Chapter 6
Explicit Min-Max MPC of Constrained Nonlinear Systems with Bounded Uncertainties

Abstract. This chapter considers two approaches to explicit min-max NMPC of general constrained nonlinear discrete-time systems in the presence of bounded disturbances and/or parameter uncertainties. The approach in Section 6.2 is based on an *open-loop* min-max NMPC formulation and constructs a piecewise linear (PWL) approximation of the optimal solution. An explicit *open-loop* min-max NMPC controller is designed for a continuous stirred tank reactor, whose heat transfer coefficient is an uncertain parameter. The approach in Section 6.3 adopts a *closed-loop* (also referred to as *feedback*) min-max NMPC formulation and builds a piecewise nonlinear (PWNL) approximation of the optimal sequence of feedback control policies. The approach is applied to design an explicit *feedback* min-max NMPC controller for a cart and spring system in the presence of bounded disturbances.

6.1 Introduction

Models are only an approximation of the real process, and therefore it is important for NMPC to be robust with respect to model uncertainties and disturbances. One approach to robust NMPC design is to optimize the *nominal* performance while guaranteeing robust feasibility and robust stability of the closed-loop system. Thus in [25], a Lyapunov-based robust NMPC design for input-affine nonlinear systems subject to uncertainty and input constraints is developed, which allows for an explicit characterization of the closed-loop stability region. Another robust NMPC strategy consists of solving a min-max problem to optimize the *robust* performance while enforcing the state and input constraints for all possible uncertainties. The min-max robust MPC was first proposed in [5]. There are two formulations of min-max NMPC: the *open-loop* and the *closed-loop* (also referred to as *feedback*) formulation (see [22] for review of the min-max NMPC approaches). The *open-loop* min-max NMPC [26, 19, 22] guarantees the robust stability and the robust feasibility of the system, but it may be very conservative since the control sequence has to ensure constraints fulfillment for all possible uncertainty scenarios without considering the fact that future measurements of the state contain information about past

A. Grancharova & T.A. Johansen: Explicit Nonlinear Model Predictive Ctrl., LNCIS 429, pp. 127–156.
springerlink.com

uncertainty values. As a result, the *open-loop* min-max NMPC controllers may have a small feasible set and sub-optimal performance. An approximate multi-parametric Nonlinear Programming (mp-NLP) approach to explicit solution of *open-loop* min-max NMPC problems has been suggested in [8]. This approach is considered in Section 6.2.

The conservativeness of the *open-loop* approaches is overcome by the *closed-loop* min-max NMPC [21, 22, 20], where the optimization is performed over a sequence of feedback control policies. With the *closed-loop* approach, the min-max NMPC problem represents a differential game where the controller is the minimizing player and the disturbance is the input of the maximizing player ('the nature') [21]. The controller chooses the control input as a function of the current state so as to ensure that the effect of the disturbance on the system output is sufficiently small for any choice made by 'the nature'. In this way, the *closed-loop* min-max NMPC would guarantee a larger feasible set and a higher level of performance compared to the *open-loop* min-max NMPC [21]. Recently, several approaches have been developed for explicit solution of min-max MPC problems for special classes of uncertain nonlinear systems. Thus, for constrained linear systems with polytopic uncertainty, approaches for explicit solution of the *open-loop* and the *closed-loop* min-max MPC problems have been developed, respectively in [6] and in [31, 4, 29]. The method in [2] applies to linear systems with polyhedral parametric uncertainty and additive bounded disturbances and both the *open-loop* and the *closed-loop* min-max control problems are solved explicitly. Approaches for explicit solution of *robust* finite horizon optimal control problems for constrained piecewise affine systems with bounded disturbances have been proposed, based on an *open-loop* formulation in [27], and on a *closed-loop* formulation in [16, 30]. Methods for explicit solution of min-max MPC or H_∞ problems for constrained linear systems with additive bounded uncertainties are suggested in [28] for the *open-loop* formulation, and in [15, 24] for the *closed-loop* formulation. In [11], an approximate mp-NLP approach to explicit solution of *closed-loop* min-max NMPC problems for general nonlinear systems with state and input constraints has been developed. This approach is considered in Section 6.3.

6.2 Explicit Open-Loop Min-Max MPC of Constrained Nonlinear Systems with Bounded Uncertainties

This section considers the approximate mp-NLP approach [8] to explicit solution of *open-loop* min-max NMPC problems for constrained nonlinear systems in the presence of model uncertainty. It is based on an orthogonal search tree structure of the state space partition and thus represents an extension of the approach in [14]. The explicit NMPC controller is designed by formulating a min-max optimization problem, i.e. by minimizing the worst-case with respect to the uncertain parameters cost function value. The controller formulation is *robust* in the sense that all constraints are attempted satisfied for all possible values of the uncertain parameters.

6.2.1 Formulation of the Open-Loop Min-Max NMPC Problem as an mp-NLP Problem

Consider the discrete-time nonlinear system:

$$x(t+1) = f(x(t), u(t), \theta) \tag{6.1}$$

$$y(t) = Cx(t) \tag{6.2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^p$ are the state, input and output variable, θ is the vector of time-invariant uncertain parameters that is assumed to belong to a bounded polyhedral set $\theta \in \Theta^A \subset \mathbb{R}^s$. It is assumed that the function f is sufficiently smooth. It is also supposed that a full measurement of the state $x(t)$ is available at the current time t. We consider the following *open-loop* robust NMPC problem: For the current $x(t)$, NMPC minimizes the worst-case cost function through the following optimization:

Problem 6.1:

$$V^*_{\max}(x(t)) = \min_{U} \max_{\theta \in \Theta^A} J(U, x(t), \theta) \tag{6.3}$$

subject to $x_{t|t} = x(t)$ and:

$$y_{\min} \leq y_{t+k|t} \leq y_{\max}\,,\ \forall \theta \in \Theta^A\,,\ k = 1, \dots, N \tag{6.4}$$

$$u_{\min} \leq u_{t+k} \leq u_{\max}\,,\ k = 0, 1, \dots, N-1 \tag{6.5}$$

$$x^T_{t+N|t} x_{t+N|t} \leq \delta\,,\ \forall \theta \in \Theta^A \tag{6.6}$$

$$x_{t+k+1|t} = f(x_{t+k|t}, u_{t+k}, \theta)\,,\ \theta \in \Theta^A\,,\ k \geq 0 \tag{6.7}$$

$$y_{t+k|t} = Cx_{t+k|t}\,,\ k \geq 0 \tag{6.8}$$

with $U = [u_t, u_{t+1}, \dots, u_{t+N-1}]$ and the cost function given by:

$$J(U, x(t), \theta) = \sum_{k=0}^{N-1} \left[x^T_{t+k|t} Q x_{t+k|t} + u^T_{t+k} R u_{t+k} \right] + x^T_{t+N|t} P x_{t+N|t} \tag{6.9}$$

Here, N is a finite horizon. The formulation implies that a direct single shooting strategy is employed, see Section 2.2.2.1, i.e. the equality constraints (6.7)–(6.8) are substituted and eliminated in the cost and constraint functions. In (6.3), the existence of the minimum and maximum are implicitly assumed. From a stability point of view it is desirable to choose δ in (6.6) as small as possible [23]. However, due to the fact that $x_{t+N|t}$ depends on the unknown θ, the feasibility of Problem 6.1 will rely on δ being sufficiently large. A part of the NMPC design will be to address this trade-off. If the system is asymptotically stable (or pre-stabilized), N is large, and possibly an integral action is introduced to account for the steady-state effect of the uncertainty, then it is more likely that the choice of a small δ will be possible.

The following assumptions are made:

Assumption 6.1. $P, Q, R \succ 0$.

Assumption 6.2. $y_{\min} < 0 < y_{\max}$.

Assumption 6.3. *θ is time-invariant uncertainty that belongs to a bounded polyhedral set, i.e. $\theta = \text{const} \in \Theta^A$. The polyhedral set Θ^A is defined by $\Theta^A = \{\theta \in \mathbb{R}^s \,|\, \theta^L \leq \theta \leq \theta^U\}$, where θ^L and θ^U represent given lower and upper bounds on θ.*

Assumption 6.4. *For each $\theta \in \Theta^A$ there exists $u_{st} \in \mathbb{R}^m$ satisfying $u_{\min} \leq u_{st} \leq u_{\max}$, and such that $f(0, u_{st}, \theta) = 0$.*

Assumption 6.4 means that the point $x = 0$, $u = u_{st}$ is a feasible steady state point for system (6.1)–(6.2). It also implies that the steady state value of the control input may be different for the different values of the uncertain parameters.

The worst-case value of cost function (6.9) with respect to the uncertain parameters is denoted by:

$$V_{\max}(U, x(t)) = \max_{\theta \in \Theta^A} J(U, x(t), \theta) \tag{6.10}$$

An optimal solution to the min-max NMPC Problem 6.1 is denoted $U^* = [u_t^*, u_{t+1}^*, \dots, u_{t+N-1}^*]$ and the control input is chosen according to the receding horizon policy $u(t) = u_t^*$. The optimization problem can be formulated in a compact form as follows:

Problem 6.2:

$$V_{\max}^*(x(t)) = \min_U \max_{\theta \in \Theta^A} J(U, x(t), \theta) \text{ subject to } G(U, x(t), \theta) \leq 0, \forall \theta \in \Theta^A \tag{6.11}$$

This min-max NMPC problem defines an mp-NLP, since it is NLP in U parameterized by x. Since the equality constraints are eliminated by the direct single shooting strategy, (6.11) contains only inequality constraints. Define the set of N-step robustly feasible initial states as follows:

$$X_f = \{x \in \mathbb{R}^n \,|\, G(U, x, \theta) \leq 0, \forall \theta \in \Theta^A \text{ for some } U \in \mathbb{R}^{Nm}\} \tag{6.12}$$

If Assumption 6.4 is satisfied and δ is chosen such that the Problem 6.1 is feasible, then X_f is a non-empty set. Then, due to Assumption 6.2, the origin is an interior point in X_f.

6.2.2 Approximate mp-NLP Approach to Explicit Open-Loop Min-Max NMPC

The numerical computations involved in constructing the approximate explicit state feedback are simplified under the following convexity assumption:

Assumption 6.5. *$G(U,x,\theta)$ is jointly convex for all $(U, x, \theta) \in U^A \times X \times \Theta^A$, where $U^A = [u_{\min}, u_{\max}]^N$ is the set of admissible inputs and $X \subseteq X_f \subseteq \mathbb{R}^n$ is a polytopic set.*

We exploit the result in [12], where it has been shown that if the constraint function $G(U,x,\theta)$ is jointly convex in U and θ, and there is U that is feasible at the vertices of Θ^A, then U is feasible for all $\theta \in \Theta^A$. This is formulated in the following lemma:

Lemma 6.1. *Suppose Assumptions 6.3 and 6.5 hold and denote the vertices of the polyhedron $\Theta^A \subset \mathbb{R}^s$ with $\{\theta_1, \theta_2, \ldots, \theta_L\}$. Denote also $\widetilde{G}^i(U,x) = G(U,x,\theta_i)$. If there exist U that satisfies the following constraints:*

$$\widetilde{G}^i(U,x) \leq 0\,,\ i \in \{1, 2, \ldots, L\} \tag{6.13}$$

then U satisfies the constraints in (6.11).

Thus, we can replace the infinite number of constraints in (6.11) with the following finite set of jointly convex constraints which are function only of U and x:

$$\widetilde{G}(U,x) \leq 0\,,\ \widetilde{G}(U,x) = \{\widetilde{G}^i(U,x)\,,\ i = 1, 2, \ldots, L\} \tag{6.14}$$

Then, the Problem 6.2 can be reformulated as:

Problem 6.3:

$$V^*_{\max}(x) = \min_U V_{\max}(U,x) \text{ subject to } \widetilde{G}(U,x) \leq 0 \tag{6.15}$$

where $V_{\max}(U,x)$ is defined by (6.10).

Problem 6.3 defines a mp-NLP problem, since it is an NLP in U parameterized by x. In case the Problem 6.3 is convex, its approximate solution can be found by applying the approximate mp-NLP approach, described in Section 1.1.5.1. Otherwise, the approximate mp-NLP approach from Section 1.1.5.2 should be used, where in addition to the set of vertices of a given hyper-rectangle in the parameter space, the optimal solution is also searched for at several interior points and global optimization methods are applied. Further, if Assumption 6.5 does not hold, then it would not be sufficient to consider the constraints $G(U,x,\theta)$ only at the vertices of the set Θ^A, i.e. it would not be possible to apply Lemma 6.1, but it would be advisable to impose these constraints also at a finite set of interior points of the set Θ^A.

6.2.3 Application 1: Min-Max MPC of a Continuous Stirred Tank Reactor

The considered approximate mp-NLP approach is applied to design an explicit min-max NMPC controller for the continuous stirred tank reactor (CSTR), described in Section 5.5. We consider the set point $\tilde{c}^* = 0.41$, $\tilde{T}^* = 3.3$. Then, the model of the reactor can be written in the form [13]:

$$\frac{dx_1}{dt} = \frac{(1-\tilde{c}^* - x_1)}{q} - k_0 e^{-\frac{E}{(\tilde{T}^*+x_2)}}(\tilde{c}^* + x_1) \tag{6.16}$$

$$\frac{dx_2}{dt} = \frac{(\tilde{T}_f - \tilde{T}^* - x_2)}{q} + k_0 e^{-\frac{E}{(\tilde{T}^*+x_2)}}(\tilde{c}^* + x_1) - \alpha u(\tilde{T}^* + x_2 - \tilde{T}_c) \tag{6.17}$$

where x_1 and x_2 denote the deviations of the dimensionless concentration and temperature from the set point values ($x_1 = \tilde{c} - \tilde{c}^*$, $x_2 = \tilde{T} - \tilde{T}^*$). The coolant flow-rate u is a real-valued control variable. The heat transfer coefficient α is an uncertain parameter that belongs to the interval:

$$1.9 \cdot 10^{-4} \leq \alpha \leq 2.5 \cdot 10^{-4} \tag{6.18}$$

The values of the other parameters are given in Section 5.5. The coolant flow-rate is constrained to be:

$$0 \leq u \leq 600 \tag{6.19}$$

We discretize the model (6.16)–(6.17) using a sampling time $T_s = 1$. The forward Euler method with step size $T_E = 0.01$ is used to integrate the equations (6.16)–(6.17).

The mp-NLP formulation described in Section 6.2.2 is applied to design an explicit *open-loop* min-max NMPC controller for this reactor. The NMPC minimizes the worst-case (maximal) value with respect to the uncertain parameter $\theta = \alpha$ of the cost function (6.9) subject to the system equations (6.16)–(6.17) and the input constraint (6.19). In (6.9), the cost matrices are $Q = P = \mathrm{diag}\{100, 300\}$, $R = 1 \cdot 10^{-6}$. The horizon is $N = 30$. In (6.6), it is chosen $\delta = 0.002$. The state space to be partitioned is defined by $X = [-0.4, 0.6] \times [-0.4, 0.5]$.

The state space partition of the approximate min-max NMPC controller resulting from the algorithms and procedures in Section 1.1.5.2 is shown in Fig. 6.1. It has 94 regions and 10 levels of search. With one scalar comparison required at each level of the $k-d$ tree, 10 arithmetic operations are required in the worst case to determine which region the state belongs to. Totally, 14 arithmetic operations are needed in real-time to compute the control input (10 comparisons, 2 multiplications and 2 additions).

The performance of the closed-loop system was simulated for initial condition $x(0) = [0.58,\ 0.3]^T$ and for three values of the uncertain parameter ($\alpha = 1.9 \cdot 10^{-4}$, $\alpha = 2.2 \cdot 10^{-4}$, $\alpha = 2.5 \cdot 10^{-4}$). The resulting closed-loop response is depicted in the state space (Fig. 6.1), as well as trajectories in time (Fig. 6.2 and Fig. 6.3). It can be seen that the explicit approximate min-max NMPC controller brings the reactor to the desired set point despite of the model uncertainty, and the constraints imposed on the system are satisfied. In order to avoid a possible offset, the dual-mode control strategy of [26] was applied and a locally stabilizing control law was used in a neighborhood of the origin.

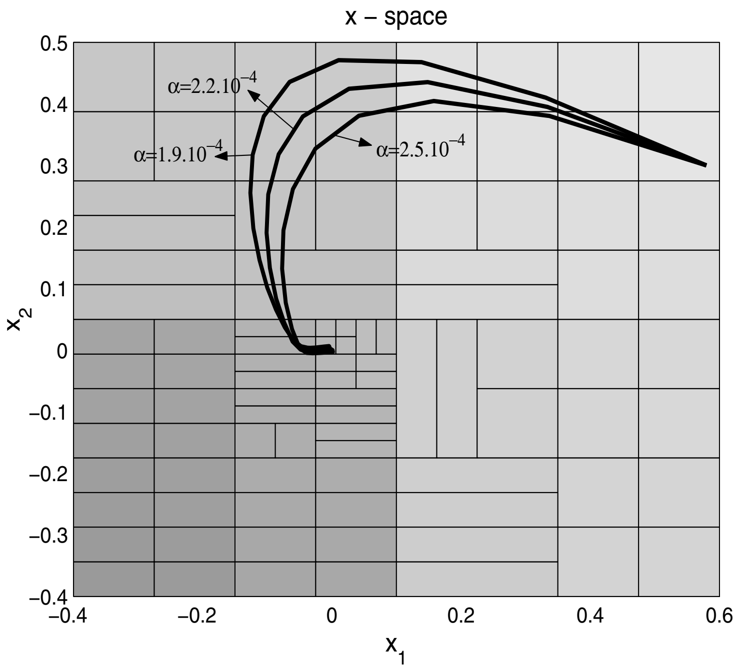

Fig. 6.1 State space partition of the approximate explicit open-loop min-max NMPC and the state trajectories corresponding to $\alpha = 1.9 \cdot 10^{-4}$, $\alpha = 2.2 \cdot 10^{-4}$, $\alpha = 2.5 \cdot 10^{-4}$.

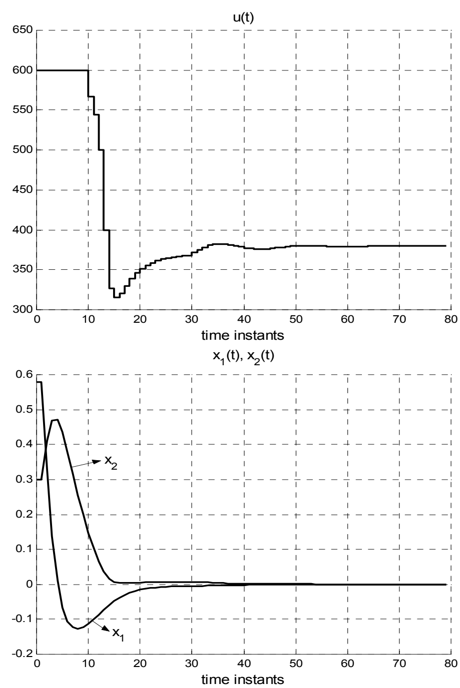

Fig. 6.2 Control input and state trajectory corresponding to $\alpha = 1.9 \cdot 10^{-4}$.

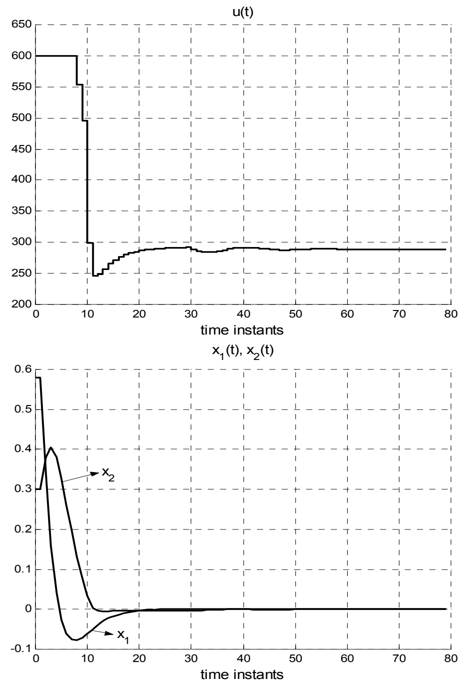

Fig. 6.3 Control input and state trajectory corresponding to $\alpha = 2.5 \cdot 10^{-4}$.

6.3 Explicit Closed-Loop Min-Max MPC of Constrained Nonlinear Systems with Bounded Uncertainties

This section considers the approximate mp-NLP approach [11] to explicit solution of *closed-loop* (*feedback*) min-max NMPC problems for general constrained nonlinear discrete-time systems in the presence of bounded disturbances and/or parameter uncertainties. The approach consists in constructing a *piecewise nonlinear* (PWNL) approximation to the optimal sequence of feedback control policies, defined on an orthogonal state space partition. Conditions guaranteeing the l_2-stability of the closed-loop system are derived.

6.3.1 Formulation of the Closed-Loop Min-Max NMPC Problem as an mp-NLP Problem

Consider the discrete-time nonlinear system:

$$\begin{aligned} x(t+1) &= f(x(t), u(t), w(t)) \\ y(t) &= h(x(t), u(t), w(t)), \end{aligned} \tag{6.20}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^r$ and $w(t) \in \mathbb{R}^q$ are the state, input, output and disturbance variable. The following constraints are imposed:

$$u_{\min} \leq u(t) \leq u_{\max}, \, y_{\min} \leq y(t) \leq y_{\max} \tag{6.21}$$

Following [21],

Assumption 6.6. *f and h are C^2 functions with $f(0,0,0) = 0$, $h(0,0,0) = 0$.*

Assumption 6.7. *$y_{\min} < 0 < y_{\max}$ and $u_{\min} < 0 < u_{\max}$.*

Assumption 6.8. *Let $\widetilde{X}$ be a non-empty set containing the origin as an interior point, and let t_0 be a positive integer. The system (6.20) is zero-state detectable in $\widetilde{X}$, i.e. $\forall x(0) \in \widetilde{X}$ and $\forall u(\cdot)$ such that constraints (6.21) are satisfied $\forall t \geq 0$ and $x(t) \in \widetilde{X}$, $\forall t \geq t_0$, we have $y(t)|_{w=0} = 0$, $\forall t \geq t_0 \Rightarrow \lim_{t \to \infty} x(t) = 0$.*

Assumption 6.9. *There exists a positive constant γ_Δ, such that the disturbance w satisfies:*

$$\|w(t)\|^2 \leq \gamma_\Delta^2 \|y(t)\|^2, t \geq t_0. \tag{6.22}$$

Let $x(t) = x$ and $u(t) = u$. Then, the space of the admissible disturbances is denoted by $W^A(u,x) \subset \mathbb{R}^q$. As mentioned in [21], inequality (6.22) can also represent a wide class of modeling errors. As in [21], first a H_∞ control problem is defined:

Definition 6.1 (H_∞ control problem). Design a state-feedback control law:

$$u = k(x) \tag{6.23}$$

guaranteeing that the closed-loop system (6.20)–(6.23) with input $w \in W^A(u,x)$ and output y has a finite l_2-gain $\leq \gamma$ in a bounded positively invariant set Ω, that is, $\forall x(t) \in \Omega$:

i. $x(t+i) \in \Omega, \forall i > 0$.
ii. $u_{\min} \leq k(x(t+i)) \leq u_{\max}$ and $y_{\min} \leq h(x(t+i), k(x(t+i)), w(t+i)) \leq y_{\max}$, $\forall i \geq 0$.
iii. There exists a positive definite function $\beta(x(t))$, such that $\forall T \geq 0$:

$$\sum_{i=0}^{T} \|y(t+i)\|^2 \leq \gamma^2 \sum_{i=0}^{T} \|w(t+i)\|^2 + \beta(x(t)) \tag{6.24}$$

for any non-zero $w \in W^A(u,x)$.

The following assumption is also made [21]:

Assumption 6.10. *Suppose that there exists an auxiliary control law $u = k_a(x)$ that solves the H_∞ control problem, with a domain of attraction Ω_a, whose boundary is assumed to be a level curve of a positive function $V_{k_a}(x)$ such that:*

$$V_{k_a}(f(x, k_a(x), w)) - V_{k_a}(x) \leq -\frac{1}{2}(\|y\|^2 - \gamma^2 \|w\|^2),$$
$$\forall x \in \Omega_a, \, \forall w \in W^A(u,x) \tag{6.25}$$

and $V_{k_a}(0) = 0$.

Definition 6.2 (Admissible disturbance realization). Let $K = \{k_0, k_1, \dots, k_{N-1}\} \triangleq \{k_0(x_{t|t}), k_1(x_{t+1|t}), \dots, k_{N-1}(x_{t+N-1|t})\}$ be a vector of feedback control policies and N be a finite horizon. Consider the closed-loop system for $i = 0, 1, 2, \dots, N-1$:

$$\begin{aligned} x_{t+i+1|t} &= f(x_{t+i|t}, k_i(x_{t+i|t}), w_{t+i}) \\ y_{t+i|t} &= h(x_{t+i|t}, k_i(x_{t+i|t}), w_{t+i}) \end{aligned} \tag{6.26}$$

with initial state $x_{t|t} = x$. Then, the disturbance realization $W = \{w_t, \dots, w_{t+N-1}\} \in \mathbb{R}^{qN}$ is admissible for the given K and x if the following holds:

$$\|w_{t+i}\|^2 \leq \gamma_\Delta^2 \|y_{t+i|t}\|^2, \; i = 0, 1, 2, \dots, N-1. \tag{6.27}$$

The space of the admissible disturbance realizations over horizon N and corresponding to the given K and x is denoted by $W^B(K, x) \subset \mathbb{R}^{qN}$.

It is supposed that a full measurement x of the state is available at the current time t. We consider the *feedback* min-max NMPC problem [22]:

Definition 6.3 (Constrained feedback min-max NMPC problem). Suppose that Assumptions 6.6–6.10 hold. For the current x, the *feedback* min-max NMPC solves the following optimization problem:

$$V^o_{\max}(x) = \min_K \max_{W \in W^B(K,x)} J(K, x, W) \tag{6.28}$$

subject to $x_{t|t} = x$ and:

$$y_{\min} \leq y_{t+i|t} \leq y_{\max}, \; i = 1, \dots, N-1 \tag{6.29}$$

$$u_{\min} \leq u_{t+i} \leq u_{\max}, \; i = 0, 1, \dots, N-1 \tag{6.30}$$

$$x_{t+N|t} \in \Omega_a \tag{6.31}$$

$$u_{t+i} = k_i(x_{t+i|t}), \; i = 0, 1, \dots, N-1 \tag{6.32}$$

$$x_{t+i+1|t} = f(x_{t+i|t}, u_{t+i}, w_{t+i}), \; w_{t+i} \in W^A(u_{t+i}, x_{t+i|t}), \; 0 \leq i \leq N-1 \tag{6.33}$$

$$y_{t+i|t} = h(x_{t+i|t}, u_{t+i}, w_{t+i}), \; w_{t+i} \in W^A(u_{t+i}, x_{t+i|t}), \; 0 \leq i \leq N-1 \tag{6.34}$$

and the cost function given by:

$$J(K, x, W) = \frac{1}{2} \sum_{i=0}^{N-1} \left[\|y_{t+i|t}\|^2 - \gamma^2 \|w_{t+i}\|^2 \right] + V_{k_a}(x_{t+N|t}) \tag{6.35}$$

Here, N is the finite horizon and γ is the l_2-gain which is interpreted as the disturbance attenuation level. Note that in (6.28)–(6.35) w_{t+i} denotes a single disturbance at time instant $t+i$, while W is an admissible disturbance realization as specified in Definition 6.2. An auxiliary control law $k_a(x)$ is typically obtained by solving the H_∞ control problem for the linearized system [26]. Thus, a practical way to compute a nonlinear control $k_a(x)$ satisfying Assumption 6.10 is suggested in [21].

An optimal solution to the *feedback* min-max NMPC problem (6.28)–(6.35) is denoted $K^o = \{k_0^o, k_1^o, \ldots, k_{N-1}^o\} \triangleq \{k_0^o(x_{t|t}), k_1^o(x_{t+1|t}), \ldots, k_{N-1}^o(x_{t+N-1|t})\}$ and the control input is chosen according to the receding horizon policy $u(x_{t|t}) = k_0^o(x_{t|t})$. It is assumed that:

Assumption 6.11. *Each feedback control policy $k_i(x_{t+i|t})$, $i = 0, 1, \ldots, N-1$, has the form:*

$$k_i(x_{t+i|t}) = \alpha_i k_a(x_{t+i|t}) + r_i(\xi_i, x_{t+i|t}) = g_i(p_i, x_{t+i|t}), \tag{6.36}$$

where $p_i = [\alpha_i^T \ \xi_i^T]^T \in \mathbb{R}^{n_i}$ are the parameters that need to be optimized, $k_a(x_{t+i|t})$ is an auxiliary control law that satisfies Assumption 6.10, and $r_i(\xi_i, x_{t+i|t})$ is a continuous function with $r_i(\xi_i, 0) = 0$.

In general, the parameterization of the form (6.36) would lead to an approximate solution to the *feedback* min-max NMPC problem (6.28)–(6.35). Denote with P the whole set of parameters that need to be determined, i.e. $P = [p_0^T \ p_1^T \ \ldots \ p_{N-1}^T]^T \in \mathbb{R}^{n_p}$, where $n_p = \sum_{i=0}^{N-1} n_i$. Then, the worst-case value of cost function (6.35) with respect to the disturbances is denoted by:

$$V_{\max}(P, x) = \max_{W \in W^B(P,x)} J(P, x, W) \tag{6.37}$$

Note that the argument K is now substituted with the argument P. Using a direct single shooting strategy to eliminate all the equality constraints (6.32)–(6.34), the optimization problem (6.28)–(6.35) can be formulated in a compact form as follows [11]:

Problem 6.4:

$$V_{\max}^o(x) = \min_P \max_{W \in W^B(P,x)} J(P, x, W) \tag{6.38}$$

$$\text{subject to } G(P, x, W) \le 0, \ \forall W \in W^B(P, x) \tag{6.39}$$

Problem 6.4 defines an mp-NLP, since it is NLP in P parameterized by x. We remark that the constraints function $G(P, x, W)$ in (6.39) is implicitly defined by (6.29)–(6.34). Define the set of N-step robustly feasible initial states:

$$X_f = \{x \in \mathbb{R}^n \,|\, G(P, x, W) \le 0, \ \forall W \in W^B(P, x) \text{ for some } P \in \mathbb{R}^{n_p}\} \tag{6.40}$$

If the problem (6.28)–(6.35) is feasible, then X_f is a non-empty set. Then, due to Assumption 6.7, the origin is an interior point in X_f.

As mentioned in Chapter 1, in parametric programming problems one seeks the solution $P^o(x)$ as an explicit function of the parameters x in some set $\underline{X} \subseteq X_f \subseteq \mathbb{R}^n$ [7]. However, in the general case, an *exact* explicit solution of Problem 6.4 with the associated shape of the state space partition can not be found. Therefore, it would be necessary to use methods for *approximate* explicit solution by preliminary specifying the structure of the partition. In [9, 11], practical computational methods for constructing an explicit *approximate* solution of *feedback* min-max NMPC problems for general constrained nonlinear systems are suggested, which are based on an orthogonal structure of the state space partition. Since the regions in the partition do not overlap (except at the boundary), the approximation corresponds to orthogonal basis-functions that form a complete basis on the space of continuous functions. This ensures an arbitrarily good approximation if the optimal solution is a continuous function. Note that this type of partition does not impose any restrictions on the class of problems that can be solved.

6.3.2 Approximate mp-NLP Approach to Explicit Closed-Loop Min-Max NMPC

In [9, 11], an approximate mp-NLP approach to explicit solution of the *feedback* (*closed-loop*) min-max NMPC problem (Definition 6.3) is proposed. In contrast to the method in [8] (considered in Section 6.2) where a sequence of control actions is optimized, here the optimization is performed over a sequence of feedback control policies. Another difference from most approximate mp-NLP approaches, where a *piecewise linear* solution is obtained, is that the presented method constructs an explicit approximate solution, which represents a *piecewise nonlinear* function.

6.3.2.1 Non-convexity and Close-to-Global Solutions

From a physical insight on the considered system (6.20), it is supposed that the disturbance w can vary in the range:

$$w_{\min} \le w(t) \le w_{\max}, \tag{6.41}$$

with known $w_{\min}$, $w_{\max}$. The procedure used to generate a discrete set of admissible disturbance realizations is the following [11]:

Procedure 6.1 (Generation of discrete set of admissible disturbance realizations). *Consider system (6.20), where* $w(t) \in [w_{\min}; w_{\max}]$. *Let* N *be a finite horizon and* $K = \{k_0, k_1, \ldots, k_{N-1}\}$ *be a vector of feedback control policies where each feedback function* $k_i(x)$, $i = 0, \ldots, N-1$ *, has the form (6.36). Suppose that the initial state of the system (6.20) is* $x_{t|t} = x$ *and let* $j_{\max}$ *be a positive integer. Then, for a given vector* $P = [p_0^T \; p_1^T \; \ldots \; p_{N-1}^T]^T$ *of parameters of* K*, a finite set* $W^0(P,x) = \{W_1, W_2, \ldots, W_{N_W}\}$ *of admissible disturbance realizations is generated where each realization* $W_s = \{w_t^s, w_{t+1}^s, \ldots, w_{t+i}^s, \ldots, w_{t+N-1}^s\}$, $s = 1, 2, \ldots, N_W$ *is determined by applying Algorithm 6.1.*

Algorithm 6.1. Generation of an admissible disturbance realization.

Input: N, $P = [p_0^T \; p_1^T \; \dots \; p_{N-1}^T]^T$, x, $j_{\max}$.
Output: $W_s = \{w_t^s, w_{t+1}^s, \dots, w_{t+i}^s, \dots, w_{t+N-1}^s\}$.
1. Let $i = 0$.
2. **while** $i \leq N-1$ **do**
3. Let $flag = 0$, $j = 0$.
4. **while** $flag = 0$ **do**
5. Generate value $w_{t+i}^s \in [w_{\min}; w_{\max}]$ by using random generator with uniform distribution.
6. $j = j+1$.
7. **if** $\|w_{t+i}^s\|^2 \leq \gamma_\Delta^2 \|h(x_{t+i|t}, k_i(x_{t+i|t}), w_{t+i}^s)\|^2$ **then**
8. Compute $x_{t+i+1|t} = f(x_{t+i|t}, k_i(x_{t+i|t}), w_{t+i}^s)$.
9. $flag = 1$.
10. **else**
11. **if** $j > j_{\max}$, **terminate** (an admissible disturbance realization is not found).
12. **end if**
13. **end while**
14. $i = i+1$.
15. **end while**

In Algorithm 6.1, the parameter $j_{\max}$ denotes the maximal allowed number of unsuccessful iterations and it is typically chosen to be $j_{\max} = 100q$, where q is the dimension of w. A special case is the case when the disturbance is of the form $w(t) = d^T y(t)$, where $d \in \mathbb{R}^r$ is a vector of uncertain parameters with $d_{\min} \leq d \leq d_{\max}$. Then, the set of the admissible disturbance realizations can be generated by simulating the closed-loop system response for different values $d^s \in [d_{\min}; d_{\max}]$, $s = 1, 2, \dots, N_W$ of d.

The procedure used to approximate Problem 6.4 is [11]:

Procedure 6.2 (Approximation of Problem 6.4). *Suppose that Assumptions 6.6–6.11 hold. Let P be a given vector of parameters of the sequence K of feedback control policies. Suppose that a finite set $W^0(P,x) = \{W_1, W_2, \dots, W_{N_W}\}$ of admissible disturbance realizations has been determined by applying Procedure 6.1. An estimate $\widetilde{V}_{\max}(P,x)$ of $V_{\max}(P,x)$ is computed as follows:*

$$\widetilde{V}_{\max}(P,x) = \max_{W_i \in W^0(P,x)} J(P,x,W_i) \tag{6.42}$$

Denote with $\widetilde{G}(P,x)$ the set of constraints functions:

$$\widetilde{G}(P,x) = \{G(P,x,W_i),\, W_i \in W^0(P,x)\} \tag{6.43}$$

Then Problem 6.4 is approximated with the following mp-NLP problem:

Problem 6.5:

$$\widetilde{V}^o_{\max}(x) = \min_P \widetilde{V}_{\max}(P,x) \text{ subject to } \widetilde{G}(P,x) \leq 0. \tag{6.44}$$

Thus, we can approximate the infinite number of constraints (6.39) with a finite amount of constraints which are functions only of P and x. For a given min-max NMPC problem it would be necessary to analyze how the size of the set of admissible disturbance realizations generated with Procedure 6.1 would effect the worst-case cost function value and the satisfaction of constraints in Problem 6.4. It should be expected that with the increase of the number of the generated disturbance sequences, the probability of satisfaction of the constraints in Problem 6.4 would be higher. On the other hand, this will lead to an increase of the computational efforts related to the design of the explicit NMPC controller. Therefore, for every specific min-max NMPC problem, a tradeoff should be made and a reasonable number of admissible disturbance realizations should be determined. Hereafter, let $X \subset \mathbb{R}^n$ be a hyper-rectangle where we seek an explicit approximate solution of Problem 6.5.

Problem 6.5 can be non-convex with multiple local minima. Therefore, it would be necessary to apply an efficient initialization of Problem 6.5 so to find a close-to-global solution. One possible way to obtain this is to find a close-to-global solution at a point $v_0 \in X_0$ (where X_0 is a hyper-rectangle in the state space) by comparing the local minima corresponding to several initial guesses and then to use this solution as an initial guess at the neighboring points $v_i \in X_0$, $i = 1, 2, \ldots, N_1$, i.e. to propagate the solution. For this purpose, Procedures 1.1 and 1.2 from Chapter 1 can be used to generate a set of points $V_0 = \{v_0, v_1, v_2, \ldots, v_{N_1}\}$, associated to X_0, and to find a close-to-global solution at these points, respectively. It should be noted that the notation used here is different from the one in Chapter 1. Thus here, the points and the set of points are denoted with v_i and V_0 (instead of w_i and W_0), the vector of optimization variables is P (instead of z), the objective function and the constraints function in the mp-NLP problem are $\widetilde{V}_{\max}(\cdot,\cdot)$ and $\widetilde{G}(\cdot,\cdot)$ (instead of $f(\cdot,\cdot)$ and $g(\cdot,\cdot)$).

6.3.2.2 Computation of Explicit Approximate Solution

We restrict our attention to a hyper-rectangle $X \subset \mathbb{R}^n$ where we seek to approximate the close-to-global sequence of control policies $K^* = \{k^*_0, k^*_1, \ldots, k^*_{N-1}\} \triangleq \{k^*_0(x_{t|t}), k^*_1(x_{t+1|t}), \ldots, k^*_{N-1}(x_{t+N-1|t})\}$. We require that the state space partition is orthogonal and can be represented as a $k-d$ tree [3]. The main idea of the approximate mp-NLP approach is to construct a *piecewise nonlinear* (PWNL) approximation $\widehat{K} = \{\widehat{k}_0, \widehat{k}_1, \ldots, \widehat{k}_{N-1}\} \triangleq \{\widehat{k}_0(x_{t|t}), \widehat{k}_1(x_{t+1|t}), \ldots, \widehat{k}_{N-1}(x_{t+N-1|t})\}$ to the close-to-global feedback $K^* = \{k^*_0, k^*_1, \ldots, k^*_{N-1}\}$ on X. The constituent sequences of *nonlinear* control policies are denoted with $\widehat{K}_{X_i} = \{\widehat{k}_{0,X_i}, \ldots, \widehat{k}_{N-1,X_i}\} \triangleq \{\widehat{k}_{0,X_i}(x_{t|t}), \ldots, \widehat{k}_{N-1,X_i}(x_{t+N-1|t})\}$ and are defined on hyper-rectangles X_i covering

X. This means that a sequence $\widehat{K}_{X_i}$ is applied for $\forall x_{t|t} \in X_i$. Let $\widehat{K}_{X_0} = \{\widehat{k}_{0,X_0}, \ldots, \widehat{k}_{N-1,X_0}\}$ be an approximation to the close-to-global solution $K^* = \{k_0^*, \ldots, k_{N-1}^*\}$, valid in X_0. Denote with $P_{X_0} = [p_{0,X_0}^T \cdots p_{N-1,X_0}^T]^T$ the parameters of $\widehat{K}_{X_0}$. According to Assumption 6.11, $\widehat{k}_{i,X_0}(x_{t+i|t}) = g_i(p_{i,X_0}, x_{t+i|t})$, $i = 0, 1, \ldots, N-1$. Let $\widehat{\widehat{V}}_{\max}(P_{X_0}, x)$ be the cost function value due to initial state $x = x_{t|t}$ and sequence $\widehat{K}_{X_0}$ of control policies, i.e.

$$\widehat{\widehat{V}}_{\max}(P_{X_0}, x) = \max_{W_i \in W^0(P_{X_0}, x)} J(P_{X_0}, x, W_i). \tag{6.45}$$

Then, the approximate sequence

$$\widehat{K}_{X_0} = \{\widehat{k}_{0,X_0}, \ldots, \widehat{k}_{N-1,X_0}\} \triangleq \{g_0(p_{0,X_0}, x_{t|t}), \ldots, g_{N-1}(p_{N-1,X_0}, x_{t+N-1|t})\}, \tag{6.46}$$

valid for $\forall x_{t|t} \in X_0$, is computed with the following procedure [11]:

Procedure 6.3 (Computation of explicit approximate solution). *Suppose that Assumptions 6.6–6.11 hold. Consider any hyper-rectangle $X_0 \subseteq X$ with a set of points $V_0 = \{v_0, v_1, v_2, \ldots, v_{N_1}\}$ determined with Procedure 1.1. Suppose that a close-to-global solution of Problem 6.5 at the points $v_i \in V_0$, $i = 0, 1, 2, \ldots, N_1$ has been obtained by applying Procedure 1.2 and let $\widetilde{V}_{\max}^*(v_i)$, $i = 0, 1, 2, \ldots, N_1$ be the close-to-global cost function values. Compute the parameters $P_{X_0} = [p_{0,X_0}^T \cdots p_{N-1,X_0}^T]^T$ of the sequence $\widehat{K}_{X_0} = \{\widehat{k}_{0,X_0}, \ldots, \widehat{k}_{N-1,X_0}\}$ by solving the NLP:*

$$\min_{P_{X_0}} \sum_{i=0}^{N_1} \left(\widehat{\widehat{V}}_{\max}(P_{X_0}, v_i) - \widetilde{V}_{\max}^*(v_i) + \mu \| g_0(p_{0,X_0}, v_i) - k_0^*(v_i) \|_2^2 \right) \tag{6.47}$$

$$\textit{subject to} \quad \widetilde{G}(P_{X_0}, v_i) \leq 0, \forall v_i \in V_0. \tag{6.48}$$

In (6.47), the parameter $\mu > 0$ is a weighting coefficient. Note that the sequence $\widehat{K}_{X_0} = \{\widehat{k}_{0,X_0}, \ldots, \widehat{k}_{N-1,X_0}\}$, computed with Procedure 6.3, satisfies the constraints in Problem 6.5 only for the discrete set of points $V_0 \subset X_0$.

6.3.2.3 Estimation of Error Bounds

Suppose that the parameters P_{X_0} of the sequence $\widehat{K}_{X_0}$, valid in X_0, has been computed with Procedure 6.3. Then, for the cost function approximation error in X_0 we have:

$$\widetilde{\varepsilon}(x) = \widehat{\widehat{V}}_{\max}(P_{X_0}, x) - \widetilde{V}_{\max}^*(x) \leq \varepsilon_0, x \in X_0. \tag{6.49}$$

The following procedure can be used to obtain an estimate $\widehat{\varepsilon}_0$ of the maximal approximation error ε_0 in X_0 [11]:

Procedure 6.4 (Computation of error bound approximation). *Consider a hyper-rectangle $X_0 \subseteq X$ with a set of points $V_0 = \{v_0, v_1, v_2, \ldots, v_{N_1}\}$ determined by applying Procedure 1.1. Compute an estimate $\widehat{\varepsilon}_0$ of the error bound ε_0 through the following maximization:*

$$\widehat{\varepsilon}_0 = \max_{i \in \{0,1,2,\ldots,N_1\}} \left(\widehat{\widetilde{V}}_{\max}(P_{X_0}, v_i) - \widetilde{V}^*_{\max}(v_i)\right). \tag{6.50}$$

The estimate $\widehat{\varepsilon}_0$ represents an approximate degree of sub-optimality, since it depends on the finite set of admissible disturbance realizations generated with Procedure 6.1.

6.3.2.4 Approximate mp-NLP Algorithm

Assume the tolerance $\bar{\varepsilon} > 0$ of the cost function approximation error is given. Denote with S_{X_0} the volume of a given hyper-rectangular region $X_0 \subset X \subset \mathbb{R}^n$, i.e. $S_{X_0} = \prod_{i=1}^{n} \Delta x_i$, where Δx_i is the size of X_0 along the state variable x_i. Let $S_{\min}$ be the minimal allowed volume of the regions in the partition of X. The following algorithm is proposed to compute the explicit approximate *feedback* min-max NMPC controller on X [11]:

Algorithm 6.2. Explicit feedback min-max NMPC.

Input: Data to Problem 6.5, the number N_0 of internal regions (used in Procedure 1.1), the parameter μ (used in Procedure 6.3), the approximation tolerance $\bar{\varepsilon}$.
Output: Partition $\Pi = \{X_1, X_2, \ldots, X_{N_X}\}$ and associated PWNL control function $\widehat{K}_\Pi = \{\widehat{K}_{X_1}, \widehat{K}_{X_2}, \ldots, \widehat{K}_{X_{N_X}}\}$.
1. Initialize the partition to the whole hyper-rectangle, i.e. $\Pi = \{X\}$.
   Mark the hyper-rectangle X as unexplored, $flag := 1$.
2. **while** $flag = 1$ **do**
3.   **while** $\exists$ unexplored hyper-rectangles in Π **do**
4.   Select any unexplored hyper-rectangle $X_0 \in \Pi$.
5.   Compute a solution to Problem 6.5 at the center point v_0 of X_0 by applying Procedure 1.2a.
6.   **if** Problem 6.5 has a feasible solution at v_0 **then**
7.   Define a set of points $V_0 = \{v_0, v_1, v_2, \ldots, v_{N_1}\}$ by applying Procedure 1.1.
8.   Compute a solution to Problem 6.5 for x fixed to each of the points v_i, $i = 1, 2, \ldots, N_1$ by applying Procedure 1.2b.
9.   **if** Problem 6.5 has a feasible solution at all points v_i, $i = 1, 2, \ldots, N_1$ **then**
10.   **if** $0 \in X_0$ **then**
11.   Let $\widehat{K}_{X_0} = k_a(x)$.
12.   **If** $X_0 \subseteq \Omega_a$, mark X_0 as explored and feasible. **Otherwise**, mark X_0 to be split.
13.   **else**

14. Compute a sequence $\widehat{K}_{X_0} = \{\widehat{k}_{0,X_0}, \dots, \widehat{k}_{N-1,X_0}\}$ of control policies using Procedure 6.3, as an approximation to be used in X_0.
15. **if** a sequence of control policies was found **then**
16. Compute an estimate $\widehat{\varepsilon}_0$ of the error bound ε_0 in X_0 by applying Procedure 6.4.
17. **If** $\widehat{\varepsilon}_0 > \bar{\varepsilon}$, mark the hyper-rectangle X_0 to be split. **Otherwise**, mark X_0 as explored and feasible.
18. **else**
19. Compute the volume S_{X_0} of the hyper-rectangle X_0. **If** $S_{X_0} < S_{\min}$, mark X_0 infeasible and explored. **Otherwise**, mark X_0 to be split.
20. **end if**
21. **end if**
22. **else**
23. Compute the volume S_{X_0} of the hyper-rectangle X_0. **If** $S_{X_0} < S_{\min}$, mark X_0 infeasible and explored. **Otherwise**, mark X_0 to be split.
24. **end if**
25. **else**
26. Compute the volume S_{X_0} of the hyper-rectangle X_0. **If** $S_{X_0} < S_{\min}$, mark X_0 infeasible and explored. **Otherwise**, mark X_0 to be split.
27. **end if**
28. **end while**
29. $flag := 0$
30. **if** $\exists$ hyper-rectangles in Π that are marked to be split **then**
31. $flag := 1$
32. **while** $\exists$ hyper-rectangles in Π that are marked to be split **do**
33. Select any hyper-rectangle $X_0 \in \Pi$ marked to be split.
34. Split X_0 into hyper-rectangles $X_1, \dots, X_{N_s}$ by applying heuristic splitting rules. Mark $X_1, \dots, X_{N_s}$ unexplored, remove X_0 from Π, and add $X_1, \dots, X_{N_s}$ to Π.
35. **end while**
36. **end if**
37. **end while**

In step 34, the heuristic splitting rules from [10] (described in details in Section 1.1.5.2) are applied to partition a given hyper-rectangle X_0. Thus, if a sequence of control policies valid in X_0 is computed, but the required accuracy is not achieved, then X_0 is split by a hyperplane through its center and orthogonal to that axis where a maximal reduction of the approximation error can be achieved. If there is no feasible solution of Problem 6.5 at the center point v_0 of X_0, or the NLP problem (6.47)–(6.48) is infeasible, then X_0 is split by a hyperplane through its center and orthogonal to an arbitrary axis. If some of the points associated to X_0 are feasible and others are not, then X_0 is split into hyper-rectangles such that some of them will include only feasible points.

6.3.3 Stability

6.3.3.1 Computation of Approximate Region of Attraction for the Sub-optimal Closed-Loop System

Let $X_\Pi = \bigcup_{i=1}^{N_X} X_i$, $X_i \in \Pi$ be the set associated to the partition Π obtained with Algorithm 6.2. Consider the suboptimal closed-loop system:

$$x(t+1) = f(x(t), \widehat{k}_0(x(t)), w(t)) \tag{6.51}$$

$$y(t) = h(x(t), \widehat{k}_0(x(t)), w(t)), \tag{6.52}$$

where $\widehat{k}_0(x(t))$ is the approximate PWNL feedback law determined with Algorithm 6.2 and is defined on the set X_Π. The fact that the explicit NMPC controller is specified for an initial condition $x(t) \in X_\Pi$ does not imply that $x(t)$ is within the region of attraction for the system (6.51)–(6.52). Therefore, the set X_Π may not be a domain of attraction for this system. In fact, although a feasible control law exists at state $x(t) \in X_\Pi$, the successor state $x(t+1)$ may go out of the set X_Π. Moreover, the set X_Π may not be convex (see the simulation example in Section 6.3.4). Therefore, first it would be useful to find a set $\Omega_1 \subseteq X_\Pi$, which is an inner convex approximation of the set X_Π. Then, a convex set $\Omega_2 \subseteq \Omega_1$ should be determined such that $\Omega_2 \supset \Omega_a$ and for every initial state that belongs to the set Ω_2, the state trajectory of the system (6.51)–(6.52) will lie in the set Ω_1. This is specified in the following definition [11].

Definition 6.4 (Approximate region of attraction for the suboptimal closed-loop system). Let $\Pi = \{X_1, X_2, \ldots, X_{N_X}\}$, $X_\Pi = \bigcup_{i=1}^{N_X} X_i$, $X_i \in \Pi$ and $\widehat{K} = \{\widehat{k}_0, \widehat{k}_1, \ldots, \widehat{k}_{N-1}\}$ be respectively the state space partition, the associated set in the state space and the approximate PWNL sequence of feedback control policies, determined with Algorithm 6.2. Let $\widehat{P} = [\widehat{p}_0^T\ \widehat{p}_1^T\ \ldots\ \widehat{p}_{N-1}^T]^T$ be the parameters of $\widehat{K}$. Assume that X_Π is a non-empty set. Suppose that there exist polyhedral sets Ω_1 and Ω_2, such that $\Omega_a \subset \Omega_2 \subseteq \Omega_1 \subseteq X_\Pi$. Let $E_{\Omega_2} = \{x^j \,|\, x^j \in \Omega_2,\ j = 1, 2 \ldots, N_{p2}\}$ denote a finite set of randomly generated points. Let the state of the system (6.51)–(6.52) at time t be $x_{t|t} = x^j \in E_{\Omega_2}$. Consider a finite set $W^0(\widehat{P}, x^j) = \{W_1, W_2 \ldots, W_{N_W}\}$ of admissible disturbance realizations $W_s = \{w_t^s, w_{t+1}^s, \ldots, w_{t+N-1}^s\}$, $s = 1, 2 \ldots, N_W$, generated by applying Procedure 6.1. Let $X^{s,j} = \{x_{t+1|t}^{s,j}, x_{t+2|t}^{s,j}, \ldots, x_{t+N|t}^{s,j}\}$ denote the state trajectory of the system (6.51)–(6.52) obtained with $\widehat{K}$ and corresponding to initial state $x^j \in E_{\Omega_2}$ and disturbance realization $W_s \in W^0(\widehat{P}, x^j)$, i.e.:

$$x_{t+i+1|t}^{s,j} = f(x_{t+i|t}^{s,j}, \widehat{k}_0(x_{t+i|t}^{s,j}), w_{t+i}^s),\ i = 0, 1, 2, \ldots, N-1, \tag{6.53}$$

Then, if:

$$X^{s,j} \in \Omega_1,\ \forall x^j \in E_{\Omega_2} \text{ and } \forall W_s \in W^0(\widehat{P}, x^j), \tag{6.54}$$

the set Ω_2 is referred to as an approximate region of attraction for the suboptimal closed-loop system (6.51)–(6.52).

Let S_{Ω_1} and S_{Ω_2} denote the volumes of the polyhedral sets Ω_1 and Ω_2 defined as their Lebesgue measures, i.e. $S_{\Omega_1} = \int_{\Omega_1} dx$ and $S_{\Omega_2} = \int_{\Omega_2} dx$. The volume of the set X_Π is $S_{X_\Pi} = \sum_{i=1}^{N_X} \int_{X_i} dx$, i.e. it represents the sum of the Lebesgue measures of all regions $X_i \in \Pi$. Then, the following procedure is applied to compute an approximate region of attraction for the closed-loop system (6.51)–(6.52) [11]:

Procedure 6.5 (Computation of approximate region of attraction for the suboptimal closed-loop system). *Let $\Pi = \{X_1, X_2, \ldots, X_{N_X}\}$, $X_\Pi = \bigcup_{i=1}^{N_X} X_i$, $X_i \in \Pi$ and $\widehat{K} = \{\widehat{k}_0, \widehat{k}_1, \ldots, \widehat{k}_{N-1}\}$ be respectively the state space partition, the associated set in the state space and the approximate PWNL sequence of feedback control policies, determined with Algorithm 6.2. Assume the set X_Π is non-empty. Suppose that there exist polyhedral sets $\Omega_1 = \{x \in X_\Pi \,|\, a_i^1 \le h_i^1 x \le b_i^1, i = 1, 2, \ldots, N_{\Omega_1}\}$ and $\Omega_2 = \{x \in X_\Pi \,|\, a_i^2 \le h_i^2 x \le b_i^2, i = 1, 2, \ldots, N_{\Omega_2}\}$, such that $\Omega_a \subset \Omega_2 \subseteq \Omega_1 \subseteq X_\Pi$. Let $E_{\Omega_1} = \{x^k \,|\, x^k \in \Omega_1, k = 1, 2, \ldots, N_{p1}\}$ and $E_{\Omega_2} = \{x^j \,|\, x^j \in \Omega_2, j = 1, 2, \ldots, N_{p2}\}$ denote finite sets of randomly generated points. Then, for specified N_{Ω_1}, N_{Ω_2}, N_{p1} and N_{p2}, the approximate region of attraction for the closed-loop system (6.51)–(6.52) is computed by implementing the following steps:*

1. *Determine the polyhedron $\Omega_1^* = \{x \in X_\Pi \,|\, a_i^{1^*} \le h_i^{1^*} x \le b_i^{1^*}, i = 1, 2, \ldots, N_{\Omega_1}\}$, where $a_i^{1^*}$, $h_i^{1^*}$, $b_i^{1^*}$, $i = 1, 2, \ldots, N_{\Omega_1}$ are computed by solving the optimization problem:*

$$\{a_i^{1^*}, h_i^{1^*}, b_i^{1^*}, i = 1, 2, \ldots, N_{\Omega_1}\} = \arg \min_{a_i^1, h_i^1, b_i^1, i=1,\ldots,N_{\Omega_1}} \left| S_{\Omega_1} - S_{X_\Pi} \right|$$
$$\textit{subject to} \;\; E_{\Omega_1} \subseteq X_\Pi. \tag{6.55}$$

2. *Determine the approximate region of attraction as the following polyhedron $\Omega_2^* = \{x \in X_\Pi \,|\, a_i^{2^*} \le h_i^{2^*} x \le b_i^{2^*}, i = 1, 2, \ldots, N_{\Omega_2}\}$, where $a_i^{2^*}$, $h_i^{2^*}$, $b_i^{2^*}$, $i = 1, 2, \ldots, N_{\Omega_2}$ are computed by solving the optimization problem:*

$$\{a_i^{2^*}, h_i^{2^*}, b_i^{2^*}, i = 1, 2, \ldots, N_{\Omega_2}\} = \arg \min_{a_i^2, h_i^2, b_i^2, i=1,\ldots,N_{\Omega_2}} \left| S_{\Omega_2} - S_{\Omega_1} \right|$$
$$\textit{subject to} \;\; E_{\Omega_2} \subseteq \Omega_1, \Omega_a \subset \Omega_2, \;\; \textit{and condition (6.54).} \tag{6.56}$$

Problems (6.55) and (6.56) are nonlinear programming problems and nonlinear programming techniques [1] can be used to solve them. Further in the paper, the sets Ω_2^* and Ω_1^* determined with Procedure 6.5 will be denoted as Ω_2 and Ω_1.

After Procedure 6.5 is implemented, a partition $\Pi^{RH} = \{R_1, R_2, \ldots, R_{N_R}\}$ is built such that $\Omega_1 = \bigcup_{i=1}^{N_R} R_i$. Each region $R_i \in \Pi^{RH}$ represents either a hyper-rectangular region, i.e. $R_i \equiv X_j$ or a polyhedral region, i.e. $R_i = X_j \cap \Omega_1$, where $X_j \in \Pi$. The PWNL function associated to the partition Π^{RH} is defined as $\widehat{K}_{\Pi^{RH}} = \{\widehat{K}_{R_1}, \widehat{K}_{R_2}, \ldots, \widehat{K}_{R_{N_R}}\}$, where $\widehat{K}_{R_i} \equiv \widehat{K}_{X_j}$, $\widehat{K}_{X_j} \in \widehat{K}_\Pi$, given that $R_i \equiv X_j$ or $R_i = X_j \cap \Omega_1$. As result, we obtain a partition Π^{RH} and an approximate PWNL sequence of feedback control policies $\widehat{K}^{RH} = \{\widehat{k}_0^{RH}, \widehat{k}_1^{RH}, \dots, \widehat{k}_{N-1}^{RH}\}$ defined on the set Ω_1.

6.3.3.2 Stability Result

This section considers the stability of the closed-loop system:

$$x(t+1) = f(x(t), \widehat{k}_0^{RH}(x(t)), w(t)) \tag{6.57}$$

$$y(t) = h(x(t), \widehat{k}_0^{RH}(x(t)), w(t)), \tag{6.58}$$

where $\widehat{k}_0^{RH}(x(t))$ is the approximate PWNL feedback law determined with Algorithm 6.2 and Procedure 6.5 and is defined on the approximate region of attraction Ω_2 computed with Procedure 6.5.

The following notation is introduced. Let N be the prediction horizon and $x_{t|t} = x$ is the initial state of the system (6.57)–(6.58). For any $x \in \Omega_1$, let $\widehat{K}_N \equiv \widehat{K}^{RH} = \{\widehat{k}_0^{RH}(x_{t|t}), \dots, \widehat{k}_{N-1}^{RH}(x_{t+N-1|t})\}$ denote the approximate solution to the optimization Problem 6.5. Let $X_N = \{x_{t+1|t}, \dots, x_{t+N|t}\}$ and $Y_N = \{y_{t|t}, \dots, y_{t+N-1|t}\}$ denote the state and output trajectories of system (6.57)–(6.58) obtained with $\widehat{K}_N$ and corresponding to a disturbance realization $W_N = \{w_t, \dots, w_{t+N-1}\} \in W^B(\widehat{K}_N, x)$ ($W^B(\widehat{K}_N, x) \subset \mathbb{R}^{qN}$ is the set of the admissible disturbance realizations over horizon N). Let $\widehat{V}_{\max}(x, N)$ be the worst-case cost function value due to initial state $x_{t|t} = x$ and sequence $\widehat{K}_N$, i.e.:

$$\widehat{V}_{\max}(x, N) = \max_{W_N \in W^B(\widehat{K}_N, x)} J(x, \widehat{K}_N, W_N, N), \tag{6.59}$$

where

$$J(x, \widehat{K}_N, W_N, N) = \frac{1}{2} \sum_{i=0}^{N-1} [\|y_{t+i|t}\|^2 - \gamma^2 \|w_{t+i}\|^2] + V_{k_a}(x_{t+N|t}). \tag{6.60}$$

Consider the sequence

$$\widehat{K}_{N+1} = \{\widehat{k}_0^{RH}(x_{t|t}), \dots, \widehat{k}_{N-1}^{RH}(x_{t+N-1|t}), k_a(x_{t+N|t})\} \tag{6.61}$$

for the Problem 6.5 with horizon $N+1$. Then, $X_{N+1} = \{x_{t+1|t}, \dots, x_{t+N|t}, x_{t+N+1|t}\}$ and $Y_{N+1} = \{y_{t|t}, \dots, y_{t+N-1|t}, y_{t+N|t}\}$ are the associated state and output trajectories of the system (6.57)–(6.58) corresponding to initial state $x_{t|t} = x$ and a disturbance realization $W_{N+1} = \{w_t, \dots, w_{t+N-1}, w_{t+N}\} \in W^C(\widehat{K}_{N+1}, x)$ ($W^C(\widehat{K}_{N+1}, x) \subset \mathbb{R}^{q(N+1)}$ is the set of the admissible disturbance realizations over horizon $N+1$). Let $\widehat{V}_{\max}(x, N+1)$ be the worst-case cost function value due to initial state $x_{t|t} = x$ and sequence $\widehat{K}_{N+1}$, i.e.:

$$\widehat{V}_{\max}(x, N+1) = \max_{W_{N+1} \in W^C(\widehat{K}_{N+1}, x)} J(x, \widehat{K}_{N+1}, W_{N+1}, N+1), \tag{6.62}$$

where $J(x,\widehat{K}_{N+1},W_{N+1},N+1) = \frac{1}{2}\sum_{i=0}^{N}[\|y_{t+i|t}\|^2 - \gamma^2\|w_{t+i}\|^2] + V_{k_a}(x_{t+N+1|t})$.

Let $\widehat{P}^{RH}$ be the parameters of $\widehat{K}^{RH}$. The following assumption is made on the solution $\widehat{K}^{RH}$ and the sets Ω_1 and Ω_2 resulting from Algorithm 6.2 and Procedure 6.5 [11]:

Assumption 6.12 (Constraints satisfaction). *The constraints $G(\widehat{P}^{RH},x,W) \leq 0$ are satisfied for all $x \in \Omega_1$ and all $W \in W^B(\widehat{P}^{RH},x)$. The sets Ω_1 and Ω_2 are such that $\Omega_a \subset \Omega_2 \subseteq \Omega_1 \subseteq X_\Pi$ and $x_{t+i+1|t} = f(x_{t+i|t},\widehat{k}_i^{RH}(x_{t+i|t}),w_{t+i}) \in \Omega_1$, $\forall x_{t|t} \in \Omega_2$, $\forall w_{t+i} \in W^A(\widehat{k}_i^{RH}(x_{t+i|t}),x_{t+i|t})$, $i = 0,1,2,\dots,N-1$.*

Here, the stability result is formulated [11]:

Theorem 6.1. *Given an auxiliary control law $k_a(x)$ and an associated invariant set Ω_a, consider two positive constants γ and γ_Δ with $\gamma_\Delta\gamma < 1$. Suppose that a non-empty region of attraction Ω_2 and associated set Ω_1 have been determined by applying Procedure 6.5. Let $\widehat{K}^{RH}$ with parameters $\widehat{P}^{RH}$ be the approximate PWNL feedback law determined with Algorithm 6.2 and Procedure 6.5. Consider the closed-loop system (6.57)–(6.58), where $\widehat{k}_0^{RH}(x(t)) = [I\ 0\ \dots\ 0]\,\widehat{K}^{RH}$. Then, under Assumptions 6.6–6.12, the following holds for the closed-loop system (6.57)–(6.58):*

i). In the absence of disturbance the origin is asymptotically stable for all $x \in \Omega_2$.
ii). In the presence of disturbance it has l_2-gain less than or equal to γ for all $x \in \Omega_2$.

Proof ([11]).

i). From Assumption 6.9 it follows that $\|y_{t+i|t}\|^2 \geq \frac{\|w_{t+i}\|^2}{\gamma_\Delta^2}$, $i \geq 0$. Then, the condition $\gamma_\Delta\gamma < 1$ leads to $\|y_{t+i|t}\|^2 > \gamma^2\|w_{t+i}\|^2$, $i \geq 0$. Therefore, the stage cost $L(y_{t+i|t},w_{t+i}) = \frac{1}{2}(\|y_{t+i|t}\|^2 - \gamma^2\|w_{t+i}\|^2)$ is a positive definite function. Then, by taking into account that $V_{k_a}(x)$ is a positive definite function too (cf. Assumption 6.10), it follows:

$$\widehat{V}_{\max}(x,N) \geq 0, \forall x \in \Omega_2. \tag{6.63}$$

In the absence of disturbance, the stage cost is $L(y,0) = L(h(x,\widehat{k}_0^{RH}(x),0),0)$ and it is a positive definite function defined on the set Ω_2 which contains the origin in its interior (according to Assumption 6.7). Then, it follows from Lemma 4.3 from [17] that there exist a $\mathscr{K}$-function $\alpha_1(\|x\|)$ such that $L(h(x,\widehat{k}_0^{RH}(x),0),0) \geq \alpha_1(\|x\|)$, $\forall x \in \Omega_2$. Similarly, there exists a $\mathscr{K}$-function $\alpha_2(\|x\|)$ such that $V_{k_a}(x) \leq \alpha_2(\|x\|)$, $\forall x \in \Omega_a$ (the reader is referred to [17] for the definition of $\mathscr{K}$-functions). Assumption 6.10 holds also in the case of absence of disturbance and therefore the set Ω_a is a positively invariant set for the nominal system (system (6.57)–(6.58) with $w(t) = 0$) in closed-loop with the auxiliary control law $k_a(x)$ and the inequality (6.25) takes the form:

$$V_{k_a}(f(x,k_a(x),0)) - V_{k_a}(x) + L(h(x,k_a(x),0),0) \leq 0, \ \forall x \in \Omega_a. \tag{6.64}$$

Therefore, according to Theorem 1 with Assumption 1 in [18] $x = 0$ is asymptotically stable for all $x \in \Omega_2$ when $w(t) = 0$.

ii). In a way similar to that in [21], it can be proved that for the worst-case cost function values defined by (6.59) and (6.62) the following holds:

$$\widehat{V}_{\max}(x,N+1) \leq \widehat{V}_{\max}(x,N), \forall x \in \Omega_2. \tag{6.65}$$

Following similar arguments as in [21] and by taking into account (6.65), for $\forall x \in \Omega_2$ and for $w_t \in W^A(\widehat{k}_0^{RH}(x),x)$, we have:

$$\begin{aligned} \widehat{V}_{\max}(x,N) &\geq \widehat{V}_{\max}(f(x,\widehat{k}_0^{RH}(x),w_t),N-1) + \frac{1}{2}\{\|h(x,\widehat{k}_0^{RH}(x),w_t)\|^2 - \gamma^2\|w_t\|^2\} \\ &\geq \widehat{V}_{\max}(f(x,\widehat{k}_0^{RH}(x),w_t),N) + \frac{1}{2}\{\|h(x,\widehat{k}_0^{RH}(x),w_t)\|^2 - \gamma^2\|w_t\|^2\}. \end{aligned} \tag{6.66}$$

Inequality (6.66) can be represented:

$$\begin{aligned} &\widehat{V}_{\max}(f(x,\widehat{k}_0^{RH}(x),w_t),N) - \widehat{V}_{\max}(x,N) \leq \\ &-\frac{1}{2}\{\|h(x,\widehat{k}_0^{RH}(x),w_t)\|^2 - \gamma^2\|w_t\|^2\}. \end{aligned} \tag{6.67}$$

Further, by considering that $x_{t+1|t} = f(x,\widehat{k}_0^{RH}(x),w_t)$ and $y_{t|t} = h(x,\widehat{k}_0^{RH}(x),w_t)$, the inequality (6.67) is written in the form:

$$\widehat{V}_{\max}(x_{t+1|t},N) - \widehat{V}_{\max}(x,N) \leq -\frac{1}{2}\{\|y_{t|t}\|^2 - \gamma^2\|w_t\|^2\}. \tag{6.68}$$

In a similar way, it can be shown that:

$$\begin{aligned} &\widehat{V}_{\max}(x_{t+i+1|t},N) - \widehat{V}_{\max}(x_{t+i|t},N) \leq -\frac{1}{2}\{\|y_{t+i|t}\|^2 - \gamma^2\|w_{t+i}\|^2\} \\ &i = 0,1,\ldots,T. \end{aligned} \tag{6.69}$$

After summing the inequalities (6.69) and by taking into account (6.63), we obtain:

$$\sum_{i=0}^{T}\frac{1}{2}\|y_{t+i|t}\|^2 \leq \gamma^2\sum_{i=0}^{T}\frac{1}{2}\|w_{t+i}\|^2 + \widehat{V}_{\max}(x,N) \tag{6.70}$$

$\forall x \in \Omega_2$, $\forall T \geq 0$, $\forall W_N \in W^B(\widehat{K}_N,x)$. Therefore, the closed-loop system (6.57)–(6.58) has l_2-gain less than or equal to γ in Ω_2. □

In the case when Assumption 6.12 does not hold, no guarantee on the l_2-gain can be given and only an estimate of its upper bound can be computed.

6.3.4 Application 2: Min-Max MPC of Cart and Spring System

Consider a cart with a mass M moving on a plane [21], shown in Fig. 6.4.

The carriage is attached to the wall via a spring with elasticity $\rho = \rho_0 e^{-x_1}$, where x_1 is the displacement of the carriage from the equilibrium position associated with

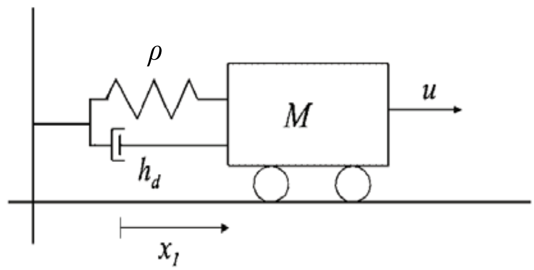

Fig. 6.4 Cart and spring system [21].

the external force $u = 0$. A damper with damping factor h_d affects the system in a resistive way. The damping factor h_d is an uncertain parameter and it is only known that $h_d = \overline{h}_d + \Delta h_d$, where $\overline{h}_d = 1.1$ and $-0.5 \leq \Delta h_d \leq 0.5$. The system is described by the nonlinear discrete-time model [21]:

$$x_1(t+1) = x_1(t) + T_s x_2(t) \tag{6.71}$$

$$x_2(t+1) = x_2(t) - T_s \frac{\rho_0}{M} e^{-x_1(t)} x_1(t) - T_s \frac{\overline{h}_d}{M} x_2(t) + T_s \frac{u(t)}{M} + T_s w(t), \tag{6.72}$$

where x_2 is the carriage velocity, $w(t) = -\frac{\Delta h_d}{M} x_2(t)$, $T_s = 0.4$ is the sampling time, $M = 1$ and $\rho_0 = 0.33$. Like in [21], we choose $y = [x_1\ x_2\ u]^T$ and it follows that $w(t) = [0\ -\frac{\Delta h_d}{M}\ 0] y(t)$. Therefore, $\|w(t)\|^2 \leq \gamma_\Delta^2 \|y(t)\|^2$ with $\gamma_\Delta = 0.5$, according to Assumption 6.9. The following input and state constraints are imposed on the system:

$$-4 \leq u \leq 4,\ -1.3 \leq x_2 \leq 1.3. \tag{6.73}$$

Therefore, the disturbances vary in the range $-1.3\gamma_\Delta \leq w \leq 1.3\gamma_\Delta$. The horizon is $N = 15$ and the terminal constraint is:

$$x_{t+N|t} \in \Omega_a,\ \Omega_a = \{x \in \mathbb{R}^n \,|\, x^T \Sigma x \leq \delta\}, \tag{6.74}$$

where $\delta = 0.001$ [21] and $\Sigma = \left[\begin{smallmatrix} 1.3 & 1.9 \\ 1.9 & 3.0 \end{smallmatrix}\right]$.

In [11], the approximate mp-NLP approach (described in Section 6.3.2) is applied to design an explicit *feedback* min-max NMPC controller for the cart. The NMPC minimizes the worst-case of the cost function (6.35) subject to the system equations (6.71)–(6.72) and the constraints (6.73)–(6.74). In (6.35), it is chosen $\gamma = 1$ and the terminal penalty is $V_{k_a} = x^T \Sigma x$ [21]. Like in [21], the feedback functions $k_i(x)$, $i = 0, 1, \ldots, N-1$ have the form:

$$k_i(p_i, x) = \alpha_i k_a(x) + \xi_{i,1} x_1^2 + \xi_{i,2} x_2^2, \tag{6.75}$$

where $p_i = [\alpha_i\ \xi_{i,1}\ \xi_{i,2}]^T$ are the parameters that need to be optimized and $k_a(x)$ is the auxiliary control law. The expression (6.75) implies that for relatively small absolute deviations from the equilibrium (small $x^2_{1,t+i|t}$ and $x^2_{2,t+i|t}$) the control input value will be generated mainly by the auxiliary control law $k_a(x_{t+i|t})$. The control law $k_a(x_{t+i|t})$ is determined by applying the method in [21]:

$$k_a(x_{t+i|t}) = -\begin{bmatrix}1 & 0\end{bmatrix} R^{-1} \begin{bmatrix}F_2^T \\ F_3^T\end{bmatrix} \Sigma f_1(x_{t+i|t}), \tag{6.76}$$

where:

$$f_1(x_{t+i|t}) = \begin{bmatrix} x_{1,t+i|t} + T_s x_{2,t+i|t} \\ x_{2,t+i|t} - T_s \frac{\rho_0}{M} e^{-x_{1,t+i|t}} x_{1,t+i|t} - T_s \frac{\bar{h}_d}{M} x_{2,t+i|t} \end{bmatrix} \tag{6.77}$$

$$F_2 = \begin{bmatrix} 0 \\ \frac{T_s}{M} \end{bmatrix}, F_3 = \begin{bmatrix} 0 \\ T_s \end{bmatrix}, R = \begin{bmatrix} F_2^T \Sigma F_2 + I & F_2^T \Sigma F_3 \\ (F_2^T \Sigma F_3)^T & F_3^T \Sigma F_3 - \alpha^2 I \end{bmatrix} \tag{6.78}$$

A set of three admissible disturbance realizations is generated which correspond to three values for the uncertain parameter Δh_d ($\Delta h_d = -0.5$, $\Delta h_d = 0$, $\Delta h_d = 0.5$). One internal region $X_0^1 \subset X_0$ is used in Procedure 1.1. In (6.47), it is chosen $\mu = 10$. The approximation tolerance is chosen to be $\bar{\varepsilon}(X_0) = \max(\bar{\varepsilon}_a, \bar{\varepsilon}_r \min_{x \in X_0} \widetilde{V}^*_{\max}(x))$, where $\bar{\varepsilon}_a = 0.003$ and $\bar{\varepsilon}_r = 0.01$ are the absolute and the relative tolerances.

The state space partition of the *feedback* min-max NMPC controller (the set X_Π) and the associated sets Ω_1 and Ω_2 are shown in Fig. 6.5. It is noticed that in some part of the set $X = [-3, 5] \times [-2, 2]$ a feasible solution does not exist. The number of

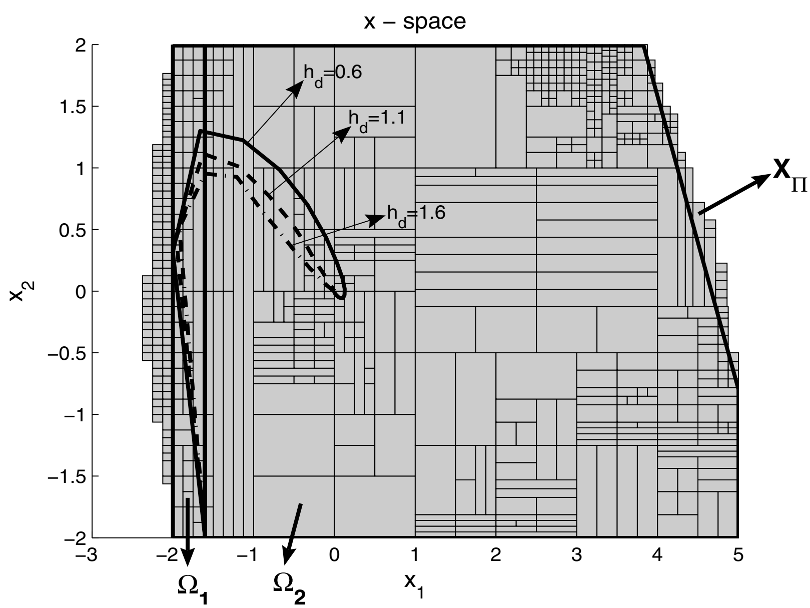

Fig. 6.5 State space partition of the explicit feedback min-max NMPC (the set X_Π), the associated sets Ω_1 and Ω_2, and the state trajectories for $h_d = 0.6$, $h_d = 1.1$, $h_d = 1.6$.

the inequalities describing the sets Ω_1 and Ω_2 is specified to be 5 and Procedure 6.5 is applied to determine them. The set Ω_1 is obtained graphically by minimizing the difference between its area and the area of the set X_Π. The computations of the state trajectories of the suboptimal closed-loop system, performed according to equation (6.53), have shown that the set Ω_2 can be determined simply by increasing the bound in one of the inequalities describing the set Ω_1. The partition (the set Ω_1 in Fig. 6.5) has 537 regions and 14 levels of search. Totally, 32 arithmetic operations are needed in real-time to compute the control input (14 comparisons, 11 multiplications, 6 additions and 1 exponential).

In Fig. 6.6, the optimal and the suboptimal feedback functions, respectively $u^*(x_1,x_2) = k_0^*(x_1,x_2)$ and $\widehat{u}(x_1,x_2) = \widehat{k}_0(x_1,x_2)$, are shown. In Fig. 6.7, the optimal cost function and the cost function approximation error, associated with the explicit approximate *feedback* min-max NMPC, are shown.

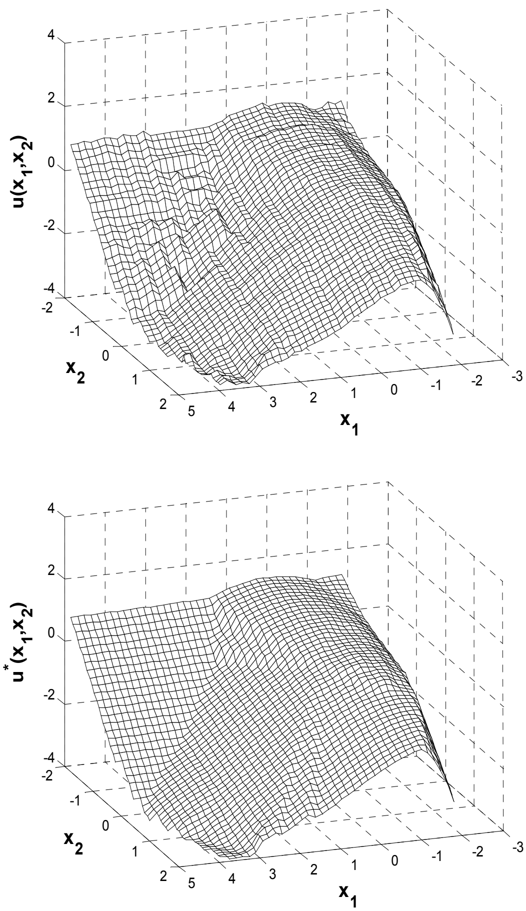

Fig. 6.6 The suboptimal (top) and the optimal (bottom) feedback functions (views rotated on 140°).

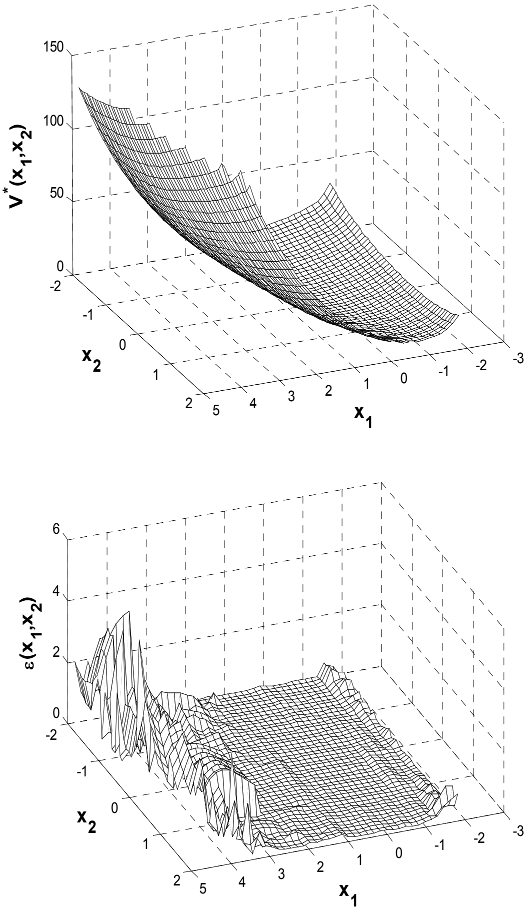

Fig. 6.7 The optimal cost function (top) and the cost function approximation error (bottom) (views rotated on 140°).

The performance of the suboptimal closed-loop system was simulated for initial state $x(0) = [-1.6 \ -2]^T$ and for three values of h_d. The response is depicted in the state space (Fig. 6.5) and as trajectories in time (Fig. 6.8 and Fig. 6.9). It can be seen that the explicit *feedback* min-max NMPC controller brings the cart to the equilibrium despite of the presence of disturbance, and the constraints imposed on the system are satisfied. It can also be observed that the state trajectory does not leave the set Ω_1.

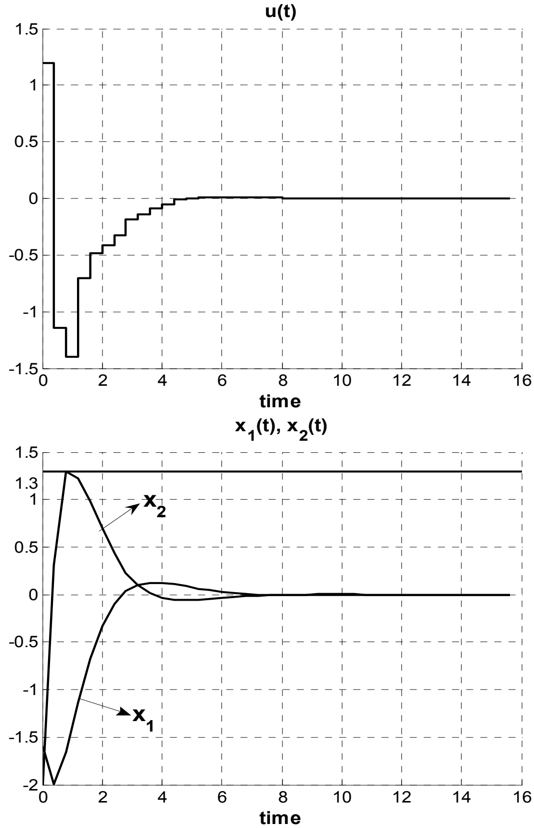

Fig. 6.8 Control input and state trajectory for $h_d = 0.6$.

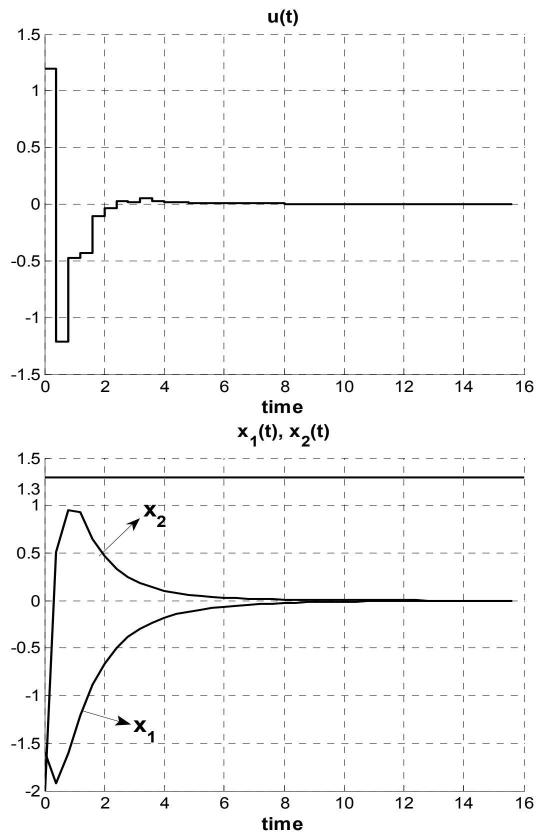

Fig. 6.9 Control input and state trajectory for $h_d = 1.6$.

References

1. Bazaraa, M.S., Sherali, H.D., Shetty, C.M.: Non-linear programming: theory and algorithms. Wiley, New York (1993)
2. Bemporad, A., Borrelli, F., Morari, M.: Min-max control of constrained uncertain discrete-time linear systems. IEEE Transactions on Automatic Control 48, 1600–1606 (2003)
3. Bentley, J.L.: Multidimensional binary search trees used for associative searching. Communications of the ACM 18, 509–517 (1975)
4. Björnberg, J., Diehl, M.: Approximate robust dynamic programming and robustly stable MPC. Automatica 42, 777–782 (2006)
5. Campo, P.J., Morari, M.: Robust model predictive control. In: Proceedings of the American Control Conference, Minneapolis, Minn., vol. 2, pp. 1021–1026 (1987)
6. Cychowski, M., O'Mahony, T.: Efficient off-line solutions to robust model predictive control using orthogonal partitioning. In: Proceedings of the 16th IFAC World Congress, Prague, Czech Republic (2005), `www.IFAC-PapersOnLine.net`
7. Fiacco, A.V.: Introduction to sensitivity and stability analysis in nonlinear programming. Academic Press, Orlando (1983)

8. Grancharova, A., Johansen, T.A.: Explicit min-max model predictive control of constrained nonlinear systems with model uncertainty. In: Proceedings of the 16th IFAC World Congress, Prague, Czech Republic (2005), www.IFAC-PapersOnLine.net
9. Grancharova, A., Johansen, T.A.: Explicit approximate approach to feedback min-max model predictive control of constrained nonlinear systems. In: Proceedings of the IEEE Conference on Decision and Control, San Diego, USA, pp. 4848–4853 (2006)
10. Grancharova, A., Johansen, T.A., Tøndel, P.: Computational aspects of approximate explicit nonlinear model predictive control. In: Findeisen, R., Allgöwer, F., Biegler, L. (eds.) Assessment and Future Directions of Nonlinear Model Predictive Control. LNCIS, vol. 358, pp. 181–192. Springer, Heidelberg (2007)
11. Grancharova, A., Johansen, T.A.: Computation, approximation and stability of explicit feedback min-max nonlinear model predictive control. Automatica 45, 1134–1143 (2009)
12. Grossmann, I.E., Halemane, K.P., Swaney, R.E.: Optimization strategies for flexible chemical processes. Computers and Chemical Engineering 7, 439–462 (1983)
13. Hicks, G., Ray, W.: Approximation methods for optimal control synthesis. The Canadian Journal of Chemical Engineering 49, 522–528 (1971)
14. Johansen, T.A.: Approximate explicit receding horizon control of constrained nonlinear systems. Automatica 40, 293–300 (2004)
15. Kerrigan, E.C., Maciejowski, J.M.: Feedback min-max model predictive control using a single linear program: Robust stability and the explicit solution. International Journal of Robust and Nonlinear Control 14, 395–413 (2004)
16. Kerrigan, E.C., Mayne, D.Q.: Optimal control of constrained piecewise affine systems with bounded disturbances. In: Proceedings of the IEEE Conference on Decision and Control, Las Vegas, NV, pp. 1552–1557 (2002)
17. Khalil, H.K.: Nonlinear systems, 3rd edn. Prentice Hall, USA (2002)
18. Lazar, M., Heemels, W.P.M.H., Bemporad, A., Weiland, S.: Discrete-time non-smooth nonlinear MPC: Stability and robustness. In: Findeisen, R., Allgöwer, F., Biegler, L. (eds.) Assessment and Future Directions of Nonlinear Model Predictive Control: Towards New Challenging Applications. LNCIS, vol. 358, pp. 93–103. Springer, Heidelberg (2007)
19. Limon, D., Alamo, T., Camacho, E.F.: Input-to-state stable MPC for constrained discrete-time nonlinear systems with bounded additive uncertainties. In: Proceedings of the IEEE Conference on Decision and Control, Las Vegas, NV, pp. 4619–4624 (2002)
20. Limon, D., Alamo, T., Salas, F., Camacho, E.F.: Input-to-state stability of min-max MPC controllers for nonlinear systems with bounded uncertainties. Automatica 42, 797–803 (2006)
21. Magni, L., de Nicolao, G., Scattolini, R., Allgöwer, F.: Robust model predictive control for nonlinear discrete-time systems. International Journal of Robust and Nonlinear Control 13, 229–246 (2003)
22. Magni, L., Scattolini, R.: Robustness and robust design of MPC for nonlinear discrete-time systems. In: Findeisen, R., Allgöwer, F., Biegler, L. (eds.) Assessment and Future Directions of Nonlinear Model Predictive Control. LNCIS, vol. 358, pp. 239–254. Springer, Heidelberg (2007)
23. Mayne, D.Q., Rawlings, J.B., Rao, C.V., Scokaert, P.O.M.: Constrained model predictive control: Stability and optimality. Automatica 36, 789–814 (2000)
24. Mayne, D.Q., Raković, S.V., Vinter, R.B., Kerrigan, E.C.: Characterization of the solution to a constrained H_∞ optimal control problem. Automatica 42, 371–382 (2006)
25. Mhaskar, P.: Robust model predictive control design for fault-tolerant control of process systems. Industrial Engineering & Chemistry Research 45, 8565–8574 (2006)

26. Michalska, H., Mayne, D.Q.: Robust receding horizon control of constrained nonlinear systems. IEEE Transactions on Automatic Control 38, 1623–1633 (1993)
27. Mukai, M., Azuma, T., Kojima, A., Fujita, M.: Approximate robust receding horizon control for piecewise linear systems via orthogonal partitioning. In: Proceedings of the European Control Conference, Cambridge, U.K. (2003)
28. Muñoz de la Peña, D., Alamo, T., Ramirez, D.R., Camacho, E.F.: Min-max model predictive control as a quadratic program. IET Control Theory and Applications 1, 328–333 (2007)
29. Muñoz de la Peña, D., Bemporad, A., Filippi, C.: Robust explicit MPC based on approximate multi-parametric convex programming. IEEE Transactions on Automatic Control 51, 1399–1403 (2006)
30. Raković, S.V., Kerrigan, E.C., Mayne, D.Q.: Optimal control of constrained piecewise affine systems with state- and input-dependent disturbances. In: Proceedings of the 16th International Symposium on Mathematical Theory of Networks and Systems, Leuven, Belgium (2004)
31. Wan, Z., Kothare, M.: An efficient off-line formulation of robust model predictive control using linear matrix inequalities. Automatica 39, 837–846 (2003)