Chapter 5
Explicit MPC of Constrained Nonlinear Systems with Quantized Inputs

Abstract. This chapter presents an approximate multi-parametric Nonlinear Integer Programming (mp-NIP) approach to design explicit MPC controllers for constrained nonlinear systems with quantized control inputs. It is organized as follows. In Section 5.2, general regulation and reference tracking quantized NMPC problems are formulated and represented as an mp-NIP problem. Then, in Section 5.3, an approximate mp-NIP approach to explicit quantized NMPC is described. The idea of the approach is to construct a piecewise constant (PWC) approximation to the optimal solution of the mp-NIP problem on a hyper-rectangle of interest by imposing an orthogonal state space partition. In Section 5.4, an explicit quantized NMPC controller for the electropneumatic clutch actuator (described in Section 4.5) is designed and its performance is compared to that of the explicit NMPC with continuous control input. In Section 5.5, the approximate mp-NIP approach is applied to design an explicit quantized NMPC controller for optimal regulation of a continuous stirred tank reactor.

5.1 Introduction

In several control engineering problems, the system to be controlled is characterized by a finite set of possible control actions. Such systems are referred to as systems with *quantized* control input and the possible values of the input represent the levels of *quantization*. For example, hydraulic systems using on/off valves are systems with *quantized* input. In order to achieve a high quality of the control system performance it would be necessary to take into account the effect of the control input *quantization*. Thus, in [10] receding horizon optimal control ideas were proposed for synthesizing *quantized* control laws for *linear* systems with *quantized* inputs and quadratic optimality criteria. Further in [1], a method for *explicit* solution of optimal control problems with *quantized* control input was developed. It is based on solving multi-parametric Nonlinear Integer Programming (mp-NIP) problems, where the cost function and the constraints depend linearly on the vector of parameters. In [4, 3], an MPC problem for constrained *nonlinear* systems with *quantized*

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input is formulated and represented as an mp-NIP problem. Then, a computational method for *explicit* approximate solution of the resulting mp-NIP problem is suggested. The benefits of the *explicit* solution consist in efficient on-line computations using a binary search tree and verifiability of the design and implementation. The mp-NIP method proposed in [4, 3] is more general compared to the mp-NIP method in [1], since it allows the cost function and the constraints to depend nonlinearly on the vector of parameters.

Note that the term Nonlinear Integer Programming is used instead of the more general Mixed-Integer Nonlinear Programming (MI-NLP) since the problem formulation contains only integer free variables. This is possible since continuous variables are eliminated using a direct single shooting strategy, and all control inputs are assumed to be quantized. The general ideas can, however, be extended to MI-NLP to account for situations with mixed continuous and integer variables.

5.2 Formulation of the Quantized NMPC Problem as an mp-NIP Problem

Consider the discrete-time nonlinear system:

$$x(t+1) = f(x(t), u(t)) \tag{5.1}$$

$$y(t) = Cx(t), \tag{5.2}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $y(t) \in \mathbb{R}^p$ is the output variable, and $u(t) \in \mathbb{R}^m$ is the control input, which is constrained to belong to the finite set of values $U^A = \{\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_L\}$, $\overline{u}_i \in \mathbb{R}^m$, $\forall i = 1, 2, \ldots, L$, i.e. $u(t) \in U^A$. Here, $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_L$ represent the levels of *quantization* of the control input u. In (5.1), $f : \mathbb{R}^n \times U^A \longmapsto \mathbb{R}^n$ is a nonlinear function. It is supposed that a full measurement of the state $x(t)$ is available at the current time t.

First, consider the optimal regulation problem where the goal is to steer the system state to the origin by minimizing a certain performance criterion. For the current $x(t)$, the *quantized* NMPC regulation solves the optimization problem:

Problem 5.1:

$$V^*(x(t)) = \min_{U \in U^B} J(U, x(t)) \tag{5.3}$$

subject to $x_{t|t} = x(t)$ and:

$$y_{\min} \le y_{t+k|t} \le y_{\max}, \;\; k = 1, \ldots, N \tag{5.4}$$

$$u_{t+k} \in U^A = \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_L\}, \;\; k = 0, 1, \ldots, N-1 \tag{5.5}$$

$$\|x_{t+N|t}\|^2 \le \delta_x \tag{5.6}$$

$$x_{t+k+1|t} = f(x_{t+k|t}, u_{t+k}), \;\; k \ge 0 \tag{5.7}$$

$$y_{t+k|t} = Cx_{t+k|t}, \;\; k \ge 0 \tag{5.8}$$

Here, $U = [u_t, u_{t+1}, \dots, u_{t+N-1}] \in \mathbb{R}^{Nm}$ is the set of free control moves, $U^B \triangleq (U^A)^N = U^A \times \dots \times U^A$. The set U^B is also represented as $U^B = \{\bar{U}_j \mid j = 1, 2, \dots, M\}$, where $\bar{U}_j \in \mathbb{R}^{Nm}$ are the levels of quantization of the control vector U and $M = L^N$. The cost function is given by:

$$J(U, x(t)) = \sum_{k=0}^{N-1} \left[\|x_{t+k|t}\|^2_{Q_x} + \|h(x_{t+k|t}, u_{t+k})\|^2_R \right] + \|x_{t+N|t}\|^2_{P_x} \tag{5.9}$$

Here, N is a finite horizon and $h : \mathbb{R}^n \times U^A \mapsto \mathbb{R}^s$ is a nonlinear function. It is assumed that $\delta_x > 0$ and P_x, Q_x, $R \succ 0$.

Now, consider the reference tracking problem where the goal is to have the output variable $y(t)$ track the reference signal $r(t) \in \mathbb{R}^p$. For the current $x(t)$, the reference tracking *quantized* NMPC solves the following optimization problem:

Problem 5.2:

$$V^*(x(t), r(t)) = \min_{U \in U^B} J(U, x(t), r(t)) \tag{5.10}$$

subject to $x_{t|t} = x(t)$ and:

$$y_{\min} \le y_{t+k|t} \le y_{\max},\ k = 1, \dots, N \tag{5.11}$$

$$u_{t+k} \in U^A = \{\overline{u}_1, \overline{u}_2, \dots, \overline{u}_L\},\ k = 0, 1, \dots, N-1 \tag{5.12}$$

$$\left\| y_{t+N|t} - r(t) \right\| \le \delta_y \tag{5.13}$$

$$x_{t+k+1|t} = f(x_{t+k|t}, u_{t+k}),\ k \ge 0 \tag{5.14}$$

$$y_{t+k|t} = C x_{t+k|t},\ k \ge 0 \tag{5.15}$$

Here, $U = [u_t, u_{t+1}, \dots, u_{t+N-1}] \in \mathbb{R}^{Nm}$ is the set of free control moves, $U^B = (U^A)^N = U^A \times \dots \times U^A$ and the cost function is given by:

$$J(U, x(t), r(t)) = \sum_{k=0}^{N-1} \left[\|y_{t+k|t} - r(t)\|^2_{Q_y} + \|h(x_{t+k|t}, u_{t+k})\|^2_R \right] + \|y_{t+N|t} - r(t)\|^2_{P_y} \tag{5.16}$$

Similar to above, N is a finite horizon and $h : \mathbb{R}^n \times U^A \longmapsto \mathbb{R}^s$ is a nonlinear function. It is assumed that $\delta_y > 0$ and P_y, Q_y, $R \succ 0$.

From a stability point of view it is desirable to choose δ_x in (5.6) or δ_y in (5.13) as small as possible [9]. However, in the case of quantized input, the equilibrium point of the closed-loop system may either have an offset from the reference, or there may be a limit cycle about the reference. Therefore, the feasibility of Problems 5.1 and 5.2 will rely on δ_x and δ_y being sufficiently large. A part of the NMPC design will be to address this tradeoff.

The optimization Problems 5.1 and 5.2 can be formulated in a compact form as follows:

Problem 5.3:

$$V^*(\tilde{x}(t)) = \min_{U \in U^B} J(U,\tilde{x}(t)) \text{ subject to } G(U,\tilde{x}(t)) \leq 0 \tag{5.17}$$

Here $\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$ and for the regulation Problem 5.1 it is:

$$\tilde{x}(t) = x(t),\ \tilde{n} = n \tag{5.18}$$

while for the reference tracking Problem 5.2 it is:

$$\tilde{x}(t) = [x(t),\, r(t)] \in \mathbb{R}^{\tilde{n}},\ \tilde{n} = n + p \tag{5.19}$$

Problem 5.3 defines a multi-parametric Nonlinear Integer Programming (mp-NIP) problem, since it is NIP in U parameterized by $\tilde{x}(t)$. An optimal solution to this problem is denoted $U^* = [u_t^*,\, u_{t+1}^*,\, \ldots,\, u_{t+N-1}^*]$ and the control input is chosen according to the receding horizon policy $u(t) = u_t^*$. Define the set of feasible parameters as follows:

$$X_f = \{\tilde{x} \in \mathbb{R}^{\tilde{n}} |\, G(U,\tilde{x}) \leq 0 \text{ for some } U \in U^B\} \tag{5.20}$$

For Problem 5.1, X_f is the set of N-step feasible initial states. If δ_x, δ_y and N are such that the Problem 5.1 or 5.2 is feasible, then X_f is a non-empty set. In parametric programming problems one seeks the solution $U^*(\tilde{x})$ as an explicit function of the parameters $\tilde{x}$ in a set $X \subseteq X_f \subseteq \mathbb{R}^{\tilde{n}}$ [2].

5.3 Approximate mp-NIP Approach to Explicit Quantized NMPC

5.3.1 Computation of Explicit Approximate Solution

We restrict our attention to a hyper-rectangle $X \subset \mathbb{R}^{\tilde{n}}$ where we seek to approximate the optimal solution $U^*(\tilde{x})$ to Problem 5.3. We require that the state space partition is orthogonal and can be represented as a $k-d$ tree. The main idea of the approximate mp-NIP approach in [4, 3] is to construct a *piecewise constant* (PWC) approximation $\widehat{U}(\tilde{x})$ to $U^*(\tilde{x})$ on X, where the constituent constant functions are defined on hyper-rectangles covering X. The solution of Problem 5.3 is computed at the $2^{\tilde{n}}$ vertices of a considered hyper-rectangle X_0, as well as at some interior points. These additional points represent the vertices and the facets centers of one or more hyper-rectangles contained in the interior of X_0. The Procedure 1.1 is used to generate a set of points $W_0 = \{w_0, w_1, w_2, \ldots, w_{N_1}\}$ associated to a hyper-rectangle X_0. Then, a close-to-global solution $U^*(w_i)$ of Problem 5.3 at a point $w_i \in W_0$ is computed by using the routine 'glcSolve' of the TOMLAB optimization environment in

Matlab [7]. The routine 'glcSolve' implements an extended version of the DIRECT algorithm [8], that handles problems with both nonlinear and integer constraints. The DIRECT algorithm (DIviding RECTangles) [8] is a deterministic sampling algorithm for searching for the global minimum of a multivariate function subject to constraints, using no derivative information. It is a modification of the standard Lipschitzian approach that eliminates the need to specify a Lipschitz constant.

Based on the close-to-global solutions $U^*(w_i)$ at all points $w_i \in W_0$, a local constant approximation $\widehat{U}_0(\tilde{x}) = K_0$ to the optimal solution $U^*(\tilde{x})$, valid in the whole hyper-rectangle X_0, is determined by applying the following procedure [4]:

Procedure 5.1 (computation of explicit approximate solution). *Consider any hyper-rectangle $X_0 \subseteq X$ with a set of points $W_0 = \{w_0, w_1, \ldots, w_{N_1}\}$ determined by applying Procedure 1.1. Compute K_0 by solving the following NIP:*

$$\min_{K_0 \in U^B} \sum_{i=0}^{N_1} \left(J(K_0, w_i) - V^*(w_i)\right) \quad \textit{subject to} \quad G(K_0, w_i) \leq 0, \forall w_i \in W_0 \tag{5.21}$$

5.3.2 *Estimation of Error Bounds*

Suppose that a constant function $\widehat{U}_0(\tilde{x}) = K_0$, associated to the hyper-rectangle X_0 has been determined by applying Procedure 5.1. Then, for the cost function approximation error in X_0 we have:

$$\varepsilon(\tilde{x}) = \widehat{V}(\tilde{x}) - V^*(\tilde{x}) \leq \varepsilon_0 \ , \ \tilde{x} \in X_0 \tag{5.22}$$

where $\widehat{V}(\tilde{x}) = J(\widehat{U}_0(\tilde{x}), \tilde{x})$ is the sub-optimal cost and $V^*(\tilde{x})$ denotes the cost corresponding to the close-to-global solution $U^*(\tilde{x})$, i.e. $V^*(\tilde{x}) = J(U^*(\tilde{x}), \tilde{x})$. The following procedure can be used to obtain an estimate $\widehat{\varepsilon}_0$ of the maximal approximation error ε_0 in X_0.

Procedure 5.2 (computation of error bound approximation). *Consider any hyper-rectangle $X_0 \subseteq X$ with a set of points $W_0 = \{w_0, w_1, \ldots, w_{N_1}\}$ determined by applying Procedure 1.1. Compute an estimate $\widehat{\varepsilon}_0$ of the error bound ε_0 through the following maximization:*

$$\widehat{\varepsilon}_0 = \max_{i \in \{0,1,2,\ldots,N_1\}} \left(\widehat{V}(w_i) - V^*(w_i)\right) \tag{5.23}$$

5.3.3 *Approximate mp-NIP Algorithm*

Assume the tolerance $\bar{\varepsilon} > 0$ of the cost function approximation error is given. The following algorithm is proposed to design explicit reference tracking *quantized* NMPC [4]:

Algorithm 5.1. Explicit reference tracking quantized NMPC.

Step 1. Initialize the partition to the whole hyper-rectangle, i.e. $\Pi = \{X\}$. Mark the hyper-rectangle X as unexplored.

Step 2. Select any unexplored hyper-rectangle $X_0 \in \Pi$. If no such hyper-rectangle exists, terminate.

Step 3. Generate a set of points $W_0 = \{w_0, w_1, w_2, \dots, w_{N_1}\}$ associated to X_0 by applying Procedure 1.1.

Step 4. Compute a solution to Problem 5.3 for $\tilde{x}$ fixed to each of the points w_i, $i = 0,1,2,\dots,N_1$ by using routine 'glcSolve' of TOMLAB optimization environment. If Problem 5.3 has a feasible solution at all these points, go to step 7. Otherwise, go to step 5.

Step 5. Compute the size of X_0 using some metric. If it is smaller than some given tolerance, mark X_0 infeasible and explored and go to step 2. Otherwise, go to step 6.

Step 6. If at least one of the points w_i, $i = 0,1,2,\dots,N_1$ is feasible, split X_0 into hyper-rectangles X_1, X_2, ... , X_{N_s} by applying the Heuristic splitting rule 1.1. Mark X_1, X_2, ... , X_{N_s} unexplored, remove X_0 from Π, add X_1, X_2, ... , X_{N_s} to Π, and go to step 2. If none of the points w_i, $i = 0,1,2,\dots,N_1$ are feasible, split X_0 into two hyper-rectangles X_1 and X_2 by a hyperplane through its center point and orthogonal to an arbitrary axis. Mark X_1 and X_2 unexplored, remove X_0 from Π, add X_1 and X_2 to Π, and go to step 2.

Step 7. Compute a constant function $\widehat{U}_0(\tilde{x})$ using Procedure 5.1, as an approximation to be used in X_0. If a feasible solution was found, go to step 8. Otherwise, split X_0 into two hyper-rectangles X_1 and X_2 by a hyperplane through its center point and orthogonal to an arbitrary axis. Mark X_1 and X_2 unexplored, remove X_0 from Π, add X_1 and X_2 to Π, and go to step 2.

Step 8. Compute an estimate $\widehat{\varepsilon}_0$ of the error bound ε_0 in X_0 by applying Procedure 5.2. If $\widehat{\varepsilon}_0 \leq \bar{\varepsilon}$, mark X_0 as explored and feasible and go to step 2. Otherwise, split X_0 into two hyper-rectangles X_1 and X_2 by applying a procedure that is similar to Procedure 1.5. Mark X_1 and X_2 unexplored, remove X_0 from Π, add X_1 and X_2 to Π, and go to step 2.

5.4 Application 1: Reference Tracking Quantized Control of an Electropneumatic Clutch Actuator Using On/Off Valves

Consider the electropneumatic clutch actuator, whose mathematical model is described in Section 4.5.1. With the *quantized* control input the two valves are only allowed to be fully open or fully closed (no pulse-width modulation is used). Thus, the control input is an integer variable which can take only three values, i.e. $u \in U^A = \{1,2,3\}$. This is related to the mass flow rate $w_v(p_A, u)$ to/from chamber A in the following way:

$$\begin{array}{l} u=1 \Rightarrow w_v(p_A,1) = -w_{v,out}, \text{ for } \theta \in [tT_s;(t+1)T_s] \\ u=2 \Rightarrow w_v(p_A,2) = 0, \text{ for } \theta \in [tT_s;(t+1)T_s] \\ u=3 \Rightarrow w_v(p_A,3) = w_{v,in}, \text{ for } \theta \in [tT_s;(t+1)T_s] \end{array} \tag{5.24}$$

where $w_{v,out}$ and $w_{v,in}$ are determined from (5.7)-(5.8), and θ is the time variable. Therefore, $u=1$ corresponds to maximal flow from chamber A, $u=2$ means no flow, and $u=3$ corresponds to maximal flow to chamber A during the *whole* sampling period T_s.

5.4.1 Design of Explicit NMPC with Quantized Control Input

In [4, 5], an explicit *quantized* NMPC controller for the clutch actuator is designed, which is based on the *simplified 3-rd order* model (4.57)–(4.59), introduced in Section 4.5.1. The *quantized* NMPC has sampling time $T_s = 0.01\,[s]$ and it minimizes the cost function (4.64) in Section 4.5.2 (with u_{t+k} and U being here the quantized control input and the quantized control input sequence, respectively), subject to the system equations (4.57)–(4.59) and the constraint (4.65). In (4.64), the horizon is $N=10$ and the weighting coefficients are $Q_y = P_y = 1$, $R = 0.1$. The extended state vector $\tilde{x}(t)$ and the state space X to be partitioned are the same as for the NMPC controller with *continuous* control input, designed in Section 4.5.2. The cost function approximation tolerance is $\bar{\varepsilon}(X_0) = \max(\bar{\varepsilon}_a, \bar{\varepsilon}_r \min\limits_{\tilde{x}\in X_0} V^*(\tilde{x}))$, where $\bar{\varepsilon}_a = 0.001$ and $\bar{\varepsilon}_r = 0.02$. The partition has 10871 regions. The performance of the explicit *quantized* NMPC was simulated for the typical clutch reference signal and the resulting response is depicted in Fig. 5.1 and Fig. 5.2. The simulations of the closed-loop system are based on the *full 5-th order* model (4.40)–(4.44) of the clutch actuator dynamics, described in Section 4.5.1.

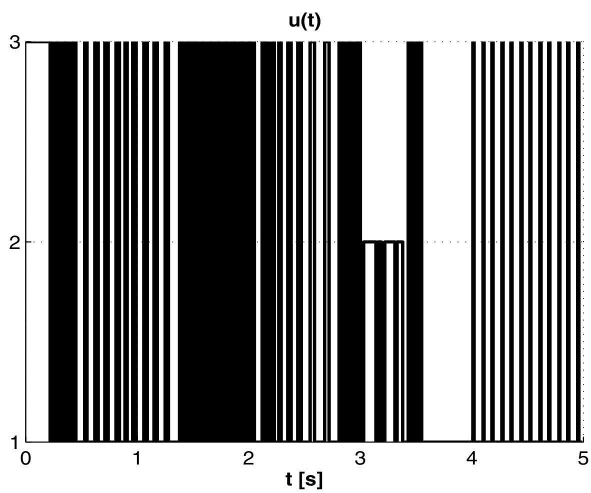

Fig. 5.1 The control input u.

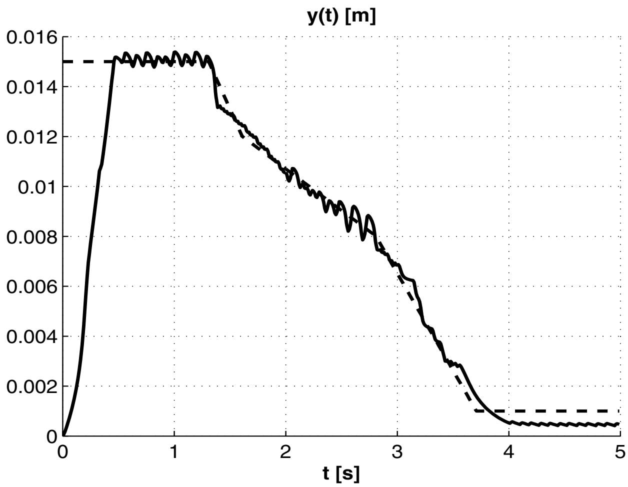

Fig. 5.2 The clutch actuator position y with the explicit *quantized* NMPC (the dashed curve is the reference signal).

5.4.2 Comparison between the Explicit NMPC with Quantized Control Input and the Explicit NMPC with Continuous Control Input

In [5], a comparative study of the explicit *quantized* NMPC controller and the explicit NMPC controller with continuous control input for reference tracking control of the electropneumatic clutch actuator is made.

5.4.2.1 Chattering

The chattering of the explicit *quantized* NMPC controller, designed in this section, and the explicit NMPC controller with continuous control input (using a PWM scheme), the SMC controller, and the PID controller, considered in Section 4.5.2, is studied. In Table 5.1, statistics about the chattering in the control input u (only for the controllers which generate a continuous control input) and in the actuator position y is given. The chattering is expressed as:

$$\Delta u(t) = |u(t) - u(t-1)|\,, \; \Delta y(t) = |y(t) - y(t-1)| \tag{5.25}$$

where $t = 2, 3, \ldots, 500$ is the discrete time instant. It can be seen that the control input chattering of the explicit NMPC with PWM is comparable to that of the PID controller and it is significantly smaller than that of the SMC controller. Also, the explicit NMPC with PWM leads to the smallest position chattering among the four studied controllers.

Table 5.1 Statistics about chattering

Controller	**Average** $\Delta u\,[\%]$	**Maximal** $\Delta u\,[\%]$	**Average** $\Delta y\,[m]$	**Maximal** $\Delta y\,[m]$
NMPC with PWM	4.65	81.99	$5.66 \cdot 10^{-5}$	$6.22 \cdot 10^{-4}$
SMC controller	12.16	168.96	$5.94 \cdot 10^{-5}$	$10.63 \cdot 10^{-4}$
PID controller	1.23	52.49	$6.63 \cdot 10^{-5}$	$10.63 \cdot 10^{-4}$
Quantized NMPC	–	–	$9.83 \cdot 10^{-5}$	$7.23 \cdot 10^{-4}$

5.4.2.2 Tracking Performance

In Table 5.2, statistics about the absolute reference tracking error $e_y(t) = |y(t) - r(t)|$ and the sum squared relative reference tracking error S_y for the four controllers are given. The error $e_y(t)$ is considered after the position settles near the first reference value $r = 0.015$ $[m]$ (after 0.8 $[s]$ of time). The reason is that the trajectories from the initial state $y = 0$ $[m]$ to a neighborhood of $r = 0.015$ $[m]$ for the four controllers are characterized by the same maximal reference tracking error $e_y = 0.015$ $[m]$. The error S_y is computed on the entire transients as:

$$S_y = \frac{1}{500} \sum_{t=1}^{500} \left(\frac{y(t) - r(t)}{r(t)} \right)^2 \tag{5.26}$$

It can be seen that the explicit NMPC with PWM provides the highest quality of tracking performance.

Table 5.2 Statistics about reference tracking error

Controller	**Average** $e_y(t)$ $[m]$	**Maximal** $e_y(t)$ $[m]$	**Sum squared error** S_y
NMPC with PWM	$2.48 \cdot 10^{-4}$	$6.89 \cdot 10^{-4}$	$0.431 \cdot 10^{-1}$
SMC controller	$3.05 \cdot 10^{-4}$	$6.30 \cdot 10^{-4}$	$0.679 \cdot 10^{-1}$
PID controller	$2.77 \cdot 10^{-4}$	$10.36 \cdot 10^{-4}$	$0.839 \cdot 10^{-1}$
Quantized NMPC	$3.27 \cdot 10^{-4}$	$10.39 \cdot 10^{-4}$	$1.063 \cdot 10^{-1}$

5.4.2.3 Real-Time Computational Complexity and Storage Requirements

The explicit approximate solutions of the two explicit NMPC controllers are implemented as binary search trees by applying the method in [11]. In Table 5.3, the real-time computational complexity (the worst-case number of arithmetic operations needed to compute the control input) and the storage requirements (in terms of numbers that have to be stored), associated to the binary search trees of the two controllers, are given. It can be observed that the number of on-line arithmetic operations is negligibly small with both controllers. The explicit NMPC with PWM requires significantly more storage in comparison to the explicit *quantized* NMPC controller. It can be explained with the fact that for each region of the partition of

Table 5.3 On-line computational complexity and storage requirements

Controller	arithmetic ops. per sample	stored reals	stored integers
Explicit NMPC with PWM	152	39960	36771
Explicit *quantized* NMPC	143	295	7831

this controller, an affine control law needs to be stored (while only one constant needs to be stored with the *quantized* controller). Further, since the total number of solutions for the *quantized* controller is only 3, merging of regions with the same solutions into one convex region leads to a significant decrease of the complexity of the search tree.

5.5 Application 2: Regulation of a Continuous Stirred Tank Reactor with Quantized Control Input

In [3], the approximate mp-NIP approach (described in Section 5.3) is applied to design an explicit *quantized* NMPC controller for optimal regulation of a continuous stirred tank reactor (CSTR). In the CSTR, a first-order irreversible reaction $A \to B$ takes place (Fig. 5.3). The mathematical model of CSTR and the values of the parameters are taken from [6]. The mass and heat balance of CSTR expressed through dimensionless concentration $\tilde{c}$ and temperature $\tilde{T}$ are [6]:

$$\frac{d\tilde{c}}{dt} = \frac{(1-\tilde{c})}{q} - k_0 e^{-\frac{E}{\tilde{T}}}\tilde{c} \tag{5.27}$$

$$\frac{d\tilde{T}}{dt} = \frac{(\tilde{T}_f - \tilde{T})}{q} + k_0 e^{-\frac{E}{\tilde{T}}}\tilde{c} - \alpha u(\tilde{T} - \tilde{T}_c) \tag{5.28}$$

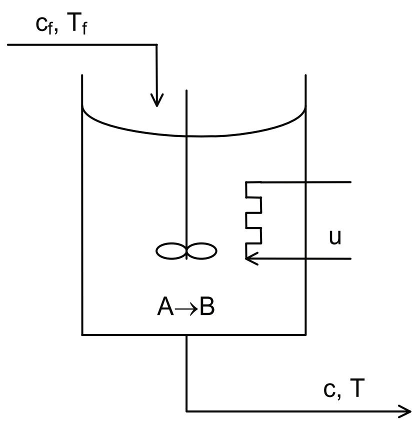

Fig. 5.3 Continuous stirred tank reactor.

where the dimensionless quantities $\tilde{c}$, $\tilde{T}$, $\tilde{T}_c$ and $\tilde{T}_f$ are defined as follows:

$$\tilde{c} = \frac{c}{c_f}, \; \tilde{T} = \frac{T}{Jc_f}, \; \tilde{T}_c = \frac{T_c}{Jc_f}, \; \tilde{T}_f = \frac{T_f}{Jc_f} \tag{5.29}$$

The coolant flowrate u is a *quantized* control variable. The values of the parameters are taken from [6] and are $q = 10$, $c_f = 1$, $T_c = 290$, $T_f = 300$, $J = 100$, $E = 25.2$, $k_0 = 300$, $\alpha = 1.95 \cdot 10^{-4}$.

We consider the set point $\tilde{c}^* = 0.41$, $\tilde{T}^* = 3.3$. Then, the model of the reactor can be written in the form:

$$\frac{dx_1}{dt} = \frac{(1 - \tilde{c}^* - x_1)}{q} - k_0 e^{-\frac{E}{(\tilde{T}^* + x_2)}} (\tilde{c}^* + x_1) \tag{5.30}$$

$$\frac{dx_2}{dt} = \frac{(\tilde{T}_f - \tilde{T}^* - x_2)}{q} + k_0 e^{-\frac{E}{(\tilde{T}^* + x_2)}} (\tilde{c}^* + x_1) - \alpha u(\tilde{T}^* + x_2 - \tilde{T}_c) \tag{5.31}$$

where x_1 and x_2 denote the deviations of the concentration and temperature from the set point values ($x_1 = \tilde{c} - \tilde{c}^*$, $x_2 = \tilde{T} - \tilde{T}^*$). The forward Euler method with step size $T_E = 0.01$ is used to integrate the equations (5.30)–(5.31).

The coolant flowrate u is *quantized* with the following levels of *quantization*:

$$u \in U^A = \{u_{\min}, \, u_{st}, \, u_{\max}\} \tag{5.32}$$

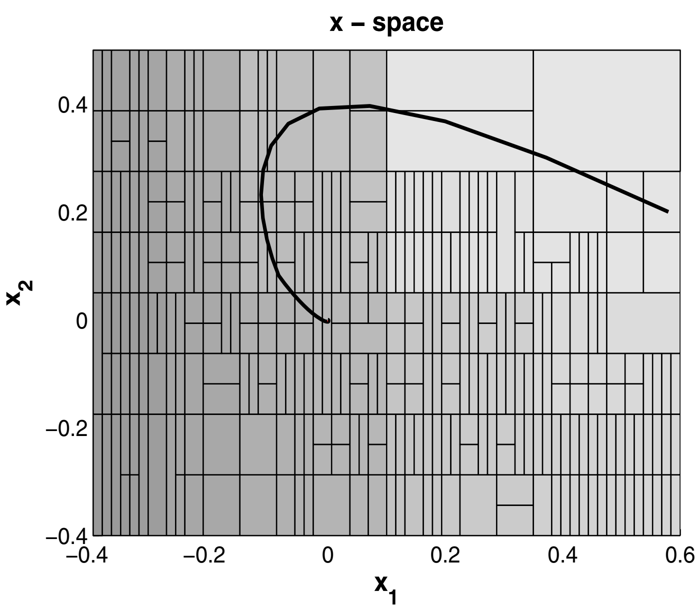

Fig. 5.4 State space partition of the explicit approximate quantized NMPC.

where $u_{\min} = 250$, $u_{\max} = 500$, and $u_{st} = 370$ is the steady state value corresponding to the set point $\tilde{c}^* = 0.41$, $\tilde{T}^* = 3.3$.

The suggested approximate mp-NIP approach is applied to design an explicit *quantized* NMPC controller for this reactor. The NMPC minimizes the cost function (5.9) subject to the system equations (5.30)–(5.31) and the input constraint

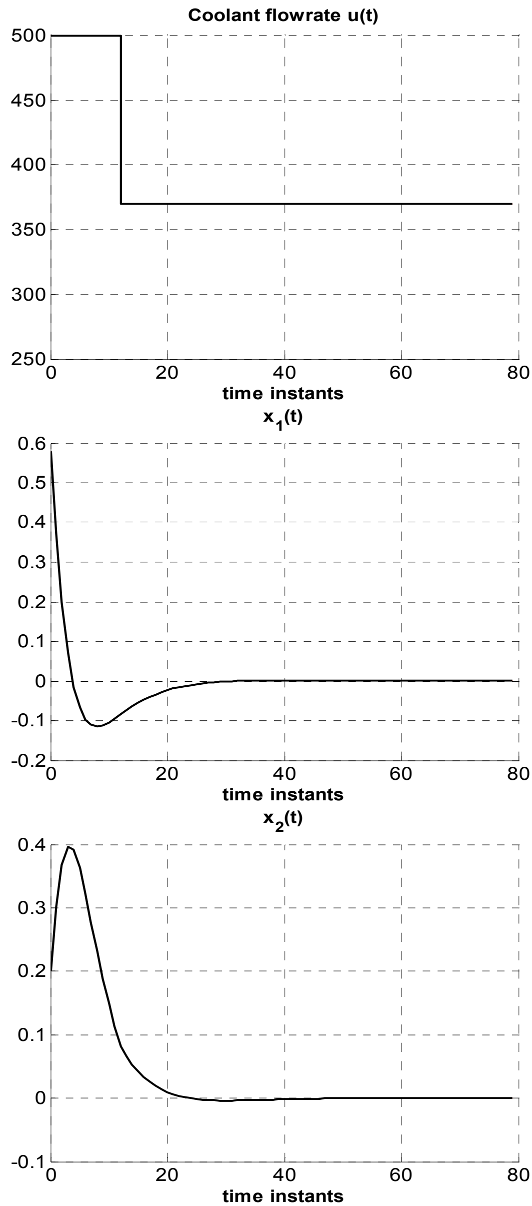

Fig. 5.5 From top to bottom: The control input, the state variable x_1, and the state variable x_2 for $\alpha = 1.95 \cdot 10^{-4}$.

(5.32). In (5.9), $h(x_{t+k|t}, u_{t+k}) \equiv u_{t+k} - u_{st}$ and the cost matrices are $Q_x = P_x = \mathrm{diag}\{100, 300\}$, $R = 1 \cdot 10^{-4}$. The horizon is $N = 30$ with a sampling time for the control input $T_s = 1$. In (5.6), it is chosen $\delta_x = 0.002$. The state space to be partitioned is defined by $X = [-0.4, 0.6] \times [-0.4, 0.5]$. The state space partition of the explicit *quantized* NMPC controller is shown in Fig. 5.4. It has 341 regions and 14 levels of search. Thus, 14 arithmetic operations are needed in real-time to compute the control input (14 comparisons). Due to *quantization*, it would be straightforward to join neighboring regions with the same solution at the first sample of the control trajectory in a postprocessing step. This would lead to a significant reduction of the complexity of the partition.

The performance of the closed-loop system was simulated for initial condition $x(0) = [0.58,\ 0.2]^T$. The resulting closed-loop response corresponding to the explicit approximate *quantized* NMPC (the solid curves) and to the exact *quantized* NMPC (the dotted curves) is depicted in the state space (Fig. 5.4), as well as trajectories in time (Fig. 5.5). The results show that the exact and the approximate solutions are indistinguishable.

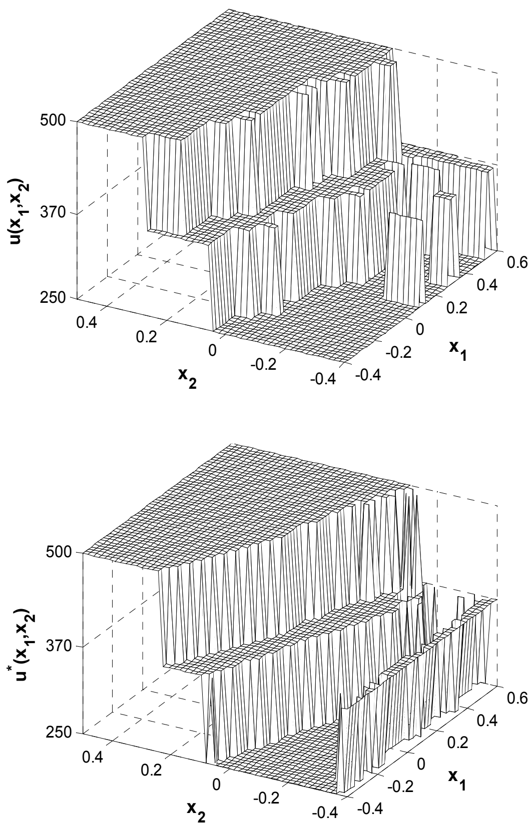

Fig. 5.6 Top: The suboptimal state feedback law $\hat{u}(x_1, x_2)$. Bottom: The optimal state feedback law $u^*(x_1, x_2)$.

The suboptimal and the optimal state feedback laws are shown in Fig. 5.6.

In order to study the robustness of the explicit controller, we assume that the real value of the heat transfer coefficient is $\alpha = 1.85 \cdot 10^{-4}$ (instead of $\alpha = 1.95 \cdot 10^{-4}$ used to design the controller). The closed-loop response corresponding to $\alpha = 1.85 \cdot 10^{-4}$ and initial condition $x(0) = [0.58,\ 0.2]^T$ is depicted in Fig. 5.7. It can be seen

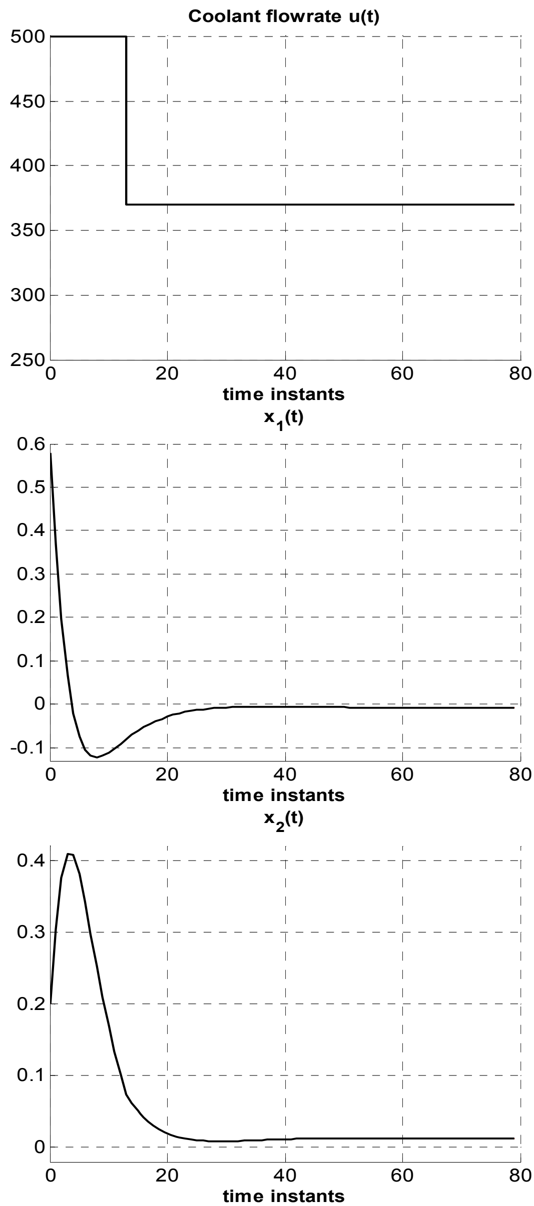

Fig. 5.7 From top to bottom: The control input, the state variable x_1, and the state variable x_2 for $\alpha = 1.85 \cdot 10^{-4}$.

that the closed-loop trajectory has an offset due to the fact that the steady state value u_{st} (cf. equation (5.32)) corresponding to the set point $\tilde{c}^* = 0.41$, $\tilde{T}^* = 3.3$ is a function of the coefficient α.

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