive control using a combination of explicit MPC and online optimization. In: Proceedings of the IEEE Conference on Decision and Control, Cancun, Mexico (2008)
22. Zhang, Y., Li, S.: Networked model predictive control based on neighbourhood optimization for serially connected large-scale processes. Journal of Process Control 17, 37–50 (2007)