Chapter 3
Continuous Approach to Concrete

Abstract. This Chapter presents continuous models to describe concrete behaviour in a quasi-static regime during monotonic and cyclic loading. In the case of monotonic loading, isotropic elasto-plastic, isotropic damage and smeared crack model, and in the case of cyclic loading elasto-plastic-damage models are described. An integral-type non-local and a second gradient approaches to model strain localization are introduced. In addition, bond-slip laws are presented.

The concrete behaviour can be modelled with different continuum models, e.g.: within non-linear elasticity (Palaniswamy and Shah 1974), linear fracture mechanics (Bažant and Cedolin 1979, Hilleborg 1985), endochronic theory (Bažant and Bhat 1976, Bažant and Shieh 1978), micro-plane theory (Bažant and Ožbolt 1990, Jirásek 1999), plasticity (Willam and Warnke 1975, Ottosen 1977, Hsieh et al. 1982, Pietruszczak et al. 1988, Pramono and Willam 1989, Etse and Willam 1994, Menétrey and Willam 1995, Winnicki et al. 2001, Lade and Jakobsen 2002, Majewski et al. 2008), damage (Dragon and Mróz 1979, Peerlings et al. 1998, Chen 1999, Ragueneau et al. 2000, Marzec et al. 2007) and discrete ones using e.g.: interface elements with cohesive fracture constitutive laws (Carol et al. 2001, Caballero et al. 2006, 2007), a lattice approach (Herrmann et al. 1989, Vervuurt et al. 1994, Schlangen and Garboczi 1997, Cusatis et al. 2003, Bolander and Sukumar 2005, Kozicki and Tejchman 2007) and a discrete element method (DEM) (Sakaguchi and Mühlhaus 1997, Donze at al. 1999, D'Addetta et al. 2002, Hentz et al. 2004).

We used different popular non-linear continuous constitutive models to simulate the concrete behaviour under monotonic loading.

3.1 Local Models for Monotonic Loading

3.1.1 Isotropic Elasto-Plastic Model

Failure for elasto-plastic materials with isotropic hardening/softening is described by a condition

$$f(\sigma_{ij}, \kappa) = 0 \tag{3.1}$$

J. Tejchman, J. Bobiński: Continuous & Discontinuous Modelling of Fracture, SSGG, pp. 49–93.
springerlink.com

with σ_{ij} - stress tensor and κ - hardening/softening parameter (in general there may be several hardening/softening parameters). If $f<0$, the material behaves elastically. If $f\geq 0$, the material behaves plastically. The stresses have to remain on the failure surface (consistency condition)

$$\dot{f}(\sigma_{ij},\kappa)=0=\frac{\partial f}{\partial \sigma_{ij}}:\dot{\sigma}_{ij}+\frac{\partial f}{\partial \kappa}:\dot{\kappa}\,. \tag{3.2}$$

Very often Equation 3.1 can be simplified by

$$f(\sigma_{ij},\kappa)=F(\sigma_{ij})-\sigma_y(\kappa)=0\,, \tag{3.3}$$

where F is the function of stress tensor invariants and σ_y is the yield stress. The strain increment is equal to the sum of elastic and plastic strain increments

$$d\varepsilon_{ij}=d\varepsilon_{ij}^{e}+d\varepsilon_{ij}^{p}\,. \tag{3.4}$$

The stress increment is related to the increment of elastic strain

$$d\sigma_{ij}=C_{ijkl}^{e}d\varepsilon_{kl}^{e}\,, \tag{3.5}$$

where C^{e}_{ijkl} is the elastic stiffness tensor

$$C_{ijkl}^{e}=\lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\,. \tag{3.6}$$

λ and μ are the Lame'a constants that are connected to the modulus of elasticity E and Poisson's ratio ν

$$\lambda=\frac{\nu E}{(1+\nu)(1-2\nu)}\,,\qquad \mu=\frac{E}{2(1+\nu)} \tag{3.7}$$

and δ_{ij} is a Kronecker delta. The increment of plastic strain is determined with the flow rule

$$d\varepsilon_{ij}^{p}=d\lambda\frac{\partial g(\sigma_{ij})}{\partial \sigma_{ij}} \tag{3.8}$$

with g as the potential function and $d\lambda$ as the positive factor of proportionality. If $f=g$, the flow rule is associated. The condition of loading and unloading is equal to

$$d\lambda\geq 0\,,\qquad f(\sigma_{ij},\kappa)\leq 0\,,\qquad d\lambda f(\sigma_{ij},\kappa)=0 \tag{3.9}$$

During plastic deformation, a stress state remains on the boundary of the elastic/plastic region

$$f(\sigma_{ij}+d\sigma_{ij},\kappa+d\kappa)=0\,. \tag{3.10}$$

Equation 3.10 may be rewritten in a rate form as (similarly to Eq. 3.2)

$$\frac{df}{d\sigma_{ij}}d\sigma_{ij}+\frac{df}{d\kappa}d\kappa=0\,. \qquad (3.11)$$

Equations 3.10 and 3.11 are known as consistency conditions and allow to determine the magnitude of the plastic strain increment.

The elasto-plastic stiffness matrix $C^{ep}{}_{ijkl}$ is calculated as

$$C^{ep}_{ijkl}=C^{e}_{ijkl}-\xi\frac{C^{e}_{ijmn}(\frac{\partial f}{\partial\sigma_{mn}})(\frac{\partial g}{\partial\sigma_{pq}})^{T}C^{e}_{pqkl}}{(\frac{\partial f}{\partial\sigma_{ij}})^{T}C^{e}_{ijkl}(\frac{\partial g}{\partial\sigma_{kl}})+H}\,, \qquad (3.12)$$

where

$$H=-(\frac{\partial f}{\partial\kappa})\,. \qquad (3.13)$$

The proportionality factor $d\lambda$ is equal to

$$d\lambda=\frac{(\frac{\partial f}{\partial\sigma_{ij}})^{T}C^{e}_{ijkl}d\varepsilon_{kl}}{(\frac{\partial f}{\partial\sigma_{ij}})^{T}C^{e}_{ijkl}(\frac{\partial g}{\partial\sigma_{kl}})\;+H}\,. \qquad (3.14)$$

The parameter ξ=1, if f=0 and κ>0, otherwise ξ=0. The stiffness matrix $C^{ep}{}_{ijkl}$ may be non-symmetric due to $f\neq g$. The stress increment can be calculated from

$$d\sigma_{ij}=C^{e}_{ijkl}(d\varepsilon_{kl}-d\lambda\frac{\partial f}{\partial\sigma_{ij}})\,. \qquad (3.15)$$

Usually

$$d\lambda=\eta d\kappa\,. \qquad (3.16)$$

when η is a constant dependent upon the model.

The constitutive models use the different stress and stress tensor invariants

$$I_1=\sigma_{11}+\sigma_{22}+\sigma_{33}\,, \qquad (3.17)$$

$$J_2=\frac{1}{2}s_{ij}s_{ij}=\frac{1}{6}[(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2]+\sigma_{12}{}^2+\sigma_{23}{}^2+\sigma_{31}{}^2\,, \qquad (3.18)$$

$$J_3=\frac{1}{3}s_{ij}s_{jk}s_{ki}\,, \qquad (3.19)$$

$$I_1^{\varepsilon} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\,, \tag{3.20}$$

$$J_2^{\varepsilon} = \frac{1}{2} e_{ij} e_{ij} = \frac{1}{6}[(\varepsilon_{11} - \varepsilon_{22})^2 + (\varepsilon_{22} - \varepsilon_{33})^2 + (\varepsilon_{33} - \varepsilon_{11})^2] + \varepsilon_{12}{}^2 + \varepsilon_{23}{}^2 + \varepsilon_{31}{}^2\,, \tag{3.21}$$

$$J_3^{\varepsilon} = \frac{1}{3} e_{ij} e_{jk} e_{ki}\,, \tag{3.22}$$

where I_1 - first stress tensor invariant, J_2 – second deviatoric stress tensor invariant, J_3 – third deviatoric stress tensor invariant, I_1^{ε} – the first strain tensor invariant, J_2^{ε} – second deviatoric strain tensor invariant and J_3^{ε} – third deviatoric strain tensor invariant. In turn, J_1 (first deviatoric stress tensor invariant) and J_1^{ε} (first deviatoric strain tensor invariant) are always

$$J_1 = s_{11} + s_{22} + s_{33} = 0\,, \tag{3.23}$$

$$J_1^{\varepsilon} = e_{11} + e_{22} + e_{33} = 0\,, \tag{3.24}$$

The stress deviator s_{ij} and strain deviator e_{ij} are calculated as

$$s_{ij} = \sigma_{ij} - \frac{I_1}{3}\delta_{ij}\,, \tag{3.25}$$

$$e_{ij} = \varepsilon_{ij} - \frac{I_1^{\varepsilon}}{3}\delta_{ij}\,. \tag{3.26}$$

To describe the behaviour of concrete, a simplified elasto-plastic model was assumed. In the compression regime, a shear yield surface based on a linear Drucker-Prager criterion and isotropic hardening and softening was used (Bobiński 2006, Marzec et al. 2007, Majewski et al. 2008) (Fig. 3.1)

$$f_1 = q + p\tan\varphi - (1 - \frac{1}{3}\tan\varphi)\sigma_c(\kappa_1)\,. \tag{3.27}$$

where q - Mises equivalent deviatioric stress, p – mean stress and φ – internal friction angle. The material hardening/softening was defined by the uniaxial compression stress $\sigma_c(\kappa_1)$, wherein κ_1 is the hardening/softening parameter corresponding to the plastic vertical normal strain during uniaxial compression. The friction angle φ was assumed as (ABAQUS 1998)

$$\tan\varphi = \frac{3(1 - r_{bc}^{\sigma})}{1 - 2r_{bc}^{\sigma}}\,, \tag{3.28}$$

wherein r_{bc}^{σ} denotes the ratio between uniaxial compression strength and biaxial compression strength (r_{bc}^{σ}=1.2). The invariants q and p were defined as

$$q = \sqrt{\frac{3}{2} s_{ij} s_{ij}}\,, \qquad p = \frac{1}{3}\sigma_{kk}\,, \tag{3.29}$$

The flow potential was assumed as

$$g_1 = q + p\tan\psi\,, \tag{3.30}$$

where ψ is the dilatancy angle ($\psi \neq \varphi$). The increments of plastic strains $d\varepsilon_{ij}^p$ were calculated as

$$d\varepsilon_{ij}^p = \frac{d\kappa_1}{1-\frac{1}{3}\tan\psi}\frac{\partial g_1}{\partial\sigma_{ij}} = \frac{d\kappa_1}{1-\frac{1}{3}\tan\psi}\left(\frac{\partial q}{\partial\sigma_{ij}} + \tan\psi\frac{\partial p}{\partial\sigma_{ij}}\right). \tag{3.31}$$

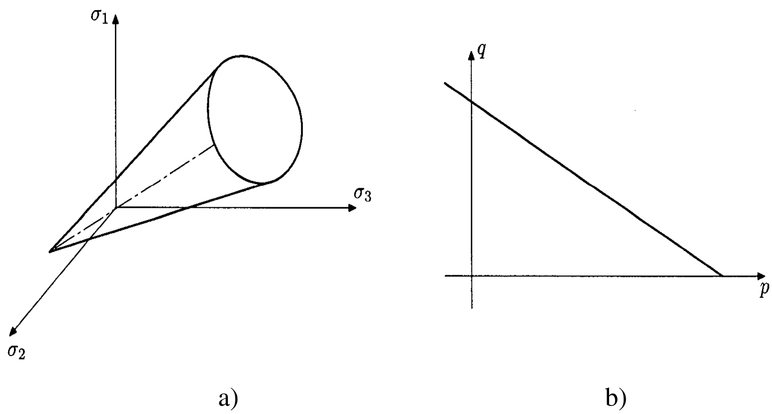

Fig. 3.1 Drucker-Prager criterion in the space of principal stresses (a) and on the plane *q-p* (b)

In turn, in the tensile regime, a Rankine criterion was used with the yield function f_2 using isotropic hardening and softening defined as (Bobiński 2006, Marzec et al. 2007, Majewski et al. 2008) (Fig. 3.2)

$$f_2 = \max\{\sigma_1, \sigma_2, \sigma_3\} - \sigma_t(\kappa_2)\,, \tag{3.32}$$

where σ_i – principal stresses, σ_t – tensile yield stress and κ_2 – softening parameter (equal to the maximum principal plastic strain ε_1^p). The associated flow rule was assumed.

The edge and vertex in the Rankine yield function were taken into account by the interpolation of 2-3 plastic multipliers according to the Koiter's rule (Pramono 1988). The same procedure was adopted in the case of combined tension (Rankine criterion) and compression (Drucker-Prager criterion).

This simple isotropic elasto-plastic model for concrete (Eqs. 3.27-3.32) requires two elastic constants: modulus of elasticity E and Poisson's ratio υ, two plastic constants: internal friction angle φ and dilatancy angle ψ, one compressive yield stress function $\sigma_c = f(\kappa_1)$ with softening and one tensile yield stress function

$\sigma_t = f(\kappa_2)$ with softening. The disadvantages of the model are the following: the shape of the failure surface in a principal stress space is linear (not paraboloidal as in reality). Thus, it is certainly not suitable in a compression regime if a large range of stress is concerned. In deviatoric planes, the shape is circular (during compression) and triangular (during tension); thus it does not gradually change from a curvilinear triangle with smoothly rounded corners to nearly circular with increasing pressure. The strength is similar for triaxial compression and extension, and the stiffness degradation due to strain localization and non-linear volume changes during loading are not taken into account.

3.1.2 Isotropic Damage Model

Continuum damage models initiated by the pioneering work of Katchanov (1986) describe a progressive loss of the material integrity due to the propagation and coalescence of micro-cracks and micro-voids. Continuous damage models (Simo and Ju 1987, Lemaitre and Chaboche 1990) are constitutive relations in which the mechanical effect of cracking and void growth is introduced with internal state variables which act on the degradation of the elastic stiffness of the material. They

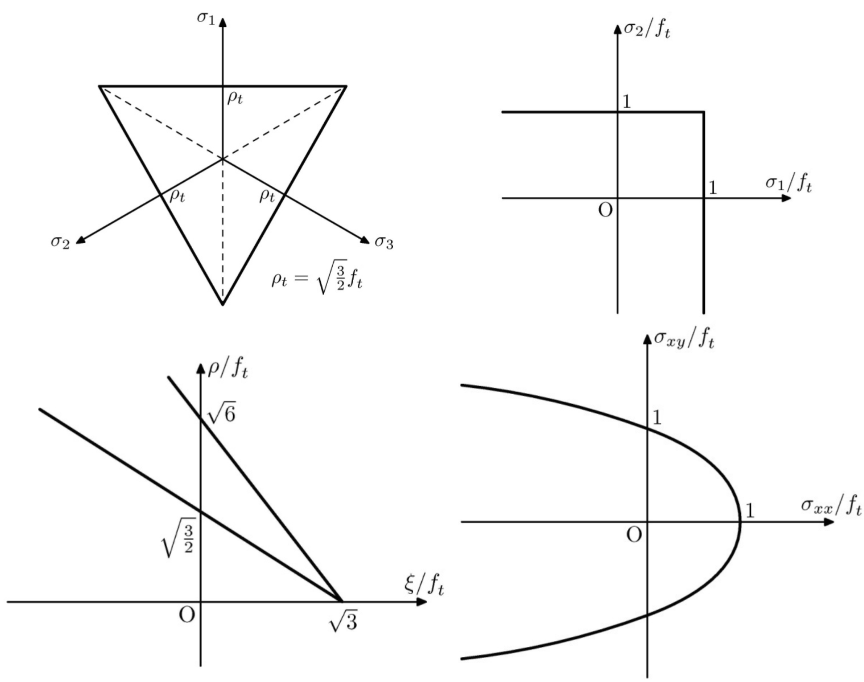

Fig. 3.2 Rankine criterion: π-plane, tensile and compressive meridian planes, σ_1-σ_2 plane, σ_{xx}-σ_{xy} plane (σ_i - principal stresses, σ_{xx} - normal stress, σ_{xy} - shear stress f_t - tensile strength, ξ - hydrostatic axis, ρ - deviatoric axis, ρ_t - deviatoric length)

can be relatively simple - isotropic (Pijaudier-Cabot 1995, Peerlings et al. 1998, Geers, et al. 1998, Huerta et al. 2003, Jirásek 2004a) or more complex - anistropic (Zhou et al. 2002, Krajcinovic and Fonseka 1981, Kuhl and Ramm 2000). The damage variable defined as the ratio between the damage area and the overall material area can be chosen as a scalar, several scalars, a second order tensor, a fourth order tensor and an eight order tensor.

A simple isotropic damage continuum model describes the material degradation with the aid of only a single scalar damage parameter D growing monotonically from zero (undamaged material) to one (completely damaged material). The stress-strain relationship is represented by

$$\sigma_{ij} = (1-D)C^{e}_{ijkl}\varepsilon_{kl}\,, \tag{3.33}$$

where C^{e}_{ijkl} is the linear elastic stiffness matrix (including modulus of elasticity E and Poisson's ratio υ) and ε_{kl} is the strain tensor. Thus, the damage parameter D acts as a stiffness reduction factor (the Poisson's ratio is not affected by damage) between 0 and 1. The growth of damage is controlled by a damage threshold parameter κ which is defined as the maximum equivalent strain measure $\tilde{\varepsilon}$ reached during the load history up to time t. The loading function of damage is

$$f(\tilde{\varepsilon},\kappa) = \tilde{\varepsilon} - \max\left\{\kappa,\kappa_0\right\}, \tag{3.34}$$

where κ_0 denotes the initial value of κ when damage begins. If the loading function f is negative, damage does not develop. During monotonic loading, the parameter κ grows (it coincides with $\tilde{\varepsilon}$) and during unloading and reloading it remains constant. To define the equivalent strain measure $\tilde{\varepsilon}$, different criteria can be used. In the book, we applied 4 different equivalent strain measures $\tilde{\varepsilon}$. First, a Rankine failure type criterion by Jirásek and Marfia (2005) was adopted

$$\tilde{\varepsilon} = \frac{\max\left\{\sigma_i^{eff}\right\}}{E}\,, \tag{3.35}$$

where σ_i^{eff} are the principal values of the effective stress

$$\sigma_i^{eff} = \sigma^{e}_{ijkl}\varepsilon_{kl}\,. \tag{3.36}$$

Second, a modified Rankine failure type criterion was applied

$$\tilde{\varepsilon} = \frac{\sigma_1^{eff} - c\left\langle -\sigma_2^{eff}\right\rangle}{E} \tag{3.37}$$

with $\sigma_1^{eff} > \sigma_2^{eff}$ and a non-negative coefficient c. This formulation is equivalent to Eq. 3.35 in a tension-tension regime, but it behaves in a different way in a mixed tension-compression regime (the coefficient c reflects the influence of the principal compressive stress). With the coefficient c=0, Eq. 3.35 is recovered.

Third, we considered a modified von Mises definition of the equivalent strain measure $\tilde{\varepsilon}$ in terms of strains (de Vree et al. 1995, Peerlings et al. 1998)

$$\tilde{\varepsilon} = \frac{k-1}{2k(1-2\nu)} I_1 + \frac{1}{2k}\sqrt{\left(\frac{k-1}{1-2\nu} I_1^{\varepsilon}\right)^2 + \frac{12k}{(1+\nu)^2} J_2^{\varepsilon}} \,. \tag{3.38}$$

The parameter k in Eq. 3.38 denotes the ratio between compressive and tensile strength of the material. A two-dimensional representation of Eq. 3.38 is given in Fig. 3.3 for k=10.

Finally, a equivalent strain measure $\tilde{\varepsilon}$ following Häuβler-Combe and Pröchtel (2005), based on the failure criterion by Hsieh-Ting-Chen (Hsieh et. al 1982), was assumed

$$\tilde{\varepsilon} = \frac{1}{2}\left(c_2\sqrt{J_2^{\varepsilon}} + c_3\varepsilon_1 + c_4 I_1^{\varepsilon} + \sqrt{\left(c_2\sqrt{J_2^{\varepsilon}} + c_3\varepsilon_1 + c_4 I_1^{\varepsilon}\right)^2 + 4c_1 J_2^{\varepsilon}}\right). \tag{3.39}$$

where ε_1 is the maximum principal total strain, c_1, c_2, c_3 and c_4 are the coefficients depending on $\alpha_1=f_t/f_c=k$, $\alpha_2=f_{bc}/f_c=r_{bc}{}^{\sigma}$ and α_3 and γ are the multipliers of the material strength in triaxial compression. The other definition of the equivalent strain measure $\tilde{\varepsilon}$ was used for concrete by Mazars and Pijaudier-Cabot (1989) using principal strains.

To describe the evolution of the damage parameter D in the tensile regime, the exponential softening law by (Peerlings et al. 1998) was mainly used (Fig. 3.4)

$$D = 1 - \frac{\kappa_0}{\kappa}\left(1 - \alpha + \alpha e^{-\beta(\kappa-\kappa_0)}\right), \tag{3.40}$$

where α and β are the material constants. The alternative forms of the damage evolution law were proposed by Geers et al. (1998), Zhou et al. (2002), Huerta et al. (2003) and Jirásek (2004a).

The damage evolution law determines the shape of the softening curve, i.e. material brittleness. The material softening starts when the when the equivalent strain measure reaches the initial threshold κ_0 (material hardening is neglected). The parameter β determines the rate of the damage growth (larger value of β causes a faster damage growth). In one dimensional problems, for $\varepsilon \rightarrow \infty$ (uniaxial tension), the stress approaches the value of $(1-\alpha)E\kappa_0$ (Fig. 3.4b).

The constitutive isotropic damage model for concrete requires the following 5 material constants: E, υ, κ_0, α and β (Eq. 3.35 and 3.40), 6 material constants: E, υ, κ_0, α, β and c (Eq. 3.37 and 3.40), 6 material constants: E, υ, k, κ_0, α and β (Eq. 3.38 and 3.40) or 9 material constants E, υ, κ_0, α, β, α_1, α_2, α_3 and γ (Eq. 3.39 and 3.40). The model is suitable for tensile failure (Marzec et al. 2007, Skarżyński et al. 2011) and mixed tensile-shear failure (Bobiński and Tejchman 2010). However, it cannot realistically describe irreversible deformations, volume changes and shear failure (Simone and Sluys 2004).

3.1.3 Anisotropic Smeared Crack Model

In a smeared crack approach, a discrete crack is represented by cracking strain distributed over a finite volume (Rashid 1968, Cope et. al. 1980, Willam et al. 1986, de Borst and Nauta 1985, de Borst 1986, Rots 1988, Rots and Blaauwendraad 1989). The model is capable of properly combining crack formation and a non-linear behaviour of concrete between cracks and of handling secondary cracking owing to rotation of the principal stress axes after primary crack formation. A secondary crack is allowed if the major principal stress exceeds tensile strength and/or if the angle between the primary crack and secondary crack exceeds a threshold angle. Since the model takes into account the crack orientation, it reflects the crack-induced anisotropy.

The total strain rate $\dot{\varepsilon}_{ij}$ is composed of a concrete strain rate $\dot{\varepsilon}_{ij}^{con}$ and several cracks strain rates $\dot{\varepsilon}_{ij}^{I}$, $\dot{\varepsilon}_{ij}^{II}$ etc. (de Borst 1986)

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{con} + \dot{\varepsilon}_{ij}^{I} + \dot{\varepsilon}_{ij}^{II} . \tag{3.41}$$

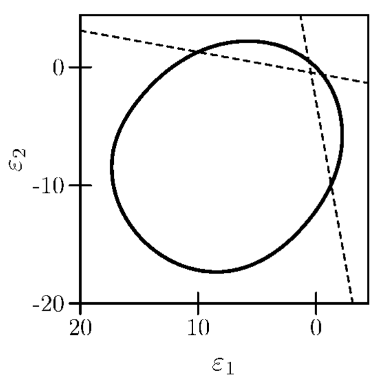

Fig. 3.3 Equivalent strain definition in principal strain space (dashed lines represent uniaxial stress paths) (Peerlings et al. 1998)

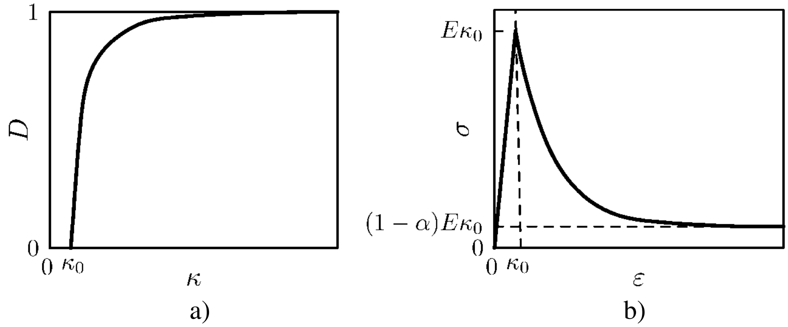

Fig. 3.4 Damage model: a) damage variable as a function of κ; b) homogeneous stress-strain behaviour during uniaxial tension (E –modulus of elasticity) (Peerlings et al. 1998)

The concrete strain rate is assumed to be related to some stress rate

$$\dot{\sigma}_{ij} = C_{ijkl}^{con} \dot{\varepsilon}_{kl}^{con} \ . \tag{3.42}$$

It may take into account elastic and plastic stress rates. The relation between the stress rate in the crack $\dot{\sigma}_{ij}^{'}$ and the crack strain rate $\dot{\varepsilon}_{kl}^{'}$ in the primary crack is given by

$$\dot{\sigma}_{ij}^{'} = C_{ijkl}^{'} \dot{\varepsilon}_{kl}^{'} \ . \tag{3.43}$$

where the primes signify that the stress rate and the crack strain components of the primary crack are taken with respect to the coordinate system of the crack. The tensor $C_{ijkl}^{'}$ represents the stress-strain relation within the primary crack. Analogously, we have for a secondary crack

$$\dot{\sigma}_{ij}^{''} = C_{ijkl}^{''} \dot{\varepsilon}_{kl}^{''} \ . \tag{3.44}$$

The double primes signify mean that the stress rate and the crack strain components of the secondary crack are taken with respect to the coordinate system of the crack. If α_{ik} are the direction cosines of the global coordination system with respect to the coordinate system of the primate crack, and if β_{ik} are the direction cosines of the global coordination system with respect to the coordinate system of the secondary crack

$$\dot{\varepsilon}_{ij}^{I} = \alpha_{ik}\alpha_{jl}\,\dot{\varepsilon}_{kl}^{'} \,, \tag{3.45}$$

$$\dot{\varepsilon}_{ij}^{II} = \beta_{ik}\beta_{jl}\,\dot{\varepsilon}_{kl}^{''} \,, \tag{3.46}$$

$$\dot{\sigma}_{ij}^{'} = \alpha_{ki}\alpha_{lj}\,\dot{\sigma}_{kl} \,, \tag{3.47}$$

$$\dot{\sigma}_{ij}^{''} = \beta_{ki}\beta_{lj}\,\dot{\sigma}_{kl} \,. \tag{3.48}$$

After a transformation of crack strain rates in global coordinates, one obtains the following relationship

$$\dot{\sigma}_{kl} = D_{klmn}^{con}(\dot{\varepsilon}_{mn} - \alpha_{mo}\alpha_{np}\,\dot{\varepsilon}_{op}^{'} - \beta_{mo}\beta_{np}\,\dot{\varepsilon}_{op}^{''}) \,. \tag{3.49}$$

Next, after some arrangements, a relationship between $\dot{\sigma}_{ij}$ and $\dot{\varepsilon}_{kl}$ is derived. A crack is initiated if the major principal stress exceeds the tensile strength. The crack direction is usually assumed to be orthogonal to the principal tensile major stress. Between stresses and strains in the crack plane z'-y' (Fig. 3.5), we have the following relationship during loading

$$\begin{bmatrix} \dot{\sigma}_{xx}^{'} \\ \dot{\sigma}_{xy}^{'} \\ \dot{\sigma}_{xz}^{'} \end{bmatrix} = \begin{bmatrix} C & 0 & 0 \\ 0 & \beta G & 0 \\ 0 & 0 & \beta G \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^{'} \\ \varepsilon_{xy}^{'} \\ \varepsilon_{xz}^{'} \end{bmatrix} \tag{3.50}$$

and during unloading

$$\begin{bmatrix} \dot{\sigma}_{xx}^{'} \\ \dot{\sigma}_{xy}^{'} \\ \dot{\sigma}_{xz}^{'} \end{bmatrix} = \begin{bmatrix} S & 0 & 0 \\ 0 & \beta G & 0 \\ 0 & 0 & \beta G \end{bmatrix} \begin{bmatrix} \varepsilon_{xx}^{'} \\ \varepsilon_{xy}^{'} \\ \varepsilon_{xz}^{'} \end{bmatrix} , \tag{3.51}$$

where the tangent modulus C represents the relation between the normal crack strain increment and normal stress increment during loading, S is the secant modulus of the unloading branch (Fig. 3.6), G is the elastic shear modulus and β is the shear stiffness reduction factor (the term βG account for effects like aggregate interlock). In addition, a threshold angle is introduced which allows new cracks to form only when the angle between the current direction of the major principal stress and the normal to the existing cracks is exceeded. When a crack fully closes, the stiffness of the uncracked concrete is again inserted.

The model has two variants: 1) the so-called fixed crack model, in which the crack orientation is fixed when the maximum principal stress attains the tensile strength (de Borst 1986), and 2) the so-called rotating crack model, in which the crack orientation is rotated so as to always remain perpendicular to the maximum principal strain direction (Rots and Blaauwendraad 1989).

In our calculations, we assumed a simplified smeared crack approach. The total strains ε_{ij} were decomposed into the elastic ε_{ij}^{e} and inelastic crack strains ε_{ij}^{cr}

$$\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{cr} . \tag{3.52}$$

The concrete stresses were related to the elastic strains via

$$\sigma_{ij} = C_{ijkl}^{e} \varepsilon_{kl}^{e} , \tag{3.53}$$

Between the concrete stresses and cracked strains, the following relationship was valid (in a local coordinate system)

$$\sigma_{ij} = C_{ijkl}^{cr} \varepsilon_{kl}^{cr} \tag{3.54}$$

with the secant cracked stiffness matrix C_{ijkl}^{cr} (defined only for open cracks).

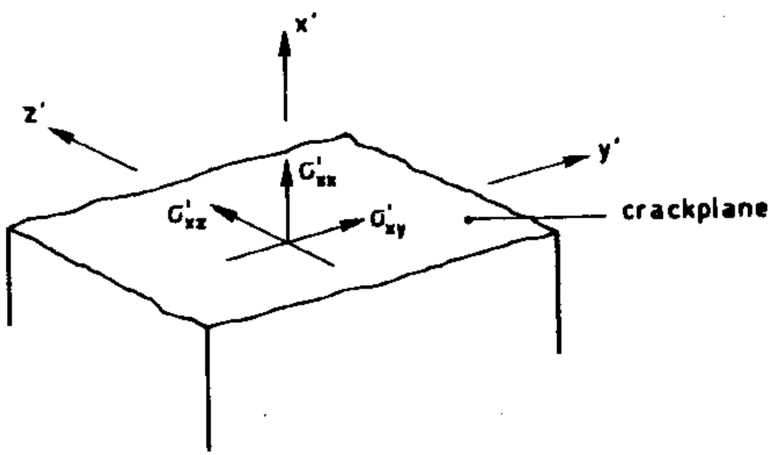

Fig. 3.5 Local coordinate system of a crack (de Borst 1986)

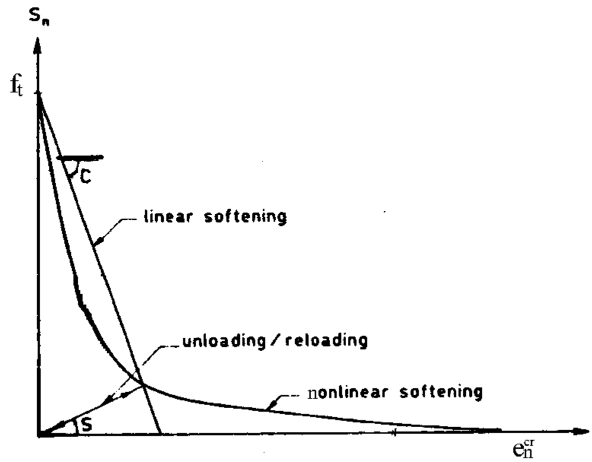

Fig. 3.6 Relationships between normal crack stress versus normal crack strain in softening range during loading, unloading and reloading (de Borst 1986)

The matrix C_{ijkl}^{cr} was assumed to be diagonal. A crack was created when the maximum tensile stress exceeded the tensile strength f_t. To define softening in a normal direction under tension, a curve by Hordijk (1991) was adopted

$$\sigma_t(\kappa) = f_t\left(\left(1 + A_1\kappa\varepsilon_i^{cr^3}\right)\exp\left(-A_2\varepsilon_i^{cr}\right) - \left(A_3\right)\varepsilon_i^{cr}\right) \tag{3.55}$$

with

$$A_1 = \frac{b_1}{\varepsilon_{nu}}, \qquad A_2 = \frac{b_2}{\varepsilon_{nu}}, \qquad A_3 = \frac{1}{\varepsilon_{nu}}(1 + b_1^3)\exp(-b_2), \tag{3.56}$$

where ε_i^{cr} is the normal cracked strain in a local i-direction, ε_{nu} denotes the ultimate cracked strain in tension and the material constants are b_1=3.0 and b_2=6.93, respectively.

The shear modulus G was reduced by the shear reduction factor β according to Rots and Blaauwendraad (1989)

$$\beta = \left(1 - \frac{\varepsilon_i^{cr}}{\varepsilon_{su}}\right)^p, \tag{3.57}$$

where ε_{su} is the ultimate cracked strain in shear and p is the material parameter. Combining Eqs. 3.51-3.54, the following relationship between stresses and total strains (in a local coordinate system) was derived

$$\sigma_{ij} = C^{s}_{ijkl}\varepsilon_{kl} \tag{3.58}$$

with the secant stiffness matrix C^{s}_{ijkl} as

$$C^{s}_{ijkl} = C^{e}_{ijkl} - C^{e}_{ijrs}(C^{e}_{rstu} + C^{cr}_{rstu})^{-1} C^{e}_{tukl}\ . \tag{3.59}$$

After cracking, the isotropic elastic stiffness matrix was replaced by the orthotropic one (in a local coordinate system). Two different formulations were investigated: a rotating crack model and a multi-fixed orthogonal crack model. In the first approach (rotating crack), only one crack was created which could rotate during deformation. To keep the principal axis of total strains and stresses aligned, the secant stiffness coefficient was calculated according to

$$C^{s}_{ijij} = \frac{\sigma_{ii} - \sigma_{jj}}{2\left(\varepsilon_{ii} - \varepsilon_{jj}\right)}\ . \tag{3.60}$$

The second formulation (fixed crack model) allowed one a creation of three mutually orthogonal cracks in 3D-problems (and two orthogonal cracks in 2D simulations, respectively). The orientation of the crack was described by its primary inclination at the onset, i.e. the crack did not rotate during loading.

The constitutive smeared crack model for concrete requires the following 8 material parameters: E, υ, p, c_1, c_2, f_t, ε_{su} and ε_{nu}.

3.2 Local Coupled Models for Cyclic Loading

An analysis of concrete elements under quasi-static cyclic loading under compression, tension and bending is complex mainly due to a stiffness degradation caused by fracture (Karsan and Jirsa 1969, Reinhardt et al. 1986, Hordijk 1991, Perdikaris and Romeo 1995). To take into account a reduction of both strength and stiffness, irreversible (plastic) strains and degradation of stiffness, a combination of plasticity and damage theories is in particular physically appealing since plasticity considers the first three properties and damage considers a loss of material strength and deterioration of stiffness. Within continuum mechanics, plasticity and damage couplings were analyzed by many researchers using different ideas (e.g. Lemaitre 1985, Mazars 1986, Simo and Ju 1987, Klisinski and Mróz 1988, Lubliner et al. 1989, Hansen and Schreyer 1994, Meschke et al. 1998, Pamin and de Borst 1999, Carol et al. 2001, Hansen and Willam 2001, Gatuingt and Pijaudier-Cabot 2002, Ibrahimbegovic et al. 2003, Salari et al. 2004, Bobiński and Tejchman 2006, Grassl and Jirásek 2006, Voyiadjis et al. 2009). An alternative to the cyclic concrete behaviour by elasto-plastic-damage models, is the application of an endochronic theory which deals with the plastic response of materials by means of memory integrals, expressed in terms of memory kernels (Bažant 1978, Khoei er al. 2003).

Below, the capability of 4 different coupled elasto-plastic damage continuum models to describe strain localization and stiffness degradation in a concrete beams subjected to quasi-static cyclic loading under tensile failure was investigated. The coupled elasto-plastic damage models proposed by Pamin and de Borst (1999), by Carol et al. (2001) and by Hansen and Willam (2001), by Meschke et al. (1998) and Marzec and Tejchman (2009, 2011) were taken into account.

The first model (Pamin and de Borst 1999) combines non-local damage with hardening plasticity based on effective stresses and a strain equivalence concept (Katchanov 1986, Simo and Ju 1987). The total strains are namely equal to strains in an undamaged skeleton between micro-cracks. Plastic flow can occur only in an undamaged specimen, therefore an elasto-plastic model is defined in terms of effective stresses. In the second model (Carol et al. 2001 and Hansen and Willam 2001) plasticity and damage are connected by two loading functions describing the behaviour of concrete in compression and tension. The onset and progression of material degradation is based upon the strain energy associated with the effective stress and strain. A damage approach (based on second-order tensors) simulates the behaviour of concrete under tension while plasticity describes the concrete behaviour under compression. A failure envelope is created by combining a linear Drucker-Prager formulation in compression with a damage formulation based on a conjugate force tensor and a pseudo-log damage rate in tension. In turn, in the third formulation (Meschke et al. 1998), an elasto-plastic criterion is enriched by new components including stiffness degradation. Degradation is written in the form of a Rankine'a criterion with hyperbolic softening. Following the partitioning concept of strain rates, an additional scalar internal variable is introduced into a constitutive formulation. Thus, the splitting of irreversible strains into components associated with plasticity and damage is obtained. Finally, based on an analysis of three initially presented formulations, an improved coupled formulation connecting plasticity and damage is presented using a strain equivalence hypothesis (Pamin and de Borst 1999). The plasticity is described with both a Drucker-Prager and a Rankine criterion in compression and tension, respectively. To describe the evolution of damage, a different definition is assumed for tension and compression. Finally to take into account a stiffness recovery at a crack closure and inelastic strains due to damage, combined damage in tension and compression based on stress weight factors is introduced.

Constitutive coupled model by Pamin and de Borst (1999)

The first formulation (called model '1') according to Pamin and de Borst (1999) combines elasto-plasticity with scalar damage assuming that total strains ε_{ij} are equal to strains in an undamaged skeleton (called effective strains ε_{ij}^{eff}). Elasto-plastic deformation occurs only in an undamaged specimen and is defined

$$\sigma_{ij}^{eff} = C_{ijkl}^{e}\varepsilon_{kl} , \tag{3.61}$$

The following failure criterion to describe a material response in an elasto-plastic regime is used

$$f_{ep} = F\left(\boldsymbol{\sigma}^{eff}\right) - \sigma_y\left(\kappa_p\right), \tag{3.62}$$

wherein σ_y - the yield stress and κ_p - the hardening parameter equal to plastic strain during uniaxial tension. As an elasto-plastic criterion in Eq. 3.62, the failure criterion by von Mises or Drucker-Prager may be used defined by effective stresses. Next, the material degradation is calculated with the aid of an isotropic damage model (3.38 and 3.40). The equivalent strain measure $\tilde{\varepsilon}$ can be defined in terms of total strains ε_{ij} or elastic strains ε_{ij}^{e}.

The local coupled elasto-plastic-damage model '1' requires the following 6 material constants to capture the cyclic tensile behaviour: E, ν, κ_0, α, β, k and one hardening yield stress function. In the case of linear hardening, 8 material constants are totally needed (in addition, the initial yield stress σ_{yt}^{o} at κ_p=0 and hardening plastic modulus H_p).

Constitutive coupled model by Carol et al (2001) and Hansen and Willam (2001)
In the second model (called model '2'), a two-surface isotropic damage/plasticity model combining damage mechanics and plasticity in a single formulation is used (Carol et al. 2001 and Hansen and Willam 2001). A plastic region in compression is described with the aid of a linear Drucker-Prager criterion. The material experiences permanent deformation under sustained loading with no loss of the material stiffness. In turn in tension, damage is formulated in the spirit of plasticity by adopting the concept of a failure condition and a total strain rate decomposition into the elastic strain rate $\mathrm{d}\varepsilon_{\mathrm{ij}}^{\;e}$ and degrading strain rate $\mathrm{d}\varepsilon_{\mathrm{ij}}^{\;d}$ (as a result of the decreasing stiffness)

$$\mathrm{d}\varepsilon_{ij} = \mathrm{d}\varepsilon_{ij}^{e} + \mathrm{d}\varepsilon_{ij}^{d}. \tag{3.63}$$

The boundary between elastic and progressive damage is governed by a failure criterion

$$f_d = f\left(\sigma_{ij}, q_d\right), \tag{3.64}$$

where q_d is the damage history variable describing the evolution of the damage surface. The stress rate is equal to as

$$\mathrm{d}\sigma_{ij} = C_{ijkl}^{s}\left(\mathrm{d}\varepsilon_{kl} - \mathrm{d}\varepsilon_{kl}^{d}\right) \tag{3.65}$$

with C^{s}_{ijkl} as the secant stiffness matrix connected with a material damage parameter D via

$$C^{s}_{ijkl} = (1-D)\,C^{e}_{ijkl}\,. \tag{3.66}$$

The application of the secant stiffness is central to the idea that the degraded strains and stresses are reversible, since the material stiffness must degrade to make this idea possible (Carol et al. 2001, Hansen and Willam 2001). The degrading strain rate was defined as the excess strain rate beyond the value that corresponded to the stress increment according to the current secant stiffness.

The effective stress and effective strain are again experienced by the undamaged material between cracks. Assuming the energy equivalence, the mutual relationship between the nominal (observed externally) and effective stress and strain is taken as

$$\sigma_{ij} = \sqrt{1-D}\,\sigma^{eff}_{ij} \qquad \text{and} \qquad \varepsilon^{eff}_{ij} = \sqrt{1-D}\,\varepsilon_{ij} \tag{3.67}$$

and

$$\sigma^{eff}_{ij}\,\varepsilon^{eff}_{ij} = \sigma_{ij}\varepsilon_{ij}\,, \tag{3.68}$$

with

$$\sigma^{eff}_{ij} = C^{e}_{ijkl}\varepsilon^{eff}_{kl} \qquad \text{and} \qquad \sigma_{ij} = (1-D)\,C^{e}_{ijkl}\varepsilon_{kl}\,. \tag{3.69}$$

The loading function (Eq. 3.64) for the Rankine-type anistropic damage model is defined in terms of the modified principal tensile conjugate forces

$$f_d = \sum_{i}^{3} f\left(-\hat{y}_{(i)}\right) - r(L)\,, \tag{3.70}$$

where $-\hat{y}_{(i)}$ - the principal components of the tensile conjugate forces tensor and $r(L)$ - the resistance function as the complementary energy. The conjugate force $-\hat{y}_{(i)}$ is a second order energy tensor written with aid of the effective stresses and strains by assuming linear isotropic elasticity

$$-\hat{y}_{(i)} = \frac{1}{2}\left\langle \sigma^{eff}_{i} \right\rangle \left\langle \varepsilon^{eff}_{i} \right\rangle, \tag{3.71}$$

where $\langle \bullet \rangle$ is the Macauley bracket. Originally, Carol et al. (2001) and by Hansen and Willam (2001) proposed the following resistance function with two parameters G_f and r_o

$$r(L) = r_o e^{-\frac{r_o}{g_f}L} , \tag{3.72}$$

with g_f - the fracture energy and r_0 – the elastic strain energy at the peak of the uniaxial tension test (E - isotropic elastic modulus)

$$r_o = \frac{(f_t)^2}{2E} . \tag{3.73}$$

The parameter L in Eq. 3.72 denotes the pseudo-log damage variable and is calculated with the aid of Eqs. 3.71 and 3.72

$$L = \ln \frac{1}{1\text{-}D} , \qquad \left(D = 1 - e^{-L} \right) . \tag{3.74}$$

The rate of L is

$$\dot{L} = \frac{\dot{D}}{1\text{-}D} . \tag{3.75}$$

However, Eq. 3.72 poorly influences the post-peak behaviour. Therefore, we proposed a new resistance function with also 2 parameters

$$r(L) = \frac{1}{2} E \kappa_0^2 \exp\left(\frac{L(2-\beta)}{\beta} \right), \tag{3.76}$$

wherein κ_0 - the threshold strain value and β - the parameter describing softening. The resistance function adopted by Nguyen (2005) was used in numerical simulations as well

$$r(L) = \frac{1}{2} \frac{f_t^2}{E} \left(\frac{E + E_{pt} e^{-L \cdot n_t}}{E e^{-L} + E_{pt} e^{-L \cdot n_t}} \right)^2 , \tag{3.77}$$

with f_t - the tensile strength, E_{pt} - the damaged stiffness modulus and n_t - the rate of the stiffness modulus.

When simultaneously considering both damage and plasticity, the total strain rate becomes the sum of the elastic, damage and plastic rate

$$\mathrm{d}\varepsilon_{ij} = \mathrm{d}\varepsilon_{ij}^e + \mathrm{d}\varepsilon_{ij}^d + \mathrm{d}\varepsilon_{ij}^p . \tag{3.78}$$

The plastic strains are permanent while elastic and damage strains were reversible. Therefore, the elastic-damage strain $\mathrm{d}\varepsilon_{ij}^{ed}$ is introduced in the total value

$$\mathrm{d}\varepsilon_{ij} = \mathrm{d}\varepsilon_{ij}^{ed} + \mathrm{d}\varepsilon_{ij}^{p}\,. \tag{3.79}$$

The local coupled elasto-plastic-damage model requires the material constants: E, ν, ϕ, ψ, g_f and r_0 (Eq. 3.72), E, ν, ϕ, ψ, κ_0 and β (Eq. 3.76), E, ν, ϕ, ψ, f_t, E_{pt}, n_t (Eq. 3.77) and one compressive hardening/softening yield stress function.

Constitutive coupled model by Meschke et al. (1998)
In the third model (called model '3'), another concept of coupling was introduced. An elasto-plastic criterion is enhanced by a new component describing the stiffness degradation (Meschke et al. 1998). The permanent strain rate decomposition is assumed as

$$\mathrm{d}\varepsilon_{ij}^{pd} = \mathrm{d}\varepsilon_{ij}^{p} + \mathrm{d}\varepsilon_{ij}^{d}\,. \tag{3.80}$$

The plastic damage strain rate $\mathrm{d}\varepsilon_{ij}^{pd}$ is calculated as in classical plasticity. The component associated with degradation and plasticity is obtained by introducing a scalar constant γ between zero and one ($0\leq\gamma\leq1$)

$$\mathrm{d}\varepsilon_{ij}^{p} = (1-\gamma)\,\mathrm{d}\varepsilon_{ij}^{pd} \qquad \text{and} \qquad \mathrm{d}\varepsilon_{ij}^{d} = \gamma\,\mathrm{d}\varepsilon_{ij}^{pd}\,. \tag{3.81}$$

The parameter γ enables one a simple splitting of effects connected with an inelastic slip process (which caused an increase of plastic strain) and a deterioration of microstructure (which contributed to an increase of the compliance tensor). The evolution law for the compliance tensor is ($d\lambda$ - proportionality factor)

$$\dot{\boldsymbol{D}} = \gamma \times d\lambda \frac{\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T}{\left(\dfrac{\partial f}{\partial \boldsymbol{\sigma}}\right)^T \boldsymbol{\sigma}}\,. \tag{3.82}$$

The stresses are updated analogously to the standard plasticity theory. To simulate concrete softening in tension, a hyperbolic softening law is chosen

$$\sigma_t(\kappa) = \frac{f_t}{\left(1+\dfrac{\kappa}{\kappa_0}\right)^2}\,, \tag{3.83}$$

where κ_0 - the parameter adjusted to the fracture energy.

This coupled elasto-plastic-damage model requires in tension the following 5 parameters: E, υ, f_t, κ_0 and γ.

Improved coupled elasto-plastic-damage model (Marzec and Tejchman 2010) In order to describe the cyclic concrete behaviour under both tension and compression, an improved coupled model (called model '4') was proposed based on the model '1' by Pamin and de Borst (1999) (which combines elasto-plasticity with a scalar damage assuming a strain equivalence hypothesis). The elasto-plastic deformation is defined in terms of effective stresses according to Eq. 3.61. Two criteria are used in an elasto-plastic regime (Marzec et al. 2007, Majewski et al. 2008): a linear Drucker-Prager criterion with a non-associated flow rule in compression and a Rankine criterion with an associated flow rule in tension defined by effective stresses (Chapter 3.1). Next, the material degradation is calculated within damage mechanics, independently in tension and compression using one equivalent strain measure $\tilde{\varepsilon}$ proposed by Mazars (1986) (ε_i - principal strains)

$$\tilde{\varepsilon} = \sqrt{\sum_i \langle \varepsilon_i \rangle^2} \,. \tag{3.84}$$

In tension, the same damage evolution function by Peerlings et al. (1998) as in the model '1' is chosen (Eq. 3.40). In turn, in compression, the definition by Geers (1997) is adopted

$$D_c = 1 - \left(1 - \frac{\kappa_0}{\kappa}\right)\left(0.01\frac{\kappa_0}{\kappa}\right)^{\eta_1} - \left(\frac{\kappa_0}{\kappa}\right)^{\eta_2} e^{-\delta(\kappa-\kappa_0)} \,, \tag{3.85}$$

where η_1, η_2 and δ are the material constants. Equation 3.85 allows for distinguishing different stiffness degradation under tension and under compression. Damage under compression starts to develop later than under tension that is consistent with experiments. The damage term '1-D' (Eq. 3.33) is defined as in ABAQUS (1998) following Lubliner et al. (1989) and Lee and Fenves (1998a)

$$(1 - D) = (1 - s_c D_t)(1 - s_t D_c) \,, \tag{3.86}$$

with two splitting functions s_t and s_c controlling the magnitude of damage

$$s_t = 1 - a_t w(\boldsymbol{\sigma}^{eff}) \qquad \text{and} \qquad s_c = 1 - a_c\left(1 - w(\boldsymbol{\sigma}^{eff})\right), \tag{3.87}$$

where a_t and a_c are the scale factors and $w(\boldsymbol{\sigma}^{eff})$ denotes the stress weight function which may be determined with the aid of principal effective stresses according to Lee and Fenves (1998a)

$$w\left(\boldsymbol{\sigma}^{eff}\right)=\begin{cases}0 & \text{if } \boldsymbol{\sigma}=0\\ \dfrac{\sum\left\langle\sigma_i^{eff}\right\rangle}{\sum\left|\sigma_i^{eff}\right|} & \text{otherwise}\end{cases}. \tag{3.88}$$

For relatively simple cyclic tests (e.g. uniaxial tension, bending), the scale factors a_t and a_c can be a_t=0 and a_c=1, respectively. Thus, the splitting functions are: $s_t = 1.0$ and $s_c = w\left(\boldsymbol{\sigma}^{eff}\right)$. For uniaxial loading cases, the stress weight function becomes

$$w\left(\sigma^{eff}\right)=\begin{cases}1 \text{ if } \sigma^{eff}>0\\ 0 \text{ if } \sigma^{eff}<0\end{cases}. \tag{3.89}$$

Thus, under pure tension the stress weight function $w = 1.0$ and under pure compression $w = 0$.

Our constitutive model with a different stiffness in tension and compression and a positive-negative stress projection operator to simulate crack closing and crack re-opening is thermodynamically consistent. It shares main properties of the model by Lee and Fenves (1998a), which was proved to not violate thermodynamic principles (plasticity is defined in the effective stress space, isotropic damage is used and the stress weight function is similar). Moreover Carol and Willam (1996) showed that for damage models with crack-closing-re-opening effects included, only isotropic formulations did not suffer from spurious energy dissipation under non-proportional loading (in contrast to anisotropic ones).

Our local coupled elasto-plastic-damage model requires the following 10 material constants E, ν, κ_0, α, β, η_1, η_2, δ, a_t, a_c and 2 hardening yield stress functions (the function by Rankine in tension and by Drucker-Prager in compression). If the tensile failure prevails, one yield stress function by Rankine can be used only.

The quantities σ_y (in the hardening function) and κ_0 are responsible for the peak location on the stress-strain curve and a simultaneous activation of a plastic and damage criterion (usually the initial yield stress in the hardening function σ_y^0=3.5-6.0 MPa and κ_0=(8-15)×10^{-5} under tension). The shape of the stress-strain-curve in softening is influenced by the constant β in tension (usually β=50-800), and by the constants δ and η_2 in compression (usually δ=50-800 and η_2=0.1-0.8). The parameter η_2 influences also a hardening curve in compression. In turn, the stress-strain-curve at the residual state is affected by the constant α (usually α=0.70-0.95) in tension and by η_1 in compression (usually η_1=1.0-1.2). Since the parameters α and η_1 are solely influenced by high values of κ, they can arbitrarily be assumed for softening materials. Thus, the most crucial material constants are σ_y^0, κ_0, β, δ and η_2. In turn, the scale factors a_t and a_c influence the damage magnitude in tension and compression. In general, they vary between zero and one. There do not exist unfortunately the experimental data allowing for determining the magnitude of a_t and a_c. Since, the compressive stiffness is recovered upon the crack closure as the load changes from tension to compression and the tensile stiffness is not recovered due to compressive micro-cracks, the parameters a_c and a_t can be taken for the sake of simplicity as a_c=1.0 and a_t=0 for many different simple loading cases as e.g. uniaxial tension and bending. The equivalent strain measure $\tilde{\varepsilon}$ can be defined in terms of total strains or elastic strains. The drawback of our formulation is the necessity to tune up constants controlling plasticity and damage to activate an elasto-plastic criterion and a damage criterion at the same moment. As a consequence, the chosen yield stress σ_y may be higher than this obtained directly in laboratory simple monotonic experiments.

The material constants E, ν, κ_0, β, α, η_1, η_2, δ and two hardening yield stress functions can be determined for concrete with the aid of 2 independent simple monotonic tests: uniaxial compression test and uniaxial tension (or 3-point bending) test. However, the determination of the damage scale factors a_t and a_c requires one full cyclic compressive test and one full cyclic tensile (or 3-point bending) test.

Table 3.1 shows a short comparison between four coupled models. The major drawback of first 3 formulations is the lack of the damage differentiation in tension and compression, stiffness recovery associated with crack closing and relationship between the tensile and compression stiffness during a load direction change. To describe these phenomena, additional material constants have to be included.

The damage hardening/softening laws assumed in constitutive models have been fully based on experimental data from uniaxial compression and uniaxial tension tests which in turn strongly depend on the concrete nature, specimen size and boundary and loading conditions. It means that they are not physically based. This fact reveals the necessity to derive macroscopic laws in a softening regime from real micro-structure evolutions in materials during homogeneous tests using e.g. a discrete element model (Widulinski et al. 2011).

The coupled model '1' can be enriched by the crack-closure effect in a similar way as our model '4'. For the models '2' and '3' due to their different structure, the crack-closure effect can be incorporated by introducing a projection operator (model '2') or by modifying the evolution law for the compliance tensor (model '3').

Table 3.1 Comparison between four local coupled elasto-plastic damage formulations to describe concrete behaviour (Marzec and Tejchman 2009, 2011)

Nr.	**Plastic strains in tension/compression**	**Stiffness degradation**	**Unique strain division**	**Stiffness recovery**	**Number of material parameters**
Model '1'	Yes	Yes	No	No	Elastic: 2 Plastic: 1 (tens.) 3 (compr.) Damage: 4
Model '2'	Yes (only in compression)	Yes (only in tension)	Yes	No	Elastic: 2 Plastic: 3 Damage: 3-4
Model '3'	Yes	Yes	Yes	No	Elastic: 2 Plastic: 2 Damage: 1
Model '4'	Yes	Yes	No	Yes	Elastic: 2 Plastic: 1 (tension), 3 (compression) Damage:2 (tension), 3 (compression) Scale factors: 2

3.3 Regularization Techniques

Classical FE-simulations of the behaviour of materials with strain localization within continuum mechanics are not able to describe properly both the thickness of localization and distance between them. They suffer from mesh sensitivity (its size and alignment) and produce unreliable results. The strains concentrate in one element wide zones and the computed force-displacement curves are mesh-dependent (especially in a post-peak regime). The reason is that differential equations of motion change their type (from elliptic to hyperbolic in static problems) and the rate boundary value problem becomes ill-posed (de Borst et al. 1992). Thus, classical constitutive continuum models require an extension in the form of a characteristic length to properly model the thickness of localized zones. Such extension can be by done within different theories: a micro-polar (Mühlhaus

1986, Sluys 1992, Tejchman and Wu 1993, Tejchman et al. 1999), a strain gradient (Zbib and Aifantis 1989, Mühlhaus and Aifantis 1991, Pamin 1994, de Borst and Pamin 1996, Pamin 2004, Sluys and de Borst 1994, Peerlings et al. 1998, Meftah and Reynouard 1998, Pamin and de Borst 1998, Chen et al. 2001, Zhou et al. 2002, Askes and Sluys 2003), a viscous (Sluys 1992, Sluys and de Borst 1994, Neddleman 1988, Loret and Prevost 1990, Lodygowski and Perzyna 1997, Winnicki et al. 2001, Pedersen et al. 2008, Winnicki 2009) and a non-local (Pijaudier-Cabot and Bažant 1987, Bažant and Lin, 1988, Brinkgreve 1994, de Vree et al. 1995, Strömberg and Ristinmaa 1996, Marcher and Vermeer 2001, Maier 2002, 2003, di Prisco et al. 2002, Bažant and Jirásek 2002, Jirásek and Rolshoven 2003, Tejchman 2004).

Other numerical technique which also enables to remedy the drawbacks of a standard FE-method and to obtain mesh-independency during formation of cracks, are approaches with strong discontinuities which enrich continuous displacement modes of the standard finite elements with additional discontinuous displacements (Belytschko et al. 2001, 2009, Simone et al. 2002, Asferg et al. 2006, Oliver et al. 2006) or approaches with cohesive (interface) elements (Ortiz and Pandolfi 1999, Zhou and Molinari 2004) (Chapters 4.1 and 4.2). In the first approaches, discontinuity paths are placed inside the elements irrespective of the size and specific orientation. In the latter approaches, discontinuity paths are defined at the edges between standard finite elements. The most realistic approach to concrete (a continuous-discontinuous approach) was used by Moonen et al. (2008).

Below two different regularization methods (integral-type non-local and explicit second-gradient) are described in detail.

3.3.1 Integral-Type Non-local Approach

A non-local model of the integral type (so called “strongly non-local model”) was used as a regularisation technique:

a) to properly describe strain localization (width and spacing),
b) to preserve the well-posedness of the boundary value problem,
c) to obtain mesh-independent results,
d) to take into account material heterogeneity and
e) to include a characteristic length of micro-structure for simulations of a deterministic size effect (Pijaudier-Cabot and Bažant 1987, Bažant and Jirásek 2002, Bobiński and Tejchman 2004).

It is based on a spatial averaging of tensor or scalar state variables in a certain neighbourhood of a given point (i.e. material response at a point depends both on the state of its neighbourhood and the state of the point itself). Thus, a characteristic length l_c can be incorporated and softening can spread over material points. It is in contrast to classical continuum mechanics, wherein the principle of local action holds (i.e. the dependent variables in each material point depend only upon the values of the independent variables at the same point), and softening at one material point does not affect directly the yield surfaces of other points. It has a physical motivation due to the fact the distribution of stresses in the interior of concrete is strongly non-uniform (Fig. 2.1b). Polizzotto et al. (1998) laid down a thermodynamic a consistent formulation of non-local plasticity. In turn, Borino et al. (2003) and Nguyen (2008) laid down a thermodynamic consistent formulation of non-local damage mechanics.

Usually it is sufficient to treat non-locally only one variable controlling material softening or degradation (Brinkgreve 1994, Bažant and Jirásek 2002, Huerta et al. 2003).

A full non-local model assumes a relationship between average stresses $\bar{\sigma}_{ij}$ and averaged strains $\bar{\varepsilon}_{ij}$ defined as

$$\bar{\sigma}_{ij}(x) = \frac{1}{\bar{V}} \iiint \omega \|x-\xi\| \sigma_{ij}(\xi) d\xi_1 d\xi_2 d\xi_3 \tag{3.90}$$

and

$$\bar{\varepsilon}_{ij}(x) = \frac{1}{\bar{V}} \iiint \omega \|x-\xi\| \varepsilon_{ij}(\xi) d\xi_1 d\xi_2 d\xi_3 \tag{3.91}$$

where $\bar{\sigma}_{ij}(\boldsymbol{x})$ and $\bar{\varepsilon}_{ij}(\boldsymbol{x})$ are the non-local softening parameters, $\boldsymbol{x}$ are the coordinates of the considered point, $\boldsymbol{\xi}$ are the coordinates of the surrounding points, ω denotes the weighting function and $\bar{V}$ denotes the weighed body volume

$$\bar{V} = \iiint \omega(\|x-\xi\|) d\xi_1 d\xi_2 d\xi_3 \ . \tag{3.92}$$

In general, it is required that the weighting function ω should not alter a uniform field which means that it must satisfy the normalizing condition (Bažant and Jirásek 2002).

As a weighting function ω (called also an attenuation function or a non-local averaging function), a Gauss distribution function was used which is in 2D calculations

$$\omega(r) = \frac{1}{l_c\sqrt{\pi}} e^{-\left(\frac{r}{l_c}\right)^2} \tag{3.93}$$

with

$$\int_{-\infty}^{\infty} \omega(r)\,\mathrm{d}r = 1 \ . \tag{3.94}$$

where the parameter l_c is a characteristic length of micro-structure and r is a distance between two material points. The averaging in Eq. 3.93 is restricted to a small representative area around each material point (the influence of points at the distance of r=3l_c is only of 0.01%) (Fig. 3.7). A characteristic length is usually related to the micro-structure of the material (e.g. maximum aggregate size and crack spacing in concrete, pore and grain size in granulates, crystal size in metals). It is determined with an inverse identification process of experimental data (Geers et al. 1996, Mahnken and Kuhl 1999, Le Bellego et al. 2003).

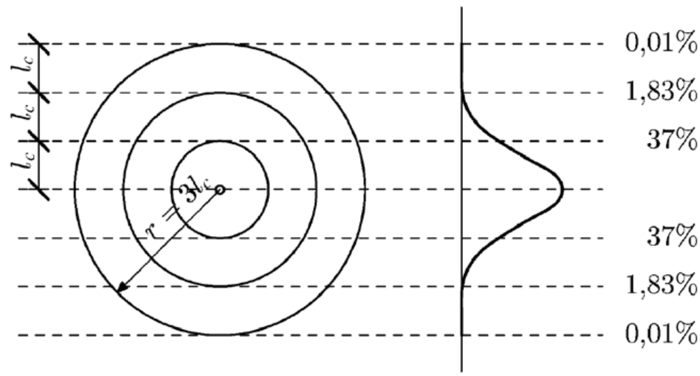

Fig. 3.7 Region of the influence of characteristic length l_c and weighting function ω (Bobiński and Tejchman 2004)

However, the determination of a representative characteristic length of micro-structure l_c is very complex in concrete since strain localization can include a mixed failure mode (cracks and shear zones) and a characteristic length (which is a scalar value) is related to the fracture process zone with a certain volume which changes during a deformation (the width of the fracture process zone increases according to e.g. Pijaudier-Cabot et al. 2004, but decreases after e.g. Simone and Sluys 2004). In turn, other researchers conclude that the characteristic length is not a constant, and it depends on the type of the boundary value problem and the current level of damage (Ferrara and di Prisco 2001). Thus, a determination of l_c requires further numerical analyses and measurements, e.g. using a Digital Image Correlation (DIC) technique (Bhandari and Inoue 2005). FE simulations of tests with measured load-displacement curves and widths of fracture process zones for different boundary value problems and specimen sizes are of importance. According to Pijaudier-Cabot and Bažant (1987), Bažant and Oh (1983), it is in concrete approximately 3×d_a^{max}, where d_a^{max} is the maximum aggregate size.

Other representations can be also used for the function ω (Ožbolt 1995, Akkermann 2000, Jirásek 2004a, di Prisco et al. 2002, Bažant and Jirásek 2002); e.g. the polynomial bell-shaped function reads

$$\omega = (1-\frac{r^2}{R^2})^2, \tag{3.95}$$

where R (interaction radius) is a parameter related to a characteristic length. To improve the behaviour of a non-local averaging in the vicinity of the boundary of a finite body, Polizzotto (2002) proposed the weight distribution preserving a uniform field and symmetry:

$$\omega = \omega(r) + [1 - \int_V \omega(r)dr]\delta(r), \tag{3.96}$$

where δ denotes the Dirac distribution. This function is corrected by a suitable multiple of the local value to compensate for boundary effects. The FE-results by Jirásek et al. (2004b) show that the type of a non-local averaging near boundaries influences the peak of the load-displacement curve; the averaging with a symmetric local correction by Eq. 3.96 results in a lower resistance.

Our FE calculations were carried out mainly with the characteristic length l_c=1.5 mm (for fine-grained concrete) and l_c=5 mm (usual concrete) based on DIC tests (Skarżynski et al. 2011, Syroka 2011).

Monotonic loading

In the calculations within elasto-plasticity (Eqs. 3.27-3.32), the softening parameters κ_i (i=1, 2) were assumed to be a linear combination of the local and non-local values (independently for both yield surfaces f_i) (so called 'over-nonlocal' formulation, Brinkgreve 1994, Bobiński and Tejchman 2004)

$$\bar{\kappa}_i(\boldsymbol{x}) = (1-m)\kappa_i(\boldsymbol{x}) + m\frac{\int_V \omega(\|\boldsymbol{x}-\boldsymbol{\xi}\|)\kappa_i(\boldsymbol{\xi})d\boldsymbol{\xi}}{\int_V \omega(\|\boldsymbol{x}-\boldsymbol{\xi}\|)d\boldsymbol{\xi}} \qquad i=1, 2, \tag{3.97}$$

where $\bar{\kappa}_i(\boldsymbol{x})$ are the non-local softening parameters and m is the additional non-local parameter controlling the size of the localized plastic zone. For m=0, a local approach is obtained and for m=1, a classical non-local model is recovered (Pijaudier-Cabot and Bažant 1987, Bažant and Lin 1988). If the parameter m>1, the influence of the non-locality increases and the localized plastic region reaches a finite mesh-independent size (Brinkgreve 1994, Bažant and Jirásek 2002, Bobiński and Tejchman 2004). Brinkgreve (1994) derived an analytical formula for the thickness of a localized zone in an one-dimensional bar during tension with necking using a modified non-local approach by Eq. 3.97. According to this formula, if the non-local parameter was m=1, the thickness of the localized zone was equal to zero (similarly as in an usual local approach). The enhanced non-local elasto-plastic model has in addition two material parameters m and l_c.

The softening non-local parameters near boundaries were calculated also on the basis of Eqs. 3.93-3.95 (which satisfy the normalizing condition). During a FE-analysis, the integral in Eq. 3.96 was replaced by a summation operator

$$\bar{\kappa}_i(\boldsymbol{x_i}) = (1-m)\kappa_i(x_i) + m\frac{\sum_j^{np}\omega(\|\boldsymbol{x_i}-\boldsymbol{\xi}_j\|)\kappa_i(\boldsymbol{\xi}_j)V_j}{\sum_j^{np}\omega(\|\boldsymbol{x_i}-\boldsymbol{\xi}_j\|)V_j}, \tag{3.98}$$

where *np* is the number of all integration points in the whole body, x_j stand for co-ordinates of the integration point in each element and V_j is the actual element volume.

In the calculations within isotropic damage mechanics (Chapters 5 and 7), the equivalent strain measure $\tilde{\varepsilon}$ (Eqs. 3.35, 3.37, 3.38 and 3.39) was replaced by its non-local definition (Marzec et al. 2007)

$$\bar{\varepsilon} = \frac{\int_V \omega(\|x-\xi\|)\tilde{\varepsilon}(\xi)d\xi}{\int_V \omega(\|x-\xi\|)d\xi}. \tag{3.99}$$

It is to note, that in some other damage formulations, the use of a this non-local variable causes problems with energy dissipation can lead to an improper solution (Jirásek 1998, Jirásek and Rolshoven 2003, Borino et al. 2003). This case occurs in the coupled elasto-plastic-damage model '2'.

In the smeared crack approach, the secant matrix C^s_{ijkl} (Eqs. 3.52-3.60) was calculated with the non-local strain tensor $\bar{\varepsilon}_{kl}$ defined (independently for all tensor components) as (Jirásek and Zimmermann 1998)

$$\bar{\varepsilon}_{kl}(x) = \frac{\int \omega(\|x-\xi\|)\varepsilon_{kl}(\xi)d\xi}{\int \omega(\|x-\xi\|)d\xi}. \tag{3.100}$$

Thus, the resulting stresses were calculated from the relationship

$$\sigma_{ij} = C^s_{ijkl}(\bar{\varepsilon}_{kl})\varepsilon_{kl}. \tag{3.101}$$

Cyclic loading

In the first coupled elasto-plastic damage model (model '1'), non-locality was applied in damage (softening was not allowed in elasto-plasticity). The equivalent strain measure was replaced by its non-local counterpart (Eq. 3.94). In the second coupled model, non-locality was prescribed in tension to the energy release Y (Marzec and Tejchman 2009)

$$Y = \frac{1}{2}\varepsilon_{ij} C^{e}_{ijkl} \varepsilon_{kl}\,, \tag{3.102}$$

which is a component of the loading function in Eq. 3.70. The non-local damage energy was composed of a local and non-local term calculated in the current (i) and previous iteration (i-1) (Strömberg and Ristinmaa 1996, Rolshoven 2003)

$$\overline{Y}^{*}_{(i)} = \left(1 - m + mA_{kl}\right) Y_{(i)} + m\left(\overline{Y}_{(i-1)} - Y_{(i-1)} A_{kl}\right), \tag{3.103}$$

wherein m – the non-local parameter controlling the size of the localized plastic zone and distribution of the plastic strain and A_{kl} - the component of a non-local matrix

$$A_{kl} = \frac{\omega\left(\left\| x^{k} - x^{l} \right\|\right) V\left(x^{l}\right)}{\sum\limits_{j=1} \omega\left(\left\| x^{k} - x^{j} \right\|\right) V\left(x^{j}\right)}\,, \tag{3.104}$$

where $V(x^l)$ is the volume associated to the integration point l. In the third model, the rates of the softening parameter were averaged according to the Brinkreve's formula (Eq. 3.97) during both tension and compression. Finally, in the improved coupled formulation (model '4'), the non-locality was introduced similarly as in the model '1' i.e. local plasticity was combined with non-local damage. However another possibility non-local plasticity combined with local damage was also considered. In this case for both tension and compression, the non-locality was applied according to the Brinkreve's formula (Eq. 3.97).

A numerical problem in non-local elasto-plastic models is the way how to calculate non-local terms since the plastic rates are unknown in advance. The plastic strain rates can be approximated by the total strain rates $d\varepsilon$ (Brinkgreve 1994) or calculated iteratively in an exact way according to the algorithm given by Strömberg and Ristinmaa (1996), and Jirásek and Rolshoven (2003). To simplify the calculations, the non-local rates were replaced by their approximation $\Delta\kappa_i^{est}$ calculated on the basis of the known total strain increment values (Brinkgreve 1994):

$$\Delta\overline{\kappa}_i\left(\boldsymbol{x}\right) \approx \Delta\kappa_i(x) + m\left(\frac{\int_V \omega(\left\| x - \xi \right\|) \Delta\kappa_i^{est}(\xi)\, d\xi}{\int_V \omega(\left\| x - \xi \right\|)\, d\xi} - \Delta\kappa_i^{est}(x)\right), \quad i=1,2 \tag{3.105}$$

The plastic strain rates can be approximated by the total strain rates $d\varepsilon$. Eq. 3.105 enables one to 'freeze' the non-local influence of the neighbouring points and to determine the actual values of the softening parameters using the same procedures as in a local formulation. The strain rates can be calculated in all integration points of the specimen, in the integration points where only plastic strains occur or only in the integration points where both plastic strains and softening simultaneously occur. The FE-results show an insignificant influence of the calculation method of non-local plastic strain rates. An approximate method proposed by Brinkgreve (1994) is less time consuming (by ca.30%) (Bobiński and Tejchman 2004).

3.3.2 Second-Gradient Approach

Second gradient models have often been used for ductile materials (metals) (Fleck and Hutchinson 1993, Menzel and Steinmann 2000), quasi-brittle materials (rock, concrete) (Sluys 1992, and Pamin 1994) and granular materials (Vardoulakis and Aifantis 1991, Chambon et al. 2001, Maier 2002, Tejchman 2004). The gradient terms are thought to reflect the fact that below a certain size scale the interaction between the micro-structural carriers of deformation is non-local (Aifantis 2003). The constitutive models capture gradients in different ways. They usually involve the second gradient of a plastic strain measure (Laplacian) in the yield or potential function (plasticity) or in the damage function (damage mechanics). The plastic multiplier which is connected to the plastic strain measure is considered as a fundamental unknown and is computed at global level simultaneously with the displacement degrees of freedom (de Borst and Mühlhaus 1992) (in the classical theory of plasticity, the plastic multiplier is determined from a simple algebraic equation, Chapter 3). Such gradient model obviously requires a C^1-continuous interpolation of the plastic multiplier field. This requirement is fulfilled by e.g. element with the 8-nodal serendipity interpolation of displacements and 4-nodal Hermitian interpolation of plastic strain with 2×2 Gaussian integration (Pamin 1994). Alternatively, all strain gradients can be taken into account (Zervos et al 2001). The stress is conjugate to the strain rate, and the so-called double stress is conjugate to its gradient. To preserve that the derivatives are continuous across two-dimensional element boundaries, a triangular element of C^1 continuity with 36 degrees of freedom can be used (Maier 2002). The degrees of freedom at each node for each displacement are the displacement itself, its both first order derivatives and all three second order derivatives. The model requires a relationship between the double stress and strain gradient. The gradient terms can be evaluated not only by using additional complex shape functions but also by applying explicit method in the form of a standard central difference scheme.

Gradient–type regularization can be derived from non-local models. By expanding an arbitrary state variable $\kappa(x+r)$ into a Taylor series around the point r=0, choosing the error function ω as the weighting function and neglecting the terms higher than the second order, the following relationship is obtained for a non-local gradient of κ for one-dimensional problems (Pamin 1994):

$$\kappa^*(x) = \frac{1}{A}[\int_{-\infty}^{\infty} \frac{1}{l_c\sqrt{\pi}} e^{-(r/l)^2} \kappa(x) dr + \int_{-\infty}^{\infty} \frac{r}{l_c\sqrt{\pi}} e^{-(r/l_c)^2} \frac{\partial \kappa(x)}{\partial x} dr +$$
$$\int_{-\infty}^{\infty} \frac{r^2}{2l_c\sqrt{\pi}} e^{-(r/l_c)^2} \frac{\partial^2 \kappa(x)}{\partial x^2} dr + \int_{-\infty}^{\infty} \frac{r^3}{6l_c\sqrt{\pi}} e^{-(r/l_c)^2} \frac{\partial^3 \kappa(x)}{\partial x^3} dr + \tag{3.106}$$
$$\int_{-\infty}^{\infty} \frac{r^4}{24l_c\sqrt{\pi}} e^{-(r/l_c)^2} \frac{\partial^4 \kappa(x)}{\partial x^4} dr + ...] = \kappa + l_c \frac{\partial \kappa}{\partial x} + \frac{l_c^2}{4} \frac{\partial^2 \kappa}{\partial x^2}.$$

For two-dimensional problems, the enhanced variable κ^* is

$$\kappa^*(x,y) = \kappa + l_c(\frac{\partial \kappa}{\partial x} + \frac{\partial \kappa}{\partial y}) + \frac{l_c^2}{4}(\frac{\partial^2 \kappa}{\partial x^2} + \frac{\partial^2 \kappa}{\partial y^2} + 2\frac{\partial^2 \kappa}{\partial x \partial y}). \tag{3.107}$$

The odd derivative can be canceled because of the implicit assumption of isotropy (de Borst et al. 1992). Thus, the enhanced variable κ^* is equal to

$$\kappa^*(x,y) = \kappa + \frac{l_c^2}{4}(\frac{\partial^2 \kappa}{\partial x^2} + \frac{\partial^2 \kappa}{\partial y^2} + 2\frac{\partial^2 \kappa}{\partial x \partial y}). \tag{3.108}$$

Instead of using complex shape functions to describe the evolution of the second gradient of κ, a central difference scheme was applied (di Prisco et al. 2002). The advantages of such approach are: simplicity of computation, little effort to modify each commercial FE-code and high computation efficiency. To take into account the effect of not only adjacent elements (as in the standard difference method), one assumed in the book a polynomial interpolation of the function κ of the fourth order in both directions:

$$\kappa(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E, \tag{3.109}$$

$$\kappa(y) = Ay^4 + By^3 + Cy^2 + Dy + E, \tag{3.110}$$

where *A, B, C, D* and *E* are constants. From the theory of a finite difference method (when the difference steps dx and dy are infinitesimal), the second derivatives of the variable κ can be approximated in each triangular element of the quadrilateral composed of 4 triangles, e.g. in the triangle '13' of Fig. 3.8 (for a mesh regular in the vertical and horizontal direction) as

$$\frac{\partial^2 \kappa_{13}}{\partial x^2} = \frac{1}{dx^2}\left[-\frac{1}{12}\kappa_3 + \frac{16}{12}\kappa_8 - \frac{30}{12}\kappa_{13} + \frac{16}{12}\kappa_{18} - \frac{1}{12}\kappa_{23}\right], \tag{3.111}$$

$$\frac{\partial^2 \kappa_{13}}{\partial y^2} = \frac{1}{dy^2}\left[-\frac{1}{12}\kappa_{11} + \frac{16}{12}\kappa_{12} - \frac{30}{12}\kappa_{13} + \frac{16}{12}\kappa_{14} - \frac{1}{12}\kappa_{15}\right], \tag{3.112}$$

$$\begin{aligned}\frac{\partial^2 \kappa_{13}}{\partial x \partial y} = \frac{1}{dxdy}\Big[&\frac{1}{12}\Big(\frac{1}{12}\kappa_1 - \frac{8}{12}\kappa_2 + \frac{8}{12}\kappa_4 - \frac{1}{12}\kappa_5\Big) - \\ &\frac{8}{12}\Big(\frac{1}{12}\kappa_6 - \frac{8}{12}\kappa_7 + \frac{8}{12}\kappa_9 - \frac{1}{12}\kappa_{10}\Big) + \\ &\frac{8}{12}\Big(\frac{1}{12}\kappa_{16} - \frac{8}{12}\kappa_{17} + \frac{8}{12}\kappa_{19} - \frac{1}{12}\kappa_{20}\Big) - \\ &\frac{1}{12}\Big(\frac{1}{12}\kappa_{21} - \frac{8}{12}\kappa_{22} + \frac{8}{12}\kappa_{24} - \frac{1}{12}\kappa_{25}\Big)\Big]\end{aligned}, \tag{3.113}$$

where the lower subscript at the variable κ denotes the number of the triangular element in the specified quadrilateral (Fig. 3.8), and *dx* and *dy* are the distances between the triangle centres in the neighbouring quadrilaterals in a horizontal and vertical direction, respectively. The calculations of the second derivatives of the variable κ in other triangles are similar. Thus, the effect of neighbouring elements near each element is taken into account. In FE calculations, the mixed derivative (Eq. 3.113) was neglected to reproduce the Laplacian of the variable κ only.

The advantage of a gradient approach is that it is suitable (as a non-local approach) for both shear and tension (decohesion) dominated applications. The explicit second-gradient strain isotropic damage approach (Eqs. 3.111 and 3.112) was used for reinforced concrete beams under monotonic loading.

The non-local and second-gradient model were implemented in the commercial finite element code ABAQUS (1998) for efficient computations. Such implementation can be performed with two methods. In the first one, two identical overlapping meshes are used. The first mesh allows to gather the information about coordinates of integration points in the entire specimen, area of all finite elements and total strain rates in each element. The elements in this mesh are defined by the user in the UEL procedure. They do not influence the results of stresses in the specimen body since they have no stiffness. The information stored is needed to calculate non-local variables with the aid of the second mesh which includes standard elements from the ABAQUS library (1998). The constitutive law is defined by the UMAT procedure. During odd iterations, the information is gathered in the elements of the first mesh. During even iterations, the stresses in the elements of the second mesh (including standard elements) are determined with taking into account non-local variables and a non-linear finite element equation is solved. Between odd and even iterations, the same element configuration is imposed. In the second method, only one mesh is used which contains user's elements (defined by the UEL procedure). During odd iterations, the information about the elements is stored, and during even iterations, the stresses within a non-local theory are determined. As compared to the first method, the second one consumes less time. However, it is less comfortable for the user due to the need of the definition of the stiffness matrix and out-of-balance load vector in finite elements.

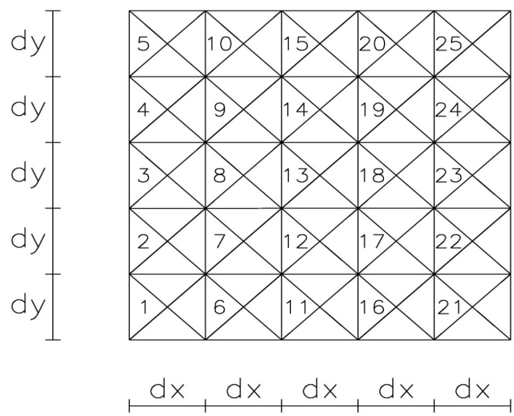

Fig. 3.8 Diagram for determination of the gradient of the constitutive variable κ in triangular finite elements using a central difference method (Tejchman 2004)

For the solution of the non-linear equation of motion governing the response of a system of finite elements, the initial stiffness method was used with a symmetric elastic global stiffness matrix instead of applying a tangent stiffness matrix (the choice was governed by access limitations to the commercial software ABAQUS (1998). To satisfy the consistency condition f=0 in elasto-plasticity, the trial stress method (linearized expansion of the yield condition about the trial stress point) using an elastic predictor and a plastic corrector with the return mapping algorithm (Ortiz and Simo 1986) was applied. The calculations were carried out using a large-displacement analysis (ABAQUS 1998). In this method, the current configuration of the body was taken into account. The Cauchy stress was taken as the stress measure. The conjugate strain rate was the rate of deformation. The rotations of the stress and strain tensor were calculated with the Hughes-Winget method (Hughes and Winget 1988). The non-local averaging was performed in the current configuration. This choice was governed again by the fact that element areas in this configuration were automatically calculated by ABAQUS (1998).

3.4 Bond-Slip Laws

Different bond-slip laws were assumed when modelling reinforced concrete elements. However, there does not exist an universal slip-bond law for reinforced concrete elements since it depends upon boundary conditions of the entire system (Chapter 2). To consider bond-slip, an interface with a zero thickness was assumed along a contact, where a relationship between the shear traction and slip was introduced. In the book, 4 different bond-slip laws were applied.

First, the simplest bond-slip proposed by Dörr (1980) without softening was used (Fig. 3.9)

$$\tau = f_t \left[0.5\left(\frac{u}{u_1}\right) - 4.5\left(\frac{u}{u_1}\right)^2 + 1.4\left(\frac{u}{u_1}\right)^3 \right] \quad \text{if} \quad 0<u\leq u_0, \tag{3.114}$$

$$\tau = \tau_{\max} = 1.9 f_t \quad \text{if} \quad u>u_1, \tag{3.115}$$

wherein τ denotes the bond stress, τ_{max} is the bond resistance, f_t is the tensile strength of concrete and u_1 is the displacement at which perfect slip occurs (u_1=0.06 mm).

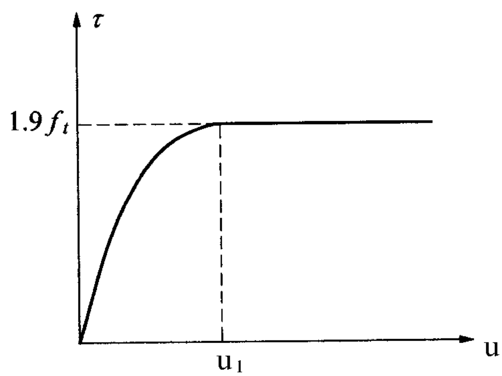

Fig. 3.9 Bond-slip law between concrete and reinforcement by Dörr (1980)

Second, the bond-slip law suggested by CEB-FIB (1992) was applied (Fig. 3.10)

$$\tau = \tau_{\max}\left(\frac{u}{u_1}\right)^{0.4}, \qquad u < u_1, \tag{3.116}$$

$$\tau = \tau_{\max}, \qquad u_1 \leq u \leq u_2, \tag{3.117}$$

$$\tau = \tau_{\max} - (\tau_{\max} - \tau_{res})\frac{u - u_2}{u_3 - u_2}, \qquad u_2 \le u \le u_3, \tag{3.118}$$

$$\tau = \tau_{res}, \qquad u_3 \le u, \tag{3.119}$$

where u_1=0.06 mm.

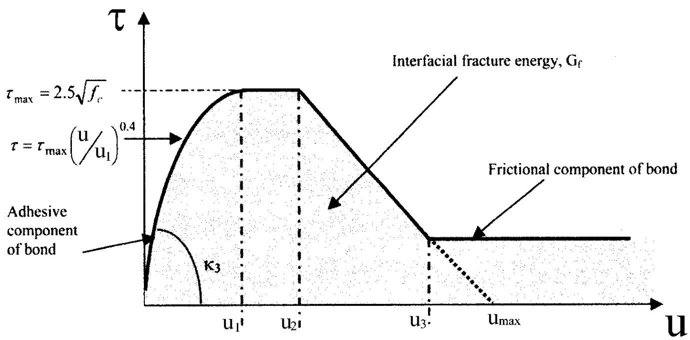

Fig. 3.10 Bond stress-slip relationship between concrete and reinforcement by CEB-FIP (1992)

Third, the bond-slip law by Haskett et al. (2008) on the basis of Eqs. 3.116-3.119 was used (Fig. 3.11)

$$\tau = \tau_{\max}\left(\frac{u}{u_1}\right)^{0.4}, \qquad 0 \le u \le u_1, \tag{3.120}$$

where u_1=1.5 mm is slip corresponding to the peak.

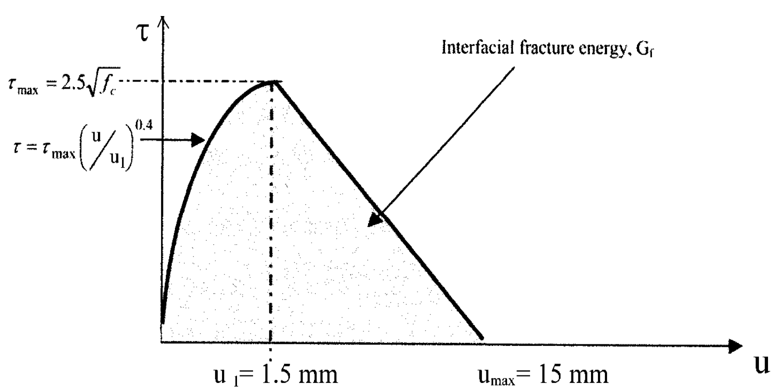

Fig. 3.11 Bond-slip relationship between concrete and reinforcement proposed by Haskett et al. (2008)

Finally, the bond-slip law was used which distinguishes pull-out failure where the bond strength is in the contact zone is exceeded, and a splitting failure which is caused by an insufficient concrete cover throughout (due to occurrence of radial cracks). Den Uijl and Bigaj (1996) and Akkermann (2000) proposed a bond model for ribbed bars based on concrete confinement. The bond model is formulated in the terms of a radial stress-radial strain relation (Fig. 3.12). The radial stresses are equal to the bond stresses. For the splitting failure, the radial strains are linear dependent on the slip, and for the pull-out failure, they are nonlinear dependent. If the radial stresses σ_r are smaller than the maximum slip stresses $\tau_{max}=5f_t$, a splitting failure takes places ($\tau_{max}/\sigma_r>1$), otherwise a pull-out failure takes place ($\tau_{max}/\sigma_r\leq 1$). The maximum radial stress and strain are at failure, respectively

$$\sigma_{r,\max} = 2 f_t (\frac{c}{d_r})^{0.88}, \qquad \varepsilon_{r,\max} = 4.22 \frac{f_t}{E_o} (\frac{c}{d_r})^{1.08}, \tag{3.121}$$

where E_o is the modulus of elasticity and c denotes the concrete cover. The bond stress is coupled with the radial stress by the friction angle.

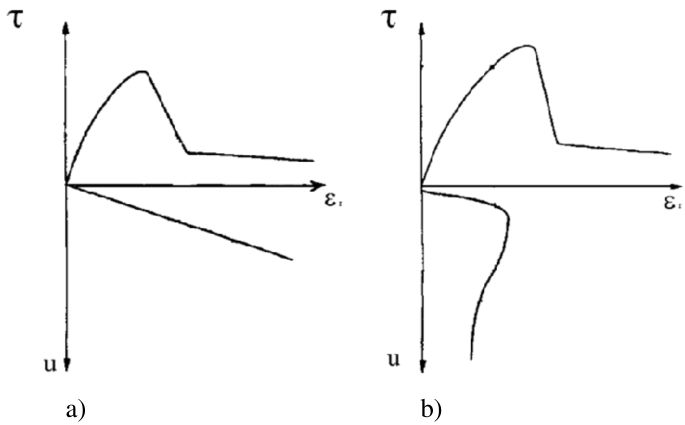

Fig. 3.12 Selected bond-slip laws between concrete and reinforcement: a) splitting failure, b) pull-out failure (τ – bond stress, u – slip, ε_r – radial strain) (den Uijl and Bigaj 1996)

References

ABAQUS. Theory Manual, Version 5.8, Hibbit, Karlsson & Sorensen Inc. (1998)

Aifantis, E.C.: Update on class of gradient theories. Mechanics of Materials 35(3), 259–280 (2003)

Akkermann, J.: Rotationsverhalten von Stahlbeton-Rahmenecken. PhD Thesis, Universität Fridericiana zu Karlsruhe, Karlsruhe (2000)

Asferg, P.N., Poulsen, P.N., Nielsen, L.O.: Modelling of crack propagation in concrete applying the XFEM. In: Meschke, G., de Borst, R., Mang, H., Bicanic, N. (eds.) Computational Modelling of Concrete Structures, EURO-C, pp. 33–42. Taylor and Francis (2006)

Askes, H., Sluys, L.J.: A classification of higher-order strain gradient models in damage mechanics. Archive of Applied Mechanics 53(5-6), 448–465 (2003)

Bažant, Z.P., Bhat, P.D.: Endochronic theory of inelasticity and failure of concrete. Journal of Engineering Mechanics Division ASCE 102(4), 701–722 (1976)

Bažant, Z.: Endochronic inelasticity and incremental plasticity. International Journal of Solids and Structures 14(9), 691–714 (1978)

Bažant, Z.P., Shieh, C.L.: Endochronic model for nonlinear triaxial behaviour of concrete. Nuclear Engineering and Design 47(2), 305–315 (1978)

Bažant, Z.P., Cedolin, L.: Blunt crack band propagation in finite element analysis. Journal of Engineering Mechanics Division ASCE 105(2), 297–315 (1979)

Bažant, Z.P., Oh, B.H.: Crack band theory for fracture of concrete. Materials and Structures RILEM 16(93), 155–177 (1983)

Bažant, Z.P., Lin, F.: Non-local yield limit degradation. International Journal for Numerical Methods in Engineering 26(8), 1805–1823 (1988)

Bažant, Z., Ožbolt, J.: Non-local microplane model for fracture, damage and size effect in structures. Journal of Engineering Mechanics ASCE 116(11), 2485–2505 (1990)

Bažant, Z., Jirásek, M.: Nonlocal integral formulations of plasticity and damage: survey of progress. Journal of Engineering Mechanics 128(11), 1119–1149 (2002)

Belytschko, T., Moes, N., Usui, S., Parimi, C.: Arbitrary discontinuities in finite elements. International Journal for Numerical Methods in Engineering 50(4), 993–1013 (2001)

Belytschko, T., Gracie, R., Ventura, G.: A review of extended/generalized finite element methods for material modeling. Modelling and Simulation in Material Science and Engineering 17(4), 1–24 (2009)

Bhandari, A.R., Inoue, J.: Experimental study of strain rates effects on strain localization characteristics of soft rocks. Soils and Foundations 45(1), 125–140 (2005)

Bobiński, J., Tejchman, J.: Numerical simulations of localization of deformation in quasi-brittle materials within non-local softening plasticity. Computers and Concrete 1(4), 433–455 (2004)

Bobiński, J., Tejchman, J.: Modelling of strain localization in quasi-brittle materials with a coupled elasto-plastic-damage model. International Journal for Theoretical and Applied Mechanics 44(4), 767–782 (2006)

Bobiński, J.: Implementacja i przykłady zastosowań nieliniowych modeli betonu z nielokalnym osłabieniem. PhD Thesis, Gdańsk University of Technology (2006)

Bobiński, J., Tejchman, J.: Continuous and discontinuous modeling of cracks in concrete elements. In: Bicanic, N., de Borst, R., Mang, H., Meschke, G. (eds.) Modelling of Concrete Structures, pp. 263–270. Taylor and Francis Group, London (2010)

Bolander, J.E., Sukumar, N.: Irregular lattice model for quasi-static crack propagation. Physical Review B 71(9), 094106(2005)

Borino, G., Failla, B., Parrinello, F.: A symmetric nonlocal damage theory. International Journal of Solids and Structures 40(13-14), 3621–3645 (2003)

Brinkgreve, R.B.J.: Geomaterial models and numerical analysis of softening. Phd Thesis, Delft University of Technology (1994)

Caballero, A., Carol, I., Lopez, C.M.: New results in 3d meso-mechanical analysis of concrete specimens using interface elements. In: Meschke, G., de Borst, R., Mang, H., Bicanic, N. (eds.) Computational Modelling of Concrete Structures, EURO-C, pp. 43–52. Taylor and Francis (2006)

Caballero, A., Carol, I., Lopez, C.M.: 3D meso-mechanical analysis of concrete specimens under biaxial loading. Fatigue and Fracture of Engineering Materials and Structures 30(9), 877–886 (2007)

Carol, I., López, C.M., Roa, O.: Micromechanical analysis of quasi-brittle materials using fracture-based interface elements. International Journal for Numerical Methods in Engineering 52(1-2), 193–215 (2001)

Carol, I., Willam, K.: Spurious energy dissipation/generation in stiffness recovery models for elastic degradation and damage. International Journal of Solids and Structures 33(20-22), 2939–2957 (1996)

Carol, I., Rizzi, E., Willam, K.: On the formulation of anisotropic elastic degradation. International Journal of Solids and Structures 38(4), 491–518 (2001)

CEB-FIP Model Code 90, London (1992)

Chambon, R., Caillerie, D., Matsuchima, T.: Plastic continuum with microstructure, local second gradient theories for geomaterials: localization studies. International Journal of Solids and Structures 38(46-47), 8503–8527 (2001)

Chen, E.: Non-local effects on dynamic damage accumulation in brittle solids. International Journal for Numerical and Analytical Methods in Geomechanics 23(1), 1–21 (1999)

Chen, J., Yuan, H., Kalkhof, D.: A non-local damage model for elastoplastic materials based on gradient plasticity theory. Report Nr.01-13, Paul Scherrer Institut, 1–130 (2001)

Cope, R.J., Rao, P.V., Clark, L.A., Norris, P.: Modeling of concrete reinforced behaviour for finite element analysis of bridge slabs. Numerical Methods for Non-linear Problems 1, 457–470 (1980)

Cusatis, G., Bažant, Z.P., Cedolin, L.: Confinement-shaer lattice model for concrete damage in tension and compression: I. Theory. Journal for Engineering Mechanics ASCE 129(12), 1439–1448 (2003)

de Borst, R., Nauta, P.: Non-orthogonal cracks in a smeared finite element model. Engineering Computations 2(1), 35–46 (1985)

de Borst, R.: Non-linear analysis of frictional materials. Phd Thesis, University of Delft (1986)

de Borst, R., Mühlhaus, H.-B., Pamin, J., Sluys, L.B.: Computational modelling of localization of deformation. In: Owen, D.R.J., Onate, H., Hinton, E. (eds.) Proc. of the 3rd Int. Conf. Comp. Plasticity, Swansea, pp. 483–508. Pineridge Press (1992)

de Borst, R., Mühlhaus, H.-B.: Gradient dependent plasticity: formulation and algorithmic aspects. International Journal for Numerical Methods in Engineering 35(3), 521–539 (1992)

de Borst, R., Pamin, J.: Some novel developments in finite element procedures for gradient-dependent plasticity. International Journal for Numerical Methods in Engineering 39(14), 2477–2505 (1996)

de Vree, J.H.P., Brekelmans, W.A.M., van Gils, M.A.J.: Comparison of nonlocal approaches in continuum damage mechanics. Computers and Structures 55(4), 581–588 (1995)

den Uijl, J.A., Bigaj, A.: A bond model for ribbed bars based on concrete confinement. Heron 41(3), 201–226 (1996) di Prisco, C., Imposimato, S., Aifantis, E.C.: A visco-plastic constitutive model for granular soils modified according to non-local and gradient approaches. International Journal for Numerical and Analytical Methods in Geomechanic 26(2), 121–138 (2002)

D'Addetta, G.A., Kun, F., Ramm, E.: On the application of a discrete model to the fracture process of cohesive granular materials. Granular Matter 4(2), 77–90 (2002)

Donze, F.V., Magnier, S.A., Daudeville, L., Mariotti, C.: Numerical study of compressive behaviour of concrete at high strain rates. Journal of Engineering Mechanics ASCE 125(10), 1154–1163 (1999)

Dörr, K.: Ein Beitag zur Berechnung von Stahlbetonscheiben unter Berücksichtigung des Verbundverhaltens. Phd Thesis, Darmstadt University (1980)

Dragon, A., Mróz, Z.: A continuum model for plastic-brittle behaviour of rock and concrete. International Journal of Engineering Science 17(2) (1979)

Etse, G., Willam, K.: Fracture energy formulation for inelastic behaviour of plain concrete. Journal of Engineering Mechanics ASCE 120(9), 1983–2011 (1994)

Ferrara, I., di Prisco, M.: Mode I fracture behaviour in concrete: nonlocal damage modeling. ASCE Journal of Engineering Mechanics 127(7), 678–692 (2001)

Fleck, N.A., Hutchinson, J.W.: A phenomenological theory for strain gradient effects in plasticity. Journal of Mechanics and Physics of Solids 41(12), 1825–1857 (1993)

Gatuingt, G., Pijaudier-Cabot, H.: Coupled damage and plasticity modelling in transient dynamic analysis of concrete. International Journal for Numerical and Analytical Methods in Geomechanics 26(1), 1–24 (2002)

Geers, M., Peijs, T., Brekelmans, W., de Borst, R.: Experimental monitoring of strain localization and failure behaviour of composite materials. Composites Science and Technology 56(11), 1283–1290 (1996)

Geers, M.G.D.: Experimental analysis and computational modeling of damage and fracture. PhD Thesis, Eindhoven University of Technology, Eindhoven (1997)

Geers, M., de Borst, R., Brekelmans, W., Peerlings, R.: Strain-based transient-gradient damage model for failure analyses. Computer Methods in Applied Mechanics and Engineering 160(1-2), 133–154 (1998)

Grassl, P., Jirásek, M.: Damage-plastic model for concrete failure. International Journal of Solids and Structures 43(22-23), 7166–7196 (2006)

Hansen, N.R., Schreyer, H.L.: A thermodynamically consistent framework for theories of elastoplasticity coupled with damage. International Journal of Solids and Structures 31(3), 359–389 (1994)

Hansen, E., Willam, K.: A two-surface anisotropic damage-plasticity model for plane concrete. In: de Borst, R. (ed.) Procceedings Int. Conf. Fracture Mechanics of Concrete Materials, Paris, Balkema, pp. 549–556 (2001)

Haskett, M., Oehlers, D.J., Mohamed Ali, M.S.: Local and global bond characteristics of steel reinforcing bars. Engineering Structures 30(2), 376–383 (2008)

Häu*β*ler-Combe, U., Prochtel, P.: Ein dreiaxiale Stoffgesetz fur Betone mit normalen und hoher Festigkeit. Beton- und Stahlbetonbau 100(1), 56–62 (2005)

Hentz, S., Donze, F.V., Daudeville, L.: Discrete element modelling of concrete submitted to dynamic loading at high strain rates. Computers and Structures 82(29-30), 2509–2524 (2004)

Herrmann, H.J., Hansen, A., Roux, S.: Fracture of disordered elastic lattices in two dimensions. Physical Review B 39(1), 637–647 (1989)

Hillerborg, A., Modeer, M., Peterson, P.E.: Analysis of crack propagation and crack growth in concrete by means of fracture mechanics and finite elements. Cem. Concr. Res. 6, 773–782 (1976)

Hillerborg, A.: The theoretical basis of a method to determine the fracture energy of concrete. Materials and Structures 18(1), 291–296 (1985)

Hordijk, D.A.: Local approach to fatigue of concrete, PhD dissertation, Delft University of Technology (1991)

Hsieh, S.S., Ting, E.C., Chen, W.F.: Plasticity-fracture model for concrete. International Journal of Solids and Structures 18(3), 181–187 (1982)

Huerta, A., Rodriguez-Ferran, A., Morata, I.: Efficient and reliable non-local damage models. In: Kolymbas, D. (ed.). Lecture Notes in Applied and Computational Mechanics, vol. 13, pp. 239–268. Springer, Heidelberg (2003)

Hughes, T.J.R., Winget, J.: Finite Rotation Effects in Numerical Integration of Rate Constitutive Equations Arising in Large Deformation Analysis. International Journal for Numerical Methods in Engineering 15(12), 1862–1867 (1980)

Ibrahimbegovic, A., Markovic, D., Gatuing, F.: Constitutive model of coupled damage-plasticity and its finite element implementatation. Revue Européenne des Eléments Finis 12(4), 381–405 (2003)

Jirásek, M., Zimmermann, T.: Analysis of rotating crack model. ASCE Journal of Engineering Mechanics 124(8), 842–851 (1998)

Jirásek, M.: Nonlocal models for damage and fracture: comparison of approaches. International Journal of Solids and Structures 35(31-32), 4133–4145 (1998)

Jirásek, M.: Comments on microplane theory. In: Pijaudier-Cabot, G., Bittnar, Z., Gerard, B. (eds.) Mechanics of Quasi-Brittle Materials and Structures, pp. 55–77. Hermes Science Publications (1999)

Jirásek, M., Rolshoven, S.: Comparison of integral-type nonlocal plasticity models for strain-softening materials. International Journal of Engineering Science 41(13-14), 1553–1602 (2003)

Jirásek, M.: Non-local damage mechanics with application to concrete. In: Vardoulakis, I., Mira, P. (eds.) Failure, Degradation and Instabilities in Geomaterials, Lavoisier, pp. 683–709 (2004a)

Jirásek, M., Rolshoven, S., Grassl, P.: Size effect on fracture energy induced by non-locality. International Journal for Numerical and Analytical Methods in Geomechanics 28(7-8), 653–670 (2004b)

Jirásek, M., Marfia, S.: Non-local damage model based on displacement averaging. International Journal for Numerical Methods in Engineering 63(1), 77–102 (2005)

Karsan, D., Jirsa, J.O.: Behaviour of concrete under compressive loadings. Journal of the Structural Division (ASCE) 95(12), 2543–2563 (1969)

Katchanov, L.M.: Introduction to continuum damage mechanics. Martimus Nijhoff, Dordrecht (1986)

Khoei, A.R., Bakhshiani, A., Mofid, M.: An implicit algorithm for hypoelasto-plastic and hypoelasto-viscoplastic endochronic theory in finite strain isotropic–kinematic-hardening model. International Journal of Solids and Structures 40(13-14), 3393–3423 (2003)

Klisiński, M., Mróz, Z.: Description of non-elastic deformation and damage of concrete. Script, Technical University of Poznan, Poznan (1988) (in polish)

Kozicki, J., Tejchman, J.: Effect of aggregate structure on fracture process in concrete using 2D lattice model. Archives of Mechanics 59(4-5), 1–20 (2007)

Krajcinovic, D., Fonseka, G.U.: The continuous damage theory of brittle materials. Journal of Applied Mechanics ASME 48(4), 809–824 (1981)

Kuhl, E., Ramm, E.: Simulation of strain localization with gradient enhanced damage models. Computational Material Sciences 16(1-4), 176–185 (2000)

Lade, P.V., Jakobsen, K.P.: Incrementalization of a single hardening constitutive model for frictional materials. International Journal for Numerical and Analytical Methods in Geomechanics 26(7), 647–659 (2002)

Le Bellego, C., Dube, J.F., Pijaudier-Cabot, G., Gerard, B.: Calibration of nonlocal damage model from size effect tests. European Journal of Mechanics A/Solids 22(1), 33–46 (2003)

Lee, J., Fenves, G.L.: Plastic-damage model for cyclic loading of concrete structures. Journal of Engineering Mechanics 124(8), 892–900 (1998a)

Lee, J., Fenves, G.L.: A plastic-damage concrete model for earthquake analysis of dams. Earthquake Engng. and Struct. Dyn. 27, 937–956 (1998b)

Lemaitre, J.: Coupled elasto-plasticity and damage constitutive equations. Computer Methods in Applied Mechanics and Engineering 51(1-3), 31–49 (1985)

Lemaitre, J., Chaboche, J.L.: Mechanics of solid materials. Cambridge University Press, Cambridge (1990)

Lodygowski, T., Perzyna, P.: Numerical modelling of localized fracture of inelastic solids in dynamic loading process. International Journal for Numerical Methods in Engineering 40(22), 4137–4158 (1997)

Loret, B., Prevost, J.H.: Dynamic strain localisation in elasto-visco-plastic solids, Part 1. General formulation and one-dimensional examples. Computer Methods in Applied Mechanics and Engineering 83(3), 247–273 (1990)

Lubliner, J., Oliver, J., Oller, S., Onate, E.: A plastic-damage model for concrete. International Journal of Solids and Structures 25(3), 229–326 (1989)

Majewski, T., Bobiński, J., Tejchman, J.: FE-analysis of failure behaviour of reinforced concrete columns under eccentric compression. Engineering Structures 30(2), 300–317 (2008)

Mahnken, T., Kuhl, E.: Parameter identification of gradient enhanced damage models. European Journal of Mechanics - A/Solids 18(5), 819–835 (1999)

Maier, T.: Numerische Modellierung der Entfestigung im Rahmen der Hypoplastizität. PhD Thesis, University of Dortmund (2002)

Maier, T.: Nonlocal modelling of softening in hypoplasticity. Computers and Geotechnics 30(7), 599–610 (2003)

Marcher, T., Vermeer, P.A.: Macro-modelling of softening in non-cohesive soils. In: Vermeer, P.A., et al. (eds.) Continuous and Discontinuous Modelling of Cohesive-Frictional Materials, pp. 89–110. Springer, Heidelberg (2001)

Marzec, I., Bobiński, J., Tejchman, J.: Simulations of crack spacing in reinforced concrete beams using elastic-plasticity and damage with non-local softening. Computers and Concrete 4(5), 377–403 (2007)

Marzec, I.: Application of coupled elasto-plastic-damage models with non-local softening to concrete cyclic behaviour. PhD Thesis, Gdańsk University of Technology (2009)

Marzec, I., Tejchman, J.: Modeling of concrete behaviour under cyclic loading using different coupled elasto-plastic-damage models with non-local softening. In: Oñate, E., Owen, D.R.J. (eds.) X International Conference on Computational Plasticity COMPLAS X, pp. 1–4. CIMNE, Barcelona (2009)

Marzec, I., Tejchman, J.: Enhanced coupled elasto-plastic-damage models to describe concrete behaviour in cyclic laboratory tests. Archives of Mechanics (2011) (in print)

Mazars, J.: A description of micro- and macro-scale damage of concrete structures. Engineering Fracture Mechanics 25(5-6), 729–737 (1986)

Mazars, J., Pijaudier-Cabot: Continuum damage theory – applications to concrete. Journal of Engineering Mechanics ASCE 115(2), 345–365 (1989)

Meftah, F., Reynouard, J.M.: A multilayered mixed beam element on gradient plasticity for the analysis of localized failure modes. Mechanics of Cohesive-Frictional Materials 3(4), 305–322 (1998)

Menétrey, P., Willam, K.J.: Triaxial failure criterion for concrete and its generalization. ACI Structural Journal 92(3), 311–318 (1995)

Menzel, A., Steinmann, P.: On the continuum formulation of higher gradient plasticity for single and polycrystals. Journal of the Mechanics and Physics of Solids 48(8), 1777–1796 (2000)

Meschke, G., Lackner, R., Mang, H.A.: An anisotropic elastoplastic-damage model for plain concrete. International Journal for Numerical Methods in Engineering 42(4), 702–727 (1998)

Moonen, P., Carmeliet, J., Sluys, L.J.: A continuous-discontinuous approach to simulate fracture processes. Philosophical Magazine 88(28-29), 3281–3298 (2008)

Mühlhaus, H.-B.: Scherfugenanalyse bei Granularen Material im Rahmen der Cosserat-Theorie. Ingen. Archiv. 56, 389–399 (1986)

Mühlhaus, H.B., Aifantis, E.C.: A variational principle for gradient plasticity. International Journal of Solids and Structures 28(7), 845–858 (1991)

Needleman, A.: A continuum model for void nucleation by inclusion debonding. Journal of Applied Mechanics 54(3), 525–531 (1987)

Needleman, A.: Material rate dependence and mesh sensitivity in localization problems. Computer Methods in Applied Mechanics and Engineering 67(1), 69–85 (1988)

Nguyen, G.D.: A thermodynamic approach to constitutive modelling of concrete using damage mechanics and plasticity theory, PhD Thesis, Trinity College, University of Oxford (2005)

Nguyen, G.D.: A thermodynamic approach to non-local damage modelling of concrete. International Journal of Solids and Structures 45(7-8), 1918–1932 (2008)

Oliver, J., Huespe, A.E., Sanchez, P.J.: A comparative study on finite elements for capturing strong discontinuities: E-FEM vs X-FEM. Computer Methods in Applied Mechanics and Engineering 195(37-40), 4732–4752 (2006)

Ortiz, M., Simo, I.C.: An analysis of a new class of integration algorithms for elastoplastic constitutive relation. International Journal for Numerical Methods in Engineering 23(3), 353–366 (1986)

Ortiz, M., Pandolfi, A.: Finite-deformation irreversible cohesive elements for three-dimensional crack-propagation analysis. International Journal for Numerical Methods in Engineering 44(9), 1267–1282 (1999)

Ottosen, N.S.: A failure criterion for concrete. Journal of Engineering Mechanics Division ASCE 103(4), 527–535 (1977)

Ožbolt, J.: Maβstabseffekt und Duktulität von Beton- und Stahlbetonkonstruktionen. Habilitation, Universität Stuttgart (1995)

Palaniswamy, R., Shah, S.P.: Fracture and stress-strain relationship of concrete under triaxial compression. Journal of the Structural Division (ASCE) 100(ST5), 901–916 (1974)

Pamin, J.: Gradient-dependent plasticity in numerical simulation of localization phenomena. PhD Thesis, University of Delft (1994)

Pamin, J., de Borst, R.: Simulation of crack spacing using a reinforced concrete model with an internal length parameter. Arch. App. Mech. 68(9), 613–625 (1998)

Pamin, J., de Borst, R.: Stiffness degradation in gradient-dependent coupled damage-plasticity. Archives of Mechanics 51(3-4), 419–446 (1999)

Pamin, J.: Gradient-enchanced continuum models: formulation, discretization and applications. Habilitation, Cracow University of Technology, Cracow (2004)

Pedersen, R.R., Simone, A., Sluys, L.J.: An analysis of dynamic fracture in concrete with a continuum visco-elastic visco-plastic damage model. Engineering Fracture Mechanics 75(13), 3782–3805 (2008)

Peerlings, R.H.J., de Borst, R., Brekelmans, W.A.M., Geers, M.G.D.: Gradient enhanced damage modelling of concrete fracture. Mechanics of Cohesive-Frictional Materials 3(4), 323–342 (1998)

Perdikaris, P.C., Romeo, A.: Size effect on fracture energy of concrete and stability issues in three-point bending fracture toughness testing. ACI Materials Journal 92(5), 483–496 (1995)

Pietruszczak, D., Jiang, J., Mirza, F.A.: An elastoplastic constitutive model for concrete. International Journal of Solids and Structures 24(7), 705–722 (1988)

Pijaudier-Cabot, G., Bažant, Z.P.: Nonlocal damage theory. Journal of Engineering Mechanics ASCE 113(10), 1512–1533 (1987)

Pijaudier-Cabot, G.: Non-local damage. In: Mühlhaus, H.-B. (ed.) Continuum Models for Materials and Microstructure, pp. 107–143. John Wiley & Sons Ltd. (1995)

Pijaudier-Cabot, G., Haidar, K., Loukili, A., Omar, M.: Ageing and durability of concrete structures. In: Darve, F., Vardoulakis, I. (eds.) Degradation and Instabilities in Geomaterials, Springer, Heidelberg (2004)

Polizzotto, C., Borino, G., Fuschi, P.: A thermodynamically consistent formulation of nonlocal and gradient plasticity. Mechanics Research Communications 25(1), 75–82 (1998)

Polizzotto, C., Borino, G.: A thermodynamics-based formulation of gradient-dependent plasticity. Eur. J. Mech. A/Solids 17, 741–761 (1998)

Polizzotto, C.: Remarks on some aspects of non-local theories in solid mechanics. In: Proc. 6th National Congr. SIMAI, Chia Laguna, Italy, CD-ROM (2002)

Pramono, E.: Numerical simulation of distributed and localised failure in concrete. PhD Thesis, University of Colorado-Boulder (1988)

Pramono, E., Willam, K.: Fracture energy-based plasticity formulation of plain concrete. Journal of Engineering Mechanics ASCE 115(6), 1183–1204 (1989)

Ragueneau, F., Borderie, C., Mazars, J.: Damage model for concrete-like materials coupling cracking and friction. International Journal for Numerical and Analytical Methods in Geomechanics 5(8), 607–625 (2000)

Rashid, Y.R.: Analysis of prestressed concrete pressure vessels. Nuclear Engineering and Design 7(4), 334–344 (1968)

Reinhardt, H.W., Cornelissen, H.A.W., Hordijk, D.A.: Tensile tests and failure analysis of concrete. Journal of Structural Engineering ASCE 112(11), 2462–2477 (1986)

Rolshoven, S.: Nonlocal plasticity models for localized failure. PhD Thesis, École Polytechnique Fédérale de Lausanne (2003)

Rots, J.G.: Computational modeling of concrete fracture. PhD Thesis, Delft University (1988)

Rots, J.G., Blaauwendraad, J.: Crack models for concrete, discrete or smeared? Fixed, Multi-directional or rotating? Heron 34(1), 1–59 (1989)

Sakaguchi, H., Mühlhaus, H.B.: Mesh free modelling of failure and localisation in brittle materials. In: Asaoka, A., Adachi, T., Oka, F. (eds.) Deformation and Progressive Failure in Geomechanics, pp. 15–21 (1997)

Salari, M.R., Saeb, S., Willam, K., Patchet, S.J., Carrasco, R.C.: A coupled elastoplastic damage model for geomaterials. Computers Methods in Applied Mechanics and Engineering 193(27-29), 2625–2643 (2004)

Schlangen, E., Garboczi, E.J.: Fracture simulations of concrete using lattice models: computational aspects. Engineering Fracture Mechanics 57, 319–332 (1997)

Simo, J.C., Ju, J.W.: Strain- and stress-based continuum damage models. Parts I and II. International Journal of Solids and Structures 23(7), 821–869 (1987)

Simone, A., Sluys, L.J.: The use of displacement discontinuities in a rate-dependent medium. Computer Methods in Applied Mechanics and Engineering 193(27-29), 3015–3033 (2004)

Skarżyński, L., Tejchman, J.: Mesoscopic modelling of strain localization in concrete. Archives of Civil Engineering LV(4), 521–540 (2009)

Skarżynski, L., Syroka, E., Tejchman, J.: Measurements and calculations of the width of the fracture process zones on the surface of notched concrete beams. Strain 47(s1), 319–322 (2011)

Simone, A., Wells, G.N., Sluys, L.J.: Discontinuous Modelling of Crack Propagation in a Gradient Enhanced Continuum. In: Proc. of the Fifth World Congress on Computational Mechanics WCCM V, Vienna, CDROM (2002)

Sluys, L.J.: Wave propagation, localisation and dispersion in softening solids, PhD thesis, Delft University of Technology, Delft (1992)

Sluys, L.J., de Borst, R.: Dispersive properties of gradient and rate-dependent media. Mechanics of Materials 18(2), 131–149 (1994)

Strömberg, L., Ristinmaa, M.: FE-formulation of nonlocal plasticity theory. Computer Methods in Applied Mechanics and Engineering 136(1-2), 127–144 (1996)

Syroka, E.: Investigations of size effects in reinforced concrete elements. Phd Thesis (2011) (under preparation)

Tejchman, J., Wu, W.: Numerical study on patterning of shear bands in a Cosserat continuum. Acta Mechanica 99(1-4), 61–74 (1993)

Tejchman, J., Herle, I., Wehr, J.: FE-studies on the influence of initial void ratio, pressure level and mean grain diameter on shear localisation. International Journal for Numerical and Analitycal Methods in Geomechanics 23(15), 2045–2074 (1999)

Tejchman, J.: Influence of a characteristic length on shear zone formation in hypoplasticity with different enhancements. Computers and Geotechnics 31(8), 595–611 (2004)

Widulinski, L., Tejchman, J., Kozicki, J., Leśniewska, D.: Discrete simulations of shear zone patterning in sand in earth pressure problems of a retaining wall. International Journal of Solids and Structures 48(7-8), 1191–1209 (2011)

Willam, K.J., Warnke, E.P.: Constitutive model for the triaxial behaviour of concrete. In: IABSE Seminar on Concrete Structures Subjected to Triaxial Stress, Bergamo, Italy, pp. 1–31 (1975)

Willam, K., Pramono, E., Sture, S.: Fundamental issues of smeared crack models. In: Proceedings SEM/RILEM International Conference of Fracture of Concrete and Rock (1986)

Winnicki, A., Pearce, C.J., Bicanic, N.: Viscoplastic Hoffman consistency model for concrete. Computers and Structures 79(1), 7–19 (2001)

Winnicki, A.: Viscoplastic and internal discontinuity models in analysis of structural concrete. Habilitation, University of Cracow (2009)

Vardoulakis, I., Aifantis, E.C.: A gradient flow theory of plasticity for granular materials. Acta Mechanica 87(3-4), 197–217 (1991)

Vervuurt, A., van Mier, J.G.M., Schlangen, E.: Lattice model for analyzing steel-concrete interactions. In: Siriwardane, Zaman (eds.) Comp. Methods and Advances in Geomechanics, Balkema, Rotterdam, pp. 713–718 (1994)

Voyiadjis, G.Z., Taqieddin, Z.N., Kattan, P.: Theoretical Formulation of a Coupled Elastic-Plastic Anisotropic Damage Model for Concrete using the Strain Energy Equivalence Concept. International Journal of Damage Mechanics 18(7), 603–638 (2009)

Zbib, H.M., Aifantis, C.E.: A gradient dependent flow theory of plasticity: application to metal and soil instabilities. Applied Mechanics Reviews 42(11), 295–304 (1989)

Zervos, Z., Papanastasiou, P., Vardoulakis, I.: A finite element displacement formulation for gradient plasticity. International Journal for Numerical Methods in Engineering 50(6), 1369–1388 (2001)

Zhou, W., Zhao, J., Liu, Y., Yang, Q.: Simulation of localization failure with strain-gradient-enhanced damage mechanics. International Journal for Numerical and Analytical Methods in Geomechanics 26(8), 793–813 (2002)

Zhou, F., Molinari, J.F.: Dynamic crack propagation with cohesive elements: a methodology to address mesh dependency. International Journal for Numerical Methods in Engineering 59(1), 1–24 (2004)

Yang, Z., Xu, X.F.: A heterogeneous cohesive model for quasi-brittle materials considering spatially varying random fracture properties. Computer Methods in Applied Mechanics and Engineering 197, 4027–4039 (2008)