Chapter 1
Computation of Green's Functions for Ocean Tide Loading

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The devil is in the details

1 Introduction

This chapter will discuss the computation of the deformation of the solid Earth due to external forces. It is a classical problem that was studied more than a century ago by famous people such as Thomson and Tait (1867) and Lamb (1895). They were followed by Love (1911) and Hoskins (1920) in the beginning of the twentieth century. Since then it has been studied extensively by seismologists who are interested in modelling the free oscillations of the Earth that occur after large earthquakes. Important contributions to this area were made by Pekeris and Jarosch (1958) and Alterman et al. (1959) which still forms the basis of what we will describe in this chapter. A thorough description of the the theory of the free oscillations of the Earth can be found in the textbook by Dahlen and Tromp (1998). An older but still good reference is the review article by Takeuchi and Saito (1972).

The reader could therefore accuse us of writing about a topic that has already been described. However, we feel that current literature does not pay much attention to the practical details of how a given profile of the density and elastic properties of the Earth are to be used to compute these deformations and it is our objective to fill this gap. We hope that a researcher or Ph.D. student who wants to learn more about this topic finds in our chapter a good starting point where all assumptions are clearly explained and where enough details are given to implement the equations into a computer program.

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We will only look at deformations caused by the varying weight of the ocean tides, also known as ocean tide loading (OTL). With the current accuracy by which these tidal deformations can be observed at the surface, we can ignore the ellipticity of the Earth and its rotation and assume that the mechanical properties of the Earth are the same for all orientations and only vary along the radius of the Earth. With sufficient accuracy we can also assume that the deformation is elastic or at least deviates only slightly from a pure elasticity.

Next, the weight of the ocean tides is normally decomposed into a sum of point loads. The advantage is that, once you know the deformation of the Earth under a single point load, and assuming that the deformations are small enough so that the principle of superposition holds, you can compute the deformation of all point loads in a similar way and add them up to get the total. The deformation due to a point load, which is a Dirac delta function, is called a Green's function. One of the first attempts to compute such a Green's function was given by Slichter and Caputo (1960) although they used a circular disc load instead of the actual limit of reducing the radius of the disc to zero and they ignored any gravity effects due to the mass distribution inside the Earth.

Longman (1962, 1963) was the first to develop the point load into a sum of Legendre polynomials and computed this sum up to degree 40. Farrell (1972) continued the work of Longman and extended the summation up to a degree of 10,000. Farrell's contribution was also a better understanding of the problem at degree 1 where the deformation is invariant with respect to a simple translation of the whole Earth. He also emphasised the use of the analytical solution of the deformation of a half-space as the asymptotic solution of the deformations of the spherical Earth. These asymptotic solutions can not only be used to check the numerical solutions but are also essential to find the value of the infinite sum of Legendre polynomials.

Longman and Farrell used the elastic properties and density profiles of the Earth that were computed by seismologists. An example is the Preliminary Reference Earth Model (PREM) published by Dziewonski and Anderson (1981). The earthquakes that are being studied by seismologists have periods of several seconds and, since tides have a period of several hours, one can wonder whether the same elastic properties should be used. So far, no observations that challenge this assumption have ever been presented.

In this chapter we explain how the these elastic properties and density profiles can be transformed in so-called Love numbers. These numbers can be used to compute the necessary Green's functions. The summation of Love numbers has already been described in detail by, among others, Farrell (1972), Francis and Mazzega (1990), Jentzsch (1997), Guo et al. (2004) and recently by Agnew (2007), and therefore will only be discussed briefly. It is the computation of the Love numbers that we will focus on. We will start at the very beginning, which means we need to start by deriving the set of differential equations that govern the deformation of the solid Earth.

2 Equations of Motions and Rheology

This section will derive the linearized equations of motion in the same way as Dahlen (1974) although we give more attention to the interpretation of all the terms.

We restrict our discussion to models of the Earth that are symmetric, non-rotating and elastic isotropic (SNREI) and everywhere in hydrostatic equilibrium. The positions of the mass particles of the Earth are denoted by $\mathbf{x}$. At the same time we will use these initial locations to label the particles. Let $\mathbf{r}(\mathbf{x}, t)$ be the position of particle with label and initial position $\mathbf{x}$ after the deformation at time t. Now we can write the Lagrangian displacement $\mathbf{s}_L(\mathbf{x}, t)$ as (Dahlen 1974)

$$\mathbf{r}(\mathbf{x}, t) = \mathbf{x} + \mathbf{s}_L(\mathbf{x}, t) \tag{1.1}$$

Instead of following the deformation of a particle with label $\mathbf{x}$ that was initially at position $\mathbf{x}$, one may describe the deformation over time one finds at the fixed position $\mathbf{r}$ inside the Earth. This is the Eulerian description of the deformation.

It will be convenient to write the changes in density ρ and potential ϕ as small perturbations from a reference state. We have (Dahlen 1974)

$$\rho_L(\mathbf{x}, t) = \rho_0(\mathbf{x}) + \rho_1^L(\mathbf{x}, t) \tag{1.2}$$

$$\rho_E(\mathbf{r}, t) = \rho_0(\mathbf{r}) + \rho_1^E(\mathbf{r}, t) \tag{1.3}$$

$$\phi_L(\mathbf{x}, t) = \phi_0(\mathbf{x}) + \phi_1^L(\mathbf{x}, t) \tag{1.4}$$

$$\phi_E(\mathbf{r}, t) = \phi_0(\mathbf{r}) + \phi_1^E(\mathbf{r}, t) \tag{1.5}$$

The subscript or superscript L and E indicate whether we are dealing with a Lagrangian or a Eulerian function. Generally, the coefficients of the functions ρ_L and ρ_E are not equal, neither those of ϕ_L and ϕ_E, because they depend on a different set of variables, the Lagrangian or Eulerian positions. Nevertheless, they describe the same changes in density and potential in the Earth. The subscript 0 represents the reference state. The subscript 1 indicates that it is a perturbed quantity.

It is good to be aware of the difference between the Lagrangian and Eulerian description, especially at the boundaries. However, we will derive here a linearized set of equations that describe small perturbations from the reference state. As a result, we will encounter many situations where this difference of description is of no importance. An example is the case where the reference density is multiplied by a small value ϵ. In these case we have $\rho_0(\mathbf{x})\epsilon \approx \rho_0(\mathbf{r})\epsilon$, where $\mathbf{x}$ and $\mathbf{r}$ are related through (1.1). In addition, for the perturbed density we have $\rho_1^E(\mathbf{x}, t) \approx \rho_1^E(\mathbf{r}, t)$. Similar relations hold for the reference potential ϕ_0 and the perturbed potential ϕ_1.

We will assume that no mass is created or destroyed which leads to the following equation of continuity:

$$\begin{aligned} \rho_1^E(\mathbf{r},t) + \rho_0(\mathbf{r})\nabla\cdot\mathbf{s}_L(\mathbf{x},t) + \mathbf{s}_L(\mathbf{x},t)\nabla\cdot\rho_0(\mathbf{r}) = 0 \\ \rho_1^E(\mathbf{r},t) = -\nabla\cdot[\rho_0(\mathbf{x})\mathbf{s}_L(\mathbf{x},t)] \end{aligned} \tag{1.6}$$

Note the change of $\rho_0(\mathbf{r})$ to $\rho_0(\mathbf{x})$ in the second line of this equation which is allowed as long as $\mathbf{s}$ is small.

In words, the first line of (1.6) states that the sum of the perturbed density in a small element plus the density change caused by the deformation of the element plus moving the element to another position where the reference density is different is constant.

Note that we have written the changes in density as the sum of the reference state plus a small perturbation. The small element can thus be considered to have a density ρ_1 and to be floating through a reference density field of ρ_0.

The gradient in density can be smooth or abrupt. At a layer interface the gradient is abrupt. A vertical displacement of the interface implies a density perturbation in the Eulerian system, and this density perturbation appears in Poisson's equation as the source of the perturbed potential to be discussed next.

Poisson's equation relates the gravitational potential to the density inside the Earth. Before we present this equation, the sign convention of the potential must be discussed. Normally, a potential ϕ_0 of a particle represents the amount of energy it contains. Thus, if we consider a particle above the Earth's surface, then the higher it is, the more gravitational potential energy it will have.

To get the reference gravitational force per unit mass, $\mathbf{g_0}$, at a fixed point inside the Earth, one must take the negative gradient of the potential ϕ_0 :

$$\mathbf{g}_0(\mathbf{r}) = -\nabla\phi_0(\mathbf{r}) \tag{1.7}$$

The perturbed gravity force per unit volume:

$$\begin{aligned} \rho_0(\mathbf{r})\mathbf{g}_1^E(\mathbf{r},t) &= -\rho_0(\mathbf{r})\nabla\phi_1^E(\mathbf{r},t) - \rho_1^E(\mathbf{r},t)\nabla\phi_0(\mathbf{r}) \\ &= -\rho_0(\mathbf{x})\nabla\phi_1^E(\mathbf{x},t) - \nabla\cdot[\rho_0(\mathbf{x})\mathbf{s}_L(\mathbf{x},t)]\mathbf{g}_0(\mathbf{x}) \end{aligned} \tag{1.8}$$

Here we have made use of (1.6) to substitute ρ_1 and again replaced $\mathbf{r}$ vectors for $\mathbf{x}$ vectors.

In geodesy, one sometimes inverses the sign of ϕ to make the force equal to the gradient of the potential, without adding a minus sign (Jekeli 2007). Depending on the sign convention of ϕ, Poisson's equation is

$$\nabla^2\phi_0(\mathbf{r}) = \pm 4\pi\rho_0(\mathbf{r})\mathrm{G} \tag{1.9}$$

$$\nabla^2\phi_1^E(\mathbf{r},\mathrm{t}) = \pm 4\pi\rho_1^E(\mathbf{r},\mathrm{t})\mathrm{G} \tag{1.10}$$

where G is the gravitational constant. Farrell (1972), Dahlen (1974), Wu and Peltier (1982) and Dahlen and Tromp (1998) all use the plus sign while Pekeris and Jarosch (1958) and Alterman et al. (1959) used the minus sign in (1.9). Since the work of Alterman et al. was very influential, their convention has been followed by many people such as Kaula (1963), Okubo (1988), Sun and Sjöberg (1999) and Guo et al. (2004). In this chapter we will follow the definition of Dahlen (1974) which means that we keep the potential energy interpretation of ϕ and use the plus sign in (1.9) and (1.10).

Next, since we assume that the Earth is in hydrostatic equilibrium, there is a uniform pressure p_0 at each depth layer in the reference state. This pressure p_0 inside the Earth increases with depth because the weight of the layers of rock above increases. A particle that is displaced to a deeper layer will therefore experience an upward buoyancy force $\mathbf{b}_L$. Remembering that we have to take the negative gradient to compute the force of our potential, the buoyancy force per unit volume to first order is

$$\begin{aligned} \mathbf{b}_L(\mathbf{x},t) &= \nabla[\mathbf{s}_L(\mathbf{x},t)\cdot\rho_0(\mathbf{x})\mathbf{g}_0(\mathbf{x})] \\ &= -\nabla[\mathbf{s}_L(\mathbf{x},t)\cdot\rho_0(\mathbf{x})\nabla\phi_0(\mathbf{x})] \end{aligned} \tag{1.11}$$

In addition, a force is required in a solid body to change the relative distances between the particles. In fact, it is the gradient of the change in distances between the particles, the strain, that relates linearly with the elastic force. This is called Hooke's law, and it is a linear law for small displacements. In three dimensions this linear relation for an isotropic material is given by the Cauchy stress tensor $\mathbf{T}_L$. It requires a constant for the change in volume, the bulk modulus κ, and another constant for the amount of shearing called μ. For our purpose we will assume that we can use the adiabatic bulk modulus. The relation of the Cauchy stress tensor $\mathbf{T}_L$ with the deformations $\mathbf{s}_L(\mathbf{x},t)$, also known as the constitutive law, is given by

$$\mathbf{T}_L(\mathbf{x},t) = \left(\kappa - \frac{2\mu}{3}\right)(\nabla\cdot\mathbf{s}_L(\mathbf{x},t))\mathbf{I} + \mu\left[\nabla\mathbf{s}_L(\mathbf{x},t) + (\nabla\mathbf{s}_L(\mathbf{x},t))^T\right] \tag{1.12}$$

where $\mathbf{I}$ is the identity tensor. We again add a subscript L to $\mathbf{T}$ to indicate it is Lagrangian: The elastic forces act on the deforming body. We implicitly assume that these deformations are so small that there is no significant change in the surface of the body. Otherwise the amount of pressure that is acting on the body would be different before and after the deformation. It is convenient to introduce another variable λ which is defined as $\lambda = \kappa - 2\mu/3$. The pair λ and μ are called the Lamé parameters. The elastic parameters are the entry point where—more generally speaking—the rheology of the Earth can enter. Rheology is the umbrella concept under which elasticity may be generalised to comprise a range of properties of solids describing how they deform, either instantaneously, by creep, or, in the extreme limit, by fluid-like flow or brittle failure. We will remain in the realm of linear laws (ignore stress-dependence of the moduli), avoid the brittle regime, and also ignore heat flow, convective instabilities and phase changes.

From (1.12) it is clear that when there are no displacements, there is no elastic force. However, the Earth is already in a strained situation even without external forcing because of the weight of the layers inside the Earth that are pressing on the layers beneath them (Love 1911). This weight causes the hydrostatic pressure p_0 discussed before for the buoyancy force. Therefore, (1.12) must be interpreted as the deviatoric stress tensor, which is the stress difference with respect to the reference stress state $\mathbf{T}_0$. Any additional stresses introduced into the Earth due to, for example, earthquakes, plate tectonics or mantle convection, which would create a $\frac{1}{3}tr(\mathbf{T}_0) \neq p_0$, are neglected.

The last equation we need is Newton's second law of motion, linearised, that states that the acceleration of a small element is determined by the sum of the gravity force of (1.8), the buoyancy force $\mathbf{b}_L$ of (1.11), the divergence of the stress tensor $\mathbf{T}_L$ and a body force $\mathbf{f}$. It is also known as the momentum equation

$$\begin{aligned} \rho_0(\mathbf{x})D_t^2\mathbf{s}_L(\mathbf{x},t) = -\rho_0(\mathbf{x})\nabla\phi_1^E(\mathbf{x},t) - \\ \nabla\cdot[\rho_0(\mathbf{x})\mathbf{s}_L(\mathbf{x},t)]\mathbf{g}_0(\mathbf{x}) - \\ \nabla[\mathbf{s}_L(\mathbf{x},t)\cdot\rho_0(\mathbf{x})\mathbf{g}_0(\mathbf{x})] + \nabla\cdot\mathbf{T}_L(\mathbf{x},t) + \mathbf{f}(\mathbf{x},t) \end{aligned} \tag{1.13}$$

The term D_t^2 on the left is the second order material (or Lagrangian) derivative with respect to time t. The $\mathbf{f}(\mathbf{x},t)$ is body force per volume and assumed to be small enough so that $\mathbf{f}(\mathbf{x},t) = \mathbf{f}(\mathbf{r},t)$.

Equations 1.10 and 1.13 are the same as those presented by Farrell (1972). Note that ϕ_1^E is the only Eulerian variable which will require some attention at the boundaries.

3 Spheroidal and Toroidal Motions

The tensor equations derived in Sect. 1.2 are concise and clear but they are not very convenient for numerical computations. To solve the tensor equations of motions we will chose a reference frame with the origin at the centre of mass of the undeformed Earth and use spherical coordinates (r, θ, λ) containing the radius, co-latitude and longitude, and unit direction vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\lambda$. This will produce expressions for the gradient, divergence and Laplacian that are more complicated than for a Cartesian coordinate system but it will facilitate the definition of the boundary conditions that will be discussed in Sect. 1.6

Since the east, north and up direction are always orthogonal to each other, one can avoid the theory of general curvilinear tensor components and use the more straightforward method described by Malvern (1969, App. II), Arfken (1985, Chap. 2) and Dahlen and Tromp (1998, App. A) to derive the desired expressions. Malvern and Dahlen and Tromp also list the expression for the Cauchy stress tensor in spherical coordinates. Hoskins (1910, 1920) and Pekeris and Jarosch (1958) present a complete set of all equations of motion expressed in spherical coordinates.

We will now repeat their derivation of these equations, but to do so we first need to put some limits on the shape of our deformation. According to Helmholtz's theorem, any differentiable vector field, thus also our deformations **s**, can be represented as the sum of an irrotational vector field which is the gradient of a scalar potential *f* plus a solenoidal (equivoluminal) vector field which is the curl of a vector potential **A**; see Arfken (1985, Chap. 1) and Malvern (1969, Chap. 8):

$$\mathbf{s} = \nabla f + \nabla \cdot \mathbf{A} \tag{1.14}$$

with $\nabla \cdot \mathbf{A} = 0$. In the presence of a body force **b** the equation of motion in terms of the potentials is

$$(\lambda + 2\mu)\nabla\, \nabla^2 f + \mu \nabla \times \nabla^2 \mathbf{A} + \rho \mathbf{b} = \rho \nabla \frac{\partial^2 f}{\partial t^2} + \rho \nabla \times \nabla \frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{1.15}$$

This equation is separable into a solenoidal part, independent of *f*, and a spheroidal part, independent of **A**, if we know how to partition the body force **b** into a curl-free and a divergence-less component (Lamb 1895). If the body force is zero, then (1.15) decouples into the two seismic wave equations, compressional waves with speed $v_\alpha = \sqrt{(\lambda + 2\mu)/\rho}$ and shear waves with speed $v_\beta = \sqrt{\mu/\rho}$.

The division of **s** into a spheroidal part which is both compressible and curl-free, and a complementary solenoidal part affords us a road fork in our story. Before we start to walk down the spheroidal road, let us remind ourselves of the decomposition of the vector potential **A** into a poloidal and a toroidal part according to Backus (1986):

$$\nabla \times \mathbf{A} = \nabla^2 (g\mathbf{r}) + \nabla \times (h\mathbf{r}) = \mathbf{S} + \mathbf{T} \tag{1.16}$$

where

$$\mathbf{S} = \nabla \left[\frac{\partial}{\partial r} (r\, g) \right] - \mathbf{r} \nabla^2 g \tag{1.17}$$

$$\mathbf{T} = -\mathbf{r} \times (\nabla h) \tag{1.18}$$

It shows that the divergence-free displacements can themselves be related to scalar potentials *g* and *h*. The poloidal part, **S**, will take part in the deformation due to a gravitating surface load with traction along the surface normal; the toroidal part, **T**, is insensitive to potential forces but susceptible to surface shear tractions.

In a radially symmetric planet the body force is due to the gravity potential of the load, and thus the curl of this force is zero. However, this part can be regarded as a particular solution of a non-homogeneous problem. The general problem with zero boundary conditions contains both the spheroidal and the toroidal part, and its solution the full array of free oscillations. We will restrict ourselves to the spheroidal part:

$$\mathbf{s} = \nabla f + \nabla^2 (g\mathbf{r}) \tag{1.19}$$

For deformations due to traction, see Merriam (1985, 1986). Expanding (1.19) into its spherical coordinates gives us

$$\mathbf{s} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \frac{df}{dr} - \frac{1}{r}\frac{d^2g}{d\theta^2} - \frac{1}{r\tan\theta}\frac{dg}{d\theta} - \frac{1}{r\sin^2\theta}\frac{d^2g}{d\lambda^2} \\ \frac{1}{r}\frac{df}{d\theta} + \frac{1}{r}\frac{dg}{d\theta} + \frac{d^2g}{drd\theta} \\ \frac{1}{r\sin\theta}\frac{df}{d\lambda} + \frac{1}{\sin\theta}\left(\frac{1}{r}\frac{dg}{d\lambda} + \frac{d^2g}{drd\lambda}\right) \end{pmatrix} \tag{1.20}$$

Owing to radial symmetry, the spheroidal deformation can be decomposed with spherical harmonics as angular base functions and radial factor functions for the depth-dependence:

$$u = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} U_n^m(r) Y_n^m(\theta, \lambda) \tag{1.21}$$

$$v = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} V_n^m(r) \frac{dY_n^m(\theta, \lambda)}{d\theta} \tag{1.22}$$

$$w = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} V_n^m(r) \frac{dY_n^m(\theta, \lambda)}{\sin\theta\, d\lambda} \tag{1.23}$$

We can see that $U(r)$ is associated with the radial deformation and $V(r)$ with the horizontal deformation. We may regard

$$\Psi = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} V_n^m(r) Y_n^m(\theta, \lambda) \tag{1.24}$$

as a potential of horizontal displacement, delivering the vectorial components when we let the horizontal gradient operator $[\hat{\theta} d_\theta,\ \hat{\lambda}(\sin\theta)^{-1} d_\lambda]$ act on it.

The perturbed potential ϕ_1 that appeared in (1.13) can also be written as the sum of spherical harmonics and, following tradition, the part containing the radial function will be represented by $P(r)$. Note that for the horizontal displacement we need to differentiate the spherical harmonics by θ or λ.

As we will argue below, we can restrict our treatment of the Spherical Harmonics of order $m = 0$, i.e. Legendre Polynomials of the first kind. At the same time we can avoid discussing normalisation and in particular the different variants that you may encounter in the literature.

The restriction to $m = 0$ comes without any sacrifice as to physics, since the physically relevant properties relate only to the spherical harmonic degree, while the spherical harmonic order carries information about such arbitrary things like pole location and azimuthal orientation; after all our model planet is radially symmetric (Phinney and Burridge 1973). Thus, for (1.21) we can equally well write

$$u = \sum_{n=0}^{\infty} U_n^m(r) \sum_{m=-n}^{n} C_{nm} Y_n^m(\theta, \lambda)$$

and mutatis mutandis for v and w, where the dimensionless coefficients C_{nm} come from the expansion of the forcing field (so the same set applies to u, v and w). We are only interested in the radial functions, so contemplating the simplest case for m, $m = 0$ suffices.

If we now fill in (1.20) for given degree n and order 0 into the equations of motion, (1.10) and (1.13), in spherical coordinates and drop the subscript n and superscript 0 from the coefficients U_n^0, V_n^0 and P_n^0, we get (Alterman et al. 1959; Wu and Peltier 1982)

$$\begin{aligned} \omega^2 \rho_0 U - \rho_0 \frac{dP}{dr} + g_0 \rho_0 X - \rho_0 \frac{d}{dr}(g_0 U) + \frac{d}{dr}(\lambda X + 2\mu \frac{dU}{dr}) + \\ \frac{\mu}{r^2}\left[4 \frac{dU}{dr} r - 4U + n(n+1)(-U - r\frac{dV}{dr} + 3V)\right] = 0 \end{aligned} \tag{1.25}$$

$$\begin{aligned} \rho_0 \omega^2 V r - \rho_0 P - g_0 \rho_0 U + \lambda X + r \frac{d}{dr}\left[\mu\left(\frac{dV}{dr} - \frac{V}{r} + \frac{U}{r}\right)\right] \\ + \frac{\mu}{r}\left[5U + 3r\frac{dV}{dr} - V - 2n(n+1)V\right] = 0 \end{aligned} \tag{1.26}$$

$$\frac{d^2 P}{dr^2} + \frac{2}{r}\frac{dP}{dr} - \frac{n(n+1)}{r^2} P = 4\pi G(\frac{d\rho_0}{dr} U + \rho_0 X) \tag{1.27}$$

with

$$X = \frac{dU}{dr} + \frac{2}{r} U - \frac{n(n+1)}{r} V \tag{1.28}$$

Equation 1.28 represents the dilatation of the material. Due to the sign difference in Poisson's equation, Alterman et al. use $-P$ in (1.25)–(1.27). In addition, we have assumed that the deformation is periodic with an angular velocity of ω. The second time derivative of the deformation $\mathbf{s}$ can in this case be written as $-\omega^2 \mathbf{s}$.

Next, (1.25) and (1.27) have been divided by Y_n^0 and (1.26) has been divided by $dY_n^0/d\theta$. This is important to remember for the case $n = 0$ which results in $dY_n^0/d\theta = 0$. For $n = 0$ one should simply set $V = 0$ and discard (1.26).

To derive (1.25)–(1.27) from (1.10) and (1.13) we not only needed the expressions of the gradient and divergence in spherical coordinates but also made use of the following relation:

$$\frac{d^2 Y_n^0}{d\theta^2} + \cot\theta \frac{dY_n^0}{d\theta} = -n(n+1)Y_n^0 \tag{1.29}$$

The result is that we have reduced the set of coupled differential equations from three to one dimensions, although one has to compute them repetitively for all values of degree n. In addition, since we are using spherical coordinates, we can more easily define the boundary conditions.

While some numerical methods, such as the spectral method discussed in Sect. 1.9, may integrate the second-order differential equations (1.25)–(1.28) with sufficient accuracy, we also give the six equations of first order in ∂_r, using the auxiliary variables

$$\begin{gathered} \alpha = \kappa + \frac{4}{3}\mu \qquad \beta = \kappa - \frac{2}{3}\mu \qquad \eta = 3\kappa + 2\mu \\ R = \tau_{rr} \qquad S = \tau_{r\theta} \end{gathered} \tag{1.30}$$

where α and β relate to the seismic longitudinal (compressional) and shear velocities

$$v_\alpha = \sqrt{\alpha/\rho} \qquad v_\beta = \sqrt{\beta/\rho} \tag{1.31}$$

respectively, parameters that are normally tabulated by seismologists for various depths of the Earth. As before, κ is the bulk modulus, which is the inverse of the compressibility, and μ is the shear modulus or rigidity. R and S are two components from our Cauchy stress tensor $\mathbf{T}_L$ and represent the radial and shear stress. Rewriting their definition provides us with two of the six first order differential equations:

$$\frac{dU}{dr} = \frac{1}{\alpha}\left(-\frac{2\beta}{r}U \quad + \frac{n(n+1)\beta}{r}V \quad + R\right) \tag{1.32}$$

$$\frac{dV}{dr} = -\frac{1}{r}U \quad + \frac{1}{r}V \quad + \frac{1}{\mu}S \tag{1.33}$$

Note that to here we deviate from (Dahlen and Tromp 1998, p. 271) who define our scalar V as $n(n+1)V$. The third equation is provided by rewriting the definition of the auxiliary variable Q which denotes the perturbed gravity plus a term $(n+1)P/r$:

$$\frac{dP}{dr} = -4\pi G\rho U \quad - \frac{n+1}{r}P \quad + Q \tag{1.34}$$

In Sect. 1.6 we will see that this auxiliary variable will facilitate defining the boundary condition at the surface. Filling in the definitions of R, S and Q into (1.25, 1.26, 1.27) gives us (Dahlen and Tromp 1998):

$$\frac{dR}{dr} = \left(-\omega^2\rho + \frac{12\kappa\mu}{\alpha r^2} - \frac{4g\rho}{r}\right)U \quad + n(n+1)\left(-\frac{6\kappa\mu}{\alpha r^2} + \frac{g\rho}{r}\right)V \\ - \frac{4\mu}{\alpha r}R \quad + \frac{n(n+1)}{r}S \quad + \rho Q \tag{1.35}$$

$$\frac{dS}{dr} = \left(-\frac{6\kappa\mu}{\alpha r^2} + \frac{g\rho}{r}\right)U \quad - \left(\omega^2\rho + \frac{\mu(n^2+n-2)}{r^2}\right)V \\ + \frac{\rho}{r}P \quad - \frac{\beta}{\alpha r}R \quad - \frac{3}{r}S \tag{1.36}$$

$$\frac{dQ}{dr} = -\frac{4\pi G\rho}{r}[(n+1)U \quad - n(n+1)V\,] \; + \frac{n-1}{r}Q \tag{1.37}$$

Gravity acceleration $g = g(r)$ can be computed from the density model. To derive these equations we also made use of the relation:

$$\frac{dg_0}{dr} = -\frac{2g_0}{r} + 4\pi G\rho_0 \tag{1.38}$$

Outside the Earth, only the first term on the right side of (1.38) would be necessary. However, inside the Earth to the second term is also necessary. With the usual notation $\mathbf{y} = [U,\ V,\ P,\ R,\ S,\ Q]^T$:

$$\frac{d\mathbf{y}}{dr} = \mathbf{A}\mathbf{y} \tag{1.39}$$

Another convention followed, for example, by Alterman et al. (1959), Longman (1962, 1963) and Farrell (1972) is to label vector $\mathbf{y}$ as $[y_1, \ldots, y_6]$. However, note that the definition of y_6 by Alterman et al. (1959) is different from our Q because it lacks the $(n+1)P/r$ part and represents the true perturbed gravity value. We prefer our semi-perturbed gravity parameter Q because it simplifies the formulation of the boundary condition at the surface.

At large n the radial functions run over many orders of magnitude, so that the equation system needs stabilisation. One method is to replace r and $\mathbf{Y}$ as follows

$$\tilde{r} = \frac{r}{a}, \quad q = (n+1)\log\tilde{r} \quad \text{and} \quad \mathbf{Y} = \mathbf{L}\mathbf{Z} \tag{1.40}$$

respectively, where a is the mean radius of the Earth and

$$\mathbf{L} = \exp\left\{\mathbf{diag}\left[a\sqrt{\tilde{r}},\ na\sqrt{\tilde{r}},\ ag(a)\sqrt{\tilde{r}},\ \frac{\kappa(0)\sqrt{\tilde{r}^3}}{(n+1)},\ \kappa(0)\sqrt{\tilde{r}^3},\ g(a)\sqrt{\tilde{r}^3}\right]\right\} \tag{1.41}$$

where κ_0 is the maximum incompressibility in the Earth, and to transform (1.39) according to Lyapunov (Gantmacher 1950) into

$$\frac{d\mathbf{Z}}{dq} = \mathbf{BZ} \tag{1.42}$$

where

$$\mathbf{B} = \mathbf{L}^{-1}\left(\frac{a}{n+1}\exp\left[\frac{q}{n+1}\right]\mathbf{AL} - \frac{d\mathbf{L}}{dq}\right) \tag{1.43}$$

in which

$$\mathbf{L}^{-1}\frac{d\mathbf{L}}{dq} = \frac{3}{2(n+1)}\mathbf{diag}[1,\ 1,\ 1,\ 3,\ 3,\ 3] \tag{1.44}$$

Matrix **B** has been given in full in Appendix 1. This scaling is particularly useful when one uses a numerical integration method such as Runga–Kutta to solve the differential equations; see Sect. 1.9

4 Fluid Core

So far we have assumed that the Earth is a solid body but seismologists tell us that the Earth has a fluid core. A fluid differs from a solid by having zero rigidity. Thus, by setting the shear modulus μ to zero in the Cauchy stress tensor, the equations presented in Sect. 1.3 continue to be applicable and we are treating the fluid as a very weak solid.

However, problems arise when the forcing period is taken to infinity to simulate static forcing. This phenomenon has received a relatively large amount of attention in the literature. We will now try to point out some main conclusions that have been derived.

It was Longman (1963) who showed that, for the case of $\omega = 0$, the (1.25) and (1.26) are no longer independent in the fluid core. This can be seen by writing these two equations in the following form:

$$\frac{d}{dr}[\rho_0(P - g_0U) + \lambda X] + \rho_0 g_0 X - (P - g_0 U)\frac{d\rho_0}{dr} = 0 \tag{1.45}$$

$$\rho_0(P - g_0U) + \lambda X = 0 \tag{1.46}$$

From (1.46) one can deduce that the term within the square brackets of (1.45) must be zero. If (1.46) is then used to rewrite (1.45) we have:

$$\left(\frac{g_0\rho_0}{\lambda} + \frac{1}{\rho_0}\frac{d\rho_0}{dr}\right)X = 0 \tag{1.47}$$

$$-\frac{N^2}{g_0}X = 0 \tag{1.48}$$

Here we made use of the definition of the Brunt–Väisälä frequency $N(r)$ that is related to the stratification of the fluid:

$$N^2(r) = -\frac{g_0^2 \rho_0}{\kappa} - \frac{g_0}{\rho_0}\frac{d\rho_0}{dr} \tag{1.49}$$

In a fluid $\lambda = \kappa$. As was explained before, X is the dilatation of the material; see (1.28). For a real Earth the dilatation is not always zero which leads us to the conclusion that $N = 0$ in (1.47) and this puts a new condition on the properties of the fluid that was not needed before. This situation corresponds to the so-called Adams–Williamson or neutral buoyancy condition. It means that the compressibility of the fluid is such that, when a small parcel of liquid is pushed to a deeper and denser layer, it will compress exactly to a volume with the same density as the surrounding fluid. If, however, the parcel afterwards rises up again, then the stratification of the fluid is stable, $N > 0$. If the parcel continues to sink the stratification is unstable, $N < 0$.

The fact that the fluid core can only be in neutral buoyancy seems strange and is called the Longman paradox (Dahlen 1974; Wunsch 1974; Chinnery 1975). One part of the solution of this paradox is that one should be careful when taking the limit of $\omega \to 0$. The result of this limit also depends on the real stratification of the fluid.

If the stratification is unstable, a boundary layer develops that gets thinner for increasing forcing period. In the extreme case of $\omega = 0$, it represents an infinitely thin layer but it still has a finite influence on the dynamics. The radial stress experiences a jump in the boundary layer and is zero in the fluid. Because in a fluid the radial stress is proportional to the dilatation, this means that X is zero in fluid after all and that the Adams–Williams condition, or neutral buoyancy, is no longer necessary to satisfy (1.47). In Sect. 1.7 we will discuss a homogeneous fluid which means $d\rho_0/dr = 0$ and $N < 0$. Thus, the stratification is unstable and, near the boundary of the fluid core with the mantle, such a boundary layer develops. Pekeris and Accad (1972) also discuss the results for a fluid with $N = 0$. In this case no boundary layer develops. For a stable stratified fluid, $N > 0$, core oscillations develop which get shorter and shorter wave-lengths for $\omega \to 0$.

Although Pekeris and Accad (1972) provide analytically correct solutions for the static deformation of the Earth with a fluid core, the fact that for an unstable stratification the horizontal displacement goes to infinity in the boundary layer and the fact that for a stable stratification an infinite amount of core oscillations are produced, indicates that there are still some problems.

Dahlen and Fels (1978) opposed the notion of trying to solve a Fourier-transformed problem in a fluid at the limit $\omega = 0$ from extrapolating solutions for small $|\omega| > 0$. Before we revisit the arguments of Dahlen and Fels (1978) we give our conclusion and recommendation. The static response cannot be obtained from sinusoidal load responses as a limit $\omega \to 0$; we endorse the use of a non-zero frequency when solving the load problem.

Stripping the problem down to the essentials, Dahlen and Fels (1978) showed that the same problem occurs in a stratified fluid in a box with hard side walls and a deformable lid exposed to a laterally homogeneous gravity field. The normal modes of this system pile up around zero frequency. The inverse Fourier transform employs the Cauchy principal value theorem for cases like this; however, as the open interval (0, Ω) contains infinitely many poles, albeit countably many, the principal value does not converge. In fact, if you expect a finite displacement to result, the Fourier integral of such a signal does not exist, since it is not square-integrable. Thus, if you expect a finite response at zero frequency (a doubtful concept per se), or, alternately, a finite response at infinite time, in Fourier the time is indistinguishable whether it is $+\infty$ or $-\infty$. Thus, you need to involve causality. Thus, Laplace transform and a Heaviside load history is the concept that is applicable, not Fourier transform.

Our task is perhaps not to estimate the time it takes for the system to reach the finite state within a given margin, but rather to determine the finite state. For that purpose, Dahlen and Fels (1978) suggest that an ad hoc viscosity be used for the core fluid. This will displace the poles of the inviscid system from the real frequency axis, giving them a slight imaginary part. The system can now be solved using the residual value theorem. The bottom line is that you would continue to exploit the 6×6 differential equations, changing the role of the shear modulus into a viscosity and Laplace-transform the equations such that the constitutive relation is expressed by

$$\sigma = 2\mu\dot{\epsilon} \quad \circ - \bullet \; \tilde{\sigma} = 2s\mu\tilde{\epsilon} \tag{1.50}$$

and the $-\omega^2$ factors are replaced by s^2, s being the Laplace transform parameter.

Farrell (1972) circumvented these difficulties by setting ω equal to the tidal period of harmonic M_2 (12.42 h). Since our main interest is to compute Green's functions for ocean tide loading, this approach is sufficient for us. Thus, it seems more instructive to represent the problem for non-vanishing ω, and again we follow Dahlen and Tromp (1998, Chap. 8).

The vanishing shear stress in a fluid region has the consequence that horizontal displacement becomes directly related to vertical displacement, potential perturbation, and vertical stress:

$$V = \frac{\rho_0 g(r)U + \rho_0 P - R}{\omega^2 \rho_0 r} \tag{1.51}$$

This equation has been derived from (1.26) by setting $\mu = 0$ and using the fact that the radial stress R is in this case equal to λX. One can use (1.51) to substitute V in (1.25) and (1.27) after which we are left with two second order differential equations.

When we use the six first order differential equations, then in the fluid we lose two rows from the differential equations, which reduce to

$$\frac{dU}{dr} = \left(\frac{\omega^2 g_0 n(n+1)}{r^2} - \frac{2}{r}\right)U + \left(\frac{1}{\kappa} - \frac{n(n+1)}{\omega^2 \rho_0 r^2}\right)R + \frac{n(n+1)}{\omega^2 r^2}P \tag{1.52}$$

$$\frac{dR}{dr} = \left(-\omega^2 \rho_0 - \frac{4\rho_0 g_0}{r} + \frac{n(n+1)\rho_0 g_0^2}{\omega^2 r^2}\right)U - \frac{n(n+1)g_0}{\omega^2 r^2}R + \left(\frac{n(n+1)\rho_0 g_0}{\omega^2 r^2} - \frac{\rho_0(n+1)}{r}\right)P + \rho_0 Q \tag{1.53}$$

$$\frac{dP}{dr} = -4\pi G\rho_0 U - \frac{n+1}{r}P + Q \tag{1.54}$$

$$\frac{dQ}{dr} = 4\pi G\rho_0\left(\frac{n(n+1)g_0}{\omega^2 r^2} - \frac{n+1}{r}\right)U - 4\pi G\frac{n(n+1)}{\omega^2 r^2}R + 4\pi G\rho_0\frac{n(n+1)}{\omega^2 r^2}P + \frac{n-1}{r}Q \tag{1.55}$$

At $n = 0$, the matrix elements on the right-hand side simplify considerably. The outcome being fairly obvious, we do not write it out. Since $V = 0$ for $n = 0$ the Earth just inflates or deflates a bit but remains spherically symmetric (Dahlen and Tromp 1998). As a result, the perturbed gravity is zero. If this is so, then we have the following relation for our semi-perturbed gravity parameter Q :

$$Q = -\frac{1}{r}P \tag{1.56}$$

which also provides us with the relation that states that no potential perturbation is possible except for the Bouguer effect due to vertical displacement:

$$\frac{dP}{dr} = -4\pi G\rho_0 U$$

If we do not suppose a solid inner core, the differential equations for the fluid interior can for $n = 0$ be shortened to a 2×2 system in U and R (Longman 1963):

$$\frac{dU}{dr} = -\frac{2}{r}U + \frac{1}{\kappa}R \tag{1.57}$$

$$\frac{dR}{dr} = -\left(\omega^2 \rho_0 + \frac{4g_0\rho_0}{r}\right)U \tag{1.58}$$

The general solution in a homogeneous sphere (constant κ) is

$$U(r) = r\,\exp\left[\frac{-\mathrm{i}\,\omega}{k}\,r\right]\left\{C_2\,L\left(-2-\frac{2\,\mathrm{i}\,g}{k\,\omega},3,\frac{2\,\mathrm{i}\,\omega}{k}\,r\right)\right.$$
$$\left. +C_1\,U\left(2+\frac{2\,\mathrm{i}\,g}{k\,\omega},4,\frac{2\,\mathrm{i}\omega}{k}\,r\right)\right\} \tag{1.59}$$

$$R(r) = \kappa\,\exp\left[\frac{-\mathrm{i}\,\omega}{k}\,r\right]\left\{2\,C_2\,\kappa\,r\,L\left(-2-\frac{2\,\mathrm{i}\,g}{k\,\omega},3,\frac{2\,\mathrm{i}\,\omega}{k}\,r\right)\right.$$
$$\left. +C_1\times\left(2-\mathrm{i}\,\frac{\omega}{k}\,r\right)U\left(2+\frac{2\,\mathrm{i}\,g}{k\,\omega},4,\frac{2\,\mathrm{i}\,\omega}{k}\,r\right)\right\} \tag{1.60}$$

where $k=\sqrt{\kappa/\rho}$ the compressional wave speed in the fluid, $L_n^{\alpha}(z)=L(n,\alpha,z)$ is the generalised Laguerre polynomial and $U(a,b,z)$ the Confluent Hypergeometric function of the second kind. The latter is singular at $r=0$ so we only need the Ls.

5 Resonance Effects

We will tacitly assume that the Earth–Moon system has reached a stationary situation. If you assume for the moment that there is no Moon and it suddenly appears, you will have some start up effects, among others starting seismic free oscillations which, owing to internal friction, slowly die out, resulting in the periodic tidal deformations that we experience today. So, when we say that we solve the tidal loading problem, we assume that the load acts on the surface with a temporal periodicity sufficiently different from the resonance frequencies that mode excitation can be neglected. In a purely elastic Earth, resonance occurs at sharply defined frequencies; however, in a visco-elastic mantle the resonance loses quality and the susceptible frequencies widen to finite intervals. As much as we are aware of this complication, we will avoid it by restricting the claims of our simplified approach to load frequencies well below one cycle per hour.

However, there is one resonance that needs attention, and it comes from the shape and fluidity of the core in a rotating planet. The core and the mantle rotate around slightly different axes, and the relative motion is known as Free Core Nutation or Nearly-Diurnal Free Wobble. Both astronomical tides of degree two and order one with a nearly-diurnal frequency and the associated ocean tides are able to excite the resonance although none of the forcing frequency exactly matches the 1 + 1/435 cycles per sidereal day frequency of the resonance. Wahr and Sasao (1981) have solved this problem by separating out the resonance in the load Love numbers and adding the effect to the normal Love numbers h_2, k_2 and l_2 (see Sect. 1.7 for their definition). This is possible since the resonance effects are primarily in the degree $n=2$, order $m=1$ spherical harmonics, and the excitation is due to the corresponding pro-grade ocean tide harmonic coefficients C_{21}^{+} for amplitude and ϵ_{21}^{+} for phase; see Lambeck and Balmino (1974) for the notations. It adds a complex-valued contribution that can conveniently be computed for different ocean models with the parametrisation in Scherneck (1991):

$$\left.\begin{array}{c}\Delta h_2(\omega)\\ \Delta l_2(\omega)\\ \Delta k_2(\omega)\end{array}\right\} = -\mathrm{i}\,\frac{4\pi G\rho_w a\Omega}{5(\omega-\Omega_R)}\frac{C_{21}^{+}}{\Phi_n}\exp(\mathrm{i}\epsilon_{21}^{+})\left\{\begin{array}{c}S_h'\\ S_l'\\ S_k'\end{array}\right. \tag{1.61}$$

where Ω_R is the angular rate of the resonance, Φ_n the potential coefficient of the luni-solar tide that generated the ocean tide whose pro-grade order-1 surface height is represented by $(C_{21}^{+},\ \epsilon_{21}^{+})$ and the S' coefficients signify the resonance strength in the respective load Love numbers (Wahr and Sasao specified $S_h' = -2.88\ \times 10^{-4}$, $S_l' = 9.16\ \times 10^{-6}$ and $S_k' = -1.45\ \times 10^{-4}$). Further modification is needed unless an observed tide at the exact frequency ω has been used to compute $(C_{21}^{+},\ \epsilon_{21}^{+})$. If we are forced to resort to frequency-domain interpolation, a factor is needed to take the effect of resonance in the body and load tide Love numbers into account at the instance of ocean tide generation, and possibly we have additional knowledge of the variation in ocean dynamics across the resonance band. These are the factors $R(\omega,\omega_0)$ and $D(\theta,\lambda,\omega,\omega_0)$ in Wahr and Sasao (1981, Eqs. 4.5 and 4.6).

6 Boundary Conditions

Now that the differential equations are in place, we will address the boundary conditions that they have to fulfil. Since our set of equations are only valid in material that shows smooth variations in density and elastic properties (their radial derivative must exist), we need to divide our Earth into spherical layers in order to cope with the jumps in density and elastic properties. As a result, we must prescribe boundary conditions at the Earth's centre, at the boundaries between the layers and at the Earth's surface. We will start with the boundary conditions at the centre of the Earth where the solutions are regular. This means that, for $n \neq 1$, the displacements and perturbed potential are zero. Mathematically this statement can be presented as

$$U(0) = 0, \quad V(0) = 0, \quad P(0) = 0 \tag{1.62}$$

In the case $n = 1$ we have a situation where displacements and potential perturbation require an additional constraint owing to the fact that a rigid translation can be added to the displacements. The only effect of this translation is a gravity term $\delta P = -g/u_c$. While this will be dealt with in detail in Sect. 1.8, we note for the conditions in the centre that the particular displacement field that causes no perturbation of gravity potential at both $r = 0$ and $r = a$ does imply a shift of the figure and thus of its centre. The relation with the normal-stress function S is as follows:

$$U(0) = -\frac{48\pi G\rho + 3\omega^2}{(8\pi G\rho)^2\rho - 80\pi G\rho^2\omega^2 - 3\rho\omega^4} S(0) \tag{1.63}$$

at $n = 1$ if $\omega \neq 0$ and

$$U(0) = \frac{3}{4G\pi\rho^2} S(0) \tag{1.64}$$

if $\omega = 0$. That $U(0) = V(0) = 0$ may be deduced from the fact that we have a symmetric loading for $n \neq 1$ which cannot affect the position of the origin. The reason $P(0)$=0 can be seen from Poisson's equation

$$\nabla^2 \left[P(r)Y_n^0(\cos\theta)\right] = -4\pi G\nabla \cdot \left(\rho\left[U(r)Y_n^0(\cos\theta)\hat{r} + V(r)\partial_\theta Y_n^0(\cos\theta)\hat{\theta}\right]\right) \tag{1.65}$$

Lifting the divergence from this equation and working out the components of the gradient, the θ-component of the equation tells us that

$$\frac{1}{r}P(r)\partial_\theta Y_n^0(\cos\theta) \rightarrow V(0)\partial_\theta Y_n^0(\cos\theta) \quad \text{for} \;\; r \rightarrow 0 \tag{1.66}$$

If $V(0)$ would some how settle at a non-zero value, the left-hand side would grow to infinity, which is a contradiction. And obviously, horizontal displacement cannot grow as $\mathcal{O}(1/r)$ when $r \rightarrow 0$. Thus, both $V(0)$ and $P(0)$ are zero.

Next, at the interface of two solid layers we have continuity in radial and horizontal displacements, in radial and horizontal stresses and in semi-perturbed gravity and potential. Mathematically this is represented as

$$\begin{aligned} R(r^+) = R(r^-), \quad S(r^+) = S(r^-), \quad Q(r^+) = Q(r^-) \\ U(r^+) = U(r^-), \quad V(r^+) = V(r^-), \quad P(r^+) = P(r^-) \end{aligned} \tag{1.67}$$

where r denotes the radius of the interface, r^+ just above it and r^- just below it. Of course the true perturbed gravity is also continous over the boundary, $y_6(r^+) = y_6(r^-)$. At the boundary of a solid and fluid layer the situation is a little different. If we indicate the radius of this mantle core boundary by c and assume the mantle lies above the core, we have $S(c^+) = 0$ while shear stresses in the fluid core are undefined because $\mu = 0$. Furthermore, continuity in the horizontal displacement V is no longer required, so this equation disappears. Another relation we have at the mantle-core boundary for $n = 0$ is

$$\begin{aligned} Q(c^+) &= \left.\frac{dP}{dr}\right|_{r=c^-} + 4\pi G\rho U(c^-) + \frac{1}{r}P(c^-) \\ &= \frac{1}{r}P(c^-) \end{aligned} \tag{1.68}$$

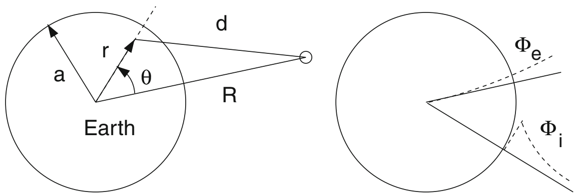

Fig. 1.1 In the *left panel* the definition of angle and distances is given. In the *right panel* we schematically show the behaviour of the external and internal perturbed potential

Finally, we need to describe the boundary conditions at the surface which depend on the type of loading is applied. Since we are interested in ocean tide loading, we assume that we have a parcel of tide-water lying on the Earth's surface. This parcel has a mass that generates a perturbation in the potential field of the solid Earth. Due to its weight, this parcel also presses on the ocean bottom. Therefore, in the ocean loading problem we must prescribe at the surface a perturbation in the potential and a normal stress. For the tidal deformation of the Earth caused by the Moon and Sun, this surface stress is zero.

Now it becomes important to distinguish between the Lagrangian and Eulerian descriptions that were explained in Sect. 1.2 The perturbed potential is a Eulerian function, evaluated at the undeformed boundary layers. Since deformation moves mass, the perturbed potential sees a 'Bouguer' effect. The stresses and displacements are evaluated at the deformed boundaries (Lagrangian) but to second order one can also just evaluate them at the undeformed boundaries.

To define our boundary conditions at the Earth's surface, it is convenient to assume that we have a unit point mass m_u at a distance R away from the Earth's centre; see Fig. 1.1. The external potential ϕ_e of this unit point mass m_u can be written as a sum of spherical harmonics:

$$\phi_e(r,\theta) = -\frac{G}{d} = -\frac{G}{a}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n Y_n^0(\cos\theta) \quad \text{for } r<R \tag{1.69}$$

We have added a minus sign because the potential should increase, become less negative, with increasing distance. Since the Earth is not completely rigid, it deforms due to the presence of this external potential, creating an additional internal potential ϕ_i. Outside the Earth this internal perturbed potential can also be written as a sum of spherical harmonics:

$$\phi_i(r,\theta) = -\frac{G}{a}\sum_{n=0}^{\infty} k_n(a)\left(\frac{a}{r}\right)^{n+1} Y_n^0(\cos\theta) \quad \text{for } r>a \tag{1.70}$$

where $k_n(a)$ are some unknown constants which will be determined later. Inside the Earth (1.70) is not valid. The total perturbed potential is $\phi_1 = \phi_e + \phi_i$. In Sect. 1.3 we have shown that for each degree n the radial part of ϕ_1 can be written as a function $P(r)$. Using the same scaling of Sect. 1.3 and setting $r=a$ we have $P_e = G/a$ and $P_i = k_n'(a)G/a$. At the surface the radial derivatives of these functions are

$$\frac{dP_e(r)}{dr} = \frac{n}{r} P_e(r) \quad \text{underneath the load} \tag{1.71}$$

$$\frac{dP_i(r)}{dr} = -\frac{n+1}{r} P_i(r) \quad \text{above the surface} \tag{1.72}$$

The perturbed gravity just below ($^-$) and above ($^+$) the surface should be equal. Remembering that the Earth's surface has been displaced due to the deformation our equation of continuity of perturbed gravity is

$$\nabla\phi_1^E(\mathbf{x},t)^- + \mathbf{s}_L(\mathbf{x},t)^- \cdot \nabla^2\phi_0(\mathbf{x},t)^- = \nabla\phi_1^E(\mathbf{x},t)^+ + \mathbf{s}_L(\mathbf{x},t)^+ \cdot \nabla^2\phi_0(\mathbf{x},t)^+ \tag{1.73}$$

Using Poisson's relation, one can replace the $\nabla^2\phi_0^-$ on the left side of the equation with $4\pi G\rho_0$ while the same term on the right is zero because we neglect the density of the atmosphere and put $\rho_0 = 0$ outside the Earth.

To first order, we will only need to consider the radial derivative and can replace the ∇ operator by d/dr. If we again decompose (1.73) into spherical harmonics, then for each degree n we have

$$\frac{dP^-}{dr} + 4\pi G\rho_0 U = \frac{dP^+}{dr} \tag{1.74}$$

If for the moment we assume that there is no external potential P_e and use (1.72) to substitute the term on the right:

$$\frac{dP_i}{dr} + \frac{n+1}{r} P_i + 4\pi G\rho_0 U = 0 \tag{1.75}$$

If we add the both the internal and external potential in (1.74), we get at the surface

$$\frac{dP}{dr} + \frac{n+1}{r} P + 4\pi G\rho_0 U \equiv Q = -\frac{2n+1}{a}\left(\frac{G}{a}\right) \tag{1.76}$$

Equation 1.76 provides the boundary condition for the semi-perturbed gravity Q at the surface. The beauty of (1.76) is that it does not contain the unknown internal potential ϕ_i explicitly.

Now we will derive the expression for a unit point load σ. According to Longman (1962), the Legendre expansion of the Dirac δ-function on a sphere with radius a is

$$\begin{aligned} \sigma &= \sum_{n=0}^{\infty} \frac{2n+1}{4\pi a^2} Y_n^0(\cos\theta) \\ &= \left(\frac{G}{a}\right) \sum_{n=0}^{\infty} \frac{2n+1}{4\pi G a} Y_n^0(\cos\theta) \end{aligned} \tag{1.77}$$

Our unit mass exerts a point load of magnitude $-g$ at the surface which means that the boundary condition for the normal stress R for degree n is

$$R = -\frac{2n+1}{a}\frac{g}{4\pi G}\left(\frac{G}{a}\right) \tag{1.78}$$

Together with the boundary condition that the horizontal stress is zero, $S = 0$, (1.76) and (1.78) provide three boundary conditions at the surface.

7 Simple Earth Models and Love Numbers

At this point it is instructive to discuss the deformation of an elastic solid Earth with constant density and constant elastic properties. For this particular situation there exist three analytical solutions for each parameter which, combined, describe the radial and horizontal deformation and perturbed potential throughout the Earth. These analytical solutions are provided by Dahlen and Tromp (1998) and are reproduced in Appendix 2. For example, the radial displacement, for degree n, is

$$\begin{aligned} U(r) &= U^{+}(r) + U^{-}(r) + U^{\oplus}(r) \\ &= y_{11}\, j_n(\gamma^{+} r) + y_{12}\, j_n(\gamma^{-} r) + y_{13}\, r^n \end{aligned} \tag{1.79}$$

Note that these solutions automatically produce zero displacements and disturbed gravity at the Earth's centre.

In the second line of (1.79), we have factored out the terms containing the spherical Bessel functions j_n and r^n and formed new coefficients y_{11}, y_{12} and y_{13} (Okubo 1988). This is not necessary but has been done to emphasise the fact that each term depends on a different function. The solutions for the horizontal displacement V and perturbed potential P can be written in the same format. All these coefficients can be grouped in a matrix:

$$\mathbf{D} = \left\{ \begin{matrix} y_{11} & y_{12} & y_{13} \\ y_{31} & y_{32} & y_{33} \\ y_{51} & y_{52} & y_{53} \end{matrix} \right\} \tag{1.80}$$

The displacement vector $\mathbf{s}$ can now be computed as $\mathbf{DJ}\boldsymbol{\theta}$ where $\mathbf{J}$ is a 3×3 matrix with our $j_n(\gamma^{+} r), j_n(\gamma^{-} r)$ and r^n terms on the diagonal. Vector $\boldsymbol{\theta}$ contains scale factors because each separate solution can be multiplied with an arbitrary constant.

From the solutions for U, V and P, we can derive the analytical solutions for the radial stress R, tangential stress S and semi-perturbed gravity Q. Again we can factor out the $j_n(\gamma^{+} r)$, $j_n(\gamma^{-} r)$ and r^n terms and form a new matrix $\mathbf{E}$ in such a way that the vector $(R, S, Q)^T$ is $\mathbf{EJ}\boldsymbol{\theta}$. Matrix $\mathbf{E}$ is defined in a similar ways as matrix $\mathbf{D}$:

$$\mathbf{E} = \begin{Bmatrix} y_{21} & y_{22} & y_{23} \\ y_{41} & y_{42} & y_{43} \\ y_{61} & y_{62} & y_{63} \end{Bmatrix} \tag{1.81}$$

Each row of matrix **E** is associated with the radial stress, tangential stress and semi-perturbed gravity. For example, the radial stress is written as

$$R(r) = y_{21}\, j_n(\gamma^+ r) + y_{22}\, j_n(\gamma^- r) + y_{23}\, r^n \tag{1.82}$$

Our next task is to estimate the scale factors $\boldsymbol{\theta}$ in order to fulfil the boundary conditions at the surface described in Sect. 1.6 Following Okubo (1988) we will compute the scaling factors for the three solutions for the body tide and load tide simultaneously. These boundary conditions, for degree n, at the surface are stored in the columns of the following matrix **x** :

$$\mathbf{x} = \frac{2n+1}{a} \begin{pmatrix} 0 & \frac{g}{4\pi G} \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \tag{1.83}$$

The first column of **x** shows that for the body tide, only the potential is non-zero at the surface. In the second column, one can see that for the load tide there is an additional radial stress. Note that the factor $-G/a$ has disappeared. Instead of a unit-mass, we are computing the deformation due to a unit-potential.

The scale factors $\boldsymbol{\theta}$ are determined with $(\mathbf{EJ}(\mathbf{a}))^{-1}\mathbf{x}$. Since the matrix **EJ** can be ill-conditioned, it makes sense to scale each row of **EJ** in such a way that the largest entry is 1. This will not change the value of $\boldsymbol{\theta}$ if vector **x** is scaled by the same factors, but will improve its numerical accuracy.

Now that these scale factors are known, we can compute the deformations U and V and the perturbed potential P at any radius r using $\mathbf{DJ}(r)\boldsymbol{\theta}$. Remember that we have computed the radial deformations $U(r)$ for a unit potential load on the Earth's surface. It was Love who represented these deformations as the product of a function $h_n(r)$ divided by g. For any other external potential ϕ_e, that again can be developed into spherical harmonics with a radial function at the surface $P_e(a)$, the radial deformations are, for degree n

$$U(r) = -h_n(r)\frac{P_e(a)}{g} \tag{1.84}$$

For the tangential displacements a similar function $l_n(r)$ is defined:

$$V(r) = -l_n(r)\frac{P_e(a)}{g} \tag{1.85}$$

The same can be done for the perturbed potential although it is customary to introduce a function k_n that is only associated to the internal perturbed potential ϕ_i :

$$P(r) = (1 + k_n(r))P_e(a) \tag{1.86}$$

Table 1.1 General constants

Constant	Unit	Value
G	$m^3kg^{-1}s^{-2}$	6.673×10^{-11}
a	m	6.371×10^{6}
ω	rad/s	1.40526×10^{-4}

Table 1.2 Properties of a homogeneous Earth (model β) and an Earth with a homogeneous mantle and a fluid core with a radius of $0.55a$ (model α)

		Model β	Model α	
Constant	Unit	All	Core	Mantle
Mean density ρ	(kg/m^3)	5517	11020	4460
Shear modulus μ	(GPa)	146	0	174
Lamé parameter λ	(GPa)	347	950	231

The minus sign in (1.84) and (1.85) is the result of our definition of the potential with the opposite sign as Love (1911) and Alterman et al. (1959). The definitions of the functions $h_n(r)$ and $l_n(r)$ already have a long tradition and it would cause too much confusion if we were to define new Love numbers with the opposite sign. Wu and Peltier (1982) follow the same sign convention of the potential as we use here but compute the deformation of the Earth due to a negative unit potential. This causes the minus sign to disappear in the definition of $h_n(r)$ and $l_n(r)$ but then it reappears in (1.86).

Normally, the values of $h_n(r), l_n(r)$ and $k_n(r)$ are only given for the Earth's surface which turns them into numbers instead of functions. The $l_n(a)$ Love number is also called the Shida number. The Love numbers are needed to compute our Green's functions to compute the ocean tide loading as we announced in Sect. 1.1 and which we will explain in more depth in Sect. 1.11

Love (1911) studied the deformation of the Earth due to the tidal force of the Moon and thus had no pressure forces on the surface. To distinguish between load Love numbers and the body tide Love numbers, the former are normally written as h'_n, l'_n and k'_n, a notation that was introduced by Munk and MacDonald (1960). As an example, we give the values of normal Love numbers and load Love numbers for a homogeneous Earth, called model β. The values for the Gravitational constant G, the mean radius of the Earth a and the angular velocity of the forcing ω (corresponding to the main tidal period of 12.42 h) are given in Table 1.1. The properties of the homogeneous Earth are listed in Table 1.2 and were taken from Alterman et al. (1959). The results are listed in Table 1.3 where we have multiplied the l, l', k and k' numbers by degree n, just to get a convenient size. The functions $h'_n(r)$ and $k'_n(r)$ are plotted in Figs. 1.2 and 1.3 for various degrees n. Note that for high values of degree n, the functions $h'_n(r)$ and $k'_n(r)$ are very small throughout the Earth and only increase near the surface. As a result, the

[1] Almost ignore; you need to assume that $g(r)/r = $ const. in order to retain the structure of the analytical solution (Vermeersen et al. 1996).

Table 1.3 The normal and load Love numbers for Earth model β for several degrees

Degree	h_n	nl_n	nk_n	h'_n	nl'_n	nk'_n
1	−18.22448	−18.22448	−18.22448	−0.18599	0.14700	0.00000
2	0.52221	0.28413	0.60384	−0.58502	−0.02167	−0.44057
10	0.10622	0.01229	0.14818	−0.88125	0.14981	−0.91403
100	0.01167	0.00014	0.01736	−1.00537	0.22378	−1.14968
1000	0.00118	0.00000	0.00177	−1.02022	0.23250	−1.17926

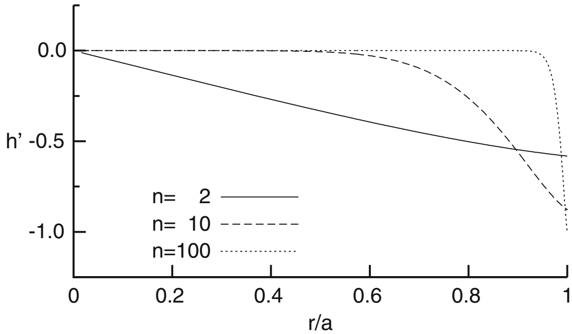

Fig. 1.2 The load love numbers h'_n for $n = 2$, 10 and 100 for a homogeneous Earth as a function of the Earth's radius

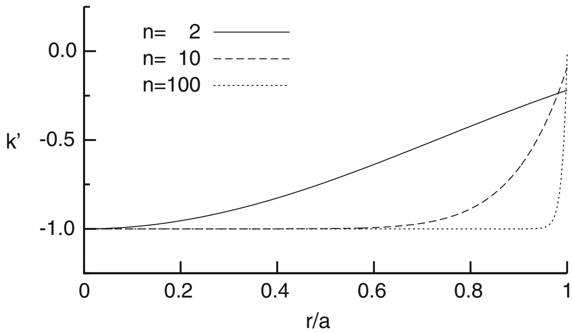

Fig. 1.3 The same as Fig. 1.2 but for k'_n

properties of the Earth just underneath the station increase in importance for increasing degree n.

Another interesting case is the deformation of an Earth with homogeneous mantle and a homogeneous liquid core. Following Alterman et al. (1959), we will call this model α. Its density and elastic properties are given in Table 1.2 . In each layer, analytical solutions for the deformation can be derived; see Appendix 2. However, in contrast to the case of the completely homogeneous Earth, in the mantle we now also need spherical Bessel functions of the second kind and solutions that contain $1/r^n$ terms. Therefore, we must extend our $\mathbf{D}, \mathbf{E}$ and $\mathbf{J}$ matrices discussed before to include these terms; see Martinec (1989).

As we discussed in Sect. 1.4 and Appendix 2, in a fluid we can derive the tangential displacement and stress from the other parameters: U, R, P and Q. In

Table 1.4 Love numbers for our model α Earth for degree $n = 2$ for different periods of forcing ($T = 2\pi/\omega$)

T	h_2	$2l_2$	$2k_2$	h'_2	$2l'_2$	$2k'_2$
6 h	0.69216	0.27579	0.72544	−0.87766	−0.08390	−0.65887
12 h	0.68037	0.27254	0.71343	−0.86444	−0.08128	−0.64730
24 h	0.67741	0.27174	0.71048	−0.86035	−0.08067	−0.64433
∞	0.67633	0.27148	0.70949	−0.85782	−0.08051	−0.64316

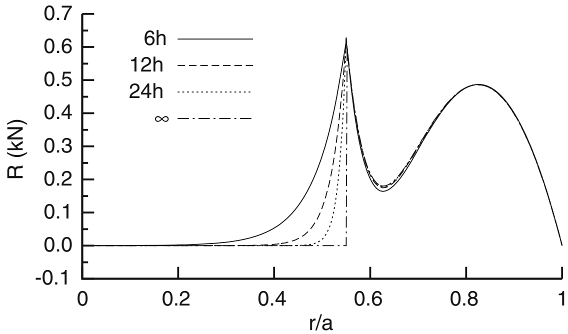

Fig. 1.4 The radial stress R inside the Earth for different values of the forcing period

addition, we only have one solution of the spherical Bessel functions of the first kind. As a result, our **E** becomes a 2×2 matrix.

We will ignore the fact that the gravity can no longer be described as $4\pi G\rho_0 r/3$ throughout the Earth, which was one of the assumptions in deriving these analytical solutions.[1]

Of course we can generalise this procedure and divide the Earth into multiple layers with constant density and constant elastic properties. Describing the problem of the deformation of the Earth as a set of propagating matrices is called the Thomson–Haskell method (Gilbert and Backus 1966) and is popular among post-glacial rebound modellers although they use something more complicated than just constant elastic properties.

Returning to our model α, the normal Love numbers for several values of the forcing period are given in Table 1.4 and the radial stress R is plotted in Fig. 1.4. In this last figure one can see that, for decreasing period, a boundary layer develops underneath the core–mantle boundary. Since in the fluid core the radial stress is related to dilatation through $R = \lambda X$, one can see that in the limit $\omega \rightarrow 0, X = 0$ throughout the core and that the Adams–Williams condition is not needed as an extra condition (Pekeris and Accad 1972).

It is interesting to see what these Love numbers would be when the limit of $\omega \rightarrow 0$ is taken. Now we should remember that the stratification of our homogeneous fluid is unstable and that a boundary layer develops (Pekeris and Accad 1972). If the jump through the boundary layer is taken into account, then we get the Love numbers listed in the last line of Table 1.4.

8 Degree-1 Response and Translational Invariance

At this point we take the opportunity to look at much discussed problem of separating displacement into a whole-body rigid translation and deformation notably at spherical harmonic degree 1; see, for instance, Blewitt (2003).

For degree 1 the situation is a little different because the load is not symmetric and this causes the Earth to move in space, in addition to deforming it. First, we will discuss the translation of the Earth in space which is equivalent to a constant $\mathbf{s}_L$. As a result, $\nabla \mathbf{s}_L = 0$. Looking at the Cauchy stress tensor, (1.12), we see that a translation of the Earth in space does not introduce any stress.

As a side note, assume for the moment that we have a homogeneous Earth with constant density. In this case, the gradient of the reference density ρ_0 is zero. From the continuity equation, (1.6), it follows that a translation of the solid Earth cannot perturb the density: $\rho_1 = 0$. Applying Poisson's equation we see that the perturbed potential ϕ_1^E is also zero and we can conclude that for a homogeneous Earth, a translation of the whole Earth does not affect our equations although it will have an effect on our boundary conditions.

For a non-homogeneous Earth, a translation will create a non-zero perturbed density ρ_1 and perturbed potential ϕ_1^E field. This is the consequence of defining a reference density ρ_0 and potential ϕ_0 field at the origin of the undeformed Earth, fixed in space, and describing the deformations as perturbations with respect to this reference field. A translation z along the $\theta = 0$ direction causes a perturbation in the potential equal to

$$\phi_1 = (r\omega^2 - g_0) z \cos\theta \tag{1.87}$$

Here we have added the potential produced by the acceleration of the translation. For tidal periods, $r\omega^2$ is much smaller than g_0 and has therefore probably been neglected by Farrell (1972).

So far we have only discussed a translation of the whole Earth. However, there also exists a degree one deformation that will generate a perturbed potential in the same way as we described in the previous sections. The only difference is that, due to the asymmetric loading, we no longer have zero displacements and a zero perturbed gravity value at the centre of the Earth and require three new boundary conditions.

To find these three new boundary conditions at the centre, we must realise that in a small ball with radius δ around this centre the Earth can be considered to be homogeneous. Repeating the results presented in Sect. 1.7 and invoking the associated mathematics from Appendix 2, we note that only the analytical solution that depends on r^n can produce displacements that are non-zero at the centre. This solution has been reproduced here (for $n = 1$) :

Table 1.5 The same as Table 1.3 but for Earth model α.

Degree	h_n	l_n	k_n	h'_n	l'_n	k'_n
1	−12.80564	−14.38607	−13.12949	−0.52853	−0.27453	−0.32385

$$
\begin{aligned}
U^{\oplus} &= c_r \frac{n}{r} r = c_r \\
V^{\oplus} &= c_r \frac{1}{r} r = c_r \\
P^{\oplus} &= c_r(\omega^2 r - \frac{4\pi}{3} G\rho r) = c_r(\omega^2 r - g_0) \\
R^{\oplus} &= c_r \frac{2n(n-1)\mu}{r^2} r = 0 \\
S^{\oplus} &= c_r \frac{2(n-1)\mu}{r^2} r = 0 \\
Q^{\oplus} &= c_r \frac{(2n+1)\omega^2 - 8\pi G\rho n(n-1)/3}{r} r = 3c_r\omega^2
\end{aligned}
\tag{1.88}
$$

where c_r is to be determined from the boundary condition at the surface. From (1.88) we can see that three new possible boundary conditions are: $U(0) = V(0), R(0) = 0$ and $S(0) = 0$. The other analytical solutions for a homogeneous sphere containing the spherical Bessel functions j_1 produce zero displacements and stresses at the Earth's centre. The solutions containing terms with $1/r$ or the spherical Bessel functions y_1 are infinite at the Earth's centre and therefore need to be set to zero.

Now that we know our new boundary conditions at the Earth's centre, let us discuss the Love numbers for our α and β Earth models discussed in Sect. 1.7 For the homogeneous Earth one can see in Table 1.3 that all normal Love numbers are the same. Because the Earth is homogeneous, no differential forces occur and the Earth does not deform but only oscillates back and forth in space. The amplitude of these oscillations is larger the longer the period of forcing. These forces produce the motion of the Earth around the solar system and are not of interest us here where we want to study tidal phenomena and our equations are only valid for small perturbations from the undeformed reference state.

The situation for the load Love numbers is different because, in addition to the gravitational attraction of the unit potential, it exerts a load on the surface in the opposite direction. That this produces a zero internal perturbed potential at the surface is just a peculiarity of a homogeneous Earth. For our Earth with a homogeneous mantle and fluid core the Love numbers for degree one are given in Table 1.5 . One can see that now the normal Love numbers are not all the same because the Earth is no longer homogeneous. Also the k'_1 load Love number is now different from zero.

It is customary to keep the origin of the reference frame fixed to the centre of mass of the deformed solid Earth. For $n \neq 1$ this always coincided with the position of the origin of the undeformed solid Earth which was the origin of our reference frame in the previous sections. However, now we must shift the frame. The centre of mass of the solid Earth has the property that it has a zero value for the perturbed potential at the surface. To achieve this we need to adjust our load Love numbers as follows (Farrell 1972):

$$\begin{aligned} [h'_1]_{CE} &= h'_1 - k'_1 \\ [l'_1]_{CE} &= l'_1 - k'_1 \\ [k'_1]_{CE} &= k'_1 - k'_1 = 0 \end{aligned} \tag{1.89}$$

For other choices for the origin of the reference frame, see Blewitt (2003). We only want to point out that all associated translations of the reference frame and modifications of the load Love numbers can be derived from our original load Love numbers h'_1, l'_1 and k'_1.

9 Numerical Methods

In Sect. 1.7 we computed the deformation of the Earth using the Thomson–Haskell method that uses the analytical solutions of the deformation inside each layer with constant density and constant elastic properties. We have already briefly mentioned that we ignored the fact that the gravity can no longer be described by $4/3\pi G\rho_0 r$ throughout the Earth. Although there are ways to minimise this last problem, one would still face problems that the deeper layers in most recent Earth models, such as PREM (Dziewonski and Anderson 1981), have density and elastic properties that vary inside each layer. Instead of also trying to minimise this problem, for example by sub-dividing these layers into layers with constant properties, we will now present methods that solve the differential equations numerically. These numerical methods are slightly more elaborate to implement than the Thomson–Haskell method but provide more flexibility. The most popular method of solving the six differential (1.32)–(1.37) is the Runge–Kutta method (Alterman et al. 1959). As with the Tomson-Haskell method, these equations are solved in each layer separately. One starts by integrating the equations from the centre of the Earth upwards to the boundary of the first layer. The computed values for the six parameters U, V, P, R, S and Q at the upper boundary are the starting values for the integration in the next layer. This process is repeated until one reaches the surface.

Starting at the centre of the Earth sounds simple. However, inspection of the differential equations shows that they are singular at $r = 0$. Secondly, we should not forget that the high spherical harmonic degrees for which we seek the load Love numbers imply extremely small deformation in the deep interior. Factoring out a scaling function and mapping the radial coordinate on a logarithmic scale helps to overcome the numerical problems. This is the Lyapunov transformation mentioned in Sect. 1.3. However, the convenience the trick gives with one hand it takes away with the other: we need starting solutions for a tiny homogeneous sphere in order to avoid the singularity problem. This has already been discussed in Sect. 1.7 but we would like to add that because of the small radius, the spherical Bessel functions of the first kind, j_n, may be approximated for radii\epsilon in the range 1–10 km,

$$j_n(\epsilon) = \frac{\sqrt{\pi}}{2\Gamma(n+3/2)}\left(\frac{\epsilon}{2}\right)^n \tag{1.90}$$

where Γ represents the Gamma function if $\epsilon \ll \sqrt{n+3/2}$, which is always fulfilled except at a few small values of n. Below $n = 10$ the spherical Bessel functions are unproblematic.

For the case of an Earth model with a fluid core at its centre we can also compute these analytical solution using a power-series ansatz. First, we replace g_0 in (1.52)–(1.55) with $4\pi G\rho_0 r/3$. With power series

$$\begin{pmatrix} U(r) \\ R(r) \\ P(r) \\ Q(r) \end{pmatrix} = \sum_{j=1}^{\infty} \begin{pmatrix} u_j/r \\ s_j \\ p_j \\ q_j/r \end{pmatrix} r^{n+j-1} \tag{1.91}$$

the differential equations produce a set of coupled recursion relations:

$$-\kappa\,\rho\,[4\pi\,G\,\rho\,n(n+1) - 3\,(j+n)\,\omega^2]\,u_j + 3\,[n\,(n+1)\,\kappa\,(s_j - \rho\,p_j)] = 3\,\rho\,\omega^2\,s_{j-2}$$

$$\begin{aligned}\rho\,[(4\pi\,G\,\rho)^2\,n(n+1) - 48\pi\,G\,\rho\,\omega^2 - 9\,\omega^4]\,u_j & \\ - 3\,[4\pi\,G\,\rho\,n(n+1)\ + 3\,(j+n-1)\,\omega^2]\,s_j & \\ - 3\rho\,[(n+1)\,(4\pi\,G\rho\,n\ - \omega^2)\,p_j + 3\,\omega^2\,q_j] &= 0\end{aligned}$$

$$4\pi\,G\,\rho\,u_j + (2n+j)\,p_j - q_j = 0$$

$$\begin{aligned}\frac{4\pi}{3}\,G\,\rho\,(n+1)\,(4\pi\,G\,\rho n\ - 3\,\omega^2)\,u_j & \\ - 4\pi\,G\,n(n+1)\,(s_j - \rho\,p_j) + (1-j)\,\omega^2\,q_j &= 0\end{aligned} \tag{1.92}$$

with starting equations

$$
\begin{aligned}
\rho\,[(4\pi\,G\,\rho)^2\,n(n+1)\ &-48\pi\,G\,\rho\,\omega^2-9\,\omega^4]\\
&-3n\,[4\pi\,G\,\rho\,(n+1)+3\,\omega^2]\,s_1\\
&-3\rho\,[(n+1)(4\pi\,G\,\rho\,n\ -\omega^2)\,p_1+3\,\omega^2\,q_1]=0
\end{aligned}
$$

$$
\rho\,(4\pi\,G\,\rho\,n\ -3\,\omega^2)\,u_1-3\,n\,(s_1-\rho\,p_1)=0 \tag{1.93}
$$

The resolved equations are shown in Appendix 3. The recursion starts with $u_1=0$, an arbitrary s_1 and a compatible $p_1=s_1/\rho$. From this, q_1 can be computed, and the recursion can step ahead to $j=3,5,\ldots$.

For degree $n\neq 1$, we know that at the Earth's centre $U=V=P=0$. However, we do not know the starting values of the radial and tangential stresses R and S, nor the starting value of the semi-perturbed gravity Q. The solution of this problem is to solve the differential equations three times and each time set another one of these three unknowns to 1 and the other two to zero. These three solutions have to be scaled afterwards to fit the boundary conditions. If we remember that in Sect. 1.3 we had written our six first order linear differential equations as, 1.39:

$$
\frac{d\mathbf{y}}{dr}=\mathbf{A}\mathbf{y} \tag{1.94}
$$

with $\mathbf{y}=[U,\ V,\ P,\ R,\ S,\ Q]^T=[y_1,\ldots,y_6]$, then for the case of ocean tide loading we can write the three solutions $\mathbf{y}^{(1)},\mathbf{y}^{(2)}$ and $\mathbf{y}^{(3)}$ at the surface as:

$$
y_2^{(1)}c_1+y_2^{(2)}c_2+y_2^{(3)}c_3=\frac{2n+1}{a}\frac{g}{4\pi G} \tag{1.95}
$$

$$
y_4^{(1)}c_1+y_4^{(2)}c_2+y_4^{(3)}c_3=0 \tag{1.96}
$$

$$
y_6^{(1)}c_1+y_6^{(2)}c_2+y_6^{(3)}c_3=\frac{2n+1}{a} \tag{1.97}
$$

Solving (1.95)–(1.97) provides us the scale factors c_1, c_2 and c_3.

We would like to emphasise that we are solving a set of non-homogeneous differential equations. In principle, we can add to these the solutions for the homogeneous differential equations that correspond to the free-oscillation of the Earth. In fact, the procedure described above is exactly how these free-oscillations of the Earth are computed. One computes the solutions of the homogeneous differential equations for various values of the forcing period $T=2\pi/\omega$ until (1.95)–(1.97) become linearly dependent, which indicates that a resonance period has been found.

As before, complications arise due to the existence of a fluid core. If we for the moment we assume that there is no solid inner core, then only need to integrate U, P, R and Q from the centre of the Earth to the core–mantle boundary as was explained in Sect. 1.4. This involves only two unknowns: R and Q. At the bottom of the mantle the tangential stress S is zero, and only the horizontal displacement V is unknown and takes the place of S in the procedure described above. If we have a solid inner core, then the situation is a little more complicated. As before, we need to compute three solutions for unit values of R, S and Q at the Earth's centre. Since we know that the tangential stress is zero at the boundary of the inner and outer core, one of the scale factors can already be written as a function of the other two. Again, we are left with three unknowns and the rest of the procedure remains the same as before.

For degree $n = 1$ we have different conditions at the Earth's centre which causes other problems because now we no longer know the values of $U(0), V(0)$ and $P(0)$. However, now $R(0) = S(0) = 0$ and we know that $U(0) = V(0)$ which is sufficient information to solve the equations.

Another numerical method that is very suitable to solve the differential equations is the spectral method where the solution is approximated by a sum of basis functions (Boyd 2000). In our case, we will use Chebychev polynomials as basis functions and our method is thus better described as the Chebychev collocation method. Its use to compute the deformation of the Earth was pioneered by Guo et al. (2001, 2004).

To explain its principles, assume that the radial displacement function $U(r)$ can be approximated by:

$$U(r) \approx U_N(r) = \sum_{i=0}^{N} a_i \psi_i(r) \tag{1.98}$$

where ψ_i is a Chebychev polynomial of degree i and a_i is a constant. Similar approximations can be made for $V(r)$ and $P(r)$. Since polynomials are easily differentiated, we can insert these approximations into our second order differential equations (1.25)–(1.27). If we evaluate these equations at $N + 1$ positions inside the Earth, called nodes, we have created $3(N + 1)$ equations with $3(N + 1)$ unknowns: the coefficients a_i of U_N, V_N and P_N. This system of linear equations can be solved and, once we know the values of the coefficients a_i, we have an approximation for the solution of the differential equations.

The distribution of these nodes should be done in such a way so as to optimize the accuracy of the approximation. It turns out that there are two good distributions, called Gauss-Radau and Gauss-Lobbato (Boyd 2000). The only significant difference between the two is that the Gauss-Radau distribution includes the end points while Gauss-Lobbato does not. Guo et al. (2001) advocate the use of the latter to avoid the singularities at the Earth's centre. However, it is convenient to replace the differential equations at these start and end nodes with the boundary conditions. At the same time singularity problems are avoided. Thus, we will use the Gauss-Radau distribution of nodes:

$$x_j = \cos\frac{\pi j}{N} \quad , \quad j = 0 \ldots N \tag{1.99}$$

where x lies between -1 and 1. We must thus scale the radius r in each layer to fit this interval. This scaling should not be forgotten when one takes the derivatives of ψ_i.

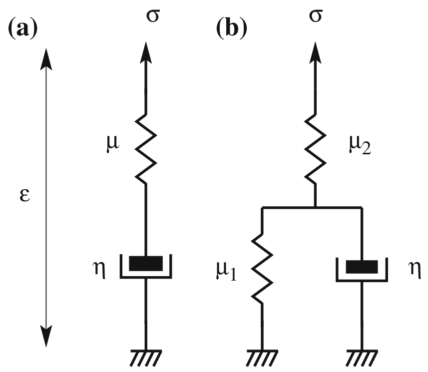

Fig. 1.5 Elementary rheological models, **a** the Maxwell body, **b** the Zener body

A disadvantage of using the three second-order differential equations is that these include the radial derivative of the elastic properties. However, these properties are normally given as low-degree polynomials, which ensures us that these derivatives can be computed easily. Another detail that might require some attention occurs again in the fluid core. Since in a fluid only four boundary conditions are needed, we must evaluate (1.26) on all nodes and not substitute the start and end points with boundary conditions. At the Earth's centre we know that for $n \neq 1, U = V = 0$ so one can make an exception and replace the differential equation on the start node at $r = 0$ with the boundary condition. For degree $n = 1$ we know that $U = V$ which removes any remaining singularities in a fluid at the Earth's centre.

As a final remark, we note that for high values of degree n, the core hardly deforms and one could in principle set the boundary conditions $U = V = P = 0$ at the core–mantle boundary.

10 Rheology: Viscosity and Anelasticity

When the temperature of rock materials is high enough yet still safely below the melting point, elastic stress will relax with time. The material will creep under stress. Whether or not this deformation recovers determines whether the material is termed anelastic or inelastic, respectively (Nowick and Berry 1972).

Basic properties can be illustrated with rheological circuit diagrams. Consider for instance the Maxwell body (Fig. 1.5a) as an example of irreversible deformation. When stress is applied, the viscous element starts creeping, but when the stress is removed it remains in the deformed position. Application of a single Maxwell model to explain delayed recovery from deformation is commonly proposed in the problem of Glacial Iostatic Adjustment (GIA). This is a phenomenon on a time scale of 1,000–100,000 years. At the time scale of tides the viscosity that is inferred from GIA studies produces very small effects of inelasticity, and we may wonder whether other approaches viable in long-period seismology might not be better suited.

We serve ourselves from developments in seismology that were set out to explain Q, the quality factor that describes the attenuation of seismic waves along their paths, or, if you wish, the decay of free oscillations over time (Knopoff 1964). The recipes that follow below will end in what is known in seismology as Generalised Maxwell rheology and the Standard Linear Solid.

Parallel connection of the dash-pot with a spring element is a simple model for recoverable strain, since now elastically stored energy is left to do work on the viscous element. By the same token the body shows stress relaxation when a strain is prescribed as a step. In the model referred to as Zener body or, alternatively, Standard Linear Solid (Fig. 1.5b), shortly after deformation, much force is put on the viscous element. As time goes by, the viscous element relaxes and stress is shared by two elastic elements in series.

We leave the solid state physics of stress relaxation or strain retardation aside and concentrate instead on how rheology enters into our differential equations.

First of all, the temporal aspect adds a phase-shifted relation between stress and strain. Fourier-transforming the shear deformation

$$\frac{\sigma}{2\mu} = \epsilon$$

(elastic), respectively

$$\frac{\dot{\sigma}}{2\mu} + \frac{\sigma}{2\eta} = \dot{\epsilon}$$

(Maxwell) gives

$$\left(\frac{\mathrm{i}\omega}{2\mu} + \frac{1}{2\eta}\right)\sigma = \mathrm{i}\omega\epsilon \tag{1.100}$$

or

$$\mu(\mathrm{i}\omega) = \frac{\mathrm{i}\omega\mu\eta}{\mathrm{i}\omega\eta + \mu} \tag{1.101}$$

which, at $\omega \to \infty$, displays unaltered elasticity μ, but at $\omega \to 0$, the resistance to shear is zero. The quantity η/μ is called relaxation time or Maxwell time. Thus, the only effect is that the equations need to be doubled with an imaginary part. The stress and gravity boundary conditions remain real-valued.

Slightly more complicated, the following exercise considers the Zener body. Here

$$\mu(\tau, \mathrm{i}\omega) = \frac{\mu_2(\mu_1 + \mathrm{i}\omega\eta)}{\mu_1 + \mu_2 + \mathrm{i}\omega\eta} = \frac{\mu_1\mu_2(1 + \mathrm{i}\omega\tau)}{\mu_2 + \mu_1(1 + \mathrm{i}\omega\tau)}$$

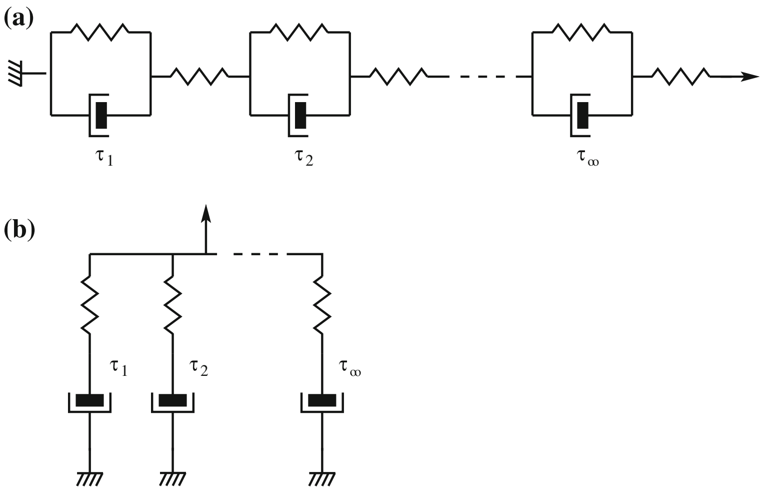

Fig. 1.6 Continuous rheological models, **a** an infinite chain of Zener elements, **b** the Generalised Maxwell Body. If the range of relaxation times is finite, the bodies may serve as models for the seismic absorption band. Extrapolation to non-seismic frequencies might be daring

with relaxation time $\tau = \eta/\mu_1$. Its zero- and infinite-frequency responses are the relaxed compliance $1/\mu_r = 1/\mu_1 + 1/\mu_2$ and the unrelaxed one $\mu_u = 1/\mu_2$, respectively.

In order to widen the discrete circuitry to a continuum, we may imagine a spectrum of relaxation times distributed over an infinite chain of Zener elements. The resulting model body is designated as the Standard Linear Solid (Fig. 1.6). Strain is now described as the integral over this infinite number of elements given the stress

$$\epsilon(\mathrm{i}\omega) = \left(\int_{\tau_1}^{\tau_2} A(\tau) \frac{d\tau}{2\mu(\tau, \mathrm{i}\omega)} \right) \sigma(\mathrm{i}\omega) \tag{1.102}$$

For normalisation of $A(\tau)$, the compliance—the integral in (1.102)—is to evaluate the relaxed and unrelaxed values at zero and infinite frequency, respectively, which amounts to demanding

$$\int_{\tau_1}^{\tau_2} A(\tau)\, d\tau = 1$$

Zschau (1983) gave an expression for the shear compliance of the absorption band model:

$$\frac{1}{\mu(\mathrm{i}\omega)} = \frac{1}{\mu_\infty}\left\{1 + \Delta^* - \frac{\mathrm{i}\omega\alpha}{1+\alpha}\,\frac{\Delta^*}{\tau_2^\alpha - \tau_1^\alpha}\left[\tau^{1+\alpha}\mathrm{F}(1+\alpha, 1; 2+\alpha; -\mathrm{i}\omega\tau)\right]_{\tau_1}^{\tau_2}\right\} \tag{1.103}$$

where F is the hypergeometric function, Δ^* the creep strength of the body

$$\Delta^* = \frac{\epsilon(t \to \infty) - \epsilon(t = 0)}{\epsilon(t = 0)}$$

(the ratio of after-effect strain to initial strain) and μ_∞ the shear modulus at ultra-seismic frequencies. The elegance of this model lies in the parsimony with three parameters, α, Δ^* and μ_∞.

Let us end this section by mentioning another generalised body, this time based on an infinite parallel coupling of Maxwell bodies with a spectrum of relaxation times, the Generalised Maxwell Body (b in Fig. 1.6). Instead of prescribing stress we now prescribe strain and formulate how the stress relaxes

$$\sigma(\mathrm{i}\omega) = 2\left(\int A(\tau)\mu(\mathrm{i}\omega,\, \tau)\, d\tau\right)\epsilon(\mathrm{i}\omega)$$

where $\mu(\mathrm{i}\omega, \tau)$ is now taken from (1.101). Closed formulas for the Generalised Maxwell Body are generally not possible; however Müller (1983) has shown such expressions for fractional integer exponents $A \propto \tau^{1/n}$. The banded nature of A as it appears limited between two finite relaxation times might appear as rather artificial. However, this so-called high-temperature background or absorption-band model has been shown to own some realism in laboratory studies of rock samples (Kampfmann and Berckhemer 1985). It can be argued that the absorption band model has an upper limit where Maxwell rheology (irreversible deformation) exceeds the relaxation that the generalised Zener body accomplishes, so that it is the lower band edge that is more critical to determine, but at the same time it is more accessible to observation owing to studies of seismic wave propagation.

11 Green's Functions

Computation of loading effects at a field point due to the tide loads distributed over the ocean surfaces of the planet is conveniently carried out using convolution with a Green's kernel function. The method and its alternative decompose the load into surface spherical harmonics and multiplication of the wave number spectrum with the load Love number spectrum has been presented comprehensively in Agnew (2007), including the extension to infinite harmonic degree in the point load problem and the generation of Green's functions using Kummer transforms. We refer to this work as seminal.

Farrell (1972) also considered disk loads in order to avoid the Gibbs phenomenon when loads are treated as if condensed to singular mass points. This creates a problem particularly when the loads are known at a few locations or on sparse grids only. As it turned out, the assumption of loading mass distributed over a circular disk of certain size attenuates the high-degree terms when summing over the load Love numbers and helps to speed up the convergence of the infinite sums in the Green's functions. The ever increasing spatial resolution of modern ocean tide models render this approach mostly obsolete today. However, the assumption of disk-distributed masses might still be necessary in a few cases.

In gravity, near-by masses exert a notable attraction effect if gravimeter and load mass are located at different heights, more specifically when a big correction ϵ is required in the equation

$$\frac{\cos\phi}{d^2} = \frac{-1}{4a^2\sin\theta/2} + \epsilon$$

where d is slope distance, ϕ the zenith angle under which the gravimeter "sees" the load, and θ the arc distance of the two points after mapping them on the sphere. An example of this effect is described by Bos et al. (2002). However, cylindrical disk loads will in general be inept to represent the actual mass distribution, and consequently, the specific geometry of the load needs to be accounted for. This amounts to have to sample the land-sea distribution with high resolution while the details of tide height in the wet areas will be uncritical.

In tilt as measured by vertical pendulums or fluid-filled tubes (Ruotsalainen 2001), loads at close distance have both a strong direct attraction effect and sensible influence due to deformation. As a third complication, the infinite sum in the Green's function converges most slowly among those considered by Farrell (1972) and Agnew (2007), even when the Kummer trick is utilised (numerical precision in the then final sums becoming critical).

We exemplify Green's function computation only in the case of tilt. The function is given as

$$\mathcal{G}_{tilt}(\theta) = \frac{GM_l}{g\,a^2} t_0 \sum_{n=1}^{\infty} (t^{-n} - h'_n + k'_n) \frac{dP_n(\cos\theta)}{d\theta}$$

with t the station height ratio $(a+h)/a$ and $t_0 = t$ if $h<0$ else $t_0 = 1$. With the disc factor included, the sum terms receive a factor $D_n = D_n(\delta)$ according to Farrell (1972):

$$D_n = -\frac{1+\cos\delta}{n(n+1)\sin\delta} \frac{dP_n(\cos\delta)}{d\delta}$$

and the sum

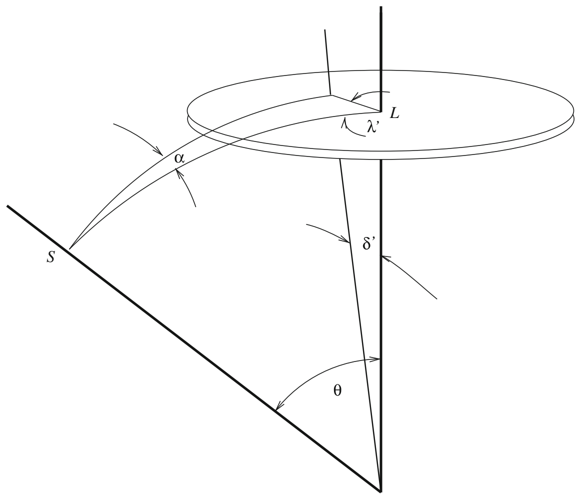

Fig. 1.7 Geometry when integrating over a disk load

$$\sum_{n=0}^{\infty} D_n P_n(\cos\theta) = \begin{cases} 0, & \theta > \delta \\ 1, & \theta \leq \delta \end{cases}$$

Ultimately, the point load problem cannot avoid dealing with the deformation in the asymptotic limit of an infinitely large pressure on an infinitely small surface area. This is the point where the analytical half-space solutions of Farrell (1972) come into play. The load Love numbers obtained are h'_∞ and $(Nl')_\infty$, and

$$(Nk')_\infty \sim \frac{3\rho_0(a)}{2\bar{\rho}}(h'_\infty + (Nl')_\infty) \tag{1.104}$$

appears to be a good approximation ($\bar{\rho}$ is the mean density of the Earth). For exercising the Kummer transformation we utilise that

$$h'_n \to h'_\infty \qquad nk'_n \to (Nk')_\infty$$

as $n \to \infty$, so we need evaluations or expressions of the infinite sums

$$\sum_{n=1}^{\infty} t^{-n} D_n \frac{dP_n(\cos\theta)}{d\theta}$$

$$\sum_{n=1}^{\infty} D_n \frac{dP_n(\cos\theta)}{d\theta}$$

$$\sum_{n=1}^{\infty} \frac{D_n}{n} \frac{dP_n(\cos\theta)}{d\theta}$$

Instead of scanning volumes of forgotten lore for analytical expressions, we can evaluate these sums without the disk factor and disk-integrate the resulting analytical expressions numerically (not forgetting the cosine of the azimuth):

$$\sum_{n=1}^{\infty} t^{-n} D_n \, P_n(\cos\theta) = \int_0^{\delta} \int_0^{2\pi} \frac{\cos\alpha \; \sin\delta' \; d\lambda' \; d\delta'}{\sqrt{1 - 2t\cos\theta' + t^2}}$$

Since we deal with short distances, the spherical trigonometric relations between the angles θ', α' and δ', λ' (see Fig. 1.7) can be approximated with plane trigonometry.

$$\sum_{n=1}^{\infty} t^{-n} \frac{dP_n(\cos\theta)}{d\theta} = -\frac{t\,\sin\theta}{(1 - 2t\cos\theta + t^2)^{3/2}} \tag{1.105}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{dP_n(\cos\theta)}{d\theta} = -\frac{1}{2}\cot(\theta/2)\,\frac{1 + 2\sin(\theta/2)}{1 + \sin(\theta/2)} \tag{1.106}$$

To be complete, we now give the definitions of the Green's function for the other type of deformations; see also Agnew (2007). In all cases we assume a distribution of mass elements dm' over the oceanic areas $\mathcal{O}$ and a solution to a deformation problem where radial symmetry of the planet's properties causes deformation under the load depending only on the the arc distance $\theta' = \angle\mathbf{r}, \mathbf{r}'$ between load and field point.

Vertical displacement

$$\begin{aligned} u(\mathbf{r}) &= \frac{G}{g\,a} \int_{\mathcal{O}} \mathcal{G}_u(\theta')\, dm' \\ \mathcal{G}_u(\theta) &= \sum_{n=0}^{\infty} h'_n \, P_n(\cos\theta) \end{aligned} \tag{1.107}$$

Horizontal displacement in the directions $\hat{n}$ (north) and $\hat{e}$ (east)

$$\begin{aligned} \left. \begin{matrix} v_n(\mathbf{r}) \\ v_e(\mathbf{r}) \end{matrix} \right\} &= \frac{G}{g\,a} \int_{\mathcal{O}} \mathcal{G}_v(\theta') \left\{ \begin{matrix} \cos\alpha' \\ \sin\alpha' \end{matrix} \right\} dm' \\ \mathcal{G}_v(\theta) &= \sum_{n=1}^{\infty} l'_n \, \frac{dP_n(\cos\theta)}{d\theta} \end{aligned} \tag{1.108}$$

Gravity on deforming surface

$$
\begin{aligned}
\Delta g(\mathbf{r}) &= \frac{G}{a^2}\int_{\mathcal{O}} \mathcal{G}_{\Delta g}(\theta')\, dm' \\
\mathcal{G}_{\Delta g}(\theta) &= \sum_{n=0}^{\infty}[2h'_n - (n+1)k'_n]\, P_n(\cos\theta) \\
&\quad - \frac{z/a + 2\sin^2\theta/2}{[(z/a)^2 + 4(1+z/a)\sin^2\theta/2]^{3/2}}
\end{aligned}
\tag{1.109}
$$

where z is the topographic height of the station, (error in Agnew (2007), who has $[\,]^3/2$ instead of $[\,]^{3/2}$ in the denominator; see also Appendix 4).

Geoid height

$$
\begin{aligned}
N(\mathbf{r}) &= \frac{G}{g\,a}\int_{\mathcal{O}} \mathcal{G}_N(\theta')\, dm' \\
\mathcal{G}_N(\theta) &= \sum_{n=0}^{\infty}(1+k'_n)\, P_n(\cos\theta)
\end{aligned}
\tag{1.110}
$$

Tide raising potential

$$
\begin{aligned}
P(\mathbf{r}) &= \frac{G}{a}\int_{\mathcal{O}} \mathcal{G}_P(\theta')\, dm' \\
\mathcal{G}_P(\theta) &= \sum_{n=0}^{\infty}(1+k'_n-h'_n)\, P_n(\cos\theta)
\end{aligned}
\tag{1.111}
$$

Tilt in the directions $\hat{n}$ (north) and $\hat{e}$ (east)

$$
\begin{aligned}
\left.\begin{matrix} t_n(\mathbf{r}) \\ t_e(\mathbf{r}) \end{matrix}\right\} &= -\frac{G}{g\,a^2}\int_{\mathcal{O}} \mathcal{G}_t(\theta')\left\{\begin{matrix} \cos\alpha' \\ \sin\alpha' \end{matrix}\right\} dm' \\
\mathcal{G}_t(\theta) &= \sum_{n=1}^{\infty}(1+k'_n-h'_n)\, \frac{dP_n(\cos\theta)}{d\theta}
\end{aligned}
\tag{1.112}
$$

Astronomical deflection of the vertical in $\hat{n}$ (north) and $\hat{e}$ (east)

$$
\begin{aligned}
\left.\begin{matrix} \eta = i_n(\mathbf{r}) \\ \xi = i_e(\mathbf{r}) \end{matrix}\right\} &= -\frac{G}{g\,a^2}\int_{\mathcal{O}} \mathcal{G}_t(\theta')\left\{\begin{matrix} \cos\alpha' \\ \sin\alpha' \end{matrix}\right\} dm' \\
\mathcal{G}_i(\theta) &= \sum_{n=1}^{\infty}(1+k'_n-l'_n)\, \frac{dP_n(\cos\theta)}{d\theta}
\end{aligned}
\tag{1.113}
$$

Strain in $\hat{n}$ (north) and $\hat{e}$ (east)

$$\begin{pmatrix} \epsilon_{nn}(\mathbf{r}) & \epsilon_{ne}(\mathbf{r}) \\ \epsilon_{ne}(\mathbf{r}) & \epsilon_{ee}(\mathbf{r}) \end{pmatrix} = \frac{G}{g\,a^2}\int_{\mathcal{O}} \mathbf{T}\begin{pmatrix} \mathcal{G}_{\theta\theta}(\theta') & 0 \\ 0 & \mathcal{G}_{\alpha\alpha}(\theta') \end{pmatrix}\mathbf{T}^{\top} dm'$$
$$\mathcal{G}_{\theta\theta}(\theta) = \sum_{n=0}^{\infty}[h'_n - n(n+1)l'_n - l'_n \cot\theta \frac{d}{d\theta}]\, P_n(\cos\theta) \tag{1.114}$$
$$\mathcal{G}_{\alpha\alpha}(\theta) = \sum_{n=0}^{\infty}(h'_n + l'_n \cot\theta)\, P_n(\cos\theta)$$

$$\mathbf{T} = \begin{pmatrix} \cos\alpha' & \sin\alpha' \\ -\sin\alpha' & \sin\alpha' \end{pmatrix} \tag{1.115}$$

We give useful formulas for arc distance θ' and azimuth α' :

$$\theta' = 2\arcsin\sqrt{\sin^2\left(\frac{\beta'-\beta}{2}\right) + \sin^2\left(\frac{\lambda'-\lambda}{2}\right)\cos\beta\ \cos\beta'} \tag{1.116}$$

$$\begin{aligned} \alpha' &= \operatorname{atan2}\left(\frac{\sin\beta' - \cos\theta'\sin\beta}{\cos\theta'},\ \cos\beta'\sin(\lambda'-\lambda)\right) \\ &= \operatorname{atan2}(\cos\alpha',\ \sin\alpha') \end{aligned} \tag{1.117}$$

where β is latitude, λ longitude, the dashed coordinates are associated with the load points and the undashed with the field point.

12 Final Remarks

We have presented here the derivation of the differential equations for the elastic-gravitational deformation of the Earth. As we have noted in our introduction, this is a classic topic that has been discussed extensively in the literature. However, we felt that current literature does not provide much information on how these equations can be implemented into a computer program. For that reason, we have tried to put more emphasis on the differences in sign conventions and definitions of variables that one may find in various publications. We also have presented here all boundary conditions explicitly and have paid particular attention to the problems related to the existence of a fluid core. Although we have focussed on ocean tide loading, where the period of the forcing is finite, we have shown which problems occur in the fluid core when the forcing period becomes infinite. We recalled the results of Pekeris and Accad (1972) who showed that for a fluid with an unstable stratification, an infinitely thin boundary layer develops that has a finite influence on the deformation. However, we pointed out that one is interested in computing the static deformation of the Earth, and some kind of dissipation should be taken into account (Wunsch 1974; Dahlen and Fels 1978).

Next, we put more emphasis than usual on the degree 1 deformation. This deformation differs from the other degrees by the fact that it causes a shift of the whole Earth in space. We also explicitly described the new boundary conditions at the centre of the Earth for this degree which is rarely done.

As examples, we presented the deformation of a homogeneous Earth and an Earth with a homogeneous mantle and fluid. For these simple cases analytical solutions exist that involve spherical Bessel functions of the first and second kind. Since we are interested in ocean tide loading where we need to compute load Love numbers up to degree 10,000, we have presented in Appendix 2 an algorithm to achieve this. The results can be used to validate the implementation of other numerical methods such as Runge–Kutta and Chebychev collocation. The case of a homogeneous Earth is also used to solve the singularity problem at the Earth's centre for the Runge–Kutta method.

In our short treatise the ocean tide loading response of the Earth is still assumed to be completely elastic. However, we showed how the elastic properties need to be changed to represent more realistic rheology. Finally, we discussed the formulas that transform the set of Love numbers into Greens functions.

Appendix 1: Lyapunov-Transformed Matrices

The Lyapunov-transformed matrix designated **B** in (1.43) is

$$\begin{pmatrix}
\frac{-2(3\kappa-2\mu)}{(n+1)(3\kappa+4\mu)} & \frac{n^2(3\kappa-2\mu)}{3\kappa+4\mu} & 0 & \frac{3\kappa_0 Z^2}{(1+n)^2(3\kappa+4\mu)} & 0 & 0 \\
\frac{-1}{n(n+1)} & \frac{1}{n+1} & 0 & 0 & \frac{\kappa_0 Z^2}{\mu n(n+1)} & 0 \\
\frac{-4\pi\rho a G Z}{g_0(n+1)} & 0 & -1 & 0 & 0 & \frac{Z^2}{n+1} \\
\frac{a^2}{\kappa_0}\left(\frac{36\kappa\mu}{a^2(3\kappa+4\mu)Z^2} - \omega^2\rho - \frac{4g\rho}{aZ}\right) & \frac{n^2(n+1)}{\kappa_0 Z^2}\left(a g\rho Z - \frac{18\kappa\mu}{3\kappa+4\mu}\right) & \frac{-a g_0(1+n)\rho}{\kappa_0 Z} & \frac{-12\mu}{(n+1)(3\kappa+4\mu)} & n(n+1) & \frac{a g_0 \rho Z}{\kappa_0} \\
\frac{1}{(n+1)\kappa_0 Z^2}\left(a g\rho Z - \frac{18\kappa\mu}{3\kappa+4\mu}\right) & \frac{a^2 n}{(n+1)\kappa_0}\left(\frac{6(2n(n+1)-1)\kappa\mu+4(n(n+1)-2)\mu^2}{a^2(3\kappa+4\mu)Z^2} - \omega^2\rho\right) & \frac{a g_0\rho}{(n+1)\kappa_0 Z} & \frac{-3\kappa+2\mu}{(n+1)^2(3\kappa+4\mu)} & \frac{-3}{n+1} & 0 \\
-\frac{4\pi\rho a G}{g_0 Z} & \frac{4\pi\rho a G n^2}{g_0 Z^2} & 0 & 0 & 0 & \frac{n-1}{n+1}
\end{pmatrix} \tag{1.118}$$

where $Z = \exp\frac{q}{n+1}$, and κ and μ have been made dimensionless by scaling with respect to κ_0 at the centre.

The matrix for the fluid case in the variables U, S, P and Q is

$$\begin{pmatrix}
-\frac{5}{2(1+n)} + \frac{g n\omega^2}{aZ} & \frac{\left(\frac{Z^2}{\kappa} - \frac{n(1+n)}{a^2\rho\omega^2}\right)}{(n+1)^2} & \frac{n}{a^2\omega^2} & 0 \\
a^2\rho\left(-\frac{4g}{aZ} + \frac{g^2 n(1+n)}{a^2 Z^2\omega^2} - \omega^2\right) & -\frac{3}{2(1+n)} - \frac{g n}{aZ\omega^2} & (1+n)\rho\left(-1+\frac{g n}{aZ\omega^2}\right) & a g_0 Z\rho \\
-\frac{4a^2 G\pi\rho}{(1+n)} & 0 & -1 - \frac{3}{2(1+n)} & \frac{a g_0 Z}{1+n} \\
\frac{4G\pi\rho(g n - aZ\omega^2)}{g_0 Z^2\omega^2} & -\frac{4Gn\pi}{a g_0(1+n)Z\omega^2} & \frac{4Gn\pi\rho}{a g_0 Z\omega^2} & 1 - \frac{7}{2(1+n)}
\end{pmatrix} \tag{1.119}$$

Appendix 2: Analytical Solution for a Homogeneous Earth

In Sect. 1.7 we discussed the computation of the Love numbers for a homogeneous Earth. In this case there exist, for each degree n, analytical solutions for the radial and tangential displacements U and V and for the perturbed potential P. These were presented for degree 2 by Love (1911) and Pekeris and Jarosch (1958). For all degrees n these were presented by Okubo (1988) and Dahlen and Tromp (1998) although they both contain small sign errors. Therefore, we will present them again, hopefully without errors, in this appendix.

The analytical solutions contain spherical Bessel functions which are defined as

$$j_n(z) = \sqrt{\frac{\pi}{2z}} J_{n+1/2}(z) \tag{1.120}$$

where n is our usual degree, z a complex number and J the normal Bessel function of the first kind. For $n = 0$ and $n = 1$, these spherical Bessel functions are

$$j_0(z) = \frac{e^{iz} - e^{-iz}}{2iz} = \frac{\sin z}{z} \tag{1.121}$$

$$j_1(z) = \frac{e^{iz} - e^{-iz}}{2iz^2} - \frac{e^{iz} + e^{-iz}}{2z} = \frac{\sin z}{z^2} - \frac{\cos z}{z} \tag{1.122}$$

As it turns out, we shall only be using values of z that are real or purely imaginary. In the first case we are dealing with fractions containing trigonometric functions. In the second we are dealing with fractions containing exponentials. To avoid numerical problems for large values of z, it is convenient to work with the logarithm of $j_n(z)$. Higher orders of $j_n(z)$ can in principle be computed using the following recurrence relation:

$$j_{n+1}(z) = \frac{2n+1}{z} j_n(z) - j_{n-1}(z) \tag{1.123}$$

However, this recursive equation is numerically unstable for increasing values of n. To compute the higher orders, we should use the algorithm of Rothwell (2008) who introduced the ratio R_n :

$$R_n = \frac{j_{n-1}(z)}{j_n(z)} = \frac{1}{(2n-1)/z - R_{n-1}} \tag{1.124}$$

$$R_n(z) = \frac{2n+1}{z} - \frac{1}{R_{n+1}(z)} \tag{1.125}$$

The continued fraction can for each degree n be computed using Lentz's method which is numerically stable (Press et al. 1988). After R_n has been computed using Lentz's method for the largest value of n, the other values of R_n can be computed using (1.125). The logarithm of $j_n(z)$, with $n \geq 1$, can now be computed as

$$\log j_n(z) = \log j_0(z) - \sum_{k=1}^{n} \log R_k \tag{1.126}$$

Note in these equations that the logarithm functions include their analytical continuation in the complex plane because R_n can be complex valued. Since the R_n are either purely real or purely complex in our application we have

$$\begin{aligned} \log i|x| &= \log|x| + i\frac{\pi}{2} \\ \log -|x| &= \log|x| + i\pi \end{aligned} \tag{1.127}$$

so the sum in (1.126) accumulates a factor of $-(\mathrm{i}^n)$. A gain in accuracy can be achieved if the summation in (1.126) is carried out separately on the characteristic and the mantissa.

We will now assume that the solutions are of the form

$$U = A_u r^{-1} j_n(\gamma r) + B_u \gamma j_{n+1}(\gamma r) \tag{1.128}$$

$$V = A_v r^{-1} j_n(\gamma r) + B_v \gamma j_{n+1}(\gamma r) \tag{1.129}$$

$$P = A_p j_n(\gamma r) \tag{1.130}$$

That the solutions contain a combination of j_n and j_{n+1} makes sense because you can write the first and second derivatives of these two functions again as a sum of parts containing j_n and j_{n+1}. Furthermore, one has to use j_n/r and j_{n+1} to ensure that both are of the same order of r. That the perturbed potential solution does not contain a j_{n+1} part seems odd at first glance. However, from Poisson's equation, (1.10), we know that the Laplacian of the perturbed potential depends on the perturbed density. The latter depends on the divergence of the displacement and one can prove that the j_{n+1} terms cancel out in the divergence (1.28).

Our task is to determine the six constants: A_u, B_u, A_v, B_v, A_p and γ. This can be done using a computer algebra program such as Mathematica or Maxima. However, we will directly present the answer in the same format as that of Dahlen and Tromp (1998, Chap. 8) who introduced the following auxiliary variables and solution for γ :

$$k = \sqrt{n(n+1)} \tag{1.131}$$

$$\gamma^2 = \frac{\omega^2}{2v_\beta^2} + \frac{\omega^2 + \frac{16}{3}\pi G\rho}{2v_\alpha^2} \pm \frac{1}{2}\left[\left(\frac{\omega^2}{v_\beta^2} - \frac{\frac{16}{3}\pi G\rho}{v_\alpha^2}\right)^2 + \left(\frac{8\pi G k\rho}{3v_\alpha v_\beta}\right)^2\right]^{1/2} \tag{1.132}$$

$$\zeta = \tfrac{3}{4}(\pi G\rho)^{-1} v_\beta^2(\gamma^2 - \omega^2/v_\beta^2) \tag{1.133}$$

$$\xi = \zeta - (n+1) \tag{1.134}$$

where v_α and v_β are the compressional and shear wave velocities, respectively. Besides solutions that contain spherical Bessel function, there exist solutions containing terms with r^n. To summarise, for each of the radial and tangential displacements U and V and the perturbed potential P we have three independent solutions:

$$U^\pm = \frac{n\xi}{r} j_n(\gamma r) - \zeta\gamma j_{n+1}(\gamma r) \quad , \quad U^\oplus = nr^{n-1} \tag{1.135}$$

$$V^\pm = \frac{\xi}{r} j_n(\gamma r) + \gamma j_{n+1}(\gamma r) \quad , \quad V^\oplus = r^{n-1} \tag{1.136}$$

$$P^\pm = -4\pi G\rho\zeta j_n(\gamma r) \quad , \quad P^\oplus = (\omega^2 - \tfrac{4}{3}\pi G\rho n) r^n \tag{1.137}$$

As explained by Dahlen and Tromp the symbol $^\pm$ needs to be understood as two solutions, one for each solution of γ. The $\oplus$ symbol is associated with solutions containing an r^n term. For the radial and tangential stresses, R and S, and perturbed gravity Q we have the following three solutions:

$$R^\pm = -\Big[(\kappa + \tfrac{4}{3}\mu)\zeta\gamma^2 - 2n(n-1)\mu\xi r^{-2}\Big] j_n(\gamma r) + 2\mu(2\zeta + k^2)\gamma r^{-1} j_{n+1}(\gamma r) \tag{1.138}$$

$$R^\oplus = 2n(n-1)\mu r^{n-2} \tag{1.139}$$

$$S^\pm = \mu\big[\gamma^2 + 2(n-1)\xi r^{-2}\big] j_n(\gamma r) - 2\mu(\zeta + 1)\gamma r^{-1} j_{n+1}(\gamma r) \tag{1.140}$$

$$S^\oplus = 2(n-1)\mu r^{n-2} \tag{1.141}$$

$$Q^\pm = -4\pi G\rho r^{-1}\big[k^2 + (n+1)\zeta\big] j_n(\gamma r) \tag{1.142}$$

$$Q^\oplus = \Big[(2n+1)\omega^2 - \tfrac{8}{3}\pi G\rho n(n-1)\Big] r^{n-1} \tag{1.143}$$

Sometimes it is assumed that the Earth is built up of spherical layers with constant properties. In this case the solutions listed above apply but in addition we need spherical Bessel functions of the second kind, also called spherical Neumann functions y_n, and solutions that depend on $1/r^n$ (represented by the $\ominus$ superscript). The two solutions associated with the spherical Neuman functions can be found by simply replacing the j_n terms with y_n. We indicate this with superscripts $\pm j$ and $\pm y$, respectively in (1.148). These functions may be computed using the relation $y_n(z) = (-1)^{n-1} j_{-n-1}(z)$. Equation 1.123 may be used to compute $j_{-n-1}(z)$ because the index of j is now decreasing. The solutions associated with $1/r$ are

$$U^{\ominus} = -\frac{n+1}{r^{n+2}} \quad V^{\ominus} = \frac{1}{r^{n+2}} \quad P^{\ominus} = -\frac{\left(\omega^2 + \frac{4}{3}\pi G\rho(n+1)\right)}{r^{n+1}} \tag{1.144}$$

$$\begin{aligned} R^{\ominus} &= 2\mu\frac{(n+1)(n+2)}{r^{n+3}} \\ S^{\ominus} &= -2\mu\frac{n+2}{r^{n+3}} \\ Q^{\ominus} &= \frac{4\pi G\rho(n+1)}{r^{n+2}} \end{aligned} \tag{1.145}$$

If the Earth is divided up into spherical layers with constant properties, the gravity inside this Earth is mostly different from that of a homogeneous Earth. To keep using the presented above equations, one therefore scales in each layer the term $\frac{4}{3}\pi G\rho$ to the mean value of g/r inside this layer (Vermeersen et al. 1996).

At this point the concept of Haskell propagator matrices (Haskell 1953) can be invoked. We have an analytical expression that relates the values of the radial solution functions between two consecutive interfaces j and $j+1$:

$$\mathbf{y}(r_{j+1}) = \mathbf{P}(r_{j+1}, r_j)\,\mathbf{y}(r_j) \tag{1.146}$$

and

$$\mathbf{y}(r) = \mathbf{A}\,\boldsymbol{\alpha} \tag{1.147}$$

where $\mathbf{A}$ is the 6×6 layer matrix composed of solutions $(U, V, P, R, S, Q)^{\top}$ where

$$\mathbf{A} = \begin{pmatrix} U^{\oplus} & U^{\ominus} & U^{+j} & U^{+y} & U^{-j} & U^{-y} \\ V^{\oplus} & V^{\ominus} & & \cdots & & \\ & & & \cdots & & \\ & & & \cdots & & Q^{-y} \end{pmatrix} \tag{1.148}$$

The propagator matrix is then

$$\mathbf{P}(r_{j+1}, r_j) = \mathbf{A}(r_{j+1})\mathbf{A}^{-1}(r_j) \tag{1.149}$$

so it can be stepped from $j = 1$ to the surface. The $\boldsymbol{\alpha}$ vector is finally determined from the boundary conditions. The reader will realise at this point that the account needs substantially more detail, for instance how the layer matrix can be inverted with elegance. We therefore refer him or her to the original source where the method is described at necessary depth, Martinec (1989). We only repeat Martinec's advice on how to treat a fluid layer. In this case, V and S are taken out of the equations, and the characteristic root is single-valued,

$$\gamma^2 = \frac{1}{v_\alpha^2}\left[\omega^2 + \frac{16\pi}{3}G\rho - \left(\frac{4\pi Gk\rho}{3\omega}\right)^2\right] \tag{1.150}$$

so we obtain a 4×4 matrix for $\mathbf{A}$.

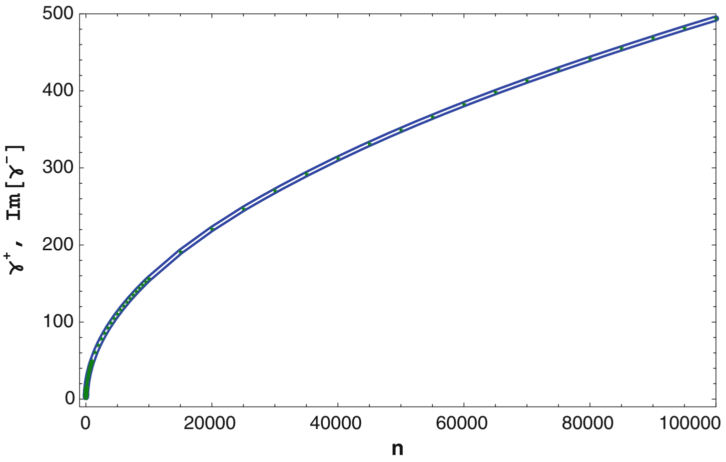

Fig. 1.8 The characteristic roots γ^+ and $\mathrm{Im}(\gamma^-)$, also providing the radial scale for the spherical Bessel functions vs spherical harmonic degree n Both functions are similar and difficult to discern in the diagram

When $\omega \to 0$, these equations no longer hold in the fluid. Pekeris and Accad (1972) showed that a boundary layer develops at the core–mantle interface that becomes infinitely thin but still has a finite effect on the dynamics. Instead of following their derivation, we will compute the jumps in U, R and Q (there is no jump in P) in the boundary layer for $\omega \to 0$ by taking the limit of our equations involving spherical Bessel functions of the first kind. To do so, it will be convenient to define

$$A = \frac{4}{3}\pi G \rho \tag{1.151}$$

Using (1.150), one can derive the limit value of γ :

$$\lim_{\omega \to 0} \gamma = \frac{kAi}{v_\alpha} \frac{1}{\omega} \tag{1.152}$$

Using this result, we can compute the limit of the spherical Bessel function:

$$\lim_{\omega \to 0} j_n(\gamma r) = \pm \frac{v_\alpha \omega}{2kAr} e^{kAr/(v_\alpha \omega)} \tag{1.153}$$

The sign of the limit depends on the degree n. Of course the exponential grows fast to infinity but we are allowed to scale our solutions with any constant so we choose it in such a way to make the $j_n(b\gamma) = 1$ at the boundary, where b is the radius of the boundary (Pekeris and Accad 1972). Below the core–mantle interface, $j_n(r\gamma) = 0$ and it thus nicely represents the jump through the boundary layer. Other results are

$$\zeta = -\frac{\omega^2}{A} \qquad \xi = -(n+1) \tag{1.154}$$

If we now collect all terms which are not zero when $\omega \to 0$, then we have the following solutions for our jump:

$$U^{\diamond}(b) = -\frac{n(n+1)}{b} \tag{1.155}$$

$$R^{\diamond}(b) = -\tfrac{4}{3}\pi G\rho k \frac{\kappa}{v_{\alpha}^2} \tag{1.156}$$

$$Q^{\diamond}(b) = -\frac{4\pi G\rho k^2}{b} \tag{1.157}$$

These equations should be added to the $\oplus$ solutions to fulfil the boundary conditions at the core–mantle interface. Inside the fluid core, only the + solutions apply.

Finally, we would like to remark that the roots $\gamma^{\pm}$ are nicely bounded. For our homogeneous Earth, model β, their values have been plotted in Fig. 1.8.

Appendix 3: Analytical Solution for a Homogeneous Fluid Inner Sphere

In a fluid the shear modulus $\mu = 0$. Here we distinguish four cases $n = 0, n > 0$, and $\omega = 0, \omega \neq 0$. First $n > 0$. Since the region includes $r = 0$, we can dismiss the irregular solutions involving the Neumann y_n-functions and the $\ominus$ functions.

Dahlen and Tromp (1998) give the solution for $n > 0$. The matrix of the differential equation simplifies to a 4×4 system in the same variable as before except for horizontal displacement and shear stress.

Dahlen and Tromp (1998) give the solution for $n > 0$. The matrix of the differential equation simplifies to a 4×4 system in the same variable as before except for horizontal displacement and shear stress.

$$\frac{d}{dr}\begin{pmatrix} U \\ R \\ P \\ Q \end{pmatrix} = \begin{pmatrix} \frac{4\pi\rho G n(n+1)}{3r\omega^2} - \frac{2}{r} & \frac{1}{\kappa} - \frac{n(n+1)}{r^2\rho\omega^2} & \frac{n(n+1)}{r^2\omega^2} & 0 \\ -\frac{\rho}{9}\left(48\pi G\rho - \frac{16\pi^2 G^2\rho^2 n(n+1)}{\omega^2} + 9\omega^2\right) & -\frac{4\pi G_{\rho} n(n+1)}{3r^2\omega^2} & \frac{(n+1)\rho(4n\pi G - 3\omega^2}{3r\omega^2} & \rho \\ -4\pi\rho & 0 & -\frac{n+1}{r} & 1 \\ \frac{-4\pi G\rho(n+1)(4n\pi G\rho - 3\omega^2)}{3r^2\omega^2} & -\frac{4\pi G n(n+1)}{r^2\omega^2} & -\frac{4\pi G\rho n(n+1)}{3r^2\omega^2} & -\frac{n+1}{r} \end{pmatrix} \begin{pmatrix} U \\ R \\ P \\ Q \end{pmatrix}. \tag{1.158}$$

with the solutions

$$U^{+} = \frac{n\xi}{r} j_n(\gamma r) - \zeta\gamma j_{n+1}(\gamma r) \tag{1.159}$$

$$R^{+} = -\kappa\zeta\gamma^2 j_n(\gamma r) \tag{1.160}$$

$$P^{+} = -4\pi G\rho\zeta j_n(\gamma r) \tag{1.161}$$

$$Q^{+} = -\frac{4\pi G\rho}{r}(n+1)(n+\zeta) j_n(\gamma r) \tag{1.162}$$

and

$$U^{\oplus} = nr^{n+1}, \quad P^{\oplus} = (\omega^2 - \frac{4\pi}{3} G\rho n) r^n \tag{1.163}$$

$$R^{\oplus} = 0, \quad Q^{\oplus} = [(2n+1)\omega^2 - \frac{8\pi}{3} G\rho n(n+1)] r^{n-1} \tag{1.164}$$

where γ was given in (1.150) and

$$\zeta = -\frac{3\omega^2}{4\pi G\rho} \qquad \xi = \zeta - (n+1) \tag{1.165}$$

These equations follow from what what discussed before, but it becomes interesting when one takes the limit of $\omega \to 0$. Unfortunately, these solutions cannot be used for $\omega \to 0$. Instead, we refer to Longman (1963) who solves a 2×2 system in the gravity variables only. At the core–mantle boundary, vertical and horizontal displacement start with arbitrary values into the mantle, horizontal shear stress is zero, vertical normal stress starts as

$$R = \rho g(r) U - \rho P \tag{1.166}$$

and the gravity variable Q as

$$Q = H - \rho U \tag{1.167}$$

The 2×2 system in the core is

$$\frac{d}{dr}\begin{pmatrix} P \\ H \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{n(n+1)}{r^2} - \frac{\rho^2}{\kappa} & -\frac{2}{r} \end{pmatrix} \cdot \begin{pmatrix} P \\ H \end{pmatrix} \tag{1.168}$$

The solution of this system is

$$P = C\sqrt{\frac{2\gamma_0}{r}} j_n(\gamma_0 r) \tag{1.169}$$

$$H = C\frac{\sqrt{2\gamma_0}}{r}\left(\frac{n}{\sqrt{r}} j_n(\gamma_0 r) + \gamma_0\sqrt{r} j_{n+1}(\gamma_0 r)\right) \tag{1.170}$$

where

$$\gamma_0 = \rho\sqrt{\frac{4\pi G}{\kappa}} \tag{1.171}$$

and C is an arbitrary constant.

Appendix 4: Tiny Fluid Sphere

The recursion relations for a tiny homogenous fluid sphere in the variables U, S, P and Q (1.91) are

$$u_1 = 0, \qquad p_1 = s_1/\rho = \gamma \tag{1.172}$$

$$q_1 = \frac{(-8\,G\,n\,(1+n)\,\pi\,\rho + (1-2\,n)\,\omega^2)\,p_1}{3\,\omega^2} \tag{1.173}$$

with an arbitrary γ as a start. Then, for $j = 3, 5, \ldots$

$$\begin{aligned} q_j = {} & 4\,G\,(1+n)\,\pi\,\rho\,\big[32\,G^2\,n^2\,(1+n)\,\pi^2\,\rho^2 - 4\,G\,n\,(4-j+j^2 \\ & +2\,n+2\,j\,n)\,\pi\,\rho\,\omega^2 + 3\,(j-1)\,(j+2\,n)\,\omega^4)\,\big]\,s_{j-2} \\ & \frac{1}{3\,(j-1)\,(j+2\,n)\,\kappa\,\omega^2\,[8\,G\,n\,(1+n)\,\pi\,\rho - (j-1)\,(j+2\,n)\,\omega^2]} \end{aligned} \tag{1.174}$$

$$\begin{aligned} u_j = {} & \Big\{32\,G^2\,n^2\,(1+n)^2\,\pi^2\,\rho^2 - 4\,G\,n\,(1+n)\,[4+j^2 \\ & -10\,n + j\,(+2\,n\,-7)]\,\pi\,\rho\,\omega^2 \\ & -3\,(j-1)\,[j^2 + 2\,(n-1)\,n + j\,(3\,n\,-1)]\,\omega^4\Big\}\,s_{j-2} \\ & \frac{1}{3\,(j-1)\,(j+2\,n)\,\kappa\,\omega^2\,[8\,G\,n\,(1+n)\,\pi\,\rho - (j-1)\,(j+2\,n)\,\omega^2]} \end{aligned} \tag{1.175}$$

$$\begin{aligned} s_j = {} & \rho\,\Big\{128\,G^3\,n^2\,(n+1)^2\,\pi^3\,\rho^3 - 16\,G^2\,n\,(n+1)\,[22+j^2 \\ & -4\,n + j\,(2\,n\,-1)]\,\pi^2\,\rho^2\,\omega^2 + 24\,G\,[2\,j^2 + 3\,j\,(n-1) \\ & -4\,n\,(2+n)]\,\pi\,\rho\,\omega^4 + 9\,(j-1)\,(j+2\,n)\,\omega^6\Big\}\,s_{j-2} \\ & \frac{1}{9\,(j-1)\,(j+2\,n)\,\kappa\,\omega^2\,[8\,G\,n\,(n+1)\,\pi\,\rho - (j-1)\,(j+2\,n)\,\omega^2]} \end{aligned} \tag{1.176}$$

$$p_j = -\frac{4\,G\,\pi\,\rho}{(j-1)\,(j+2\,n)\,\kappa}\,s_{j-2} \tag{1.177}$$

Appendix 5: Gravity Green's Function and Kummer Transform

We noted another error in Agnew (2007) where he specified the Kummer transformation of the Green's function for surface gravity. With our notation,

$$\mathcal{G}_{\Delta g}(\theta) = \sum_{n=0}^{\infty} [2h'_n - (n+1)k'_n]\, P_n(\cos\theta) - \frac{z/a + 2\sin^2\theta/2}{[(z/a)^2 + 4(1+z/a)\sin^2\theta/2]^{3/2}} \tag{1.178}$$

Since

$$\lim_{n\to\infty} nk'_n = (Nk')_\infty \neq 0$$

the Kummer transformation of the sum term should read

$$\begin{aligned} &2h'_0 + \sum_{n=1}^{\infty}\left[2(h'_n - h'_\infty) - (n+1)\left(k'_n - \frac{(Nk')_\infty}{n}\right)\right] P_n(\cos\theta) \\ &\quad + \frac{h'_\infty}{\sin\theta/2} - (Nk')_\infty\left[\frac{1}{2\sin\theta/2} - 1 - \log\left(\sin\frac{\theta}{2} + \sin^2\frac{\theta}{2}\right)\right] \end{aligned} \tag{1.179}$$

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