# Multi-class Network with Phase Type Service Time and Group Deletion Signal

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**Abstract.** We consider networks with multiple classes of customers which receive service with a Phase type distribution. The service discipline is Last In First Out. We consider negative signal and a new type of signal: the group deletion signal. Negative signals eliminate a customer in service (if there are any) and group deletion signal delete all consecutive customers in the same class and same phase at the back-end of the buffer. We prove that the network has product form solution.

**Keywords:** Queue, Network, phase type, quasi-reversible, LIFO, product form.

## 1 Introduction

Traditional queueing networks model systems are used to represent contention among customers for a set of resources. Customers moves form server to server, waiting for service. But the customers do not interact among themselves or modify the queue or the server. G-network models overcome some of the limitations of conventional queueing network models adding signals and interactions between signals and customers. Despite this deep modification of the model, G-networks still preserve the product form property of some Markovian queueing networks. In his seminal paper [10], Gelenbe introduced negative customer, the first type of signal. A negative customer is never queued. A negative customer deletes a positive customer at its arrival at a backlogged queue. Positive customers are usual customers which are queued and receive service or are deleted by negative customers. Under typical assumptions for Markovian queueing networks (Poisson arrival for both types of customers, exponential service time for positive customers, Markovian routing of customers, open topology, independence) Gelenbe proved that such a network has a product form solution for its steady-state behavior. The results are more complex than Jackson's networks. The G-networks flow equations exhibit some uncommon properties: they are neither linear as in closed queueing networks nor contracting as in Jackson queueing networks. Therefore the existence of a solution had to be proved [11] by new techniques, a numerical algorithm was also developed [8].

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G-networks had been extended in many directions. First many signals were introduced and shown to lead to product form solution: triggers which redirect other customers among the queues, catastrophes which flush all the customers out of a queue [12,13] and resets [14]. Multiple class versions of these models have also been derived [6,9,15].

Most of the signals studied so far have a globally negative effect on the queues. Indeed, the balance of customers in the queues involved in a signal is negative (triggers, deletion, catastrophes) or negative in expectation (resets). Recently more complex interactions were introduced: change the class of the customer in service [4], change the phase of the customer in service for Phase type service distribution, synchronised arrivals in a set of queues [5]. For a review, one can see these books [3,16], and many references therein.

G-networks had also motivated new important results in the theory of queues. As negative customers lead to customer deletions, the original description of quasi-reversibility does not hold anymore and new versions have been proposed. At the time being, the description proposed by Chao and his co-authors in [3] looks sufficient to study queues with customers and signals. At the same time a completely different approach, based on Stochastic Process Algebra, was proposed by Harrison [17,18]. The main results (CAT and RCAT theorems and their extensions [1,17,18,19]) give some sufficient conditions for product form stationary distributions. Thus Harrison's technique clearly has a different range of applications as it allows to represent component based models which are much more general than networks of queues. An interesting open question is to mix both results to obtain a quasi-reversibility characterisation directly from a SPA specification using a master-slave description of the system (like in RCAT) rather than arrivals, departure and internal transitions as proposed in [3].

Here we introduce a new type of signal which deletes several customers of the same type (same class and same phase) at the back end of a LIFO queue. Batch deletion were studied by Gelenbe in [12] for single class model. To the best of our knowledge the multiclass problem was not considered until now, except the catastrophe in a PS queue studied in [7] which is easier to model. Moreover, the group-deletion signal is not like the previously studied batch which was studied as we seek to delete all customers of the same type, which means that it will depend on the type of customers, while the effect of a batch or a catastrophe does not depend on class of customers. We also assume that the service time distributions are Phase type.

The following of the paper is as follows. Section 2 is devoted to the definition of quasi-reversibility as it has been generalised by Miyazawa and his co-authors to take into account signals. In section 3, we show that the queues are quasi-reversible. Finally in section 4, using quasi-reversibility we prove that the steady-state has a product form solution.

## 2 Preliminaries

In this section, we will introduce the network of quasi-reversible queues which is introduced by Chao, Miyazawa and Pinedo in [3]. All the results presented in this section comes form [3] and have been already used in the context of G-networks in [4] to prove that networks with Phase type services and another type of signal also have a product form steady-state solution. We summarize the results of [3] for the readers.

### 2.1 Definition of Quasi-reversibility of Chao, Miyazawa and Pinedo

Let we introduce the definition of quasi-reversibility of a queue with signals and instantaneous movement.

Consider a queue where the queue-content evolves as a continuous time Markov chain on state space $\mathcal{S}$. For a pair of states $(\boldsymbol{x}, \boldsymbol{y})$, we decompose the transition rate function $q(\boldsymbol{x}, \boldsymbol{y})$ of queue into three types of rates: $q_u^A(\boldsymbol{x}, \boldsymbol{y}), u \in T$; $q_u^D(\boldsymbol{x}, \boldsymbol{y}), u \in T$; $q^I(\boldsymbol{x}, \boldsymbol{y})$, where $T$ is the set of the classes of arrivals and departures, which is countable. The transition rate of the queue can be written as:

$$q(\boldsymbol{x}, \boldsymbol{y}) = \sum_{u \in T} q_u^A(\boldsymbol{x}, \boldsymbol{y}) + \sum_{u \in T} q_u^D(\boldsymbol{x}, \boldsymbol{y}) + q^I(\boldsymbol{x}, \boldsymbol{y}), \quad \boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}.$$

The transition rate functions $q_u^A$, $q_u^D$ and $q^I$ generate the point processes corresponding to class $u$ arrivals, class $u$ departures and the internal transitions, respectively. "A", "D" and "I" stand for "arrival", "departure" and "internal".

Suppose that $q$ admits a stationary distribution $\pi$. Furthermore, assume that when a class $u$ arrives and changes the state of the queue from $\boldsymbol{x}$ to $\boldsymbol{y}$, it instantaneously triggers a class $v$ departure with probability $f_{u,v}(\boldsymbol{x}, \boldsymbol{y})$, where:

$$\sum_v f_{u,v}(\boldsymbol{x}, \boldsymbol{y}) \leq 1, \ u \in T, \ \boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}.$$

With probability $1 - \sum_v f_{u,v}(\boldsymbol{x}, \boldsymbol{y})$ the class $u$ arrival does not trigger any departure. The function $f_{u,v}(\boldsymbol{x}, \boldsymbol{y})$ is the *triggering probability.*

The quasi-reversibility of instantaneous movement is defined as follows.

**Definition 2.1.** *If there exist two sets of non-negative numbers* $\{\alpha_u, u \in T\}$ *and* $\{\beta_u, u \in T\}$ *such that: for all* $\boldsymbol{x} \in \mathcal{S}$, $u \in T$,

$$\sum_{\boldsymbol{y} \in \mathcal{S}} q_u^A(\boldsymbol{x}, \boldsymbol{y}) = \alpha_u, \tag{1}$$

$$\sum_{\boldsymbol{y} \in \mathcal{S}} \pi(\boldsymbol{y}) \Big[ q_u^D(\boldsymbol{y}, \boldsymbol{x}) + \sum_{v \in T} q_v^A(\boldsymbol{y}, \boldsymbol{x}) f_{v,u}(\boldsymbol{y}, \boldsymbol{x}) \Big] = \beta_u \pi(\boldsymbol{x}), \tag{2}$$

*then the queue with signal is said to be* quasi-reversible with respect to $\{q_u^A, f_{u,v}, u, v \in T\}$, $\{q_u^D, u \in T\}$ and $\{q^I\}$ .

The non-negative numbers $\alpha_u$ and $\beta_u$ are called the arrival rate and departure rate of class $u$ customers.

*Example 2.1.* Let us give an example of a G-queue $M/M/1/\infty$ with negative signal to introduce the definition. Customers arrive with rate $\lambda$, service rate is $\mu$. Negative signals arrive with rate $\lambda^-$. A negative signal will eliminate a customer if there is any in the queue.

We use the indices $c$ for customers and $-$ for negative signals. The transition rates of the queue are given by:

$$\begin{aligned} q_c^A(n, n+1) &= \lambda, \quad n \geq 0, \\ q_-^A(n, n-1) &= \lambda^-, \quad n \geq 1, \\ q_-^A(0, 0) &= \lambda^-, \\ q_c^D n, n-1 &= \mu, \quad n \geq 1. \end{aligned}$$

We add the rate $q_-^A(0,0)$ for equation (1) to hold for both $c$ and $-$, where $\alpha_c = \lambda$ and $\alpha_- = \lambda^-$. Note that $q_-^A(0,0)$ is a dummy transition. Therefore it is possible to add such a transition rate.

The stationary distribution $\pi$ is given by

$$\pi(n) = \pi(0)\big(\frac{\lambda}{\mu + \lambda^-}\big)^n.$$

Equation (2) is satisfied for $c$ with

$$\beta_c = \frac{\lambda\mu}{\mu + \lambda^-}.$$

We now consider the negative signals. We add the triggering probability $f_{-,-}(n+1, n) = 1$ for $n \geq 0$ in the queue. Hence, one obtains equation (2) for $-$ with

$$\beta_- = \frac{\lambda\lambda^-}{\mu + \lambda^-}.$$

Chao et al. proved that this definition of queue without instantaneous movements is equivalent to the quasi-reversible definition of Kelly in [20]. This implies that the arrival processes and the departure (triggered and non-triggered) of class $u$ customers are Poisson.

We use the definition of Chao, Miyazawa and Pinedo as it is more convenient for G-networks with instantaneous movements.

### 2.2 Network of Quasi-reversible Queues with Signals and Instantaneous Movement

Consider a network of $N$ queues. Each queue is a quasi-reversible queue with signals as described above. The set of arrival and departure classes is $T$ (we may have a set $T_i$ for each queue $i$, however, for the sake of simplicity, we take $T = \cup_i T_i$).

Let $\boldsymbol{x}_i$ be the state of queue $i$ with state space $\mathcal{S}_i$. The Poisson source has index 0 and for the sake of simplification, we assume that the source has only one state which is denoted as 0.

For queue $i$, we introduce functions $p_{iu}^A$, $q_{iu}^D$, $q_i^I$ and $f_{iu,v}$ on the state space $\mathcal{S}_i$:

- $p_{iu}^A(\boldsymbol{x}_i, \boldsymbol{y}_i)$ = the probability that a class $u$ arrival at queue $i$ changes the state from $\boldsymbol{x}_i$ to $\boldsymbol{y}_i$, where it is assumed that $\sum_{\boldsymbol{y}\in\mathcal{S}_i} p_{iu}^A(\boldsymbol{x}_i, \boldsymbol{y}_i) = 1$, $\boldsymbol{x}_i \in \mathcal{S}_i$;
- $q_{iu}^D(\boldsymbol{x}_i, \boldsymbol{y}_i)$ = the rate at which class $u$ departures change the state of queue $i$ from $\boldsymbol{x}_i$ to $\boldsymbol{y}_i$;
- $q_i^I(\boldsymbol{x}_i, \boldsymbol{y}_i)$ = the rate at which internal transitions change the state of queue $i$ from $\boldsymbol{x}_i$ to $\boldsymbol{y}_i$;
- $f_{iu,v}(\boldsymbol{x}_i, \boldsymbol{y}_i)$ = the triggering probability that when a class $u$ arrival occurs at queue $i$ and the state changes from $\boldsymbol{x}_i$ to $\boldsymbol{y}_i$, it simultaneously induces a class $v$ departure, where $\sum_{v\in T} f_{iu,v}(\boldsymbol{x}_i, \boldsymbol{y}_i) \leq 1,\ i \leq N,\ u \in T,\ \boldsymbol{x}_i, \boldsymbol{y}_i \in \mathcal{S}_i$.

For source 0, we set $p_{0u}^A(0,0) = 1$, $p_{0u}^A(0,0) = \beta_{0u}$, $q_0^I(0,0) = 0$ and $f_{0u,v} \equiv 0$. Here, $\beta_{0u}^A$ is the arrival rate to the network from the outside (the source).

In Chao's model, a queue is defined by three rates $q_u^A$, $q_u^D$ and $q^I$. In that case, the arrival effect function may be defined as:

$$p_u^A(\boldsymbol{x}, \boldsymbol{y}) = \frac{q_u^A(\boldsymbol{x}, \boldsymbol{y})}{\sum_z q_u^A(\boldsymbol{x}, \boldsymbol{z})} ,$$

and $q_u^D$ and $q^I$ are the departure and internal transition functions.

The dynamics of the network are described as follows. Customers of class $u$ arrive to the network from outside (the source) according to a Poisson process with rate $\beta_{0u}$, and are routed to queue $i$ as a class $v$ arrival with probability $r_{0u,iv}$. A class $u$ departure from queue $i$, either trigger or non-trigger, enters queues $j$ as a class $v$ arrival with probability $r_{iu,jv}$. It is assumed that:

$$\sum_{j=0}^{N} \sum_{v} r_{iu,jv} = 1,\ i = 0, 1, \ldots N,\ u \in T.$$

Furthermore, whenever there is a class $u$ arrival at queue $i$, either from the outside or from other queues, it makes the state of the queue change from $\boldsymbol{x}_i$ to $\boldsymbol{y}_i$ with probability $p_{iu}^A(\boldsymbol{x}_i, \boldsymbol{y}_i)$, it also triggers a class $u$ departure with probability $f_{iu,v}(\boldsymbol{x}_i, \boldsymbol{y}_i)$, and it triggers no departure from queue $i$ with probability $1 - \sum_{v\in T} f_{iu,v}(\boldsymbol{x}_i, \boldsymbol{y}_i)$, $i = 0, 1, \ldots, N$.

The transition rate function of the network is denoted by $q(\boldsymbol{x}, \boldsymbol{y})$, $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_N$ (note that we accept the case where $q(\boldsymbol{x}, \boldsymbol{x}) \neq 0$).

Consider for each queue $i$ the following auxiliary process:

$$q_i^{(\boldsymbol{\alpha}_i)}(\boldsymbol{x}_i, \boldsymbol{y}_i) = \sum_{u\in T} \Big( \alpha_{iu} p_{iu}^A(\boldsymbol{x}_i, \boldsymbol{y}_i) + q_{iu}^D(\boldsymbol{x}_i, \boldsymbol{y}_i) \Big) + q_i^D(\boldsymbol{x}_i, \boldsymbol{y}_i),$$

where $(\boldsymbol{\alpha}_i) = (\alpha_{iu}, u \in T)$ are considered as dummy parameters and their values are determined by the traffic equations.

Suppose that $q_i^{(\boldsymbol{\alpha}_i)}$ has a stationary distribution $\pi_i^{(\boldsymbol{\alpha}_i)}$. Note that this is always true for the source 0, as for all $\boldsymbol{\alpha}_0$, $\pi_0^{(\boldsymbol{\alpha}_0)(0)} = 1$. We now require that $q_i^{(\boldsymbol{\alpha}_i)}$ be quasi-reversible.

We always have

$$\sum_{\boldsymbol{y}_i \in \mathcal{S}_i} \alpha_{iu} p_{iu}^A(\boldsymbol{x}_i, \boldsymbol{y}_i) = \alpha_{iu}, \qquad i = 1, \ldots N,\ u \in T.$$

Hence, the quasi-reversibility of $q_i^{(\boldsymbol{\alpha}_i)}$ for $i = 1, \ldots, N$ is equivalent to the existence of a set of non-negative numbers $\beta_{iu}, u \in T$ such that:

$$\sum_{\boldsymbol{y}_i} \pi_i^{(\boldsymbol{\alpha}_i)}(\boldsymbol{y}_i)\Big[q_{iu}^D(\boldsymbol{y}_i, \boldsymbol{x}_i) + \sum_{v \in T} \alpha_{iv} p_{iv}^A(\boldsymbol{y}_i, \boldsymbol{x}_i) f_{iv,u}(\boldsymbol{y}_i, \boldsymbol{x}_i)\Big] = \beta_{iu} \pi_i^{(\boldsymbol{\alpha}_i)}(\boldsymbol{x}_i), \qquad (3)$$

for all $\boldsymbol{x}_i \in \mathcal{S}_i$, $i = 1, \ldots, N$ and $u \in T$.

Queue $i$ in isolation is said to be quasi-reversible with $\boldsymbol{\alpha}_i$ if (3) is satisfied.

Since $\alpha_{iu}$ and $\beta_{iu}$ are the arrival and the departure rates of class $u$ customers at queue $i$, we have the following traffic equations:

$$\alpha_{iu} = \sum_{j=0}^{N} \sum_{v} \beta_{jv} r_{jv,iu}\ , \qquad i = 0, 1, \ldots, N. \qquad (4)$$

We need the following condition to ensure that the network process is regular:

$$\sum_{i=1}^{N} \sum_{x_i \in \mathcal{S}_i} \pi_i^{(\boldsymbol{\alpha}_i)} \sum_{\boldsymbol{y}_i \in \mathcal{S}_i} q_i^{(\boldsymbol{\alpha}_i)}(\boldsymbol{x}_i, \boldsymbol{y}_i) < \infty.$$

We have the theorem:

**Theorem 2.1.** *If each queue $i$ with signals, $i = 1, \ldots, N$, is quasi-reversible with $\boldsymbol{\alpha}_i$ which are the solution to the traffic equations (4), then the queueing network with signal has the product form stationary distribution*

$$\pi(\boldsymbol{x}) = \prod_{i=1}^{N} \pi_i^{(\boldsymbol{\alpha}_i)}(\boldsymbol{x}_i),$$

*where $\pi_i^{(\boldsymbol{\alpha}_i)}$ is the stationary distribution of $q_i^{(\boldsymbol{\alpha}_i)}$, $i = 1, \ldots, N$.*

## 3 A LIFO Multi-class PH Queue with Signal Deletion Customers of Same Sub-class

The goal is to model a generalized network of multiple classes of (positive) customers where the service times of each class are assumed to be Phase type and 2 types of signal: negative signal and group deletion signal.

In [2], Bonald and Tran modelled the Phase type service by considering a change of class inside the queue after service. More precisely, the phase is modelled by an absorbing DTMC with $k + 1$ states where 0 is the only absorbing state. The transition probability matrix of this DTMC is denoted as $H$. It can be viewed as follow: each "phase" $ph$ demands an exponential service time. After "phase" $ph$, a customer can change to some "phase" $ph'$ with some probability (given by matrix $H$) and still stay in the queue. The arriving rate of each "phase" will respect the ratio of the initial probability of the Phase-type.

*Example 3.1.* Consider a queue of 2 classes of customers:

- Class 1 which arrives with rate $\lambda_1$ and asks for exponential service of rate $\mu_1$;
- Class 2 which arrives with rate $\lambda_2$ and asks a PH service with two transient states (and an absorbing state): $ph$,$ph'$ and initial distribution $\nu$. State $ph$ and $ph'$ demands exponential services time $\mu_2$ and $\mu_2'$.

Hence, the matrix $H$ is given by

$$H = \left(\begin{array}{c|cc|c} 0 & 0 & 0 & \mathrm{H[1,0]} \\ \hline 0 & 0 & H[ph,ph'] & H[ph,0] \\ 0 & H[ph',ph] & 0 & H[ph',0] \\ \hline 0 & 0 & 0 & 0 \end{array}\right),$$

where

- $H[1,0] = 1$, which means that if a customer reach phase 1, then after service completion, it will reach the absorbing state.
- $H[ph,ph']$ and $H[ph',ph]$ are the probabilities in which after service completion of phase $ph$ and $ph'$, respectively, customer of class 2 can ask for another "phase" $ph'$, $ph$, respectively.
- $H[ph,0]$ and $H[ph',0]$ are the probabilities in which after of phase $ph$ and $ph'$, respectively, customer of class 2 will reach the absorbing state.

Then, inside the queue, there are changes of class after service (in this case, $ph$ to $ph'$ and $ph'$ to $ph$). The arrival rate for phase 1 is $\lambda_1$, while it is $\lambda_2\nu(ph)$ and $\lambda_2\nu(ph')$ for phase 2 ($ph$) and 3 ($ph'$).

We will use this presentation to model our network. A multi-class network of queues with phase type service times can be modelled as a multi-class network of queues with exponential service times where the customers can change class inside the queue after completion of an exponential service.

### 3.1 Description

Consider the LIFO multi-class queue with the set of customers given by $\mathcal{C}$ and a special class index 0 ($0 \notin \mathcal{C}$) which denotes the "absorbing state". There are two types of signals: negative signals and group-deletion signals. The set of negative signals is given by $\mathcal{S}^- = \{s_c^-\}_{c\in\mathcal{C}}$ and the set of group-deletion signals is given by $\mathcal{S} = \{s_c\}_{s\in\mathcal{C}}$.

*Remark 3.1.* We can consider only one class of negative signal and one class of group-deletion signal, with different probability of success for each class of customer. However, we consider different classes to have more flexible routing while connecting the networks.

Customers of class $c$ arrive according to a Poisson process of rate $\lambda^{(c)}$, they require exponential service times with mean $1/\mu^{(c)}$, for $c \in \mathcal{C}$. Denote by $\lambda$ the sum of all $\lambda^c$: $\lambda = \sum_{c \in \mathcal{C}} \lambda(c)$. A customer of class $c$ after completion of service will change to class $k$ (it demands another service) with probability $H[c, k]$ or reach the state 0 (it quits the queue at the completion of its service) with probability $H[c, 0]$. The following condition is satisfied:

$$H[c, 0] + \sum_{k \in \mathcal{C}} H[c, k] = 1.$$

Negative signals of class $s_c^-$ arrive according to a Poisson process of rate $\lambda^-(c)$. Arriving signal of type $s_c^-$ will eliminate a customer $c$ in service (customer at the back-end of the buffer).

Group deletion signals of class $s_c$ arrive according to a Poisson process of rate $\lambda^s(c)$. If customer in service is of class $c$, then the arriving signal of class $s_c$ will cancel all consecutive customers of class $c$ at the back-end of the buffer (it will eliminate all customers of class $c$ until it finds a customer of another class).

If the queue length is $n$, then the state of queue is

$$\boldsymbol{x} = (x(1), x(2), \cdots, x(n)),$$

where $x(l)$ is the class of customer in position $l$.

*Remark 3.2.* We consider also triggering effect in the queue to have instantaneous movements when considering network model. However, details of the triggering effect will be given later when we study the quasi-reversibility.

### 3.2 Stationary Distribution

We first give the system of equation which plays an important role in calculating the stationary distribution.

**Definition 3.1.** *The* PH Group-deletion Equations *associated to the considered LIFO queue are the equations of the variable* $\boldsymbol{\rho} = \{\rho(c)\}_{c \in \mathcal{C}} \in \mathbb{R}_+^{\mathcal{C}}$*, defined by*

$$\lambda(c) + \sum_{k \in \mathcal{C}} \rho(k)\mu(k)H[k, c] = \frac{\rho(c)\lambda^s(c)}{1 - \rho(c)} + \rho(c)\mu(c) + \rho(c)\lambda^-(c), \quad \text{if } \lambda^s(c) > 0, \quad (5)$$

$$\lambda(c) + \sum_{k \in \mathcal{C}} \rho(k)\mu(k)H[k, c] = \rho(c)\big(\mu(c) + \lambda^-(c)\big), \quad \text{if } \lambda^s(c) = 0. \quad (6)$$

Let $\mathcal{C}^+$ be a subset of $\mathcal{C}$: $\mathcal{C}^+ = \{c \in \mathcal{C} \mid \lambda^s(c) > 0\}$.

*Remark 3.3.* If $\boldsymbol{\rho}$ is a solution of (5,6), then taking the sum over all $c$ in $\mathcal{C}$, and using the fact that $H[c,0] + \sum_{k \in \mathcal{C}} H[c,k] = 1$, one has

$$\begin{aligned}\lambda &= \sum_{c \in \mathcal{C}} (\rho(c)\mu(c) + \rho(c)\lambda^-(c)) + \sum_{c \in \mathcal{C}^+} \frac{\rho(c)\lambda^s(c)}{1-\rho(c)} - \sum_{c,k \in \mathcal{C}} \rho(k)\mu(k)H[k,c] \\ &= \sum_{c \in \mathcal{C}} \rho(c)\mu(c)H[c,0] + \sum_{c \in \mathcal{C}} \rho(c)\lambda^-(c)) + \sum_{c \in \mathcal{C}^+} \frac{\rho(c)\lambda^s(c)}{1-\rho(c)}.\end{aligned}$$

We now give lemmas which prove the existence of the solution of *PH Group-deletion Equations*. We first prove this property in the extreme cases: $\mathcal{C}^+ = \mathcal{C}$ or $\mathcal{C}^+ = \emptyset$. Then we treat the general case.

**Lemma 3.1.** *If $\mathcal{C}^+ = \emptyset$ (which means that there is no deletion-group signal in the queue), then there exists a solution $(\rho(c))_{c \in \mathcal{C}}$ of the system (5,6).*

*Proof.* The system can be rewritten as follows:

$$\boldsymbol{\rho} = \boldsymbol{\eta} + \boldsymbol{\rho} M,$$

where $\boldsymbol{\eta} = (\eta(c))_{c \in \mathcal{C}}$ is a vector and $M = M[c,k]_{c,k \in \mathcal{C}}$ is a matrix the entries of which are given by:

$$\eta(c) = \frac{\lambda(c)}{\mu(c) + \lambda^-(c)}\ , \quad M[c,k] = \frac{\mu(k)H[k,c]}{\mu(c) + \lambda^-(c)}\ .$$

One has that $rank(Id - M) = |\mathcal{C}|$ as at least one variable $H[c,0] > 0$. Hence, there exist a unique solution given by

$$\boldsymbol{\rho} = \boldsymbol{\eta}(Id - M)^{-1}.$$

This completes the proof.

**Lemma 3.2.** *If $\mathcal{C}^+ = \mathcal{C}$ (which means that $H[c,k] = 0$ for all $c, k \in \mathcal{C}$), then there exists a solution $(\rho(c))_{c \in \mathcal{C}}$ of the system (5,6) which satisfies*

$$0 \leq \rho(c) < 1.$$

*Proof.* $\rho(c)$ is a root of a polynomial of degree 2:

$$\begin{aligned}P^c(\rho(c)) = \rho(c)^2\big(\mu(c) + \lambda^-(c) - \mu(c)H[c,c]\big) - \rho(c)\big(\lambda^s(c) + \mu(c) + \lambda^-(c) \\ + \sum_{k \neq c} \rho(k)\mu(k)H[k,c] - \mu(c)H[c,c]\big) + \big(\lambda[c] + \sum_{k \neq c} \rho(k)\mu(k)H[k,c]\big)\end{aligned}$$

Polynomial $P^c(X) = 0$ has a solution in [0,1) as $P^c(0) \geq 0$ and $P^c(1) < 0$. As the multiplication of the two solutions of $P^c(X)$ is positive $(\lambda(c)/\big(\mu(c) + \lambda^-(c)\big))$, we have that $P^c(X) = 0$ has 2 positive roots. It implies that $P^c(X)$ has a unique solution in [0,1) (which is depending on $\rho(k)_{k \neq c}$).

Consider the function $\Phi(\rho) : [0,1]^{\mathcal{C}^+} \to [0,1]^{\mathcal{C}^+}$ where $\Phi(\rho)(c)$ is the unique solution in [0,1) of $P^c(X)$. As $[0,1]^{\mathcal{C}^+}$ is a non-empty compact, then applying Brower's fixed point theorem, one has that there exists a fixed point solution in $[0,1]^{\mathcal{C}^+}$ of the equation: $\Phi(\rho) = \rho$, or $\Phi(\rho)(c) = \rho(c)$. Clearly the solution will be in the set $[0,1)^{\mathcal{C}^+}$.

This completes the proof.

We now have the result in the general case.

**Lemma 3.3.** *There exists a solution $(\rho(c))_{c\in\mathcal{C}}$ of the system (5,6) which satisfies*

$$0 \le \rho(c) < 1, \quad \textit{for all } c \in \mathcal{C}^+.$$

*Proof.* Consider $c \in \mathcal{C} \setminus \mathcal{C}^+$. Similarly to Lemma 3.1, one has that there exists a solution $\boldsymbol{\rho}_1 = (\rho(c))_{c\in\mathcal{C}\setminus\mathcal{C}^+}$ which depends on $\boldsymbol{\rho}_2 = \rho(c)_{c\in\mathcal{C}^+}$ determined by:

$$\boldsymbol{\rho}_1 = \boldsymbol{\eta}_1 (Id - M_1)^{-1},$$

where

$$\eta_1(c) = \frac{\lambda(c) + \sum_{k\in\mathcal{C}^+} \mu(k) H[k,c]}{\mu(c) + \lambda^-(c)}, \quad M_1[c,c'] = \frac{\mu(c')H[c',c]}{\mu(c)+\lambda^-(c)}, \quad \text{for } c, c' \notin \mathcal{C}^+.$$

This implies that for $c \notin \mathcal{C}$, $\rho(c)$ is a linear combination of $(\rho(c)_{c\in\mathcal{C}^+})$, which is denoted by $\Phi_1(c)$.

For $c \in \mathcal{C}^+$, consider the polynomial

$$\begin{aligned} P^c(X) = X^2\big(\mu(c) + \lambda^-(c)\big) - X\Big(\lambda^s(c) + \mu(c) + \lambda^-(c) + \sum_{k\in\mathcal{C}^+} \rho(k)\mu(k)H[k,c] \\ + \sum_{k\notin\mathcal{C}^+} \Phi_1(k)\mu(k)H[k,c]\Big) + \Big(\lambda(c) + \sum_{k\in\mathcal{C}^+} \rho(k)\mu(k)H[k,c] \\ + \sum_{k\notin\mathcal{C}^+} \Phi_1(k)\mu(k)H[k,c]\Big). \end{aligned}$$

Similarly to Lemma 3.2, one has that there is a unique solution in $[0,1)$ of $P^c(X)$.

Consider the function $\Phi_2(\boldsymbol{\rho}_2) : [0,1]^{\mathcal{C}^+} \to [0,1]^{\mathcal{C}^+}$ where $\Phi(\rho)(c)$ is the unique solution in [0,1) of $P^c(X)$. Similarly to Lemma 3.2, we have a solution to the fixed point equation: $\Phi_2(\rho_2)(c) = \rho_2(c)$ which satisfies $\rho_2(c) \in [0,1)$.

This completes the proof.

**Lemma 3.4.** *Let $\rho$ be a solution to the system of equations (5,6). The considered LIFO queue has an invariant measure $p$ defined by*

$$p(x(1), x(2), \cdots, x(n)) = \prod_{l\le n} \rho(x(l)). \tag{7}$$

*Proof.* To prove that $p$ is an invariant measure, one has to check that for all $\boldsymbol{x} = x(1), x(2), \cdots, x(n)$, one has

$$\sum_{\boldsymbol{y}} p(\boldsymbol{x})Q(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{y}} p(\boldsymbol{y})Q(\boldsymbol{y}, \boldsymbol{x}),$$

where $Q$ is the infinitesimal generator of the queue.

The left-hand side is given by

$$L = \lambda, \quad \text{if } n = 0,$$

or

$$L = p((x))(\lambda + \mu(x(n)) + \lambda^-(x(n)) + \lambda^s(x(n))), \quad \text{if } n > 0.$$

Using the equation $\sum_{k>0} \rho(c)^k = \rho(c)/(1-\rho(c))$ when $\rho(c) < 1$, the right-hand side is given by

$$R = \sum_{c \in \mathcal{C}} \rho(c)\mu(c)H[c, 0] + \sum_{c \in \mathcal{C}} \rho(c)\lambda^-(c) + \sum_{c \in \mathcal{C}^+} \frac{\rho(c)\lambda^s(c)}{1-\rho(c)}, \quad \text{if } n = 0$$

or

$$R = \frac{p(\boldsymbol{x})}{\rho(x(n))} \Big\{ \lambda(x(n)) + \rho(x(n)) \sum_{k \in \mathcal{C}} \rho(k)\lambda^-(k) + \rho(x(n)) \sum_{k \in \mathcal{C}} \rho(k)\mu(k)H[k, 0]$$
$$+ \sum_{k \in \mathcal{C}^+, k \neq x(n)} \frac{\rho(k)\lambda^s(k)}{1-\rho(k)} \rho(x(n)) + \sum_{k \in \mathcal{C}} \rho(k)\mu(k)H[k, x(n)] \Big\}, \quad \text{if } n > 0.$$

Let us comment about the expression of $R$. In the sum, the first term corresponds to an arrival; the second term corresponds to an elimination caused by a negative signal; the third term corresponds to a completion of service of phase $k$ to reach absorbing state 0 and leave the queue; the fourth term corresponds to an elimination caused by a group deletion signal of class $s_k$, for $k \neq x(n)$; and the last term corresponds to a service completion of phase $k$ which provokes another service of phase $x(n)$.

When $n = 0$, we have $L = R$ as in remark 3.3. Using this equation, for $n > 0$, one has

$$R\frac{\rho(x(n))}{p(\boldsymbol{x})} = \lambda(x(n)) + \sum_{k \in \mathcal{C}} \rho(k)\mu(k)H[k, x(n)]$$
$$+\rho(x(n))\big(\lambda - \mathbb{1}_{x_n \in \mathcal{C}^+} \frac{\rho(x(n)\lambda^s(x(n)))}{1-\rho(x(n))}\big)$$

The equations (5,6) imply that

$$R\frac{\rho(x(n))}{p(\boldsymbol{x})} = \rho(x(n))\big(\mu(x(n)) + \lambda^-(x(n)) + \lambda\big) + \mathbb{1}_{x(n) \in \mathcal{C}^+} \rho(x(n)\lambda^s(x(n))),$$

which yields that $L = R$ when $n > 0$.

This completes the proof.

We now have the main result for one queue.

**Theorem 3.1.** *Consider the LIFO multi-class PH queue with negative signal and group-deletion signal where the variables are* $(\lambda(c), \mu(c), \lambda^-(c), \lambda^s(c))$. *Let* $\boldsymbol{\rho}$ *be a solution of the* PH Group-deletion Equations *(6,5) which satisfies* $\rho(c) < 1$ *for* $c \in \mathcal{C}^+$.

*If we have*

$$\sum_{c \in \mathcal{C}} \rho(c) < 1$$

*then the queue is stable and the stationary distribution is given by: for* $\boldsymbol{x} = \{x(1), \cdots, x(n)\}$

$$\pi(\boldsymbol{x}) = K \prod_{1}^{n} \rho(x(l)), \tag{8}$$

*where* $K$ *is the normalization constant given by* $K = 1 - \sum_{c \in \mathcal{C}} \rho(c)$.

*Proof.* If $\sum_{c \in \mathcal{C}} \rho(c) < 1$, then one has that $\sum_{x} \pi(\boldsymbol{x}) = 1$. Using Lemma 3.4, one has that $\pi$ is the stationary distribution. This completes the proof.

### 3.3 Quasi-reversibility

In this section, we will study the quasi-reversibility as defined by Chao et. al.

Consider the departure process of customer of class $c$, one has

$$\frac{\sum_{\boldsymbol{y}} q_c^D(\boldsymbol{y}, \boldsymbol{x}) \pi(\boldsymbol{y})}{\pi(\boldsymbol{x})} = \rho(c)\mu(c)H[c, 0].$$

Consider the negative signal of class $s_c^-$, one has

$$\frac{\sum_{\boldsymbol{y}} q_{s_c^-}^A(\boldsymbol{y}, \boldsymbol{x}) \pi(\boldsymbol{y})}{\pi(\boldsymbol{x})} = \rho(c)\lambda^-(c).$$

The problem is more complicated while considering the group-deletion signal of class $s_c$, one has

$$\frac{\sum_{\boldsymbol{y}} q_{s_c}^A(\boldsymbol{y}, \boldsymbol{x}) \pi(\boldsymbol{y})}{\pi(\boldsymbol{x})} = \begin{cases} \sum_{k \geq 1} \rho(c)^k \lambda^s(c) = \dfrac{\lambda^s(c)\rho(c)}{1 - \rho(c)}, & \text{if } x(n) \neq c; \\ 0, & \text{if } x(n) = c. \end{cases}$$

Hence, we have the quasi-reversibility for customers which reach the absorbing state after finishing service. We can add triggering probability for negative signal when there are a successful elimination

$$f_{s_c^-, s_c^-}(x(1) \cdots x(n)c, x(1) \cdots x(n)) = 1,$$

then we also have the quasi-reversibility for negative signal.

To have the quasi-reversibility for group-deleting signal, one needs to modify the network as follows:

- Triggering probability for group-deleting signal when there are a successful group-deletion: $f_{s_c,s_c}(x(1)\cdots x(n)c^k, x(1)\cdots x(n)) = 1$,
- When the state of the queue is $x(1)\cdots x(n)$, then there is a process of group-deletion signal of class $s_{x(n)}$ is active to departure with rate $\lambda^s(c)\rho(c)/(1-\rho(c))$.

We now have all ingredients to study the network.

## 4 Network of LIFO Multi-class PH Queues with Negative Signal and Group-Deletion Signal

### 4.1 Description

Consider the network of $N$ LIFO multi-class Ph queues with negative signal and group-deletion signal.

At queue $i$, the set of classes of customers is given by $\mathcal{C}^i$. The arrival rate and service rate of customer of type $c$ are $\lambda^i(c)$ and $\mu^i(c)$, respectively. The matrix which determines the service rate is given by $H^i[c,k]$ and $H^i[c,0]$ for $c,k \in \mathcal{C}^i$. The arrival rate of negative signal of type $s_c^-$ is $\lambda^{i,-}(c)$. The arrival rate of group-deletion signal of type $s_c$ is $\lambda^{i,s}(c)$.

At queue $i$, after service completion at phase $c$, a customer asks another service of phase $k$ with probability $H^i[c,k]$; or reaches absorbing state 0 at queue $i$ with probability $H^i[c,0]$, and then

- leaves to queue $j$ as a customer of class $c'$ with probability $P^{i,j}(c,c')$, as a negative signal of class $s^-c'$ with probability $P^{i,j}(c,s_{c'}^-)$, as a group-deletion signal of class $s_{c'}$ with probability $P^{i,j}(c,s_{c'})$,
- or quits the network with probability $d^i(c)$.

The following condition is satisfied:

$$\sum_j \sum_{c'\in\mathcal{C}^j} (P^{i,j}(c,c')P^{i,j}(c,s_{c'}^-)P^{i,j}(c,s_{c'})) + d^i(c) = 1.$$

A negative signal of class $s_c^-$ arrives to queue $i$ finding customer of class $c$ in service will eliminate this customer, and then

- moves to queue $j$ as a customer of class $c'$ with probability $P^{i,j}(s_c^-,c')$, as a negative signal of class $s^-c'$ with probability $P^{i,j}(s_c^-,s_{c'}^-)$, as a group-deletion signal of class $s_{c'}$ with probability $P^{i,j}(s_c^-,s_{c'})$,
- or quits the network with probability $d^i(s_c^-)$.

The following condition is satisfied:

$$\sum_j \sum_{c'\in\mathcal{C}^j} (P^{i,j}(s_c^-,c')P^{i,j}(s_c^-,s_{c'}^-)P^{i,j}(s_c^-,s_{c'})) + d^i(s_c^-) = 1.$$

A group-deletion signal of class $s_c$ arrive to queue $i$ finding customer of class $c$ in service will cancel all customers of class $c$ at the back-end of the buffer.

*Remark 4.1.* We know that we can generalize this mechanism of the group-deletion signal with probability of successful deletion in the network as we will have the quasi-reversibility. However, for the sake of simplicity, we will not introduce the generalized version of group-deletion signal.

### 4.2 Stationary Distribution

**Definition 4.1.** *The* Traffic Equations *of the network are given by*

$$\Lambda^i(c) = \lambda^i(c) + \sum_{k \in \mathcal{C}^i} \rho^i(k)\mu^i(k)H^i[k,c] + \sum_{j,k \in \mathcal{C}^j} \rho^j(k)\{\mu^j(k)H[k,0]P^{j,i}(k,c) + \Lambda^{j,-}(k)P^{j,i}(s_k^-,c)\}, \quad (9)$$

$$\Lambda^{i,-}(c) = \lambda^{i,-}(c) + \sum_{j,k \in \mathcal{C}^j} \rho^j(k)\{\mu^j(k)H[k,0]P^{j,i}(k,s_c^-) + \Lambda^{j,-}(k)P^{j,i}(s_k^-,s_c^-)\}, \quad (10)$$

$$\Lambda^{i,s}(c) = \lambda^{i,s}(c) + \sum_{j,k \in \mathcal{C}^j} \rho^j(k)\{\mu^j(k)H[k,0]P^{j,i}(k,s_c) + \Lambda^{j,-}(k)P^{j,i}(s_k^-,s_c)\}, \quad (11)$$

*where* $\boldsymbol{\rho}^i$ *is the solution to the* PH group-deletion Equations *associated to the variables* $(\Lambda^i(c), \Lambda^{i,-}(c), \Lambda^{i,s}(c), \mu^i(c))$.

*Remark 4.2.* As mentioned in section 3.3, we can consider a modified network to have the quasi-reversible property for group deletion signal. However, we do not give the detail in this paper. We only consider the "simple" network as presented.

We now have the main result to the paper.

**Theorem 4.1.** *Consider the network of N LIFO multi-class PH queues with negative signals and group-deletion signals. If there exists a solution to the traffic equations (9,10,11) which satisfies: for all i*

$$\sum_{c \in \mathcal{C}^i} \rho^i(c) < 1,$$

*then the network is stable and the stationary distribution has a product form given by*

$$\pi(\boldsymbol{x}^1, \cdots, \boldsymbol{x}^N) = \prod_{i=1}^{N} \pi^i(\boldsymbol{x}^i) = K \prod_{i=1}^{N} \rho^i(x^i(1) \cdots x^i(n_i)),$$

*where* $\boldsymbol{x}^i = (x^i(1) \cdots x^i(n_i))$ *and* $K$ *is the normalisation constant.*

The theorem can be deduced by using the result in Theorem 2.1.

## 5 Concluding Remarks

We hope that this new result will improve the applicability of G-network to model systems with complex destruction mechanism of customers. Extension to other queueing discipline is not that trivial because of the combinatorial problem for description of set of customers to be deleted when the queueing discipline is a general symmetric discipline as defined by Kelly.

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