Performance Evaluation of Noise Subspace Methods of Frequency Estimation Techniques

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Abstract. Frequency Estimation methods have the ability to resolve complex exponentials that are closely spaced in frequency. The estimation of the frequencies is based on the eigen decomposition of the autocorrelation matrix of the input data. The autocorrelation matrix after eigen decomposition produces two subspaces, namely noise subspace and signal subspace. The methods that are based on the estimation of frequencies using noise subspace of the autocorrelation matrix are called Noise subspace methods of Frequency Estimation. Pisarenko Harmonic Decomposition, MUSIC method, Eigen Vector method and the Minimum Norm methods belongs to the category of Noise subspace methods. This paper investigates the performance evaluation of all the Noise Subspace methods of frequency estimation techniques for a common Synthetic Power signal having harmonics at 600Hz, 900Hz and 1500Hz with a sampling frequency of 3000Hz. Extensive Monte-Carlo simulation is carried out for ten numbers of times and the simulated figures are shown. The values obtained after the application of Noise subspace methods are compared with that of the actual inputs and are tabulated. The simulation of all methods is performed by using MATLAB software.

Keywords: Autocorrelation matrix, Eigen decomposition, Eigen Vector method, Minimum Norm method, MUSIC method, Noise Subspace, Pisarenko Harmonic Decomposition.

1 Introduction

The methods of Spectrum Estimation which have the ability to resolve complex exponentials that are closely spaced in frequency are known as Harmonic or Frequency Estimation methods [1, 2]. These methods use models in estimating the power spectrum of a WSS random process. The estimation of frequencies depends on the eigen decomposition of the autocorrelation matrix into subspaces, a signal subspace and a noise subspace. The Pisarenko Harmonic Decomposition method, MUlti SIgnal Classification (MUSIC) method, Eigen Vector method and Minimum Norm method belongs to the category of Noise subspace methods of frequency estimation. Section 1.1 describes the eigen decomposition of the autocorrelation matrix.

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In order to motivate the use of an eigendecomposition of the autocorrelation matrix as an approach that may be used for frequency estimation, consider the first-order process

$$x(n) = A_1 e^{jn\omega_1} + \omega(n) \tag{1}$$

That consists of a single complex exponential in white noise. The amplitude of the complex exponential is $A_1 = |A_1| e^{j\phi_1}$ where ϕ_1 is a uniformly distributed random variable, and $\omega(n)$ is white noise that has a variance of σ_ω^2, the autocorrelation sequence of $x(n)$ is

$$r_x(k) = P_1 e^{jk\omega_1} + \sigma_\omega^2 \delta(k) \tag{2}$$

where $P_1 = |A_1|^2$ is the power in the complex exponential. Therefore, the $M \times M$ autocorrelation matrix for $x(n)$ is a sum of an autocorrelation matrix due to the signal, $\mathbf{R}_s$, and an autocorrelation matrix due to the noise, $\mathbf{R}_n$,

$$\mathbf{R}_x = \mathbf{R}_s + \mathbf{R}_n \tag{3}$$

It is possible to extract all of the parameters of interest about $x(n)$ from the eigenvalues and eigenvectors of $\mathbf{R}_x$ as follows:

1. Perform an eigendecomposition of the autocorrelation matrix, $\mathbf{R}_x$. The largest eigenvalue will be equal to $MP_1 + \sigma_w^2$ and the remaining eigenvalues will be equal to σ_w^2.
2. Use the eigenvalues of $\mathbf{R}_x$ to solve for the power P_1 and the noise variance as follows:

$$\sigma_w^2 = \lambda_{\min}$$

$$P_1 = \frac{1}{M}(\lambda_{\max} - \lambda_{\min}) \tag{4}$$

3. Determine the frequency ω_1 from the eigenvector $\mathbf{v}_{\max}$ that is associated with the largest eigenvalue using, for example, the second coefficient of $\mathbf{v}_{\max}$,

$$\omega_i = \arg\{v_{\max}(1)\} \tag{5}$$

2 Mathematical Modeling

The Frequency Estimation methods use models in estimating the power spectrum of a WSS random process. Various models are used for estimating the frequencies of complex exponentials in noise using the noise subspace of the eigen decomposed autocorrelation matrix. The following sections give the detailed mathematical modeling of the noise subspace methods of frequency estimation techniques.

2.1 Pisarenko Harmonic Decomposition

This method is based on the determination of frequencies that are derived from the eigenvector corresponding to the minimum eigenvalue of the autocorrelation matrix. The steps involved in the determination of frequencies using Pisarenko Harmonic Decomposition method are summarized as follows:

Step 1: Given that a process consists of p complex exponentials in white noise, find the minimum eigenvalue $\lambda_{\min}$ and the corresponding eigenvector $\mathbf{v}_{\min}$ of the $(p+1)\times(p+1)$ autocorrelation matrix $\mathbf{R}_x$.

Step 2: Set the white noise power equal to the minimum eigenvalue, $\lambda_{\min} = \sigma_w^2$, and set the frequencies equal to the angles of the roots of the eigenfilter

$$V_{\min}(z) = \sum_{k=0}^{p} v_{\min}(k) z^{-k} \tag{6}$$

or the location of the peaks in the frequency estimation function

$$\hat{P}_{PHD}(e^{j\omega}) = \frac{1}{\left|\mathbf{e}^H \mathbf{v}_{\min}\right|^2} \tag{7}$$

Step 3: Compute the powers of the complex exponentials by solving the linear equations (8).

$$\begin{bmatrix} \left|V_1(e^{j\omega_1})\right|^2 & \left|V_1(e^{j\omega_1})\right|^2 & \left|V_1(e^{j\omega_1})\right|^2 & \left|V_1(e^{j\omega_1})\right|^2 \\ \left|V_2(e^{j\omega_1})\right|^2 & \left|V_2(e^{j\omega_1})\right|^2 & \left|V_2(e^{j\omega_1})\right|^2 & \left|V_2(e^{j\omega_1})\right|^2 \\ \vdots & \vdots & \vdots & \vdots \\ \left|V_p(e^{j\omega_1})\right|^2 & \left|V_p(e^{j\omega_1})\right|^2 & \cdots & \left|V_p(e^{j\omega_1})\right|^2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_p \end{bmatrix} = \begin{bmatrix} \lambda_1 - \sigma_w^2 \\ \lambda_2 - \sigma_w^2 \\ \vdots \\ \lambda_p - \sigma_w^2 \end{bmatrix} \tag{8}$$

2.2 MUSIC Method

This method determines the frequencies of complex exponentials in noise by reducing the effects of spurious peaks. To see how the MUSIC algorithm works, assume that $x(n)$ is a random process consisting of p complex exponentials in white noise with a variance of σ_w^2, and let $\mathbf{R}_x$ be the $M \times M$ autocorrelation matrix with $M > p+1$. If the eigenvalues of $\mathbf{R}_x$ are arranged in decreasing order, $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_M$, and if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_M$ are the corresponding eigenvectors, then we may divide these eigenvectors into two groups: the p signal eigenvectors corresponding to the p largest eigenvalues, and the *M-p* noise eigenvectors that, ideally, have eigenvalues equal to σ_w^2.

Although we could consider estimating the white noise variance by averaging the *M-p* smallest eigenvalues

$$\sigma_w^2 = \frac{1}{M-p}\sum_{k=p+1}^{M}\lambda_k \tag{9}$$

Estimating the frequencies of the complex exponentials is a bit more difficult. Since the eigenvectors of $\mathbf{R}_x$ are of length *M*, each of the noise subspace eigenfilters

$$V_i(z) = \sum_{k=0}^{M-1} v_i(k) z^{-k}; \quad i = p+1,\ldots,M \tag{10}$$

will have *M*-1 roots (zeros). Ideally, *p* of these roots will lie on the unit circle at the frequencies of the complex exponentials, and the eigen spectrum

$$\left|V_i(e^{j\omega})\right|^2 = \frac{1}{\left|\sum_{k=0}^{M-1} v_i(k) e^{-jk\omega}\right|^2} \tag{11}$$

associated with the noise eigenvector $\mathbf{v}_i$ will exhibit sharp peaks at the frequencies of the complex exponentials. However, the remaining (*M-p*-1) zeros may lie anywhere and, infact, some may lie close to the unit circle, giving rise to spurious peaks In the eigenspectrum. Furthermore, with inexact autocorrelations, the zeros of $V_i(z)$ that are on the unit circle may not remain on the unit circle. Therefore, when only one noise eigenvector is used to estimate the complex exponential frequencies, there may be some ambiguity in distinguishing the desired peaks from the spurious ones[1,4]. In the MUSIC algorithm, the effects of these spurious peaks are reduced by averaging, using the frequency estimation function

$$\hat{P}_{MU}(e^{j\omega}) = \frac{1}{\sum_{i=p+1}^{M} \left|\mathbf{e}^H \mathbf{v}_i\right|^2} \tag{12}$$

The frequencies of the complex exponentials are then taken as the locations of the *p* largest peaks in $\hat{P}_{MU}(e^{j\omega})$. Once the frequencies have been determined the power of each complex exponential may be found using Eq.(12).

2.3 Eigen Vector Method

The Frequency Estimation method in which the frequencies of complex exponentials in noise are determined by reducing the effects of spurious peaks by averaging and this procedure also involves multiplication of the inverse of eigenvalues associated with the eigen vectors is known as Eigen Vector method. Specifically, the EV method estimates the exponential frequencies from the peaks of the eigenspectrum

$$\hat{P}_{EV}(e^{j\omega}) = \frac{1}{\sum_{i=p+1}^{M} \frac{1}{\lambda_i} \left| \mathbf{e}^H \mathbf{v}_i \right|^2} \tag{13}$$

where λ_i is the eigenvalue associated with the eigenvector $\mathbf{v}_i$

2.4 Minimum Norm Method

The minimum norm algorithm uses a single vector $\mathbf{a}$ that is constrained to lie in the noise subspace, and the complex exponential frequencies are estimated from the peaks of the frequency estimation function,

$$\hat{P}_{MN}(e^{j\omega}) = \frac{1}{\left| \mathbf{e}^H \mathbf{a} \right|^2} \tag{14}$$

With $\mathbf{a}$ constrained to lie in the noise subspace, if the autocorrelation sequence is known exactly, then $\left| \mathbf{e}^H \mathbf{a} \right|^2$ will have nulls at the frequencies of each complex exponential. Therefore, the z-transform of the coefficients in $\mathbf{a}$ may be factored as follows:

$$A(z) = \sum_{k=0}^{M-1} a(k) z^{-k} = \prod_{k=1}^{p} (1 - e^{j\omega_k} z^{-1}) \prod_{k=p+1}^{M-1} (1 - z_k z^{-1}) \tag{15}$$

where z_k for $k = p+1, \ldots, M-1$ are the spurious roots that do not, in general, lie on the unit circle. The problem then is to determine which vector in the noise subspace minimizes the effects of the spurious zeros on the peaks of $\hat{P}_{MN}(e^{j\omega})$. The approach that is used in the minimum norm algorithm is to find the vector $\mathbf{a}$ that satisfies the following three constraints:

1. The vector $\mathbf{a}$ lies in the noise subspace.
2. The vector $\mathbf{a}$ has minimum norm.
3. The first element of $\mathbf{a}$ is unity.

3 Selection Criteria for Performance Evaluation

An important factor in the selection of a spectrum estimation technique is the performance of the estimator. In comparing one non-parametric method to another, there is a trade-off between resolution and variance. The variability, ν of the estimate is represented as,

$$\nu = \frac{\operatorname{var}\left\{\hat{P}_x(e^{j\omega})\right\}}{E^2\left\{\hat{P}_x(e^{j\omega})\right\}} \tag{16}$$

The variability must be as low as possible in order to determine the given non-parametric method as the best method.

Resolution, Δw of the estimate is represented as,

$$\Delta w = f_2 - f_1 \tag{17}$$

where $f_2 - f_1$ is the bandwidth of the mainlobe [4,5].

The resolution must be high in order to determine the given non-parametric method as the best method.

The overall figure of merit μ is defined as the product of the variability, ν and the resolution Δw .

$$\mu = \nu \Delta w \tag{18}$$

As the figure of merit decreases the performance of the non-parametric method increases, so the figure of merit should be as low as possible [6].

4 Monte-Carlo Simulation of a Synthetic Signal Consisting of Harmonics

For the purpose of simulation a signal $x(n)$ consisting of three complex exponentials in white noise is considered. It is represented as,

$$x(n) = \sum_{k=1}^{3} A_k e^{j(n\omega_k + \phi_k)} + w(n) \tag{19}$$

where the amplitudes A_k are equal to one, the frequencies ω_k are 0.2π, 0.3π and 0.5π (the denormalized frequencies are 200Hz, 300Hz and 500Hz) , the phases are uncorrelated random variables that are uniformly distributed over the interval $[0, 2\pi]$, and the variance of the white noise is $\sigma_w^2 = 0.5$. Using ten different realizations of $x(n)$ with $N = 64$ values, overlay plots of the frequency estimation functions using Pisarenko's method, the MUSIC algorithm, the eigenvector method, and the minimum norm algorithm are shown in the Fig.s 1(a), 2(a), 3(a) and 4(a) respectively. The average of the Monte-Carlo simulated plots are shown in Fig.s 1(b), 2(b), 3(b) and 4(b) respectively.

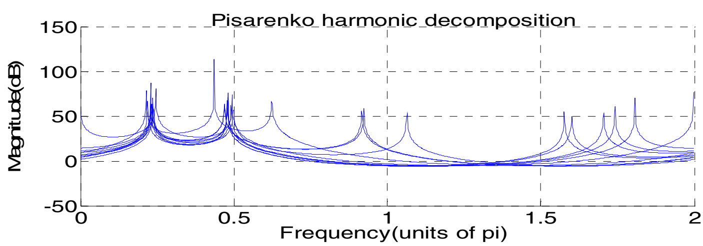

Fig. 1(a). Monte-Carlo simulated Pisarenko's estimates of x(n)

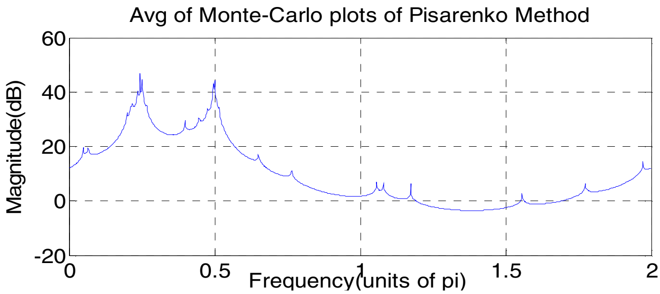

Fig. 1(b). Avg of Monte-Carlo simulated Pisarenko's estimates of x(n)

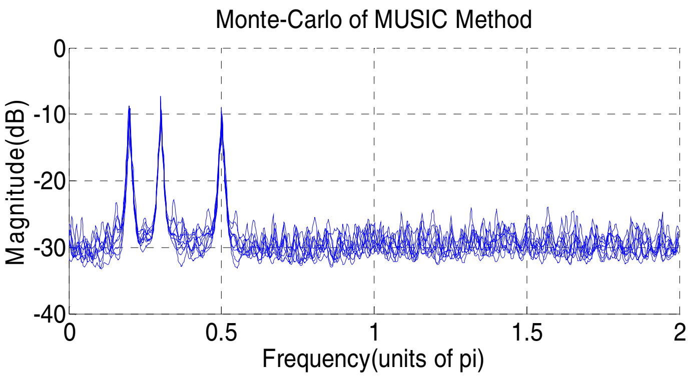

Fig. 2(a). Monte-Carlo simulated MUSIC estimates of x(n)

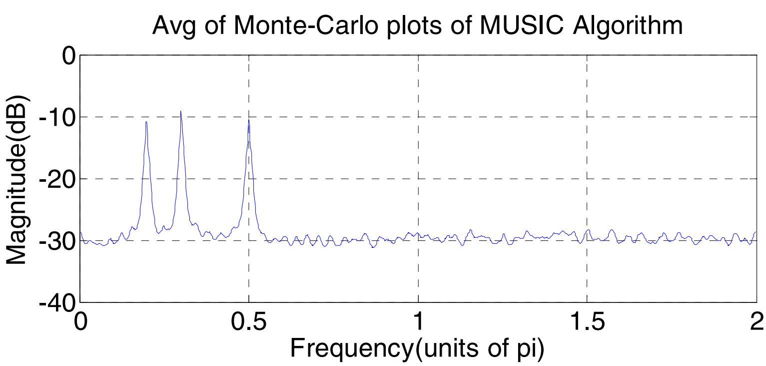

Fig. 2(b). Avg of Monte-Carlo simulated MUSIC estimates of x(n)

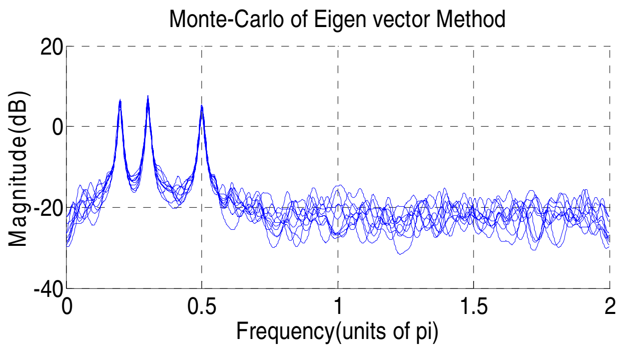

Fig. 3(a). Monte-Carlo simulated Eigen Vector estimates of x(n)

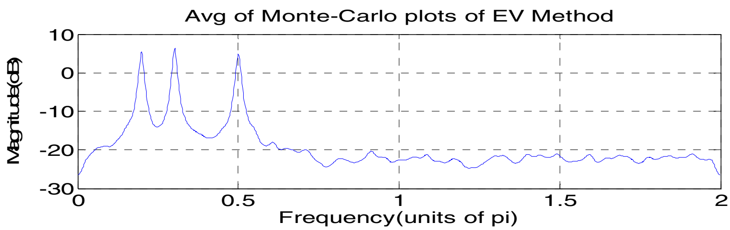

Fig. 3(b). Avg of Monte-Carlo simulated Eigen Vector estimates of x(n)

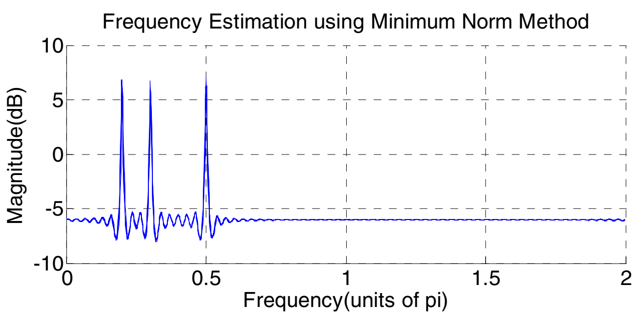

Fig. 4(a). Monte-Carlo simulated Minimum Norm estimates of x(n)

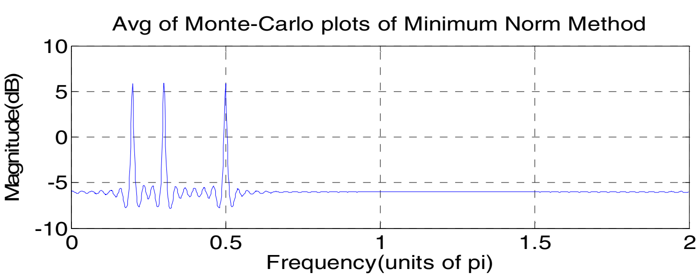

Fig. 4(b). Avg of Monte-Carlo simulated Minimum Norm estimates of x(n)

The estimated frequencies are compared with the true values and are represented in Table 1.

Table 1. Comparison of Estimated Vs True Frequencies for Various Noise Subspace based Methods

Method Used	Frequency (Hz)		Frequency (Hz)		Frequency (Hz)	
	True	Estimated	True	Estimated	True	Estimated
Pisarenko Harmonic Decomposition	600	732.3000	900	750	1500	1500
MUSIC	600	597.6000	900	908.4000	1500	1500
Eigen Vector	600	591.9000	900	906.1000	1500	1500
Minimum Norm	600	597.6000	900	902.4000	1500	1500

Performance Evaluation of Noise subspace methods of Frequency Estimation is done according to the selection criteria and is represented in Table 2. Since the estimation of frequencies can be compared with the true values, one more parameter named accuracy is added in the evaluation process which gives the closeness of the estimated result to the true value.

Table 2. Performance Evaluation of Noise subspace based Frequency Estimation methods

Method Used	Variability	Resolution	Figure of merit	Accuracy
Pisarenko	1.4876	0.6	0.8926	33.33%
MUSIC	0.0108	0.9	0.0097	92%
Eigen Vector	0.0901	0.75	0.0676	92%
Minimum Norm	0.0471	0.7	0.0330	94%

5 Conclusion

A synthetic power signal having harmonics at 600Hz, 900Hz and 1500Hz with a sampling frequency of 3000Hz is simulated using extensive Monte-Carlo simulation for ten times. It is observed from the simulated results and Tabular forms 1.0 and 2.0 that the performance of MUSIC method is best when compared to all other Noise subspace based Frequency Estimation techniques, as it produced least variability, figure of merit, good accuracy and highest resolution. It is also observed from the simulated results that the effect of spurious peaks which gives ambiguity regarding the detection of exact harmonic frequencies is least with the MUSIC method and highest with the Pisarenko Harmonic Decomposition. Therefore, the MUSIC method exactly suits in predicting the presence of harmonic frequencies as well as magnitudes.

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