Chapter 9
Viscoelastic Finite Element Formulation

The finite element method is the most popular numerical procedure for the analysis of solids and structures, including those with time dependent properties. In this chapter, we present an incremental viscoelastic finite element formulation for problems with geometrical nonlinearity characterized by large displacements and rotations with small strains. The formulation is based on a total Lagrangian kinematic description. We begin with a brief presentation on the principle of virtual displacements for geometrically nonlinear problems. Procedures used for the computational implementation of the nonlinear viscoelastic model are also presented. We assume that the reader has a basic knowledge of the finite element method and of nonlinear continuum mechanics.

9.1 Principle of Virtual Displacements

Let us consider the motion of a body with arbitrary large displacements and rotations. Figure 9.1 shows the body configurations C^0, C^t and $C^{t+\Delta t}$ at instants τ_0, t and $t + \Delta t$, respectively, and the fixed coordinate system used as reference for the static and kinematic variable. We are interested in evaluating the body equilibrium in a finite sequence of configurations corresponding to times $t_1, t_2, \ldots, t_n$ within the analysis time range. As strategy used in this evaluation, we assume that the variable fields in the configuration $C^{t+\Delta t}$ can be completely determined if the solutions at times $\tau \leq t$ are already known.

The equilibrium condition of the body at time $t + \Delta t$ can be established by the principle of virtual displacements, as follows

$$\int_{\Omega^{t+\Delta t}} \boldsymbol{\sigma}^{t+\Delta t} : \delta\boldsymbol{\varepsilon} d\Omega^{t+\Delta t} = \int_{\Omega^{t+\Delta t}} \left(\mathbf{b}^{t+\Delta t}\right)^T \delta\mathbf{u} d\Omega^{t+\Delta t} + \int_{\Gamma^{t+\Delta t}} \left(\mathbf{t}^{t+\Delta t}\right)^T \delta\mathbf{u} d\Gamma^{t+\Delta t} \quad (9.1)$$

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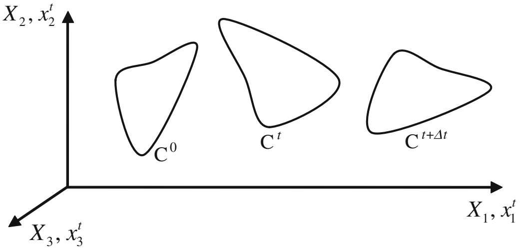

Fig. 9.1 Body configurations and coordinate systems

where the first member is the virtual work of the internal forces, whereas the second member represents the virtual work of the external forces, i.e., body forces $\mathbf{b}$ and surface forces $\mathbf{t}$. In (9.1), $\delta\boldsymbol{\varepsilon}$ represents a variation in the infinitesimal strains associated to the virtual increment $\delta\mathbf{u}$ in the displacement $\mathbf{u}^{t+\Delta t}$. The superscripts stand for the instant of time at which the quantities are determined. The integrals appearing in (9.1) are evaluated over the domain $\Omega^{t+\Delta t}$ and its boundary $\Gamma^{t+\Delta t}$ corresponding to configuration $C^{t+\Delta t}$.

For geometrically nonlinear analyses different definitions of stress and strain tensors are used depending on the characteristics of the problem. In the present development we use the *second Piola–Kirchhoff stress tensor* $\mathbf{S}$ and the *Green–Lagrange strain tensor* $\mathbf{E}$ (see Appendix B) that are energetically conjugated (1), i. e.,

$$\int_{\Omega^{t+\Delta t}} \boldsymbol{\sigma}^{t+\Delta t} : \delta\boldsymbol{\varepsilon} d\Omega^{t+\Delta t} = \int_{\Omega^0} \mathbf{S}_0^{t+\Delta t} : \delta\mathbf{E}_0^{t+\Delta t} d\Omega^0 \tag{9.2}$$

where the index 0 is used to indicate that the quantities are referred to the initial configuration C^0. Substituting (9.2) into (9.1), we have

$$\int_{\Omega^0} \mathbf{S}_0^{t+\Delta t} : \boldsymbol{\delta}\mathbf{E}_0^{t+\Delta t} d\Omega^0 = \int_{\Omega^{t+\Delta t}} \left(\mathbf{b}^{t+\Delta t}\right)^T \boldsymbol{\delta}\mathbf{u} d\Omega^{t+\Delta t} + \int_{\Gamma^{t+\Delta t}} \left(\mathbf{t}^{t+\Delta t}\right)^T \boldsymbol{\delta}\mathbf{u} d\Gamma^{t+\Delta t} \tag{9.3}$$

As the Second Piola–Kirchhoff stress tensor and the Green–Lagrange strain tensor are independent from the rigid body rotations, we may write

$$\mathbf{S}_0^{t+\Delta t} = \mathbf{S}_0^t + \Delta\mathbf{S}_0 \tag{9.4}$$

$$\mathbf{E}_0^{t+\Delta t} = \mathbf{E}_0^t + \Delta\mathbf{E}_0 = \mathbf{E}_0^t + \Delta\mathbf{e}_0 + \Delta\boldsymbol{\eta}_{\boldsymbol{0}}$$

where $\Delta\mathbf{S}_0$ and $\Delta\mathbf{E}_0$ are the increments of the stress and strain measures between t and $t + \Delta t$, respectively. In (9.4), $\Delta\mathbf{E}_0$ is decomposed in a linear part $\Delta\mathbf{e}_0$ and nonlinear part $\Delta\boldsymbol{\eta}_0$, which in index notation are defined by

$$\Delta e_{0ij} = \frac{1}{2}\left(\frac{\partial \Delta u_i}{\partial X_j} + \frac{\partial \Delta u_j}{\partial X_i} + \frac{\partial u_k^t}{\partial X_i}\frac{\partial \Delta u_k}{\partial X_j} + \frac{\partial u_k^t}{\partial X_j}\frac{\partial \Delta u_k}{\partial X_i}\right) \tag{9.5}$$

$$\Delta\eta_{0ij} = \frac{1}{2}\frac{\partial \Delta u_k}{\partial X_i}\frac{\partial \Delta u_k}{\partial X_j}$$

where $\Delta\mathbf{u} = \mathbf{u}^{t+\Delta t} - \mathbf{u}^{\mathrm{t}}$ is the displacement increment vector and $\mathbf{X} = (X_1, X_2, X_3)$ is the particle position in the initial configuration.

Substituting (9.4) into (9.3) and considering that the external loading is independent of the deformation, we obtain the total Lagrangian formulation of the incremental principle of virtual displacements as

$$\begin{aligned}\int_{\Omega^0} \Delta\mathbf{S}_0 : \delta(\Delta\mathbf{E}_0) d\Omega^0 + \int_{\Omega^0} \mathbf{S}_0^t : \delta(\Delta\boldsymbol{\eta}_0) d\Omega^0 = \int_{\Omega^0} \left(\mathbf{b}_0^{t+\Delta t}\right)^T \delta\mathbf{u} d\Omega^0 \\ + \int_{\Gamma^0} \left(\mathbf{t}_0^{t+\Delta t}\right)^T \delta\mathbf{u} d\Gamma^0 - \int_{\Omega^0} \mathbf{S}_0^t : \delta(\Delta\mathbf{e}_0) d\Omega^0\end{aligned} \tag{9.6}$$

being $\mathbf{b}_0^{t+\Delta t}$ and $\mathbf{t}_0^{t+\Delta t}$ the body and surface forces at time $t + \Delta t$, respectively, measured with respect to the initial configuration.

9.2 Linearization of the Principle of Virtual Displacements

We consider a viscoelastic body subjected to both mechanical and hygrothermal loads. For this case the total increment of the Green–Lagrange strain tensor at time interval $[t, t + \Delta t]$ is given by

$$\Delta\mathbf{E}_0 = \Delta\mathbf{E}_0^e + \Delta\mathbf{E}_0^V + \Delta\mathbf{E}_0^{HT} \tag{9.7}$$

where the superscripts e, V and HT are used to indicate the elastic, viscoelastic and hygrothermal contributions, respectively. Neglecting the effect of the nonlinear part $\Delta\boldsymbol{\eta}_0$ as an approximation to obtain the increment of the second Piola–Kirchhoff stress tensor, we may write

$$\Delta\mathbf{S}_0 \cong \mathbf{C}^e(\Delta\mathbf{e}_0 - \Delta\mathbf{e}_0^V - \Delta\mathbf{e}_0^{HT}) \tag{9.8}$$

where $\mathbf{C}^e$ is the 4th order elastic stiffness tensor of the material. Thus, using (9.8), the incremental principle of virtual displacements (9.6) can be rewritten in the form

$$\begin{aligned}\int_{\Omega^0} \mathbf{C}^e\Delta\mathbf{e}_0 : \delta(\Delta\mathbf{e}_0) d\Omega^0 + \int_{\Omega^0} \mathbf{S}_0^t : \delta(\Delta\boldsymbol{\eta}_0) d\Omega^0 = \int_{\Omega^0} \left(\mathbf{b}_0^{t+\Delta t}\right)^T \delta\mathbf{u} d\Omega^0 + \int_{\Gamma^0} \left(\mathbf{t}_0^{t+\Delta t}\right)^T \delta\mathbf{u} d\Gamma^0 \\ + \int_{\Omega^0} \mathbf{C}^e\Delta\mathbf{e}_0^V : \delta(\Delta\mathbf{e}_0) d\Omega^0 + \int_{\Omega^0} \mathbf{C}^e\Delta\mathbf{e}_0^{HT} : \delta(\Delta\mathbf{e}_0) d\Omega^0 - \int_{\Omega^0} \mathbf{S}_0^t : \delta(\Delta\mathbf{e}_0) d\Omega^0\end{aligned} \tag{9.9}$$

Equation (9.9) is the linearized form of the incremental principle of virtual displacements which will be used to derive the nonlinear finite element formulation in Sect. 9.3. It is worth noticing that to obtain (9.9) the approximation $\delta(\Delta\mathbf{E}_0) \cong \delta(\Delta\mathbf{e}_0)$was used.

From relations (9.5), we may show that

$$\delta\left(\Delta e_{0ij}\right) = \frac{1}{2}\left[\delta\left(\frac{\partial \Delta u_i}{\partial X_j}\right) + \delta\left(\frac{\partial \Delta u_j}{\partial X_i}\right) + \frac{\partial u_k^t}{\partial X_i}\delta\left(\frac{\partial \Delta u_k}{\partial X_j}\right) + \frac{\partial u_k^t}{\partial X_j}\delta\left(\frac{\partial \Delta u_k}{\partial X_i}\right)\right] \tag{9.10}$$

$$\delta\left(\Delta \eta_{0ij}\right) = \frac{1}{2}\left[\frac{\partial \Delta u_k}{\partial X_i}\delta\left(\frac{\partial \Delta u_k}{\partial X_j}\right) + \frac{\partial \Delta u_k}{\partial X_j}\delta\left(\frac{\partial \Delta u_k}{\partial X_i}\right)\right]$$

9.3 Nonlinear Viscoelastic Finite Element Formulation

In an incremental geometrically nonlinear analysis, the total displacements in the current configuration $\mathbf{u}^{t+\Delta t}$ are obtained by adding the displacement increments $\Delta\mathbf{u}$ to the point coordinates $\mathbf{x}^{\mathrm{t}}$ corresponding to the last configuration

$$\mathbf{u}^{t+\Delta t} = \mathbf{x}^t + \Delta\mathbf{u} \tag{9.11}$$

This is why it is convenient to use the same interpolation functions for displacement and coordinates (or geometry). The same interpolation functions used in the linear isoparametric finite element formulation can be employed for the nonlinear approach. Thus, for the three-dimensional case, the coordinate vector $\mathbf{X} = [\,X_1 \quad X_2 \quad X_3\,]^T$of a finite element, with N nodal points, are in general defined in the initial configuration as

$$\mathbf{X}^{(e)} = \mathbf{H}(\boldsymbol{\xi})\tilde{\mathbf{X}}^{(e)} \tag{9.12}$$

where $\tilde{\mathbf{X}}^{(e)} = \left[\tilde{\mathbf{X}}_1^{(e)T} \quad \tilde{\mathbf{X}}_2^{(e)T} \quad \ldots \quad \tilde{\mathbf{X}}_N^{(e)T}\right]^T$ is the element nodal coordinate vector being $\tilde{\mathbf{X}}_k^{(e)} = \left[\tilde{X}_1^{(e)} \quad \tilde{X}_2^{(e)} \quad \tilde{X}_3^{(e)}\right]_k^T$ the coordinate vector of the element k-node. In (9.12), $\mathbf{H}(\boldsymbol{\xi})$ represents the interpolation function matrix which has the general form

$$\mathbf{H}(\boldsymbol{\xi}) = [\,\mathbf{H}_1(\boldsymbol{\xi}) \quad \mathbf{H}_2(\boldsymbol{\xi}) \quad \ldots \quad \mathbf{H}_N(\boldsymbol{\xi})\,] \tag{9.13}$$

with the diagonal submatrices

$$\mathbf{H}_k(\boldsymbol{\xi}) = N_k(\boldsymbol{\xi})\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{9.14}$$

where $N_k(\boldsymbol{\xi})$, (k = 1,2,…,N), indicate the element interpolation functions whose argument is the natural coordinates $\boldsymbol{\xi}$.

Similarly, the element displacement vector $\mathbf{u}^{(e)} = \begin{bmatrix} u_1^{(e)} & u_2^{(e)} & u_3^{(e)} \end{bmatrix}^T$ is given by the approximation

$$\mathbf{u}^{(e)} = \mathbf{H}(\boldsymbol{\xi})\tilde{\mathbf{u}}^{(e)} \tag{9.15}$$

where $\tilde{\mathbf{u}}^{(e)} = \begin{bmatrix} \tilde{\mathbf{u}}_1^{(e)T} & \tilde{\mathbf{u}}_2^{(e)T} & \ldots & \tilde{\mathbf{u}}_N^{(e)T} \end{bmatrix}^T$ represents the element nodal displacement vector and its components $\tilde{\mathbf{u}}_k^{(e)} = \begin{bmatrix} \tilde{u}_1^{(e)} & \tilde{u}_2^{(e)} & \tilde{u}_3^{(e)} \end{bmatrix}_k^T$ are the nodal displacement of the element k-node.

To simplify the finite element equations we use in this section Voigt notation. Then, for the three-dimensional case, the second Piola–Kirchhoff stress vector is given by

$$\hat{\mathbf{S}} = [\, S_{11} \quad S_{22} \quad S_{33} \quad S_{23} \quad S_{13} \quad S_{12} \,]^T \tag{9.16}$$

and the Green–Lagrange strain vector by

$$\hat{\mathbf{E}} = [\, E_{11} \quad E_{22} \quad E_{33} \quad 2E_{23} \quad 2E_{13} \quad 2E_{12} \,]^T \tag{9.17}$$

In Voigt notation, the shear components in (9.17) are doubled to allow writing the internal virtual work per volume unit as $\hat{\mathbf{S}}^T\delta\hat{\mathbf{E}}$. Then, the equilibrium equation (9.9) is expressed as

$$\begin{aligned} \int_{\Omega^0} (\Delta\hat{\mathbf{e}}_0)^T \hat{\mathbf{C}} \delta(\Delta\hat{\mathbf{e}}_0) d\Omega^0 + \int_{\Omega^0} \left(\hat{\mathbf{S}}_0^t\right)^T \delta(\Delta\hat{\boldsymbol{\eta}}_0) d\Omega^0 = \int_{\Omega^0} \left(\hat{\mathbf{b}}_0^{t+\Delta t}\right)^T \delta\hat{\mathbf{u}} d\Omega^0 + \int_{\Gamma^0} \left(\hat{\mathbf{t}}_0^{t+\Delta t}\right)^T \delta\hat{\mathbf{u}} d\Gamma^0 \\ + \int_{\Omega^0} \left(\Delta\hat{\mathbf{e}}_0^V\right)^T \hat{\mathbf{C}} \delta(\Delta\hat{\mathbf{e}}_0) d\Omega^0 + \int_{\Omega^0} \left(\Delta\hat{\mathbf{e}}_0^{HT}\right)^T \hat{\mathbf{C}} \delta(\Delta\hat{\mathbf{e}}_0) d\Omega^0 \\ - \int_{\Omega^0} \left(\hat{\mathbf{S}}_0^t\right)^T \delta(\Delta\hat{\mathbf{e}}_0) d\Omega^0 \end{aligned} \tag{9.18}$$

where $\hat{\mathbf{C}}$ is the elastic constitutive matrix and the strain increment vectors are defined by

$$\Delta\hat{\mathbf{e}}_0 = [\, \Delta e_{011} \quad \Delta e_{022} \quad \Delta e_{033} \quad 2\Delta e_{023} \quad 2\Delta e_{013} \quad 2\Delta e_{012} \,]^T \tag{9.19}$$

$$\Delta\hat{\boldsymbol{\eta}}_0 = [\, \Delta\eta_{011} \quad \Delta\eta_{022} \quad \Delta\eta_{033} \quad 2\Delta\eta_{023} \quad 2\Delta\eta_{013} \quad 2\Delta\eta_{012} \,]^T$$

with $\Delta\hat{\mathbf{E}}_0 = \Delta\hat{\mathbf{e}}_0 + \Delta\hat{\boldsymbol{\eta}}_0$. Similar definitions are employed for the viscoelastic and hygrothermal strain increment vectors $\Delta\hat{\mathbf{e}}_0^v$ and $\Delta\hat{\mathbf{e}}_0^{HT}$. Using the interpolation functions to express the displacements and increment displacements in (9.10), we obtain the variations

$$\delta(\Delta\hat{\mathbf{e}}_0) = \mathbf{B}_L \delta\left(\Delta\tilde{\mathbf{u}}^{(e)}\right) \qquad \delta(\Delta\hat{\boldsymbol{\eta}}_0) = \mathbf{B}_{NL} \delta\left(\Delta\tilde{\mathbf{u}}^{(e)}\right) \tag{9.20}$$

where $\mathbf{B}_L$ and $\mathbf{B}_{NL}$ are the linear and nonlinear strain–displacement matrices [1, 7]. $\delta\left(\Delta\tilde{\mathbf{u}}^{(e)}\right)$ is the variation in the nodal displacement increment vector of the element.

Introducing the strain–displacement relations into (9.18) and using the interpolation functions to express the displacements appearing in this equation, we obtain the following incremental equilibrium relationship for an element (3)

$$\left(\mathbf{k}_L^t + \mathbf{k}_{NL}^t\right)\Delta\tilde{\mathbf{u}}^{(e)} = \mathbf{r}^{t+\Delta t} - \mathbf{f}_0^t + \Delta\mathbf{f}^V + \Delta\mathbf{f}^{HT} \tag{9.21}$$

being $\Delta\tilde{\mathbf{u}}^{(e)}$and $\mathbf{r}^{t+\Delta t}$ the vector of nodal displacement increments and the vector of external nodal loading at time $t + \Delta t$ respectively, and

$$\mathbf{k}_L^t = \int_{\Omega^{0(e)}} \left(\mathbf{B}_L^t\right)^T \hat{\mathbf{C}}\mathbf{B}_L^t d\Omega^{0(e)} \quad \text{(linear stiffness matrix at time } t\text{)} \tag{9.22}$$

$$\mathbf{k}_{NL}^t = \int_{\Omega^{0(e)}} \left(\mathbf{B}_{NL}^t\right)^T \hat{\mathbf{S}}_0^t \mathbf{B}_{NL}^t d\Omega^{0(e)} \quad \text{(nonlinear stiffness matrix at time } t\text{)} \tag{9.23}$$

$$\mathbf{f}_0^t = \int_{\Omega^{0(e)}} \left(\mathbf{B}_L^t\right)^T \hat{\mathbf{S}}_0^t d\Omega^{0(e)} \\ \text{(vector of nodal forces equivalent to the element stresses at time } t\text{)} \tag{9.24}$$

$$\Delta\mathbf{f}^V = \int_{\Omega^{o(e)}} \left(\mathbf{B}_L^t\right)^T \hat{\mathbf{C}}\Delta\hat{\mathbf{e}}^V d\Omega^{o(e)} \quad \text{(viscoelastic load increment vector)} \tag{9.25}$$

$$\Delta\mathbf{f}^{HT} = \int_{\Omega^{o(e)}} \left(\mathbf{B}_L^t\right)^T \hat{\mathbf{C}}\Delta\hat{\mathbf{e}}^{HT} d\Omega^{o(e)} \quad \text{(hygrothermal load increment vector)} \tag{9.26}$$

In these last equations, the integrals are determined on the element domain in the initial configuration $\Omega^{0(e)}$. The matrices $\mathbf{B}_L^t$ and $\mathbf{B}_{NL}^t$ are the linear and nonlinear strain displacement matrices at time t, respectively. The present approach, for which the kinematic and static variables and integration domains are referred to the initial configuration, is known as total Lagrangian formulation. An alternative and equivalent approach consists of the updated Lagrangian formulation that, for each incremental step $t + \Delta t$, adopts C^t as reference configuration [1].

For the case of small displacements, the incremental equilibrium equation (9.21) becomes

$$\mathbf{k}_L^t\Delta\tilde{\mathbf{u}}^{(e)} = \mathbf{r}^{t+\Delta t} - \mathbf{r}^t + \Delta\mathbf{f}^V + \Delta\mathbf{f}^{HT} \tag{9.27}$$

9.4 Numerical Solution of the Equilibrium Equation

The numerical solution of the geometrically nonlinear problem (9.21) can be obtained using an iterative procedure in which the element equilibrium equation at time $t + \Delta t$ is given by

$$\left(\mathbf{k}_L^{t+\Delta t(i-1)} + \mathbf{k}_{NL}^{t+\Delta t(i-1)}\right)\Delta\tilde{\mathbf{u}}^{(e)(i)} = \mathbf{r}^{t+\Delta t(i)} - \mathbf{f}_0^{t+\Delta t(i-1)} + \Delta\mathbf{f}^{V(i)} + \Delta\mathbf{f}^{HT(i)} \quad (9.28)$$

where the superscripts i and i−1 indicate iterative steps. In this iterative approach, the element viscoelastic and hygrothermal load increment vectors, $\Delta\mathbf{f}^{V(i)}$ and $\Delta\mathbf{f}^{HT(i)}$, are taken as null for $i \geq 2$. For the first iteration $i = 1$, this last vector is computed by using (9.26), with

$$\Delta\hat{\mathbf{e}}^{HT(1)} = \boldsymbol{\alpha}\Delta\Theta^{(1)} + \boldsymbol{\beta}\Delta H^{(1)} \quad (9.29)$$

being $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ the vectors of the temperature expansion and hygroscopic expansion coefficients, respectively. $\Delta\Theta^{(1)}$ and $\Delta H^{(1)}$ are the temperature and moisture changes, respectively, for the first iterative step at time $t + \Delta t$. The element viscoelastic load increment vector $\Delta\mathbf{f}^{V(1)}$ is obtained for the first iteration at time $t + \Delta t$ using the viscoelastic strains computed by the equilibrated stresses corresponding to time t (see Chap. 3).

For an assemblage of finite elements, the global equilibrium equation can be written as

$$\left(\mathbf{K}_L^{t+\Delta t(i-1)} + \mathbf{K}_{NL}^{t+\Delta t(i-1)}\right)\Delta\tilde{U}^{(i)} = \mathbf{R}^{t+\Delta t(i)} - \mathbf{F}_0^{t+\Delta t(i-1)} + \Delta\mathbf{F}^{V(i)} + \Delta\mathbf{F}^{HT(i)} \quad (9.30)$$

where the variables have analogous meanings to those appearing in element equilibrium equation (9.28), but referred to the global coordinates. An alternative form of writing this global equilibrium equation is

$$\left(\mathbf{K}_L^{t+\Delta t(i-1)} + \mathbf{K}_{NL}^{t+\Delta t(i-1)}\right)\Delta\tilde{\mathbf{U}}^{(i)} = \Delta\lambda^{(i)}\bar{\mathbf{P}} + \mathbf{F}_d^{t+\Delta t(i-1)} + \Delta\mathbf{F}^{V(i)} + \Delta\mathbf{F}^{HT(i)} \quad (9.31)$$

where $\Delta\lambda^{(i)}$ is the loading factor corresponding to the iteration i at time $t + \Delta t$, $\bar{\mathbf{P}}$ is the reference load vector and $\mathbf{F}_d^{t+\Delta t(i-1)}$ is the unbalanced force vector at the iteration (i-1) of the step $t + \Delta t$. Using (9.30), the vector $\Delta\mathbf{F}^{HT(i)}$, for each time step, must be computed for the temperature and moisture increments $\Delta\lambda^{(1)}\bar{\Theta}$ and $\Delta\lambda^{(1)}\bar{H}$, being $\bar{\Theta}$and $\bar{H}$ reference temperature and moisture values, respectively.

As solution algorithm to solve (9.31) we may use, for instance, the well-known Newton–Raphson method [2]. In this method, the loading factor value $\Delta\lambda^{(i)}$ is adopted in the beginning of the first iteration ($i = 1$) of each incremental step and is null for i $\geq$ 2. One limitation of the Newton–Raphson method is the numerical instability that occurs near the limit points. To overcome this problem, we may use a displacement control algorithm, such as the Generalized Displacement Control Method [6]. The application of this method to viscoelastic problems can be found in Pavan et al. [5] and Oliveira and Creus [4].

9.5 Procedures of the Viscoelastic Finite Element Analysis

The implementation of the above geometrically nonlinear finite element formulation for the analysis of viscoelastic problems consists of the following main steps:

(1) Input the data for geometry, control parameters, mesh discretization, boundary conditions;
(2) Input the mechanical loads, temperature and moisture changes in each loading stage;
(3) Input the material properties corresponding to the temperature and moisture values;
(4) Assemble the strain–displacement matrices ($\mathbf{B}_L^{t(i)}$,$\mathbf{B}_{NL}^{t(i)}$) in the integration points of the elements;
(5) Assemble the element stiffness matrices ($\mathbf{k}_L^{t(i)}$,$\mathbf{k}_{NL}^{t(i)}$) and global stiffness matrices ($\mathbf{K}_L^{t(i)}$,$\mathbf{K}_{NL}^{t(i)}$);
(6) If there are temperature and moisture changes in the current loading stage and $i = 1$, assemble the element and global hygrothermal load increment vectors ($\Delta\mathbf{f}^{HT(1)}$,$\Delta\mathbf{F}^{HT(1)}$). For i ≥ 2, $\Delta\mathbf{f}^{HT(\mathrm{i})} = 0$ and $\Delta\mathbf{F}^{HT(\mathrm{i})} = 0$;
(7) If the external loading was already applied at the current loading stage, assemble the element and global viscoelastic load increment vectors corresponding to the time interval of the incremental step ($\Delta\mathbf{f}^{V(1)}$,$\Delta\mathbf{F}^{V(1)}$). The viscoelastic strains can be computed by the state variables approach, as seen in Chaps. 3 and 4. For $i \geq 2$, $\Delta\mathbf{f}^{V(\mathrm{i})} = 0$ and $\Delta\mathbf{F}^{V(\mathrm{i})} = 0$;
(8) Compute the nodal displacement increments $\Delta\tilde{\mathbf{U}}^{(i+1)}$;
(9) Update the nodal displacement $\tilde{\mathbf{U}}^{t(i+1)} = \tilde{\mathbf{U}}^{t(i)} + \Delta\tilde{\mathbf{U}}^{(i+1)}$and nodal coordinates;
(10) Assemble the strain–displacement matrices ($\mathbf{B}_L^{t(i+1)}$,$\mathbf{B}_{NL}^{t(i+1)}$) for the integration points of the elements in the updated configuration;
(11) Compute the stresses in the element integration points and vectors of nodal forces equivalent to these stresses (9.24), $\mathbf{F}_0^{t(i+1)}$, for the updated configuration;
(12) Determine the unbalanced force vector $\mathbf{F}_d^{t(i+1)}$;
(13) If the convergence criterion is not satisfied, then, do $i = i$+1 and return to step 4;
(14) If the convergence criterion is satisfied, two additional conditions must be checked: (a) if the time interval corresponding to the current loading stage is not complete, do $t + \Delta t$, $i = 1$ and go to step 4; (b) if the time interval is complete, then return to the new loading stage (step 2), if it exists, continuing the analysis.

Applications of these procedures to the analysis of viscoelastic laminated plates and shells may be found in Marques and Creus [3] and applications to viscoelastic thin-walled composite beams in Oliveira and Creus [4].

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