Distributed $(\Delta + 1)$-Coloring in the Physical Model

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Abstract. In multi-hop radio networks, such as wireless ad-hoc and sensor networks, nodes employ a MAC (Medium Access Control) protocol such as TDMA to coordinate accesses to the shared medium and to avoid interference of close-by transmissions. These protocols can be implemented using standard node coloring. The $(\Delta + 1)$-coloring problem is to color all nodes in as few timeslots as possible using at most $\Delta + 1$ colors such that any two nodes within distance R are assigned different colors, where R is a given parameter and Δ is the maximum degree of the modeled unit disk graph using the scaling factor R. Being one of the most fundamental problems in distributed computing, this problem is well studied and there are a long chain of algorithms for it. However, all previous work are based on models that are highly abstract, such as message passing models and graph based interference models, which limit the utility of these algorithms in practice.

In this paper, for the first time, we consider the distributed $\Delta + 1$-coloring problem under the more practical SINR interference model. In particular, without requiring any knowledge about the neighborhood, we propose a novel randomized $(\Delta + 1)$-coloring algorithm with time complexity $O(\Delta \log n + \log^2 n)$. For the case where nodes can not adjust their transmission power, we give an $O(\Delta \log^2 n)$ randomized algorithm, which only incurs a logarithmic multiplicative factor overhead.

1 Introduction

The node coloring problem underpins the design of interference avoidance mechanisms in many multi-hop radio networks including wireless ad-hoc and sensor networks. In these networks, radio communications are subject to interference, and messages may be lost due to interference. Without any interference avoidance mechanism, coordinating the nodes to achieve efficient and reliable communication is a time consuming task. Traditionally, nodes employ MAC (Medium Access Control) protocols to coordinate their accesses to the shared medium and to avoid interference of close-by transmissions, such as TDMA (Time Division Multiple-Access). These MAC protocols can all be reduced to the classical node coloring problem. For example, by assigning different colors to different time

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slots in a TDMA scheme, a proper coloring with parameter d corresponds to a MAC layer without "close-by" interference, i.e., no two nodes within distance d of each other transmit at the same time. In [3], it is shown that even under the complicated (but more realistic) SINR model, we can still implement an interference free TDMA-like MAC protocol by computing a proper coloring for a well defined d if we adopt a uniform power assignment. Conventionally, the node coloring problem is one of the most fundamental problems related to symmetry breaking, and therefore has attracted a great deal of attention in the distributed computing community.

Almost all previous work to derive distributed node coloring algorithms assume the graph based model in which interference is represented by a localized function—a message can be correctly received only if there are no other simultaneously transmitting senders in the receiver's neighborhood. However, in multi-hop radio networks, interference is cumulative and is caused by all simultaneously transmitting nodes, near by and far away. The physically based Signal-to-Interference-plus-Noise-Ratio (SINR) model [7] captures this reality in wireless networks more closely. Under the SINR model, the signal strength fades with distance to the power of some path-loss exponent α and a message can be successfully received if the ratio of the received signal strength and the sum of the interference caused by simultaneously transmitting nodes plus noise is above a certain hardware-defined threshold β.

1.1 Related Work

In the absence of global knowledge, to derive a $(\Delta+1)$-coloring in a distributed manner is challenging and has attracted much attention in the distributed computing community for more than two decades. The traditional message passing model was first considered. Since Cole and Vishkin presented the first distributed $(\Delta+1)$-coloring for rings in [2], a long line of papers were devoted to this problem. The state-of-the-art results are the $O(\Delta) + \frac{1}{2}\log^* n$ algorithm for arbitrary graphs in [1] and the optimal $O(\log^* n)$ algorithm for bounded-independence graphs in [14]. However, the message passing model abstracts away some crucial elements of wireless networks, such as interference, collision and asynchrony. Taking interference into account and assuming a locally synchronous circumstance, Schneider and Wattenhofer [15] proposed a distributed $(\Delta+1)$-coloring algorithm with running time $O(\Delta + \log \Delta \log n)$ and $O(\Delta + \log^2 n)$ with and without knowledge of Δ respectively. When further considering asynchrony, assuming prior knowledge of n and Δ, Moscibroda and Wattenhofer [10] gave an $O(\Delta \log n)$ distributed coloring algorithm for the simple unit disk graph model which only considers direct interferences from neighbors. In an extended version [11], the result was generalized for the bounded-independence graph. In a recent paper [3], Derbel and Talbi showed that the algorithm in [11] also works under the SINR model within the same time bound. However, all the above three algorithms need $O(\Delta)$ colors instead of at most $\Delta + 1$ colors.

In the SINR model, the interference is modeled as a global function, which makes the design of efficient distributed algorithms with global performance guarantee difficult. In spite of this, there have been many attempts in recent years. In [13], assuming that all nodes can perform physical carrier sensing, Scheideler et al. derived an $O(\log n)$ distributed algorithm for computing a constant approximate dominating set. The first distributed local broadcasting algorithm was presented by Goussevskaia et al. in [4] and the result is improved in a recent paper [17]. Kesselheim and Vöcking [8] considered the contention resolution problem and showed that their distributed algorithm is asymptotically optimal up to a $\log^2 n$ factor.

1.2 Our Contribution

To the best of our knowledge, this work is the first one considering the distributed $(\Delta + 1)$-coloring problem under the physical model. Without any knowledge on neighborhood, we give an $O(\Delta \log n + \log^2 n)$ time randomized distributed $(\Delta + 1)$-coloring algorithm for asynchronous wake-up multi-hop radio networks under the physical model. Our result even matches the coloring algorithm in [3] for large Δ, e.g., $\Delta \geq \log n$, which needs a linear estimate of Δ and uses $O(\Delta)$ colors. In our algorithm, we adopt a clustering coloring strategy, i.e., a Maximal Independent Set (MIS) is first computed, and then the nodes in the MIS assign colors for their neighbors. To make the strategy available, we first show that the MIS algorithm in [12] still works under the SINR model by carefully tuning the parameters. This algorithm is of independent interest, since it is the first MIS algorithm in the physical model.

Furthermore, if nodes can not adjust their transmission powers, we also give a distributed $(\Delta+1)$-coloring algorithm with time complexity $O(\Delta \log^2 n)$ by iteratively carrying out the MIS algorithm, which also does not need any knowledge on neighborhood.

2 Problem Definitions and Model

For two nodes u and v, we use $d(u, v)$ to denote the Euclidian distance between u and v. Given a distance parameter R, we say two nodes u and v are neighbors if $d(u, v) \leq R$. The neighborhood of a node v is the set of all its neighbors, denoted by $N(v)$. Additionally, we use $N[v]$ to denote the set $N(v) \cup \{v\}$. For a node v, we denote by Δ_v the number of nodes in v's neighborhood. We write $\Delta = \max_{v \in V} \Delta_v$. A set of nodes S is called an independent set if any two nodes of S are not in each other's neighborhood. A node coloring is proper if each set of nodes with the same color is an independent set, i.e., the distance between any two nodes with the same color is larger than R. Then the $(\Delta + 1)$-*coloring problem* is to color all nodes properly in as few timeslots as possible using at most $\Delta + 1$ colors.

In this work, we deal with unstructured radio networks [9]. In particular, nodes may wake up asynchronously or be woken up by incoming messages without access to a global clock. After waking up, nodes may start executing the algorithm at any time and nodes can not perform physical carrier sensing. The only prior knowledge given to nodes is a polynomial estimate of the number n of nodes in the network and nodes are clueless about the number of nodes in its close proximity. We also assume that every node v has a unique ID_v. Additionally, we assume that nodes are placed arbitrarily on the plane. We define a node v's running time as the length of the interval from the timeslot when v starts executing the algorithm to the timeslot when v quits the algorithm. The time complexity of the algorithm is the maximum of all the nodes' running times.

We adopt the practical physical model (or the SINR model) [7] in this paper. In particular, a message sent by node u to node v can be correctly received at v iff

$$\frac{\frac{P_u}{d(u,v)^\alpha}}{N + \sum_{w \in V \setminus \{u,v\}} \frac{P_w}{d(w,v)^\alpha}} \geq \beta, \tag{1}$$

where P_u (P_w) is the transmission power for node u (w), α is the path-loss exponent whose value is normally between 2 and 6, β is a hardware determined threshold value which is greater than 1, N is the ambient noise, and $\sum_{w \in V \setminus \{u,v\}} \frac{P_w}{d(w,v)^\alpha}$ is the interference experienced by the receiver v caused by all simultaneously transmitting nodes in the network.

The transmission range R_T of a node v can be defined as the maximum distance at which a node u can receive a clear transmission from v ($SINR \geq \beta$) when there are no other simultaneous transmissions in the network. From the SINR condition (1), $R_T \leq R_{max} = (\frac{P}{\beta \cdot N})^{1/\alpha}$ for the given power level P. We further assume that $R_T < R_{max}$ and define $R_T = (P/cN\beta)^{1/\alpha}$, where $c > 1$ is a constant determined by the environment.

In subsequent sections, when we say "an event occurs with high probability" we mean that the event occurs with probability $1 - n^{-c}$ for a constant $c > 0$, and "a node correctly get a color" means that the resulting coloring of the network is proper. Greek letters represent constants. The following Definition 1 and Lemma 1 will be used in the analysis of algorithms.

Definition 1. *For a node $v \in V$, the probabilistic interference at v, Ψ_v, is defined as the expected interference experienced by v in a certain timeslot t.*

$$\Psi_v = \sum_{u \in V \setminus \{v\}} \frac{P_u p_u}{d(u,v)^\alpha}, \tag{2}$$

where P_u is the transmission power and p_u is the sending probability of node u in timeslot t.

Lemma 1 ([4]). *Consider two disks D_1 and D_2 of radii R_1 and R_2, $R_1 > R_2$, we define $\chi(R_1, R_2)$ to be the smallest number of disks D_2 needed to cover the larger disk D_1. It holds that: $\chi(R_1, R_2) \leq \frac{2\pi}{3\sqrt{3}} \cdot \frac{(R_1+2R_2)^2}{R_2^2}$.*

3 An $O(\Delta \log n + \log^2 n)$ $(\Delta + 1)$-Coloring Algorithm

In this section, we give a distributed randomized coloring algorithm as described in Algorithm 1. It is assumed that every node v possesses a color list from which it chooses a color. Without loss of generality, we assume that all nodes' color lists are $\{0, 1, \ldots, n-1\}$, where n is the estimate of the number of nodes. Algorithm 1 has two main steps. A Maximal Independent Set (MIS) in terms of $3R$, i.e., every pair of nodes in the MIS has distance larger than 3R, is first computed; the nodes in this MIS are the leaders of their neighbors. Then by communicating with their neighbors within distance R, each leader decides when their neighbors can choose an available color. Without confusion, we will just call Algorithm 1 excluding the MIS algorithm as the coloring algorithm. In order to compute a maximal independent set, we first show that the distributed MIS algorithm in [12] still works under the SINR model by carefully tuning the parameters. Due to asynchrony, when some nodes execute the MIS algorithm, other nodes may be carrying out the coloring algorithm. Here we show that under such an asynchronous circumstance, the MIS algorithm can still correctly output an independent set in any timeslot with high probability. Due to the space limit, we put the description and the analysis of the MIS algorithm in the full version [16]. In addition, nodes adopt different transmission powers when executing different operations in Algorithm 1. Generally speaking, nodes adopt the transmission power of $P_M = c \cdot 3^\alpha N \beta R^\alpha$ when they execute the MIS algorithm and transmit a $StartTransmit$ message in state $\mathcal{G}$, while nodes adopt the transmission power of $P_C = cN\beta R^\alpha$ when they perform other operations. By the definition in Section 2, the transmission ranges of nodes are $3R$ and R for P_M and P_C, respectively.

There are four states in the coloring algorithm. After executing the MIS algorithm, all leaders in the computed independent set join state $\mathcal{G}$, while all nodes within distance $3R$ from these leaders join state $\mathcal{S}$. Then each node in $\mathcal{G}$ makes its neighbors within distance R join state $\mathcal{C}_1$. By continuously transmitting an *AskColor* message, each node in state $\mathcal{C}_1$ endeavors to acquire a *Grant* message from its leader. After receiving the *Grant* message from the leader, a node in state $\mathcal{C}_1$ joins state $\mathcal{C}_2$, in which it chooses a color that has not been chosen by its neighbors, and transmits its choice to all neighbors. Nodes still in state $\mathcal{S}$ keep silence so that they do not interfere with the coloring process of their neighbors. Next we describe Algorithm 1 in more details.

After waking up, a node v will first wait for at most $2\mu \log n$ timeslots. During the process, if v received a message $DoNotTransmit_u$, it enters state $\mathcal{S}$ and adds u into its forbidden set F_v. Otherwise, it starts executing the MIS algorithm after waiting for $2\mu \log n$ timeslots. After executing the MIS algorithm, each node will either join state $\mathcal{M}$ meaning that it is a member of the computed independent set, or join state $\mathcal{S}$. Here we must point out a difference of our MIS algorithm from that in [12] in state $\mathcal{M}$. In our algorithm, when a node v joins state $\mathcal{M}$, it first uses $\mu \log n$ timeslots to wake up all nodes within distance $3R$ by transmitting a message with constant probability. Then v transmits a $DoNotTransmit_v$ message forcing all nodes within distance $3R$ to join state $\mathcal{S}$. After doing this, v will join state $\mathcal{G}$ and start executing the coloring algorithm.

In the coloring algorithm, the leaders in state $\mathcal{G}$ first choose color 0 as its own color. Then they transmit a *StartColoring* message making their neighbors within distance R join state $\mathcal{C}_1$. While in state $\mathcal{G}$, a node v adds each of its neighbors that send an *AskColor* message to v into a set Q_v. If Q_v is not empty, it deletes the first node u from Q_v and transmits a $Grant_u$ message with constant probability for $2\mu \log n$ timeslots. We assign two counters c_v and b_v to each node v in state $\mathcal{G}$. In particular, c_v is used to count the number of timeslots that v has not received any *AskColor* message since the last one, while b_v is for counting the number of *Grant* messages that have been transmitted by v. These two counters are set for guaranteing that with high probability, v will not quit the algorithm until all neighbors have been colored. Then if Q_v is empty and $c_v > b_v \cdot 5\mu \log n + 3\mu \log^2 n + \mu \log n$, v quits the algorithm after transmitting a $StartTransmit_v$ message for $\mu \log n$ timeslots adopting power P_M. By doing so, v removes its restriction on nodes within distance $3R$ caused by the message $DoNotTransmit_v$.

For each node u in state $\mathcal{S}$, it will do nothing except listening. When u stays in state S, it adds the nodes that send *DoNotTransmit* messages to u into its forbidden set F_u, and it removes a node v from F_u if it receives a message $StartTransmit_v$. Node u will not leave state $\mathcal{S}$ until F_u is empty or it receives a *StartColoring* message from a leader node v. For the first case, u starts executing the MIS algorithm. For the second case, it joins state $\mathcal{C}_1$ and starts competing for the right of choosing a color. After joining state $\mathcal{C}_1$, node u starts transmitting an $AskColor_u$ message with a small initial transmission probability. Then if u did not receive any *Grant* message and did not change its transmission probability for $3\mu \log n$ timeslots, it doubles the transmission probability. While in state $\mathcal{C}_1$, if u receives a *Grant* message and the *Grant* message is not for u, it halves the transmission probability. By doing this, it is guaranteed that the sum of transmission probabilities in any local region of the network can be bounded with high probability, which helps bound the interference caused by simultaneously transmitting nodes. If the received *Grant* message is for u, it joins state $\mathcal{C}_2$. After joining $\mathcal{C}_2$, u chooses the first color remaining in its color list except color 0 and transmits a $Color_u$ message with constant probability for $\mu \log n$ timeslots to inform its neighbors of its choice. After waking up, each node will delete the color in the received *Color* message from its color list; hence it will not choose a color that has been chosen by its neighbors. In order to make sure that Algorithm 1 is correct with high probability, we assign $\mu = \frac{2^{\omega+8} \cdot 4^{3 \cdot 2^{1-\omega}} \cdot \chi(3^{1+2/(\alpha-2)} R_I + 3R, 0.5R)}{1-1/\rho}$, where ρ and R_I (Equation (3) below) are constants defined in the following analysis.

3.1 Analysis

In this section, we will show that with high probability, each node can correctly get a color after executing Algorithm 1 for $O(\Delta \log n + \log^2 n)$ timeslots, and the

Algorithm 1. $(\Delta + 1)$-Coloring

Initially, $p_v = \frac{2^{-\omega-1}}{n}; c_v = 0; b_v = 0; t_v = 0; Q_v = \emptyset; T_v = \emptyset; \omega = 6.4;$

Upon node v wakes up

wait for $2\mu \log n$ timeslots
if Received $DoNotTransmit_u$ from node u **then** add u into F_v; $state = \mathcal{S}$;
Else execute the MIS algorithm adopting transmission power P_M **end if**

Message Received

if Received $Color_w$ **then** delete the color in $Color_w$ from its color list **end if**

Node v in state $\mathcal{G}$

choose color 0;
for $\mu \log n$ timeslots **do** transmit $StartColoring_v$ adopting power P_C with probability $2^{-\omega}$ **end for**
if Q_v is not empty **then**
  $b_v = b_v + 1$;
  for $2\mu \log n$ timeslots **do** delete the first node u from Q_v and transmit $Grant_u$ adopting power P_C with probability $2^{-\omega}$; $c_v = c_v + 1$ **end for**
else $c_v = c_v + 1$ **end if**
if Q_v is empty and $c_v > b_v \cdot 5\mu \log n + 3\mu \log^2 n + \mu \log n$ **then**
  for $\mu \log n$ timeslots **do** transmit $StartTransmit_v$ adopting power P_M with probability $2^{-\omega}$ **end for**
  quit
end if

Message Received

if Received $AskColor_u$ **then** add u into Q_v; $c_v = 0$ **end if**

Node v in state $\mathcal{S}$

if F_v is empty **then** execute the MIS algorithm with power P_M **else** listen **end if**

Message Received

if Received $DoNotTransmit_w$ from node w **then** add w into F_v **end if**
if Received $Color_w$ **then** delete the color in $Color_w$ from its color list **end if**
if Received $StartTransmit_w$ from node w **then** delete w from F_v **end if**
if Received $StartColoring_w$ from node w **then** $state = \mathcal{C}_1$ **end if**

Node v in state $\mathcal{C}_1$

$t_v = t_v + 1$
if $t_v > 3\mu \log n$ **then** $p_v = 2P_v$; $t_v = 0$ **end if**
transmit $AskColor_v$ adopting power P_C with probability p_v;

Message Received

if received $Grant_v$ **then** $state = \mathcal{C}_2$ **end if**
if received $Grant_w$ for some node w that has not been received before **then** $p_v = p_v/2$; $t_v = 0$ **end if**
if Received $Color_w$ **then** delete the color in $Color_w$ from its color list **end if**

Node v in state $\mathcal{C}_2$

choose the first available color from its color list;
for $\mu \log n$ timeslots **do** transmit a message $Color_v$ containing its color adopting power P_C with probability $2^{-\omega}$ **end for**
quit;

total number of colors used is at most $\Delta+1$. We first give some definitions and notations that will be used in the subsequent analysis. A new parameter R_I is defined as follows, for bounding the interference.

$$R_I = R\left(2^{7-\omega}3^{\alpha+1}\sqrt{3}\pi\rho\beta \cdot \frac{1}{1-1/c} \cdot \frac{\alpha-1}{\alpha-2}\right)^{1/(\alpha-2)}, \tag{3}$$

where ρ is a constant larger than 1. We choose ρ such that $R_I > 2R$. Furthermore, we denote T_i, D_i and I_i as the disks centered at node i with radius R, $\frac{R}{2}$ and R_I, respectively. By E_i^r we denote the disk centered at node i with radius r. Without confusion, we also use T_i, D_i, I_i and E_i^r to denote the set of nodes in T_i, D_i, I_i and E_i^r, respectively.

Before analyzing Algorithm 1, we first give a lemma on the time complexity and the correctness of the MIS Algorithm, which is proved in the full version [16].

Lemma 2. *With probability $1-O(n^{-3})$, every node $v \in V$ decides whether it joins the computed independent set or state $\mathcal{S}$ after executing the MIS algorithm for at most $O(\log^2 n)$ timeslots. Furthermore, with probability at least $1-O(n^{-3})$, in any timeslot t, the independent set computed by the MIS algorithm is correct.*

The following property is also proved to be correct with probability at least $1-O(n^{-3})$ in the analysis of the MIS algorithm which is put in the full version [16].

Property 1. For any disk D_i and in any timeslot t throughout the execution of the algorithm, the sum of transmission probabilities of nodes that are executing the MIS algorithm is at most $3 \cdot 2^{-\omega}$.

In order to bound the interference, we present Property 2 which can be proved to be correct with probability at least $1-O(n^{-1})$ in Lemma 9.

Property 2. For any disk D_i and in any timeslot t throughout the execution of the algorithm,

(i) There is at most one node in state $\mathcal{C}_2$;

(ii) The sum of transmission probabilities of nodes in state $\mathcal{C}_1$ is at most $\sum_{u\in\mathcal{C}_1} \leq 2^{-\omega}$;

(iii) There is at most one node in state $\mathcal{G}$.

Based on Property 1, Property 2 and the transmission probability in each state, we can bound the sum of transmission probabilities as follows.

Lemma 3. *Assume that Property 1 and Property 2 hold. For any disk D_i and in any timeslot t throughout the execution of the algorithm, the sum of transmission probabilities can be bounded as $\sum_{v\in D_i} p_v \leq 3 \cdot 2^{1-\omega}$.*

In the subsequent lemma 4, we show that the interference by far-away nodes can be bounded by a constant, and then in Lemma 5, we give a sufficient condition for a successful transmission. The proofs of Lemma 4 and Lemma 5 are put in the full version [16].

Lemma 4. *Assume that Property 1 and Property 2 hold. Then for every node* u*, the probabilistic interference caused by nodes outside* I_u *can be bounded as:* $\Psi_u^{v \notin I_u} \leq \frac{(1-1/c)P_C}{\rho \beta R^\alpha}$.

Lemma 5. *Assume that Property 1 and Property 2 hold. If node* v *is the only sending node in* $E_v^{R_I+R}$*, with probability* $1 - \frac{1}{\rho}$*, the message sent by* v *will be received successfully by all nodes in* T_v*.*

Based on the sufficient condition for a successful transmission in Lemma 5, in the following Lemma 6, we show the successful transmissions of messages used in the algorithm in given timeslots with high probability. Then in Lemma 7, we state that with high probability, a leader will not quit the algorithm until all its neighbors have been colored.

Lemma 6. *Assume that Property 1 and Property 2 hold. Then with probability at least* $1 - \frac{1}{n^4}$*, the following results are correct:*

(i) *After entering state* $\mathcal{G}$*, a node* v *can successfully send a message StartColoring to all its neighbors in* $\mu \log n$ *timeslots.*

(ii) *A node* v *in state* $\mathcal{G}$ *can successfully send a message Grant to all its neighbors in* $\mu \log n$ *timeslots.*

(iii) *A node* v *in state* $\mathcal{G}$ *can successfully send a message StartTransmit to all nodes within distance* $3R$ *in* $\mu \log n$ *timeslots.*

(iv) *A node* v *in state* $\mathcal{C}_2$*, after choosing a color, can successfully send a message* $Color_v$ *to all neighbors in* $\mu \log n$ *timeslots.*

Proof. We only prove (i) here. (ii), (iii), (iv) can be proved similar to (i).

Proof of (i): As shown in Lemma 5, if v is the only sending node in $E_v^{R_I+R}$, with probability $1 - \frac{1}{\rho}$, the message *StartColoring* sent by v can be received successfully by all nodes in T_v. Let P_1 denote the event that v is the only sending node in $E_v^{R_I+R}$, then

$$\begin{aligned} P_1 = 2^{-\omega} \prod_{u \in E_v^{R_I+R} \setminus \{v\}} (1 - p_u) &\geq 2^{-\omega} \prod_{u \in E_v^{R_I+R}} (1 - p_u) \\ \geq 2^{-\omega} \cdot \left(\frac{1}{4}\right)^{\sum_{u \in E_v^{R_I+R}} p_u} &\geq 2^{-\omega} \cdot \left(\frac{1}{4}\right)^{3 \cdot 2^{1-\omega} \cdot \chi(R_I+R, 0.5R)} \end{aligned} \tag{4}$$

The last inequality is by Lemma 1 and Lemma 3. Then the probability P_{no} that v fails to transmit the message *StartColoring* to all nodes in T_v is at most

$$\begin{aligned} P_{no} &\leq \left(1 - (1 - 1/\rho) 2^{-\omega} \cdot \left(\frac{1}{4}\right)^{3 \cdot 2^{1-\omega} \cdot \chi(R_I+R, 0.5R)}\right)^{\mu \log n} \\ &\leq e^{-(1-1/\rho) 2^{-\omega} \mu \log n \cdot \left(\frac{1}{4}\right)^{3 \cdot 2^{1-\omega} \cdot \chi(R_I+R, 0.5R)}} \in n^{-4}. \end{aligned} \tag{5}$$

Lemma 7. *Assume that Property 1 and Property 2 hold. Then with probability at least* $1 - \frac{1}{n^4}$*, a node* v *in state* $\mathcal{G}$ *will not quit the algorithm until all its neighbors have been colored.*

Proof. Assume that v quits the algorithm in timeslot t when there are $d > 0$ neighbors staying in state $\mathcal{C}_1$. Denote the set of these d nodes as T. We further assume that v forces d_v neighbors joining state $\mathcal{C}_1$ after transmitting the $StartColoring_v$ message. Thus before time t, v has transmitted $(d_v - d)$ *Grant* messages. Then by Algorithm 1, v has not receive an *AskColor* message since the timeslot $t - ((d_v - d) \cdot 5\mu \log n + 3\mu \log^2 n + \mu \log n)$. Next we show that during the interval $[t - ((d_v - d) \cdot 5\mu \log n + 3\mu \log^2 n + \mu \log n), t)$, there is at least one node that can successfully transmit an *AskColor* message to v with high probability. Then v will not quit the algorithm in timeslot t. This contradiction completes the proof.

By Algorithm 1, the initial transmission probability of each node in T is assigned as $\frac{2^{-\omega-1}}{n}$, and each node in T will either doubles its transmission probability every $3\mu \log n$ timeslots, or received a *Grant* message from v and halves the transmission probability. Because v received the last *AskColor* message before the timeslot $t - ((d_v - d) \cdot 5\mu \log n + 3\mu \log^2 n + \mu \log n)$ and v transmits each *Grant* message for $2\mu \log n$ timeslots, v have completed the transmission of $(d_v - d)$ *Grant* messages by the timeslot $t - ((d_v - d) \cdot 5\mu \log n + 3\mu \log^2 n + \mu \log n) + 2(d_v - d)\mu \log n - 1$. So in timeslot $t^* = t - ((d_v - d) \cdot 3\mu \log n + 3\mu \log^2 n + \mu \log n)$, each node in T has transmission probability at least $\frac{2^{-\omega-1-d_v+d}}{n}$. From t^*, each node in T doubles its transmission probability every $3\mu \log n$ timeslots. In timeslot $t - \mu \log n$, each node in T has a constant transmission probability of $2^{-\omega-1}$. Then using a similar argument as in the proof of Lemma 6, we can get that with probability at least $1 - n^{-4}$, there is at least one node in T that can successfully transmit an *AskColor* message to v by the timeslot $t - 1$. □

Lemma 8. *Assume Property 1 and Property 2 hold. A node v will correctly get a color after waking up for $O(\Delta \log n + \log^2 n)$ timeslots with probability $1 - O(n^{-2})$.*

Proof. After waking up for at most $2\mu \log n$ timeslots, v enters state $\mathcal{S}$ or starts executing the MIS algorithm. If v takes part in the MIS algorithm, by Lemma 2, with probability $1 - O(n^{-3})$, it will correctly enter state $\mathcal{S}$ or state $\mathcal{G}$ after $O(\log^2 n)$ timeslots. Next we bound the time v stays in state $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{G}$.

We first bound the time that node v would stay in state $\mathcal{C}_1$. Assume that u is the leader of v. By Algorithm 1, during every $3\mu \log n$ timeslots, either v receives at least one new *Grant* messages from u, or it doubles its transmission probability. If the received *Grant* message is not for v, it means that a node in $N(u)$ will join state $\mathcal{C}_2$. By Lemma 2, with probability $1 - O(n^{-3})$, when u stays in state $\mathcal{M}$, there is not another node in E_u^{3R} staying in state $\mathcal{M}$. By the MIS algorithm and the analysis for the MIS algorithm in the full version [16], with probability $1 - O(n^{-4})$, u can force all other nodes in E_u^{3R} to join state $\mathcal{S}$ and not to restart competing for joining state $\mathcal{M}$ until receiving a $StartTransmit_u$ message from u. Thus, with probability $1 - O(n^{-3})$, there are no other nodes in E_u^{3R} joining state $\mathcal{G}$ when u stays in state $\mathcal{G}$. Additionally, only nodes in $N(u)$ and $E_u^{3R} \setminus E_u^{2R}$ may join state $\mathcal{C}_1$ by receiving a *StartColoring* message before u quits. Thus all nodes in $E_u^{2R} \setminus N(u)$ will stay in state $\mathcal{S}$ while u stays in state $\mathcal{G}$.

Then after at most $(\Delta - 1 + \log n)3\mu \log n$ timeslots, either v receives a $Grant_v$ message and joins state $\mathcal{C}_2$, or v has transmission probability of $2^{-1-\omega}$, since v can receive at most $\Delta - 1$ *Grant* messages not for v and each of which would halve v's transmission probability. Then by a similar argument as in Lemma 6, v will successfully transmit an *AskColor* message to u in $2\mu \log n$ timeslots with probability $1 - n^{-4}$. Furthermore, by Lemma 7, with probability $1 - n^{-4}$, u did not quit the algorithm before receiving the *AskColor* message from v. After successfully transmitting message $AskColor_v$ to u, by Algorithm 1 and Lemma 6 *(ii)*, with probability $1 - n^{-4}$, v will receive a $Grant_v$ message from u in at most $2\mu\Delta \log n$ timeslots. So each node will stay in state $\mathcal{C}_1$ for at most $5\mu\Delta \log n + 3\mu \log^2 n$ timeslots with probability at least $1 - O(n^{-3})$. By Algorithm 1, it is easy to see that each node stays in state $\mathcal{C}_2$ for $\mu \log n$ timeslots.

Next we bound the time that a node v stays in state $\mathcal{G}$. By Lemma 6 *(i)*, after entering state $\mathcal{G}$ for $\mu \log n$ timeslots, v will successfully send a *StartColoring* message to all its neighbors with probability $1 - n^{-4}$. Then all nodes in $N(v)$ without choosing their colors will enter state $\mathcal{C}_1$. As shown above, with probability at least $(1 - O(n^{-3}))^{\Delta} \in 1 - O(n^{-2})$, each node in $N(v)$ will join state $\mathcal{C}_2$ after joining state $\mathcal{C}_1$ for at most $O(\Delta \log n + \log^2 n)$ timeslots. Then by the algorithm, v will quit from the algorithm after waiting for additional $O(\Delta \log n + \log^2 n)$ timeslots by noticing that b_v is at most Δ. So with probability at least $1 - O(n^{-2})$, the total time that v stays in state $\mathcal{G}$ is at most $O(\Delta \log n + \log^2 n)$.

Next we bound the time from v waking up to it next entering state $\mathcal{C}_1$ or $\mathcal{G}$. By the algorithm, after waking up for at most $2\mu \log n$ timeslots, either v starts executing the MIS algorithm or there comes a node in E_v^{3R} joining state $\mathcal{G}$. If v starts executing the MIS algorithm, by Lemma 2, with probability at least $1 - O(n^{-3})$, there will be a node in E_v^{3R} joining state $\mathcal{G}$. So after waking up for at most $O(\log^2 n)$ timeslots, a node in E_v^{3R} will join state $\mathcal{G}$. From then on, by Algorithm 1 and the analysis above, with probability at least $1 - O(n^{-2})$, after every $O(\Delta \log n + \log^2 n)$ timeslots, there will be at least one node u in E_v^{3R} joining state $\mathcal{G}$ and all nodes in $N[u]$ quit from the algorithm. We can see that all nodes joining state $\mathcal{G}$ are independent in terms of R. So there are only a constant number of nodes in E_v^{3R} being able to join state $\mathcal{G}$, denoted by c'. Then after at most $c' O(\Delta \log n + \log^2 n)$ timeslots, there will be a node in $N[v]$ joining state $\mathcal{G}$. Thus, with probability at least $1 - O(n^{-2})$, the total time that v spends before entering state $\mathcal{C}_1$ or $\mathcal{G}$ after waking up is at most $O(\Delta \log n + \log^2 n)$.

Combining all the above, with probability $1 - O(n^{-2})$, every node stays in the algorithm for at most $O(\Delta \log n + \log^2 n)$ timeslots. Finally, we prove that each node can correctly get a color with probability at least $1 - O(n^{-2})$. As shown before, with probability $1 - O(n^{-3})$, when a node v is in state $\mathcal{G}$, there is not another node in E_v^{3R} staying in state $\mathcal{G}$ as well. By Lemma 7, with probability $1 - O(n^{-4})$, v will not leave state $\mathcal{G}$ until all its neighbors get colored. Thus, with probability $1 - O(n^{-2})$, all nodes with color 0, i.e., all nodes used to join state $\mathcal{G}$, are independent in terms of R. If v chooses another color, by the algorithm, it will choose an available color and broadcast the chosen color to its neighbors as soon as it receives the *Grant* message from its leader. By Property 2 *(i)*, there is not a node in $N(v)$ staying in state $\mathcal{C}_2$ when v is in state $\mathcal{C}_2$. By Lemma 6 (iv), when staying in state $\mathcal{C}_2$, v can successfully send its color to its neighbors with probability $1-n^{-4}$. Note also that in Algorithm 1, v has been woken up before the first node in its neighborhood starts choosing a color with probability $1-n^{-4}$. Thus when v chooses a color in state $\mathcal{C}_2$, with probability $1-n^{-3}$, v has received all the colors chosen by its neighbors and there are no other nodes in $N(v)$ choosing a color at the same time. So v will correctly select a color with probability $1-O(n^{-2})$. □

Lemma 9. *Property 2 holds with probability* $1-O(n^{-1})$.

Proof (Sketch proof). We prove Property 2 by showing that with high probability, none of (i) (ii) and (iii) is the first property to be violated.

Claim. With probability at least $1-O(n^{-1})$, Property 2 (i) is not the first property to be violated.

Proof. Otherwise, assume that D_i is the disk violating Property 2 (i) in timeslot t. We further assume that node $v \in D_i$ joins state $\mathcal{C}_2$ in timeslot t and another node u also stays in state $\mathcal{C}_2$ in timeslot t. Assume that w is u's leader. We can still assume that all properties are correct before t. Then it can be shown that w must also be v's leader with probability $1-O(n^{-4})$. Furthermore, w must have started transmitting $Grant_v$ before the timeslot t. Hence, by Algorithm 1, w must have started transmit $Grant_u$ by the timeslot $t-2\mu\log n$. Then by Lemma 6 (ii), u have received $Grant_u$ from w by $t-\mu\log n-1$ with probability $1-n^{-4}$. Noting that u stays in state C_2 for $\mu\log n$ timeslots, u have quit from the algorithm before t with probability $1-n^{-4}$. This contradiction shows that Property 2 (i) is not the first violated property when u stays in state C_2 with probability $1-O(n^{-3})$. Then for D_i, the Claim is true with probability $1-O(n^{-2})$. And the Claim is correct for every disk with probability $1-O(n^{-1})$. □

Claim. With probability at least $1-n^{-1}$, Property 2 (ii) is not the first property to be violated.

Proof. Otherwise, assume that D_i is the first disk violating Property 2 (ii) in timeslot t^*. Before timeslot t^*, we can still assume that all properties hold. Assume that v is the leader of some nodes of D_i that stays in $\mathcal{C}_1$. Denote $C_{v1}(t)$ as the set of node in $N(v)$ that are in state $\mathcal{C}_1$ in timeslot t. Then it can be shown that in timeslot t^*, with probability at least $1-O(n^{-4})$, all nodes in D_i that are in state $\mathcal{C}_1$ have the same leader v. Next we prove a little stronger result: with probability at least $1-O(n^{-2})$, in any timeslot t, the sum of transmission probability of all nodes in $C_{v1}(t)$ is at most $2^{-\omega}$. Otherwise, assume that in timeslot t, $\sum_{u\in C_{v1}(t)} p_u > 2^{-\omega}$. Denote $I=[t-3\mu\log n, t)$. By Algorithm 1, every node in C_{v1} doubles its transmission probability at most once during the interval. Furthermore, the sum of transmission probabilities of newly joined nodes is at most $\frac{2^{-\omega-1}}{n}\cdot n = 2^{-\omega-1}$. Hence, it holds that in timeslot $t-3\mu\log n$, the sum of transmission probabilities is at least $2^{-2-\omega}$. Consequently, during the interval I, $2^{-2-\omega} \le \sum_{u\in C_{v1}} p_u < 2^{-\omega}$. Furthermore, during the interval I, for any disk D_j,

$j \neq i$, $\sum_{v \in D_j} p_v \leq 3 \cdot 2^{1-\omega}$. Then using these transmission probability bounds, it can be shown that at least one node in C_{v1} can send a message *AskColor* to v during the interval $I_1 = [t - 3\mu \log n, t - 2\mu \log n - 1]$ with probability $1 - n^{-4}$. Then in the interval $(t - 3\mu \log n, t - 1]$, with probability $1 - n^{-4}$, all nodes in C_{v1} receives a new $Grant_w$ message and halve their transmission probability except w which enters state $\mathcal{C}_2$. Thus with probability $1 - O(n^{-4})$, D_i will not violate Property 2 (*ii*) in timeslot t. By Lemma 8, v stays in state $\mathcal{G}$ for at most $O(\Delta \log n + \log^2 n)$ timeslots with probability $1 - O(n^{-2})$. Thus when v stays in state $\mathcal{G}$, there is not a violation timeslot for D_i with probability $1 - O(n^{-2})$. Additionally, when there are nodes in D_i which are in state $\mathcal{C}_1$, it means that there is a node staying in state $\mathcal{G}$ in $E_i^{\frac{3R}{4}}$. From Algorithm 1, we know that all nodes that joined state $\mathcal{G}$ during executing the algorithm are independent in terms of R. Hence, there are at most constant nodes in $E_i^{\frac{3R}{4}}$ which can join state $\mathcal{G}$. Thus D_i is not the first disk violating Property 2 (*ii*) with probability $1 - O(n^{-2})$. Then the Claim is true for all disks with probability $1 - O(n^{-1})$. □

Claim. With probability at least $1 - O(n^{-2})$, Property 2 (*iii*) is not the first property to be violated.

Proof. Otherwise, assume that D_i violates it in timeslot t for the first time. Then there is a new node u in D_i joining state $\mathcal{G}$ in timeslot t, while there has been another node v in D_i staying in state $\mathcal{G}$ in timeslot t. Before t, we can still assume that all properties are correct. By Algorithm 1, each node in E_v^{3R} will not try to join state $\mathcal{G}$ until it receives the $StartTransmit_v$ from v. By Algorithm 1, v has not started transmitting $StartTransmit_v$ by the timeslot $t - \mu \log n$, since v still stays in state $\mathcal{G}$ in timeslot t. Also noticing that each node need $\Omega(\log^2 n)$ timeslots to join state $\mathcal{G}$ by executing the MIS algorithm. So there will not come up another node in E_v^{3R} joining state $\mathcal{G}$ by the timeslot $t + \Omega(\log^2 n)$ with probability $1 - O(n^{-4})$. This contradicts with the fact that u joins state $\mathcal{G}$ in timeslot t. Thus when v stays in state $\mathcal{G}$, there is not such a violation timeslot t with probability $1 - O(n^{-4})$. Then with probability $1 - O(n^{-3})$, there is not a timeslot such that Property 2 (*iii*) is first violated in D_i. This is true for every disk with probability $1 - O(n^{-2})$. □

Theorem 1. *After waking up for* $O(\Delta \log n + \log^2 n)$ *timeslots, every node* v *will correctly get a color from* $\{0, 1, \ldots, \Delta_v\}$ *with probability at least* $1 - O(n^{-1})$.

Proof. Since Properties 1 and 2 have been shown to be correct with probability $1 - O(n^{-1})$, by Lemma 8, with probability at least $1 - O(n^{-1})$, every node v will correctly choose a color after executing Algorithm 1 for at most $O(\Delta \log n + \log^2 n)$ timeslots. Furthermore, when v chooses a color, either v chooses color 0, or it chooses the first available color in its color list by Algorithm 1. Because v receives at most $\Delta_v - 1$ colors from its neighbors (one of its neighbors is a leader), v can still choose a color from $\{0, 1, \ldots, \Delta_v\}$. □

4 Distributed $(\Delta + 1)$-Coloring for Uniform Power Assignment

In some multi-hop radio networks, nodes may not be able to adjust their transmission powers. In such a case, assuming that nodes adopt uniform power assignment, i.e., all nodes transmit with the same power level, we can obtain a distributed $(\Delta+1)$-coloring algorithm by iteratively carrying out the MIS algorithm. We only need to change the operations in the last state $\mathcal{M}$ in the MIS algorithm. Each node in state $\mathcal{M}$ first chooses an available color that has not been chosen by its neighbors, and then transmits a message m_C containing its choice to its neighbors for $\mu \log n$ timeslots with constant probability after waking up all its neighbors. Then all the nodes having received the message m_C delete the received color from their color list and restart executing the algorithm. By Lemma 2, we know that with high probability, in any timeslot, all nodes in state $\mathcal{M}$ form an independent set. Furthermore, similar to the proof of Lemma 6, we can show that with high probability, each node can successfully transmit its choice to its neighbors before any neighbor starts choosing a color. These two points ensure the correctness of the computed coloring. We assume that all nodes transmit with power $P = cN\beta R^{\alpha}$. Then we can get the following lemma, based on which the theorem on the correctness and the time complexity of the proposed coloring algorithm can be proved.

Lemma 10. *With probability at least $1 - O(n^{-2})$, a node v will correctly get a color in $O(\Delta_v^{2R} \log^2 n)$ timeslots after starting executing the algorithm, where Δ_v^{2R} is the number of nodes in E_v^{2R}. Furthermore, v will choose a color from $\{0, 1, \cdots, \Delta_v\}$.*

Proof. Using a similar argument as in the analysis of the MIS algorithm (in the full version[16]), we can get that after a node v starts or restarts the algorithm for $O(\log^2 n)$ timeslots, there will be a node in E_v^{2R} joining state $\mathcal{M}$ with probability $1-O(n^{-3})$. Thus after at most $O(\Delta_v^{2R} \log^2 n)$ timeslots, v will join state $\mathcal{M}$ with probability at least $1-O(n^{-2})$. Furthermore, using a similar manner for proving Lemma 6, we can show that all neighbors of v which have chosen colors before v have informed v their choices with probability $1 - O(n^{-3})$. And by Lemma 2, when v is in state $\mathcal{M}$, with probability $1 - O(n^{-3})$, none of v's neighbors stay in state $\mathcal{M}$ simultaneously. Thus v will correctly choose a color different from all its neighbors with probability at least $1 - O(n^{-3})$. Putting all together, we know that with probability at least $1 - O(n^{-2})$, v will correctly get a color in $O(\Delta_v^{2R} \log^2 n)$ timeslots after starting executing the algorithm. Finally, since there are Δ_v nodes in v's neighborhood, v have deleted at most Δ_v different colors from its color list when v chooses a color. Thus v can choose a color from $\{0, 1, \cdots, \Delta_v\}$. □

Theorem 2. *If the nodes adopt the uniform power assignment, there exists a distributed algorithm such that with probability at least $1 - O(n^{-1})$, each node will correctly get a color after executing the algorithm for $O(\Delta \log^2 n)$ timeslots. Furthermore, the total number of colors used is at most $\Delta + 1$.*

Proof. By Lemma 10, for a node v, with probability at least $1 - O(n^{-2})$, it will correctly get a color in $O(\Delta_v^{2R} \log^2 n)$ timeslots after starting executing the algorithm, where Δ_v^{2R} is the number of nodes in E_v^{2R}. Furthermore, v will choose a color from $\{0, 1, \cdots, \Delta_v\}$. Thus the theorem is correct for all nodes with probability $1 - O(n^{-1})$ by noting that $\Delta_v^{2R} \leq \chi(2R, R)\Delta \in O(\Delta)$.

5 Conclusion

In this paper, we study the distributed $\Delta + 1$-coloring problem in unstructured multi-hop radio networks under the SINR interference model. Without any knowledge of the neighborhood, our proposed new distributed ($\Delta + 1$)-coloring algorithm has time complexity $O(\Delta \log n + \log^2 n)$. Our result even matches the $O(\Delta)$-coloring algorithm in [3] for large Δ; their algorithm needs a prior estimate of Δ. For networks in which the nodes can not adjust their transmission powers, we give a ($\Delta + 1$)-coloring algorithm with time complexity $O(\Delta \log^2 n)$. Furthermore, by carefully tuning the parameters, we show that the maximal independent set algorithm in [12] still works under the SINR constraint, which is of independent interest.

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