Chapter 5
Portfolio Risk Management Modelling by Bi-level Optimization

Todor Stoilov and Krasimira Stoilova

Bulgarian Academy of Sciences,
Institute of Information and Communication Technologies,
1113 Sofia, Acad. G. Bontchev str. BL.2, Bulgaria
todor@hsi.iccs.bas.bg

Abstract. The portfolio optimization theory targets the optimal resource allocation between sets of securities, available at the financial markets. Thus, the investment process is a task, which targets the maximization of the portfolio return and minimization of the portfolio risk. Because such an optimization problem becomes multi-criterion optimization one it lacks an unique solution. A balance between the portfolios risk and portfolio return has to be integrated in a common scalar criterion for the risk management. The book chapter considers a bi-level optimization paradigm for the investment process. The optimization process evaluates the optimal Sharp ratio of risk versus the return to identify the parameter of the investor's preferences to risk at the upper level. At the lower level of optimization the optimal portfolio is evaluated using the upper level defined investor's preferences. In that manner, the portfolio optimization results in an unique solution, which is determined according to the objective considerations and it is not based on subjective assumptions of the portfolio problem. As a result, the portfolio risk is minimized according to two arguments: the content of the portfolio with appropriate assets and by the parameter of investor's preferences to risk.

1 Introduction

The estimation and the forecast of the financial risk is currently one of the major tasks of the investment's process management. This problem is in the scope of statistics and probability modelling. The financial risk is always related with the portfolio management [20]. The uncertainty about future events makes the market behaviour unpredictable and prevents the assessment of the parameters of the financial markets under dynamical changes. The analysis of the market is performed under predefined assumptions, which are taken into consideration when allocating financial resources. Generally, such assumptions concern uncertainty in ideal mathematical behavior, constant and not changing environment influences. The formal models in investment apply mathematical analytical tools, which formalize both the behavior of the market players and future events associated with financial markets. The allocation of investment resources is formalized and the resulting mathematical methods strongly influence the working practice of financial institutions [4].

J. Lu, L.C. Jain, and G. Zhang: Handbook on Decision Making, ISRL 33, pp. 91–110.
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According to the portfolio theory the decision maker makes his decisions taking into account the risk of the investment. The risk has a meaning of uncertainty. The term "risk" is used when the future is not determined and predictable. Currently, the portfolio optimization models are based on probability theory. However, the probabilistic approaches cannot fully formalize the real market behaviour. Another attempt for handling uncertainty of the financial market is the application of the fuzzy set theory [4, 21].

The most monumental contribution for the application of the modern mathematical models in finance and particularly in risk assessment gives the work of Markowitz [9]. The portfolio selection is the most impact-making development in modern mathematical finance management. The Markowitz theory of portfolio management deals with the individual investor. This theory makes combination of the probability theory and optimization. The investor's goal is to maximize the return and to minimize the risk of the investment decisions. The investor's return is formalized as the mean value of a random behaved function of the portfolio securities returns. The risk is formalized as a variance of these portfolio securities. These mathematical representations of return and risk allow defining a simple optimization problem which formalizes the portfolio management. The two important goals of the investor are to maximize the profit and to minimize the risk of the investment. The exact portfolio solution depends on the level of risk the investor can bear in comparison with the level of portfolio return. Thus, the relation between return and risk is always a major parameter, which has to be identified by the investor for practical utilization of the portfolio theory. In that manner, the decision making process of the investment is generally managed by the subjective assumptions of the investor for the risk/return relation of the portfolio. A decreasing of the subjective influence of the investment process can be achieved if the unknown investor's coefficient for undertaking risk is calculated according to the optimization problem. A formal model proposing a new bi-level optimization problem for the portfolio optimization is presented in the book chapter. The upper level evaluates the parameter of the investor's risk preference. Then, this parameter is used for the optimal resource allocation by minimizing risk and maximizing the portfolio return. Thus, in a common formal problem the portfolio management is performed with a lack of subjective influence in the process of resource allocation. The portfolio risk is minimized according to two types of arguments: the portfolio content and the parameter of the investor's risk preference.

2 Taxonomy of the Risk

The content of the term "risk" is hidden in the uncertainty of the future process, which influences the return or costs of the financial assets. The term "risk" is addressed to several categories of the financial world [3].

Market Risk. It is defined as a risk to financial portfolio, related to the dynamical changes of the market prices of equity, foreign exchange rates, interest rates, commodity prices. The financial firms generally take a market risk to receive profits. Particularly, they try to take a risk they intent to have and they actively manage the market risk.

Liquidity risk is defined as the particular risk from conducting transactions in markets with low liquidity as evidence low trading volume and large bid-ask spread. Under such conditions the attempt to sell assets may push prices lower and the assets have to be sold below their fundamental values or within a time frame, which is longer than expected.

Operational risk is defined as a risk of loss due to physical catastrophe, technical failure and human error in the operation of the firm.

Credit risk is defined as the risk that counterparty may become less likely to fulfill its obligations upon date.

Business risk is defined as the risk that changes in variables of a business plan. It will destroy that plan, including quantifiable risks for business cycle and demands, changes in the competitive behaviour and/or technology.

The term of the risk that is usually employed and easily formalized is the variance of the dynamically changed costs of the financial assets. Introduced by Markovitz, the assets characteristics are defined by their average return E_i and their risk, evaluated as the variance τ_i . The book chapter considers the market risk, which results in different values of the variances of the average returns. As an example, data for the rates of three currencies, USD, GBP and CHF, taken from the Bulgarian site http://econ.bg for a period of 15 days is given in Fig.1.

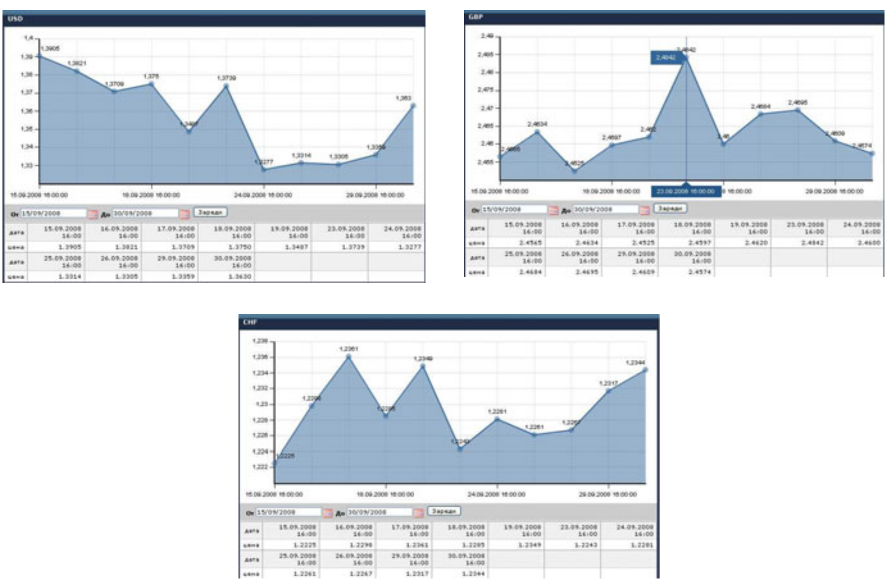

Fig. 1. Rates of USD, GBP CHF currency, taken from http://econ.bg

The data is employed to define the portfolio problem (see Table 1; BGN is the Bulgarian currency).

Table 1. Daily returns of three currencies

DATE	Rate of USD [BGN]	Rate of GBP [BGN]	Rate of CHF [BGN]	R_1 - daily return USD %	R_2 - daily return GBP %	R_3 - daily return CHF %
30.9.2008	1,3630	2,4574	1,2344	2,029	-0,142	0,219
29.9.2008	1,3359	2,4609	1,2317	0,406	-0,348	0,408
26.9.2008	1,3305	2,4695	1,2267	-0,068	0,045	0,049
25.9.2008	1,3314	2,4684	1,2261	0,279	0,341	-0,163
24.9.2008	1,3277	2,4600	1,2281	-3,363	-0,974	0,31
23.9.2008	1,3739	2,4842	1,2243	1,868	0,902	-0,858
19.9.2008	1,3487	2,4620	1,2349	-1,913	0,094	0,521
18.9.2008	1,3750	2,4597	1,2285	0,299	0,294	0,-615
17.9.2008	1,3709	2,4525	1,2361	-0,81	-0,688	0,512
16.9.2008	1,3821	2,4695	1,2298	-0,6	0,529	0,392
15.9.2008	1,3905	2,4565	1,2225			

These initial values are noted as R_i $(i=1,n;\ n=3)$, where i is the index of the asset. It is necessary to evaluate the average return E_i and the risk τ_i for each asset. The evaluation of the average return is found as the weighted sum $E_i = \sum_t P^t R_i^t$, where P^t is the probability that i has a return R_i^t at time t [17]. The values R_i^t for the USD currency are assumed to be probably equal to the probability $P^t = \frac{1}{N}$ = 0,1; N=10 – number of days, used for the currency rate. Hence, the average return of the USD currency is calculated as

$$E_1 = 0{,}1(2{,}029 + 0{,}406 + \ldots - 0{,}6) = -0{,}1873 .$$

In the same fashion,

$$E_2 = 0{,}1(-0{,}142 - 0{,}348 + \ldots + 0{,}529) = 0{,}0053$$
$$E_3 = 0{,}1(0{,}219 + 0{,}408 + \ldots + 0{,}392) = 0{,}15544 \quad ,$$

and the return vector of the three assets is

$$E^T = \left| -0{,}1873 \quad 0{,}0053 \quad 0{,}1544 \right| .$$

The risk of each asset is defined by the variance of the daily returns [3]:

$$\tau_i^2 = \sum_t P^t (E_i - R_t^i)^2 \quad ,$$

which results in $\tau_i^T = \left| 1{,}5355 \quad 0{,}5357 \quad 0{,}4147 \right| .$

These data are the input for the definition of the portfolio optimization problem. The average values E_i represent the mean value around which the daily returns R_i fluctuate. The risk τ_i is a quantitative assessment of the diapason in which R_i varies. Larger diapasons imply higher risk levels.

3 Portfolio Optimization Problem

The portfolio theory was developed as a decision support tool for the allocation of investments for the sells of financial assets (securities, bounds) from the stock exchange [1]. Such an allocation is called "investment" decision making. The investor treats each asset as a prospect for future income. Thus, the better combination of financial assets (securities) in the portfolio, the better return for the investor. The portfolio contains a set of securities. The portfolio optimization problem is defined as problem for optimal allocation of financial resources for trading financial assets. The problem of portfolio optimization targets the optimal resource allocation in the investment process [12]. The resource allocation is done by investing capital in financial assets (or goods), which will generate return for the investor after a period of time. The objective of the investment process is to maximize the return while keeping risk at minimum [11]. In 1952, Harry Markowitz suggested a simple and powerful approach to quantify risk. According to the portfolio theory [12] the analytical relations between the portfolio risk V_p, portfolio return E_p and the values of the investment per type of assets x_i are

$$E_p = \sum_{i=1}^{n} E_i x_i = E^T x$$

$$V_p = \sum_{j}^{n} \sum_{i=1}^{n} x_i x_j \operatorname{cov}(i, j) = x^T \operatorname{cov}(.) x \ ,$$

where

E_i - average value of the return of asset i;

$E^T = (E_1, \ldots, E_n)^T$ - vector with dimension 1 x n;

$\operatorname{cov}(i,j)$ – co-variation coefficient between the assets i and j.

The component $V_p = x^T \operatorname{cov}(.)\, x$ formalizes the quantitative assessment of the portfolio risk. The component $E_p = E^T x$ is the quantitative evaluation of the portfolio return. The portfolio problem solutions x_i , $i=1,n$ determine the relative amounts of the investment per security i.

The co-variation is calculated from previously available statistical data for the returns of assets i and j and it takes the form of a symmetrical matrix

$$\text{cov}(.) = \begin{vmatrix} \text{cov}(1,1) & \text{cov}(1,2) & \cdots & \text{cov}(1,n) \\ \text{cov}(2,1) & \text{cov}(2,2) & \cdots & \text{cov}(2,n) \\ \vdots & & & \\ \text{cov}(n,1) & \text{cov}(n,2) & \cdots & \text{cov}(n,n) \end{vmatrix}_{nxn}.$$

The components cov(i,j) are evaluated from the values $R_i^{(1)}, R_i^2, \cdots, R_i^{(N)}$ and $R_j^{(1)}, R_j^2, \cdots, R_j^{(N)}$, which concern the profit of assets i and j for discrete time moments (1), (2),…, (N). The co-variation coefficient between assets i and j is calculated as

$$\text{cov}(i,j) = \frac{1}{N}\begin{bmatrix} (R_i^{(1)} - E_i)(R_j^{(1)} - E_j) + (R_i^{(2)} - E_i)(R_j^{(2)} - E_j) + \\ + \cdots (R_i^{(N)} - E_i)(R_j^{(N)} - E_j) \end{bmatrix},$$

where

$$E_i = \frac{1}{N}\left[R_i^{(1)} + R_i^{(2)} + \cdots + R_i^{(N)}\right], \quad E_j = \frac{1}{N}\left[R_j^{(1)} + R_j^{(2)} + \cdots + R_j^{(N)}\right]$$

are the average profits of the assets i and j for the period $T = [1,2,....,N]$. Particularly, the value $\text{cov}(i,i) = \tau_i^2$ gives the variation of the return of asset i. The portfolio theory defines the so-called "standard" problem of optimization [12]:

$$\min_x [\frac{1}{2} x^T \text{cov}(.) x - \sigma E^T x], \tag{1}$$

$$x^T.\mathbf{1} = 1,$$

where cov(.) – a symmetric positively defined n x n square matrix,
E – a (n x 1) vector of the average profits of the assets for the period of time $T = [1,2,....,N]$;

$$1 = \begin{vmatrix} 1 \\ \vdots \\ 1 \end{vmatrix}, \qquad \text{is a unity vector, } n \times 1;$$

σ – a parameter of the investor's preferences to undertake a risk in the investment process.

The constraint of the optimization problem presents the equation $x_1 + x_2 + \cdots + x_n = 1$, which formalizes the fact that the investment is not partly implemented and the full amount of the resources are devoted for the investments. If the right side of the constraint is less than 1, this means that the amount of the investment is not effectively used. The investment per different assets has to be performed for the total amount of the available investment resources, numerically presented as a relative value of 1. The solutions x_i , i=1, 2, …, n give the relative values of the investment, which are allocated for the assets i, i=1, 2, …, n.

The component of the target function $V_p = x^T \operatorname{cov}(.)x$ is the quantitative assessment of the portfolio risk. The component $E_p = E^T x$ is the quantitative value of the portfolio return. The target function of problem (1) aims to minimize the portfolio risk V_p and also maximize its return E_p. The parameter σ has a numerical value from the range $[0, +\infty]$. This coefficient quantitatively formalizes the investor's ability to undertake risk. If $\sigma=0$, the investor is very cautious (even a coward) and his general task is to decrease the risk of the investment, $\min_x [x^T \operatorname{cov}(.)x]$. If $\sigma=+\infty$, the investor has forgotten the existence of the risk in the investments. His target is to obtain a maximal return from the investment. For that case, the relative weight of the return in the target function is most important, and then the optimization problem has an analytical form: $\min_x [-\sigma E^T x] \equiv \max_x [E^T x]$.

Thus, in the portfolio problem a new unknown parameter σ is introduced , which assesses the investor's preferences for undertaking risk in decision making. This parameter influences the portfolio problem, making it a parametric one. Respectively, for a new value of σ, the portfolio problem (1) has to be solved again. The trivial case when σ is not properly estimated the optimization problem has to be solved for a set of σ. The values σ introduce strong subjective influence to the solutions of the portfolio problem. Additionally, for practical reasons, the portfolio problem has to be solved multiple times with a set of values for the coefficient of the investor's preferences σ to undertake risk. Thus, for real time applications of investment, the estimation of σ and the solution of (1) become quite important.

The numerical assessment of σ is a subjective task for the financial analyzer. This coefficient strongly influences the definition and respectively the solutions of the portfolio problem. Respectively, σ also changes the final investment decision.

The portfolio theory uses the space risk-return $V_p=V_p(E_p)$ for the assessment of the portfolio characteristics found as combinations of admissible assets. The investors have to choose optimal portfolios from the upper set of admissible solutions named "efficiency frontier". This "efficiency frontier" is not evidently found. Points from this curve can be found by solving the portfolio optimization problem with different values of the parameter σ. The "efficient frontier" is evaluated point by point according to an iterative numerical procedure:

1. An initial value of σ for the investor's preferences is chosen. The zero value $\sigma=0$. is a good starting point and this corresponds to the case of an investor, who is not keen on risky decisions;
2. the portfolio problem is solved with the chosen σ

$$\min_x [\frac{1}{2} x^T \operatorname{cov}(.) x - \sigma E^T x]$$
$$x^T \times 1 = 1$$

and the optimal solution $x(\sigma)$ is found;

3. evaluation of the portfolio risk and portfolio return:

$$V_p = x^T(\sigma)\mathrm{cov}(.)x^T(\sigma) , \qquad E_p = E^T x(\sigma) .$$

These values give a point into the space $V_p = V_p(E_p)$, which belongs to the efficient frontier;

4. new value of $\sigma_{new}=\sigma_{old}+\Delta$ is chosen, where Δ is determined by considerations for completeness in moving into the set $\sigma = [0,+\infty]$. Then, go to point 2.

Hence, for each solution of the portfolio optimization problem one point into the space $V_p = V_p(E_p)$, belonging to the curve of the efficiency frontier is found (see Fig.2).

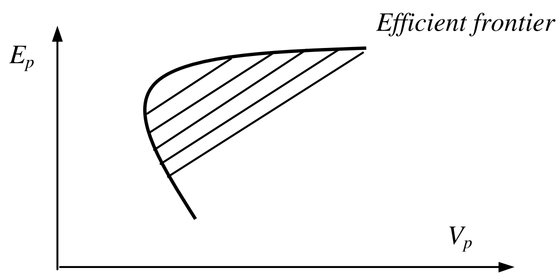

Fig. 2. Efficiency frontier of the portfolio optimization

For practical cases of individual investor, problem (1) is solved with a set of values of σ. Having a set of solutions $x(\sigma)$ the final value of σ^* for that investor is empirically estimated, which gives also the final optimal portfolio solution $x(\sigma^*)$ as well. However, such an approach generates a contradiction between the manner of quantitative definition of problem (1) and the final decision for the investment. The portfolio theory insists that the value of σ^* has to be estimated before solving the problem. However, in practice σ^* is estimated after evaluating a set of portfolio problems (1) with different values for σ. Respectively, the subjective influence in definition of σ^* is quite obvious.

The formal model, which is developed in this book chapter targets at the decrease of the subjective influence in evaluating and assessing the parameter of investor's preference to risk σ. The idea of the model is to formalize the decision making process by two hierarchically interconnected optimization problems (Fig.3).

The optimization problem for evaluating σ is stated at the upper hierarchical level. This problem can be defined from considerations, which are not subjectively influenced. For example, this optimization problem, can target the evaluation of such a σ, which consequently will result in a "well" ratio between the portfolio risk and return.

On the lower hierarchical level the standard portfolio optimization problem is solved using $\sigma_{,}$ estimated from the upper optimization problem. Unfortunately, both optimization problems are interconnected by their arguments. The solution of the upper level problem influences as parameter the corresponding low level optimization problem and vice versa. Hence, a bi-level optimization problem is stated, which represent the decision making process in portfolio optimization.

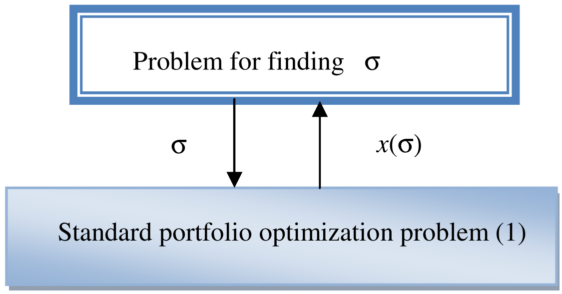

Fig. 3. Definition of bi-level portfolio optimization problem

4 Bi-level Hierarchical Optimization Problems

A general peculiarity of bi-level optimization problems is that by solving an appropriate optimization problem on the upper level, the evaluated solutions are used to define a set of parameters in the lower level optimization problems. The solutions of the last in turn define a set of parameters for the upper level problem. Thus, an interrelation between the solutions at the upper and lower level optimization problems influence the exact form of the optimization problems.

The general bi-level hierarchical optimization problem is made by the formulation of the Stackelberg game [16]. The Stackelberg problem can be interpreted as a game between two players, each of them making decisions [13, 14, 15]. The decisions of the leader (upper level problem) answer the questions: which is the best strategy for the leader, if he knows the goal function and the constraints of the follower (lower level problem) and how the leader has to choose his next decisions? When the leader evaluates his decisions, the follower chooses his own strategy for decision making for minimization of his target function. Respectively, the follower solves an optimization problem of mathematical programming form.

The formal presentation of the Stackelberg game in bi-level hierarchical forms is given as interconnected optimization sub-problems. The lower level optimization problem is in the form

$$\min_{y \in Y} f(x, y) \tag{2}$$

$$g(x, y) \le 0 , \tag{3}$$

where $x \in R^n$ is a coordination parameter, defined from the solutions of the upper level optimization problem, $y \in Y \subseteq R^m$ is the solution of the lower level optimization sub-problem, $f: R^n \times R^m \to R^1$ and $g: R^n \times R^m \to R^q$. This sub-problem is parameterized by the values of *x*. Let *P(x)* denotes the optimal solution of problem (2) for given x:

$$P(x) = \left\{ y^* \in S(x) \mid f(x, y^*) = \min_{y \in Y} f(x, y^*), \quad g(x, y) \le 0 \right\},$$

where

$$S(x) = \{ y \in Y \mid g(x, y) \le 0 \}.$$

The optimal problem of the upper level for given lower level solution $y^* \in P(x)$ is

$$\min_{x \in X} F(x, y^*) \tag{4}$$

$$G(x, y^*) \le 0 \tag{5}$$

$$y^* \in P(x) \tag{6}$$

where $F: R^n \times R^m \to R^1, \quad G: R^n \times R^m \to R^p, \quad X \subseteq R^n$.

This problem is solved by the leader. The bi-level hierarchical problem, titled as Stackelberg game, is formulated as hierarchical system with two levels. The optimization sub-problem (2-3) is a slave one to the coordination problem (4)-(6). The particular constraint *P*(*x*) determines the rational set of reactions of the slaver player. The feasible area of the coordination (4)-(6) is non-explicitly analytically defined

$$IR = \{ (x, y^*) \mid G(x, y^*) \le 0, \quad y^* \in P(x) \}.$$

The reaction of the slaver is evaluated from the set of rational reactions *P*(*x*), while *IR* represents the feasible set for the decisions of the leader, among which he can search the optimal solution.

The book chapter considers a special form of the Stackelberg's problem:

$$\min_{x \in X, y^*} \{ F(x, y^*) / G(x, y^*) \le 0 \} \tag{7}$$

$$y^* \in P(x) = \left\{ y^* \in S(x) \mid f(x, y^*) = \min_{y \in Y} f(x, y^*), \quad g(x, y) \le 0 \right\}, \tag{8}$$

where the upper level is influenced by the reaction of the lower level by the minimal function *w*(*x*), defined as

$$w(x) = \min_{y \in Y} f(x, y) \tag{9}$$

satisfying the definition set

$$g(x, y) \le 0. \tag{10}$$

For this model the notation $w(x)$ refers to the minimal value of the goal function of the lower level $f(x,y)$, where the optimization is performed towards the argument y. The upper level problem can be formulated in a way, excluding y, substituting it in the target function and constraints explicitly with the minimal valued function $w(x)$:

$$\min_{x \in X} F(x, w(x)) \tag{11}$$

$$G(x, w(x)) \leq 0, \tag{12}$$

where $F : R^n \times R^1 \to R^1$ and $G : R^n \times R^1 \to R^p$. By the combination of relations (9)-(10) and (11)-(12) the bi-level hierarchical optimization problem is stated in the form

$$\min_{x \in X} F(x, w(x)) \tag{13}$$

$$G(x, w(x)) \leq 0 \tag{14}$$

$$w(x) = \min_{y \in Y} f(x, y) \tag{15}$$

$$g(x, y) \leq 0 . \tag{16}$$

Both (7)-(8) and (13)-(16) are general nonlinear optimization problems. Due to methodological difficulties for the solution of hierarchically interconnected optimization problems, the classical application of the portfolio theory currently lacks a solution of the bi-level optimization problems. The portfolio problem is solved by quantitative assessment of σ^* in advance, without applying interconnected hierarchical optimization. The value of σ^* is estimated intuitively or empirically by an expert. Here, a methodology for the solution of bi-level portfolio problem is applied, derived as non-iterative coordination [17, 19]. The methodology for non-iterative coordination in hierarchical systems defines analytical approximations of the inexplicit function $w(x)$, used by the upper and lower optimization problems. Thus, analytical relations between the investor's preferences for the risk σ and the solutions x_i are derived [18]. Such relations support fast solution of the bi-level problem and respectively support real time decision making. The upper level problem is defined with a target function, which minimizes the Sharp ratio: portfolio risk versus portfolio return. The argument of this optimization problem is the investor's preferences for the risk σ. Applying the non-iterative methodology [17, 19] analytical relations between the portfolio problem's parameters E_p, V_p , the portfolio solutions x_i and the parameter of the investor's preference σ are derived. These relations speed up the decision making process and the investment decisions can be made in real time.

5 Solution of Portfolio Bi-level Problem

The solutions of the initial problem (1) x_i have to be described as analytical functions of the σ parameter. For that case the initial problem (1) is rewritten in the form

$$\min_x [\frac{1}{2} x^T Qx + R^T x] \quad (17)$$
$$Ax = C \; ,$$

where the correspondence between problems (1) and (17) is:

$$Q=\mathrm{cov}(.), \qquad R=-\sigma E, \qquad A=1 , \quad C=1 .$$

If the value of the coefficient σ is asserted, problem (1) has a solution, denoted like $x(\sigma)$. For the case when σ changes, the solution of the portfolio problem x is an inexplicit analytical function of σ:

$$x=x(\sigma).$$

The portfolio risk

$$V_p(\sigma) = x^T(\sigma)\mathrm{cov}(.)x(\sigma)$$

and the portfolio return

$$E_p(\sigma) = E^T x(\sigma)$$

are also implicit functions of σ.

Problem (2) can be solved using the method of the non-iterative coordination, which gives possibility to derive approximations of the implicit analytical relations of the portfolio parameters $V_p(\sigma)$, $E_p(\sigma)$, $x(\sigma)$ towards the argument σ. Using relation (15) from [19], the analytical solution of problem (2) is

$$x^{opt} = -Q^{-1}[R - A^T(AQ^{-1}A^T)^{-1}(AQ^{-1}R + C)] \, . \quad (18)$$

Using this relation, the analytical descriptions of the portfolio risk and return become

$$V_p = x^{Topt} Q x^{opt} =$$
$$= \left\{-\left[(C^T + R^T Q^{-1}A^T)(-AQ^{-1}A^T)^{-1}A + R^T\right]Q^{-1}\right\}Q$$
$$\left\{-Q^{-1}\left[R - A^T(AQ^{-1}A^T)^{-1}(AQ^{-1}R + C)\right]\right\} .$$

After several transformations it follows

$$V_p = R^T Q^{-1}\left[R - A^T(AQ^{-1}A^T)^{-1}A\right]Q^{-1}R + C^T(AQ^{-1}A^T)^{-1}C \, .$$

The analytical relation of the portfolio return is obtained as the linear relation towards x_{opt} or

$$E_p = E^T x = R^T x^{opt} = R^T \left\{-Q^{-1}\left[R - A^T (AQ^{-1}A^T)^{-1}(AQ^{-1}R + C)\right]\right\} =$$
$$= -R^T Q^{-1}[R - A^T (AQ^{-1}A^T)^{-1} AQ^{-1}R] + R^T Q^{-1} A^T (AQ^{-1}A^T)^{-1} C$$

Finally

$$V_p = R^T Q^{-1}\left[R - A^T (AQ^{-1}A^T)^{-1} A\right] Q^{-1} R + C^T (AQ^{-1}A^T)^{-1} C \tag{19}$$

$$E_p = -R^T Q^{-1}[R - A^T (AQ^{-1}A^T)^{-1} AQ^{-1}R] + R^T Q^{-1} A^T (AQ^{-1}A^T)^{-1} C \tag{20}$$

Relations (19) and (20) can be expressed in terms of the initial portfolio problem (1). Thus, explicit analytical relations for the portfolio risk V_p, portfolio return E_p and the optimal solution of the portfolio problem x_{opt} are derived towards the coefficient of the investor's risk preference σ. For the current problem (1), taking into account the correspondence between problems (1) and (17), it follows

$$x^{opt}(\sigma) = Q^{-1}\{\left[E - A^T (AQ^{-1}A^T)^{-1} AQ^{-1}E\right]\sigma + A^T (AQ^{-1}A^T)^{-1} C\} \tag{21}$$

$$V_p(\sigma) = E^T Q^{-1}[R - A^T (AQ^{-1}A^T)^{-1} A] Q^{-1} E\sigma^2 + C^T (AQ^{-1}A^T)^{-1} C \tag{22}$$

$$E_p(\sigma) = E^T x^{opt}(\sigma) = E^T Q^{-1}\left\{\left[R - A^T (AQ^{-1}A^T)^{-1} A\right] Q^{-1} E\sigma + A^T (AQ^{-1}A^T)^{-1} C\right\}. \tag{23}$$

To simplify the notations, the following coefficients are introduced:

$$\alpha = E^T Q^{-1}\left[R - A^T (AQ^{-1}A^T)^{-1} A\right] Q^{-1} E \tag{24}$$
$$\beta = C^T (AQ^{-1}A^T)^{-1} C$$
$$\gamma = E^T Q^{-1} A^T (AQ^{-1}A^T)^{-1} C \quad ,$$

where the parameters α, β and γ are scalars. Relations (22) and (23) become

$$V_p(\sigma) = \alpha\sigma^2 + \beta, \qquad E_p(\sigma) = \alpha\sigma + \gamma. \tag{25}$$

The new derived relations (21)-(24) describe in analytically explicit form the functional relations between the portfolio parameters for risk, return and optimal solution towards the coefficient of the investor's preferences to risk σ. Hence, the solution of the portfolio problem (1) is calculated using relations (21)-(23) without the implementation of optimization algorithms for the solution of the low level optimization problem. This considerably speeds up the problem solution of (1). Hence, the portfolio optimization problem can be solved in real time, with no iterative calculations, which benefits the decision making in the fast dynamic environment of the stock exchange.

On the upper optimization level it is necessary to evaluate the parameter of investor's preferences σ, under which the better (minimal) value of Sharp ratio (the relation Risk/Return) is optimized. The problem for the evaluation of σ in a formal way is stated as

$$\min_{\sigma\geq 0} \left\{ \frac{Risk(\sigma)}{Portfolio\_return(\sigma)} = \frac{V_p(\sigma)}{E_p(\sigma)} \right\} .$$

According to relation (25) the analytical form of the problem is

$$\min_{\sigma} \left\{ \frac{V_p(\sigma)}{E_p(\sigma)} = \frac{\alpha\sigma^2+\beta}{\alpha\sigma+\gamma} = \eta(\sigma) \right\} . \tag{26}$$

This problem evaluates the parameter of the investor's preferences σ according to objective considerations. Thus, the portfolio optimization problem is stated as bi-level optimization procedure (Fig. 4). The advantage for the evaluation of σ comes from the fact that the estimation of σ is done by overcoming the subjective influences of the investor, and it is found from a real optimization problem.

The solution σ_{opt} of such a problem is found according to the relations

$$\sigma_{opt} \equiv \arg\left\{ \min\left[0, \quad \frac{d\eta(\sigma)}{d\sigma} = 0 \right] \right\}$$

$$\sigma_{opt} = \min\left[0, \quad \frac{d}{d\sigma}\left(\frac{\alpha\sigma^2+\beta}{\alpha\sigma+\gamma}\right) = 0 \right]$$

$$\frac{d\eta(\sigma)}{d\sigma} = \frac{2\alpha\sigma(\alpha\sigma+\gamma)-\alpha(\alpha\sigma^2+\beta)}{(\alpha\sigma+\gamma)^2} = 0 .$$

The following condition must hold:

$$\alpha\sigma+\gamma = E_p > 0, \qquad \alpha \neq 0 . \tag{27}$$

Then

$$\sigma_1^{opt} = \frac{-\gamma+\sqrt{\gamma^2+\alpha\beta}}{\alpha} = \frac{-1+\sqrt{1+\frac{\alpha}{\gamma}\frac{\beta}{\gamma}}}{\frac{\alpha}{\gamma}} .$$

For the particular case when the value of *C* is a digital number (*C*=1 for relative assessment of the investment), then

$$\sigma^{opt} = \max\left[0, \quad \frac{\gamma}{2}\left(-1+\sqrt{\frac{E^TQ^{-1}E}{E^TQ^{-1}A^T(AQ^{-1}A^T)^{-1}AQ^{-1}E}}\right) \right] \tag{28}$$

This relation gives analytical way of calculation of the optimal parameter for risk preferences of the investor. For that reason the solution of the upper level optimization problem is reduced to analytical relation (28), applied for the calculation of σ_{opt}.

6 Assessment of the Bi-level Calculations

An illustration of the solution of a set of bi-level optimization problems is given below. A set of optimization problems is defined with a maximal amount of 13 securities, traded at the Bulgarian stock exchange, n=13. The portfolio optimization problems has been defined and solved with variable number of securities n (n=2, 3, …,13). Respectively, the corresponding matrices for the portfolios problems were chosen from the largest matrices $Q|_{13x13}$ and $E|_{13x1}$, which were defined from the Bulgarian stock exchange data as

Q_{13}=	[0.786167	-0.162559	-0.071959	-0.164858	0.047331
	-0.603850	0.166243	-1.047479	-0.539301	-0.603588
	-0.891402	0.008453	-0.133797;		
	-0.162559	2.986604	0.793027	0.007331	0.026117
	0.098366	0.045948	-0.111006	0.655404	-0.071133
	0.035753	-0.293635	-0.410321;		
	-0.071959	0.793027	0.555667	0.039004	-0.008898
	0.174713	0.013785	-0.060925	0.161520	0.066703
	0.170407	-0.029705	-0.249643;		
	-0.164858	0.007331	0.039004	0.848372	-0.727659
	0.746201	0.241088	-1.953395	-1.625136	-0.313008
	-1.802728	-0.786455	0.865357;		
	0.047331	0.026117	-0.008898	-0.727659	1.819902
	-1.231908	0.170827	-1.672948	-0.841580	-1.216701
	-1.527534	0.263455	-0.134094;		
	-0.603850	0.098366	0.174713	0.746201	-1.231908
	5.813142	0.356443	0.135327	-1.384723	-0.966858
	0.370034	1.723733	-0.238691;		
	0.166243	0.045948	0.013785	0.241088	0.170827
	0.356443	1.200181	0.075195	0.246796	0.080846
	1.063820	0.504612	-0.220170;		
	-1.047479	-0.111006	-0.060925	-1.953395	-1.672948
	0.135327	0.075195	12.288469	0.077632	-0.607134
	3.229930	-1.407467	-0.200097;		

-0.539301	0.655404	0.161520	-1.625136	-0.841580
-1.384723	0.246796	0.077632	5.300921	1.053221
2.486395	1.651404	-3.039378;		
-0.603588	-0.071133	0.066703	-0.313008	-1.216701
-0.966858	0.080846	-0.607134	1.053221	2.517762
0.653272	-0.035333	-0.194460;		
-0.891402	0.035753	0.170407	-1.802728	-1.527534
0.370034	1.063820	3.229930	2.486395	0.653272
29.446252	1.644501	0.462070;		
0.008453	-0.293635	-0.029705	-0.786455	0.263455
1.723733	0.504612	-1.407467	1.651404	-0.035333
1.644501	11.792594	1.208723;		
-0.133797	-0.410321	-0.249643	0.865357	-0.134094
-0.238691	-0.220170	-0.200097	-3.039378	-0.194460
0.462070	1.208723	16.782235];		

E_{13}^T = [1.147143 1.805000 1.084717 1.239130 1.713784
2.571667 1.099146 2.230377 1.554639 1.217075
2.512333 2.510000 1.844951].

The problems with lower dimension n<13 are defined by sequential removal of the leading row and column from Q. Respectively, the lower order matrices E were generated by removal of the leading component of vector E.

The target of the experiments was to evaluate 30 points of the efficient frontier for each optimization problem. Then, having the efficient frontier, the optimization procedure continues with finding the portfolio, which has minimal Sharp ratio (risk versus return). For that case the parameter of the investor's preferences for risk σ_{opt} is calculated, using (28).

The sequence of the solution of the portfolio problem is the following:

- Analytical definition of the portfolio problem (1) with n=13;
- Evaluation of the scalar values of the intermediate parameters $\alpha(n)$, $\beta(n)$, $\gamma(n)$ from (24);
- Starting the calculations of the efficient frontier with initial value $\sigma^*=0$;
- Evaluation of the portfolio parameters $V_p=V_p(\sigma^*, \alpha(n), \beta(n), \gamma(n)$, $E_p=E_p(\sigma^*, \alpha(n), \beta(n), \gamma(n))$, according to (25). Thus, one point from the efficient frontier in the space risk/return $V_p^*(E_p^*)$ is found;
- New value of the coefficient σ is chosen, $\sigma^{**}=\sigma^*+1/30$.

These steps are performed for 30 points of the graphics $V_p = V_p(E_p)$.

Following problem (26), the optimal value of the parameter of the investor's preferences σ_{opt} is identified, evaluated as a solution of an upper level optimization problem:

$$\min_{\sigma}\left\{\frac{Risk(\sigma)}{\mathrm{Re}\,turn(\sigma)}\right\} = \min_{\sigma}\left\{\frac{V_p(\sigma)}{E_p(\sigma)} = \frac{x^T(\sigma)Qx(\sigma)}{E^T x(\sigma)}\right\}, \tag{29}$$

where $x(\sigma)$ is an implicit function, defined by the solution of the portfolio optimization problem (1) for different values of σ. Problem (29) introduces an objective criterion for the choice and estimation of the coefficient of the investor's preferences. This problem makes advantages for the estimation of σ in comparison with its subjective choice from the financial analyzer, which is performed according to the classical model of the portfolio optimization.

The optimal value of σ_{opt} is numerically calculated using (28).

Figure 4 presents the graphics $\frac{V_p(\sigma)}{E_p(\sigma)}$ for different optimization problems with varying dimensions n=2,3,..7. These graphics explicitly demonstrate the minimum (towards σ) of the ratio of portfolio risk versus return. The corresponding value σ_{opt} is found according to objective considerations, coming from the upper level optimization problem for minimization of Sharp ratio:

$$\sigma^{opt} = \arg\left\{\min_{\sigma}\left\{\frac{V_p(\sigma)}{E_p(\sigma)} = \frac{x^T(\sigma)Qx(\sigma)}{E^T x(\sigma)}\right\}\right\}. \tag{30}$$

Problem (30) uses objective target function, which is the Sharp ratio. Thus, the argument σ is calculated as a solution of a well defined and consistent optimization problem. In comparison with the classical portfolio theory the value of σ is not assessed by subjective consideration of the financial analyzer, which is an advantage of the bi-level portfolio problem. The solution of problem (30) can be expressed also analytically, according to relation (28).

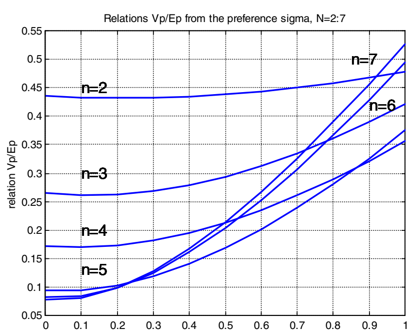

Fig. 4. Relation of the ratio risk/return from σ for different problem dimensions (n=2, 3, …, 7)

As illustration the corresponding values of risk $V_p(x(\sigma_{opt}))$, portfolio return $E_p(x(\sigma_{opt}))$ and its optimal value ratio $\frac{V_p(\sigma)}{E_p(\sigma)}$ are given in figures 5-7. For the case of portfolio problems with dimensions n=[2; 3; 4; 5; 6; 7], the optimal values of σ_{opt}, risk V_p and return E_p are the following

σ_{opt} = [0.2161; 0.1307; 0.0851; 0.0464; 0.0406; 0.0384];

$V_p(\sigma^{opt})$ = [0.5715; 0.2903; 0.197; 0.1199; 0.1118; 0.1083];

$E_p(\sigma^{opt})$ = [1.3223; 1.1103; 1.1568; 1.292; 1.3767; 1.4091];

$V_p/(E_p)$ = [0.4322; 0.2614; 0.1703; 0.0928; 0.0812; 0.0769].

The graphical interpretation of these results is given in figures 5 - 8, where the notation sigma-opt is used for the value σ_{opt} .

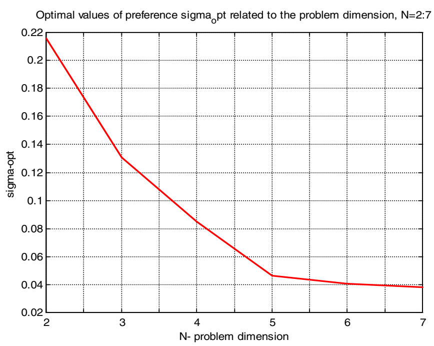

Fig. 5. Relation of σ_{opt} and the problem dimension n=2,3,…,7

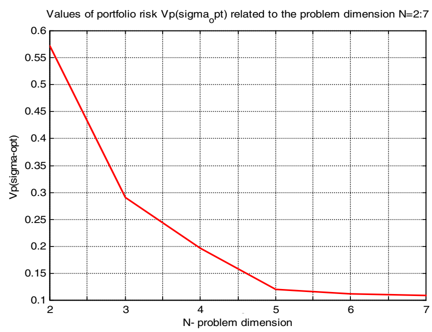

Fig. 6. Relation of $V_p(\sigma^{opt})$ and the problem dimension n=2,3,…,7

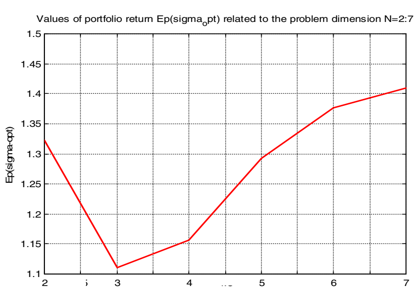

Fig. 7. Relation of $E_p(\sigma^{opt})$ and the problem dimension n=2,3,…,7

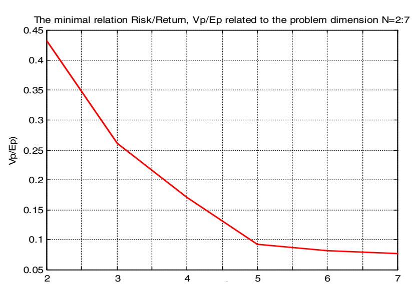

Fig. 8. Relation of $V_p/(E_p)$ and the problem dimension n=2,3,…,7

These results prove the consistency of the definition of the portfolio optimization problem as a bi-level optimization one. On the upper level the optimal value of the parameter of the investor's preferences is calculated, according to objective optimization criteria. In the current case it an optimization problem for the minimization of the ratio risk/return (Sharpe Ratio) has been chosen. The optimal value of the parameter σ was derived analytically as a solution of the upper level optimization problem. This overcomes the weakness of the classical definition of the portfolio optimization problem, which assumes subjective estimation of σ.

7 Conclusion

The book chapter developed a new formal model of the portfolio problem, which was presented as bi-level optimization one. The risk of the investment was minimized twice by optimal content of portfolio securities and optimal assessment of the parameter of risk preference. The classical description of the portfolio problem is like single level optimization with predefined parameter for risk preference σ. This parameter has to be estimated by the financial analyzer and the portfolio theory insists σ to be given before solving the portfolio problem. The estimation of σ is a source of subjective influence for the problem definition and the evaluated optimal solution. Currently, the portfolio problem is solved for a set of values of σ by means to estimate the influence of σ to the problem solutions. In this research the process of decision making was presented as a two level optimization system. The upper level defined the optimal value of the parameter of risk preferences of the investor σ by minimizing the Sharp ratio (portfolio risk versus portfolio return). The lower optimization level used σ and solved the portfolio optimization problem. The bi-level formalism defined in an unique way the most appropriate value of σ by optimizing the Sharp ratio. In that manner, the bi-level formalism achieved two benefits: suppressed the subjective assessment of the investor's risk preferences and calculated and applied the optimal value of σ by minimizing the Sharp ratio. These two outcomes considerably improved the bi-level definition of the portfolio problem in comparison with the classical single level optimization problem.

Additionally, this work developed and applied a special method for solving the optimization problem, titled non-iterative coordination. It allowed to define explicitly and analytically the upper level optimization problem for solving σ and to derive explicit analytical relations between the portfolio problem solutions and σ, $x(\sigma)$. These relations speed up the optimal problem solution and the definition of the efficient frontier of portfolios. Thus, the decision making process can be performed in real time which can respond to the fast dynamic changes of the security market and reduce the risk of the investment.

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