Parameterized Complexity in Multiple-Interval Graphs: Domination

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Abstract. We show that several variants of the problem k-DOMINATING SET, including k-CONNECTED DOMINATING SET, k-INDEPENDENT DOMINATING SET, k-DOMINATING CLIQUE, d-DISTANCE k-DOMINATING SET, k-PERFECT CODE and d-DISTANCE k-PERFECT CODE, when parameterized by the solution size k, remain W[1]-hard in either multiple-interval graphs or their complements or both.

1 Introduction

We introduce some basic definitions. The *intersection graph* $\Omega(\mathcal{F})$ of a family of sets $\mathcal{F} = \{S_1, \ldots, S_n\}$ is the graph with $\mathcal{F}$ as the vertex set and with two different vertices S_i and S_j adjacent if and only if $S_i \cap S_j \neq \emptyset$; the family $\mathcal{F}$ is called a *representation* of the graph $\Omega(\mathcal{F})$. Let $t \geq 2$ be an integer. A *t-interval graph* is the intersection graph of a family of t-intervals, where each *t-interval* is the union of t disjoint intervals in the real line. A *t-track interval graph* is the intersection graph of a family of t-track intervals, where each *t-track interval* is the union of t disjoint intervals on t disjoint parallel lines called tracks, one interval on each track. Note that the t disjoint tracks for a t-track interval graph can be viewed as t disjoint "host intervals" in the real line for a t-interval graph. Thus t-track interval graphs are a subclass of t-interval graphs. If a t-interval graph has a representation in which all intervals have unit lengths, then the graph is a *unit t-interval graph*. If a t-interval graph has a representation in which the t disjoint intervals of each t-interval have the same length (although the intervals from different t-intervals may have different lengths), then the graph is a *balanced t-interval graph*. Similarly we define unit t-track interval graphs and balanced t-track interval graphs.

As generalizations of the ubiquitous interval graphs, multiple-interval graphs such as t-interval graphs and t-track interval graphs have numerous applications, traditionally to scheduling and resource allocation [13,1], and more recently to bioinformatics [4,8]. For this reason, a systematic study of various classical optimization

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problems in multiple-interval graphs has been undertaken by several groups of researchers. In terms of approximability, Bar-Yehuda et al. [1] presented a $2t$-approximation algorithm for MAXIMUM INDEPENDENT SET in t-interval graphs, and Butman et al. [2] presented approximation algorithms for MINIMUM VERTEX COVER, MINIMUM DOMINATING SET, and MAXIMUM CLIQUE in t-interval graphs with approximation ratios $2 - 1/t$, t^2, and $(t^2 - t + 1)/2$, respectively.

Fellows et al. [7] initiated the study of multiple-interval graph problems from the perspective of parameterized complexity. In general graphs, the four problems k-VERTEX COVER, k-INDEPENDENT SET, k-CLIQUE, and k-DOMINATING SET, parameterized by the solution size k, are exemplary problems in parameterized complexity theory [6]: it is well-known that k-VERTEX COVER is in FPT, k-INDEPENDENT SET and k-CLIQUE are W[1]-hard, and k-DOMINATING SET is W[2]-hard. Since t-interval graphs are a special class of graphs, all FPT algorithms for k-VERTEX COVER in general graphs immediately carry over to t-interval graphs. On the other hand, the parameterized complexities of k-INDEPENDENT SET, k-CLIQUE, and k-DOMINATING SET in t-interval graphs are not at all obvious. Indeed, in general graphs, k-INDEPENDENT SET and k-CLIQUE are essentially the same problem (the problem k-INDEPENDENT SET in any graph G is the same as the problem k-CLIQUE in the complement graph $\overline{G}$), but in t-interval graphs, they manifest different parameterized complexities. Fellows et al. [7] showed that k-INDEPENDENT SET in t-interval graphs is W[1]-hard for any $t \geq 2$, then, in sharp contrast, gave an FPT algorithm for k-CLIQUE in t-interval graphs parameterized by both k and t. Fellows et al. [7] also showed that k-DOMINATING SET in t-interval graphs is W[1]-hard for any $t \geq 2$. Recently, Jiang [9] strengthened the two hardness results for t-interval graphs, and showed that k-INDEPENDENT SET and k-DOMINATING SET remain W[1]-hard even in unit t-track interval graphs for any $t \geq 2$. In particular, we have the following theorem on the parameterized complexity of k-DOMINATING SET in unit 2-track interval graphs:

Theorem 1 (Jiang 2010 [9]). k-DOMINATING SET *in unit 2-track interval graphs is W[1]-hard with parameter k.*

The lack of symmetry in the parameterized complexities of k-INDEPENDENT SET and k-CLIQUE in multiple-interval graphs and their complements leads to a natural question about k-DOMINATING SET, which is known to be W[1]-hard in multiple-interval graphs: Is it still W[1]-hard in the complements of multiple-interval graphs? Our following theorem (here "co-3-track interval graphs" denotes "complements of 3-track interval graphs") gives a positive answer:

Theorem 2. k-DOMINATING SET *in co-3-track interval graphs is W[1]-hard with parameter k.*

A *connected dominating set* in a graph G is a dominating set S in G such that the induced subgraph $G(S)$ is connected. An *independent dominating set* in a graph G is both a dominating set and an independent set in G. A *dominating clique* in a graph G is both a dominating set and a clique in G. With connectivity taken in account, the problem k-DOMINATING SET has three important variants: k-CONNECTED DOMINATING SET, k-INDEPENDENT DOMINATING SET, and k-DOMINATING CLIQUE. Recall the sharp contrast in parameterized complexities of the two problems k-INDEPENDENT SET and k-CLIQUE in multiple-interval graphs and their complements. This leads to more natural questions about k-DOMINATING SET: Are the two problems k-INDEPENDENT DOMINATING SET and k-DOMINATING CLIQUE still W[1]-hard in multiple-interval graphs and their complements? Also, without veering to either extreme, how about k-CONNECTED DOMINATING SET?

We show that our FPT reduction for the W[1]-hardness of k-DOMINATING SET in co-3-track interval graphs in Theorem 2 also establishes the following theorem:

Theorem 3. k-CONNECTED DOMINATING SET *and* k-DOMINATING CLIQUE *in co-3-track interval graphs are both W[1]-hard with parameter* k*.*

Similarly, it is not difficult to verify that the FPT reduction for the W[1]-hardness of k-DOMINATING SET in unit 2-track interval graphs [9] also establishes the following theorem:

Theorem 4. k-INDEPENDENT DOMINATING SET *in unit 2-track interval graphs is W[1]-hard with parameter* k*.*

For the two problems k-CONNECTED DOMINATING SET and k-DOMINATING CLIQUE in multiple-interval graphs, we obtain a weaker result:

Theorem 5. k-CONNECTED DOMINATING SET *and* k-DOMINATING CLIQUE *in unit 3-track interval graphs are both W[1]-hard with parameter* k*.*

Another important variant (indeed a generalization) of k-DOMINATING SET is called d-DISTANCE k-DOMINATING SET, where each vertex is able to dominate all vertices within a threshold distance d. Note that k-DOMINATING SET is simply d-DISTANCE k-DOMINATING SET with $d = 1$. For this distance variant of k-DOMINATING SET, we obtain the following theorem:

Theorem 6. d-DISTANCE k-DOMINATING SET *for any* $d \geq 2$ *in balanced 3-interval graphs is W[1]-hard with parameter* k*.*

The last variant of k-DOMINATING SET that we study in this paper is called k-PERFECT CODE. A *perfect code* in a graph $G = (V, E)$, also known as a *perfect dominating set* or an *efficient dominating set*, is a subset of vertices $V' \subseteq V$ that includes exactly one vertex from the closed neighborhood of each vertex $u \in V$. Recall that the *open neighborhood* of u is $N(u) = \{v \mid \{u, v\} \in E\}$, and that the *closed neighborhood* of u is $N[u] = N(u) \cup \{u\}$. The problem k-PERFECT CODE is that of deciding whether a given graph G has a perfect code of size exactly k. It is known to be W[1]-complete with parameter k in general graphs [5,3]. Since every graph of maximum degree 3 is the intersection graph of a family of unit 2-track intervals [10, Theorem 4], it follows that k-PERFECT CODE is NP-complete in unit 2-track interval graphs. In the following theorem, we show that k-PERFECT CODE is indeed W[1]-hard in unit 2-track interval graphs:

Theorem 7. k-PERFECT CODE *in unit 2-track interval graphs is W[1]-hard with parameter* k*.*

The distance variant of k-PERFECT CODE, denoted as d-DISTANCE k-PERFECT CODE, is also studied in the literature [12]. We show that d-DISTANCE k-PERFECT CODE is also W[1]-hard in unit 2-track interval graphs:

Theorem 8. d-DISTANCE k-PERFECT CODE *for any* $d \geq 2$ *in unit 2-track interval graphs is W[1]-hard with parameter* k.

We refer to [11] for some related results. All proofs of W[1]-hardness in this paper are based on FPT reductions from the W[1]-complete problem k-MULTICOLORED CLIQUE [7]: Given a graph G of n vertices and m edges, and a vertex-coloring $\kappa : V(G) \to \{1, 2, \ldots, k\}$, decide whether G has a clique of k vertices containing exactly one vertex of each color. Without loss of generality, we assume that no edge in G connects two vertices of the same color.

2 Dominating Set

In this section we prove Theorem 2. We show that k-DOMINATING SET in co-3-track interval graphs is W[1]-hard by an FPT reduction from the W[1]-complete problem k-MULTICOLORED CLIQUE [7].

Let (G, κ) be an instance of k-MULTICOLORED CLIQUE. We will construct a family $\mathcal{F}$ of 3-track intervals such that G has a clique of k vertices containing exactly one vertex of each color if and only if the complement of the intersection graph $G_{\mathcal{F}}$ of $\mathcal{F}$ has a dominating set of k' vertices, where $k' = k + \binom{k}{2}$.

Vertex selection: Let $v_1, \ldots, v_n$ be the set of vertices in G, sorted by color such that the indices of all vertices of each color are contiguous. For each color i, $1 \leq i \leq k$, let $V_i = \{v_p \mid s_i \leq p \leq t_i\}$ be the set of vertices v_p of color i. For each vertex v_p, $1 \leq p \leq n$, let $\langle v_p \rangle$ be a *vertex 3-track interval* consisting of the following three intervals on the three tracks:

$$\langle v_p \rangle = \begin{cases} \text{track 1} : (p-1, p) \\ \text{track 2} : (p-1+m+1, p+m+1) \\ \text{track 3} : (p-1+m+1, p+m+1). \end{cases}$$

For each color i, $1 \leq i \leq k$, let $\langle V_i \rangle$ be the following 3-track interval:

$$\langle V_i \rangle = \begin{cases} \text{track 1} : (t_i, m+n+1) \\ \text{track 2} : (0, s_i - 1 + m + 1) \\ \text{track 3} : (m, m+1). \end{cases}$$

Edge selection: Let $e_1, \ldots, e_m$ be the set of edges in G, also sorted by color such that the indices of all edges of each color pair are contiguous. For each pair of distinct colors i and j, $1 \leq i < j \leq k$, let $E_{ij} = \{e_r \mid s_{ij} \leq r \leq t_{ij}\}$ be the set of edges $v_p v_q$ such that v_p has color i and v_q has color j. For each edge e_r, $1 \leq r \leq m$, let $\langle e_r \rangle$ be an *edge 3-track interval* consisting of the following three intervals on the three tracks:

$$\langle e_r \rangle = \begin{cases} \text{track 1} : (r-1+n+1, r+n+1) \\ \text{track 2} : (r-1, r) \\ \text{track 3} : (r-1, r). \end{cases}$$

For each pair of distinct colors i and j, $1 \leq i < j \leq k$, let $\langle E_{ij} \rangle$ be the following 3-track interval:

$$\langle E_{ij} \rangle = \begin{cases} \text{track 1} : (0, s_{ij} - 1 + n + 1) \\ \text{track 2} : (t_{ij}, n + m + 1) \\ \text{track 3} : (m, m + 1). \end{cases}$$

Validation: For each edge $e_r = v_p v_q$ such that v_p has color i and v_q has color j, let $\langle v_p e_r \rangle$ and $\langle v_q e_r \rangle$ be the following 3-track intervals:

$$\langle v_p e_r \rangle = \begin{cases} \text{track 1} : (p, s_{ij} - 1 + n + 1) \\ \text{track 2} : (t_{ij}, p - 1 + m + 1) \\ \text{track 3} : (r - 1, r), \end{cases} \quad \langle v_q e_r \rangle = \begin{cases} \text{track 1} : (q, s_{ij} - 1 + n + 1) \\ \text{track 2} : (t_{ij}, q - 1 + m + 1) \\ \text{track 3} : (r - 1, r). \end{cases}$$

Let $\mathcal{F}$ be the following family of $n + m + k + \binom{k}{2} + 2m$ 3-track intervals:

$$\begin{aligned} \mathcal{F} = \big\{ \langle v_p \rangle \mid 1 \leq p \leq n \big\} &\cup \big\{ \langle e_r \rangle \mid 1 \leq r \leq m \big\} \\ &\cup \big\{ \langle V_i \rangle \mid 1 \leq i \leq k \big\} \cup \big\{ \langle E_{ij} \rangle \mid 1 \leq i < j \leq k \big\} \\ &\cup \big\{ \langle v_p e_r \rangle, \langle v_q e_r \rangle \mid e_r = v_p v_q \in E_{ij}, 1 \leq i < j \leq k \big\}. \end{aligned}$$

This completes the construction. We refer to Figure 1 for an example. The following five properties of the construction can be easily verified:

1. For each color i, $1 \leq i \leq k$, all 3-track intervals $\langle v_p \rangle$ for $v_p \in V_i$ are pairwise-disjoint.
2. For each color i, $1 \leq i \leq k$, $\langle V_i \rangle$ intersects all other 3-track intervals except the vertex 3-track intervals $\langle v_p \rangle$ for $v_p \in V_i$.
3. For each pair of distinct colors i and j, $1 \leq i < j \leq k$, all 3-track intervals $\langle e_r \rangle$ for $e_r \in E_{ij}$ are pairwise-disjoint.
4. For each pair of distinct colors i and j, $1 \leq i < j \leq k$, $\langle E_{ij} \rangle$ intersects all other 3-track intervals except the edge 3-track intervals $\langle e_r \rangle$ for $e_r \in E_{ij}$.
5. For each pair of distinct colors i and j, $1 \leq i < j \leq k$, for each edge $e_r \in E_{ij}$ and each vertex v_p incident to e_r, $\langle v_p e_r \rangle$ intersects all other 3-track intervals except the vertex 3-track interval $\langle v_p \rangle$ and the edge 3-track intervals for the edges in E_{ij} other than $\langle e_r \rangle$.

Lemma 1. *G has a k-multicolored clique if and only if $\overline{G_\mathcal{F}}$ has a k'-dominating set, where $k' = k + \binom{k}{2}$.*

Proof. For the direct implication, if $K \subseteq V(G)$ is a k-multicolored clique in G, then the following subset $\mathcal{D} \subseteq \mathcal{F}$ of 3-track intervals is a k'-dominating set in $\overline{G_\mathcal{F}}$:

$$\mathcal{D} = \big\{ \langle v_p \rangle \mid v_p \in K \big\} \cup \big\{ \langle e_r \rangle \mid v_p, v_q \in K, e_r = v_p v_q \big\}.$$

To verify this, check that each $\langle v_p \rangle \notin \mathcal{D}$ is dominated by $\langle v_{p'} \rangle \in \mathcal{D}$ for some vertex $v_{p'}$ of the same color as v_p (Property 1), each $\langle e_r \rangle \notin \mathcal{D}$ is dominated by $\langle e_{r'} \rangle \in \mathcal{D}$ for some edge $e_{r'}$ of the same color pair as e_r (Property 3), each $\langle V_i \rangle$ is dominated by $\langle v_p \rangle \in \mathcal{D}$ for some $v_p \in V_i$ (Property 2), each $\langle E_{ij} \rangle$ is dominated by $\langle e_r \rangle \in \mathcal{D}$ for some $e_r \in E_{ij}$

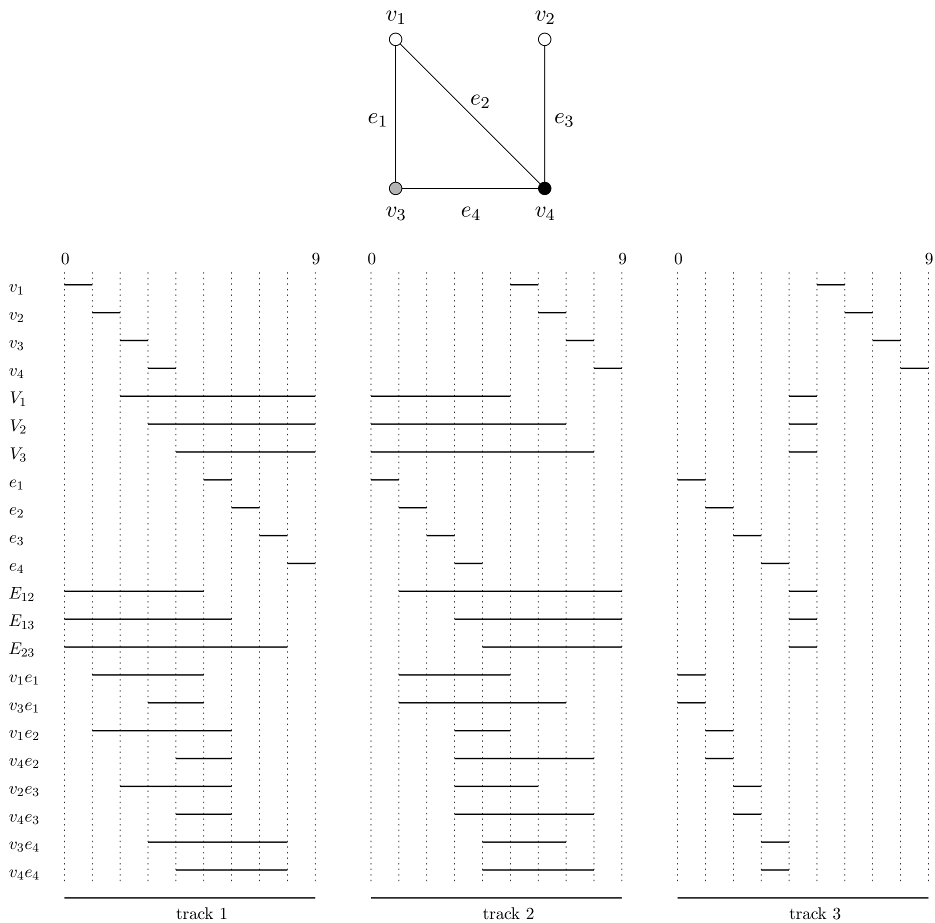

Fig. 1. Top: A graph G of $n = 4$ vertices v_1, v_2, v_3, v_4 and $m = 4$ edges $e_1 = v_1v_3, e_2 = v_1v_4, e_3 = v_2v_4, e_4 = v_3v_4$, with $k = 3$ colors $\kappa(v_1) = \kappa(v_2) = 1$, $\kappa(v_3) = 2$, and $\kappa(v_4) = 3$. $V_1 = \{v_1, v_2\}, V_2 = \{v_3\}, V_3 = \{v_4\}$; $E_{12} = \{e_1\}, E_{13} = \{e_2, e_3\}, E_{23} = \{e_4\}$. $K = \{v_1, v_3, v_4\}$ is a 3-multicolored clique. Bottom: A family $\mathcal{F}$ of $n + m + k + \binom{k}{2} + 2m = 22$ 3-track intervals. $\mathcal{D} = \{\langle v_1\rangle, \langle v_3\rangle, \langle v_4\rangle, \langle e_1\rangle, \langle e_2\rangle, \langle e_4\rangle\}$ is a 6-dominating set in the complement of the intersection graph of $\mathcal{F}$.

(Property 4), and each $\langle v_p e_r\rangle$ is dominated either by $\langle v_p\rangle \in \mathcal{D}$, when $v_p \in K$, or by $\langle e_{r'}\rangle \in \mathcal{D}$ for some edge $e_{r'}$ of the same color pair as e_r, when $v_p \notin K$ (Property 5).

For the reverse implication, suppose that $\mathcal{D} \subseteq \mathcal{F}$ is a k'-dominating set in $\overline{G_{\mathcal{F}}}$. We will find a k-multicolored clique $K \subseteq V(G)$ in G. For each color i, $1 \le i \le k$, $\mathcal{D}$ must include either $\langle V_i\rangle$ or at least one of its neighbors in $\overline{G_{\mathcal{F}}}$. Thus by Properties 1 and 2, we can assume without loss of generality that $\mathcal{D}$ does not include $\langle V_i\rangle$ but includes at least one vertex 3-track interval $\langle v_p\rangle$ for some $v_p \in V_i$. Similarly, for each pair of distinct colors i and j, $1 \le i < j \le k$, we can assume by Properties 3 and 4 that $\mathcal{D}$ does not include $\langle E_{ij}\rangle$ but includes at least one edge 3-track interval $\langle e_r\rangle$ for some $e_r \in E_{ij}$.

Since $k' = k + \binom{k}{2}$, it follows that $\mathcal{D}$ includes exactly one vertex 3-track interval of each color, and exactly one edge 3-track interval of each color pair. For each pair of distinct colors i and j, $1 \le i < j \le k$, let $e_r = v_p v_q$ be the edge whose 3-track interval $\langle e_r \rangle$ is included in $\mathcal{D}$. By Property 5 of the construction, the two 3-track intervals $\langle v_p e_r \rangle$ and $\langle v_q e_r \rangle$ cannot be dominated by $\langle e_r \rangle$ and hence must be dominated by $\langle v_p \rangle$ and $\langle v_q \rangle$, respectively. Therefore the vertex selection and the edge selection are consistent, and the set of k vertex 3-track intervals in $\mathcal{D}$ corresponds to a k-multicolored clique K in G.

3 Connected Dominating Set, Independent Dominating Set, and Dominating Clique

In this section we prove Theorems 3, 4, and 5.

For Theorem 3, to show the W[1]-hardness of k-CONNECTED DOMINATING SET and k-DOMINATING CLIQUE in co-3-track interval graphs, let us review our FPT reduction for Theorem 2, in particular, the proof of Lemma 1, in the previous section. Observe that for the direct implication of Lemma 1, our proof composes a dominating set $\mathcal{D}$ of pairwise-disjoint 3-track intervals, and that for the reverse implication of Lemma 1, our proof uses only the fact that $\mathcal{D}$ is a dominating set without any assumption about its connectedness. This implies that our FPT reduction for Theorem 2 also establishes Theorem 3. By a similar argument, it is not difficult to verify that the FPT reduction for the W[1]-hardness of k-DOMINATING SET in unit 2-track interval graphs [9] also establishes the W[1]-hardness of k-INDEPENDENT DOMINATING SET in unit 2-track interval graphs in Theorem 4.

For Theorem 5, to show the W[1]-hardness of k-CONNECTED DOMINATING SET and k-DOMINATING CLIQUE in unit 3-track interval graphs, we use the same construction as in the previous FPT reduction for the W[1]-hardness of k-DOMINATING SET in unit 2-track interval graphs [9] for the first two tracks. Then, on track 3, we use the same (coinciding) unit interval for all multiple-intervals in

$$\mathcal{F}' = \left\{ \widehat{u_i} \mid u \in V_i, 1 \le i \le k \right\} \cup \left\{ \widehat{u_i v_j}_{\text{left}}, \widehat{u_i v_j}_{\text{right}} \mid uv \in E_{ij}, 1 \le i < j \le k \right\},$$

and use a distinct unit interval disjoint from all other unit intervals for each of the remaining multiple-intervals. Now the dominating set composed in the direct implication of the proof in [9] becomes a clique. Since the reverse implication of the proof in [9] does not depend on the additional intersections between the multiple-intervals in $\mathcal{F}'$, the modified reduction establishes Theorem 5.

4 Distance Dominating Set

In this section we prove Theorem 6. We show that d-DISTANCE k-DOMINATING SET is W[1]-hard in 3-interval graphs for any $d \ge 2$ by an FPT reduction again from k-MULTICOLORED CLIQUE.

First we consider the case $d = 2$. Let (G, κ) be an instance of k-MULTICOLORED CLIQUE. We will construct a family $\mathcal{F}$ of 3-intervals as illustrated in Figure 2 such that G has a k-multicolored clique if and only if the intersection graph $G_{\mathcal{F}}$ of $\mathcal{F}$ has a 2-distance k'-dominating set, where $k' = k + \binom{k}{2}$. For convenience, we specify some 3-intervals in $\mathcal{F}$ as 2-intervals or intervals, and assume an implicit extension of each 2-interval or interval to a 3-interval by adding extra intervals that are disjoint from the other intervals in $\mathcal{F}$. We use (u, v, w) to denote a 3-interval that is the union of three disjoint intervals u, v, w, in no particular order. Similarly, we use (u, v) for a 2-interval.

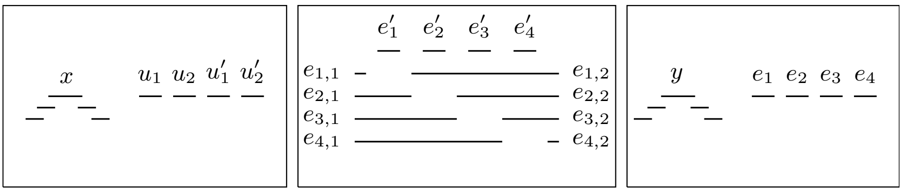

Fig. 2. The vertex gadget for V_i (left) is connected to the edge gadget for E_{ij} (right) by a validation gadget (middle)

Vertex selection: For each color i, $1 \leq i \leq k$, let V_i be the set of vertices of color i. Write $|V_i| = \phi$. There are $2\phi+1$ disjoint intervals labeled with $x, u_1, \ldots, u_\phi, u'_1, \ldots, u'_\phi$ in the vertex selection gadget for V_i. For each vertex $u = u_s \in V_i$, we add two 2-intervals $\langle u \rangle_1 = (x, u_s)$ and $\langle u \rangle_2 = (u_s, u'_s)$ to $\mathcal{F}$. We also add four dummy intervals to $\mathcal{F}$: two dummy intervals intersect with x; the other two dummy intervals intersect with the first two dummy intervals, respectively.

Edge Selection: For each pair of distinct colors i and j, $1 \leq i < j \leq k$, let E_{ij} be the set of edges uv such that u has color i and v has color j. Write $|E_{ij}| = \psi$. There are $\psi + 1$ disjoint intervals labeled with $y, e_1, \ldots, e_\psi$ in the edge selection gadget for E_{ij}. For each edge $e = e_s \in E_{ij}$, we add a 2-interval $\langle e \rangle = (y, e_s)$ to $\mathcal{F}$. We also add four dummy intervals to $\mathcal{F}$ in the same way as in each vertex selection gadget.

Validation: For each pair of distinct colors i and j, $1 \leq i < j \leq k$, we construct two validation gadgets that connect the two vertex gadgets for V_i and V_j, respectively, to the edge gadget for E_{ij}. In the following we describe the validation gadget between the vertex gadget for V_i and the edge gadget for E_{ij}; the construction of the other validation gadget is similar. Write $|E_{ij}| = \psi$. There are 3ψ intervals in this validation gadget. First, there are ψ disjoint intervals labeled with $e'_1, \ldots, e'_\psi$. Then, for each e'_s, there are two disjoint intervals $e_{s,1}$ and $e_{s,2}$ intersecting with all intervals e'_t with $t \neq s$. For each edge $e = e_s \in E_{ij}$, we add a 2-interval $\langle e, i \rangle = (e_s, e'_s)$ to $\mathcal{F}$. For each vertex $u = u_t \in V_i$ incident to some edge $e = e_s \in E_{ij}$, we add a 3-interval $\langle u, e \rangle = (u'_t, e_{s,1}, e_{s,2})$ to $\mathcal{F}$.

In summary, the construction gives us the following family $\mathcal{F}$ of 3-intervals:

$$\mathcal{F} = \big\{\langle u \rangle_1, \langle u \rangle_2 \mid u \in V_i, 1 \leq i \leq k\big\} \cup \big\{\langle e \rangle \mid e \in E_{ij}, 1 \leq i < j \leq k\big\}$$
$$\cup \big\{\langle e, i \rangle, \langle e, j \rangle, \langle u, e \rangle, \langle v, e \rangle \mid e = uv \in E_{ij}, 1 \leq i < j \leq k\big\} \cup \text{DUMMIES},$$

where DUMMIES is the set of $4k + 4\binom{k}{2}$ dummy intervals.

Observe that for each pair of disjoint intervals $e_{s,1}$ and $e_{s,2}$ in the validation gadgets, we can extend $e_{s,1}$ to the left and extend $e_{s,2}$ to the right until they have the same length. This does not change the intersection pattern of the intervals. Therefore $\mathcal{F}$ can be transformed into a family of balanced 3-intervals, where $\langle e\rangle$, $\langle e, i\rangle$, $\langle e, j\rangle$ and DUMMIES use intervals of length 1, and $\langle u\rangle_1, \langle u\rangle_2, \langle u, e\rangle$ use intervals of length m, where m is the number of edges in G.

Lemma 2. *G has a k-multicolored clique if and only if $G_{\mathcal{F}}$ has a 2-distance k'-dominating set, where $k' = k + \binom{k}{2}$.*

Proof. We first prove the direct implication. Suppose G has a k-multicolored clique $K \subseteq V(G)$, then it is easy to verify the following subfamily $\mathcal{D}$ of 3-intervals is a 2-distance k'-dominating set in $G_{\mathcal{F}}$:

$$\mathcal{D} = \big\{\langle u\rangle_1 \mid u \in K\big\} \cup \big\{\langle e\rangle \mid e = uv,\, u, v \in K\big\}.$$

We next prove the reverse implication. Suppose that $\mathcal{D}$ is a 2-distance k'-dominating set in $G_{\mathcal{F}}$. In order to dominate the dummies we can assume without loss of generality that $\mathcal{D}$ includes at least one $\langle u\rangle_1$ from each vertex gadget and at least one $\langle e\rangle$ from each edge gadget. Since $\mathcal{D}$ has size $k' = k + \binom{k}{2}$, we must have exactly one $\langle u\rangle_1$ from each vertex gadget and exactly one $\langle e\rangle$ from each edge gadget in $\mathcal{D}$. Consider $\langle e\rangle$ from the edge gadget for E_{ij}, where $e = uv$. Note that $\langle e\rangle$ dominates all multiple-intervals in the two validation gadgets for E_{ij} except $\langle u, e\rangle$ and $\langle v, e\rangle$, which must be dominated by $\langle u\rangle$ and $\langle v\rangle$, respectively, in the corresponding vertex gadgets. Therefore the subset of vertices $K = \{v \in V(G) \mid \langle v\rangle_1 \in \mathcal{D}\}$ is a k-multicolored clique in G. □

The above construction can be easily generalized to handle the case $d > 2$. To do this, extend each vertex gadget to include d pairs of dummy intervals instead of two pairs, and to include d disjoint intervals for each vertex u (instead of only the two labeled with u and u') such that there is a path of length $d-1$ from $\langle u\rangle_1$ to $\langle u\rangle_d$ in $G_{\mathcal{F}}$. Extend each edge gadget in a similar way. Then the same argument applies.

5 Perfect Code

In this section we prove Theorem 7. We show that k-PERFECT CODE in unit 2-track interval graphs is W[1]-hard by a reduction from k-MULTICOLORED CLIQUE.

Let (G, κ) be an instance of k-MULTICOLORED CLIQUE. We will construct a family $\mathcal{F}$ of unit 2-track intervals such that G has a k-multicolored clique if and only if the intersection graph $G_{\mathcal{F}}$ of $\mathcal{F}$ has a k'-perfect code, where $k' = k + 2\binom{k}{2}$.

Vertex selection: For each color i, $1 \le i \le k$, let V_i be the set of vertices of color i. We construct a vertex-selection gadget for V_i as illustrated in Figure 3. Write $|V_i| = \phi$. On each track, we start with 2ϕ unit intervals arranged in ϕ rows and two (slanted) columns. The ϕ intervals in each column are pairwise-intersecting. The two intervals in each row slightly overlap such that each interval in the left column intersects with all intervals in the same or higher rows in the right column. For the rth vertex u in V_i, $1 \le r \le \phi$, we add a *vertex 2-track interval* $\langle u\rangle = (u_1, u_2)$ to $\mathcal{F}$, where u_1 and u_2

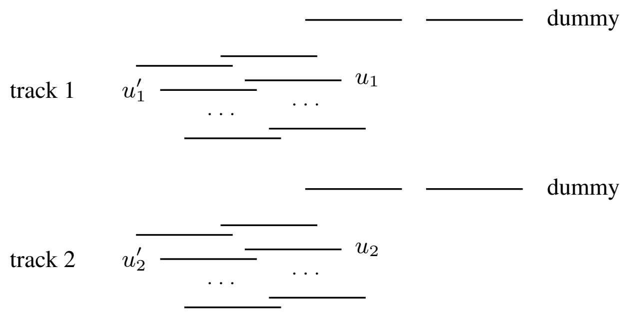

Fig. 3. An illustration of a vertex-selection gadget

are the intervals in the rth row and the right column on tracks 1 and 2, respectively. Denote by u'_1 and u'_2 the intervals in the rth row and the left column on tracks 1 and 2, respectively; they will be used for validation. Besides the ϕ vertex 2-track intervals $\langle u \rangle$, we also add two dummy 2-track intervals to $\mathcal{F}$. The first (resp. second) dummy 2-interval consists of a unit interval on track 1 (resp. track 2) that intersects all intervals in the right column and no interval in the left column, and a unit interval on track 2 (resp. track 1) that is disjoint from all other intervals.

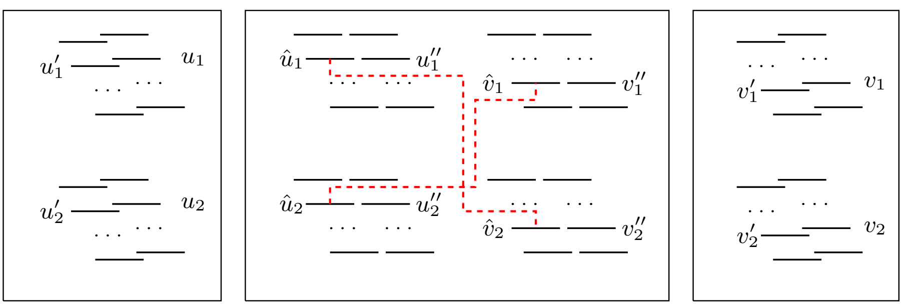

Fig. 4. An illustration of an edge-selection gadget (middle) and the corresponding vertex-selection gadgets (left and right). Two edge 2-track intervals $(\hat{u}_1, \hat{v}_2)$ and $(\hat{u}_2, \hat{v}_1)$ are represented by dashed lines. Dummy 2-track intervals are omitted from the figure.

Edge selection: For each pair of distinct colors i and j, $1 \leq i < j \leq k$, let E_{ij} be the set of edges uv such that u has color i and v has color j. We construct an edge selection gadget for E_{ij} as illustrated in Figure 4. We start with four disjoint groups of intervals, two groups on each track, with two columns of intervals in each group. Write $|V_i| = \phi_i$ and $|V_j| = \phi_j$. The two groups on the left correspond to color i and have ϕ_i rows; the two groups on the right correspond to color j and have ϕ_j rows. Different from the formation in the vertex selection gadgets, here in each group each interval in the left column intersects with all intervals in higher rows in the right column but not the interval in the same row. In the two groups on the left, for the rth vertex $u \in V_i$, $1 \le r \le \phi_i$, denote by $\hat{u}_1$ and $\hat{u}_2$ the intervals in the rth row and the left column on tracks 1 and 2, respectively, and denote by u_1'' and u_2'' the intervals in the rth row and the right column on tracks 1 and 2, respectively. Similarly, for each vertex $v \in V_j$, denote by $\hat{v}_1, \hat{v}_2, v_1', v_2'$ the corresponding intervals in the two groups on the right. For each edge $uv \in E_{ij}$, we add two *edge 2-track intervals* $\langle uv\rangle_1 = (\hat{u}_1, \hat{v}_2)$ and $\langle uv\rangle_2 = (\hat{u}_2, \hat{v}_1)$ to $\mathcal{F}$. Besides these edge 2-track intervals, we also add four dummy 2-track intervals to $\mathcal{F}$, one for each group of intervals. The dummy 2-track interval for each group consists of a unit interval that intersects all intervals in the left column and no interval in the right column in the group, and a unit interval on the other track that is disjoint from all other intervals.

Validation: For each pair of distinct colors i and j, $1 \le i < j \le k$, we add $2|V_i| + 2|V_j|$ *validation 2-track intervals* to $\mathcal{F}$ as illustrated in Figure 4. Specifically, for each vertex $u \in V_i$, we add $\langle u{*}_{ij}\rangle_1 = (u_1', u_2'')$ and $\langle u{*}_{ij}\rangle_2 = (u_2', u_1'')$, and for each vertex $v \in V_j$, we add $\langle {*}v_{ij}\rangle_1 = (v_1', v_2'')$ and $\langle {*}v_{ij}\rangle_2 = (v_2', v_1'')$.

In summary, the construction gives us the following family $\mathcal{F}$ of unit 2-track intervals:

$$\mathcal{F} = \big\{\langle u\rangle \mid u \in V_i, 1 \le i \le k\big\} \cup \big\{\langle uv\rangle_1, \langle uv\rangle_2 \mid uv \in E_{ij}, 1 \le i < j \le k\big\} \\ \cup \big\{\langle u{*}_{ij}\rangle_1, \langle u{*}_{ij}\rangle_2, \langle {*}v_{ij}\rangle_1, \langle {*}v_{ij}\rangle_2 \mid u \in V_i, v \in V_j, 1 \le i < j \le k\big\} \\ \cup \text{DUMMIES},$$

where DUMMIES is the set of $2k + 4\binom{k}{2}$ dummy 2-track intervals.

Lemma 3. *G has a k-multicolored clique if and only if $G_{\mathcal{F}}$ has a k'-perfect code, where $k' = k + 2\binom{k}{2}$.*

Proof. We first prove the direct implication. Suppose G has a k-multicolored clique $K \subseteq V(G)$, then it is easy to verify that the following subfamily $\mathcal{D}$ of unit 2-track intervals is a k'-perfect code in $G_{\mathcal{F}}$:

$$\mathcal{D} = \big\{\langle u\rangle \mid u \in K\big\} \cup \big\{\langle uv\rangle_1, \langle uv\rangle_2 \mid u, v \in K\big\}.$$

We next prove the reverse implication. Suppose $\mathcal{D}$ is a k'-perfect code in $G_{\mathcal{F}}$. Observe that the dummy 2-track intervals in our construction are pairwise-disjoint. Moreover, the two dummies in each vertex gadget share the same open neighborhood which is not empty, and the same is true about the two dummies associated with the two groups of intervals, the left group on track 1 and the right group on track 2 (resp. the right group on track 1 and the left group on track 2) of each edge gadget. It follows that these dummies cannot be included in $\mathcal{D}$. In order to perfectly dominate the dummies, $\mathcal{D}$ must include exactly one vertex 2-track interval $\langle u\rangle$ from each vertex selection gadget and two edge 2-track intervals $\langle uv\rangle_1$ and $\langle xy\rangle_2$ from each edge selection gadget. Consider an edge 2-track interval $\langle uv\rangle_1 = (\hat{u}_1, \hat{v}_2)$ from the edge selection gadget for E_{ij}, and observe the validation 2-track intervals dominated by $\langle uv\rangle_1$. To perfectly dominate the validation 2-track intervals $\langle w{*}_{ij}\rangle_2$ for all $w \in V_i$, $\mathcal{D}$ must include $\langle u\rangle$ from the vertex selection gadget for V_i. Similarly, to perfectly dominate the validation 2-track intervals $\langle {*}w_{ij}\rangle_1$

for all $w \in V_j$, $\mathcal{D}$ must include $\langle v \rangle$ from the vertex selection gadget for V_j. Then, to perfectly dominate the validation 2-track intervals $\langle w*_{ij} \rangle_1$ for all $w \in V_i$, and $\langle *w_{ij} \rangle_2$ for all $w \in V_j$, the two intervals $\hat{u}_2$ and $\hat{v}_1$ must be used. This implies that the other edge 2-track interval from the same edge selection gadget must be $\langle uv \rangle_2 = (\hat{u}_2, \hat{v}_1)$. Therefore the subset of vertices $K = \{u \in V(G) \mid \langle u \rangle \in \mathcal{D}\}$ is a k-multicolored clique in G. □

6 Distance Perfect Code

In this section we prove Theorem 8. We show that d-DISTANCE k-PERFECT CODE is W[1]-hard in unit 2-interval graphs for any $d \geq 2$ by an FPT reduction from k-MULTICOLORED CLIQUE.

We consider the case $d = 2$ first. Let (G, κ) be an instance of k-MULTICOLORED CLIQUE. We will construct a family $\mathcal{F}$ of unit 2-intervals as illustrated in Figure 5 such that G has a k-multicolored clique if and only if the intersection graph $G_{\mathcal{F}}$ of $\mathcal{F}$ has a 2-distance k'-perfect code, where $k' = k + \binom{k}{2}$.

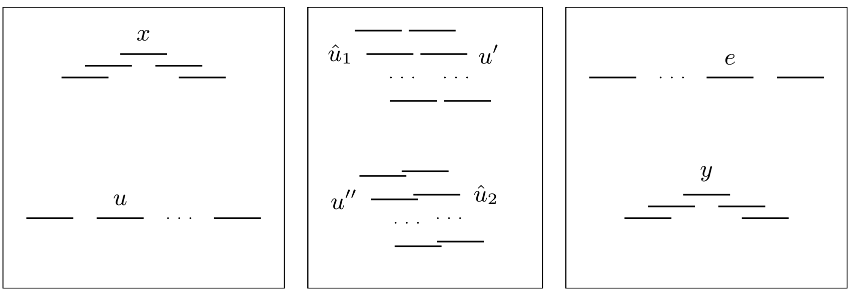

Fig. 5. The vertex gadget for V_i (left) is connected to the edge gadget for E_{ij} (right) by a validation gadget (middle)

Vertex selection: For each color i, $1 \leq i \leq k$, let V_i be the set of vertices of color i. We construct a vertex-selection gadget for V_i as illustrated in Figure 5. Write $|V_i| = \phi$. On track 1 there is an interval labeled by x. On track 2 there are ϕ disjoint intervals, one for each vertex in V_i. For the rth vertex u in V_i, $1 \leq r \leq \phi$, we add a 2-track interval $\langle u \rangle = (x, u)$ to $\mathcal{F}$. We also add four dummy 2-track intervals to $\mathcal{F}$: two dummy 2-track intervals intersect with x; the other two dummy 2-track intervals intersect with the first two dummy 2-track intervals, respectively. In figure 5, only one interval (on track 1) of each dummy 2-track intervals is drawn.

Edge selection: For each pair of distinct colors i and j, $1 \leq i < j \leq k$, let E_{ij} be the set of edges uv such that u has color i and v has color j. Write $|E_{ij}| = \psi$. There are ψ disjoint intervals on track 1, one for each edge in E_{ij}. There is an interval labeled by y on track 2. For each edge $e \in E_{ij}$, add a 2-track interval $\langle e \rangle = (y, e)$ to $\mathcal{F}$. We also add four dummy 2-track intervals to $\mathcal{F}$ in the similar way as in each vertex selection gadget.

Validation selection: For each pair of distinct colors i and j, $1 \le i < j \le k$, we construct two validation gadgets that connect the two vertex gadgets for V_i and V_j, respectively, to the edge gadget for E_{ij}. First we describe the validation gadget between the vertex gadget for V_i and the edge gadget for E_{ij}. Write $|V_i| = \phi$ and $|E_{ij}| = \psi$. On track 1, there are 2ϕ interval arranged in ϕ rows and two (slanted) columns. The ϕ intervals in each column are pairwise-intersecting. Moreover, each interval in the left column intersects with all intervals in higher rows in the right column but not the interval in the same row. For the rth vertex $u \in V_i$, $1 \le r \le \phi$, denote by $\hat{u}_1$ and u' the left and right intervals, respectively, in the rth row. On track 2, the arrangement of the 2ϕ intervals are similar except that each interval in the left column intersects with all intervals in the higher rows *and* the interval in the same row. Denote by u'' and $\hat{u}_2$ the left and right intervals, respectively, in the rth row. We add $2\phi + \psi$ validation 2-track intervals to $\mathcal{F}$. For each vertex $u \in V_i$, add $\langle u*_{ij}\rangle_1 = (u, u')$ and $\langle u*_{ij}\rangle_2 = (\hat{u}_1, \hat{u}_2)$ to $\mathcal{F}$. For each edge $e = uv \in E_{ij}$, add $\langle u, e\rangle = (e, u'')$ to $\mathcal{F}$.

The validation gadget between the vertex gadget for V_j and the edge gadget for E_{ij} (not shown in Figure 5) is constructed similarly. For each vertex $v \in V_j$, we add $\langle *v_{ij}\rangle_1 = (v, v')$ and $\langle *v_{ij}\rangle_2 = (\hat{v}_1, \hat{v}_2)$ to $\mathcal{F}$. For each edge $e = uv \in E_{ij}$, we add $\langle v, e\rangle = (e, v'')$ to $\mathcal{F}$.

In summary, the construction gives us the following family $\mathcal{F}$ of unit 2-track intervals:

$$\begin{aligned}\mathcal{F} = &\big\{\langle u\rangle \mid u \in V_i, 1 \le i \le k\big\} \cup \big\{\langle e\rangle \mid e \in E_{ij}, 1 \le i < j \le k\big\}\\ &\cup \big\{\langle u*_{ij}\rangle_1, \langle u*_{ij}\rangle_2, \langle *v_{ij}\rangle_1, \langle *v_{ij}\rangle_2 \mid u \in V_i, v \in V_j, 1 \le i < j \le k\big\}\\ &\qquad \cup \big\{\langle u, e\rangle, \langle v, e\rangle \mid e = uv \in E_{ij}, 1 \le i < j \le k\big\} \cup \text{DUMMIES},\end{aligned}$$

where DUMMIES is the set of $4k + 4\binom{k}{2}$ dummy 2-track intervals.

Lemma 4. *G has a k-multicolored clique if and only if $G_{\mathcal{F}}$ has a 2-distance k'-perfect code, where $k' = k + \binom{k}{2}$.*

Proof. We first prove the direct implication. Suppose G has a k-multicolored clique $K \subseteq V(G)$, then one can verify that the following subfamily $\mathcal{D}$ of 2-track intervals is a 2-distance k'-perfect code in $G_{\mathcal{F}}$:

$$\mathcal{D} = \big\{\langle u\rangle \mid u \in K\big\} \cup \big\{\langle e\rangle \mid e = uv,\ u, v \in K\big\}.$$

We next prove the reverse implication. Suppose that $\mathcal{D}$ is a 2-distance k'-perfect code in $G_{\mathcal{F}}$. By a similar argument as in the proof of Lemma 3, the dummies cannot be included in $\mathcal{D}$. In order to perfectly dominate the dummies, $\mathcal{D}$ must include exactly one $\langle u\rangle$ from each vertex gadget and exactly one $\langle e\rangle$ from each edge gadget. For the rth vertex u and tth vertex w in V_i, we write $u \le_i w$ if $r \le t$ and $u >_i w$ if $r > t$. Consider $\langle e\rangle$ from the edge gadget for E_{ij}, where $e = uv$. Observe that in the validation gadget between the vertex gadget for V_i and the edge gadget for E_{ij}, the 2-track intervals $\{\langle w*_{ij}\rangle_2 \mid w \in V_i, w \le_i u\}$ are within distance 2 from $\langle e\rangle$. Then, to perfectly dominate the 2-track intervals $\{\langle w*_{ij}\rangle_2 \mid w \in V_i, w >_i u\}$, the 2-track interval $\langle u\rangle$ from the vertex gadget for V_i must be included in $\mathcal{D}$. Similarly, to perfectly dominate the 2-track intervals $\langle *w_{ij} \rangle_2$ in the other validation gadget, the 2-track interval $\langle v \rangle$ from the vertex gadget for V_j must also be included in $\mathcal{D}$. Therefore the subset of vertices $K = \{u \in V(G) \mid \langle u \rangle \in \mathcal{D}\}$ is a k-multicolored clique in G. □

The above construction can be generalized to handle the case $d > 2$. We postpone the details to the full version of this paper.

Concluding Remarks. A general direction for extending our work is to strengthen the existing W[1]-hardness results for more restricted graph classes. For example, we showed in Theorem 2 that k-DOMINATING SET in co-3-track interval graphs is W[1]-hard with parameter k. Is it still W[1]-hard in co-2-track interval graphs or co-unit 3-track interval graphs? Many questions can be asked in the same spirit. In particular, are k-INDEPENDENT DOMINATING SET and k-PERFECT CODE W[1]-hard in co-t-interval graphs for some constant t?

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