Closure Spaces of Isotone Galois Connections and Their Morphisms

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Abstract. We present results on closure spaces induced by isotone fuzzy Galois connections. Such spaces play a fundamental role in the analysis of relational data such as formal concept analysis or relational factor analysis. We provide a characterization of such closure spaces and study their morphisms. The results contribute to foundations of a matrix calculus over relational data.

1 Introduction

Closure structures are among the fundamental mathematical structures that naturally appear in many areas of pure and applied mathematics. In particular, closure structures are the fundamental structures behind formal concept analysis and other data analysis methods that are based on attribute sharing (rather than attribute distance). The results in this paper are motivated by the recent results on decompositions of matrices over residuated lattices and factor analysis of relational data described by such matrices, see e.g. [3–5]. These results reveal a fundamental role of closure and interior structures for the decompositions and motivate us to further investigate the calculus of matrices over residuated lattices. Such matrices include Boolean matrices as a particular case but have much richer structure. An important concept, studied in this paper, is that of a closure space of isotone and antitone Galois connections induced by such matrices. Such spaces are in fact the spaces of optimal factors for matrix decompositions [3, 4]. In the setting of Boolean matrices, there exists a natural bijective mapping between the spaces of isotone and antitone Galois connections. Moreover, these spaces exhaust all closure spaces. This is no longer true in the setting of matrices over residuated lattices. While it is known from the previous results that the closure spaces of antitone fuzzy Galois connections exhaust all fuzzy closure spaces, we show in this paper that the closure spaces of isotone fuzzy Galois connections are particular fuzzy closure spaces. We provide a characterization of such spaces. Moreover, we study morphisms of such spaces and show a correspondence between such morphisms and matrices (matrices induce morphisms and vice versa). The results contribute to the foundations of analysis of qualitative data, namely to the development of a matrix calculus for such data.

2 Preliminaries: Matrices, Decompositions, Concept Lattices

Matrices. We deal with matrices whose degrees are elements of residuated lattices. Note that instead of matrices, we could consider fuzzy relations (with degrees in complete residuated lattices) between possibly infinite sets. The results would then be more

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general (matrices correspond to relations between finite sets). Recall that a (complete) residuated lattice [1, 10, 16] is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a (complete) lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist (the lattice order is denoted by $\leq$; 0 and 1 denote the least and greatest element, respectively);
(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$;
(iii) $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leq c$ iff $a \leq b \rightarrow c$.

Throughout the paper, $\mathbf{L}$ denotes an arbitrary (complete) residuated lattice. Common examples of complete residuated lattices include those defined on the real unit interval, i.e. $L = [0, 1]$, or on a finite chain in a unit interval, e.g. $L = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$. For instance, for $L = [0, 1]$, we can use any left-continuous t-norm for $\otimes$, such as minimum, product, or Łukasewicz, and the corresponding residuum $\rightarrow$ [1, 10, 16]. Residuated lattices are commonly used in fuzzy logic [1, 9, 10]. Elements $a \in L$ are called grades (degrees of truth). Operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of a (truth function of) conjunction and implication, respectively.

We deal with compositions $I = A * B$ which involve an $n \times m$ matrix I, an $n \times k$ matrix A, and a $k \times m$ matrix B. We assume that $I_{ij}, A_{il}, B_{lj} \in L$. That is, all the matrix entries are elements of a given residuated lattice $\mathbf{L}$. Therefore, examples of matrices I which are subject to the decomposition are

$$\begin{pmatrix} 1.0 & 1.0 & 0.0 & 0.0 & 0.6 & 0.4 \\ 1.0 & 0.9 & 0.0 & 0.0 & 1.0 & 0.8 \\ 1.0 & 1.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 1.0 & 0.5 & 0.0 & 0.7 & 1.0 & 0.4 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

The second matrix makes it apparent that binary matrices are a particular case for $L = \{0, 1\}$.

For convenience and since there is no danger of misunderstanding, we take the advantage of identifying $n \times m$ matrices over residuated lattices (the set of all such matrices is denoted by $L^{n\times m}$) with binary fuzzy relations between X and Y (the set of all such relations is denoted by $L^{X\times Y}$). Also, we identify vectors with n components over residuated lattices (the set of all such vectors is denoted by L^n) with fuzzy sets in X (the set of all such fuzzy sets is denoted by L^X). As usual, we identify vectors with n components with $1 \times n$ matrices.

Composition Operators. We use three matrix composition operators, $\circ$, $\triangleleft$, and $\triangleright$. In the decompositions $I = A * B$, I_{ij} is interpreted as the degree to which the object i has the attribute j; A_{il} as the degree to which the factor l applies to the object i; B_{lj} as the degree to which the attribute j is a manifestation (one of possibly several manifestations) of the factor l. The composition operators are defined by

$$(A \circ B)_{ij} = \bigvee_{l=1}^{k} A_{il} \otimes B_{lj}, \tag{1}$$

$$(A \triangleleft B)_{ij} = \bigwedge_{l=1}^{k} A_{il} \rightarrow B_{lj}, \tag{2}$$

$$(A \triangleright B)_{ij} = \bigwedge_{l=1}^{k} B_{lj} \rightarrow A_{il}. \tag{3}$$

Note that these operators were extensively studied by Bandler and Kohout, see e.g. [12] to which we refer for an overview of knowledge processing applications. The operators have natural verbal descriptions. For instance, $(A \triangleleft B)_{ij}$ is the truth degree of "for every factor l, if l applies to object i then attribute j is a manifestation of l". One may easily see that $\triangleright$ can be defined in terms of $\triangleleft$ and vice versa. Note also that for $L = \{0,1\}$, $A \circ B$ coincides with the well-known Boolean product of matrices [11].

Concept Lattices Associated to I. For a positive integer n, we denote

$$\hat{n} = \{1,\ldots,n\}.$$

In addition, we put

$$X = \{1,\ldots,n\}, \quad Y = \{1,\ldots,m\}.$$

Recall that L^U denotes the set of all L-sets in U, i.e. all mappings from U to L. Consider the following pairs of operators between L^X and L^Y induced by matrix $I \in L^{n\times m}$:

$$C^{\uparrow}(j) = \textstyle\bigwedge_{i=1}^{n}(C(i) \rightarrow I_{ij}), \quad D^{\downarrow}(i) = \textstyle\bigwedge_{j=1}^{m}(D(j) \rightarrow I_{ij}), \tag{4}$$

$$C^{\cap}(j) = \textstyle\bigvee_{i=1}^{n}(C(i) \otimes I_{ij}), \quad D^{\cup}(i) = \textstyle\bigwedge_{j=1}^{m}(I_{ij} \rightarrow D(j)), \tag{5}$$

$$C^{\wedge}(j) = \textstyle\bigwedge_{i=1}^{n}(I_{ij} \rightarrow C(i)), \quad D^{\vee}(i) = \textstyle\bigvee_{j=1}^{m}(D(j) \otimes I_{ij}), \tag{6}$$

for $C \in L^X$, $D \in L^Y$, $j \in \{1,\ldots,m\}$, and $i \in \{1,\ldots,n\}$. Furthermore, denote the corresponding sets of fixpoints by $\mathcal{B}(X^{\uparrow},Y^{\downarrow},I)$, $\mathcal{B}(X^{\cap},Y^{\cup},I)$, and $\mathcal{B}(X^{\wedge},Y^{\vee},I)$, i.e.

$$\mathcal{B}(X^{\uparrow},Y^{\downarrow},I) = \{\langle C,D\rangle \mid C^{\uparrow} = D,\ D^{\downarrow} = C\},$$
$$\mathcal{B}(X^{\cap},Y^{\cup},I) = \{\langle C,D\rangle \mid C^{\cap} = D,\ D^{\cup} = C\},$$
$$\mathcal{B}(X^{\wedge},Y^{\vee},I) = \{\langle C,D\rangle \mid C^{\wedge} = D,\ D^{\vee} = C\}.$$

The sets of fixpoints are complete lattices, called concept lattices associated to I, and their elements are called formal concepts. These structures are the fundamental structures of formal concept analysis [6]. For a formal concept $\langle C,D\rangle$, C and D are called the extent and the intent and they represent the collection of objects and attributes to which the formal concept applies. The sets of all extents and intents of the respective concept lattices are denoted by $\mathrm{Ext}(X^{\uparrow},Y^{\downarrow},I)$, $\mathrm{Int}(X^{\uparrow},Y^{\downarrow},I)$, $\mathrm{Ext}(X^{\cap},Y^{\cup},I)$, $\mathrm{Int}(X^{\cap},Y^{\cup},I)$, $\mathrm{Ext}(X^{\wedge},Y^{\vee},I)$, and $\mathrm{Int}(X^{\wedge},Y^{\vee},I)$. It may be shown that

$$\mathrm{Ext}(X^{\uparrow},Y^{\downarrow},I) = \{C \in L^X \mid C = C^{\uparrow\downarrow}\},$$
$$\mathrm{Int}(X^{\uparrow},Y^{\downarrow},I) = \{D \in L^Y \mid D = D^{\downarrow\uparrow}\},$$

and the same for the other cases.

The above-defined operators and their sets of fixpoints have extensively been studied, see e.g. [2, 7, 14]. Clearly, $\langle C,D\rangle \in \mathcal{B}(X^{\cap},Y^{\cup},I)$ iff $\langle D,C\rangle \in \mathcal{B}(Y^{\wedge},X^{\vee},I^{\mathrm{T}})$, where I^{T} denotes the transpose of I; so one could consider only one pair, $\langle^{\cap},^{\cup}\rangle$ or $\langle^{\wedge},^{\vee}\rangle$, and obtain the properties of the other pair by a simple translation. Note that if $L = \{0,1\}$, $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ coincides with the ordinary concept lattice of the formal context consisting of X, Y, and the binary relation (represented by) I; and that $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$ is isomorphic to $\mathcal{B}(X^{\cap}, Y^{\cup}, \overline{I})$ with $\langle A, B\rangle \mapsto \langle A, \overline{B}\rangle$ being an isomorphism ($\overline{U}$ denotes the complement of U). Therefore, as is well known, for $L = \{0,1\}$, each of the three operators is definable by any of the remaining two. The mutual definability fails for general L because it is based on the law of double negation which does not hold for general residuated lattices. A simple framework that enables us to consider all the three operators as particular types of a more general operator is provided in [4], cf. also [7] for another possibility. For simplicity, we do not work with the general approach and use the three operators because they are well known.

3 Closure Spaces Induced by $\langle {}^{\wedge}, {}^{\vee}\rangle$

The following results are well known [1]. $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ forms an (antitone) **L**-Galois connection [1], ${}^{\uparrow\downarrow}$ and ${}^{\downarrow\uparrow}$ are **L**-closure operators in X and Y, and $\mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I)$ and $\mathrm{Int}(X^{\uparrow}, Y^{\downarrow}, I)$ are **L**-closure systems in X and Y, respectively. Moreover, any **L**-closure system in X is in the form of $\mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I)$ (same for Y).

Recall $V \subseteq L^U$ is called an **L**-*closure system* (in the context of fuzzy sets; or *c-subspace*, in the context of matrices) if

- V is closed under *left* $\rightarrow$*-multiplications*, i.e. $a \rightarrow C \in V$ for each $a \in L$ and $C \in V$ (here, $a \rightarrow C$ is defined by $(a \rightarrow C)(i) = a \rightarrow C(i)$ for $i = 1, \dots, n$);
- V is closed under $\bigwedge$*-intersections*, i.e. for $C_j \in V$ $(j \in J)$ we have $\bigwedge_{j\in J} C_j \in V$ (here, $\bigwedge_{j\in J} C_j$ is defined by $(\bigwedge_{j\in J} C_j)(i) = \bigwedge_{j\in J} C_j(i)$).

For $\langle {}^{\wedge}, {}^{\vee}\rangle$, it is known that $\langle {}^{\wedge}, {}^{\vee}\rangle$ forms an isotone **L**-Galois connection [7], ${}^{\wedge\vee}$ and ${}^{\vee\wedge}$ are **L**-interior and **L**-closure operators in X and Y, and $\mathrm{Ext}(X^{\wedge}, Y^{\vee}, I)$ and $\mathrm{Int}(X^{\wedge}, Y^{\vee}, I)$ are **L**-interior and **L**-closure systems in X and Y, respectively. The situation might seem completely dual to that of $\langle {}^{\uparrow}, {}^{\downarrow}\rangle$ (which is the case when $L = \{0,1\}$, see above). However, as the next example shows, it is not. Namely, there exist **L**-closure systems that are not of the form $\mathrm{Int}(X^{\wedge}, Y^{\vee}, I)$.

Example 1. Let **L** be the standard Gödel algebra, $U = \{u\}$, $\mathcal{S} = \{\{{}^{0.5}/u\}, \{{}^{1}/u\}\}$. Therefore, $L = [0,1]$ and $a \rightarrow b = 1$ if $a \le b$ and $a \rightarrow b = b$ of $a > b$. Clearly, $\mathcal{S}$ is closed under intersections and $\rightarrow$-shifts, hence it is an **L**-closure system. However, $\mathcal{S}$ is not of the form $\mathcal{S} = \mathrm{Int}(X^{\wedge}, Y^{\vee}, I)$. (This claim is justified at the end of this section.)

Therefore, **L**-closure systems that are of the form $\mathrm{Int}(X^{\wedge}, Y^{\vee}, I)$ are just particular **L**-closure systmes. Below, we provide their characterization. For a system $\mathcal{S} \subseteq L^U$, put

$$[\mathcal{S}]_{\bigwedge} = \{\bigwedge \mathcal{T} \mid \mathcal{T} \subseteq \mathcal{S}\},$$
$$[\mathcal{S}]_{\rightarrow} = \{a \rightarrow A \mid a \in L,\ A \in \mathcal{S}\},$$
$$[\mathcal{S}]^{\rightarrow} = \{A \rightarrow a \mid a \in L,\ A \in \mathcal{S}\}.$$

Note that $A \rightarrow a$ is defined by $(A \rightarrow a)(u) = A(u) \rightarrow a$ and call $A \rightarrow a$ the *right* $\rightarrow$*-multiple* of A by a. Therefore, $[\mathcal{S}]_{\bigwedge}$ is the system of all intersections of fuzzy sets from

$\mathcal{S}$, $[\mathcal{S}]_{\rightarrow}$ is the system of all left →-multiplications of fuzzy sets from $\mathcal{S}$, and $[\mathcal{S}]^{\rightarrow}$ is the system of all right →-multiplications of fuzzy sets from $\mathcal{S}$. It is known that for any $\mathcal{S} \subseteq L^U$, $[[\mathcal{S}]_{\rightarrow}]_{\bigwedge}$ is the least, w.r.t. inclusion, **L**-closure system containing $\mathcal{S}$. $[[\mathcal{S}]_{\rightarrow}]_{\bigwedge}$ is called the **L**-closure system generated by $\mathcal{S}$, or the *c-span* of $\mathcal{S}$.

Note that in fuzzy logic, $b \rightarrow 0$ is called the negation of the truth degree b. Correspondingly, the fuzzy set $A \rightarrow 0$ is called the complement of A. Clearly, in the above terms, $A \rightarrow 0$ is the right multiple of A by 0. From this point of view, the right multiples $A \rightarrow a$ generalize the concept of a complement of a fuzzy set. $A \rightarrow a$ could naturally be called the *a-complement* of A.

In the classical case ($L = \{0, 1\}$), every A is a complement of some B; namely, of $B = A \rightarrow 0$. This is no longer true for the general setting of residuated lattices (not even for $a = 0$). We only have:

Lemma 1. *A is an a-complement of some fuzzy set if and only if $A = (A \rightarrow a) \rightarrow a$.*

Proof. Easy, follows from $((b \rightarrow a) \rightarrow a) \rightarrow a = b \rightarrow a$. □

This lemma is, in a sense, the key observation in characterizing the **L**-closure systems $\mathrm{Int}(X^{\wedge}, Y^{\vee}, I)$. We are going to show that $\mathrm{Int}(X^{\wedge}, Y^{\vee}, I)$ are just the **L**-closure systems that are generated by a-complements of some collection $\mathcal{T}$ of fuzzy sets. Such systems are conveniently characterized by the following theorem.

Theorem 1. *For any $\mathcal{T} \subseteq L^U$, $[[\mathcal{T}]^{\rightarrow}]_{\bigwedge}$ is an **L**-closure system. It is the least, w.r.t. inclusion, **L**-closure system containing all a-complements (i.e., right →-multiplications) of fuzzy sets from $\mathcal{T}$.*

Proof. Sketch: Clearly, $[[\mathcal{T}]^{\rightarrow}]_{\bigwedge}$ contains all a-complements of fuzzy sets from $\mathcal{T}$. Essential to the proof is to check that $[[\mathcal{T}]^{\rightarrow}]_{\bigwedge}$ is closed under left →-multiplications (this follows from $a \rightarrow (b \rightarrow c) = b \rightarrow (a \rightarrow c)$). The rest is by standard arguments. □

The following theorem provides our characterization.

Theorem 2. *For any $\mathcal{S} \subseteq L^U$, $\mathcal{S} = \mathrm{Int}(X^{\wedge}, Y^{\vee}, I)$ for some I if and only if $\mathcal{S} = [[\mathcal{T}]^{\rightarrow}]_{\bigwedge}$ for some $\mathcal{T} \subseteq L^U$, i.e. $\mathcal{S}$ is an **L**-closure system generated by a system of all a-complements of fuzzy sets from $\mathcal{T}$.*

Proof. Sketch: "⇒" is done by checking the conditions and using standard properties of residuated lattice.

"⇐": Let $X = \mathcal{T}$, $Y = U$, $I(A, u) = A(u)$ for $A \in \mathcal{S}$, $u \in U$. One can show that $\mathcal{S} = \mathrm{Int}(X^{\wedge}, Y^{\vee}, I)$. □

Definition 1. *We call the systems $\mathcal{S}$ satisfying the condition of Theorem 2* c-closure spaces *("c" for "complement").*

Example 1 (continued). Suppose, by contradiction, that $\mathcal{S} = \mathrm{Int}(X^{\wedge}, Y^{\vee}, I)$. Then $U = X$ and by Theorem 2, $\mathcal{S}$ is a system generated by a system of all a-complements of fuzzy sets from some $\mathcal{T}$. According to Theorem 1, $[[\mathcal{T}]^{\rightarrow}]_{\bigwedge} = \{\{{}^{0.5}/u\}, \{{}^{1}/u\}\}$. Then, $\{{}^{0.5}/u\}$ needs to be an intersection of other fuzzy sets from $[\mathcal{T}]^{\rightarrow}$ or $\{{}^{0.5}/u\} \in [\mathcal{T}]^{\rightarrow}$. Clearly, $\{{}^{0.5}/u\} \in [\mathcal{T}]^{\rightarrow}$ must be the case. Therefore, $\{{}^{0.5}/u\} = \{{}^{a}/u\} \rightarrow b$ for some b. Clearly, $a > b = 0.5$ must be the case. But then, we also have $\{{}^{a}/u\} \rightarrow 0.4 = \{{}^{0.4}/u\} \in [\mathcal{T}]^{\rightarrow}$, a contradiction to $[\mathcal{T}]^{\rightarrow} \subseteq [[\mathcal{T}]^{\rightarrow}]_{\bigwedge} = \{\{{}^{0.5}/u\}, \{{}^{1}/u\}\}$.

4 Morphisms of c-Closure Spaces

In this section we define morphisms of c-closure spaces, i.e. the particular **L**-closure spaces characterized in Section 3, and show that they are induced by matrices over residuated lattices via the $\triangleright$-product.

Definition 2. *A mapping $h : V \to W$ from a c-closure space $V \subseteq L^p$ into a c-closure space $W \subseteq L^q$ is called a* complement-preserving c-morphism *if*

- *h is an c-morphism, i.e. $h(a \to C) = a \to h(C)$ and $h(\bigwedge_{k\in K} C_k) = \bigwedge_{k\in K} h(C_k)$ for any $a \in L$, $C, C_k \in L^p$;*
- *if C is an a-complement then $h(C)$ is an a-complement.*

A complement-preserving c-morphism $h : V \to W$ from a c-subspace $V \subseteq L^p$ into a c-subspace $W \subseteq L^q$ is called an extendable *if there is a complement-preserving c-morphism $h' : L^p \to L^q$ such that $h'(C) = h(C)$ for each $C \in V$.*

A complement-preserving c-morphism h is called a complement-preserving c-isomorphism *if h is bijective and both h and h^{-1} are extendable complement-preserving c-morphisms.*

In what follows we assume only extendable complement-preserving c-morphisms.

First, every matrix induces a morphism:

Lemma 2. *For every matrix $A \in L^{p\times q}$, the mapping $h_A : L^p \to L^q$ defined by*

$$h_A(C) = C \triangleright A \quad (= C^{\wedge_A})$$

is a complement-preserving c-morphism.

Proof. Sketch: Being a c-morphism follows easily from the properties of residuated lattices. Let $C = D \to a$, then

$$[(D \to a) \triangleright A](j) = \bigwedge_i A_{ij} \to (D(i) \to a) = \bigwedge_i ((A_{ij} \otimes D(i)) \to a) =$$
$$= \bigvee_i (A_{ij} \otimes D(i)) \to a = (D \circ A)(j) \to a$$

Whence, if C is an a-complement then $C \triangleright A$ is a-complement. □

Second, every morphism is induced by some matrix.

Lemma 3. *If $h : V \to L^q$ is a complement-preserving c-morphism of a c-closure space V, then there exists a matrix $A_h \in L^{p\times q}$ such that $h(C) = C \triangleright A$ for every $C \in V$.*

Proof. Let $A \in L^{p\times q}$ be defined by

$$A_{ij} = \textstyle\bigwedge_{C\in V} ((h(C))(j) \to C(i)).$$

That is, $A_{i\_} = \bigwedge_{C\in V} (h(C) \to C(i))$, i.e. the row $A_{i\_}$ contains a vector of degrees that can be interpreted as the intersection of images of those vectors C from V for which the corresponding fuzzy set contains i (in Boolean case: for which the ith component is 1).

We now check $h(C) = C \triangleright A$ for every $C \in L^p$. First,

$$(C \triangleright A)(j) = \bigwedge_{i=1}^{p} [A_{ij} \to C(i)] = \\ = \bigwedge_{i=1}^{p} [(\bigwedge_{C' \in V} (h(C'))(j) \to C'(i))) \to C(i)] \geq (h(C))(j).$$

We omit the second part ($(C \triangleright A)(j) \leq (h(C))(j)$), which is technically more involved, due limited space. □

As a corollary, we get the following characterization of morphisms:

Theorem 3. *$h : V \to L^q$ is a complement-preserving c-morphism of a c-closure space V if and only if there exists a matrix $A_h \in L^{p \times q}$ such that $h(C) = C \triangleright A$ for every $C \in L^p$.*

Proof. Directly from Lemma 2 and Lemma 3. □

5 Isomorphic c-Closure Spaces

The aim of this section is to provide a criterion of isomorphism of c-closure spaces.

Lemma 4. *Let $I, J \in L^{p \times q}$. We have $B^{\wedge_I} = B^{\wedge_J}$ for each $B \in L^p$ iff $I = J$.*

Proof. "⇒": Suppose $B^{\wedge_I} = B^{\wedge_J}$. Assume $I_{ij} \neq J_{ij}$ for some i, j. Without loss of generality, we may assume $I_{ij} \not\leq J_{ij}$. Let

$$B(l) = \begin{cases} J_{ij} & \text{if } i = l, \\ 1 & \text{otherwise} . \end{cases}$$

Then $B^{\wedge_I}(j) = \bigwedge_l I_{lj} \to B(l) = I_{ij} \to J_{ij} \neq 1$, and $B^{\wedge_J}(j) = \bigwedge_l J_{lj} \to B(l) = J_{ij} \to J_{ij} = 1$, which is a contradiction.

"⇐": Obvious. □

We need to recall the following notions. $V \subseteq L^n$ is called an *i-subspace* if

– V is closed under $\otimes$-multiplication, i.e. for every $a \in L$ and $C \in V$, $a \otimes C \in V$ (here, $a \otimes C$ is defined by $(a \otimes C)(i) = a \otimes C(i)$ for $i = 1, \ldots, n$); and
– V is closed under $\bigvee$-union, i.e. for $C_j \in V$ ($j \in J$) we have $\bigvee_{j \in J} C_j \in V$ (here, $\bigvee_{j \in J} C_j$ is defined by $(\bigvee_{j \in J} C_j)(i) = \bigvee_{j \in J} C_j(i)$).

A mapping $h : V \to W$ from an i-subspace $V \subseteq L^p$ into an i-subspace $W \subseteq L^q$ is called an *i-morphism* if it is a $\otimes$- and $\bigvee$-morphism, that is, $h(a \otimes C) = a \otimes h(C)$ and $h(\bigvee_{k \in K} C_k) = \bigvee_{k \in K} h(C_k)$ for any $a \in L$, $C, C_k \in L^p$. An i-morphism $V \to W$ is called

– an *extendable i-morphism* if h can be extended to an i-morphism of $L^p \to L^q$.
– an *i-isomorphism* if h is bijective and both h and h^{-1} are extendable i-morphisms.

Theorem 4. *Let* $I \in L^{n\times m}$ *and* $J \in L^{p\times r}$ *be matrices. Then there exist a complement-preserving c-isomorphism*

$$h : \mathrm{Int}(\hat{n}^{\wedge}, \hat{m}^{\vee}, I) \to \mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J)$$

if and only if there exists a matrix $K \in L^{p\times m}$ *such that* $\mathrm{Int}(\hat{n}^{\wedge}, \hat{m}^{\vee}, I) = \mathrm{Int}(\hat{p}^{\wedge}, \hat{m}^{\vee}, K)$ *and* $\mathrm{Ext}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J) = \mathrm{Ext}(\hat{p}^{\wedge}, \hat{m}^{\vee}, K)$.

Proof. "⇒": Let $h : \mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J) \to \mathrm{Int}(\hat{n}^{\wedge}, \hat{m}^{\vee}, I)$ be a complement-preserving c-isomorphism. According to Lemma 3, there exist matrices $X \in L^{r\times m}$ and $Y \in L^{m\times r}$ such that

$$h(C) = C \triangleright X \text{ and } h^{-1}(D) = D \triangleright Y$$

for every $C \in \mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J)$ and $D \in \mathrm{Int}(\hat{n}^{\wedge}, \hat{m}^{\vee}, I)$.

Thus we have $C = h(h^{-1}(C)) = (C \triangleright X) \triangleright Y$. Now, since $(C \triangleright X) \triangleright Y = C \triangleright (X \circ Y)$ and since $B^{\wedge_J} \in \mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J)$ for every $B \in L^{\hat{p}}$,

$$B^{\wedge_J} = B^{\wedge_J \wedge_{(X\circ Y)}} = B^{\wedge_{J\circ X\circ Y}}$$

for every $B \in L^{\hat{p}}$. $J = J \circ X \circ Y$ now follows from Lemma 4. From that we have $\mathrm{Ext}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J) = \mathrm{Ext}(\hat{p}^{\wedge}, \hat{m}^{\vee}, K)$. Furthermore, if $D \in \mathrm{Int}(\hat{n}^{\wedge}, \hat{m}^{\vee}, I)$, then $D \triangleright Y = h^{D} \in \mathrm{Int}(\hat{n}^{\wedge}, \hat{m}^{\vee}, I)$. Since $D = (D \triangleright Y) \triangleright X$, we get $D = (C \triangleright J) \triangleright X = C \triangleright (J \circ X)$ showing $D \in \mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J \circ X)$. We established $\mathrm{Int}(\hat{n}^{\wedge}, \hat{m}^{\vee}, I) \subseteq \mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J \circ X)$. If $D \in \mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J \circ X)$ then $D = C \triangleright (J \circ X) = (C \triangleright J) \triangleright X$ for some $C \in L^{p}$. Since $C \triangleright J \in \mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J)$, we get

$$D = (C \triangleright J) \triangleright X = h(C \circ J) \in \mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, I),$$

proving $\mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J \circ X) \subseteq \mathrm{Int}(\hat{n}^{\wedge}, \hat{m}^{\vee}, I)$. Summing up, we proved $\mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J \circ X) = \mathrm{Int}(\hat{n}^{\wedge}, \hat{m}^{\vee}, I)$. Now, $J \circ X$ yields the required matrix K.

"⇐": Since $\mathrm{Ext}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J) = \mathrm{Ext}(\hat{p}^{\wedge}, \hat{m}^{\vee}, K)$, there exists a matrix $S \in L^{m\times r}$ for which $K \circ S = J$ and a matrix $T \in L^{m\times r}$ for which $J \circ T = K$, respectively. Consider now mappings $f : \mathrm{Int}(\hat{p}^{\wedge}, \hat{m}^{\vee}, K) \to \mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J)$ and $g : \mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J) \to \mathrm{Int}(\hat{p}^{\wedge}, \hat{m}^{\vee}, K)$ defined for $D \in \mathrm{Int}(\hat{p}^{\wedge}, \hat{m}^{\vee}, K)$ and $F \in \mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J)$ by

$$f(D) = D \triangleright S \qquad \text{and} \qquad g(F) = F \triangleright T.$$

Notice that every $D \in \mathrm{Int}(\hat{p}^{\wedge}, \hat{m}^{\vee}, K)$ is in the form $D = C \triangleright K$ for some $C \in L^{p}$ and that every $F \in \mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J)$ is in the form $F = E \triangleright J$ for some $E \in L^{p}$. The mappings f and g are defined correctly. Indeed,

$$f(D) = D \triangleright S = (C \triangleright K) \triangleright S = C \triangleright (K \circ S) = C \triangleright J$$

for some C, and because $C \circ J \in \mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J)$, we have $f(D) \in \mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J)$. In a similar way one obtains $g(F) \in \mathrm{Int}(\hat{p}^{\wedge}, \hat{m}^{\vee}, K)$.

Next, observe that for D, which is the form $D = C \triangleright K$ for some C,

$$\begin{aligned} g(f(D)) &= ((C \triangleright K) \triangleright S) \circ T = (C \triangleright (K \circ S)) \triangleright T = \\ &= (C \triangleright J) \triangleright T = C \triangleright (J \circ T) = C \triangleright K = D \end{aligned}$$

and, similarly, $f(g(F)) = F$, proving that f and g are mutually inverse bijections. Finally, due to Lemma 2, f (and g) is a complement-preserving c-morphism. This shows that $\mathrm{Int}(\hat{p}^{\wedge}, \hat{m}^{\vee}, K) \cong \mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J)$, and hence $\mathrm{Int}(\hat{n}^{\wedge}, \hat{m}^{\vee}, I) \cong \mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J)$. $\square$

Note that switching h for its inverse h^{-1} in Theorem 4 brings a matrix $K' \in L^{p\times m}$ such that $\mathrm{Ext}(\hat{n}^{\wedge}, \hat{m}^{\vee}, I) = \mathrm{Ext}(\hat{p}^{\wedge}, \hat{m}^{\vee}, K')$ and $\mathrm{Int}(\hat{p}^{\wedge}, \hat{r}^{\vee}, J) = \mathrm{Int}(\hat{p}^{\wedge}, \hat{m}^{\vee}, K')$. The matrix K and K' does not need to be equal. As an counterexample consider L being a chain $0 < a < b < 1$ with $\otimes$ defined as follows

$$x \otimes y = \begin{cases} x \wedge y & \text{if } x = 1 \text{ or } y = 1, \\ 0 & \text{otherwise,} \end{cases}$$

for each $x, y \in L$. One can easily see that $x \otimes \bigvee_j y_j = \bigvee_j (x \otimes y_j)$ and thus an adjoint operation $\rightarrow$ exists such that $\langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ is a complete residuated lattice. Namely, $\rightarrow$ is given as follows:

$$x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x = 1, \\ b & \text{otherwise,} \end{cases}$$

for each $x, y \in L$. Now, matrices $I = \begin{pmatrix} a \end{pmatrix}, J = \begin{pmatrix} b \end{pmatrix} \in L^{1\times 1}$ have the same set of intents, namely $\{[b], [1]\}$. It is easy to check, that the identity on L^1 is complement-preserving c-isomorphism. We get that $K = I$ and $K' = J$; on the other hand, there is no such matrix which could stand for both K and K'. This is contrary to analogous theorem for extendable i-morphisms.

The following theorem shows that i-isomorphism between extents of two concept lattices defines concept-preserving c-isomorphism between intents of the concept lattices.

Theorem 5. *If* $h_{\mathrm{Ext}} : \mathrm{Ext}(X_1^{\wedge}, Y_1^{\vee}, I_1) \rightarrow \mathrm{Ext}(X_2^{\wedge}, Y_2^{\vee}, I_2)$ *is i-isomorphism then corresponding mapping* $h_{\mathrm{Int}} : \mathrm{Int}(X_1^{\wedge}, Y_1^{\vee}, I_1) \rightarrow \mathrm{Int}(X_2^{\wedge}, Y_2^{\vee}, I_2)$ *is complement-preserving c-isomorphism.*

We omit the proof of Theorem 5 because of lack of space.

An analogy of Theorem 5 which would read that complement-preserving c-isomorphism between intents defines an i-isomorphisms between extents does not hold. The example following Theorem 4 can be used as the counterexample.

6 Conclusions

We investigated the closure spaces induced by isotone Galois connections, i.e. mappings induced by a matrix describing a graded relationship between objects and attributes. Such mappings naturally appear in analysis of relational data. We showed that unlike the bivalent case, these spaces are just particular closure spaces, we called c-closure spaces. We provided a characterization of such closure spaces: they are exactly the closure spaces generated by a-complements of fuzzy sets. Furthermore, we defined the notion of a morphism between such closure spaces and showed that these morphisms are just the mappings generated by matrices over residuated lattices by triangular product projections. In addition, we provided a criterion of isomorphism of two c-closure spaces in terms of row and column spaces of matrices over residuated lattices. The results show that behind the methods of relational data analysis, there is a reasonable calculus of matrices over residuated lattices. The role of this calculus is analogous to the role of ordinary matrix calculus for the analysis of real-valued data using the methods based on linear algebra.

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References

1. Belohlavek, R.: Fuzzy Relational Systems: Foundations and Principles. Kluwer, Academic/Plenum Publishers, New York (2002)
2. Belohlavek, R.: Concept lattices and order in fuzzy logic. Annals of Pure and Applied Logic 128(1-3), 277–298 (2004)
3. Belohlavek, R.: Optimal triangular decompositions of matrices with entries from residuated lattices. Int. J. Approximate Reasoning 50(8), 1250–1258 (2009)
4. Belohlavek, R.: Optimal decompositions of matrices with entries from residuated lattices. J. Logic and Computation (2011), doi:10.1093/logcom/exr023
5. Belohlavek, R., Vychodil, V.: Factor Analysis of Incidence Data via Novel Decomposition of Matrices. In: Ferré, S., Rudolph, S. (eds.) ICFCA 2009. LNCS, vol. 5548, pp. 83–97. Springer, Heidelberg (2009)
6. Ganter, B., Wille, R.: Formal Concept Analysis. Mathematical Foundations. Springer, Berlin (1999)
7. Georgescu, G., Popescu, A.: Non-dual fuzzy connections. Archive for Mathematical Logic 43, 1009–1039 (2004)
8. Green, J.A.: On the structure of semigroups. Annals of Mathematics 54(1), 163–172 (1951)
9. Gottwald, S.: A Treatise on Many-Valued Logics. Research Studies Press, Baldock (2001)
10. Hájek, P.: Metamathematics of Fuzzy Logic. Kluwer, Dordrecht (1998)
11. Kim, K.H.: Boolean Matrix Theory and Applications. M. Dekker (1982)
12. Kohout, L.J., Bandler, W.: Relational-product architectures for information processing. Information Sciences 37(1-3), 25–37 (1985)
13. Markowsky, G.: The factorization and representation of lattices. Transactions of the AMS 203, 185–200 (1975)
14. Pollandt, S.: Fuzzy Begriffe. Springer, Heidelberg (1997)
15. Popescu, A.: A general approach to fuzzy concepts. Mathematical Logic Quarterly 50(3), 1–17 (2004)
16. Ward, M., Dilworth, R.P.: Residuated lattices. Trans. Amer. Math. Soc. 45, 335–354 (1939)