1 Introduction

1.1 Elasto-Plastic Finite Elements

The finite element method (FEM) has now become recognized as a general method with wide applications in engineering and applied mechanics. FEM was originally developed in the field of structural analysis. All the problems were linear in the sense that they involve the solution of sets of linear algebraic equations. It is the linear elastic FEM.

The elasto-plastic FEM, non-linear material problems or computational plasticity, has also been widely accepted, and many excellent books on computational plasticity have been written. Overviews and analysis can be found in Zienkiewicz (1971; 1989), Cook et al. (1989), Reddy (1993), Bathe (1996), Han and Reddy (1999), Belytschko et al. (2000), Smith and Griffiths (2004), Reddy (2009), Anandarajah (2010), etc. The theories and implementation of the plasticity FEM are described by Oden (1972), Hinton and Owen (1979), Owen and Hinton (1980), Miyoshi (1985), Kobayashi et al. (1989), Strin (1993), Pan (1995), Crisfield (1997), Bonet and Wood (1997), Simo and Hughes (1998), Belytschko et al. (2000), Smith and Griffiths (2004), Kojic and Bathe (2005) and Neto et al. (2009). *Plasticity and Geotechnics* was written by HS Yu (2006). *Lecture Notes on Computational Geomechanics: Inelastic Finite Elements for Pressure Sensitive Materials* was presented by Jeremić, et al. (2010). A detailed introduction to the plasticity FEM program (2D) can be found in Owen and Hinton (1980) and the proceedings of Owen et al. (1989), and in (Neto et al., 2009), in which a computer program of approximately 11,000 lines of FORTRAN codes is given. The 2D non-linear thermo-elastoplastic consolidation program PLASCON was given by Lewis and Schrefler (1987). The Mohr-Coulomb Theory and the Critical State Models (Roscoe et al., 1963, 1968; Schofield and Wroth, 1968) were used. The theory and implementation of nonlinear analysis in soil mechanics was described by Chen and

M.-H. Yu et al., *Computational Plasticity*

Mizuno (1990). The Drucker-Prager criterion and cap models were implemented for studying soil mechanics problems.

Advances in computational nonlinear mechanics before 1999 were described and edited by Wunderlich et al. (1981), Doltsinis (1989), Smith (1990) and Inoue et al. (1990), Desai and Gioda (1990), Ehlers (1999). Computational plasticity applied to metal forming can be found in Kobayashi et al. (1989), Khoei (2005), Dixit and Dixit (2008). Concrete plasticity and finite element analysis for limit-state design of concrete structures can be found in Chen (1982), Nielsen (1991), Kotsovos and Pavlovic (1995). A textbook on the combination of plasticity and geomechanics was written by Davis and Selvadural (2002). Materials including metals, soils and others are idealized as a continuum. Most engineering theories of metals and soil behavior of practical interest have depended on the continuum assumption, as indicated by Davis and Selvadural (2002). Ten serial International Conferences on Computational Plasticity (COMPLAS) have been successfully held in Barcelona, Spain since 1987. *Computational Plasticity* was published (Oñate and Owen, 2007), which contains 14 invited contributions written by distinguished authors who participated in the VIII International Conference on Computational Plasticity.

Nonlinearities can be introduced by either geometric or material property effects. Geometric nonlinearities often arise in problems involving solid media in which the strains are sufficiently large to significantly affect the shape of the solution domain. Material nonlinearities include elasto-plastic deformation characterized by an irreversible straining which can only be sustained once a certain level of stress, known as the yield limit, yield function, strength theory or material model, has been reached. Material nonlinearities also include nonlinearly elastic solids, whose properties are functions of the local state of deformation. Elasto-plastic nonlinearities are studied and applied widely in mechanics and engineering. The nonlinear elasto-plastic material model is of great importance for computational plasticity.

Elasto-plastic programs have been used for many years in the world. Material models are usually implemented in terms of the Tresca criterion and Huber-von Mises yield criterion for metallic materials and the Mohr-Coulomb criterion or the Drucker-Prager criterion for geomaterials. These material models (the single model) are suited to one kind of material. A new material model, the unified strength theory (UST), is implemented in computer codes and used for computational plasticity in this book.

The material parameters of the unified strength theory (UST) are the same as the material parameters of the Mohr-Coulomb theory and the Drucker-Prager criterion. Most parts of the computer codes are also the same as the other elasto-plastic computer codes, only the yield criteria and its associated flow rule are different. the result obtained by using the Mohr-Coulomb theory is a special case of the result using UST. The two results are identical. More results, however, can be obtained by using UST.

Unified strength theory has been applied in many research and engineering fields. UST and its implementation can be reliably employed therefore in engineering and R & D applications. UST can be adapted for more materials and structures. It has provided more choices for researchers and engineers.

1.2 Bounds and Region of the Convex Yield Surface

As a matter of primary importance, the bounds and the region of the failure criteria have to be determined before research is started on the effect of failure criteria. There are hundreds of yield and failure criteria that can be seen (Yu, 2002, 2004). Various yield criteria and failure criteria have been proposed in the past; however, all of them must be situated between the bounds if the convexity-is considered. The lower bound is the single-shear strength theory (the Mohr-Coulomb strength theory) and the single-shear yield criterion (the Tresca yield criterion or the maximum shear stress criterion), as shown in Figs. 1.1 and 1.2. The upper bound is the twin-shear strength theory (Yu et al., 1985) for SD materials (strength difference of material in tension and in compression), as shown in Fig. 1.1, and the twin-shear yield criterion (Yu, 1961a; 1961b; 1983) or the maximum deviatoric stress criterion (Haythornthwaite, 1961) for non-SD materials, respectively (Fig. 1.2). Other yield criteria are situated between these two bounds.

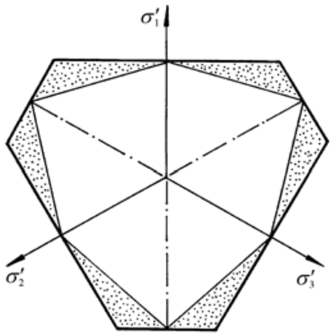


Fig. 1.1 Bounds and region of yield loci for SD materials

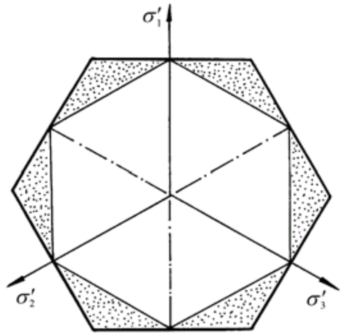
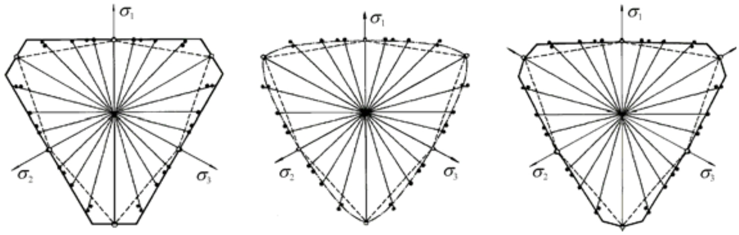


Fig. 1.2 Bounds and region of yield loci for non-SD materials

Most of the experimental results are situated between these two bounds. Figure 1.3(a) shows the experimental result for sand, given by Nakai and Matsuoka (1980). It is in good agreement with the Matsuoka-Nakai criterion, as shown in Fig. 1.3(b). It is interesting that the piece-wise linear criterion is also in very good agreement with this experimental result, as shown in Fig. 1.3(c). The piece-wise linear loci in Fig. 1.3(c) is the unified strength theory with b=3/4.

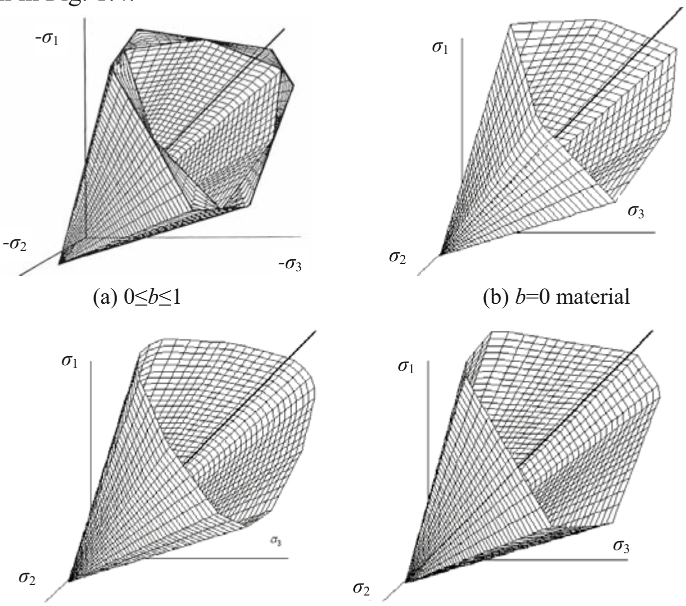


(a) Two bounds (b) Matsuoka-Nakai criterion (c) Unified strength theory with b=3/4

Fig. 1.3 Comparisons of test results with curve criterion and piece-wise linear criterion

1.3 Unified Strength Theory and its Implementation in Computer Codes

A unified strength theory stating that the yield loci covered the entire region from the lower bound to upper bound was proposed by Yu (1991). The details of the unified strength theory can be seen in (Yu, 1992; 2004). It is the natural development of the twin-shear idea and twin-shear yield criterion for non-SD materials (Yu, 1961) and the twin-shear strength theory for SD materials (Yu, 1985). The serial limit surfaces and three special cases in stress space of the unified strength theory are shown in Fig. 1.4.

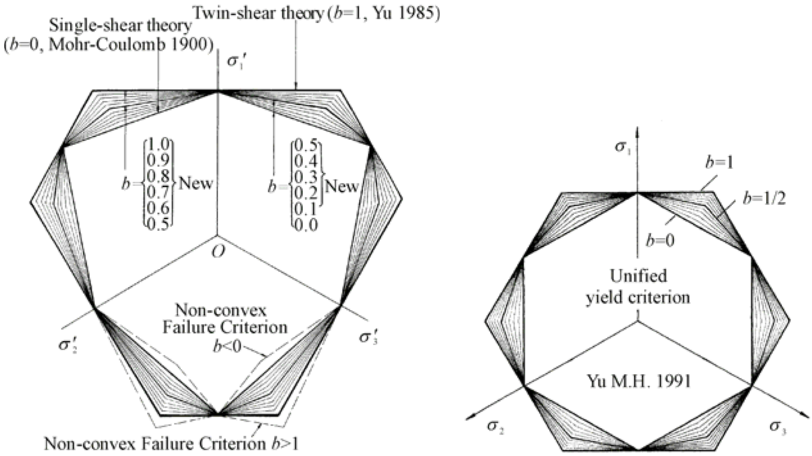


(a) $0 \le b \le 1$ (b) b=0 material

(c) b=1/2 material (d) b=1 material

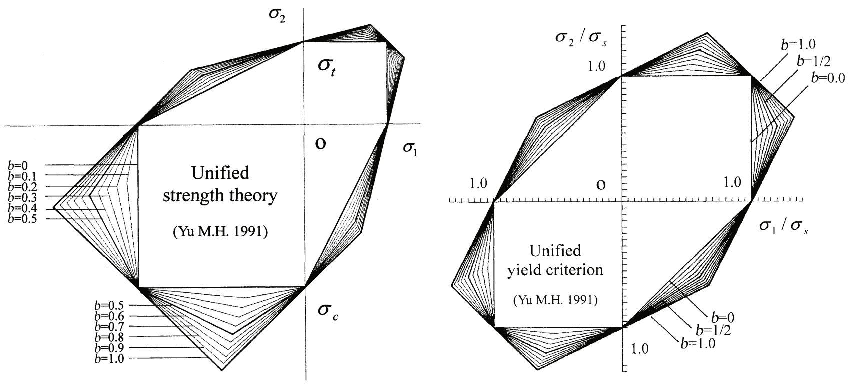
Fig. 1.4 Serial limit surfaces of the unified strength theory in stress space

The serial limit loci in the deviatoric plane of the unified strength theory are shown in Fig. 1.5. The yield criteria of the unified strength theory can be extended to the non-convex criteria, as shown in Fig. 1.5(a). Some well-known yield criteria and a lot of new criteria can be deduced from the unified strength theory, as shown in Fig. 1.5. The serial limit loci in the deviatoric plane of the unified strength theory can regenerate to serial yield loci for non-SD materials, as shown in Fig. 1.5(b). Limit loci for SD materials and yield loci for non-SD materials of the unified strength theory in plane stress state are shown in Fig. 1.6.



(a) Limit loci for SD materials (b) Yield loci for non-SD materials

Fig. 1.5 Serial limit loci of the unified strength theory in deviatoric plane



(a) SD materials (b) Non-SD materials

Fig. 1.6 Serial limit loci of the unified strength theory in plane stress state

A review of "Unified Strength Theory and its Applications. Springer, Berlin, 2004" was written by Teodorescu (2006) in *Zentralblatt MATH*. "Here, starting from the idea of twin-shear and twin-shear yield criterion, the author sets up a twin-shear strength theory and then a unified strength theory, the limit loci of which cover all regions of the convex limit loci and can be extended to the region of non-convex limit loci." As pointed out by Teodorescu (2006), the serial yield criteria of the unified strength theory are piece-wise linear criteria that consist of two expressions, as follows

$$F = \sigma_1 - \frac{\alpha}{1+b}(b\sigma_2 + \sigma_3) = \sigma_t, \qquad \text{when} \quad \sigma_2 \le \frac{\sigma_1 + \alpha\sigma_3}{1+\alpha} \tag{1.1a}$$

$$F' = \frac{1}{1+b}(\sigma_1 + b\sigma_2) - \alpha\sigma_3 = \sigma_t, \qquad \text{when} \quad \sigma_2 \ge \frac{\sigma_1 + \alpha\sigma_3}{1+\alpha} \tag{1.1b}$$

where σ_1, σ_2, σ_3 are three principal stresses; σ_t is tensile yield point of material; α is the ratio of tensile yield point to compressive yield point of material $\alpha=\sigma_t/\sigma_c$. The relations of the serial loci are shown in Fig. 1.7.

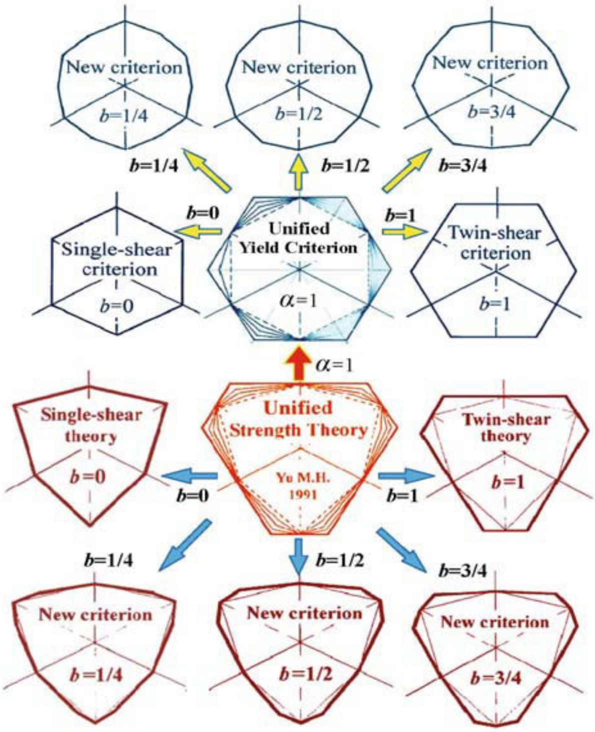


Fig. 1.7 Special cases of the unified strength theory

UST (Unified Strength Theory) has been implemented in some elasto-plastic programs including some commercial FEM codes and applied to engineering problems (Yu, 1992;Yu and Li, 1991; Yu et al., 1992; Yu and Zeng, 1994; Yu et al., 1993; 1998; Yu et al,. 1997; 1999; Fan and Qiang, 2001; Zhang and Loo, 2001; Sun et al., 2004; Li and Ishii, 1998; Zhang et al., 2008; Wang et al., 2008; Li , 2008). The singularities at the corners of single-shear theory, twin-shear theory and the unified strength theory have been overcome by using a unified numerical procedure. A unified elasto-plastic program (UEPP) has been established, which was applied to some engineering problems (Yu and Zeng, 1994; Yu et al., 1997; 1999; Yu, 1998). UST has also been implemented in ABQUSE by Wang JQ in 2008.

The twin-shear strength theory, unified strength theory and its unified elasto-plastic constitutive model are implemented in FLAC-3D by Zhang (2008), Li (2008), Qiao and Li (2010), and Ma (2010). Before unified strength theory, the twin-shear yield criterion (for non-SD materials) and the twin-shear strength theory (for SD materials) were implemented in some FEM codes by An et al. (1991), Yu and Li (1991), Quint (1993; 1994), Shen (1993).

Therefore, unified strength theory that can be used in finite difference computation, has also developed FLAC-3D mechanical analysis serviceability. The confirmation of unified strength theory has been tested, finally demonstrating that this model is very good at taking into account the effect of intermediate principal stress. Based on unified strength theory and its constitutive relationship, as well as finite difference computation developed in FLAC-3D, the stability and protection of the underground caves at the Huanren power plant were calculated. The excavation, the spread of the plastic region around the cave area and the distribution and change in displacement were obtained by Li and Qiao. The effects of the irregular surface, the in-situ stress field's distribution and different constitutive relations concerning stability have been studied.

1.4 The Effect of Yield Criteria on the Numerical Analysis Results

The Tresca yield criterion and the Huber-von Mises criterion were described in most textbooks about metal plasticity and computational plasticity. A great deal of research has been dedicated to showing the effects of failure criteria on the numerical results of load-carrying capacities of structures. A famous example was given by Humpheson and Naylor (1975), and was further studied by Zienkiewicz and Pande (1977). Shapes of loading surfaces of concrete models and their influence on the peak load and failure mode in structural analyses were given by Pivonka et al. (2003). Figure 1.8 shows some differences between the results obtained for plane strain flexible footing on a weightless material (Humpheson and Naylor, 1975). The forms of different limit surfaces on the deviatoric plane are shown in Fig. 1.9.

The influence of different forms of yield surfaces on load-bearing capacity is obvious. The Mohr-Coulomb strength theory, the Williams-Warnke criterion, the Gudehus-Argyris criterion and various circular cone approximations, i.e. extension cone, compromise cone, compression cone and the Drucker-Prager criterion (inscribed cone of the Mohr-Coulomb semi-infinite hexagonal cone with unequal sides) have been used. They show a great difference between results obtained using various failure criteria. Obviously, the question arises as to which one of these results should be preferred, because there is only one reasonable result for a given material and structure.

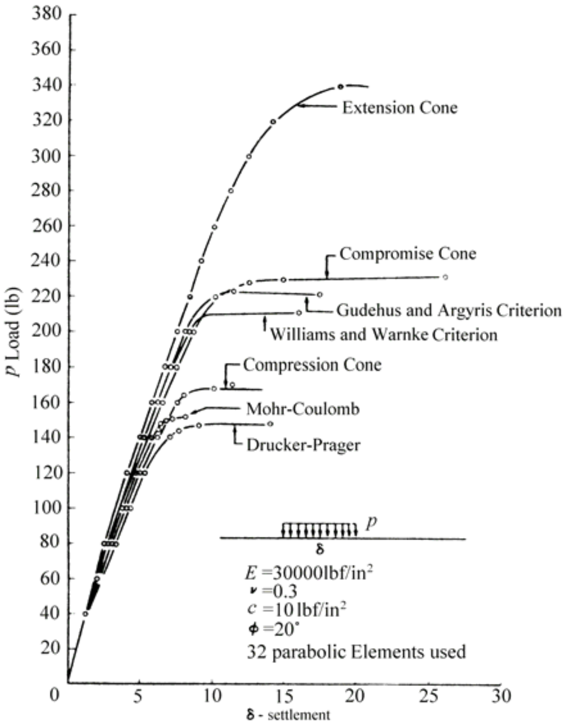


Fig. 1.8 Load-displacement curve

In this example, most of the limit surfaces of different failure criteria are cones in the stress space. The limit loci in the meridian plane are linear. This means that the strength of materials is linearly dependent on the hydrostatic stress, as has been demonstrated in a number of tests. The differences between the limit loci of various failure criteria in the deviatoric plane are shown in Fig. 1.10. Some smooth limit loci of various approximations to the Mohr-Coulomb failure criteria can be presented.

In Fig. 1.10, the limit locus 1 is the Mohr-Coulomb strength theory (1900), locus 2 is the twin-shear strength theory (Yu et al., 1985), locus 3 is the William-Warnke criterion (1975), locus 4 is the twin-shear smooth model (Yu and Liu, 1990a; 1990b) and locus 5 is the Gudehus-Argyris criterion (1973; 1974). Other smooth models can be found in the literature. Most limit loci match the two basic experimental points *a* and *b*. The circular loci cannot be matched with these two basic experimental points, as shown in Fig. 1.9.

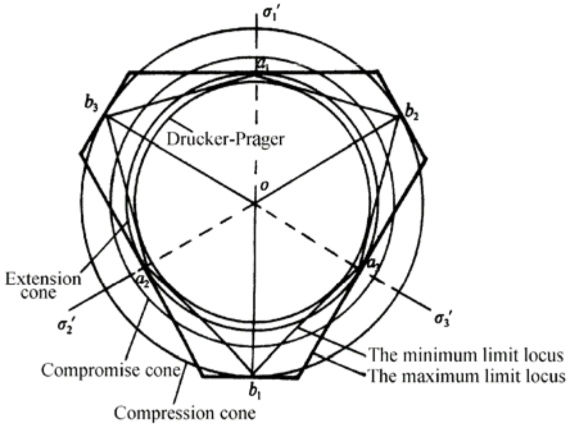
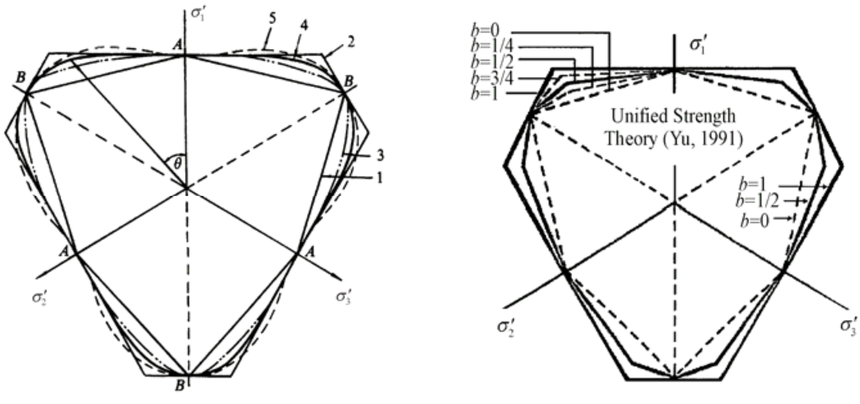


Fig. 1.9 Different limit loci on deviatoric plane

In general, the five typical limit loci of the unified strength theory with b=0, b=0.25, b=0.5, b=0.75 and b=1, which cover all the region of the convex area of the limit loci, can be adapted for different materials. The five typical limit loci of the unified strength theory are shown in Fig. 1.10 (b). Sometimes, the three loci (lower bound, median loci and upper bound shown in Fig. 1.10 (b). are used for analysis of structures.



(a) (b)

Fig. 1.10 The linear criteria and curve criteria

Nayak, Zienkiewicz (1972), Zienkiewicz and Pande (1977) have pointed out that the choice of the best limit surface is still in the hands of the analyst who has modeled the strength behavior in the best possible manner. They also indicated that the Drucker-Prager criterion and the limit loci of extensive circular cones give a very poor approximation to the real failure conditions.

The effect of the yield criterion was studied by Humpheson and Naylor (1975), Zienkiewicz and Pande (1977), Li et al. (1994; 1998), Moin and Pankaj (1998), Wang and Fan (1998), American Institute of Aeronautics and Astronautics (1999), Yu (2004), Scheunemann (2004) and others. The choice of yield criteria has a marked effect on the analytical results of load-bearing capacities of structures, on the prediction of the forming limit diagram (FLD), on deformation, discontinuous bifurcation and localization behavior, and dynamic behavior of structures. This conclusion was also given by Chen and Baladi (1985), Wagoner and Knibloe (1989), Frieman and Pan (2000), Cao et al. (2000), Kuroda and Tvergaard (2000), Wang and Lee (2006), Huang and Cui (2006), Haderbache and Laouami (2010). Effects of the yield criterion on local deformations in numerical simulation were studied by Hopperstad (1998). The effect of failure criterion on slope stability analysis was studied by Haderbache and Laouami. The results show that the effect of failure criterion on slope stability analysis and the Mohr-Coulomb theory do not consider the intermediate principal stress, overshadowing the real behavior of soil. It is also shown that the results are correct because the intermediate principal stress exists really in the soil and may have a direct effect on the stability of a sliding slope under external actions. Wang and Lee (2006) pointed out that many factors can influence the final simulation result, the most important of which is a suitable yield criterion.

The results obtained by using the unified slip-line field theory for plane strain problems, the unified characteristics line theory for plane stress problems and spatial axisymmetric problems (Yu et al., 2006), as well as every example of unified solutions for limit, shakedown and dynamic plastic analyses of structures (Yu et al., 2009) show the serial difference. Results indicate that predictions of the limit capacity of a structure are sensitive to the selection of yield criteria. The application and choice of strength theory has a significant influence on the results.

A large number of materials models have been proposed throughout the years. So far, no general model can simulate the strength behavior of materials under complex stress. Therefore, several models are normally implemented in commercial programs to allow for simulations of different material types under various conditions. It is obviously of great importance to choose a constitutive model suitable for the material and the problem under consideration, as well as to assign proper values to the parameters included in this model. Of course, there is still a need for new models. A general but simple model that is thereby suitable for more materials may be developed.

The unified strength theory with b=0, b=0.25, b=0.5, b=0.75 and b=1, or the unified strength theory with b=0, b=0.5, and b=1 will be applied for plastic analyses of different structures in our monograph. The unified strength theory with b=0.6 for concrete material is used for static and dynamic analyses by Zhou at Nanyang Technological University, Singapore, and by Zhang et al. at Griffith University, Australia, which will be described in Chapters 13-15 and Chapter 22.

A slope problem is shown in Fig. 1.11. The single-shear theory of Mohr-Coulomb or the three-shear theory of Drucker-Prager do not completely match experimental data for geomaterials. It has been shown that the yield criteria of geomaterials depend not only on the maximum shear stress, but also on the intermediate principal shear stress and also on the intermediate principal stress σ_2 and the third invariant of the deviatoric stress tensor J_3. The reason that

Mohr-Coulomb theory and the Drucker-Prager criterion are not in good agreement with the experimental data is that the effect of σ_2 and the effect of J_3 are neglected. The unified strength theory with b=0, b=0.25, b=0.5, b=0.75 and b=1 is used. The plastic displacements of the slope with different yield criteria under the same conditions are shown in Fig. 1.12. The 3D simulation of a landslip using unified strength theory will be described in Chapter 20.

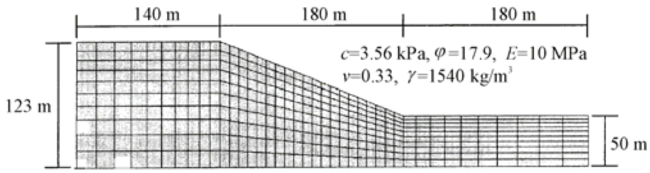


Fig. 1.11 A slope problem

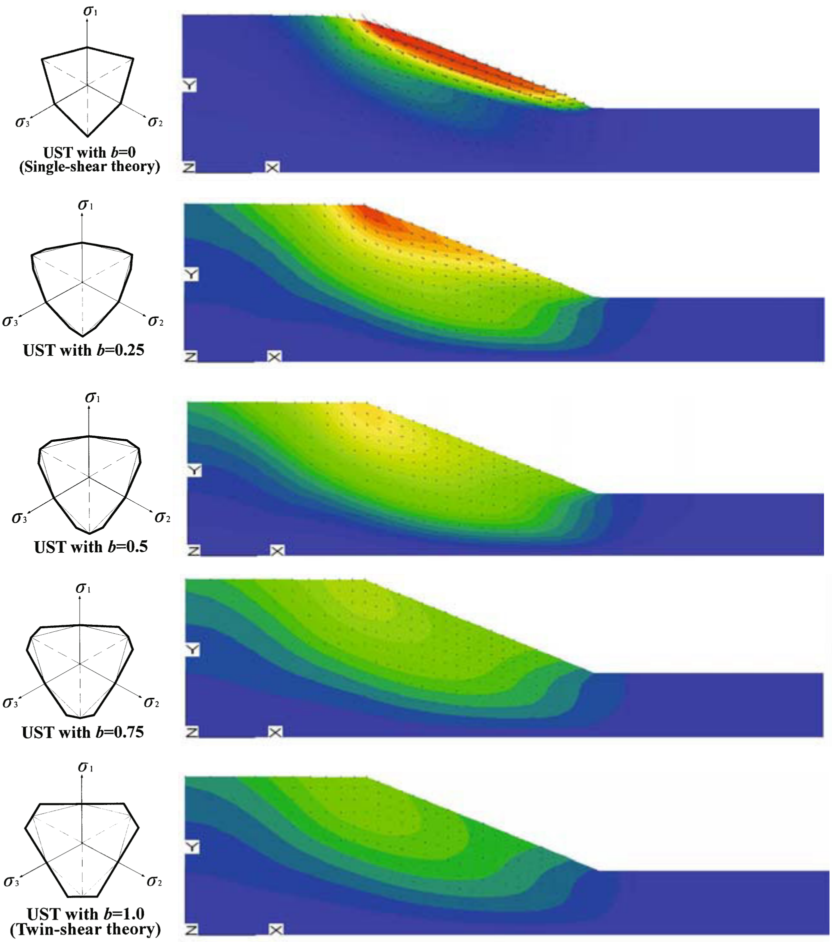


Fig. 1.12 Displacements of the slope with different yield criteria

It is seen that the difference in the results is obvious, however, the result obeying the unified strength theory with b=0 and the result obeying the Mohr-Coulomb theory are identical. That is due to the fact that the unified strength theory encompasses the Mohr-Coulomb theory as a special case. On the other hand, the Mohr-Coulomb theory can be deduced from the unified strength theory when b=0.

1.5 Historical Review: with Emphasis on the Implementation and Application of Unified Strength Theory

Finite element methods originated in the field of structural analysis and were widely developed and exploited in civil structures and aerospace industries during the 1950s and 1960s. Such methods are firmly established in civil and aeronautical engineering.

Strength theory (yield criterion and failure criterion, or the material model) as one of the most important constitutive relations has been implemented in various computational codes, especially nonlinear computer codes based on the finite element method (FEM). Elasto-plastic programs have been used for many years around the world. In general, these material models are the Tresca-Mohr-Coulomb single-shear series and the Huber-von Mises-Drucker-Prager three-shear series of strength theories. A reference book on the topic is available (Brebbia, 1985). Several excellent textbooks and monographs devoted to computational plasticity have been published. Related books and a brief history of nonlinear finite elements before 2000 were described by Belytschko, Liu and Moran (2000).

The form of yield surfaces of the single-shear series of strength theories is angular in the π-plane. However, the flow vector of the plastic strain is not uniquely defined at the corners of the Tresca and Mohr-Coulomb criteria and the direction of the plastic strain there is indeterminate. Koiter (1953) has provided limits within which the incremental plastic strain vector must lie. These singularities give rise to constitutive models that are difficult to implement numerically. To avoid such singularity, Drucker and Prager (1952) introduced an indented Huber-von Mises criterion in which the ridge corners have been rounded. The Drucker-Prager criterion has been widely implemented in nonlinear FEM codes and is widely used for geomechanics and in geotechnical engineering. Unfortunately, this gives a very poor approximation to the real failure conditions (Humpheson and Nayalor, 1975; Zienkiewicz and Pande, 1977; Chen, 1985, Chen and Baladi, 1985).

Therefore, a lot of smooth ridge models were proposed. They include the Gudehus-Argyris criterion, William-Warnke criterion, Lade-Duncan criterion, Matsuoka-Nakai criterion, Dafalias criterion, Burd criterion, Menetrey-Willam criterion, Zhao-Song criterion, JJ Jiang criterion and others. Most of them are of the octahedral-shear type (J2 theory) function expressed by three shear stresses. Various forms of smooth models were summarized in Chapter 3 of this monograph and Chapter 11 of another monograph (Yu, 2004).

At the same time, the singularities of the Tresca and Mohr-Coulomb yield criteria can also be overcome by rounding off the corners of the surface or by employing a simple mathematical artifice in the numerical procedure (Owen, 1980). The accurate treatments of corners in yield surfaces were studied by Marques (1984), Ortiz et al. (1985; 1987; 1994), Yin (1984), Yin and Zhou (1985), Sloan and Booker (1986), de Borst (1987; 1989), Simo et al. (1988), Runesson et al. (1988), Pramono and Willam (1989), Pankaj and Bicanic (1991), Khan and Huang (1995), Larsson and Runesson (1996), Jeremic and Sture (1997) and others. So single-shear-type yield criteria are easy to use and easily implemented in computational codes. The singularity of Tresca plasticity at finite strains was studied by Peric and de Neto (1999).

A course on "Advanced Numerical Applications and Plasticity in Geomechanics" was held by the International Centre for Mechanical Sciences (Le Centre International des Sciences et Mecaniques (CISM) in Udine, Italy). Eight papers were edited by Griffiths and Gioda in 2000. The material spans a remarkable range of topics, from theoretical developments involving novel algorithms and constitutive models to practical applications involving prediction of stresses and deformations in tunnel linings.

A monograph on the "Introduction to Computational Plasticity" was given by Dunne and Petrinic (2005). A range of plasticity models including those for superplasticity, porous plasticity, creep, cyclic plasticity and thermo-mechanical fatigue are introduced. Microplasticity and continuum plasticity, the implementation of constitutive equations, and associated material Jacobian into finite element software are addressed. The Huber-von Mises yield criterion implemented in the commercial code ABAQUS is described. In the Proceedings of Computational Plasticity: Models, Software and Applications, which was edited by Owen et al. (1989), 101 papers were presented.

The yield criteria have been implemented in the most current commercial FEM systems, such as ABAQUS, ADINA, ANSYS, ASKA, ELFEN (Univ. of Wales Swansea), MSC-NASTRAN, MARC, NonSAP and AutoDYN, DYNA and DY-PLAS (Dynamic Plasticity). In some systems, only the Huber-von Mises criterion, Drucker-Prager criteria, Mohr-Coulomb criterion and some other single curve criteria were implemented. The functions and the applied fields of many powerful commercial FEM codes were limited to the choice of failure criteria. More effective and systematic models of materials under complex stress are needed.

As pointed out by Humpheson and Naylor (1975), Zienkiewicz and Pande (1977), and Chen (1982, 1984), there is basically a shortcoming in the Drucker-Prager surface in connection with rock-soil strength modeling: the independence of τ_8 on the angle of similarity θ. It is known that the trace of the failure surface on the deviatoric planes is not circular (Chen, 1982; 1984; 1994).

To facilitate the choice of a model and to determine in an organized way the parameter values based on all the performed tests in a constitutive driver (i.e., a computer program containing a library of models where the tests can be simulated on the constitutive level and where parameter optimization can be performed), four soil plasticity models have been proposed by Mattsson et al. (1999). These models have, so far, been included in the constitutive driver. The main idea was that the concept could be used for constructing constitutive drivers as a supplement to commercial programs with their constitutive models, as well as for researchers verifying and developing such models. A practical finite element code for plane and axisymmetric modeling of soil and rock plasticity, called PLAXIS, was provided by Vermeer (1998).

The twin-shear strength theory has been implemented in some special finite element programs. such as An and Yu (1991) for solving the hydropower structure, Yu and Li (1991), Yu et al. (1992) for a mechanical structure, Shen (1993) for studying soil mechanics problems, Yu and Meng (1992; 1993) for studying the stability of the ancient city wall in Xi'an, China; Li, et al. (1994) for composite, Li and Ishii (1998) for the structural analysis of a dam, etc. The elasto-visco-plastic finite element analysis of a self-enhanced thick cylinder using the twin-shear strength theory was given by Liu et al. (1994). The twin-shear yield criterion and the twin-shear strength theory have been implemented in three commercial FEM codes by Quint Co. (1993; 1994). The twin-shear strength theory was implemented in an FEM code and applied to analyze the stability of the high slopes of the Three-Gorges Lock by the Yangzhi River Science Academy, China.

3D finite element numerical modeling of large underground caves and the stability of the excavated rock mass of the Tai'an Pumped Storage Hydraulic Plant was done by Professors Sun, Shang, Zhang et al. at Zhejiang University, Hangzhou, China and East China Investigation and Design Institute, the State Power Corporation of China (Sun et al., 2004a; 2004b).

Unified strength theory was also used for dynamic response and blast-resistance analysis of a tunnel subjected to blast loading by Zhejiang University (Liu and Wang, 2004). A new failure criterion introduced from unified strength theory when the strength parameter b=1/2, was used by professor Liu for a railroad tunnel. It was also used for the analysis of the stability of a slope (Bai, 2005), the failure analysis of a concrete road (Liang, 2004) and 3D failure process analysis of rock and associated numerical tests by Liang (2005) at Northeastern University, China.

Recently, analysis on textural stress and rock failure in diversion tunnels using the twin-shear strength theory was given by Yang and Zhang (2008; 2009). The twin-shear theory is also used for studying the sudden-crack phenomenon and simulation of the surrounding rock mass in a diversion tunnel (Yang et al., 2008). The adaptive arithmetic of arch dam cracking analysis using the twin-shear strength theory was given by Yang et al. (2009). The singularity has been overcome, and it is easy to use. The twin-shear yield criterion and the twin-shear strength theory have been implemented in three commercial FEM codes by Quint Co. (1993; 1994) in Tokyo, Japan

The unified yield criterion and the unified strength theory have been implemented and applied to some plasticity and engineering problems (Yu et al., 1992; Yu and Zeng, 1994; Yu et al., 1997; 1999). The singularities at the corners of the single-shear series of strength theory, twin-shear series of strength theory and unified strength theory have been overcome using a unified numerical procedure, i.e., UEPP Code (Yu et al., 1993; Yu and Zeng, 1994; Yu et al., 1997; 1999; Yu, 1998).

Unified strength theory was used to study structural reliability analysis by Wang et al. (2008) at Sichuan University, Sichuan Province, China. It was also used for nonlinear finite element analysis of an RC plate and shell by Wang (1998) at Nanyang Technological University, Singapore.

Unified strength theory is also implemented in the general commercial code, such as ABAQUS and AutDYN by Fan and Qiang (2001) and Zhang et al. (2001) at Griffith University, Australia, for research on the punch of concrete and dynamic problems. Normal high-velocity impact on concrete slabs was simulated using unified strength theory (Fan and Qiang, 2001). Unified strength theory was implemented in non-linear FEM at Nanyang Technological University, Singapore, by Zhou (2002) for the numerical analysis of reinforced concrete subjected to dynamic load. Recently, the secondary development and application of unified strength theory and associated elastoplastic constitutive model to ABAQUS were presented by the North China Electric Power University, Beijing, China and the Institute of Water Resources and Hydropower Research of China (Wang and Lu, 2009) as well as by Tongji University (Pan et al., 2010).

Based on the finite element theoretical scheme of a unified elastoplastic constitutive model, and according to the UMAT interface requirement of ABAQUS, the corresponding UMAT codes are programmed, which will be called the main analytical module of ABAQUS. Adopting a degenerative model of the unified strength (b=0, Mohr-Coulomb model) and the built-in Mohr-Coulomb model of ABAQUS, the uniaxial tests and circular chamber are analyzed to verify the correctness and efficiency of the developed material subroutine. Finally, considering the general form of the unified elastoplastic constitutive model ($b \neq 0$) and the hard condition of yield surface, which are not available in ABAQUS software, a circular chamber is simulated and the variational discipline of the stress field is obtained. The basic procedures provided and the programming essentials of UMAT redefined in ABAQUS are universal and can offer a reference for other developers (Pan et al., 2010).

Unified strength theory was used to study topology optimization of evolutionary structures by Li et al. (2008). The abstract of the paper shows that: "Based on the traditional evolutionary structural optimization method and considering wide applications of unified strength theory for all kinds of engineering structures, this paper presents a bi-directional evolutionary structural topology optimization method based on the unified strength criterion. It can be used not only for isotropic materials, but also for many kinds of anisotropic materials. Finally, some numerical examples are given and the results show that this method has wide use in topology optimization design for structures of fragile materials, anisotropic material processing and design fields".

Recently, two papers concerning the applications of the three-parameter unified strength theory to an FEM program, and the Monte-Carlo 3D nonlinear stochastic FEM model for structure reliability analysis were constructed by Wang, et al. (2008a; 2008b). The three-parameter and five parameter unified strength theory are used for the load bearing capability of a concrete-filled steel tube component considering the effect of intermediate principal stress and plastic seismic damage of a concrete structure by Shao and Qian (2007) and Shao et al. (2007).

UST (unified strength theory) and slip line field theory are also implemented in ANSYS by Li and Chen (2010). The results can be employed to analyze the differences in safety factors and the positions of the critical slip surfaces for unified yield criteria.

A new effective three-dimensional finite difference method (FDM) computer program, FLAC-3D (Fast Lagrangian Analysis of Continua in 3D) was presented (FLAC-3D, 1997). Stability analysis on the high slopes of the Three Gorges Lock using FLAC-3D was given (Kou et al., 2001). It is a pity, however, that only two failure criteria, the Mohr-Coulomb criterion and the Drucker-Prager criterion were implemented in FLAC-3D code.

Unified strength theory is implemented in FLAC-3D by Zhang et al. (2008) for the analysis of structures at the National Key Laboratory of Geomechanics and Geotechnical Engineering, Institute of Rock and Soil Mechanics, Chinese Academy of Science. The abstract of the paper of Zhang, et al. (2008) shows that: "Unified strength theory is a new theory system which can almost describe the strength characteristics of most geomaterials and has been applied widely. And FLAC-3D is an excellent geotechnical program. If the former can be integrated into the later, many complex problems in engineering will be settled. So, according to this problem, the numerical scheme for an elastoplastic unified constitutive model in FLAC-3D was studied. And the numerical format of the elastoplastic constitutive model based on the unified strength theory was derived….The merits of unified strength theory and the FLAC-3D program will be utilized well in geo-engineering after their combination."

In-situ stress measurement and stability analysis based on unified strength theory in large scale underground caverns was presented at Beijing Scientific and Technical University (Li, 2008). Unified strength theory was also implemented in FLAC-3D for stability analysis in large scale underground caverns. Recently, unified strength theory was implemented in FLAC by Hohai University for the dynamic analysis of the 500 kV underground transformer substation for Shanghai World EXPO (Fen and Du, 2010).

Table 1.1 gives some applications of yield criteria in FEM codes.

These implementations and applications can be classified in three types, as follows:

(1) A special elasto-plastic FE program referred as the UEPP—Unified Elasto-Plastic Program. The first version of UEPP was used by Yu's research group at Xi'an Jiaotong University in 1990-1991. The third version of UEPP was used in 1998. WB Zheng, GW Ma, SY Yang, Y Wang, LN He, and N Lu made their contributions to UEPP. At the same time, the twin-shear failure criterion was implemented in several FE codes by ZJ Shen (1989), Yu and Li (1991), An et al. (1991), Yu and Meng (1992), Quint Co. (1993; 1994), Liu et al. (1994), Li and Ishii (1998).

(2) Implementation in several nonlinear FE codes written by researchers at some universities.

(3) Implementation in several commercial nonlinear FE codes by researchers.

Table 1.1 Some applications of yield criteria in FEM codes

Yield criteria	FEM codes														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Tresca criterion	√	√	√	√	√		√	√					√		√
Mises criterion	√	√	√	√	√		√	√		√			√		√
Mohr-Coulomb	√	√	√	√	√	√	√	√	√	√			√		√
Drucker-Prager	√		√		√	√									√
Twin-shear criterion for Non-SD materials	√	√			√	√	√		√	√			√		√
Twin-shear strength theory for SD materials	√	√			√	√	√	√	√	√			√		√
Unified yield criterion for Non-SD materials	√				√	√	√	√			√	√		√	√
Unified strength theory	√					√	√	√			√	√		√	√
Others	√	√	√	√	√	√	√	√	√	√	√	√	√	√	√

Notes:
1—HAJIF (The Aircraft Strength Research Institute of China)
2—COMPMAT and STAMPS (Quint Co., Japan)
3—MARC (USA)
4—NASTRAN (USA)
5—ANSYS used in Xi'an Jiaotong University and Hohai University, Nanjing, China
6—FLAC-3D used in Beijing Sci. Tech. University and Xi'an Jiaotong University, as well as Wuhan Rock-Soil Mechanics Institute of Chinese Academy
7—UEPP(Xi'an Jiaotong U., Xi'an, China)
8—UEPP (Nanyang Technological University, Singapore)
9—Academy of Yangzhi River
10—North-West Institute for Investigation and Research in Hydraulic-Power
11—Zhejiang University and East China Investigation and Design Institute, State Power Corporation of China
12—Griffith University, Australia; Sichuan University, China; North-East University, China
13—Jinan University, Jinan, Shandong Province, China
14—Zhejiang University and Railway Co.
15—AutDYN at Nanyang Technological University, Singapore

1.6 Brief Summary

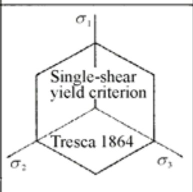
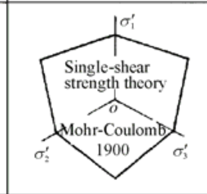
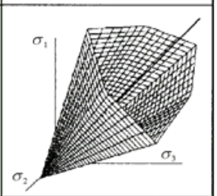
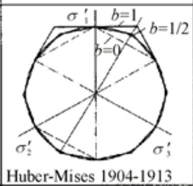
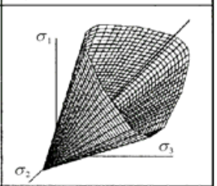
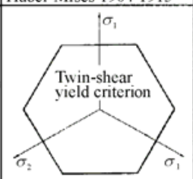
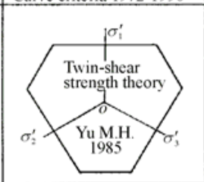
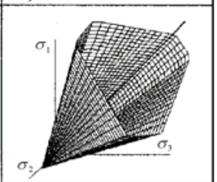
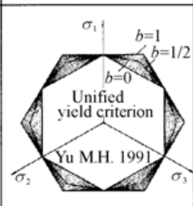
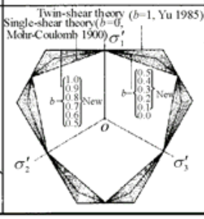
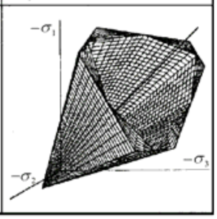
Most materials in structures are acted under the complex stress states, i.e., biaxial and multiaxial stresses. Strength theory provides a yield (or failure) criterion, a limiting stress state for elasticity, or an initial deformation for plasticity. Sometimes it is also used as an associated or non-associated flow rule for plastic constitutive relations.

A series of research works were carried out to show the effects of strength theory on the results of elastoplastic analysis, the load-carrying capacities of structures. The unified yield criterion and unified strength theory provide us with an effective approach to study these effects. Unified strength theory has been implemented in several computational plasticity codes. It is possible for us to adopt a different value of the unified strength theory parameter b to meet the requirements of different materials and structures.

The effects of failure criteria on the analytical results of the slip field for plane strain problems, the characteristic fields of plane stress problems and spatial axisymmetric problems using unified strength theory are researched by Yu et al. Systematic results can be seen in (Yu et al., 2006). The choice of strength theory has a significant influence on these results. Interested readers may refer to the book entitled *Generalized Plasticity* published by Springer in 2006. Comments on the model of Maohong Yu are given by Altenbach and Kolupaev (2008).

Advances in strength theories are briefly summarized in Table 1.2.

Table 1.2 Advance in strength theories

Advances in Strength theories	Non-SD materials (one-parameter)	SD materials (two-parameter)	Limit surfaces in stress space
Single-shear strength theory (inner bound)			
Three-shear strength theory (octahedral stress theory)			
Twin-shear strength theory (outer bound)			
Unified strength theory (serial criterial) (Yu, 1991)			

The twin-shear idea was proposed in 1961. Since then, the twin-shear yield criterion for non-SD materials, the generalized twin-shear strength theory for SD materials and the unified strength theory were successfully presented in 1961, 1985 and 1991. It can be seen that the development was very slow, covering 30 years of the development of unified strength theory.

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