Chapter 7
Explicit Stochastic NMPC

Abstract. This chapter considers two approaches to explicit stochastic NMPC of general constrained nonlinear discrete-time systems in the presence of disturbances and/or parameter uncertainties with known probability distributions. In Section 7.2, an approach to explicit solution of *closed-loop* (*feedback*) stochastic NMPC problems for constrained nonlinear systems, described by stochastic *parametric* models, is considered. The approach constructs a piecewise nonlinear (PWNL) approximation to the optimal sequence of feedback control policies. It is applied to design an explicit *feedback* stochastic NMPC controller for the cart and spring system. In Section 7.3, an explicit approximate approach to *open-loop* stochastic NMPC based on Gaussian process models is presented. The Gaussian process models are *non-parametric* probabilistic black-box models, whose advantage in comparison to the stochastic *parametric* models is that they provide information about the prediction uncertainty. The approach in Section 7.3 constructs a piecewise linear (PWL) approximation to the optimal control sequence and it is applied to design an explicit stochastic NMPC reference tracking controller for a combustion plant.

7.1 Introduction

Mathematical models of engineering systems usually contain some amount of uncertainty (typically unknown additive disturbances and/or uncertain model parameters). In the robust MPC problem formulation, the model uncertainty is taken into account. In some applications, the system to be controlled is described by a stochastic model where the probabilistic distribution of the uncertainty is assumed to be known. Several approaches for constrained *open-loop* MPC based on stochastic *parametric* models are proposed in [32, 44, 46, 31, 7, 8, 5, 6, 24]. The approaches [32, 44, 46] are based on linear state space models with stochastic parameters and/or additive noise and they optimize the expected value of the cost function subject to hard input constraints [32] or probabilistic constraints [44, 46]. In [31, 7, 8, 5], stochastic linear MPC approaches incorporating a probabilistic cost and probabilistic constraints are developed. The method suggested in [31] is based on a moving

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average (MA) model with random coefficients. It was further extended to linear time-varying MA models [8] and to state space models with stochastic uncertainty in the output or the input map [7, 5]. Methods for *open-loop* stochastic MPC for nonlinear systems have been proposed in [6, 24].

The stochastic MPC methods mentioned above employ an *open-loop* formulation, which guarantees the robust stability and the robust feasibility of the system, but it may be conservative. This is related to the fact that the control sequence has to ensure constraints fulfillment for all possible uncertainty scenarios without considering the fact that future measurements of the state contain information about past uncertain values. Similar to the *closed-loop* min-max NMPC approaches [35, 34, 36], the conservativeness of the *open-loop* stochastic NMPC can be overcome by a *closed-loop* stochastic NMPC formulation, where the optimization is performed over a sequence of feedback control policies. In [18, 1], methods for *closed-loop* stochastic NMPC based on on-line optimization have been proposed and an approximate mp-NLP approach to explicit *closed-loop* min-max NMPC has been suggested in [21]. Based on the approach in [21], in [22] the explicit solution of *closed-loop* (*feedback*) stochastic NMPC problems for constrained nonlinear systems in the presence of uncertainty is considered by employing stochastic *parametric* models. The approach [22] constructs a piecewise nonlinear (PWNL) approximation to the optimal sequence of feedback control policies for efficient on-line implementation. This approach is considered in Section 7.2.

The stochastic MPC approaches [32, 44, 46, 31, 7, 8, 5, 6, 24, 18, 1, 22] are based on *parametric* probabilistic models. Alternatively, the stochastic systems can be modeled with *non-parametric* models which can offer a significant advantage compared to the *parametric* models. This is related to the fact that the *non-parametric* probabilistic models provide information about prediction uncertainties which are difficult to evaluate appropriately with the *parametric* models. The Gaussian process model is an example of a *non-parametric* probabilistic black-box model and up to now it has been applied to model mainly static nonlinearities. Its use and properties for modeling are reviewed in [41]. The use of Gaussian processes in the modeling of dynamic systems is a relatively recent development e.g. [17, 43, 26, 2, 29, 28, 39]. An on-line optimization approach and an approximate explicit approach to *open-loop* stochastic NMPC based on Gaussian process models have been proposed in [38, 30, 33] and in [19, 20], respectively. The approach [19, 20] constructs a piecewise linear (PWL) approximation to the optimal control sequence and it is considered in Section 7.3. A recent state-of-the-art survey of control algorithms based on Gaussian process models is provided in [27].

7.2 Explicit Stochastic NMPC Based on Parametric Probabilistic Models

This section considers the approximate mp-NLP approach [22] to explicit solution of *closed-loop* (*feedback*) stochastic NMPC problems for constrained nonlinear systems, described by stochastic *parametric* models. It is assumed that the dis-

crete probability distribution of the uncertainty is known. The approach constructs a piecewise nonlinear (PWNL) approximation to the optimal sequence of feedback control policies, defined on an orthogonal partition of the state space.

7.2.1 Formulation of the Feedback Stochastic NMPC Problem as an mp-NLP Problem

Consider the discrete-time nonlinear system:

$$\begin{aligned} x(t+1) &= f(x(t),u(t),w(t)) \\ y(t) &= h(x(t),u(t),w(t)), \end{aligned} \tag{7.1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^s$ and $w(t) \in \mathbb{R}^n$ are the state, input, output and uncertainty variables, and $t \in \mathbb{Z}_{\geq 0}$ is the discrete time. The input and the output variables are required to fulfill the following constraints:

$$u_{\min} \leq u \leq u_{\max},\ y_{\min} \leq y \leq y_{\max}. \tag{7.2}$$

The following assumptions are made [22]:

Assumption 7.1. *f and h are C^2 functions with $f(0,0,0) = 0$, $h(0,0,0) = 0$.*

Assumption 7.2. *The uncertainty $w(t) = [w_1(t), w_2(t), \ldots, w_n(t)]$ includes both internal (state-dependent) and external uncertainty, i.e. it has the form $w_i(t) = \lambda_i x_i(t) + \gamma_i(t)$, where $x_i(t)$ is the i-th element of the state vector $x(t)$. Here λ_i is a discrete random parameter, associated to the internal (model) uncertainty. It takes values in the set $\Lambda_i = \{\lambda_i^1, \lambda_i^2, \ldots, \lambda_i^{n_{\lambda_i}}\}$ and is characterized with the probability mass function $\varphi_i : \Lambda_i \to [0,1]$:*

$$\varphi_i(\lambda_i^j) = \Pr(\lambda_i = \lambda_i^j),\ j = 1,2,\ldots,n_{\lambda_i}\ \ with\ \sum_{j=1}^{n_{\lambda_i}} \varphi_i(\lambda_i^j) = 1. \tag{7.3}$$

The external uncertainty $\gamma_i(t)$ is stochastic and piecewise constant with infrequent changes in the sense that $\gamma_i(t) = const$ for periods of time, which are not less than N_γ ($N_\gamma \in \mathbb{N}$ is supposed to be sufficiently large). It is assumed that γ_i takes values in the set $\Gamma_i = \{\gamma_i^1, \gamma_i^2, \ldots, \gamma_i^{n_{\gamma_i}}\}$, which contains the nominal value $\gamma_i = 0$, and it is characterized with the probability mass function $\psi_i : \Gamma_i \to [0,1]$:

$$\psi_i(\gamma_i^j) = \Pr(\gamma_i = \gamma_i^j),\ j = 1,2,\ldots,n_{\gamma_i}\ \ with\ \sum_{j=1}^{n_{\gamma_i}} \psi_i(\gamma_i^j) = 1. \tag{7.4}$$

The overall vector of uncertain model parameters is denoted $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n] \in \Lambda$ with $\Lambda = \Lambda_1 \times \Lambda_2 \times \ldots \times \Lambda_n$. Given $\lambda^j = [\lambda_1^{i_1}, \lambda_2^{i_2}, \ldots, \lambda_n^{i_n}]$, $i_1 \in \{1,2,\ldots,n_{\lambda_1}\}$, ... , $i_n \in \{1,2,\ldots,n_{\lambda_n}\}$, the probability mass function $\varphi : \Lambda \to [0,1]$ is:

$$\varphi(\lambda^j) = \Pr(\lambda = \lambda^j) = \varphi_1(\lambda_1^{i_1})\varphi_2(\lambda_2^{i_2})...\varphi_n(\lambda_n^{i_n}) \tag{7.5}$$

with $j = 1, \ldots, n_\lambda$ and $n_\lambda = n_{\lambda_1} n_{\lambda_2} ... n_{\lambda_n}$. Similarly, the overall vector of external uncertainty is denoted $\gamma = [\gamma_1, \gamma_2, \ldots, \gamma_n] \in \Gamma$ with $\Gamma = \Gamma_1 \times \Gamma_2 \times \ldots \times \Gamma_n$. Given $\gamma^j = [\gamma_1^{j_1}, \gamma_2^{j_2}, \ldots, \gamma_n^{j_n}]$, $i_1 \in \{1, 2, \ldots, n_{\gamma_1}\}$, ... , $i_n \in \{1, 2, \ldots, n_{\gamma_n}\}$, the probability mass function $\psi : \Gamma \to [0, 1]$ is:

$$\psi(\gamma^j) = \Pr(\gamma = \gamma^j) = \psi_1(\gamma_1^{j_1})\psi_2(\gamma_2^{j_2})...\psi_n(\gamma_n^{j_n}) \tag{7.6}$$

with $j = 1, \ldots, n_\gamma$ and $n_\gamma = n_{\gamma_1} n_{\gamma_2} ... n_{\gamma_n}$.

Then, the overall uncertainty is:

$$w = \mathrm{diag}(\lambda)x + \gamma. \tag{7.7}$$

Further, the following assumption is made:

Assumption 7.3. $y_{\min} < 0 < y_{\max}$ *and* $u_{\min} < 0 < u_{\max}$.

As in [35], first a H_∞ control problem is defined:

Problem 7.1 (H_∞ control problem):
Design a state-feedback control law:

$$u = k(x) \tag{7.8}$$

guaranteeing that the closed-loop system (7.1)–(7.8) with input $w = \mathrm{diag}(\lambda)x + \gamma$, $\lambda \in \Lambda$, $\gamma \in \Gamma$, and output y has a finite l_2-gain $\leq \alpha$ in a bounded positively invariant set Ω, that is, $\forall x(t) \in \Omega$:

i. $x(t+i) \in \Omega, \forall i > 0$.
ii. $u_{\min} \leq k(x(t+i)) \leq u_{\max}$ and $y_{\min} \leq h(x(t+i), k(x(t+i)), w(t+i)) \leq y_{\max}$, $\forall i \geq 0$.
iii. There exists a positive definite function $\beta(x(t))$, such that $\forall T \geq 0$:

$$\sum_{i=0}^{T} \|y(t+i)\|^2 \leq \alpha^2 \sum_{i=0}^{T} \|w(t+i)\|^2 + \beta(x(t)) \tag{7.9}$$

for any non-zero w.

The following assumption is also made [35]:

Assumption 7.4. *There exists an auxiliary control law $u = k_a(x)$ that solves the H_∞ control problem, with a domain of attraction $\Omega_a(k_a, \alpha)$, whose boundary is assumed to be a level curve of a positive function $V_{k_a}(x)$ such that:*

$$\begin{aligned} &V_{k_a}(f(x, k_a(x), w)) - V_{k_a}(x) < -\frac{1}{2}(\|y\|^2 - \alpha^2\|w\|^2) \\ &\forall x \in \Omega_a(k_a, \alpha)\,,\ \forall \lambda \in \Lambda\,,\ \forall \gamma \in \Gamma \end{aligned} \tag{7.10}$$

and $V_{k_a}(0) = 0$.

Denote with:

$$K = \{k_0, k_1, \ldots, k_{N-1}\} \triangleq \{k_0(x_{t|t}), k_1(x_{t+1|t}), \ldots, k_{N-1}(x_{t+N-1|t})\} \tag{7.11}$$

a vector of feedback control policies. It is supposed that a full measurement x of the state is available at the current time t. We consider the following feedback stochastic NMPC problem [22]:

Problem 7.2 (Constrained feedback stochastic NMPC problem):
Suppose that Assumptions 7.1–7.4 hold. For the current x, the feedback stochastic NMPC solves the following optimization problem:

$$V_E^*(x) = \min_K \underset{\lambda\in\Lambda,\,\gamma\in\Gamma}{E}\{J(K,x,\lambda,\gamma)\} \tag{7.12}$$

subject to $x_{t|t} = x$ and:

$$y_{\min} \le y_{t+i|t} \le y_{\max}, \forall\lambda \in \Lambda, \forall\gamma \in \Gamma, i = 1, \ldots, N \tag{7.13}$$
$$u_{\min} \le u_{t+i} \le u_{\max}, i = 0, 1, \ldots, N-1 \tag{7.14}$$
$$x_{t+N|t} \in \Omega_a(k_a, \alpha), \forall\lambda \in \Lambda, \forall\gamma \in \Gamma \tag{7.15}$$
$$u_{t+i} = k_i(x_{t+i|t}), i = 0, 1, \ldots, N-1 \tag{7.16}$$
$$x_{t+i+1|t} = f(x_{t+i|t}, u_{t+i}, w_{t+i}), i \ge 0 \tag{7.17}$$
$$y_{t+i|t} = h(x_{t+i|t}, u_{t+i}, w_{t+i}), i \ge 0 \tag{7.18}$$

and the cost function given by:

$$J(K,x,\lambda,\gamma) = \frac{1}{2}\sum_{i=0}^{N-1}\left[\|y_{t+i|t}\|^2 - \alpha^2\|w_{t+i}\|^2\right] + V_{k_a}(x_{t+N|t}). \tag{7.19}$$

Here, N is a finite horizon, α is the l_2-gain which is interpreted as the uncertainty attenuation level, and $E\{.\}$ means mathematical expectation. It is supposed that $N \ll N_\gamma$, and by Assumption 7.2 it can be accepted that $\gamma_{t+i} = const, i = 0, 1, \ldots, N-1$. Then by Assumption 7.2, the expectation can be expressed:

$$\underset{\lambda\in\Lambda,\,\gamma\in\Gamma}{E}\{J(K,x,\lambda,\gamma)\} = \sum_{i=1}^{n_\lambda}\sum_{j=1}^{n_\gamma} J(K,x,\lambda^i,\gamma^j)\varphi(\lambda^i)\psi(\gamma^j) \tag{7.20}$$

An auxiliary control law $k_a(x)$ is typically obtained by solving the H_∞ control problem for the linearized system [35]. Thus, a practical way to compute a nonlinear control $k_a(x)$ satisfying Assumption 7.4 for nonlinear input-affine systems is suggested in [35].

An optimal solution to the feedback stochastic NMPC problem (7.12)–(7.19) is denoted $K^* = \{k_0^*, k_1^*, \ldots, k_{N-1}^*\}$ and the control input is chosen according to the receding horizon policy $u(x_{t|t}) = k_0^*(x_{t|t})$.

The following assumption is made:

Assumption 7.5. *Each feedback control policy* $k_i(x_{t+i|t})$, $i = 0, \ldots, N-1$ *has the form:*

$$k_i(x_{t+i|t}) = \eta_i k_a(x_{t+i|t}) + r_i(\xi_i, x_{t+i|t}) = g_i(p_i, x_{t+i|t}), \tag{7.21}$$

where $p_i = [\eta_i^T \ \xi_i^T]^T \in \mathbb{R}^{n_i}$ *are the parameters that need to be optimized,* $k_a(x_{t+i|t})$ *is an auxiliary control law that satisfies Assumption 7.4, and* $r_i(\xi_i, x_{t+i|t})$ *is a parameterized continuous function with* $r_i(\xi_i, 0) = 0$.

In general, the parameterization of the form (7.21) would lead to an approximate solution to the feedback stochastic NMPC problem (7.12)–(7.19). Denote with P the whole set of parameters that need to be determined, i.e.:

$$P = [p_0^T \ p_1^T \ \ldots \ p_{N-1}^T]^T \in \mathbb{R}^{n_p}, \ n_p = \sum_{i=0}^{N-1} n_i. \tag{7.22}$$

Then, the expected value (7.20) of the cost function is:

$$V_E(P, x) = \underset{\lambda \in \Lambda, \gamma \in \Gamma}{E} \{J(P, x, \lambda, \gamma)\}. \tag{7.23}$$

It should be noted that the argument K in the cost function (7.19) is now replaced with the argument P.

Using the ideas of direct single shooting to eliminate the equality constraints, the optimization problem (7.12)–(7.19) can be formulated in a compact form as follows [22]:

Problem 7.3:

$$V_E^*(x) = \min_P \sum_{i=1}^{n_\lambda} \sum_{j=1}^{n_\gamma} J(P, x, \lambda^i, \gamma^j) \varphi(\lambda^i) \psi(\gamma^j) \tag{7.24}$$

$$\text{subject to: } G(P, x, \lambda, \gamma) \leq 0, \ \forall \lambda \in \Lambda, \forall \gamma \in \Gamma. \tag{7.25}$$

Problem 7.3 defines a multi-parametric Nonlinear Programming (mp-NLP) problem, since it is NLP in P parameterized by x. We remark that the constraints function $G(P, x, \lambda, \gamma)$ in (7.25) is implicitly defined by (7.13)–(7.18). Also, since Λ and Γ are discrete sets, (7.25) represents a finite number of constraints. It should be noted that the number of constraints (7.25) increases rapidly with the increase of the horizon and the sizes n_λ and n_γ of the uncertainty sets Λ and Γ. Thus, as the horizon increases from N_1 to N_2 and the sizes of the sets Λ and Γ increase from n_λ^1 to n_λ^2, and from n_γ^1 to n_γ^2, respectively, the number of constraints will increase $(N_2 n_\lambda^2 n_\gamma^2)/(N_1 n_\lambda^1 n_\gamma^1)$ times. This leads to a stronger motivation for an explicit approach, where the computational complexity is handled in off-line.

Define the set of N-step robustly feasible initial states:

$$X_f = \{x \in \mathbb{R}^n \,|\, G(P, x, \lambda, \gamma) \leq 0, \forall \lambda \in \Lambda, \forall \gamma \in \Gamma \ \text{ for some } \ P \in \mathbb{R}^{n_p}\}. \tag{7.26}$$

In parametric programming problems one seeks the solution $P^*(x)$ as an explicit function of the parameters x in some set $X \subseteq X_f \subseteq \mathbb{R}^n$ [14].

7.2.2 Approximate mp-NLP Approach to Explicit Feedback Stochastic NMPC

The approximate mp-NLP approach [22] to explicit feedback stochastic NMPC is similar to the approach to explicit feedback min-max NMPC (described in Section 6.3.2). Thus, we restrict our attention to a hyper-rectangle $X \subset \mathbb{R}^n$ where we seek to approximate the optimal sequence of control policies $K^* = \{k_0^*, k_1^*, \ldots, k_{N-1}^*\}$. The associated optimal control input is

$$u_{t+i}^* = k_i^*(x_{t+i|t}) = g_i(p_i^*, x_{t+i|t}),\ i = 0, 1, \ldots, N-1, \tag{7.27}$$

where $P^* = [p_0^{*T}\ p_1^{*T}\ \ldots\ p_{N-1}^{*T}]^T$ is determined by solving Problem 7.3. We require that the state space partition is orthogonal and can be represented as a $k-d$ tree. The main idea of the approximate mp-NLP approach [22] is to construct a *piecewise nonlinear* (PWNL) approximation $\hat{K} = \{\hat{k}_0, \hat{k}_1, \ldots, \hat{k}_{N-1}\}$ to the optimal feedback $K^* = \{k_0^*, k_1^*, \ldots, k_{N-1}^*\}$ on X, where the constituent *nonlinear* control functions $K_{X_i} = \{k_{0,X_i}, k_{1,X_i}, \ldots, k_{N-1,X_i}\}$ are defined on hyper-rectangles X_i covering X. Let $K_{X_0} = \{k_{0,X_0}, \ldots, k_{N-1,X_0}\}$ be an approximation to the optimal solution $K^* = \{k_0^*, \ldots, k_{N-1}^*\}$, valid in the whole hyper-rectangle X_0. Denote with $P_{X_0} = [p_{0,X_0}^T \ldots p_{N-1,X_0}^T]^T$ the parameters of K_{X_0}. The corresponding approximate value of the control input is

$$\hat{u}_{t+i} = k_{i,X_0}(x_{t+i|t}) = g_i(p_{i,X_0}, x_{t+i|t}),\ i = 0, 1, \ldots, N-1. \tag{7.28}$$

Let $\hat{V}_E(P_{X_0}, x)$ be the cost function value due to initial state $x = x_{t|t}$ and control function K_{X_0}, i.e.

$$\hat{V}_E(P_{X_0}, x) = \underset{\lambda \in \Lambda,\, \gamma \in \Gamma}{E} \left\{ J(P_{X_0}, x, \lambda, \gamma) \right\}. \tag{7.29}$$

Then, the approximate control function K_{X_0} is determined by applying the following procedure [22]:

Procedure 7.1 (Computation of explicit approximate solution). *Suppose Assumptions 7.1–7.5 hold. Consider any hyper-rectangle $X_0 \subseteq X_f$ with a set of points $V_0 = \{v_0, v_1, v_2, \ldots, v_{N_1}\} \subseteq X_0$. Compute the parameters $P_{X_0} = [p_{0,X_0}^T \ldots p_{N-1,X_0}^T]^T$ of the control function $K_{X_0} = \{k_{0,X_0}, \ldots, k_{N-1,X_0}\}$ by solving the following NLP:*

$$\min_{P_{X_0}} \sum_{i=0}^{N_1} \left(\hat{V}_E(P_{X_0}, v_i) - V_E^*(v_i) + \mu \| g_0(p_{0,X_0}, v_i) - g_0(p_0^{*i}, v_i) \|^2 \right) \tag{7.30}$$

$$\textit{subject to}\ \ G(P_{X_0}, v_i, \lambda, \gamma) \leq 0, \forall v_i \in V_0, \forall \lambda \in \Lambda, \forall \gamma \in \Gamma. \tag{7.31}$$

We remark that the optimal parameters p_0^{*i} in the feedback function $k_0^*(v_i) = g_0(p_0^{*i}, v_i)$ in (7.30) are determined by solving Problem 7.3 for $x = v_i$, and the parameter $\mu > 0$ is a weighting coefficient.

Note that the control function $K_{X_0} = \{k_{0,X_0}, \ldots, k_{N-1,X_0}\}$, computed with Procedure 7.1, satisfies the constraints in Problem 7.3 only for the discrete set of points V_0 in the hyper-rectangle X_0 and for the discrete sets of values Λ and Γ of the internal and external uncertainties.

Suppose that the parameter vector P_{X_0} of the control function K_{X_0}, valid in X_0, has been determined by applying Procedure 7.1. Then, for the cost function approximation error in X_0 we have:

$$\varepsilon(x) = \hat{V}_E(P_{X_0}, x) - V_E^*(x) \leq \varepsilon_0,\ x \in X_0. \tag{7.32}$$

An estimate $\hat{\varepsilon}_0$ of the error bound ε_0 is computed as:

$$\hat{\varepsilon}_0 = \max_{i \in \{0,1,2,\ldots,N_1\}} (\hat{V}_E(P_{X_0}, v_i) - V_E^*(v_i)). \tag{7.33}$$

If $\hat{\varepsilon}_0 > \bar{\varepsilon}$, where $\bar{\varepsilon} > 0$ is the specified tolerance of the cost function approximation error, the region X_0 is divided and the procedure is repeated for the new regions.

The approximate mp-NLP algorithm for design of explicit feedback stochastic NMPC represents a slight modification of the algorithm, described in Section 6.3.2.4.

It should be noted that in case of non-convexity of Problem 7.3, it can not be guaranteed that the approximation error $\varepsilon(x)$ associated to the explicit feedback stochastic NMPC will satisfy the requirement $\varepsilon(x) \leq \bar{\varepsilon}$ for all $x \in X$. The non-convexity may also imply that the constraints (7.25) are violated at some points of the state space. In this respect, the described computational method does not necessarily lead to guaranteed properties, but when combined with verification and analysis methods gives a practical tool for development and implementation of explicit feedback stochastic NMPC. The possibility for implementation verification is a significant advantage of the explicit NMPC in comparison to NMPC based on real-time optimization.

7.2.3 *Application 1: Stochastic MPC of the Cart and Spring System*

Consider the cart and spring system, described in Section 6.3.4. The damping factor h_d is uncertain, but it is known that $h_d = \bar{h}_d + \lambda$. Here, $\bar{h}_d = 1.1$ and λ is a stochastic parameter. It is supposed that $-0.5 \leq \lambda \leq 0.5$ and the following discrete set of values is considered $\lambda \in \Lambda = \{-0.5, 0, 0.5\}$ with the corresponding values

of the probability mass function $\varphi(-0.5) = 0.2$, $\varphi(0) = 0.6$, $\varphi(0.5) = 0.2$. The external uncertainty for this system is $\gamma = 0$. Recall that the system is described by the following nonlinear discrete-time model [35]:

$$x_1(t+1) = x_1(t) + T_s x_2(t) \tag{7.34}$$

$$x_2(t+1) = x_2(t) - T_s \frac{\rho_0}{M} e^{-x_1(t)} x_1(t) - T_s \frac{\overline{h}_d}{M} x_2(t) + T_s \frac{u(t)}{M} + T_s w(t), \tag{7.35}$$

where x_2 is the carriage velocity, $w(t) = -\frac{\lambda}{M} x_2(t)$ is a state dependent (internal) uncertainty, $T_s = 0.4$ is the sampling time, $M = 1$ and $\rho_0 = 0.33$. The following input and state constraints are imposed on the system:

$$-4 \leq u \leq 4, \; -1.3 \leq x_2 \leq 1.3. \tag{7.36}$$

The horizon is $N = 15$ and the terminal constraint is:

$$x_{t+N|t} \in \Omega_a \,, \; \Omega_a = \{x \in \mathbb{R}^n \,|\, x^T \Sigma x \leq \delta\}, \tag{7.37}$$

where $\delta = 0.001$ [35] and $\Sigma = \left[\begin{smallmatrix} 1.3 & 1.9 \\ 1.9 & 3.0 \end{smallmatrix}\right]$.

The mp-NLP approach described in Section 7.2.2 is applied to design an explicit feedback stochastic NMPC controller for the cart [22]. The NMPC minimizes the mathematical expectation (7.20) of the cost function (7.19) subject to the system equations (7.34)–(7.35) and the constraints (7.36)–(7.37). In (7.19), it is chosen $\alpha = 1$ and the terminal penalty is given by $V_{k_a} = x^T \Sigma x$ [35]. Like in Section 6.3.4, the feedback functions $k_i(x_{t+i|t})$, $i = 0, \ldots, N-1$ have the form:

$$k_i(p_i, x_{t+i|t}) = \eta_i k_a(x_{t+i|t}) + \xi_{i,1} x_{1,t+i|t}^2 + \xi_{i,2} x_{2,t+i|t}^2, \tag{7.38}$$

where $p_i = [\eta_i \; \xi_{i,1} \; \xi_{i,2}]^T$ are the parameters that need to be optimized and $k_a(x_{t+i|t})$ is the auxiliary control law. The control law $k_a(x_{t+i|t})$ is determined in the way described in Section 6.3.4 (respectively in [35]).

In [23], a condition on the approximation tolerance has been derived such that the asymptotic stability of the nonlinear system in closed-loop with the approximate explicit NMPC is guaranteed. According to this condition, the tolerance is chosen to be dependent on the state, which would lead to a state space partition with less complexity in comparison to that corresponding to an uniform tolerance. In [22], a similar approach is applied and the approximation tolerance is chosen to be dependent on the state as $\bar{\varepsilon}(X_0) = \max(\bar{\varepsilon}_a, \bar{\varepsilon}_r \min_{x \in X_0} V_E^*(x))$, where $\bar{\varepsilon}_a = 0.005$ and $\bar{\varepsilon}_r = 0.03$ are the absolute and the relative tolerances.

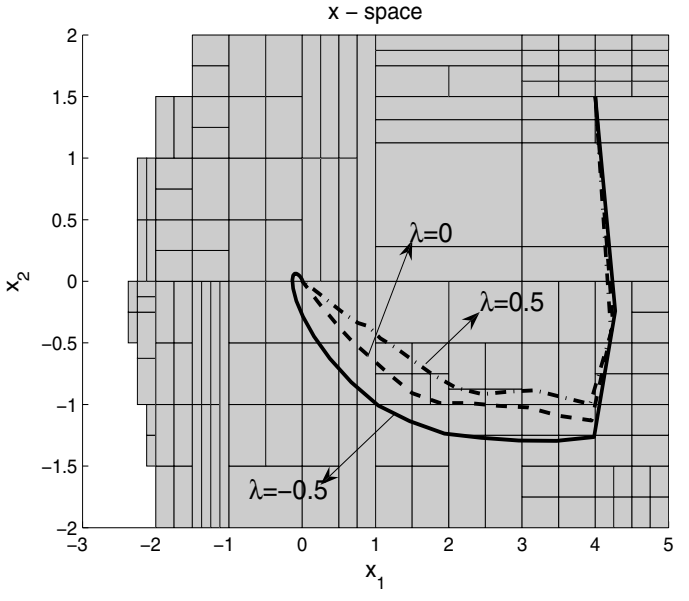

Fig. 7.1 State space partition of the explicit approximate feedback stochastic NMPC and the state trajectories corresponding to $\lambda = -0.5$, $\lambda = 0$, $\lambda = 0.5$.

The state space partition of the explicit approximate feedback stochastic NMPC controller is shown in Fig. 7.1. The partition has 150 regions and 11 levels in a binary search tree representation. Totally, 27 arithmetic operations are needed in real-time to compute the control input by traversing the binary search tree (11 comparisons, 10 multiplications, 5 additions and 1 exponential).

The performance of the closed-loop system was simulated for initial state $x(0) = [4\ 1.5]^T$ and for the three values of the stochastic parameter λ. The response is depicted in the state space (Fig. 7.1), as well as trajectories in time (Fig. 7.2 and Fig. 7.3). In Fig. 7.2 and Fig. 7.3, the control and state trajectories obtained with the explicit min-max NMPC controller (designed in Section 6.3.4) are given for comparison. The cost function values corresponding to the closed-loop trajectories associated to the explicit stochastic NMPC and to the explicit min-max NMPC are $\hat{V}_E = 121.64$ and $\hat{V}_{\min-\max} = 122.57$ (for $h_d = 0.6$), and $\hat{V}_E = 141.59$ and $\hat{V}_{\min-\max} = 141.70$ (for $h_d = 1.6$). Therefore, the explicit min-max NMPC appears to be slightly more conservative, since it is characterized with larger values of the cost function in comparison to the explicit stochastic NMPC. It can be seen that the explicit feedback stochastic NMPC controller brings the cart to the equilibrium despite of the presence of stochastic uncertainty, and the constraints imposed on the system are satisfied.

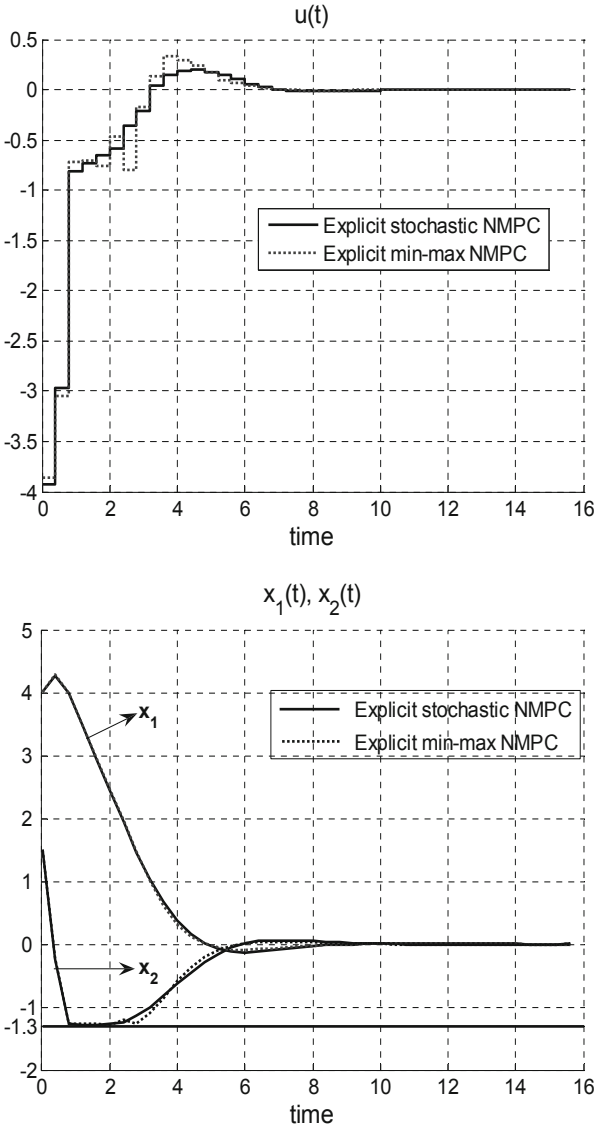

Fig. 7.2 Control input and state trajectory for $h_d = 0.6$.

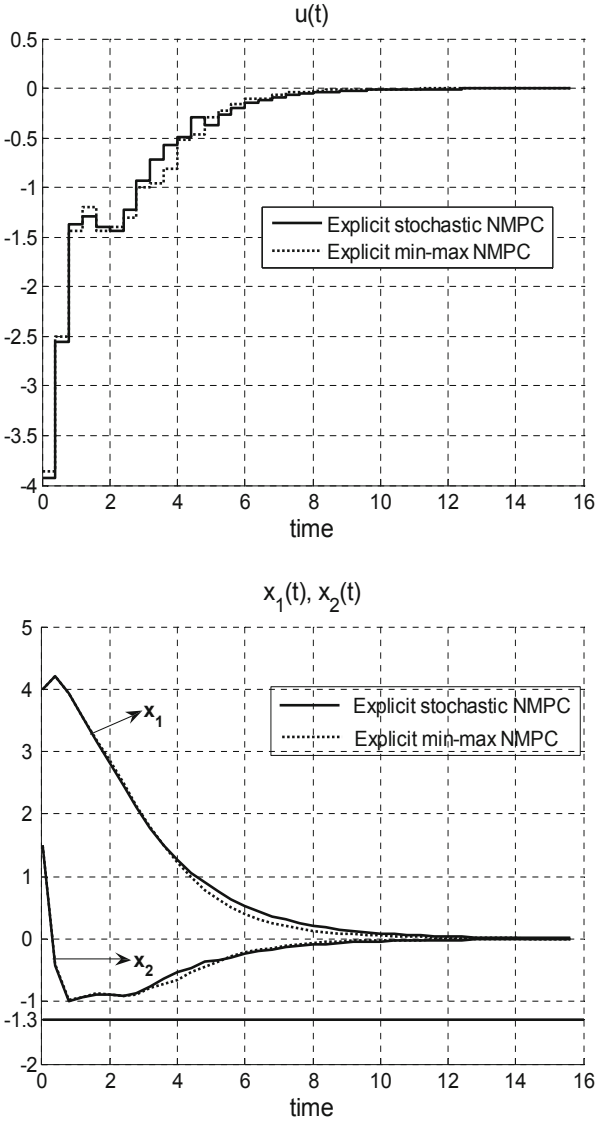

Fig. 7.3 Control input and state trajectory for $h_d = 1.6$.

7.3 Explicit Stochastic NMPC Based on Gaussian Process Models

In this section, the approximate mp-NLP approach [19, 20] to explicit *open-loop* stochastic NMPC based on Gaussian process models (abbreviated as GP-NMPC) is presented. The approach constructs a piecewise linear (PWL) approximation to the optimal control sequence, defined on an orthogonal partition of the state space.

7.3.1 Modeling of Dynamic Systems with Gaussian Processes

A Gaussian process is an example of the use of a flexible, probabilistic, nonparametric model which directly provides us with quantification of the uncertainty of predictions. Its use and properties for modeling are reviewed in [41].

A Gaussian process is a collection of random variables which have a joint multivariate Gaussian distribution. Assuming a relationship of the form $y = f(z)$ between an input $z \in \mathbb{R}^D$ and output $y \in \mathbb{R}$, we have $y(1), y(2), \ldots, y(M) \sim \mathcal{N}(0, \mathbf{K})$, where $K_{pq} = \mathrm{Cov}(y(p), y(q)) = C(z(p), z(q))$ gives the covariance between the output points $y(p)$ and $y(q)$ corresponding to the input points $z(p)$ and $z(q)$. Thus, the mean $\mu(z)$ (usually assumed to be zero) and the covariance function $C(z(p), z(q))$ fully specify the Gaussian process. Note that the covariance function $C(z(p), z(q))$ can be any function with the property that it generates a positive definite covariance matrix. A common choice is the Gaussian covariance function [45, 41]:

$$C(z(p), z(q)) = v_1 \exp\left[-\frac{1}{2}\sum_{i=1}^{D} w_i (z_i(p) - z_i(q))^2\right] + v_0 \alpha_{pq} \tag{7.39}$$

where $\Theta = [w_1, \ldots, w_D, v_0, v_1]$ is a vector of parameters called hyperparameters and z_i denotes the i-th component of the D-dimensional input vector z. The hyperparameter v_1 controls the magnitude of the covariance and the hyperparameters w_i represent the relative importance of each component z_i of vector z. The part $v_0 \alpha_{pq}$ represents the covariance between outputs due to white noise, where α_{pq} is the Kronecker operator and v_0 is the white noise variance (when assuming different kinds of noise the covariance function should be changed appropriately, e.g. [15]). For a given problem, the hyperparameters are learned (identified) using the data at hand. After the learning, one can use the w parameters as indicators of 'how important' the corresponding input components (dimensions) are: if w_i is zero or near zero it means that the inputs in dimension i contain little information and could possibly be removed.

Consider a set of M D-dimensional input vectors $\mathbf{Z} = [z(1), z(2), \ldots, z(M)]^T$ and a vector of output data $Y = [y(1), y(2), \ldots, y(M)]^T$. Based on the data $(\mathbf{Z}, Y)$, and given a new input vector z^*, we wish to estimate the probability distribution of the corresponding output y^*. Unlike other models, there is no model parameter determination as such, within a fixed model structure. With this model, most of the effort consists in *tuning* the parameters of the covariance function. This is done by maximizing the log-likelihood with the vector of hyperparameters Θ:

$$\mathscr{L}(\Theta) = \log(p(Y \mid \mathbf{Z})) = -\frac{1}{2}\log(|\mathbf{K}|) - \frac{1}{2}Y^T\mathbf{K}^{-1}Y - \frac{M}{2}\log(2\pi) \tag{7.40}$$

where $\mathbf{K}$ is the $M \times M$ training covariance matrix with determinant $|\mathbf{K}|$ and the hyperparameters distribution $p(\Theta \mid Y, \mathbf{Z})$ is approximated with their most likely values. The optimization requires the computation of the derivative of $\mathscr{L}$ with respect to each of the parameters:

$$\frac{\partial \mathscr{L}(\Theta)}{\partial \theta_i} = -\frac{1}{2}\text{trace}\left(\mathbf{K}^{-1}\frac{\partial \mathbf{K}}{\partial \theta_i}\right) + \frac{1}{2}Y^T\mathbf{K}^{-1}\frac{\partial \mathbf{K}}{\partial \theta_i}\mathbf{K}^{-1}Y \tag{7.41}$$

Here, it involves the computation of the inverse of the $M \times M$ covariance matrix $\mathbf{K}$ at every iteration, which can be computationally demanding for large M. The reader is referred to [41] for a detailed description of the parameter optimization methods.

Given that the hyperparameters are known, we can estimate the probability distribution of the corresponding output y^* at some new input vector z^*:

$$p(y^* \mid Y, \mathbf{Z}, z^*) = \frac{p(Y, y^*, \mathbf{Z}, z^*)}{p(Y \mid \mathbf{Z}, z^*)} \tag{7.42}$$

It can be shown that this distribution is Gaussian with mean and variance [45]:

$$\mu(z^*) = k(z^*)^T\mathbf{K}^{-1}Y \tag{7.43}$$

$$\sigma^2(z^*) = k_0(z^*) - k(z^*)^T\mathbf{K}^{-1}k(z^*) \tag{7.44}$$

where $k(z^*) = [C(z(1), z^*), \ldots, C(z(M), z^*)]^T$ is the $M \times 1$ vector of covariances between the test input and the training inputs and $k_0(z^*) = C(z^*, z^*)$ is the autocovariance of the test input. The vector $k(z^*)^T\mathbf{K}^{-1}$ in (7.43) can be interpreted as a vector of smoothing terms which weights the training outputs Y to make a prediction at the test point z^*. If the new input is far away from the data points, the term $k(z^*)^T\mathbf{K}^{-1}k(z^*)$ in (7.44) will be small, so that the predicted variance $\sigma^2(z^*)$ will be large. Thus, from the system identification point of view equation (7.43) provides the model prediction and equation (7.44) its confidence.

Gaussian processes can be used to model static nonlinearities and can therefore be used for modeling of dynamic systems if delayed input and output signals are used as regressors [26]. In such cases an autoregressive model is considered, such that the current predicted output depends on previous estimated outputs, as well as on previous control inputs:

$$\begin{aligned} z(t) &= [\hat{y}(t-1), \hat{y}(t-2), \ldots, \hat{y}(t-L), u(t-1), u(t-2), \ldots, u(t-L)]^T \\ \hat{y}(t) &= f(z(t)) + \eta(t) \end{aligned} \tag{7.45}$$

where t denotes consecutive number of data sample, L is a given lag, and $\eta(t)$ is the prediction error. The quality of the predictions with a Gaussian process model is assessed by computing the average squared error (ASE):

$$\text{ASE} = \frac{1}{M}\sum_{i=1}^{M}[\mu(\hat{y}(i)) - y(i)]^2 \tag{7.46}$$

and by the log predictive density error (LD) [26]:

$$\text{LD} = \frac{1}{2}\log(2\pi) + \frac{1}{2M}\sum_{i=1}^{M}\left(\log[\sigma^2(\hat{y}(i))] + \frac{[\mu(\hat{y}(i)) - y(i)]^2}{\sigma^2(\hat{y}(i))}\right) \tag{7.47}$$

In (7.46), (7.47), $\mu(\hat{y}(i))$ and $\sigma^2(\hat{y}(i))$ are the prediction mean and variance, $y(i)$ is the system's output and M is the number of the training points.

The Gaussian process model now not only describes the dynamic characteristics of the non-linear system, but at the same time provides information about the confidence in the predictions. The Gaussian process can highlight areas of the input space where prediction quality is poor, due to the lack of data, by indicating the higher variance around the predicted mean. The Gaussian process modelling approach in [26] has been applied to model the dynamics of various systems e.g. [2, 29, 20, 28, 39].

7.3.2 Formulation of the Stochastic GP-NMPC Problem as an mp-NLP Problem

Consider a stochastic nonlinear discrete-time system:

$$x(t+1) = f(x(t), u(t)) + \xi(t) \tag{7.48}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and input variables, $\xi(t) \in \mathbb{R}^n$ are Gaussian disturbances, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear continuous function. Suppose that a Gaussian process model of the system (7.48) is obtained by applying the approach described in Section 7.3.1. Suppose the initial state $x(t) = x_{t|t}$ and the control inputs $u(t+k) = u_{t+k}$, $k = 0, 1, \ldots, N-1$ are given. Then, the probability distribution of the predicted states $x_{t+k+1|t}$, $k = 0, 1, \ldots, N-1$ which correspond to the given initial state $x_{t|t}$ and control inputs u_{t+k}, $k = 0, 1, \ldots, N-1$ can be obtained [16]:

$$x_{t+k+1|t}\,|\,x_{t+k|t}, u_{t+k} \sim \mathcal{N}(\mu(x_{t+k+1|t}), \sigma^2(x_{t+k+1|t})),\ k = 0, 1, \ldots, N-1 \tag{7.49}$$

The 95% confidence interval of the random variable $x_{t+k+1|t}$ is $[\mu(x_{t+k+1|t}) - 2\sigma(x_{t+k+1|t});\ \mu(x_{t+k+1|t}) + 2\sigma(x_{t+k+1|t})]$, where $\sigma(x_{t+k+1|t})$ is the standard deviation.

In [19, 20], a reference tracking NMPC problem based on a Gaussian process model (GP-NMPC) is considered, where the goal is to have the state vector $x(t)$ track the reference signal $r(t) \in \mathbb{R}^n$. In the problem formulation, the type of the cost function is like the one used in [3]. Suppose that a full measurement of the state $x(t)$ is available at the current time t. For the current $x(t)$, the reference tracking GP-NMPC solves the following optimization problem [19, 20]:

Problem 7.4:

$$V^*(x(t), r(t), u(t-1)) = \min_U J(U, x(t), r(t), u(t-1)) \tag{7.50}$$

subject to $x_{t|t} = x(t)$ and:

$$\mu(x_{t+k|t}) - 2\sigma(x_{t+k|t}) \geq x_{\min},\ k = 1, \ldots, N \tag{7.51}$$
$$\mu(x_{t+k|t}) + 2\sigma(x_{t+k|t}) \leq x_{\max},\ k = 1, \ldots, N \tag{7.52}$$
$$u_{\min} \leq u_{t+k} \leq u_{\max},\ k = 0, 1, \ldots, N-1 \tag{7.53}$$
$$\Delta u_{\min} \leq \Delta u_{t+k} \leq \Delta u_{\max},\ k = 0, 1, \ldots, N-1 \tag{7.54}$$
$$\max\{\|\mu(x_{t+N|t}) - 2\sigma(x_{t+N|t}) - r(t)\|, \\ \|\mu(x_{t+N|t}) + 2\sigma(x_{t+N|t}) - r(t)\|\} \leq \delta \tag{7.55}$$
$$\Delta u_{t+k} = u_{t+k} - u_{t+k-1},\ k = 0, 1, \ldots, N-1 \tag{7.56}$$
$$x_{t+k+1|t} \,|\, x_{t+k|t}, u_{t+k} \sim \mathcal{N}(\mu(x_{t+k+1|t}), \sigma^2(x_{t+k+1|t})) \\ k = 0, 1, \ldots, N-1 \tag{7.57}$$

with $U = [u_t, u_{t+1}, \ldots, u_{t+N-1}]$ and the cost function given by:

$$J(U, x(t), r(t), u(t-1)) = \sum_{k=0}^{N-1} \left[\|\mu(x_{t+k|t}) - r(t)\|_Q^2 + \|\Delta u_{t+k}\|_R^2\right] \\ + \|\mu(x_{t+N|t}) - r(t)\|_P^2 \tag{7.58}$$

Here, N is a finite horizon and $P, Q, R \succ 0$. From a stability point of view it is desirable to choose δ in the terminal constraint (7.55) sufficiently small [37]. If the horizon N is large and the Gaussian process model has a small prediction uncertainty, then it is more likely that the choice of a small δ will be possible.

It should be noted that a more general stochastic MPC problem is formulated in [31, 7, 8, 5], where a probabilistic formulation of the cost includes the probabilistic bounds of the predicted variable. The stochastic MPC problem considered here (Problem 7.4) is of a more special form since the cost function (7.58) includes the mean value of the random variable. However, the approximate approach to the explicit solution of Problem 7.4 (which is based on the approximate mp-NLP algorithms, given in Section 1.1.5) can be easily extended to the more general case of stochastic MPC problem formulation where the optimization is performed on the expected value of the cost function.

We introduce a parameter vector:

$$\tilde{x}(t) = [x(t), r(t), u(t-1)] \in \mathbb{R}^{\tilde{n}},\ \tilde{n} = 2n + m \tag{7.59}$$

Let $\tilde{x}$ be the value of the parameter vector at the current time t. Using a direct single shooting strategy, the equality constraints are eliminated and the optimization Problem 7.4 can be formulated in a compact form as follows [19, 20]:

Problem 7.5:

$$V^*(\tilde{x}) = \min_U J(U,\tilde{x}) \text{ subject to } G(U,\tilde{x}) \leq 0 \tag{7.60}$$

The GP-NMPC problem defines an mp-NLP, since it is NLP in U parameterized by $\tilde{x}$. An optimal solution to this problem is denoted $U^* = [u_t^*, u_{t+1}^*, \ldots, u_{t+N-1}^*]$ and the control input is chosen according to the receding horizon policy $u(t) = u_t^*$. Define the set of feasible parameter vectors as follows:

$$X_f = \{\tilde{x} \in \mathbb{R}^{\tilde{n}} \,|\, G(U,\tilde{x}) \leq 0 \text{ for some } U \in \mathbb{R}^{Nm}\} \tag{7.61}$$

If δ in (7.55) is chosen such that the Problem 7.4 is feasible, then X_f is a non-empty set. In parametric programming problems one seeks the solution $U^*(\tilde{x})$ as an explicit function of the parameters $\tilde{x}$ in some set $X \subseteq X_f \subseteq \mathbb{R}^{\tilde{n}}$ [14]. In case the Problem 7.5 is convex, its approximate solution can be found by applying the approximate mp-NLP approach, described in Section 1.1.5.1. Otherwise, the approximate mp-NLP approach from Section 1.1.5.2 should be used, where in addition to the set of vertices of a given hyper-rectangle in the parameter space, the optimal solution is also computed at several interior points and global optimization methods are applied.

7.3.3 Application 2: Reference Tracking Control of a Combustion Plant

Energy production is one of the largest sources of air pollution and CO_2. Therefore a rational and ecological use of energy is the main task of the thermoelectric power plants. A feasible method to reduce the NO_x, CO, CO_2 emissions and to increase the efficiency is to improve the control strategies of existing power plants, i.e. to optimize the combustion process [11]. The objectives for the improvement of the power plant combustion process are energy saving, pollution reduction, longer plant lifetime, less downtime and maintenance effort, increased safety in operation, i.e. overall cost reduction. These goals can be achieved through application of modern control algorithms with low on-line computational complexity and high reliability of the implementation. Feedback combustion control is possible since continuous flue gases analyzers are available [11]. For control purposes it would be ideal to measure all flue gases components. But the price for such a realization would currently be too high in comparison with the savings achieved. Therefore the control of the oxygen fraction in the flue gases, measured on-line by the well known in-situ ZrO_2 analyzers, is often the best solution [10]. Based on that, different algorithms for combustion control have been studied in [42, 9, 4, 10, 25] and more recent in e.g. [40, 13]. It should be noted that these methods assume that the combustion model is known exactly. However, the mathematical models are only an approximation of the real process and they usually contain some amount of uncertainty (unknown additive disturbances and/or uncertain model parameters). In order to achieve a robust performance of the control system it would be necessary to take into account the uncertainty when designing the controller.

In [20], a Gaussian process model of a combustion plant (a steam boiler PK 401 at Cinkarna Celje Company, Celje, Slovenia) is obtained. Then, the approximate mp-NLP approach (described in Section 1.1.5.2) is applied to design an explicit reference tracking GP-NMPC controller that brings the air factor of the combustion plant to its optimal value with every change of the load factor. Thus, an efficient on-line optimization of the combustion plant is achieved where both the economic and the environmental aspects are taken into account. Because of the operation safety of the considered combustion plant and because interrupts in plant operation are not favored by company management, the results obtained in [20] are based on simulation data to show the potential use of the approximate mp-NLP approach to the optimal control of industrial combustion plants.

7.3.3.1 Optimal Operation of Combustion Plants

Fuel composition can be expressed with percentage of carbon C, hydrogen H, oxygen O, nitrogen N, sulphur S, ash A and water W [12]:

$$\mathrm{C} + \mathrm{H} + \mathrm{O} + \mathrm{N} + \mathrm{S} + \mathrm{A} + \mathrm{W} = 100\% \tag{7.62}$$

Composition of the air is usually expressed only with the percentage of oxygen O_2 and nitrogen N_2:

$$\mathrm{O}_2 + \mathrm{N}_2 = 21\% + 79\% = 100\% \tag{7.63}$$

The combustion process is schematically shown in Fig. 7.4. The limited fuel sources, considerable increase in the fuel prices and the enormous environment pollution require decreasing the fuel use, the heat losses and the amount of harmful flue gases emissions, i.e. to optimize the combustion process [9]. It has been shown in [9] that in order to achieve an optimal operation of the combustion plants, it is necessary to optimize the air factor λ defined as:

$$\lambda = \frac{V_{\mathrm{air}}}{V_{\mathrm{air,stoich}}} \tag{7.64}$$

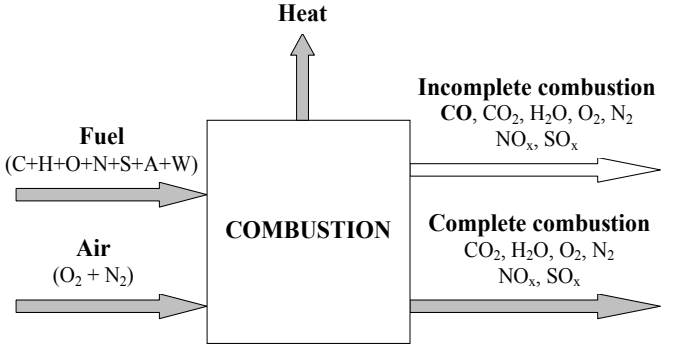

Fig. 7.4 Input and output flows of the combustion process.

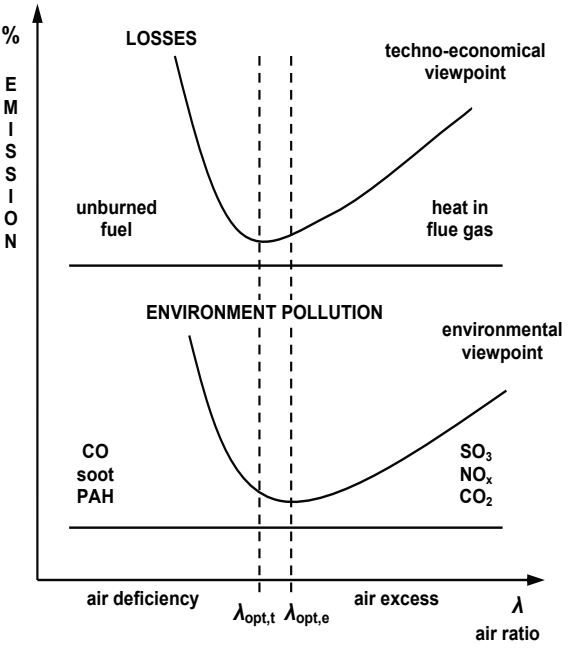

Fig. 7.5 Techno-economical and environmental viewpoints of combustion [9].

where V_{air} is the volume of the air which goes into the combustion chamber and $V_{\text{air,stoich}}$ is the stoichiometrically required volume of the air necessary for complete combustion of 1 kg fuel. The combustion plant is working with air deficiency when $\lambda < 1$, and with air excess when $\lambda > 1$. Fig. 7.5 from [9] shows the aspects of the optimal combustion of fuel. From techno-economical viewpoint, the losses of the combustion can be reduced in two ways: 1) by reducing the quantity of the unburned fuel and 2) by reducing the quantity of the flue gases, i.e. of the heat losses. This leads to the optimal value $\lambda_{\text{opt,t}}$ of the air factor (cf. Fig. 7.5). From environmental viewpoint, it is desired to minimize the quantity of the harmful emissions and the corresponding optimal value of the air factor is $\lambda_{\text{opt,e}}$ (cf. Fig. 7.5). By taking into account both the techno-economical and the environmental aspects of combustion operation, it follows that the value λ of the air factor should be kept within the interval $[\lambda_{\text{opt,t}};\ \lambda_{\text{opt,e}}]$.

It has been also shown in practice that the optimal air factor λ_{opt} depends on the load factor β defined as:

$$\beta = \frac{\Phi_{\text{fuel}}}{\Phi_{\text{fuel,max}}} \tag{7.65}$$

where Φ_{fuel} and $\Phi_{\text{fuel,max}}$ are respectively the current and the maximal allowed fuel flowrate. The relation $\lambda_{\text{opt}} = f(\beta)$ is shown in Fig. 7.6, where it can be seen that the optimal operation of the combustion plant is achieved with an air excess.

Therefore, the goal is to apply control algorithms that will maintain the air factor on its optimal value with every change of the load factor. Due to the importance of

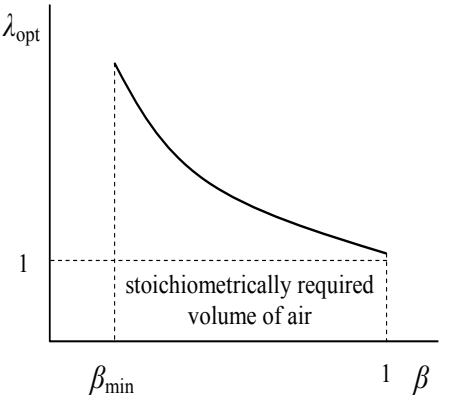

Fig. 7.6 The dependence of the optimal air factor on the load factor [9].

the described issue from economic and also environmental aspect, the combustion control is the field of constant development and research. This is also the driver for the development of the modeling and control approaches presented in the next sections.

7.3.3.2 Gaussian Process Model of a Combustion Plant

In [20], the system under investigation is a process of combustion in a steam boiler PK 401 at Cinkarna Celje Company, Celje, Slovenia. It was not possible to perform experiments on this plant during its operation because of plant safety and because interrupts in plant operation are not favored by company management. Therefore, the Gaussian process model identification was based on simulation data generated by adding a Gaussian disturbance to the analytical model developed in [12].

The fuel composition is expressed with the percentages of carbon C, hydrogen H, oxygen O, nitrogen N, sulphur S, ash A and water H_2O (denoted respectively with $x_C^{fuel}, x_H^{fuel}, x_O^{fuel}, x_N^{fuel}, x_S^{fuel}, x_A^{fuel}, x_{H_2O}^{fuel}$):

$$x_C^{fuel} + x_H^{fuel} + x_O^{fuel} + x_N^{fuel} + x_S^{fuel} + x_A^{fuel} + x_{H_2O}^{fuel} = 100\% \tag{7.66}$$

The composition of the air is assumed to be 21% oxygen and 79% nitrogen. The equations of the developed analytical model [12] are based on the stoichiometric chemical reactions of combustion:

$$C + O_2 \rightarrow CO_2 + Q_1 \tag{7.67}$$

$$C + \frac{1}{2}O_2 \rightarrow CO + Q_2 \tag{7.68}$$

$$2H_2 + O_2 \rightarrow 2H_2O + Q_3 \tag{7.69}$$

$$S + O_2 \rightarrow SO_2 + Q_4 \tag{7.70}$$

where Q_1, Q_2, Q_3, Q_4 are the heats of the reactions. The composition of the flue gases, resulting from the combustion process, is expressed in the following way [12]:

$$x_{O_2} + x_{CO} + x_{CO_2} + x_{SO_2} + x_{N_2} + x_{H_2O} = 100\% \tag{7.71}$$

where x_{O_2}, x_{CO}, x_{CO_2}, x_{SO_2}, x_{N_2} and x_{H_2O} are the volume percentages of oxygen, carbon monoxide, carbon dioxide, sulphur dioxide, nitrogen and water. Then, the volume balances for the separate components of the flue gases are described by the following equations [12]:

$$\frac{dx_{O_2}}{dt} = \frac{1}{V_k}\{-x_{O_2}[\Phi_{air} + \Phi_{fuel}(V_d - V_o)] + 21\Phi_{air} - 100V_o\Phi_{fuel}\} \tag{7.72}$$

$$\frac{dx_{CO}}{dt} = \frac{1}{V_k}\{-x_{CO}[\Phi_{air} + \Phi_{fuel}(V_d - V_o)] + 1.866ax_C^{fuel}\Phi_{fuel}\} \tag{7.73}$$

$$\frac{dx_{CO_2}}{dt} = \frac{1}{V_k}\{-x_{CO_2}[\Phi_{air} + \Phi_{fuel}(V_d - V_o)] + 1.866(1-a)x_C^{fuel}\Phi_{fuel}\} \tag{7.74}$$

$$\frac{dx_{SO_2}}{dt} = \frac{1}{V_k}\{-x_{SO_2}[\Phi_{air} + \Phi_{fuel}(V_d - V_o)] + 0.699x_S^{fuel}\Phi_{fuel}\} \tag{7.75}$$

$$\frac{dx_{N_2}}{dt} = \frac{1}{V_k}\{-x_{N_2}[\Phi_{air} + \Phi_{fuel}(V_d - V_o)] + 79\Phi_{air} + 0.8x_N^{fuel}\Phi_{fuel}\} \tag{7.76}$$

$$\begin{aligned}\frac{dx_{H_2O}}{dt} = \frac{1}{V_k}\{&-x_{H_2O}[\Phi_{air} + \Phi_{fuel}(V_d - V_o)] + 11.117x_H^{fuel}\Phi_{fuel} \\ &+1.244x_{H_2O}^{fuel}\Phi_{fuel}\}\end{aligned} \tag{7.77}$$

In (7.72)–(7.77), V_k is the volume of the combustion chamber [m^3], Φ_{fuel} is the normalized total flow of fuel [kg s^{-1}], Φ_{air} is the normalized total flow of air [N m^3 s^{-1}], V_o is the theoretically required oxygen volume for the combustion of one unit of fuel [N m^3 kg^{-1}], V_d is the theoretically obtained gas volume from one unit of fuel [N m^3 kg^{-1}], a is the relative portion of carbon converted into CO.

The model (7.72)–(7.77) enables the simulation of the six flue gases components. However, for control design purposes only its O_2-part (equation (7.72)) named also O_2-model is used [9, 10]. The input to the O_2-model is the angular position of the damper, which is used to control the air flow Φ_{air}. The model output is the oxygen concentration in the flue gases. As the damper is a part of the closed-loop, it has to be modeled and added to the O_2-model (7.72). The dependence of the air flow Φ_{air} on the angle ϕ of the damper is given by the following relation [9]:

$$\Phi_{air} = \frac{\Phi_{air,max}}{2}\exp\left(\frac{3(\phi - 45)}{45}\right),\ 0^\circ \le \phi \le 45^\circ \tag{7.78}$$

$$\Phi_{air} = \frac{\Phi_{air,max}}{2}\left(2 - \exp\left(\frac{-3(\phi - 45)}{45}\right)\right),\ 45^\circ \le \phi \le 90^\circ \tag{7.79}$$

where $\Phi_{air,max}$ is the maximum flow of air.

The O_2-model (7.72) is a deterministic model, which does not take into account the stochastic disturbances (e.g. change in the fuel composition, change of the humidity of the air flow) that may influence the combustion process. In order to consider the stochastic nature of plant operation, the dynamics of x_{O_2} is represented by the following stochastic discrete-time model [20]:

$$x_{O_2}(t+1) = f(x_{O_2}(t), \Phi_{\text{fuel}}(t), \phi(t)) + \xi(t) \tag{7.80}$$

Here, $\xi(t) \in \mathbb{R}$ is a Gaussian disturbance which represents the additive effect of the unmeasured stochastic disturbances. The sampling time, determined according to system dynamics, was selected to be $T_s = 1$ $[s]$.

In [20], the signals ϕ and Φ_{fuel} for identification were generated by random number generators with normal distributions. The signal x_{O_2} was computed from the O_2-model (7.72) and a Gaussian disturbance ξ with zero mean and variance 0.05 was added to it. The ϕ signal blocking was $T_\phi = 5T_s$, i.e. it is kept constant for 5 time instants. The Φ_{fuel} signal blocking was $T_{\Phi_{\text{fuel}}} = 100T_s$. The number M of the signals samples used for the identification determines the dimension of the covariance matrix. In our case, $M = 1000$. Based on the generated data set, the discrete-time system (7.80) is approximated with a Gaussian process with the following hyperparameters [20]:

$$\Theta = [w_1, w_2, w_3, v_0, v_1] = [0.01346, 0.02847, 0.00036, 0.21984, 55.56554] \tag{7.81}$$

The maximum likelihood framework was used to determine the hyperparameters. The optimization method applied for identification of the Gaussian process model was the conjugate gradient method with line searches [16]. The response of the Gaussian process model to the identification signal is shown in Fig. 7.7. The associated average squared error and log density error are respectively ASE=0.6051 and LD=143.4835.

In [20], the signals ϕ and Φ_{fuel} for validation were generated by random number generator with normal distribution and rate of change that is different from the one used for the identification signals. The mean and the variance of x_{O_2} predicted with the identified Gaussian process model are obtained by iterative one-step ahead predictions, where at each step the predicted mean of x_{O_2} is fed back to the input. The response of the Gaussian process model to the validation signals is shown in Fig. 7.8. The associated prediction errors are ASE=0.9177 and LD=188.8626.

7.3.3.3 Design and Performance of Explicit Stochastic Reference Tracking Controller for the Combustion Plant

In [20], an explicit stochastic reference tracking GP-NMPC controller for the combustion plant considered in Section 7.3.3.2 is designed. The block-scheme of the control system is shown in Fig. 7.9. The controller brings the air factor (respectively the concentration of oxygen in the flue gases) on its optimal value with every change of the load factor and thus an optimal operation of the combustion plant is achieved.

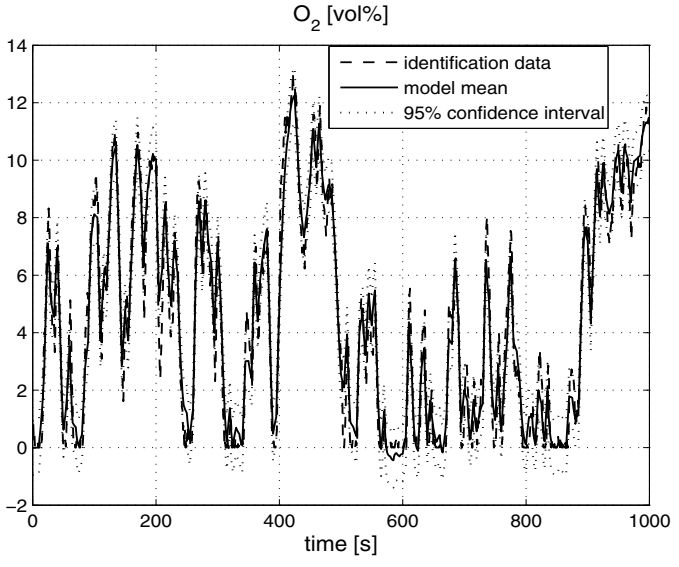

Fig. 7.7 Response of the Gaussian process model to the excitation signal used for identification.

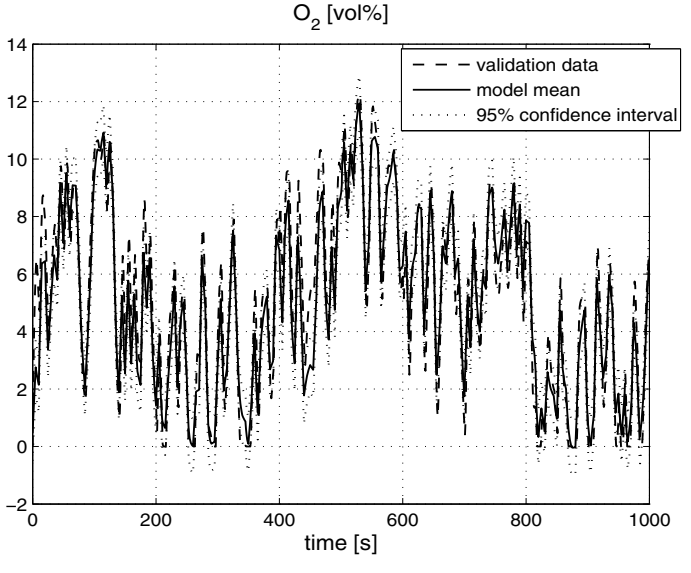

Fig. 7.8 Response of the Gaussian process model to the excitation signal used for validation.

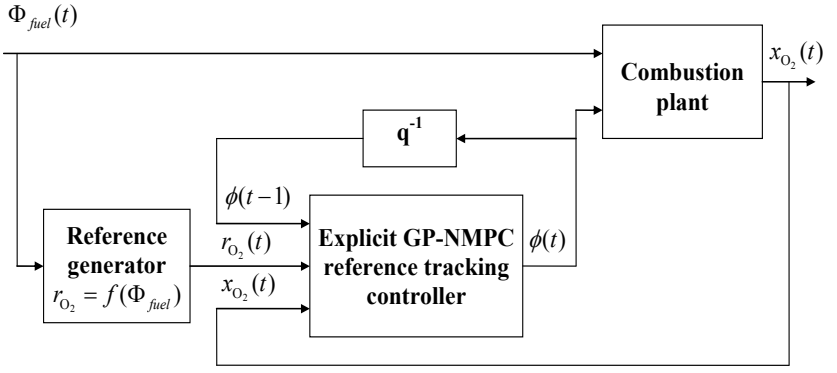

Fig. 7.9 Block scheme of the control system.

Table 7.1 Reference values for the percentage of O_2 in the flue gases

Φ_{fuel} [kg s^{-1}]	0.7	0.8	0.9	1.0	1.1	1.2	1.3
r_{O_2} [vol%]	4.5	4.1	3.7	3.4	3.2	3.0	2.8

The control input is $u = \phi$ (the angle of the damper for the air flow), the state variable is $x = x_{O_2}$ (the percentage of O_2 in the flue gases), and the reference signal is $r = r_{O_2}$ (the required percentage of O_2 in the flue gases). For this particular combustion plant, the reference values r_{O_2} corresponding to different values of the fuel flowrate Φ_{fuel} have been obtained by experiments and are given in Table 7.1 [9]. In case the fuel flowrate Φ_{fuel} does not take a value from this table, then the reference value r_{O_2} is computed through linear interpolation between the neighboring points in the table.

The mp-NLP approach described in Section 1.1.5.2 is applied to design an explicit stochastic reference tracking GP-NMPC controller for the combustion plant based on its Gaussian process model obtained in Section 7.3.3.2:

$$x_{O_2}(t+1)\,|\,x_{O_2}(t), \Phi_{\text{fuel}}(t), \phi(t) \sim \mathcal{N}(\mu(x_{O_2}(t+1)), \sigma^2(x_{O_2}(t+1))) \tag{7.82}$$

The following control input and rate constraints are imposed on the plant:

$$30^\circ \leq \phi \leq 60^\circ\,,\ -3^\circ \leq \Delta\phi \leq 3^\circ \tag{7.83}$$

The prediction horizon is $N = 10$ and the terminal constraint is:

$$\left|\mu(x_{O_2}(t+N)) - r_{O_2}(t)\right| \leq 0.001 \tag{7.84}$$

The weighting matrices in the cost function (7.58) are $Q = 20$, $R = 1$, $P = 20$. The GP-NMPC minimizes the cost function (7.58) subject to the Gaussian process model (7.82) and the constraints (7.83)–(7.84). The parameter vector is $\tilde{x}(t) = [x_{\mathrm{O}_2}(t), \Phi_{\mathrm{fuel}}(t), \phi(t-1)] \in \mathbb{R}^3$, which leads to a 3-dimensional parameter space to be partitioned. The latter is defined by $X = [0; 7] \times [0.7; 1.3] \times [30; 60]$. The cost function approximation tolerance is chosen as $\bar{\varepsilon}(X_0) = \max(\bar{\varepsilon}_a, \bar{\varepsilon}_r \min_{\tilde{x} \in X_0} V^*(\tilde{x}))$, where $\bar{\varepsilon}_a = 0.005$ and $\bar{\varepsilon}_r = 0.1$ are the absolute and the relative tolerances, respectively. The partition of the explicit GP-NMPC controller is shown in Fig. 7.10. It has 513 regions and 12 levels of a binary search tree representation. Totally, 18 arithmetic operations are needed in real-time to compute the control input by traversing the binary search tree (12 comparisons, 3 multiplications and 3 additions).

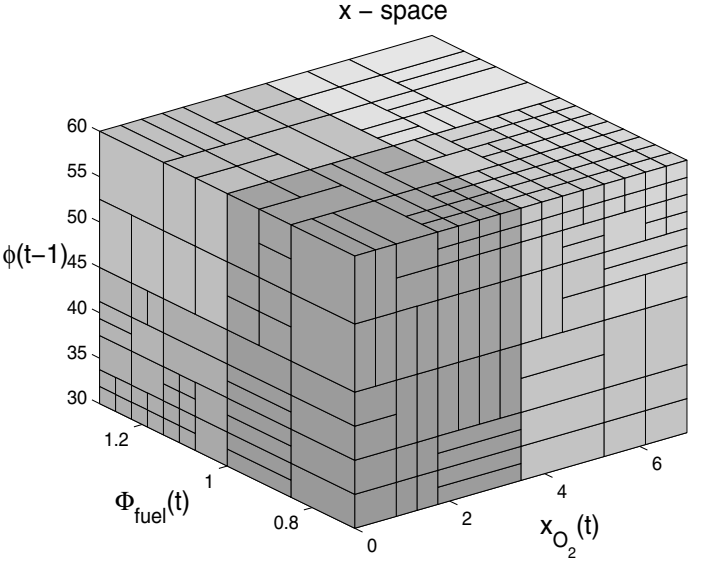

Fig. 7.10 Parameter space partition of the explicit approximate GP-NMPC controller.

The performance of the closed-loop system was simulated for the following change in the fuel flowrate:

$$\begin{aligned} \Phi_{\mathrm{fuel}}(t) &= 1.1\,[\mathrm{kg\,s^{-1}}], t \in [0; 50]; \\ \Phi_{\mathrm{fuel}}(t) &= 1.25\,[\mathrm{kg\,s^{-1}}], t \in [51; 100]; \\ \Phi_{\mathrm{fuel}}(t) &= 1.05\,[\mathrm{kg\,s^{-1}}], t \in [101; 150] \end{aligned} \tag{7.85}$$

and initial conditions for the state and control variable $x_{\mathrm{O}_2}(0) = 3.3\,[\mathrm{vol\%}]$ and $\phi(0) = 46^\circ$, respectively. The resulting closed-loop response is depicted in Fig. 7.11 and Fig. 7.12.

The results show that the exact and the approximate solutions are almost indistinguishable.

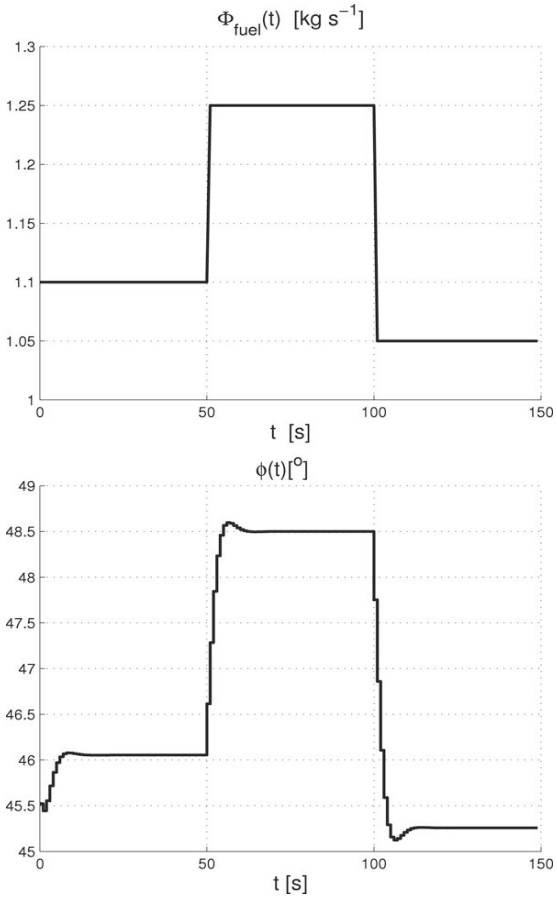

Fig. 7.11 Top: Change of the fuel flowrate. Bottom: The control input with the approximate explicit GP-NMPC (the solid curve) and with the exact GP-NMPC (the dotted curve).

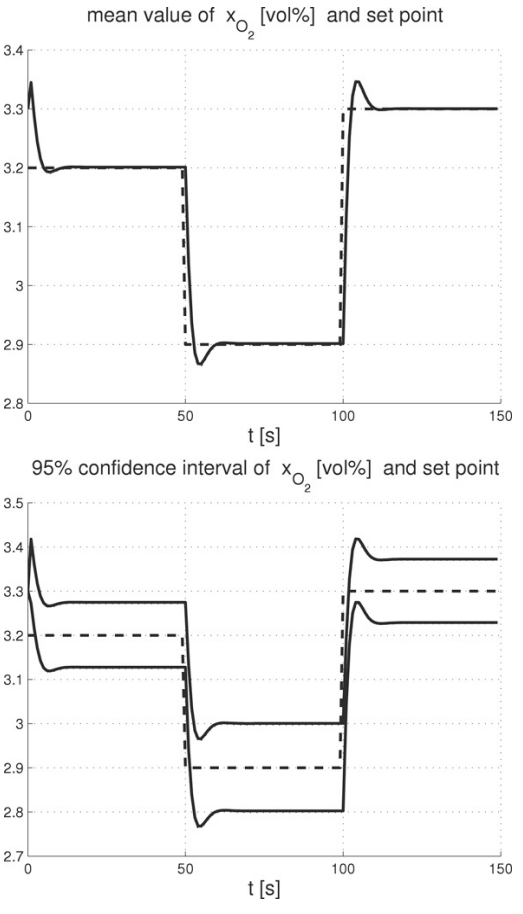

Fig. 7.12 Top: The mean value of the state variable predicted with the Gaussian process model. Bottom: The 95% confidence interval of the state variable predicted with the Gaussian process model. The solid curves are with the approximate explicit GP-NMPC, the dotted curves are with the exact GP-NMPC and the dashed curve is the set point.

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