2

Stress and Strain

2.1 Introduction

In applied mechanics and engineering, materials and structures are generally regarded as continua. This permits us to describe the behaviour and consequences of the use of materials and structures by means of continuous functions. A material is a point (element), and a structure is a body. The structure may be considered as a partly ordered set of material elements (points) filling a structure (body). The cube is often used as an element.

An element that can fill a space without gaps and overlapping is called spatial equipartition. Various polyhedra used in continuum mechanics result in spatial equipartition, such as cubic elements and regular hexagonal elements. The dodecahedron element, orthogonal octahedron element (twin-shear model) and pentahedron element will be described in this chapter.

It is assumed that the reader is familiar with the basic concepts of the mechanics of materials and the theory of elasticity, including the definitions of stress and strain. We shall, however, briefly review some of these basic concepts. In addition, some new concepts are also described in this chapter.

2.2 Stress at a Point, Stress Invariants

A general state of stress at a point can be determined by a stress tensor σ_{ij}, which stands for nine components, as shown in Fig. 2.1, and can be expressed as

M.-H. Yu et al., *Computational Plasticity*

$$\sigma_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \tag{2.1}$$

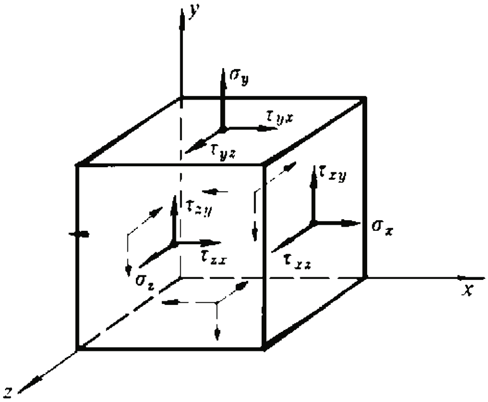

Fig. 2.1 Nine stresses on element

Stress and strain are second-order tensors. The concepts of tensor and tensor notation are useful in derivations and in the proof of theorems.

It can be seen in the course of elasticity, mechanics of solids or plasticity, using three-dimensional transformations, that there exists a coordinate system σ_1, σ_2, σ_3 where the state of stress at the same point can be described by the following:

$$\sigma_i = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \tag{2.2}$$

The stresses σ_1, σ_2, σ_3 are referred to as the principal stresses, as shown in Fig. 2.2.

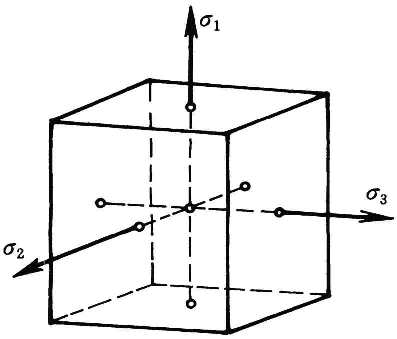

Fig. 2.2 Principal stress element

An element of material subjected to principal stresses σ_1, σ_2 and σ_3 acting in mutually perpendicular directions (Fig. 2.2) is said to be in a state of triaxial stress or three-dimensional stress. If one of the principal stresses equals zero, this is referred to as the plane stress state or biaxial stress state. The triaxial stress and biaxial stress are called the polyaxial stresses, multiaxial stresses or complex stress. The principal planes are the planes on which the principal stresses occur on mutually perpendicular planes.

The principal stresses are the three roots of the equation:

$$\begin{aligned}&\sigma^3-(\sigma_x+\sigma_y+\sigma_z)\sigma^2+(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x-\tau_{xy}{}^2+\tau_{yz}{}^2+\tau_{zx}{}^2)\sigma\\&-(\sigma_x\sigma_y\sigma_z+2\tau_{xy}\tau_{yz}\tau_{zx}-\sigma_x\tau_{yz}{}^2-\sigma_y\tau_{zx}{}^2-\sigma_z\tau_{xy}{}^2)=0\end{aligned} \tag{2.3}$$

which can be rewritten as

$$\sigma^3-I_1\sigma^2+I_2\sigma-I_3=0 \tag{2.4}$$

where I_1, I_2, I_3 are

$$\begin{aligned}I_1&=\sigma_x+\sigma_y+\sigma_z\\I_2&=\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x-\tau_{xy}^2-\tau_{yz}^2-\tau_{zx}^2\\I_3&=\sigma_x\sigma_y\sigma_z+2\tau_{xy}\tau_{yz}\tau_{zx}-(\sigma_x\tau^2{}_{yz}+\sigma_y\tau^2{}_{zx}+\sigma_z\tau^2{}_{xy})\end{aligned} \tag{2.5}$$

The quantities I_1, I_2 and I_3 are independent of the direction of the axes chosen. They are called the first, second and third invariants of the stress at a point (or invariant quantities).

If we choose the principal directions as the directions of the coordinate axes, then the three stress invariants take on the simple form

$$\begin{aligned}I_1&=\sigma_1+\sigma_2+\sigma_3\\I_2&=\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1\\I_3&=\sigma_1\sigma_2\sigma_3\end{aligned} \tag{2.6}$$

The three invariants I_1, I_2 and I_3 are three independent quantities which specify the state of stress just as well as the three principal stresses σ_1, σ_2 and σ_3.

2.3 Deviatoric Stress Tensor and its Invariants

It is convenient in the study of strength theory and plasticity to split the stress tensor into two parts, one called the deviatoric stress tensor S_{ij} and the other the spherical stress tensor p_{ij}. The relation is

$$\sigma_{ij} = S_{ij} + p_{ij} = S_{ij} + \sigma_m \delta_{ij} \tag{2.7}$$

The spherical stress tensor is the tensor whose components are $\sigma_m \delta_{ij}$, where σ_m is the mean stress, i.e.,

$$p_{ij} = \sigma_m \delta_{ij} = \sigma_m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} \tag{2.8}$$

where

$$\sigma_m = (\sigma_x + \sigma_y + \sigma_z)/3 = (\sigma_1 + \sigma_2 + \sigma_3)/3 = I_1/3 \tag{2.9}$$

It is apparent that σ_m is the same for all possible orientations of the axes. Hence σ_m is named spherical stress. Also, since σ_m is the same in all directions, it can be considered to act as a hydrostatic stress or hydrostatic pressure, denoted by p. It is equal to one-third of the first invariant, $p = \sigma_m = I_1/3$.

The deviatoric stress tensor S_{ij} can be determined as

$$S_{ij} = \sigma_{ij} - p_{ij} = \sigma_{ij} - \sigma_m \delta_{ij} = \begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma_m \end{bmatrix} \tag{2.10}$$

The invariants of the deviatoric stress tensor are denoted by J_1, J_2, J_3 and can be obtained as follows:

$$\begin{aligned} J_1 &= S_1 + S_2 + S_3 = 0,\ J_2 = \frac{1}{2} S_{ij} S_{ij} = \frac{2}{3}(\tau_{13}^2 + \tau_{12}^2 + \tau_{23}^2) \\ J_3 &= \left| S_{ij} \right| = S_1 S_2 S_3 = \frac{1}{27}(\tau_{13} + \tau_{12})(\tau_{21} + \tau_{23})(\tau_{31} + \tau_{32}) \end{aligned} \tag{2.11}$$

The invariants of the deviatoric stress tensor J_2, and J_3 can be written in terms of the principal stresses

$$\begin{aligned} J_2 &= \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \\ J_3 &= (\sigma_1 - \sigma_m)(\sigma_2 - \sigma_m)(\sigma_3 - \sigma_m) \end{aligned} \tag{2.12}$$

2.4 Stresses on the Oblique Plane

If the three principal stresses σ_1, σ_2, σ_3 are acting on three principal planes, respectively, at a given point, we can determine the stresses acting on any plane through this point. This can be done by consideration of the static equilibrium of an infinitesimal tetrahedron formed by this plane and the principal planes, as shown in Fig. 2.3. In this figure, we have shown the principal stresses acting on the three principal planes. These stresses are assumed to be known. We wish to find the stresses σ_α and τ_α acting on the oblique plane whose normal direction cosines are l, m and n.

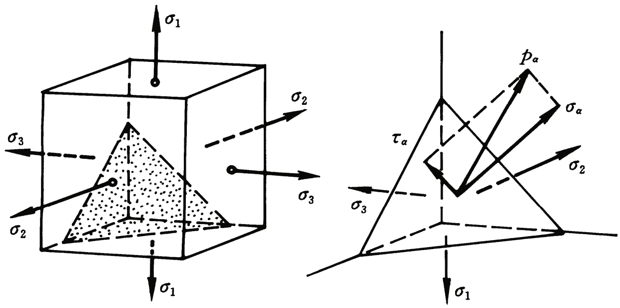

Fig. 2.3 Stress on an infinitesimal tetrahedron

2.4.1 Stresses on the Oblique Plane

The normal stress σ_α and shear stress τ_α acting on this plane can be determined as follows:

$$\begin{aligned}
\sigma_\alpha &= \sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2 \\
\tau_\alpha &= \sigma_1^2 l^2 + \sigma_2^2 m^2 + \sigma_3^2 n^2 - (\sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2) \\
\boldsymbol{p}_\alpha &= \boldsymbol{\sigma}_\alpha + \boldsymbol{\tau}_\alpha
\end{aligned} \tag{2.13}$$

2.4.2 Principal Shear Stresses

The three principal shear stresses τ_{13}, τ_{12} and τ_{23} can be obtained as

$$\tau_{13} = \frac{1}{2}(\sigma_1 - \sigma_3), \quad \tau_{12} = \frac{1}{2}(\sigma_1 - \sigma_2), \quad \tau_{23} = \frac{1}{2}(\sigma_2 - \sigma_3) \tag{2.14}$$

The maximum shear stress acts on the plane bisecting the angle between the largest and smallest principal stresses and is equal to half of the difference between these principal stresses

$$\tau_{\max} = \tau_{13} = \frac{1}{2}(\sigma_1 - \sigma_3) \tag{2.15}$$

The corresponding normal stresses σ_{13}, σ_{12} and σ_{23} acting on the sections where τ_{13}, τ_{12} and τ_{23} are acting, respectively, are

$$\sigma_{13} = \frac{1}{2}(\sigma_1 + \sigma_3), \quad \sigma_{12} = \frac{1}{2}(\sigma_1 + \sigma_2), \quad \sigma_{23} = \frac{1}{2}(\sigma_2 + \sigma_3) \tag{2.16}$$

The three principal shear stresses τ_{13}, τ_{12} and τ_{23} and corresponding normal stresses σ_{13}, σ_{12} and σ_{23} acting on the principal shear stresses sections form a rhomboidal- dodecahedron (τ_{13}, τ_{12}, τ_{23}; σ_{13}, σ_{12}, σ_{23}), as shown in Fig. 2.4.

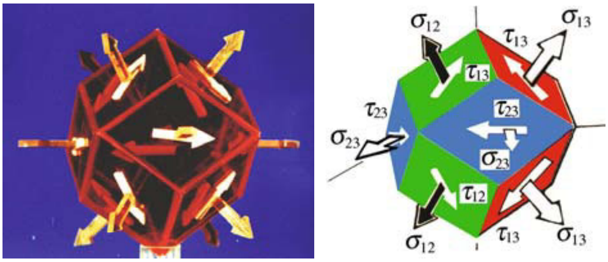

Fig. 2.4 Dodecahedron multi-shear model (τ_{13}, τ_{12}, τ_{23}; σ_{13}, σ_{12}, σ_{23})

The three principal stresses, three principal shear stresses and the three normal stresses acting on the principal shear stresses sections can be illustrated by three stress circles. This is referred to as the Mohr circle, as shown in Fig. 2.5.

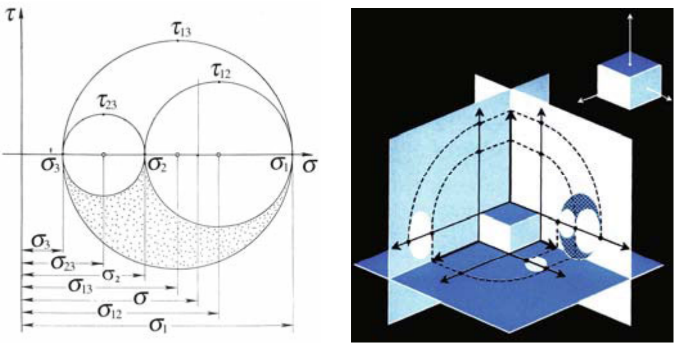

Fig. 2.5 The principal stresses, principal shear stresses and stress circles

The magnitude of the normal and shear stresses of any plane are equal to the distance of the corresponding stress point on the stress circle. The three principal shear stresses are evidently equal to the radius of the three Mohr circles. A detailed description of the stress circle can be found in Johnson and Mellor (1962), Kussmaul (1981), Chakrabarty (1987), Davis and Selvadurai (2002) and others. Figure 2.6 shows the relations between the stress circles and different planes, where the stresses are acted on.

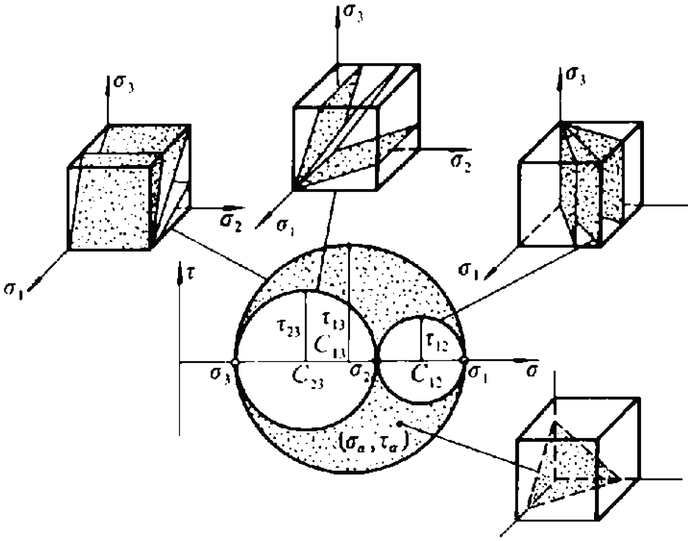

Fig. 2.6 Relations between the stress circles and different planes

2.4.3 *Octahedral Shear Stress*

If the normal of the oblique plane makes equal angles with all the principal axes, and

$$l=m=n=\pm\frac{1}{\sqrt{3}} \tag{2.17}$$

then these planes are called the octahedral plane and the shear stresses acting on it are called the octahedral shear stresses. The normal stress, called the octahedral normal stress σ_8 (or σ_{oct}), acting on this plane equals the mean stress

$$\sigma_8=\frac{1}{3}(\sigma_1+\sigma_2+\sigma_3)=\sigma_m \tag{2.18}$$

A tetrahedron similar to this one can be constructed in each of the four quadrants above the x–y plane and in each of the four quadrants below the x–y plane. On the oblique face of each of these eight tetrahedra the condition $l^2=$

$m^2=n^2=1/3$ will apply. The difference between the tetrahedra will be in the signs attached to l, m and n. The eight tetrahedra together form an isoclinal octahedron element, as shown in Fig. 2.7, and each of the eight planes form the face of this octahedron.

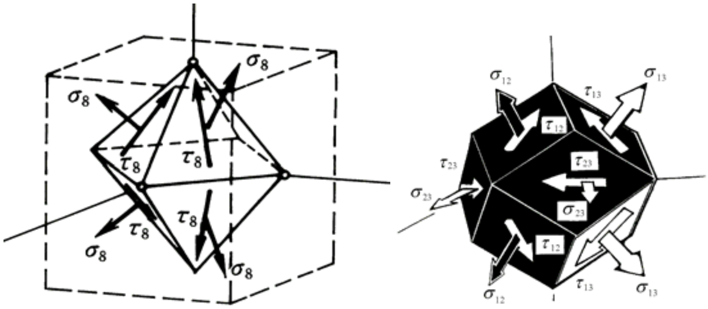

Fig. 2.7 Isoclinal octahedron element and dodecahedron element

The octahedral normal stress is given by Eq. (2.18) and the octahedral shear stress τ_8 (sometimes denoted as τ_{oct}) acting on the octahedral plane is

$$\begin{aligned} \tau_8 &= \frac{1}{3}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]^{1/2} \\ &= \frac{1}{\sqrt{3}}[(\sigma_1-\sigma_m)^2+(\sigma_2-\sigma_m)^2+(\sigma_3-\sigma_m)^2]^{1/2} \end{aligned} \tag{2.19}$$

The direction cosines l, m and n of principal planes, principal shear stress planes and the octahedral plane, as well as the normal stresses and shear stresses, are listed in Table 2.1.

Table 2.1 Direction cosines of the principal planes, the principal shear stress planes and the octahedral planes

	Principal plane			Principal shear stress planes			Octa. plane
l	± 1	0	0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{3}}$
m	0	± 1	0	$\pm\frac{1}{\sqrt{2}}$	0	$\pm\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{3}}$
n	0	0	± 1	0	$\pm\frac{1}{\sqrt{2}}$	$\pm\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{3}}$
σ	σ_1	σ_2	σ_3	$\sigma_{12}=\frac{\sigma_1+\sigma_2}{2}$	$\sigma_{13}=\frac{\sigma_1+\sigma_3}{2}$	$\sigma_{23}=\frac{\sigma_2+\sigma_3}{2}$	σ_8
τ	0	0	0	$\tau_{12}=\frac{\sigma_1-\sigma_2}{2}$	$\tau_{13}=\frac{\sigma_1-\sigma_3}{2}$	$\tau_{23}=\frac{\sigma_2-\sigma_3}{2}$	τ_8

2.5 From Single-Shear Element to Twin-Shear Element

The cubic element (σ_1, σ_2, σ_3), i.e. the principal stress element, is commonly used. The three principal stresses σ_1, σ_2, σ_3 act on this element, as shown at the top of Fig. 2.8. According to the concept of stress state, various polyhedral elements can be drawn.

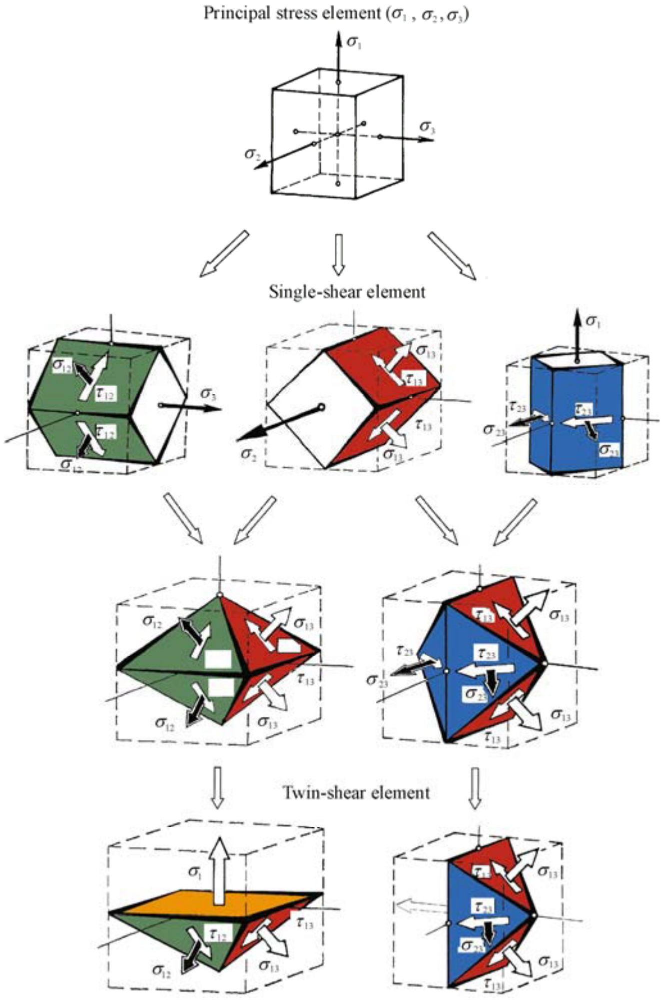

Fig. 2.8 From single-shear element to twin-shear element

Three quadrangular prism elements can be obtained from the cubic element, as shown in Fig. 2.8. The single shear stresses, τ_{13}, τ_{12}, or τ_{23}, act respectively.This may be referred to as the single-shear element. The first single-shear element is the maximum shear stress element in which the maximum shear stress τ_{13} and respective normal stress σ_{13}, as well as the intermediate principal stress σ_2, act on this element. Another single-shear element is the quadrangular prism element (τ_{12}, σ_{12}, σ_3 when $\tau_{12} \geq \tau_{23}$). The intermediate principal shear stress element, the intermediate principal shear stress τ_{12} and the respective normal stress σ_{12}, as well as the minimum principal stress σ_3 act on this element. Other one is the quadrangular prism element (τ_{23}, σ_{23}, σ_1, when $\tau_{12} \geq \tau_{23}$). The minimum principal shear stress element, the minimum principal shear stress τ_{23} and the respective normal stress σ_{23}, as well as the maximum principal stress σ_1 act on this element.

Two orthogonal octahedron elements (τ_{13}, τ_{12}; σ_{13}, σ_{12}) and (τ_{13}, τ_{23}; σ_{13}, σ_{23}) can be obtained from the single-shear element, as shown in Fig. 2.8 This may be referred to as the twin-shear element. The principal shear stresses τ_{13}, τ_{12} and the respective normal stresses σ_{13}, σ_{12} act on the first twin-shear element. The principal shear stresses τ_{13}, τ_{23} and the respective normal stresses σ_{13}, σ_{23} act on the second twin-shear element. These two twin-shear elements form a spatial equipartition in continuum mechanics.

2.6 Stress Space

The stress point $P(\sigma_1, \sigma_2, \sigma_3)$ in stress space can be expressed in other forms, such as $P(x, y, z)$, $P(r, \theta, \xi)$ or $P(J_2, \theta, \xi)$. The geometrical representation of these transfers can be seen in Fig. 2.9.

For the straight line OZ passing through the origin and making the same angle with each of the coordinate axes, the equation is

$$\sigma_1 = \sigma_2 = \sigma_3 \tag{2.20}$$

The equation for the π_0-plane is

$$\sigma_1 + \sigma_2 + \sigma_3 = 0 \tag{2.21}$$

The stress tensor σ_{ij} can be divided into the spherical stress tensor and deviatoric stress tensor. The stress vector σ can also be divided into two parts, the hydrostatic stress vector σ_m and the mean shear stress vector τ_m.

$$\sigma = \sigma_m + \tau_m \tag{2.22}$$

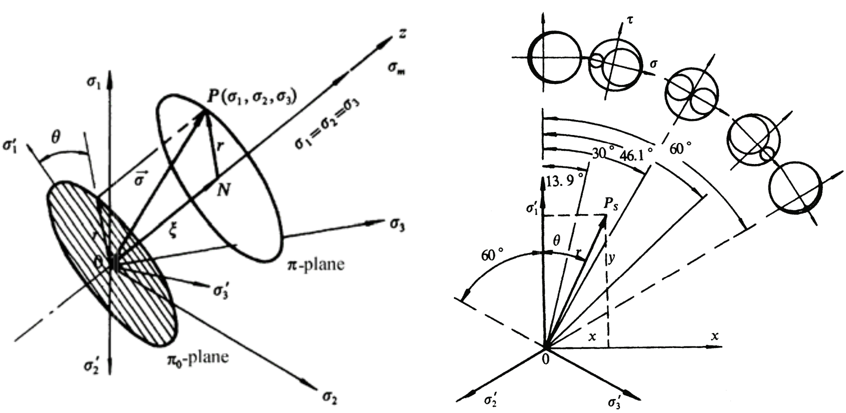

Fig. 2.9 Cylindrical coordinates and stress state in the π-plane

Their magnitudes are given by

$$\xi = \frac{1}{\sqrt{3}}(\sigma_1 + \sigma_2 + \sigma_3) \tag{2.23}$$

$$r = \sqrt{\frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} \tag{2.24}$$

$$\tau_m = \sqrt{\frac{\tau^2{}_{13} + \tau^2{}_{12} + \tau^2{}_{23}}{3}} = \sqrt{\frac{1}{12}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} \tag{2.25}$$

The π-plane is parallel to the π_0-plane and is given by

$$\sigma_1 + \sigma_2 + \sigma_3 = C \tag{2.26}$$

where C is a constant. The spherical stress tensor σ_m is the same for all points in the π-plane of the stress space and

$$\sigma_m = \frac{C}{3} \tag{2.27}$$

The projections of the three principal stress axes σ_1, σ_2, σ_3 in the stress space are σ_1', σ_2', σ_3'. The relationship between them is

$$\sigma_1' = \sigma_1 \cos\beta = \sqrt{\frac{2}{3}}\sigma_1, \sigma_2' = \sigma_2 \cos\beta = \sqrt{\frac{2}{3}}\sigma_2, \sigma_3' = \sigma_3 \cos\beta = \sqrt{\frac{2}{3}}\sigma_3 \tag{2.28}$$

where β is the angle between $O'A$, $O'B$, $O'C$ and the three coordinates, as shown in

Fig. 2.10.

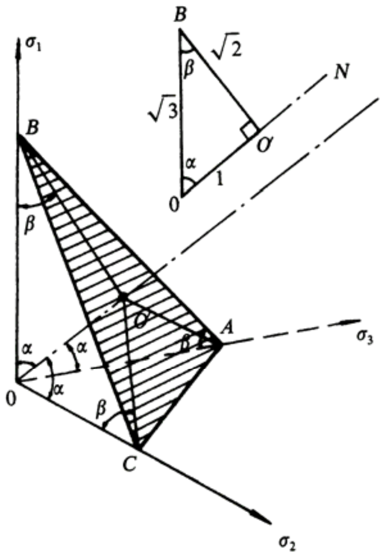

Fig. 2.10 Deviatoric plane

In the following, we will introduce Relationship between (σ_1, σ_2, σ_3) and (x, y, z) The relationship between the coordinates of the deviatoric plane and the principal stresses are

$$x=\frac{1}{\sqrt{2}}(\sigma_3-\sigma_2),\ y=\frac{1}{\sqrt{6}}(2\sigma_1-\sigma_2-\sigma_3),\ z=\frac{1}{\sqrt{3}}(\sigma_1+\sigma_2+\sigma_3) \tag{2.29}$$

$$\sigma_1=\frac{1}{3}(\sqrt{6}y+\sqrt{3}z), \tag{2.30}$$

$$\sigma_2=\frac{1}{6}(2\sqrt{3}z-\sqrt{6}y-3\sqrt{2}x)$$

$$\sigma_3=\frac{1}{6}(3\sqrt{2}x-\sqrt{6}y+2\sqrt{3}z)$$

The relationship between the cylindrical coordinates (ξ, r, θ) and the principal stresses (σ_1,σ_2,σ_3) are

$$\xi=|ON|=\frac{1}{\sqrt{3}}(\sigma_1+\sigma_2+\sigma_3)=\frac{I_1}{3}=\sqrt{3}\sigma_m \tag{2.31}$$

$$\begin{aligned} r=|NP|&=\frac{1}{\sqrt{3}}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]^{\frac{1}{2}} \\ &=\left(S_1^2+S_2^2+S_3^2\right)^{\frac{1}{2}}=\sqrt{2J_2}=\sqrt{3}\tau_8=2\tau_m \end{aligned} \tag{2.32}$$

$$\theta = \tan^{-1}\left(\frac{x}{y}\right) \tag{2.33}$$

From Eq. (2.25) and Eq. (2.28) we can obtain

$$\cos\theta = \frac{y}{r} = \frac{\sqrt{6}S_1}{\sqrt{2J_2}} = \frac{\sqrt{3}}{2}\frac{S_1}{\sqrt{J_2}} = \frac{2\sigma_1 - \sigma_2 - \sigma_3}{2\sqrt{3}\sqrt{J_2}} \tag{2.34}$$

The second and third invariants of the deviatoric stress tensor are

$$J_2 = -(S_1S_2 + S_2S_3 + S_3S_1) \tag{2.35}$$

$$J_3 = S_1S_2S_3 \tag{2.36}$$

Three principal deviatoric stresses can be deduced

$$S_1 = \frac{2}{\sqrt{3}}\sqrt{J_2}\cos\theta,\ S_2 = \frac{2}{\sqrt{3}}\sqrt{J_2}\cos\left(\frac{2\pi}{3} - \theta\right),\ \ S_3 = \frac{2}{\sqrt{3}}\sqrt{J_2}\cos\left(\frac{2\pi}{3} + \theta\right) \tag{2.37}$$

These relationships are suitable for the conditions $\sigma_1 \geq \sigma_2 \geq \sigma_3$ and $0 \leq \theta \leq \pi/3$. The limit loci in the π-plane have threefold symmetry, so if the limit loci in the range of 60° are given, the limit loci in the π-plane can be obtained.

The three principal stresses can be expressed as

$$\begin{aligned} \sigma_1 &= \frac{1}{\sqrt{3}}\xi + \sqrt{\frac{2}{3}}r\cos\theta \\ \sigma_2 &= \frac{1}{\sqrt{3}}\xi + \sqrt{\frac{2}{3}}r\cos(\theta - 2\pi/3) \\ \sigma_3 &= \frac{1}{\sqrt{3}}\xi + \sqrt{\frac{2}{3}}r\cos(\theta + 2\pi/3) \end{aligned} \tag{2.38}$$

The principal stresses can also be expressed in terms of the first invariant I_1 of the stress tensor and the second invariant of the deviatoric stress J_2, as

$$\begin{aligned} \sigma_1 &= \frac{I_1}{3} + \frac{2}{\sqrt{3}}\sqrt{J_2}\cos\theta \\ \sigma_2 &= \frac{I_1}{3} + \frac{2}{\sqrt{3}}\sqrt{J_2}\cos(\theta - \frac{2\pi}{3}) \end{aligned} \tag{2.39}$$

$$\sigma_3 = \frac{I_1}{3} + \frac{2}{\sqrt{3}}\sqrt{J_2}\cos(\theta + \frac{2\pi}{3})$$

The principal shear stresses can also be obtained

$$\begin{aligned} \tau_{13} &= \sqrt{J_2}\sin\left(\theta + \frac{\pi}{3}\right) = \sqrt{2}\tau_{\mathrm{m}}\sin\left(\theta + \frac{\pi}{3}\right) \\ \tau_{12} &= \sqrt{J_2}\sin\left(\frac{\pi}{3} - \theta\right), \quad \tau_{23} = \sqrt{J_2}\sin(\theta) \end{aligned} \tag{2.40}$$

2.7 Stress State Parameters

The stress state at a point (element) is determined by the combination of the three principal stresses (σ_1, σ_2, σ_3). Based on the characteristics of the stress state and by introducing a certain parameter, it can be divided into several types. Lode (1926) introduced a stress parameter μ_σ as

$$\mu_\sigma = (2\sigma_2 - \sigma_1 - \sigma_3)/(\sigma_1 - \sigma_3) \tag{2.41}$$

μ_σ is referred to as the Lode stress parameter. The Lode parameter can be expressed in terms of principal shear stress as

$$\mu_\sigma = \frac{2\sigma_2 - \sigma_1 - \sigma_3}{\sigma_1 - \sigma_3} = \frac{\tau_{23} - \tau_{12}}{\tau_{13}} \tag{2.42}$$

In fact, there are three principal shear stresses τ_{13}, τ_{12} and τ_{23} in the three-dimensional principal stress state. However, the three principal shear stresses τ_{13}, τ_{12} and τ_{23} are not independent and only two principal shear stresses are dependent variables, because the maximum principal shear stress τ_{13} is equal to the sum of the other two shear stresses. This relationship is expressed as

$$\tau_{13} = \tau_{12} + \tau_{23} \tag{2.43}$$

Hence, the twin-shear idea was proposed by Yu (1961). The twin-shear function can be established as (Yu, 1983; Yu and He, 1983; 1985)

$$f = \tau_{13} + \tau_{12} \tag{2.44}$$

$$f' = \tau_{13} + \tau_{23} \tag{2.45}$$

Subsequently, Yu (1991; 1992) introduced the “twin shear stress” concept into the analysis of the stress state and offered two twin-shear stress parameters as

$$\mu_\tau = \frac{\tau_{12}}{\tau_{13}} = \frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_3} = \frac{S_1 - S_2}{S_1 - S_3} \tag{2.46a}$$

$$\mu'_\tau = \frac{\tau_{23}}{\tau_{13}} = \frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3} = \frac{S_2 - S_3}{S_1 - S_3} \tag{2.46b}$$

$$\mu_\tau + \mu'_\tau = 1, \quad 0 \le \mu_\tau \le 1, \ 0 \le \mu'_\tau \le 1 \tag{2.46c}$$

The twin-shear stress parameters are simpler and have an explicit physical meaning. They can reflect the state of the intermediate principal stress and can represent the status of the stress state.

The twin-shear stress parameters have nothing to do with the hydrostatic stress. They instead represent the status of the deviatoric stress state and the stress angle on the deviatoric plane in the stress space, as shown in Fig. 2.12. Five different stress states are shown in Fig. 2.12. They are

1) θ=0° (μ_τ=1);
2) θ=13.9° (μ_τ=3/4, μ_τ'=3/);
3) θ=30° (μ_τ=μ_τ'=0.5);
4) θ=46.1° (μ_τ=1/4, μ_τ'=3/4) ;
5) θ=60° (μ_τ=0, μ_τ'=1).

According to the meaning of the twin-shear stress parameters, we know that if μ_τ=1 (μ_τ'=0, stress angle equals θ=0°), the stress states include the three following cases:

1) σ_1>0, σ_2=σ_3=0, uniaxial tension stress state;
2) σ_1=0, σ_2=σ_3<0, equal biaxial compression stress state;
3) σ_1>0, σ_2=σ_3<0, uniaxial tension, equal biaxial compression stress state.

If μ_τ=μ_τ'=0.5 (stress angle equals θ=30°), the corresponding stress states are as follows:

1) σ_2=(σ_1+σ_3)/2=0, pure shear stress state;
2) σ_2=(σ_1+σ_3)/2>0, biaxial tension and uniaxial compression stress state;
3) σ_2=(σ_1+σ_3)/2<0, uniaxial tension and biaxial compression stress state.

If μ_τ=0 (μ_τ'=1, stress angle equals θ=60°), then the corresponding stress states are as follows:

1) σ_1=σ_2=0, σ_3<0, uniaxial compression stress state;
2) σ_1=σ_2>0, σ_3=0, equal biaxial tension stress state;
3) σ_1=σ_2>0, σ_3<0, equal biaxial tension and uniaxial compression stress state.

According to the twin-shear stress parameters and the magnitude of the two smaller principal shear stresses, the stress state can be divided into three kinds of conditions as follows:

1) Extended tension stress state, τ_{12} >τ_{23}, 0≤μ_τ'<0.5<μ_τ≤1. The stress state (uniaxial tension and biaxial compression) can be expressed by deviatoric stress, and the absolute magnitude of the tensile stress is a maximum, so it can be called the

extended tension stress state. When the intermediate principal stress σ_2 equals the minimum principal stress σ_3, then $\mu_\tau=1$ ($\mu_\tau'=0$). If $\sigma_2=\sigma_3=0$, the extended tension stress state becomes the uniaxial tension stress state.

2) Extended shear stress state, $\tau_{12}=\tau_{23}$, $\sigma_2=(\sigma_1+\sigma_3)/2$. The two smaller stress circulars are equal, the second deviatoric stress $S_2=0$ and the magnitude of the other two deviatoric stresses are identical, but one is tensile and the other is compressive. The two twin-shear stress parameters are identical, $\mu_\tau=\mu_\tau'=0.5$. If $\sigma_2=(\sigma_1+\sigma_3)/2=0$, the extended shear stress state becomes the pure shear stress state.

3) Extended compression stress state, $\tau_{12}<\tau_{23}$, $0\le\mu_\tau<0.5<\mu_\tau'\le1$. If $\sigma_1=\sigma_2=0$, $\sigma_3<0$, this stress state becomes the uniaxial compression stress state.

The twin-shear parameters simplify the Lode parameter and have a clear physical meaning. Their relationships are

$$\mu_\tau=\frac{1-\mu_\sigma}{2}=1-\mu'_\tau\,,\quad \mu'_\tau=\frac{1+\mu_\sigma}{2}=1-\mu_\tau \tag{2.47}$$

Some types of stress states and stress state parameters including the Lode parameter and the twin-shear stress parameters are summarized in Table 2.2.

Table 2.2 Principal stresses, shear stresses and stress state parameters

Stress state		Principal stress	Principal shear stress	Deviatoric stress	Stress angle	Parameter of stress state		
						μ_τ	μ'_τ	μ_σ
Extended tension	Pure tension, Equal Biaxial compression	$\sigma_2=\sigma_3$	$\tau_{12}=\tau_{13}$ $\tau_{23}=0$	$S_2=S_3$ $S_1=S_2+S_3$	0°	1	0	−1
	$\tau_{23}=\frac{\tau_{12}}{3}$, $\tau_{13}=4\tau_{23}$	$\sigma_2<\frac{\sigma_1+\sigma_3}{2}$	$\tau_{12}>\tau_{23}$	$S_1=S_2+S_3$	13.9°	$\frac{3}{4}$	$\frac{1}{4}$	$\frac{-1}{2}$
Pure shear		$\sigma_2=\frac{\sigma_1+\sigma_3}{2}$	$\tau_{12}=\tau_{23}$	$S_1=\lvert S_3\rvert$ $S_2=0$	30°	0.5	0.5	0
Extended compression	$\tau_{12}=\frac{\tau_{23}}{3}$, $\tau_{13}=4\tau_{12}$	$\sigma_2>\frac{\sigma_1+\sigma_3}{2}$	$\tau_{12}<\tau_{23}$	$\lvert S_3\rvert=S_1+S_2$	46.1°	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{2}$
	Pure compression equal biaxial compression	$\sigma_2=\sigma_1$	$\tau_{12}=0$ $\tau_{23}=\tau_{13}$	$S_1=S_2$ $\lvert S_3\rvert=S_1+S_2$	60°	0	1	+1

The relationships between various shear stresses are listed in Table 2.3. It is convenient to compare various textbooks on plasticity. Different symbols or expressions may be used on different courses.

Table 2.3 Relationships between various shear stresses and J_2

	q_τ	τ_8	τ_s	$\tau_\pi = r$	J_2	S_{ij}
Generalized shear stress q_τ	q_τ	$\frac{3}{\sqrt{2}}\tau_8$	$\sqrt{3}\tau_s$	$\sqrt{\frac{3}{2}}\tau_\pi$	$\sqrt{3J_2}$	$\sqrt{\frac{3}{2}S_{ij}S_{ij}}$
Octrahedral shear stress τ_8	$\frac{\sqrt{2}}{3}q_\tau$	τ_8	$\sqrt{\frac{2}{3}}\tau_s$	$\frac{1}{\sqrt{3}}\tau_\pi$	$\sqrt{\frac{2}{3}J_2}$	$\sqrt{\frac{1}{3}S_{ij}S_{ij}}$
Pure shear stress τ_s	$\frac{1}{\sqrt{3}}q_\tau$	$\sqrt{\frac{3}{2}}\tau_8$	τ_s	$\frac{1}{\sqrt{2}}\tau_\pi$	$\sqrt{J_2}$	$\sqrt{\frac{1}{2}S_{ij}S_{ij}}$
Shear stress on deviatoric plane $\tau_\pi=r$	$\sqrt{\frac{2}{3}}q_\tau$	$\sqrt{3}\tau_8$	$\sqrt{2}\tau_s$	τ_π	$\sqrt{2J_2}$	$\sqrt{S_{ij}S_{ij}}$
Second invariant J_2 of deviatoric stress	$\frac{1}{3}q_\tau^2$	$\frac{3}{2}\tau_8^2$	τ_s^2	$\frac{1}{2}\tau_\pi^2$	J_2	$\frac{1}{2}S_{ij}S_{ij}$

2.8 Strain Components

When a continuum is deformed, a generic point experiences a displacement $\{U\}$ with components u, v, w, with respect to Cartesian orthogonal axes x, y, z, respectively. For very small strains, the axial strains ε_x, ε_y, ε_z and shear strains γ_{xy}, γ_{yz}, γ_{zx} can be expressed by the displacement differentiation as follows

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z} \tag{2.48}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{2.49}$$

The six strain components ε_x, ε_y, ε_z, γ_{xy}, γ_{yz}, γ_{zx} can describe completely the state of strain at the considered point. Similar to the stress tensor, there also exist principal strains ε_1, ε_2, ε_3 with the companion shear strains equal to zero. For a plane strain state, the third principal strain ε_3 vanishes and the principal strains can be expressed as follows

$$\begin{aligned} \varepsilon_1 &= \frac{\varepsilon_x + \varepsilon_y}{2} + \sqrt{\frac{1}{4}(\varepsilon_x - \varepsilon_y)^2 + (\gamma_{xy}/2)^2} \\ \varepsilon_2 &= \frac{\varepsilon_x + \varepsilon_y}{2} - \sqrt{\frac{1}{4}(\varepsilon_x - \varepsilon_y)^2 + (\gamma_{xy}/2)^2} \end{aligned} \tag{2.50}$$

The principal direction is given by

$$\tan 2\alpha = -\frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} \tag{2.51}$$

Equation (2.51) still holds when $\varepsilon_z \equiv \varepsilon_3 \neq 0$, provided ε_z is a principal strain.

2.9 Equations of Equilibrium

The following three differential equations of equilibrium in the direction of the coordinate axes are

$$\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0 \\
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0 \\
&\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0
\end{aligned} \tag{2.52}$$

where X, Y, Z are the components of the body force per unit volume. For a body in an equilibrium state, the variation in stresses is governed by the above equations of equilibrium.

2.10 Generalized Hooke's Law

Equations relating to stress, strain, stress-rate (increase in stress per unit time) and strain-rate are called constitutive equations, which are determined by the material properties under consideration. In the case of elastic solids, the constitutive equations take the form of the generalized Hooke's law, which involves stress and strain instead of the stress-rate and strain-rate.

In a general three-dimensional stress state, the generalized Hooke's law has the form of

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \tag{2.53a}$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \tag{2.53b}$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \tag{2.53c}$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}, \quad \gamma_{yz} = \frac{1}{G}\tau_{yz}, \quad \gamma_{xz} = \frac{1}{G}\tau_{xz} \tag{2.53d}$$

where E and ν are the modulus of elasticity and the Poisson's ratio, respectively. G is the modulus of rigidity. Only two of them are independent and there is

$$G = \frac{E}{2(1+\nu)}$$

Equations (2.53a)–(2.53d) may be rewritten conversely as

$$\begin{aligned}
\sigma_x &= 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) \\
\sigma_y &= 2G\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) \\
\sigma_z &= 2G\varepsilon_z + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) \\
\tau_{xy} &= G\gamma_{xy}, \quad \tau_{yz} = G\gamma_{yz}, \quad \tau_{zx} = G\gamma_{zx}
\end{aligned} \tag{2.54}$$

where the constants G and λ are called Lame's constants and

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{2.55}$$

Another important elastic constant is called the bulk modulus of elasticity K, which defines the dilatation (volumetric strain) ε_v as the unit change in volume

$$\varepsilon_v = \varepsilon_x + \varepsilon_y + \varepsilon_z \tag{2.56}$$

with the hydrostatic component of stress, or spherical component of stress σ_m,

$$\sigma_m = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) \tag{2.57}$$

As such

$$\varepsilon_v = \frac{1}{K}\sigma_m \tag{2.58}$$

From the generalized Hooke's law, K is derived as

$$K = \frac{E}{[3(1-2\nu)]} \tag{2.59}$$

2.11 Compatibility Equations

Equations (2.48) and (2.49) implicitly show that the strain components are functions of the three displacement components. Differentiate the first equation within Eq. (2.48) twice with respect to y and the second equation within Eq. (2.48) with respect to x and add the results,

$$\frac{\partial \varepsilon_x^2}{\partial y^2} + \frac{\partial \varepsilon_y^2}{\partial x^2} = \frac{\partial^3 u}{\partial y^2 \partial x} + \frac{\partial^3 v}{\partial x^2 \partial y} \tag{2.60}$$

Differentiating the first equation within Eq. (2.20) with respect to x and y yields

$$\frac{\partial \gamma_{xy}^2}{\partial x \partial y} = \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{2.61}$$

And since the order of differentiation for single-value, continuous functions is immaterial

$$\frac{\partial \varepsilon_x^2}{\partial y^2} + \frac{\partial \varepsilon_y^2}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Similarly, we can derive the following additional equations

$$\frac{\partial \varepsilon_x^2}{\partial y^2} + \frac{\partial \varepsilon_y^2}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{2.62a}$$

$$\frac{\partial \varepsilon_y^2}{\partial z^2} + \frac{\partial \varepsilon_z^2}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \tag{2.62b}$$

$$\frac{\partial \varepsilon_z^2}{\partial x^2} + \frac{\partial \varepsilon_x^2}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \tag{2.62c}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \tag{2.62d}$$

$$2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \tag{2.62e}$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) \tag{2.62f}$$

Equations (2.62a)–(2.62f) are called Saint-Venant compatibility equations, or compatibility equations in terms of strain.

In total, there are fifteen governing equations, including three equilibrium equations (Eqs. (2.52)), six strain displacement relations (Eq. (2.48) and Eq. (2.49)), and six stress-strain relations (Eqs. (2.53)) for solving the fifteen variables (the six stress components σ_x, σ_y, σ_z, τ_{xy}, τ_{yz} and τ_{xz}, the six strain components ε_x, ε_y, ε_z, γ_{xy}, γ_{yz} and γ_{xz}, and the three displacements u, v, w). The compatibility equations are derived from the strain-displacement equations and, therefore, cannot be counted as governing equations. The compatibility equations will be satisfied automatically if the fifteen governing equations are satisfied.

2.12 Governing Equations for Plane Stress Problems

For plane stress problems, the stress components are simplified as

$$\sigma_x = \sigma_x(x, y), \quad \sigma_y = \sigma_y(x, y), \quad \tau_{xy} = \tau_{xy}(x, y) \tag{2.63a}$$

$$\tau_{xz} = \tau_{yz} = \sigma_z = 0 \tag{2.63b}$$

The equilibrium equations become

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \tag{2.64a}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0 \tag{2.64b}$$

where the body forces X and Y are functions of x and y only, and Z equals zero. The strain-stress relations take the form of

$$\varepsilon_x = \varepsilon_x(x, y) = \frac{1}{E}[\sigma_x - \nu\sigma_y] \tag{2.65a}$$

$$\varepsilon_y = \varepsilon_y(x, y) = \frac{1}{E}[\sigma_y - \nu\sigma_x)] \tag{2.65b}$$

$$\varepsilon_z = \varepsilon_z(x, y) = \frac{1}{E}[-\nu(\sigma_x + \sigma_y)] \tag{2.65c}$$

$$\gamma_{xy} = \gamma_{xy}(x, y) = \frac{1}{G}\tau_{xy} \tag{2.65d}$$

The two shear strains γ_{xz} and γ_{yz} and the normal strain ε_z vanish. Finally, the strain-displacement relations are simplified as

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{2.66}$$

There are eight equations in total to correlate the eight unknown quantities of σ_x, σ_y, τ_{xy}, ε_x, ε_y, γ_{xy}, u and v. Again, the governing equations can only be solved with specific stress and displacement boundary conditions.

2.13 Governing Equations in Polar Coordinates

For analysis of a circular ring and plate, rotating disk, curved bars of a narrow rectangular cross section with a circular axis, etc., it is advantageous to use polar coordinates. If the external forces are also rotationally symmetric, the stress state can be assumed to be the plane stress independent of the z-axis which is perpendicular to the polar coordinates plane. The position of a point in the middle plane of a plate is then defined by the distance from the origin O and the angle θ between r and a certain axis O_x fixed in the plane. Denoting σ_r and σ_θ as the normal stress components in the radial and circumferential directions, respectively, and $\tau_{r\theta}$ as the shear stress component, the equation of equilibrium takes the form of

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0 \tag{2.67a}$$

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + S = 0 \tag{2.67b}$$

where R and S are the components of body force per unit volume in the radial and tangential directions, respectively.

The corresponding stress components are derived as

$$\sigma_r = \frac{A}{r^2} + B(1 + 2\log r) + 2C \tag{2.68a}$$

$$\sigma_\theta = -\frac{A}{r^2} + B(3 + 2\log r) + 2C \tag{2.68b}$$

$$\tau_{r\theta} = 0 \tag{2.68c}$$

A, B and C are constants that can be determined by boundary conditions.

Denoting the displacements in the radial and tangential directions as u_r and u_θ, respectively, the strain components in the polar coordinates are derived as

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\theta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \quad \text{and} \quad \gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \tag{2.69}$$

The generalized Hooke's law is then expressed by

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) \tag{2.70a}$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) \tag{2.70b}$$

$$\gamma_{r\theta} = \frac{1}{G}\tau_{r\theta} \tag{2.70c}$$

Thus, based on the equilibrium equations, the strain-displacement relations, the compatibility equations and Hooke's law plus relative boundary conditions, the stress and displacement fields of the rotational symmetrical body can be solved. Detailed derivations can be referred to in *Theory of Elasticity* by Timoshenko and Goodier (1970) and in *Elasticity: Tensor, Dynamic and Engineering Approaches* by Chou and Pagano (1967).

2.14 Brief Summary

This chapter presents the fundamentals of solid mechanics. Some basic concepts with respect to the stress tensors, stress tensor invariants, deviatoric stress tensors, deviatoric stress tensor invariants, octahedral shear and normal stresses, principal stresses and principal shear stresses, strain components, and some new concepts regarding twin-shear stresses, the twin-shear element and the twin-shear stress parameter are introduced. They are used in the following chapters.

Stress states can be studied on many courses, such as elasticity, plasticity, mechanics of solids, rock mechanics and soil mechanics. The basic formulae are only given here.

The relationships between various shear stresses and J_2 are listed in Table 2.3. Different notations may be used in different textbooks. It is useful to refer to other textbooks.

Governing equations for general stress state solids, plane stress solids and rotationally symmetrical solids are given.

It should be mentioned that only the governing equations in the elastic range of solids are considered. Based on the elastic solutions, by adopting proper yield criterion, the elastic limit load of the solid body can be derived. For elasto-plastic analysis and plastic limit analysis, a yield criterion and a relevant flow law should be applied. The following two chapters will introduce conventional yield criteria and a unified strength theory developed by Yu (1991; 1992; 2004).

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