Chapter 10
The Boundary Element Method for Viscoelasticity Problems

The Boundary Element Method (BEM) is derived through the discretization of an integral equation (the classical Somigliana identity, first published in 1886). An interesting account of BEM early development may be found in [2]. This formulation can only be derived for certain classes of problems and hence, is not as widely applicable as the finite element method. However, when applicable, it often results in numerical methods that are easier to use and computationally more efficient. The advantages of the BEM arise from the fact that only the boundary of the domain requires sub-division. In cases where the domain is exterior to the boundary (e.g. the atmosphere surrounding an airplane, the soil surrounding a tunnel, the material surrounding a crack tip) the advantages of the BEM are even greater as the equation governing the infinite domain is reduced to an equation over the (finite) boundary. In this chapter we shortly review two alternative procedures for the solution of problems in linear viscoelasticity: the solution in the Laplace transformed domain and the use of a general inelastic formulation. For the latter, we make reference to the use of the Dual Reciprocity Method (DRM) that allows a pure boundary formulation.

10.1 Linear Elastic Problems and Somigliana Identity

We begin with a short summary of the classical boundary element formulation [1]. The boundary element method for linear elasticity may be established beginning with the Somigliana identity. Let us consider a body of volume Ω and surface Γ subjected to body forces b_k and surface forces p_i (following a tradition in the area, $\boldsymbol{p}$ in place of $\boldsymbol{t}$ will be used in this Chapter to denote tractions). Then, the Somigliana identity

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$$u_l^i + \int_\Gamma p_{lk}^* u_k d\Gamma = \int_\Gamma u_{lk}^* p_k d\Gamma + \int_\Omega u_{lk}^* b_k d\Omega \tag{10.1}$$

gives the value of the displacements at any internal points in terms of the boundary values of u_k and p_k, the domain forces b_k and the fundamental solutions u_{lk}^* and p_{lk}^*. p_{lk}^* are the tractions in the k direction due to a unit force at i acting in the l direction, and u_{lk}^* are the displacements in the k direction due to a unit force at i on the l direction. An updated derivation of the Somigliana identity may be found in [1], where (10.1) is obtained by reciprocity with a singular solution of the Navier equation for body force components modeled as unit point loads

$$Gu_{l,kk}^* + \frac{G}{1-2v} u_{k,kl}^* + \Delta^i e_l = 0 \tag{10.2}$$

where Δ^i represents the Dirac delta function at i. For a boundary point, (10.1) transforms to

$$c_{lk}^i u_l^i + \int_\Gamma p_{lk}^* u_k d\Gamma = \int_\Gamma u_{lk}^* p_k d\Gamma + \int_\Omega u_{lk}^* b_k d\Omega \tag{10.3}$$

where the integrals are in the sense of Cauchy principal value. For Γ smooth at point i it is $c_{lk}^i = \delta_{lk}/2$.

10.1.1 Boundary Element Formulation for the Linear Elastic Case

In order to obtain a numerical procedure, the boundary is discretized in elements, over which displacements and tractions are expressed in terms of their values at the nodal points. Using now matrix notation,

$$\mathbf{u} = \Phi \mathbf{u^j} \quad \mathbf{p} = \Phi \mathbf{p^j} \tag{10.4}$$

where $\mathbf{u}^j$ and $\mathbf{p}^j$ are the element nodal displacements and tractions and the interpolation functions $\mathbf{\Phi}$ are the standard finite element type functions. Then, writing (10.3) in matrix form we have

$$\mathbf{c^i u^i} + \int_{\mathbf{\Gamma}} \mathbf{p}^* \mathbf{u d \Gamma} = \int_{\mathbf{\Gamma}} \mathbf{u}^* \mathbf{p d \Gamma} + \int_{\mathbf{\Omega}} \mathbf{u}^* \mathbf{b d}\Omega \tag{10.5}$$

and using (10.4)

$$\mathbf{c}^i \mathbf{u}^i + \sum_{j=1}^{N} \left\{ \int_{\Gamma_j} \mathbf{p}^* \Phi \mathbf{d}\Gamma \right\} \mathbf{u}^j = \sum_{j=1}^{N} \left\{ \int_{\Gamma_j} \mathbf{u}^* \Phi \mathbf{d}\Gamma \right\} \mathbf{p^j} + \sum_{s=1}^{M} \left\{ \int_{\Omega_s} \mathbf{u}^* \mathbf{b} d\Omega \right\} \tag{10.6}$$

The sum from $j = 1$ to N indicate summation over all the N elements, Γ_j is the surface of element j and $\mathbf{u}^j$ and $\mathbf{p}^j$ the corresponding displacements and tractions. The domain was divided into M internal cells of volume Ω_s over which the body forces integral have to be computed. After integration we have for a given node i

$$\mathbf{c}^{\mathbf{i}}\mathbf{u}^{\mathbf{i}}+\sum_{j=1}^{NE}\mathbf{H}^{ij}\mathbf{u}^{j}=\sum_{j=1}^{NE}\mathbf{G}^{ij}\mathbf{p}^{j}+\sum_{s=1}^{M}\mathbf{B}^{is} \tag{10.7}$$

The contribution for all the NE nodes may be written in matrix form

$$\mathbf{HU}=\mathbf{GP}+\mathbf{B} \tag{10.8}$$

After the boundary conditions are introduced, all unknowns are set into a vector $\mathbf{X}$ leading to a system of equations

$$\mathbf{AX}=\mathbf{F} \tag{10.9}$$

10.2 Viscoelastic Solutions in the Laplace Transform Domain

If the correspondence principle (see Chap. 5) is applied to the quasi-static problem, the relevant boundary integral equation in the Laplace transformed domain is written

$$c^i_{lk}\bar{u}^i_l(s)+\int_\Gamma p^*_{lk}(s)\bar{u}_k(s)d\Gamma=\int_\Gamma u^*_{lk}(s)\bar{p}_k(s)d\Gamma+\int_\Omega u^*_{lk}(s)\bar{b}_k(s)d\Omega \tag{10.10}$$

where now $u^*_{lk}(s)$ and $p^*_{lk}(s)$ are the elastic fundamental solutions for displacements and tractions in which the elastic constants have been replaced by the corresponding functions in the transformed space according to Sect. 5.2. A discussion of this type of approach may be found in Syngellakis [9], Gaul and Schanz [3]. The main difficulty is the inversion from the Laplace to the real (time) domain.

10.3 Formulation Considering Inelastic Strains

The general boundary integral equation including the effect of inelastic strains may be written in incremental form as [1]

$$c^i_{lk}\dot{u}^i_l=\int_\Gamma u^*_{lk}(\dot{p}_k+\dot{p}^v_k)d\Gamma-\int_\Gamma p^*_{lk}u_k d\Gamma+\int_\Omega u^*_{lk}(\dot{b}_k+\dot{b}^v_k)d\Omega \tag{10.11}$$

where

$$\dot{p}^v_i = \dot{\sigma}^v_{ij} n_j; \quad \dot{b}^v_i = -\dot{\sigma}^v_{ij,j}; \quad \dot{\sigma}^v_{ij} = E_{ijkl}\dot{\varepsilon}^v_{kl} \tag{10.12}$$

and ε^v_{ij} is the deferred part of strain as defined in Chap. 2. Equation (10.11) is known as the *pseudo-surface traction, pseudo-body force approach*; the inelastic forces are included adding $\dot{p}^v_k$ to $\dot{p}_k$ in the surface traction boundary integral and $\dot{b}^v_k$ to $\dot{b}_k$ in the body force domain integral. This formulation has been applied to time-dependent problems by a series of authors; see for example Brebbia et al. [1]. The domain integral has to be computed using cells defined over the domain. There are alternatives that avoid the domain integration, one of which is the Dual Reciprocity Formulation (DRM) [6].

10.3.1 DRM Applied to Viscoelasticity

With reference to (10.11), we define $\dot{w}^v$ so that $\dot{w}^v_{,i} = \dot{b}^v_j$. Using the DRM strategy, we expand $\dot{w}^v$ as the sum of known approximating functions with initially unknown coefficients

$$\dot{w}^v \simeq \sum_{j=1}^{M} f^m \dot{\alpha}^m \tag{10.13}$$

where M is the number of DRM collocation points. Differentiating (10.13) we obtain

$$\dot{w}^v_{,j} = \dot{b}^v_j \simeq \sum_{j=1}^{M} f^m_{,j} \dot{\alpha}^m \tag{10.14}$$

Considering (10.11) and making the regular body forces $b_j = 0$, we can now substitute $\dot{b}^v_j$ given by (10.14) obtaining

$$c^i_{lk}\dot{u}^i_l = \int_\Gamma u^*_{lk}(\dot{p}_k + \dot{p}^v_k)d\Gamma - \int_\Gamma p^*_{lk}\dot{u}_k d\Gamma + \sum_{m=1}^{M}\left(\int_\Omega u^*_{lk} f^m_{,k}\right)\dot{\alpha}^m \tag{10.15}$$

The DRM particular solutions $\hat{u}_j$ should satisfy the Navier equation

$$G\hat{u}^j_{k,ll} + \frac{G}{1-2\nu}\hat{u}^j_{l,lk} = f^j_{,k} \tag{10.16}$$

Taking the domain term to the boundary with DRM we obtain

$$c_{lk}^i \dot{u}_l^i = \int_\Gamma u_{lk}^*(\dot{p}_k + \dot{p}_k^v)d\Gamma - \int_\Gamma p_{lk}^* \dot{u}_k d\Gamma + \sum_{m=1}^{M}\left(c_{lk}\hat{u}_l^m - \int_\Gamma u_{lk}^* \hat{p}_l^m d\Gamma + \int_\Gamma p_{lk}^* \hat{u}_l^m d\Gamma\right)\dot{\alpha}^m \tag{10.17}$$

After discretization and approximation of the above equation to all boundary nodes, the following system of equations is obtained

$$\mathbf{H}\dot{\mathbf{U}} - \mathbf{G}\dot{\mathbf{P}} = (\mathbf{H}\hat{\mathbf{U}} - \mathbf{G}\hat{\mathbf{P}})\dot{\boldsymbol{\alpha}} \tag{10.18}$$

or, substituting from (10.13) $\dot{\boldsymbol{\alpha}} = \boldsymbol{F}^{-1}\dot{\mathbf{w}}$

$$\mathbf{H}\dot{\mathbf{U}} - \mathbf{G}\dot{\mathbf{P}} = (\mathbf{H}\hat{\mathbf{U}} - \mathbf{G}\hat{\mathbf{P}})\mathbf{F}^{-1}\dot{\mathbf{w}}^v \tag{10.19}$$

or

$$\mathbf{H}\dot{\mathbf{U}} - \mathbf{G}\dot{\mathbf{P}} = \dot{\mathbf{D}} \tag{10.20}$$

Applying the usual BEM procedure we set the system of equations in the form

$$\mathbf{A}\dot{\mathbf{X}} = \dot{\mathbf{Y}} + \dot{\mathbf{D}} \tag{10.21}$$

From its solution we obtain $\dot{u}_i, \dot{p}_i$, and we can determine the boundary and internal stress tensors and advance in time. Additional and numerical examples may be found in [8].

10.4 Other Procedures

Other different and complementary procedures may be seen in Liu and Antes [4]; Mesquita and Coda [5]; Schanz and Antes [7].

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