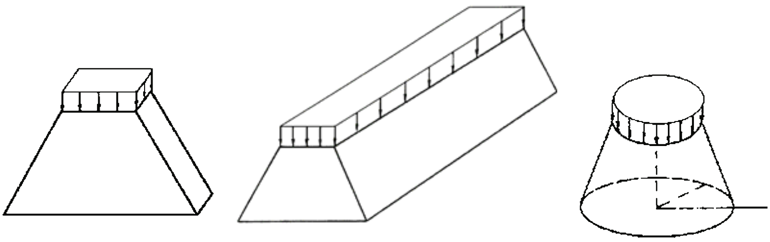
7

Examples of the Application of Unified Elasto-Plastic Constitutive Relations

7.1 Introduction

The unified strength theory, its singularity and the process of the piece-wise linear yield criterion, the implementation of the unified strength theory, and the subroutine of yield criteria (subroutine "INVAR" to calculate equivalent stresses), the subroutine of flow vector and the subroutine of the corner (subroutine "YIELD" and "FOLWPL" to calculate flow vector) have been introduced in previous chapters. Some simple examples including the plane stress, plane strain and spatial axial-symmetry problems will be described in this chapter. It is easy for the reader to find out about the possibility of adding new functions to the procedure (Yu, 1992; Yu et al., 1992; 1994; UEPP User's Manual, 1998).



(a) Plane stress problem (b) Plane strain problem (c) Spatial axisymmetric problem

Fig. 7.1 Three kinds of engineering structures

Plane stress, plane strain and spatial axisymmetric problems are three important problems in plasticity and engineering. Figure 7.1 shows an example of these three kinds of structures. Figure 7.1(a) is a plane stress structure with a

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uniform thickness of thin lamina, deformed under the action of the force which lies in its median plane. Figure 7.1(b) is a plane strain problem with zero strain in the z direction (length) in a structure of very great thickness. Figure 7.1(c) is a spatial axisymmetrical problem which is symmetrical in terms of geometry, boundary conditions and external loading about an axis.

These three kinds of structure shown in Fig. 7.1 have an identical section, a trapezoid, but different stress states. The stresses normal to the solution domain σ_z in the plane stress state are zero principal stress and nonzero principal stress in the plane strain; the hoop stress σ_θ in special axisymmetrical problems is also a principal stress.

Several examples are calculated using the UEPP (Yu et al., 1992; Yu et al., 1993; Yu and Zeng, 1994; UEPP User's Manual, 1998) based on the unified strength theory (for SD materials) or the unified yield criterion (for non-SD materials) with b=0, b=1/2 and b=1 in this chapter. Some examples are the same as the previous examples presented in the literature, for comparison. These three special cases of the unified strength theory are three basic criteria, which are the lower criterion, the upper criterion and the median criterion for all the convex criteria. So, three basic results can be obtained.

Sometimes, five types of criteria of the unified strength theory with b=0, b=1/4, b=1/2, b=3/4 and b=1 are used. Eleven results can be obtained using the unified strength theory with b=0, b=0.1, b=0.2, b=0.3, b=0.4, b=0.5, b=0.6, b=0.7, b=0.8, b=0.9 and b=1, if needed.

7.2 Plane Stress Problems

7.2.1 Elasto-Plastic Analysis of a Cantilever Beam

A plastic analysis of a simple, plane stress, cantilever beam can be seen in the book by Zienkiewicz in 1971. Ideal plasticity behaviour of a Huber von Mises material model was assumed. The spread of plastic zones for different ratios of q/q_p when q_p is calculated as from plastic beam theory (q_p = collapse load) is shown in Fig. 7.2. The loads are given in terms of the collapse load estimated on the basis of elementary plastic hinge theory.

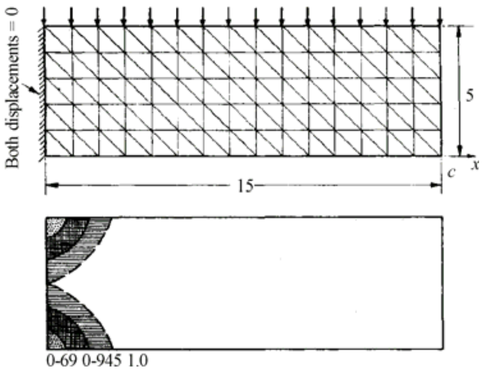


Fig. 7.2 The spread of plastic zones for cantilever beam (Zienkiewicz, 1971)

Figure 7.3 shows the increase in displacements with load. As the collapse load is approached progressively, larger numbers of iterations are required and indeed at P/P_p=1 no convergence was achieved (Zienkiewicz, 1971). Thus, although the non-linear solution allows a lower bound in collapse load to be found (by satisfying equilibrium and yield conditions) the actual collapse load cannot be found by incrementing the loads. To obtain a better picture of collapse behavior it is simpler to apply specified displacements at the load point and to increment these until no further increase in the reaction at that point is achieved.

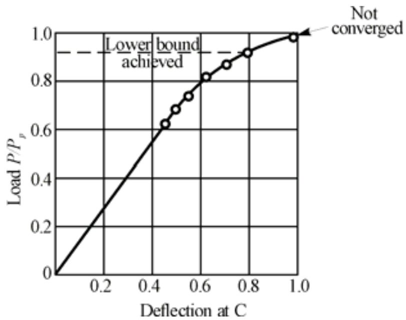


Fig. 7.3 Displacements versus load P/P_p

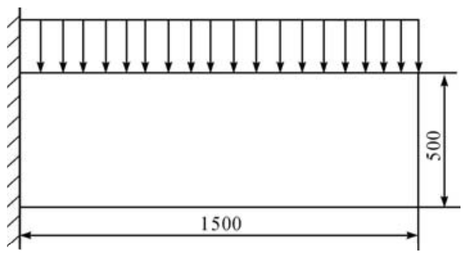
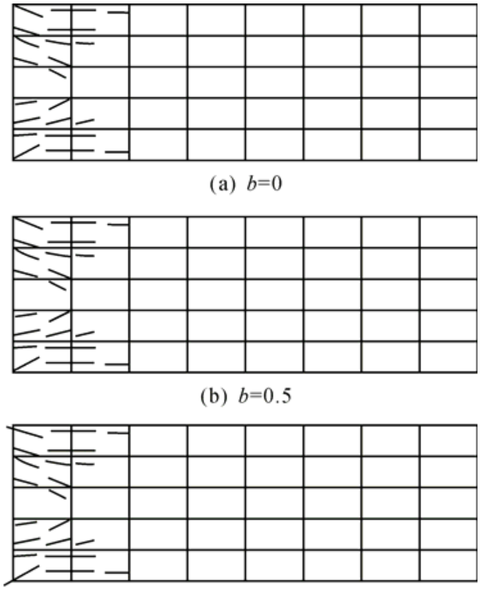


Fig. 7.4 A similar cantilever beam (Zienkiewicz, 1971)

A similar cantilever beam under the act of a uniform load shown in Fig. 7.4 is studied again in terms of the unified strength theory and UEPP (Yu, 1993). In Fig. 7.4 the mechanical behaviour of material is E=2.1×10^5 MPa, ν=0.3, ideal plasticity, yield stress σ_y=240 MPa. Calculate the elastic limit of the beam, the plastic region and the load-displacement relationship using the unified yield criterion with different parameter b.

The beam is considered as a plane stress problem. The configuration and the division into 40 isoparametric elements are shown in Fig. 7.5. An 8-nodes isoparametric element is used. The number of the nodes is 147. After the first iteration the elastic limits of the cantilever beam in terms of the unified yield criterion with different value of b (b=0, b=1/2, b=1) are obtained respectively, as follows.



(a) b=0

(b) b=0.5

(b) b=1

Fig. 7.5 Plastic zones of cantilever beam under the same load w_e=15 N/mm^2

1) q_e=9.03 N/mm^2 (unified yield criterion with b=0, single-shear criterion, i.e. the single-shear yield criterion or the Tresca criterion);

2) q_e=9.36 N/mm^2 (unified yield criterion with b=1/2, linear Mises criterion);

3) q_e=9.54 N/mm^2 (unified yield criterion with b=1, twin-shear criterion).

The plastic zones for different values of the unified yield criterion parameter b under the same load w_e=15 N/mm^2 are different, as shown in Fig. 7.5. Figure 7.5(a) is the plastic zone of the unified yield criterion with b=0. It is also the plastic zone of the single-shear criterion or the Tresca yield criterion. Figure 7.5(b) is the plastic zone of the unified yield criterion with b=1/2 It approximates to the plastic zone of the Huber-von Mises yield criterion. Figure 7.5(c) is the plastic zone of the unified yield criterion with b=1. It is the plastic zone of the twin-shear yield criterion.

The increase in displacements with load in terms of different values of the unified strength theory parameter b is displayed in Fig. 7.6. It requires a large number of iterations of non-linear computation, until no further increase in the reaction at that point is achieved (furthermore, the numerical solution process will be divergent). The plastic limit load is approached progressively to the point where no convergence is achieved, as shown in Fig. 7.3 and Fig. 7.6. The convergent results using the three basic criteria can be obtained respectively, as shown in Fig. 7.6.

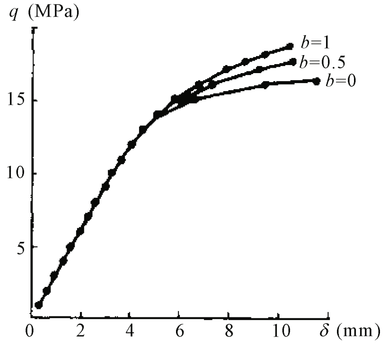


Fig. 7.6 Increase of displacements with load for different yield criteria

The numerical results achieved by using the unified strength theory with b=0 are very close to the results of Zienkiewicz (1971). Some new results, moreover, are obtained by using the unified strength theory.

7.2.2 Elasto-Plastic Analysis of a Trapezoid Structure under Uniform Load

The trapezoid structure is an important structure in engineering. It can be processed as a unified problem in FEM-2D codes.

On a symmetrical trapezoidal plate, the vertex angle formed by the extension line of its two bevels is 2ξ (ξ=45°), the upper side of the trapezoidal plate (Fig. 7.7) exerts a uniform pressure q. The parameters of the material are elastic modulus, E=2.06$\times$10^4 MPa, Poisson's ratio μ=0.167. The uniaxial tensile strength of the material is σ_t=2.4 MPa, the uniaxial compressive stress is σ_c=24 MPa, i.e. the strength ratio of extension to compression is α=0.1. Then calculate the elastic limit load and plastic limit load by using the unified yield criterion with different parameter b.

This problem can be considered as a plane stress problem. We analyze a half of the trapezoidal structure because of the symmetry. The isoparametric elements can be chosen as eight nodes and a quadrilateral, as shown in Fig. 7.8. And there are 128 element and 433 nodes in total. The elastic limit can be calculated under different yield criteria.

The elastic limit of a trapezoidal plate in terms of the unified yield criterion with a different value of b (b=0, b=0.5, b=1, i.e. different yield criteria) under plane stress state can be obtained as follows.

1) q_e=23.9 MPa (unified strength theory with b=0, single-shear theory, i.e. the Mohr-Coulomb theory);

2) q_e=25.2 MPa (unified strength theory with b=1/2, a new criterion);

3) q_e=28.1 MPa (unified strength theory with b=1, i.e. Twin-shear theory).

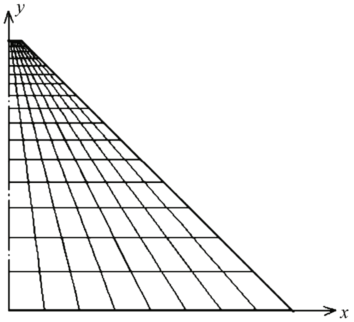


Fig. 7.7 Mesh of a half-trapezoidal structure

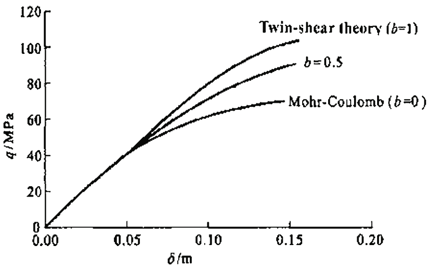


Fig. 7.8 load-displacement relations

As shown in Fig. 7.8, the load-displacement curve of node 2 in the middle of the plate can be obtained by increasing the uniform load step-by-step. The solution is obtained until the limit load is reached. The limit load is approached progressively. A larger number of iterations are required until no convergence is achieved.

The solution is convergent when q=70 MPa (for single-shear theory or the unified strength theory with b=0), q=92 MPa (for the unified strength theory with b=1/2), q=104 MPa (for the unified strength theory with b=1). Increase the load again and the solution process will be divergent; therefore three kinds of limit load can be obtained, respectively.

The plastic limit of a trapezoidal structure in terms of the unified yield criterion with different values of b (b=0, b=0.5, b=1) under plane stress state are

1) q_p=70 MPa (unified strength theory with b=0, single-shear theory, i.e. the Mohr-Coulomb theory);

2) q_p=92 MPa (unified strength theory with b=1/2, a new criterion);

3) q_p=104 MPa (unified strength theory with b=1, twin-shear theory).

7.3 Plane Strain Problems

Figure 7.9 shows a typical structure in railway and high road engineering and the city wall in Xi'an, China. It can be simplified to a plane strain problem. The uniform distributed load is applied on the top. A symmetrical trapezoidal structure with a top angle 2ξ and the slip line field of the trapezoid structure are also shown in Fig. 7.9. The slip line field was constructed as shown in Fig. 7.9 (Yu et al.). The stability of the city wall in Xi'an, China, can be seen in Chapter 18.

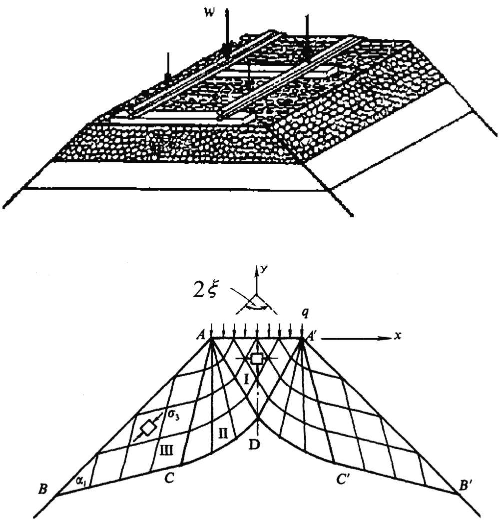


Fig. 7.9 Base of a railroad and the plane strain problem

The mesh of a trapezoidal structure under plane strain is identical to the mesh of a plane stress problem shown in Fig. 7.8. The isoparametric elements with eight nodes and a quadrilateral are used.

The elastic limit of a symmetrical trapezoidal structure in terms of the unified yield criterion with different values of b (b=0, b=0.5, b=1) under plane strain state can be obtained as follows.

1) q_e=28.8 MPa（unified strength theory with b=0, single-shear theory, i.e. the Mohr-Coulomb theory);

2) q_e=33.1 MPa (unified strength theory with b=1/2, a new criterion);

3) q_e=35.8 MPa (unified strength theory with b=1, twin-shear theory).

The load-displacement relation of node 2 in the middle of the trapezoidal structure under plane strain condition using the three yield criteria can be obtained, as shown in Fig. 7.10.

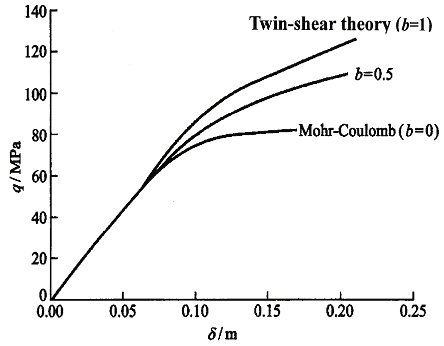


Fig. 7.10 Load-displacement relation of a trapezoidal structure under plane strain condition

The convergent result can be obtained when q=81.8 MPa (unified strength theory with b=0, i.e. the Mohr-Coulomb single-shear theory), q=109 MPa (unified strength theory, b=0.5), q=125.8 MPa (unified strength theory, b=1). Increasing the loads further, the numeric solution process will be divergent. Then the limit load can be obtained by using the three yield criteria respectively.

The plastic limit of a trapezoidal structure in terms of the unified yield criterion with different values of b (b=0, b=0.5, b=1) under the plane strain state can be obtained as follows.

1) q_p=81.8 MPa (unified strength theory with b=0, single-shear theory, i.e. the Mohr-Coulomb theory);

2) q_p=109 MPa (unified strength theory with b=1/2, a new criterion);

3) q_p=125.8 MPa (unified strength theory with b=1, twin-shear theory).

7.4 Spatial Axisymmetric Problems

7.4.1 Analysis of Plastic Zone for Thick-Walled Cylinder

Another example of using the unified strength theory is a thick-walled cylinder shown in Fig. 7.11 and Fig. 7.12. When the internal pressure exceeds p_e, a plastic zone will begin at the inner surface and spread outwards toward the outer surface. The elastic–plastic boundary at any stage has a radius r_c. In the elastic region, ($r_c \le r \le r_b$), the radial and circumferential stresses are obtained from Lame's equations using the boundary condition σ_r=0 at r=r_b and the fact that the material at r=r_c is stressed to the yield point. The pressure reaches its maximum value when the plastic zone reaches the outer surface of the thick-walled tube.

The elastic part of the elastic-plastic thick-walled tube may be considered as a new tube with inner radius r_c and outer radius r_b, with an internal pressure p_e. The stress distribution in the elastic region for an incompressible material is easily obtained.

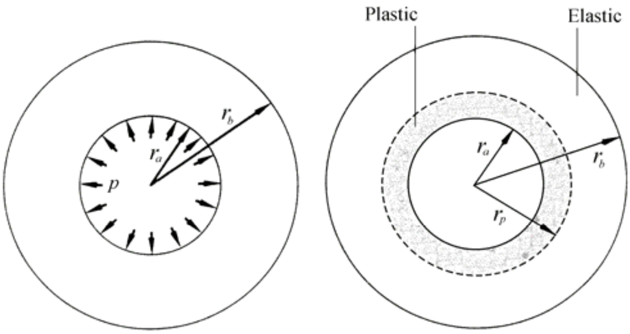


Fig. 7.11 Thick-walled cylinder

Due to symmetry, only a quarter of the thick-walled cylinder is shown. The cylinder has an inner radius r_a=100 mm, and an outer radius r_b=200 mm. The elastic modulus is E =2.1×10^5 MPa, Poisson ratio μ=0.3, tensile yield stress σ_y =240 MPa. This example is the same as the example in Chapter 7 of the book (Owen and Hinton, 1980) for comparison.

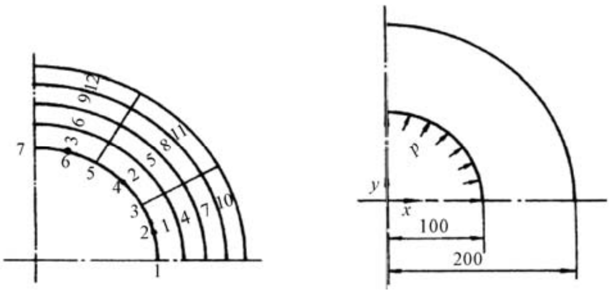


Fig. 7.12 Thick-walled cylinder and finite element mesh

Four elastic limit pressures using different yield criteria are obtained as follows.

1) p_e=97.9 MPa (unified yield criterion with b=0, single-shear criterion or the Tresca criterion);

2) p_e=111.0 MPa (unified yield criterion with $b=1/(1+\sqrt{3})$, a new criterion);

3) p_e=111.6 MPa (Huber-von Mises yield criterion);

4) p_e=125.8 MPa (unified yield criterion with b=1, twin-shear yield criterion).

Figure 7.13 shows the distribution of the circumferential stress in a thick-walled cylinder with the twin-shear yield criterion (b=1), the Huber-Mises criterion ($b=1/(1+\sqrt{3})$) and the Tresca criterion (b=0) respectively. The curves are the analytical solution and the dots are the numerical solution. The third curve in Fig. 7.13 agrees with the previous result (Johnson and Mellor, 1962; Mendelson, 1968; Chakrabarty, 1987).

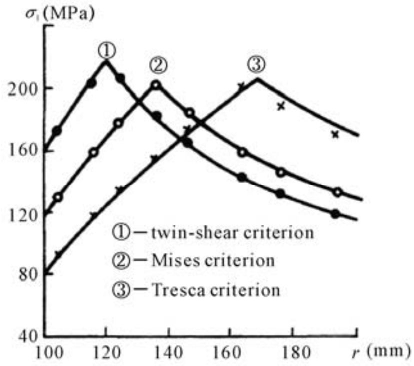
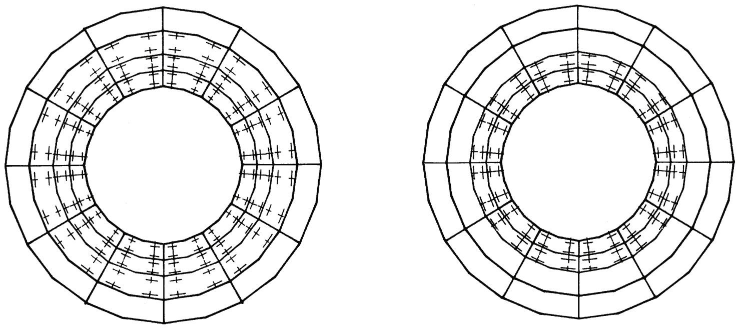


Fig. 7.13 Distribution of circumferential stress in cylinder with different yield criteria

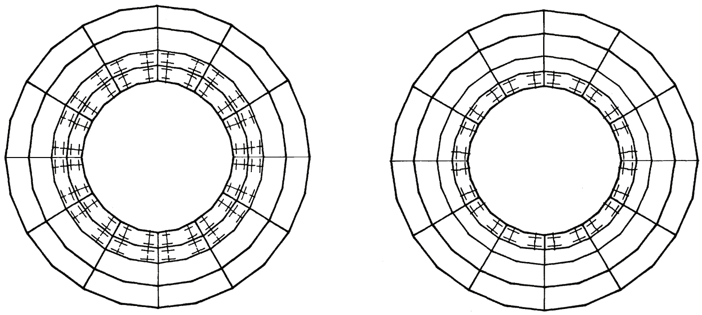
The distributions of circumferential stresses with different yield criteria in the elasto-plastic thick-walled cylinder subjected to internal pressure p=160 MPa are shown in Fig. 7.13. Three curves are drawn according to the analytical solutions in terms of the unified strength theory (Yu, 2004). The points are obtained from the numerical calculations of the thick-walled cylinder that obey the unified yield criterion with b=0, $b=1/(1+\sqrt{3})$ and b =1, respectively.

The plastic zones in a thick-walled cylinder with different yield criteria under the same load are shown in Fig. 7.14. The radius of the plastic zone with the unified yield criterion when b=0 (Tresca yield criterion) is larger than that obtained from $b=1/(1+\sqrt{3})$ and b=1. As a comparison, the distribution of circumferential stress σ_θ in the elasto-plastic thick-walled cylinder obeying the Huber-von Mises yield criterion is also shown in Fig. 7.14(c).

It is seen that the results obeying the Huber-von Mises yield criterion and the unified yield criterion with $b=1/(1+\sqrt{3})$ are identical both in analytical solution and numerical calculation.



(a) UST with b=0 (Single-shear criterion,) (b) UST with $b = 1/(1+\sqrt{3})$



(c) Huber-von Mises criterion (d) UST with b=1 (Twin-shear criterion)

Fig. 7.14 Distribution of plastic zone in thick-walled cylinder with different parameter b

7.4.2 Analysis for Limit-Bearing Capacity of a Circular Plate

The numerical result of a simply supported circular plate can be seen in the book of Owen and Hinton (1980) in terms of the Huber-von Mises criterion. A uniformly loaded simply supported circular plate and finite element (FE) mesh are shown in Fig. 7.15. Only one-half of the plate is analyzed due to the symmetry. The plate is modeled by five axisymmetric elements and loading takes the form of a progressively increasing uniformly distributed load. The isoparametric element with eight nodes in the element family of UEPP is chosen for analysis. It is the same as in Owen and Hinton (1980).

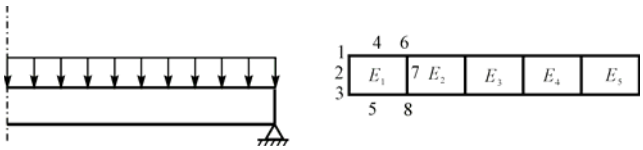


Fig. 7.15 Simply supported plate and finite element mesh of plate

The elastic limit of the simply supported circular plate in terms of different values of b of the unified yield criterion (i.e. different yield criteria) are obtained by using the unified strength theory as follows.

1) b=0 (Tresca criterion), q_e=2.2707 N/mm^2;

2) b=1/2 (linear Mises criterion), q_e=2.2733 N/mm^2;

3) b=1 (twin-shear criterion), q_e=2.2747 N/mm^2.

It is worth noting that the three elastic limits of a simply supported circular plate for different yield criteria are identical. This is because the maximum stress point is situated at the centre point of the plate, which has a special stress state, i.e. σ_r=σ_θ. All the three yield loci of the Tresca criterion, Huber-von Mises criterion and the twin-shear criterion cross at the same point A, as shown in Fig. 7.16.

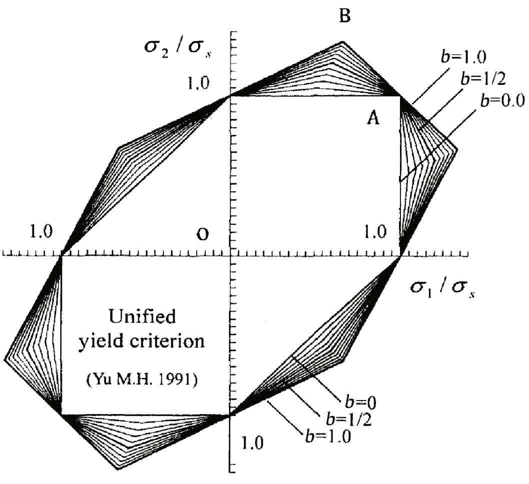


Fig. 7.16 Yield loci of the unified yield criterion in plane stress state

The plastic zone spreads from the centre point to the neighboring area around the centre point of the simply supported circular plate after a further increase in the load, in which the two stresses are not equal, i.e. $\sigma_r \neq \sigma_\theta$. The yield point at the yield loci spreads from A to other points, where the yield states of the three yield criteria are not identical.

The elasto-plastic load-displacement curves at the center of the plate with different yield criteria are shown in Fig. 7.17. Ideal plastic behaviour with a different parameter b of the unified yield criterion is assumed. The median elasto-plastic load-deflection curve of the plate is the same as that of Owen and

Hinton (1980).

The plastic limit of a uniform loading simply supported circular plate using the three basic criteria, i.e., the unified yield criterion with b=0, b=1/2 and b=1, can be obtained as follows.

1) b=0 (Tresca criterion), q_p=2.7 N/mm^2;
2) b=1/2 (a new criterion), q_p=3.0 N/mm^2;
3) b=1 (twin-shear criterion), q_p=3.18 N/mm^2.

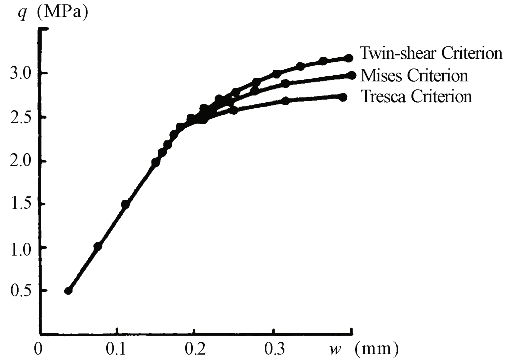
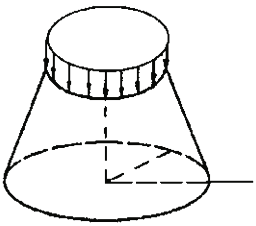


Fig. 7.17 Load-deflection curve of plate with the unified yield criterion

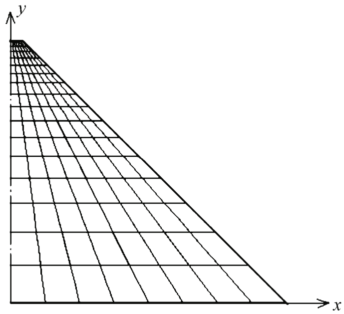
The differences between the three curves show the effect of the yield criterion on the plastic limit bearing capacity of circular plates. The limit-bearing capacity of a plate with the twin-shear criterion (unified yield criterion when b=1) is the maximum, and the limit-bearing capacity with the single-shear criterion (Tresca criterion or the unified yield criterion when b=0) is the minimum. The limit-bearing capacity with the Huber-von Mises criterion is median. The median result is equivalent to the result obeying the unified yield criterion with b=1/2 or $b=1/(1+\sqrt{3})$. The numerical results using the UEPP are very close to the analytical results described in (Yu, Ma and Li JC, 2009)

7.4.3 Truncated Cone under the Uniform Load on the Top

A truncated cone under the uniform load on the top is shown in Fig. 7.18(a). Similar to the plane stress problem and plane strain problem, this spatial symmetric structure has the identical section with the plane problem as shown in Fig. 7.18(b).



(a) Spatial symmetric problem



(b) FEM mesh

Fig. 7.18 Truncated cone and its FEM mesh

The elastic limit of the truncated cone in terms of the unified yield criterion with three basic values of b ($b=0, b=0.5, b=1$)in a spatial axisymmetric stress state can be obtained as follows.

1) $q_e = 33.6$ MPa (unified strength theory with b=0, single-shear theory, i.e. the Mohr-Coulomb theory);

2) $q_e = 40.6$ MPa (unified strength theory with b=1/2, a new criterion);

3) $q_e = 43.0$ MPa (unified strength theory with b=1, the twin-shear theory).

The load-displacement relationship of node 2 in the middle of the truncated cone structure under uniform load on the top, using the unified strength theory with three parameters b (b=0, b =0.5, b =1), is obtained as shown in Fig. 7.19.

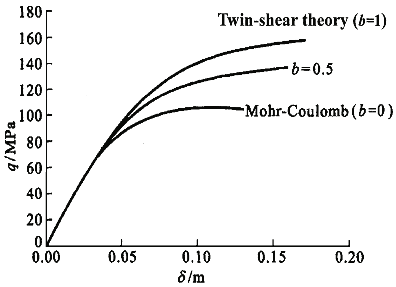


Fig. 7.19 Load-displacement relation of a truncated cone

The convergent result can be obtained when q=105 MPa (unified strength theory with b=0, i.e., the Mohr-Coulomb single-shear theory), q=136 MPa (unified strength theory, b=0.5), q=158.1 MPa (unified strength theory, b=1). Increasing the loads further, the numeric solution process will be divergent. Then the limit load can be obtained by using the unified strength theory with three parameters respectively.

The plastic limit of a cone in terms of the unified yield criterion with three basic values of b (b=0, b =0.5, b =1) under the plane strain state can be obtained as follows.

1) q_p=105 MPa (unified strength theory with b=0, the single-shear theory, i.e. the Mohr-Coulomb theory);

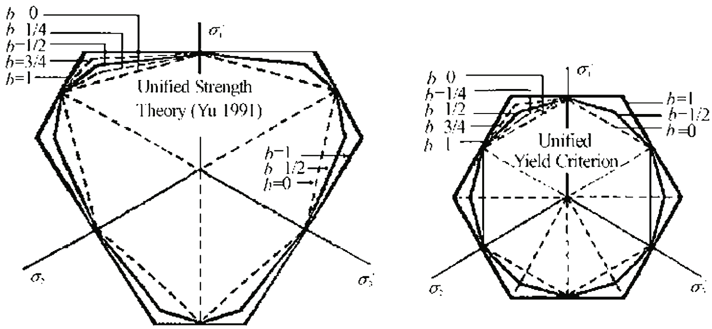
2) q_p=136 MPa (unified strength theory with b=1/2, a new criterion);

3) q_p=158.1 MPa (unified strength theory with b=1, the twin-shear theory).

7.5 Brief Summary

The unified strength theory has been implemented in a non-linear FE program. Several examples are calculated by using the unified strength theory and associated flow rule. A series of results can be obtained for every example, which may be adapted for most materials and structures.

Plane stress, plane strain and spatial axisymmetric problems are three important problems in plasticity and engineering. Some typical examples, described in textbooks and monographs relating to computational plasticity are calculated again for comparison. The results show that the result of the unified strength theory with b=0 is in good agreement with the result using the Mohr-Coulomb theory. The result of the unified strength theory with α=1 and b=0 is in good agreement with the result using the Tresca criterion. The result of the unified strength theory with α=1 and b=1/2 is equivalent to the result using the Huber-von Mises criterion. The result of the unified strength theory with b=1 is in good agreement with the result using the twin-shear theory and the result of the unified strength theory with α=1 and b=0 is in good agreement with the result using the twin-shear yield criterion, or the maximum deviatoric stress criterion. A series of new results can be also obtained by using the unified strength theory. The Tresca-Mohr–Coulomb single-shear strength theory, the twin-shear strength theory and a new median criterion can be deduced from the unified strength theory when b=0, b=1 and b=1/2. They are all piecewise linear yield criteria. The lower bound, upper bound and the median criterion situated between these two bounds may be considered as three basic criteria for SD materials ($\alpha \neq 1$) and non-SD materials (α=1). The yield loci of the three criteria are shown in Fig. 7.20.



(a) SD materials ($\alpha \neq 1$) (b) Non-SD materials (α=1)

Fig. 7.20 Yield loci of several typical criteria of the unified strength theory

The numerical results obtained by using the unified strength theory are in good agreement with the results of the unified solution described in (Yu et al., 2009).

The unified strength theory and associated flow rule have also been implemented in several commercial non-linear FE codes and applied to engineering problems, which will be described in the next chapters.

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