Deciding Conditional Termination*

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Abstract. This paper addresses the problem of conditional termination, which is that of defining the set of initial configurations from which a given program terminates. First we define the dual set, of initial configurations, from which a non-terminating execution exists, as the greatest fixpoint of the pre-image of the transition relation. This definition enables the representation of this set, whenever the closed form of the relation of the loop is definable in a logic that has quantifier elimination. This entails the decidability of the termination problem for such loops. Second, we present effective ways to compute the weakest precondition for non-termination for difference bounds and octagonal (non-deterministic) relations, by avoiding complex quantifier eliminations. We also investigate the existence of linear ranking functions for such loops. Finally, we study the class of linear affine relations and give a method of under-approximating the termination precondition for a non-trivial subclass of affine relations. We have performed preliminary experiments on transition systems modeling real-life systems, and have obtained encouraging results.

1 Introduction

The termination problem asks whether every computation of a given program ends in a halting state. The universal termination asks whether a given program stops for every possible input configuration. Both problems are among the first ever to be shown undecidable, by A. Turing [24]. In many cases however, programs will terminate when started in certain configurations, and may[1] run forever, when started in other configurations. The problem of determining the set of configurations from which a program terminates on all paths is called *conditional termination*.

In program analysis, the presence of non-terminating runs has been traditionally considered faulty. However, more recently, with the advent of *reactive systems*, accidental termination can be an equally serious error. For instance, when designing a web server,

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[1] If the program is non-deterministic, the existence of a single infinite run, among other finite runs, suffices to consider an initial configuration non-terminating.

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a developer would like to make sure that the main program loop will not exit unless a stopping request has been issued. These facts lead us to considering the *conditional non-termination* problem, which is determining the set of initial configurations which guarantee that the program will not exit.

In this paper we focus on programs that handle integer variables, performing linear arithmetic tests and (possibly non-deterministic) updates. A first observation is that the set of configurations guaranteeing non-termination is the greatest fixpoint of the pre-image of the program's transition relation[2] R. This set, called the *weakest recurrent set*, and denoted $wrs(R)$ in our paper, can be defined in first-order arithmetic, provided that the closed form of the infinite sequence of relations $\{R^i\}_{i \geq 0}$, obtained by composing the transition relation with itself $0, 1, 2, \ldots$ times, can also be defined using first-order arithmetic. Moreover, if the fragment of arithmetic we use has quantifier elimination, the weakest recurrent set can be expressed in a quantifier-free decidable fragment of arithmetic. This also means that the problem $wrs(R) \overset{?}{=} \emptyset$ is decidable, yielding universal termination decidability proofs for free.

Contributions of this Paper. The main novelty in this paper is of rather theoretical nature: we show that the non-termination preconditions for integer transition relations defined as either *octagons* or *linear affine loops with finite monoid property* are definable in quantifier-free Presburger arithmetic. Thus, the universal termination problem for such program loops is decidable. However, since quantifier elimination in Presburger arithmetic is a complex procedure, we have developed alternative ways of deriving the preconditions for non-termination, and in particular:

- for *difference bounds*, we reduce the problem of finding the weakest recurrent set to finding the maximal solution of a system of inequalities in the complete lattice of integers extended with $\pm\infty$, where the right-hand sides use addition and $\min$ operators. Efficient algorithms for finding such maximal solutions are based on policy iteration [14]. This encoding gives us a worst-case time complexity of $\mathcal{O}(n^2 \cdot 2^n)$ in the number of variables n, for the computation of the weakest recurrent set for difference bounds relations.
- for *octagonal relations*, we use a result from [5], namely that the sequence $\{R^i\}_{i \geq 0}$ is, in some sense, periodic. We give here a simple quantifier elimination method, targeted for the class of formulae defining weakest recursive sets. Moreover, we investigate the existence of linear ranking functions, and prove that, for each well-founded octagonal relations, there exists an effectively computable witness relation i.e., a well-founded relation that has a linear ranking function.
- for *linear affine relations*, weakest recurrent sets can be defined in Presburger arithmetic if we consider several restrictions concerning the transformation matrix. If the matrix A defining R has eigenvalues which are either zeros or roots of unity, all non-zero eigenvalues being of multiplicity one (these conditions are equivalent to the finite monoid property of [2,12]), then $wrs(R)$ is Presburger definable. Otherwise, if all non-zero eigenvalues of A are roots of unity, of multiplicities greater

[2] This definition is the dual of the *reachability set*, needed for checking safety properties: the reachability set is the least fixpoint of the post-image of the transition relation.

or equal to one, $wrs(R)$ can be expressed using polynomial terms. In this case, we can systematically issue termination preconditions, which are of significant practical importance, as noted in [11].

For space reasons, all proofs are deferred to [7].

Practical Applications. Unfortunately, in practice, the cases in which the closed form of the sequence $\{R^i\}_{i \geq 0}$ is definable in a logic that has quantifier elimination, are fairly rare. All relations considered so far are conjunctive, meaning that they can represent only simple program loops of the form `while(condition){body}`, where the loop body contains no further conditional constructs. In order to deal with more complicated program loops, one can use the results from this paper in several ways:

- use the decision procedures as a back-end of a termination analyzer, in order to detect spurious non-termination counterexamples consisting of a finite prefix (stem) and a conjunctive loop body (lasso). The spurious counterexamples can be discarded by intersecting the program model with the complement of the weak deterministic Büchi automaton representing the counterexample, as in [17].
- abstract a disjunctive loop body $R_1 \vee \ldots \vee R_n$ by a non-deterministic difference bounds or octagonal[3] relation $R^{\#} \supseteq R_{1,\ldots,n}$ and compute the weakest recurrent set of the latter. The complement of this set is a set of configurations from which the original loop terminates.
- attempt to compute a *transition invariant* i.e., an overapproximations of the transitive closure of the disjunctive loop body $(R_1 \vee \ldots \vee R_n)^+$ (using e.g., the semi-algorithmic unfolding technique described in [6]) and overapproximate it by a disjunction $R_1^{\#} \vee \ldots \vee R_m^{\#}$ of difference bounds or octagonal relations. Then compute the weakest recurrent set of each relation in the latter disjunction. If $wrs(R_1^{\#}) = \ldots = wrs(R_m^{\#}) = \emptyset$, the original loop terminates on any input, following the principle of transition invariants [20].

1.1 Related Work

The literature on program termination is vast. Most work focuses however on universal termination, such as the techniques for synthesizing linear ranking functions of Sohn and Van Gelder [22] or Podelski and Rybalchenko [19], and the more sophisticated method of Bradley, Manna and Sipma [9], which synthesizes lexicographic polynomial ranking functions, suitable when dealing with disjunctive loops. However, not every terminating program (loop) has a linear (polynomial) ranking function. In this paper we show that, for an entire class of non-deterministic linear relations, defined using octagons, termination is always witnessed by a computable octagonal relation that has a linear ranking function.

Another line of work considers the decidability of termination for simple (conjunctive) linear loops. Initially Tiwari [23] shows decidability of termination for affine linear loops interpreted over *reals*, while Braverman [10] refines this result by showing

[3] The linear affine relations considered in this paper are deterministic, which makes them unsuitable for abstraction.

decidability over *rationals* and over *integers*, for homogeneous relations of the form $C_1\mathbf{x} > 0 \ \wedge \ C_2\mathbf{x} \geq 0 \ \wedge \ \mathbf{x}' = A\mathbf{x}$. The non-homogeneous integer case seems to be much more difficult as it is closely related to the open *Skolem's Problem* [16]: given a linear recurrence $\{u_i\}_{i \geq 0}$, determine whether $u_i = 0$ for some $i \geq 0$.

To our knowledge, the first work on proving non-termination of simple loops is reported in [15]. The notion of *recurrent sets* occurs in this work, however without the connection with fixpoint theory, which is introduced in the present work. Finding recurrent sets in [15] is complete with respect to a predefined set of templates, typically linear systems of rational inequalities.

The work which is closest to ours is probably that of Cook et al. [11]. In this paper, the authors develop an algorithm for deriving termination preconditions, by first guessing a ranking function candidate (typically the linear term from the loop condition) and then inferring a supporting assertion, which guarantees that the candidate function decreases with each iteration. The step of finding a supporting assertion requires a fixpoint iteration, in order to find an invariant condition. Unlike our work, the authors of [11] do not address issues related to completeness: the method is not guaranteed to find the weakest precondition for termination, even in cases when this set can be computed. On the other hand, it is applicable to a large range of programs, extracted from real-life software. To compare our method with theirs, we tried all examples available in [11]. Since most of them are linear affine relations, we used our under-approximation method and have computed termination preconditions, which turn out to be slightly more general than the ones reported in [11].

2 Preconditions for Non-termination

In the rest of this paper we denote by $\mathbf{x} = \{x_1, \ldots, x_n\}$ the set of working variables, ranging over a domain of values denoted as $\mathcal{D}$. A *state* is a valuation $s : \mathbf{x} \to \mathcal{D}$, or equivalently, an n-tuple of values from $\mathcal{D}$. An *execution step* is a relation $R \subseteq \mathcal{D}^n \times \mathcal{D}^n$ defined by an *arithmetic formula* $\mathcal{R}(\mathbf{x}, \mathbf{x}')$, where the set $\mathbf{x}' = \{x'_1, \ldots, x'_n\}$ denotes the values of the variables after executing R once. If s and s' are valuations of the sets $\mathbf{x}$ and $\mathbf{x}'$, we denote by $\mathcal{R}(s, s')$ the fact that $(s, s') \in R$. A relation R is said to be *consistent* if there exist states s, s' such that $\mathcal{R}(s, s')$.

Relational composition is defined as $R_1 \circ R_2 = \{(s, s') \in \mathcal{D}^n \times \mathcal{D}^n \mid \exists s'' \in \mathcal{D}^n \ . \ \mathcal{R}_1(s, s'') \ \wedge \ \mathcal{R}_2(s'', s')\}$. For any relation $R \in \mathcal{D}^n \times \mathcal{D}^n$, we consider R^0 to be the identity relation, and we define $R^{i+1} = R^i \circ R$, for all $i \geq 0$. The pre-image of a set $S \subseteq \mathcal{D}^n$ via R is the set $pre_R(S) = \{s \in \mathcal{D}^n \mid \exists s' \in S \ . \ \mathcal{R}(s, s')\}$. It is easy to check that $pre^i_R(S) = pre_{R^i}(S)$, for any $S \subseteq \mathcal{D}^n$ and for all $i \geq 0$. For any $i \geq 0$, we write $\mathcal{R}^i$ for the formula defining the relation R^i and $\mathcal{R}^{-i}(\top)$ for the formula defining the set $pre_{R^i}(\mathcal{D}^n)$.

Definition 1. *A relation R is said to be* $*$-consistent *if and only if, for any $k > 0$, there exists a sequence of states $s_1, \ldots, s_k$, such that $\mathcal{R}(s_i, s_{i+1})$, for all $i = 1, \ldots, k-1$. R is said to be* well-founded *if and only if there is no infinite sequence of states $\{s_i\}_{i>0}$, such that $\mathcal{R}(s_i, s_{i+1})$, for all $i > 0$.*

Notice that if a relation is not $*$-consistent, then it is also well-founded. However the dual is not true. For instance, the relation $R = \{(n, n-1) \mid n > 0\}$ is both $*$-consistent and well-founded.

Definition 2. *A set $S \subseteq \mathcal{D}^n$ is said to be a* non-termination precondition *for R if, for each state $s \in S$ there exists an infinite sequence of states $s_0, s_1, s_2, \ldots$ such that $s = s_0$ and $\mathcal{R}(s_i, s_{i+1})$, for all $i \geq 0$.*

If $S_0, S_1, \ldots$ are all non-termination preconditions for R, then the (possibly infinite) union $\bigcup_{i=0,1,\ldots} S_i$ is a non-termination precondition for R as well. The set $wnt(R) = \bigcup\{S \in \mathcal{D}^n \mid S$ is a non-termination precondition for $R\}$ is called the *weakest non-termination precondition* for R. A relation R is well-founded if and only if $wnt(R) = \emptyset$. A set S such that $S \cap wnt(R) = \emptyset$ is called a *termination precondition.*

Definition 3. *A set $S \subseteq \mathcal{D}^n$ is said to be* recurrent *for a relation $R \in \mathcal{D}^n \times \mathcal{D}^n$ if and only if $S \subseteq pre_R(S)$.*

Proposition 1. *Let $S_0, S_1, \ldots \in \mathcal{D}^n$ be a (possibly infinite) sequence of sets, all of which are recurrent for a relation $R \in \mathcal{D}^n \times \mathcal{D}^n$. Then their union $\bigcup_{i=0,1,\ldots} S_i$ is recurrent for R as well.*

The set $wrs(R) = \bigcup\{S \in \mathcal{D}^n \mid S$ is a recurrent set for $R\}$ is called the *weakest recurrent set* for R. By Proposition 1, $wrs(R)$ is recurrent for R. Next we define the weakest recurrent set as the greatest fixpoint of the transition relation's pre-image.

Lemma 1. *Given a relation $R \in \mathcal{D}^n \times \mathcal{D}^n$, the weakest recurrent set for R is the greatest fixpoint of the function $X \mapsto pre_R(X)$.*

As a consequence, we obtain $wrs(R) = \bigcap_{i>0} pre^i_R(\mathcal{D}^n)$, by the Kleene Fixpoint Theorem. Since $pre^i_R = pre_{R^i}$, we have $wrs(R) = \bigcap_{i>0} pre_{R^i}(\mathcal{D}^n)$. In other words, from any state in the weakest recurrent set for a relation, an iteration of any finite length of the given relation is possible. The following lemma shows that in fact, this is exactly the set of states from which an infinite iteration is also possible.

Lemma 2. *Given a relation $R \in \mathcal{D}^n \times \mathcal{D}^n$, the weakest recurrent set for R equals its weakest non-termination precondition.*

The characterization of weakest recurrent sets as greatest fixpoints of the pre-image function suggests a method for computing such sets. In this section we show that, for certain classes of relations, these sets are definable in Presburger arithmetic, which gives a decision procedure for the well-foundedness problem for certain classes of relations, and consequently, for the termination problem for several classes of program loops.

Definition 4. *Given a relation $R \in \mathcal{D}^n \times \mathcal{D}^n$ defined by an arithmetic formula $\mathcal{R}(\mathbf{x}, \mathbf{x}')$, the* closed form *of R is a formula $\mathcal{R}^{(k)}(\mathbf{x}, \mathbf{x}')$, with free variables $\mathbf{x} \cup \mathbf{x}' \cup \{k\}$, such that for every integer valuation $i > 0$ of k, $\mathcal{R}^{(i)}(\mathbf{x}, \mathbf{x}')$ defines the relation R^i.*

Since, by Lemma 1, we have $wrs(R) = \text{gfp}(pre_R) = \bigcap_{i>0} pre_{R^i}(\mathcal{D}^n)$, using the closed form of R, one can now define:

$$wrs(R) \equiv \forall k > 0 \; \exists \mathbf{x}' \, . \, \mathcal{R}^{(k)}(\mathbf{x}, \mathbf{x}') \tag{1}$$

Because Presburger arithmetic has quantifier elimination, $wrs(R)$ can be defined in Presburger arithmetic[4] whenever $\mathcal{R}^{(k)}$ can. In [5] we show three classes of relations for which $\mathcal{R}^{(k)}$ is Presburger definable: difference bounds, octagonal and finite-monoid affine relations (the formal definitions of these classes are given in the next section). For each of these classes of loops termination is decidable, by the above argument.

3 Difference Bounds Relations

In this and the following sections, we assume that the variables $\mathbf{x} = \{x_1, \ldots, x_n\}$ range over integers i.e., that $\mathcal{D} = \mathbb{Z}$.

Definition 5. *A formula $\phi(\mathbf{x})$ is a* difference bounds constraint *if it is equivalent to a finite conjunction of atomic propositions of the form $x_i - x_j \leq a_{ij}$, for $1 \leq i, j \leq n, i \neq j$, where $a_{ij} \in \mathbb{Z}$.*

Given a difference bounds constraint ϕ, a *difference bounds matrix* (DBM) representing ϕ is a matrix $m_\phi \in \mathbb{Z}_\infty^{n \times n}$ such that $(m_\phi)_{ij} = a_{ij}$, if $x_i - x_j \leq a_{ij}$ is an atomic proposition in ϕ, and ∞, otherwise.

If ϕ is inconsistent (logically equivalent to false) we also say that m_ϕ is inconsistent. The next definition gives a canonical form for consistent DBMs.

Definition 6. *A consistent DBM $m \in \mathbb{Z}_\infty^{n \times n}$ is said to be* closed *if and only if $m_{ii} = 0$ and $m_{ij} \leq m_{ik} + m_{kj}$, for all $1 \leq i, j, k \leq n$.*

Given a consistent DBM m, we denote by m^* the (unique) closed DBM equivalent with it. It is well-known that, if m is consistent, then m^* is unique, and can be computed from m in time $\mathcal{O}(n^3)$, by the classical Floyd-Warshall algorithm. The closure of DBM provides an efficient means to compare difference bounds constraints.

Proposition 2 ([18]). *Given two consistent difference bounds constraints $\varphi(\mathbf{x})$ and $\psi(\mathbf{x})$, the following conditions are equivalent:*

- $\forall \mathbf{x} \,.\, \varphi(\mathbf{x}) \rightarrow \psi(\mathbf{x})$
- $(m^*_\varphi)_{ij} \leq (m^*_\psi)_{ij}$*, for all* $1 \leq i, j \leq n$

In the following, let R be a relation defined by a difference bounds constraint. It is easy to show that, for any $i \geq 0$, the relation R^i is a difference bounds relation as well – in other words, difference bounds relations are closed under composition. Moreover, if S is a set defined by a difference bounds constraint, then the set $pre_{R^i}(S)$ is defined by a difference bounds constraint as well. But since $wrs(R) = \bigcap_{i>0} pre_{R^i}(\mathbb{Z}^n)$, it turns out that $wrs(R)$ can be defined by a difference bounds constraint, since the class of difference bounds constraints is closed under (possibly infinite) intersections.

We are now ready to describe the procedure computing the weakest recurrent set for a difference bounds relation R. Since $wrs(R)$ is a (possibly inconsistent) difference bounds constraint, we use the template $\mu(\mathbf{x}, \mathbf{p}) \equiv \bigwedge_{1 \leq i \neq j \leq n} x_i - x_j \leq p_{ij}$, where p_{ij} are parameters ranging over $\mathbb{Z}_{\pm\infty}$ (we clearly do not need to track the constraints of the

[4] Or, for that matter, in any theory that has quantifier elimination.

form $x_i - x_i \leq p_{ii}$). Moreover, we assume that the template is closed (Definition 6), which can be encoded as a system of inequalities of the form:

$$p_{ij} \leq \min \left\{ p_{ik} + p_{kj} \mid k \neq i, k \neq j \right\} \tag{2}$$

Next, we compute the (symbolic) difference bounds constraint corresponding to the set $pre_R(\mu) \equiv \exists \mathbf{x}' \,.\, \mathcal{R}(\mathbf{x}, \mathbf{x}') \wedge \mu(\mathbf{x}', \mathbf{p})$. This step requires computing the closure of the DBM corresponding to $\mathcal{R} \wedge \mu$, and elimination of the $\mathbf{x}'$ variables. The result is a closed symbolic DBM π, whose entries are $\min$-terms consisting of sums of p_{ij} and integer constants. Further, we encode the recurrence condition $\mu \subseteq pre_R(\mu)$, again as a system of inequalities (Proposition 2) of the form:

$$p_{ij} \leq \pi_{ij}, i \neq j \tag{3}$$

By conjoining the inequalities (2) and (3), we obtain a system of inequalities with variables p_{ij}, whose right-hand sides are linear combinations of p_{ij} with addition and $\min$. We are interested in the maximal solution of this system, which can be obtained using an efficient policy iteration algorithm [14] in the complete lattice of $\mathbb{Z}_{\pm\infty}$ with addition, $\min$ and $\max$ operators. This solution defines the weakest recurrent set for R, and consequently, the weakest precondition for non-termination of the R loop. Since $wrs(R)$ is a difference bounds constraint, for any relation R definable by a difference bounds constraint, the maximal solution of the system is unique. It is to be noted that, if for some $1 \leq i \neq j \leq n$ we obtain $p_{ij} = -\infty$, then the weakest recurrent set is empty i.e., the relation R is well-founded.

Lemma 3. *Computing the weakest recurrent set of a difference bounds relation can be done in time $\mathcal{O}(n^2 \cdot 2^n)$, where n is the number of variables.*

4 Octagonal Relations

Octagonal relations are a generalization of difference bounds relations.

Definition 7. *A formula $\phi(\mathbf{x})$ is an* octagonal constraint *if it is equivalent to a finite conjunction of terms of the form $\pm x_i \pm x_j \leq a_{ij}$, where $a_{ij} \in \mathbb{Z}$ and $1 \leq i, j \leq n$.*

We represent octagons as difference bounds constraints over the *dual* set of variables $\mathbf{y} = \{y_1, y_2, \ldots, y_{2n}\}$, with the convention that y_{2i-1} stands for x_i and y_{2i} for $-x_i$, respectively. For example, the octagonal constraint $x_1 + x_2 = 3$ is represented as $y_1 - y_4 \leq 3 \wedge y_2 - y_3 \leq -3$. To handle the dual variables in the following, we define $\bar{\imath} = i - 1$, if i is even, and $\bar{\imath} = i + 1$ if i is odd. We say that a DBM $m \in \mathbb{Z}_\infty^{2n \times 2n}$ is *coherent* iff $m_{ij} = m_{\bar{\jmath}\bar{\imath}}$ for all $1 \leq i, j \leq 2n$. The coherence property is needed because any atomic proposition $x_i - x_j \leq a$, in ϕ can be represented as both $y_{2i-1} - y_{2j-1} \leq a$ and $y_{2j} - y_{2i} \leq a$, $1 \leq i, j \leq n$. We denote by $\overline{\phi}$ the difference bounds formula $\phi[y_1/x_1, y_2/-x_1, \ldots, y_{2n-1}/x_n, y_{2n}/-x_n]$ with free variables $\mathbf{y}$. The following equivalence relates ϕ and $\overline{\phi}$:

$$\phi(\mathbf{x}) \Leftrightarrow (\exists y_2, y_4, \ldots, y_{2n} \,.\, \overline{\phi} \wedge \bigwedge_{i=1}^{n} y_{2i-1} + y_{2i} = 0)[x_1/y_1, \ldots, x_n/y_{2n-1}] \tag{4}$$

Given a coherent DBM m representing $\overline{\phi}$, we say that m is *octagonal-consistent* if and only if ϕ is consistent. The following definition gives the canonical form of a DBM representing an octagonal-consistent constraint.

Definition 8. *An octagonal-consistent coherent DBM* $m \in \mathbb{Z}^{2n \times 2n}$ *is said to be* tightly closed *if and only if the following hold:*

1. $m_{ii} = 0, \forall 1 \leq i \leq 2n$
2. $m_{i\bar{i}}$ *is even*, $\forall 1 \leq i \leq 2n$
3. $m_{ij} \leq m_{ik} + m_{kj}, \forall 1 \leq i,j,k \leq 2n$
4. $m_{ij} \leq \lfloor \frac{m_{i\bar{i}}}{2} \rfloor + \lfloor \frac{m_{\bar{j}j}}{2} \rfloor, \forall 1 \leq i,j \leq 2n$

Given an octagonal-consistent DBM m, we denote by m^t the equivalent tightly closed DBM. The tight closure of an octagonal-consistent DBM m is unique and can be computed in time $\mathcal{O}(n^3)$ as $m^t_{i,j} = min\left\{m^*_{i,j}, \left\lfloor \frac{m^*_{i,\bar{i}}}{2} \right\rfloor + \left\lfloor \frac{m^*_{\bar{j},j}}{2} \right\rfloor\right\}$ [1]. This generalizes to unbounded finite compositions of octagonal relations [4]:

$$\forall k \geq 0 \,.\, (m^t_{R^k})_{i,j} = min\left\{(m^*_{\overline{\mathcal{R}}^k})_{i,j}, \left\lfloor \frac{(m^*_{\overline{\mathcal{R}}^k})_{i,\bar{i}}}{2} \right\rfloor + \left\lfloor \frac{(m^*_{\overline{\mathcal{R}}^k})_{\bar{j},j}}{2} \right\rfloor\right\} \tag{5}$$

Notice that the above relates the entries of the tightly closed DBM representation of R^k with the entries of the closed DBM representation of the relation defined by $\overline{\mathcal{R}}^k$.

We are now ready to introduce a result [5] that defines the "shape" of the closed form $\mathcal{R}^{(k)}$ for an octagonal relation R. Intuitively, for each $i \geq 0$, R^i is an octagon, whose bounds evolve in a periodic way. The following definition gives the precise meaning of periodicity for relations that have a matrix representation.

Definition 9. *An infinite sequence of matrices* $\{M_k\}_{k=1}^{\infty} \in \mathbb{Z}_{\infty}^{m \times m}$ *is said to be* ultimately periodic *if and only if:*

$$\exists b > 0 \; \exists c > 0 \; \exists \Lambda_0, \Lambda_1, \ldots, \Lambda_{c-1} \in \mathbb{Z}_{\infty}^{m \times m} \,.\, M_{b+(k+1)c+i} = \Lambda_i + M_{b+kc+i}$$

for all $k \geq 0$ *and* $i = 0, 1, \ldots, c-1$. *The smallest* b, c *for which the above holds are called* prefix *and* period *of the* $\{M_k\}_{k=1}^{\infty}$ *sequence, respectively.*

A result reported in [5] is that the sequence $\{m^t_{R^i}\}_{i \geq 0}$ (5) of tightly closed matrices representing the sequence $\{R^i\}_{i \geq 0}$ of powers of a $*$-consistent octagonal relation R is ultimately periodic, in the sense of the above definition. The constants b and c from Definition 9 will also be called the *prefix and period of the octagonal relation* R, throughout this section.

For a set $\mathbf{v}$ of variables, let $U(\mathbf{v}) = \{\pm v_1 \pm v_2 \mid v_1, v_2 \in \mathbf{v}\}$ denote the set of octagonal terms over $\mathbf{v}$. As a first remark, by the periodicity of the sequence $\{m^t_{R^i}\}_{i \geq 0}$, the closed form of the subsequence $\{R^{b+c\ell}\}_{\ell \geq 0}$ (of $\{R^i\}_{i \geq 0}$) can be defined as:

$$\mathcal{R}^{(\ell)}_{b,c} \equiv \bigwedge_{u \in U(\mathbf{x} \cup \mathbf{x}')} u \leq a_u \ell + d_u \tag{6}$$

where $a_u = (\Lambda_0)_{ij}$, $d_u = (m^t_{R^b})_{ij}$ for all octagonal terms $u = y_i - y_j$. This is indeed the case, since the matrix sequence $\{m^t_{R^{b+c\ell}}\}_{\ell \geq 0}$ is ultimately periodic i.e., $m^t_{R^{b+c\ell}} = m^t_{R^b} + \ell\Lambda_0$, for all $\ell \geq 0$.

Lemma 4. *Let R be a $*$-consistent octagonal relation with prefix b, period c and let $\mathcal{R}_{b,c}^{(\ell)}$ be the closed form of $\{R^{b+c\ell}\}_{\ell\geq 0}$ as defined in* (6)*. Then, R is well-founded iff there exists $u \in U(\mathbf{x})$ s.t. $a_u < 0$. Moreover, if R is not well-founded, $wrs(R) \equiv \mathcal{R}^{-b}(\top)$.*

The above lemma can be used to compute $wrs(R)$ for an octagonal relation R. First we need to check the $*$-consistency of R, using the method reported in [6]. Second, we compute the closed form (6) and check for the existence of a term $u \in U(\mathbf{x})$ such that $a_u < 0$, in which case R is well-founded. Finally, if this is not the case, then we compute $wrs(R) = \mathcal{R}^{-b}(\top)$.

4.1 On the Existence of Linear Ranking Functions

A ranking function for a given relation R constitutes a proof of the fact that R is well-founded. We distinguish here two cases. If R is not $*$-consistent, then the well-foundedness of R is witnessed simply by an integer constant $i > 0$ such that $R^i = \emptyset$. Otherwise, if R is $*$-consistent, we need a better argument for well-foundedness. In this section we show that, for any $*$-consistent well-founded octagonal relation R with prefix b, the (strenghtened) relation defined by $\mathcal{R}^{-b}(\top) \wedge \mathcal{R}$ is well-founded and has a linear ranking function, even when R alone does not have one. For space reasons, we do not give here all the details of the construction of such a function. However, the existence proof suffices, as one can use *complete* ranking function extraction tools (such as e.g. RankFinder [19]) in order to find them.

Definition 10. *Given a relation $R \subseteq \mathbb{Z}^n \times \mathbb{Z}^n$, a* linear ranking function *for R is a term $f(\mathbf{x}) = \sum_{i=1}^{n} a_i x_i$ such that, for all states $s, s' : \mathbf{x} \to \mathbb{Z}$:*

1. *f is* decreasing*: $\mathcal{R}(s, s') \to f(s) > f(s')$*
2. *f is* bounded*: $\mathcal{R}(s, s') \to (f(s) > h \ \wedge \ f(s') > h)$, for some $h \in \mathbb{Z}$.*

The main result of this section is the following:

Theorem 1. *Let $R \subseteq \mathbb{Z}^n \times \mathbb{Z}^n$ be a $*$-consistent and well-founded octagonal relation, with prefix $b \geq 0$. Then, the relation defined by $\mathcal{R}^{-b}(\top) \wedge \mathcal{R}$ is well founded and has a linear ranking function.*

The first part of the theorem is proved by the following lemma:

Lemma 5. *Let $R \subseteq \mathbb{Z}^n \times \mathbb{Z}^n$ be a relation, and $m > 0$ be an integer. Then $wrs(R) = \emptyset$ if and only if $wrs(R_m) = \emptyset$, where R_m is the relation defined by $\mathcal{R}^{-m}(\top) \wedge \mathcal{R}$.*

It remains to prove that the witness relation defined by $\mathcal{R}^{-b}(\top) \wedge \mathcal{R}$ has a linear ranking function, provided that it is well-founded. The proof is organized as follows. First we show that well-foundedness of an octagonal relation R is equivalent to the well-foundedness of its difference bounds representation $\overline{\mathcal{R}}$ (Lemma 6). Second, we use a result from [8], that the constraints in the sequence of iterated difference bounds relations $\{\overline{\mathcal{R}}^i\}_{i\geq 0}$ can be represented by a finite-state weighted automaton, called the *zigzag automaton* in the sequel. If the relation defined by $\overline{\mathcal{R}}$ is well-founded, then this weighted automaton must have a cycle of negative weight. The structure of this cycle, representing several of the constraints in $\overline{\mathcal{R}}$, is used to show the existence of the linear ranking function for the witness relation $\mathcal{R}^{-b}(\top) \wedge \mathcal{R}$.

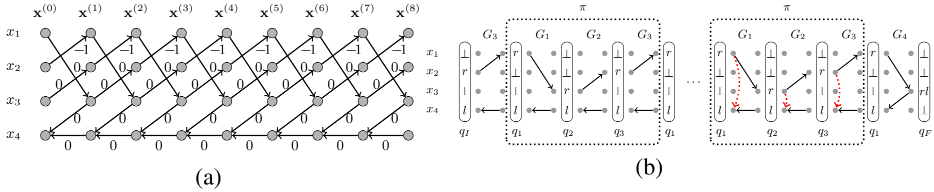
$\mathbf{x}^{(0)}$ $\mathbf{x}^{(1)}$ $\mathbf{x}^{(2)}$ $\mathbf{x}^{(3)}$ $\mathbf{x}^{(4)}$ $\mathbf{x}^{(5)}$ $\mathbf{x}^{(6)}$ $\mathbf{x}^{(7)}$ $\mathbf{x}^{(8)}$
x_1
x_2
x_3
x_4
−1
0
(a)
π
G_1 G_2 G_3 G_4
$\bot$ r l rl
q_I q_1 q_2 q_3 q_F
(b)

Fig. 1. (a) Unfolding of the constraint graph of the difference bounds relation $\mathcal{R} \equiv x_2 - x_1' \leq -1 \wedge x_3 - x_2' \leq 0 \wedge x_1 - x_3' \leq 0 \wedge x_4' - x_4 \leq 0 \wedge x_3' - x_4 \leq 0$. (b) A run of the zigzag automaton $\mathcal{A}_R$ over a path in the unfolded constraint graph of R.

Lemma 6. *Let $R \subseteq \mathbb{Z}^n \times \mathbb{Z}^n$ be an octagonal relation and R_{db} be the difference bounds relation defined by $\overline{\mathcal{R}}$. Then R is well-founded if and only if R_{db} is well-founded.*

The above lemma reduces the problem of showing existence of a ranking function for an octagonal relation $\mathcal{R}(\mathbf{x}, \mathbf{x}')$ to showing existence of a ranking function for its difference bounds encoding $\overline{\mathcal{R}}(\mathbf{y}, \mathbf{y}')$. Assume that $f(\mathbf{y})$ is a ranking function for $\overline{\mathcal{R}}$. Then $f[x_i/y_{2i-1}, -x_i/y_{2i}]_{i=1}^{n}$ is a linear ranking function for R. Hence, in the rest of this section, we consider without loss of generality that R is a difference bounds relation.

Zigzag Automata. For the later developments, we need to introduce the *zigzag automaton* corresponding to a difference bounds relation R. Intuitivelly, for any $i > 0$, the relation R^i can be represented by a constraint graph which is the i-times repetition of the constraint graph of R. The constraints induced by R^i can be represented as shortest paths in this graph, and can be recognized (in the classical automata-theoretic sense) by a weighted automaton $\mathcal{A}_R$ (see Fig. 1 for an example). The structure of this automaton is needed to show the existence of a linear ranking function.

The following lemma proves the existence of a negative weight cycle in the zigzag automata corresponding to well-founded difference bounds relation. The intuition behind this fact is that the rates of the DBM sequence $\{m_{R^i}\}_{i>0}$ are weights of optimal ratio (weight per length) cycles in the zigzag automaton. According to the previous section, if R is well-founded, there exists a negative rate for $\{m_{R^i}\}_{i>0}$, which implies the existence of a negative cycle in the zigzag automaton.

Lemma 7. *If R is a $*$-consistent well-founded difference bounds relation of prefix $b \geq 0$, and $\mathcal{A}_R$ is its corresponding zigzag automaton, then there exists a cycle π from a state q to itself, such that $w(\pi) < 0$ and there exists paths π_i from an initial state to q, and π_f from q to a final state, such that $|\pi_i| + |\pi_f| = b$.*

Next we prove the existence of a linear decreasing function, based on the existence of a negative weight cycle in the zigzag automaton.

Lemma 8. *If R is a $*$-consistent well-founded difference bounds relation of prefix $b \geq 0$, then there exists a linear function $f(\mathbf{x})$ such that, for all states $s, s' : \mathbf{x} \to \mathbb{Z}$ we have $\mathcal{R}^{-b}(\top)(s) \wedge \mathcal{R}(s, s') \to f(s) > f(s')$.*

Last, we prove that the function f of Lemma 8 is bounded, concluding that it is indeed a ranking function. Since each run in the zigzag automaton recognizes a path from some x_i to some x_j, a run that repeats a cycle can be decomposed into a prefix, the cycle itself and a suffix. The recognized path may traverse the cycle several times, however each exit point from the cycle must match a subsequent entry point. These paths from the exit to the corresponding entries give us the necessary lower bound. In fact, these paths appear already on constraint graphs that represent unfoldings of R^i, for any $i \geq b$. Hence the need for a strenghtened witness $\mathcal{R}^{-b}(\top) \wedge \mathcal{R}$, as $\mathcal{R}$ alone is not enough for proving boundedness of f.

Lemma 9. *If R is a $*$-consistent well-founded difference bounds relation of prefix b, and $f(\mathbf{x})$ is the linear decreasing function from Lemma 8, there exists an integer h such that, for all states $s, s' : \mathbf{x} \to \mathbb{Z}$, $(\mathcal{R}^b(\top)(s) \ \wedge \ \mathcal{R}(s,s')) \to (f(s) \geq h \ \wedge \ f(s') \geq h)$.*

As an experiment, we have tried the RANKFINDER [19] tool (complete for linear ranking functions), which failed to discover a ranking function on the relation $\mathcal{R}$ from Fig. 1. This comes with no surprise, since no linear decreasing function that is bounded after the first iteration exists. However, RANKFINDER finds a ranking function for the witness relation $\mathcal{R}^{-b}(\top) \wedge \mathcal{R}$ instead.

5 Linear Affine Relations

Let $\mathbf{x} = \langle x_1, \ldots, x_n \rangle^\top$ be a column vector of variables ranging over integers. A linear affine relation is a relation of the form $\mathcal{R}(\mathbf{x}, \mathbf{x}') \equiv C\mathbf{x} \geq \mathbf{d} \ \wedge \ \mathbf{x}' = A\mathbf{x} + \mathbf{b}$, where $A \in \mathbb{Z}^{n \times n}$, $C \in \mathbb{Z}^{p \times n}$ are matrices, and $\mathbf{b} \in \mathbb{Z}^n$, $\mathbf{d} \in \mathbb{Z}^p$ are column vectors of integer constants. Notice that we consider linear affine relations to be deterministic, unlike the octagonal relations considered in the previous. In the following, it is convenient to work with the equivalent homogeneous form:

$$\begin{gathered} \mathcal{R}(\mathbf{x}, \mathbf{x}') \equiv C_h \mathbf{x}_h \geq \mathbf{0} \ \wedge \ \mathbf{x}'_h = A_h \mathbf{x}_h \\ A_h = \begin{pmatrix} A & \mathbf{b} \\ 0 & 1 \end{pmatrix} \ C_h = \begin{pmatrix} C & -\mathbf{d} \end{pmatrix} \ \mathbf{x}_h = \begin{pmatrix} \mathbf{x} \\ x_{n+1} \end{pmatrix} \end{gathered} \tag{7}$$

The closed form of a linear affine relation is defined by the following formula:

$$\mathcal{R}^{(k)}(\mathbf{x}, \mathbf{x}') \ \equiv \ \exists x_{n+1}, x'_{n+1} . \mathbf{x}'_h = A_h^k \mathbf{x}_h \wedge \forall 0 \leq \ell < k . C A_h^\ell \mathbf{x} \geq \mathbf{0} \wedge x_{n+1} = 1 \tag{8}$$

Intuitively, the first conjunct defines the (unique) outcome of iterating the relation $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$ for k steps, while the second (universally quantified) conjunct ensures that the condition ($C\mathbf{x} \geq \mathbf{d}$) has been always satisfied all along the way. The definition of the weakest recursive set of a linear affine relation is (after the elimination of the trailing existential quantifier):

$$wrs(R)(\mathbf{x}) \ \equiv \ \exists x_{n+1} \forall k > 0 \ . \ C_h A_h^k \mathbf{x} \geq \mathbf{0} \ \wedge \ x_{n+1} = 1 \tag{9}$$

The main difficulty with the form (9) comes from the fact that the powers of a matrix A cannot usually be defined in a known decidable theory of arithmetic. In the following, we discuss the case of A having the finite monoid property [2,25], which leads to

$wrs(R)$ being Presburger definable. Further, we relax the finite monoid condition and describe a method for generating sufficient termination conditions, i.e. sets $S \in \mathbb{Z}^n$ such that $S \cap wrs(R) = \emptyset$.

Some basic notions of linear algebra are needed in the following. If $A \in \mathbb{Z}^{n \times n}$ is a square matrix, and $\mathbf{v} \in \mathbb{Z}^n$ is a column vector of integer constants, then any complex number $\lambda \in \mathbb{C}$ such that $A\mathbf{v} = \lambda\mathbf{v}$, for some complex vector $\mathbf{v} \in \mathbb{C}^n$, is called an *eigenvalue* of A. The vector $\mathbf{v}$ in this case is called an *eigenvector* of A. It is known that the eigenvalues of A are the roots of the *characteristic polynomial* $\det(A - \lambda I_n) = 0$, which is an effectively computable univariate polynomial in λ. A complex number r is said to be a *root of the unity* if $r^d = 1$ for some integer $d > 0$.

In the previous work of Weber and Seidl [25], Boigelot [2], and Finkel and Leroux [12], a restriction of linear affine relations has been introduced, with the goal of defining the closed form of relations in Presburger arithmetic. A matrix $A \in \mathbb{Z}^{n \times n}$ is said to have the *finite monoid property* if and only if its set of powers $\{A^i \mid i \geq 0\}$ is finite. A linear affine relation has the finite monoid property if and only if the matrix A defining the update has the finite monoid property.

Lemma 10 ([12,2]). *A matrix $A \in \mathbb{Z}^{n \times n}$ has the finite monoid property iff:*

1. *all eigenvalues of A are either zero or roots of the unity, and*
2. *all non-zero eigenvalues are of multiplicity one.*

Both conditions are decidable.

In the following, we drop the second requirement of Lemma 10, and consider only linear relations, such that all non-zero eigenvalues of A are roots of the unity. In this case, $\mathcal{R}^{(k)}$ cannot be defined in Presburger arithmetic any longer, thus we renounce defining $wrs(R)$ precisely, and content ourselves with the discovery of *sufficient conditions for termination*. Basically given a linear affine relation R, we aim at finding a disjunction $\phi(\mathbf{x})$ of linear constraints on $\mathbf{x}$, such that $\phi \wedge wrs(R)$ is inconsistent, without explicitly computing $wrs(R)$.

Lemma 11. *Given a square matrix $A \in \mathbb{Z}^{n \times n}$, whose non-zero eigenvalues are all roots of the unity. Then $(A^m)_{i,j} \in \mathbb{Q}[m]$, for all $1 \leq i, j \leq n$, are effectively computable polynomials with rational coefficients.*

We turn now back to the problem of defining $wrs(R)$ for linear affine relations R of the form (9). First notice that, if all non-zero eigenvalues of A are roots of the unity, then the same holds for A_h (7). By Lemma 11, one can find rational polynomials $p_{i,j}(k)$ defining $(A_h^k)_{i,j}$, for all $1 \leq i, j \leq n$. The condition (9) resumes to a conjunction of the form:

$$wrs(R)(\mathbf{x}) \equiv \bigwedge_{i=1}^{n} \forall k > 0 \;.\; P_i(k, \mathbf{x}) \geq 0 \tag{10}$$

where each $P_i = a_{i,d}(\mathbf{x}) \cdot k^d + \ldots + a_{i,1}(\mathbf{x}) \cdot k + a_{i,0}(\mathbf{x})$ is a polynomial in k whose coefficients are the linear combinations $a_{i,d} \in \mathbb{Q}[\mathbf{x}]$. We are looking after a sufficient condition for termination, which is, in this case, any set of valuations of $\mathbf{x}$ that would invalidate (10). The following proposition gives sufficient invalidating clauses for each conjunct above. By taking the disjunction of all these clauses we obtain a sufficient termination condition for R.

Lemma 12. *Given a polynomial* $P(k,\mathbf{x}) = a_d(\mathbf{x}) \cdot k^d + \ldots + a_1(\mathbf{x}) \cdot k + a_0(\mathbf{x})$*, there exists* $n > 0$ *such that* $P(n,\mathbf{x}) < 0$ *if, for some* $i = 0, 1, \ldots, d$*, we have* $a_{d-i}(\mathbf{x}) < 0$ *and* $a_d(\mathbf{x}) = a_{d-1}(\mathbf{x}) = \ldots = a_{d-i+1}(\mathbf{x}) = 0$.

Example Consider the following program [11], and its linear transformation matrix A.

```
while (x ≥ 0)
    x' = x + y
    y' = y + z
```

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad A^k = \begin{pmatrix} 1 & k & \frac{k(k-1)}{2} \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$$

The characteristic polynomial of A is $\det(A - \lambda I_3) = (1 - \lambda)^3$, hence the only eigenvalue is 1, with multiplicity 3. Then we compute A^k (see above), and $x' = x + k \cdot y + \frac{k(k-1)}{2} z$ gives the value of x after k iterations of the loop. Hence the (precise) non-termination condition is: $\forall k > 0 \;.\; \frac{z}{2} \cdot k^2 + (y - \frac{z}{2}) \cdot k + x \geq 0$. A sufficient condition for termination is: $(z < 0) \vee (z = 0 \wedge y < 0) \vee (z = 0 \wedge y = 0 \wedge x < 0)$ □

We can generalize this method further to the case where all eigenvalues of A are of the form $q \cdot r$, with $q \in \mathbb{R}$ and $r \in \mathbb{C}$ being a root of the unity. The main reason for not using this condition from the beginning is that we are, to this point, unaware of its decidability status. With this condition instead, it is sufficient to consider only the eigenvalues with the maximal absolute value, and the polynomials obtained as sums of the polynomial coefficients of these eigenvalues. The result of Lemma 11 and the sufficient condition of Lemma 12 carry over when using these polynomials instead.

6 Experimental Evaluation

We have validated the methods described in this paper by automatically verifying termination of all the octagonal running examples, and of several integer programs synthesized from (i) programs with lists [3] and (ii) VHDL models [21]. We have first computed automatically their strongest summary relation $\mathcal{T}$, by adapting the method for reachability analysis for integer programs, described in [6], and implemented in the FLATA tool [13]. Then we automatically proved that $\mathcal{T}$ is contained in a disjunction of octagonal relations, which are found to be well-founded by the procedure described in Section 4.

We have first verified the termination of the LISTCOUNTER and LISTREVERSAL programs, which were obtained using the translation scheme from [3], which generates an integer program from a program manipulating dynamically allocated single-selector linked lists. Using the same technique, we also verified the COUNTER and SYNLIFO programs, obtained by translating VHDL designs of hardware counter and synchronous LIFO [21]. These models have infinite runs for any input values, which is to be expected, as they encode the behavior of synchronous reactive circuits.

Second, we have compared (Table 1) our method for termination of linear affine loops with the examples given in [11], and found the same termination preconditions

Table 1. Termination preconditions for several program fragments from [11]

Program	Cook et. al [11]	Linear Affine Loops

if (lvar $\geq$ 0)
while (lvar $< 2^{30}$)
lvar = lvar $\ll$ 1;
	$lvar > 0 \vee lvar < 0 \vee lvar \geq 2^{30}$	$\neg(lvar{=}0) \vee lvar \geq 2^{30}$

while (x $\leq$ N)
if (*) { x=2*x+y;
y=y+1; }
else x ++;
	$x > N \vee x + y \geq 0$	$x > N \vee x + y \geq 0$

while (x $\geq$ N)
x = -2*x + 10;
	$x > 5 \vee x + y \geq 0$	$x \neq \frac{10}{3} \iff$ true

//@ requires $n > 200$
x = 0;
while (1)
if (x < n) { x=x+y;
if (x $\geq$ 200) break; }
	$y > 0$	$y > 0$

as they do, with one exception, in which we can prove universal termination in integer input values (row 3 of Table 1). The last example from [11] is the Euclidean Greatest Common Divisor algorithm, for which we infer automatically the correct termination preconditions using a disjunctively well-founded octagonal abstraction of the transition invariant.

7 Conclusions

We have presented several methods for deciding conditional termination of several classes of program loops manipulating integer variables. The universal termination problem has been found to be decidable for octagonal relations and linear affine loops with the finite monoid property. In other cases of linear affine loops, we give sufficient termination conditions. We have implemented our method in the FLATA tool [13] and performed a number of preliminary experiments.

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References

1. Bagnara, R., Hill, P.M., Zaffanella, E.: An Improved Tight Closure Algorithm for Integer Octagonal Constraints. In: Logozzo, F., Peled, D.A., Zuck, L.D. (eds.) VMCAI 2008. LNCS, vol. 4905, pp. 8–21. Springer, Heidelberg (2008)
2. Boigelot, B.: Symbolic Methods for Exploring Infinite State Spaces, PhD Thesis, vol. 189. Collection des Publications de l'Université de Liège (1999)
3. Bouajjani, A., Bozga, M., Habermehl, P., Iosif, R., Moro, P., Vojnar, T.: Programs with Lists Are Counter Automata. In: Ball, T., Jones, R.B. (eds.) CAV 2006. LNCS, vol. 4144, pp. 517–531. Springer, Heidelberg (2006)
4. Bozga, M., Gîrlea, C., Iosif, R.: Iterating Octagons. In: Kowalewski, S., Philippou, A. (eds.) TACAS 2009. LNCS, vol. 5505, pp. 337–351. Springer, Heidelberg (2009)

5. Bozga, M., Iosif, R., Konečný, F.: Fast Acceleration of Ultimately Periodic Relations. In: Touili, T., Cook, B., Jackson, P. (eds.) CAV 2010. LNCS, vol. 6174, pp. 227–242. Springer, Heidelberg (2010)
6. Bozga, M., Iosif, R., Konečný, F.: Relational Analysis of Integer Programs. Technical Report TR-2011-14, Verimag, Grenoble, France (2011)
7. Bozga, M., Iosif, R., Konečný, F.: Deciding Conditional Termination. Technical Report TR-2012-1, Verimag, Grenoble, France (2012)
8. Bozga, M., Iosif, R., Lakhnech, Y.: Flat parametric counter automata. Fundamenta Informaticae 91, 275–303 (2009)
9. Bradley, A.R., Manna, Z., Sipma, H.B.: Linear Ranking with Reachability. In: Etessami, K., Rajamani, S.K. (eds.) CAV 2005. LNCS, vol. 3576, pp. 491–504. Springer, Heidelberg (2005)
10. Braverman, M.: Termination of Integer Linear Programs. In: Ball, T., Jones, R.B. (eds.) CAV 2006. LNCS, vol. 4144, pp. 372–385. Springer, Heidelberg (2006)
11. Cook, B., Gulwani, S., Lev-Ami, T., Rybalchenko, A., Sagiv, M.: Proving Conditional Termination. In: Gupta, A., Malik, S. (eds.) CAV 2008. LNCS, vol. 5123, pp. 328–340. Springer, Heidelberg (2008)
12. Finkel, A., Leroux, J.: How to Compose Presburger-Accelerations: Applications to Broadcast Protocols. In: Agrawal, M., Seth, A.K. (eds.) FSTTCS 2002. LNCS, vol. 2556, pp. 145–156. Springer, Heidelberg (2002)
13. http://www-verimag.imag.fr/FLATA.html
14. Gawlitza, T., Seidl, H.: Precise Fixpoint Computation Through Strategy Iteration. In: De Nicola, R. (ed.) ESOP 2007. LNCS, vol. 4421, pp. 300–315. Springer, Heidelberg (2007)
15. Gupta, A., Henzinger, T.A., Majumdar, R., Rybalchenko, A., Xu, R.-G.: Proving non-termination. SIGPLAN Not. 43, 147–158 (2008)
16. Halava, V., Harju, T., Hirvensalo, M., Karhumaki, J.: Skolem's problem – on the border between decidability and undecidability (2005)
17. Iosif, R., Rogalewicz, A.: Automata-Based Termination Proofs. In: Maneth, S. (ed.) CIAA 2009. LNCS, vol. 5642, pp. 165–177. Springer, Heidelberg (2009)
18. Miné, A.: The octagon abstract domain. Higher-Order and Symbolic Computation 19(1), 31–100 (2006)
19. Podelski, A., Rybalchenko, A.: A Complete Method for the Synthesis of Linear Ranking Functions. In: Steffen, B., Levi, G. (eds.) VMCAI 2004. LNCS, vol. 2937, pp. 239–251. Springer, Heidelberg (2004)
20. Podelski, A., Rybalchenko, A.: Transition invariants. In: LICS 2004, pp. 32–41 (2004)
21. Smrcka, A., Vojnar, T.: Verifying Parametrised Hardware Designs Via Counter Automata. In: Yorav, K. (ed.) HVC 2007. LNCS, vol. 4899, pp. 51–68. Springer, Heidelberg (2008)
22. Sohn, K., van Gelder, A.: Termination detection in logic programs using argument sizes. In: PODS 1991 (1991)
23. Tiwari, A.: Termination of Linear Programs. In: Alur, R., Peled, D.A. (eds.) CAV 2004. LNCS, vol. 3114, pp. 70–82. Springer, Heidelberg (2004)
24. Turing, A.M.: On computable numbers, with an application to the entscheidungsproblem. Proceedings of the London Mathematical Society 42, 230–265 (1936)
25. Weber, A., Seidl, H.: On finitely generated monoids of matrices with entries in n. In: ITA 1991, 19–38 (1991)