Oscillatory Behavior for a Class of Recurrent Neural Networks with Time-Varying Input and Delays*

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Abstract. In this paper, the existence of oscillations for a recurrent neural network with time delays between neural interconnections is investigated. Several simple and practical criteria to determine the oscillatory behavior are obtained.

Keywords: Recurrent neural network, time-varying input, delay, oscillation.

1 Introduction

In [9], Hu and Liu have studied the stability of a class of continuous-time recurrent neural networks (RNNs) defined by the following model:

$$x_i'(t) = \sum_{j=1}^{n} w_{ij} g_j(x_j(t)) + u_i(t), \quad x_i(0) = x_{i0}, \quad i = 1, 2, \cdots, n. \tag{1}$$

The equivalent form of (1) in matrix format is given by

$$x'(t) = Wg(x(t)) + u(t), \quad x(0) = x_0. \tag{2}$$

where $x = [x_1, x_2, \cdots x_n]^T$ is the state vector, $W = (w_{ij})_{n\times n}$ is a constant connection weight matrix, $u(t) = [u_1(t), u_2(t), \cdots, u_n(t)]^T$ is a continuous input vector function which is called the time-varying input, $g(x) = [g_1(x_1), g_2(x_2), \cdots, g_n(x_n)]^T$ is a nonlinear vector-valued activation function. Assume that time-varying input $u_i(t)$ tend to constants u_i as t tends to infinity, and $g(\cdot)$ belongs to the class of globally Lipschitz continuous and monotone increasing activation functions. The authors established two sufficient conditions for global output convergence of this class of neural networks.

It is known that time-varying inputs $u(t)$ can drive quickly $x(t)$ to some desired region of the activation space [7]. Apart from discussing of convergence of neural

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network models, their oscillatory behaviors have also been exploited in many applications. For example, combustion-instability control [4], robotic control [10], the design of associative memory [13], sleep and walking oscillation modeling [16], wall oscillation in wall-bounded turbulent flows [15], oscillations of brain activity [14], mixed-mode oscillations in chemistry, physics and neuroscience [2], heat flow oscillation [17], oscillation in a network model of neocortex [3], neuronal population oscillations during epileptic seizures [11], oscillation in population model [18]. biochemical oscillations [5], oscillations in power systems [12]. In reality, time-delay neural networks are frequently encountered in various areas, and a time delay is often a source of instability and oscillations in a system. This is due to the finite switching speed of amplifiers in electronic neural networks or to the finite signal propagation time in biological networks. Therefore, in this paper, we discuss the oscillatory behavior of the time delay neural network with the time-varying input as follows:

$$x_i'(t) = \sum_{j=1}^{n} w_{ij} g_j(x_j(t-\tau_j)) + u_i(t),\ x_i(0) = x_{i0},\ i = 1, 2, \cdots, n. \tag{3}$$

where w_{ij}, τ_j are constants, in which *n* corresponds to the number of units in the networks, $x_i(t)\,(i = 1, 2, \cdots, n)$ correspond to the state vectors of the *i*th neural unit at time *t*, w_{ij} are the synaptic connection strengths, $\tau_j > 0\,(j = 1, 2, \cdots, n)$ represent delays. The equivalent form of (3) in matrix format is given by

$$x'(t) = Wg(x(t-\tau)) + u(t), \quad x(0) = x_0 .\mathrm{x} + \mathrm{y} = \mathrm{z}\ . \tag{4}$$

where $x(t-\tau) = [x_1(t-\tau_1), x_2(t-\tau_2), \cdots, x_n(t-\tau_n)]^T$. We assume that the function $g_j(x(t-\tau_j))\,(j = 1, 2, \cdots, n)$ in (3) belong to the class of Lipschitz continuous and monotone activation functions; that is, for each g_j, there exist $k_j > 0$ such that

$$0 < [g_j(x_j) - g_j(y_j)]/(x_j - y_j) \le k_j \quad (j = 1, 2, \cdots, n). \tag{5}$$

It should be note that such activation functions may be unbounded. There are many frequently used activation functions that satisfy this condition, for example, tanh(x), $0.5(|x+1| - |x-1|)$, max(0, x), and so on. We also assume that the time-varying input u_i satisfy the following conditions

$$\lim_{t\to\infty} u_i(t) = u_i\ . \tag{6}$$

where $u_i\ (i = 1, 2, \cdots, n)$ are constants, i.e. we assume that $\lim_{t\to\infty} u(t) = u.$

Definition 1. A solution of system (3) is called oscillatory if the solution is neither eventually positive nor eventually negative. If $x(t) = [x_1(t), x_2(t), \cdots x_n(t)]^T$ is an oscillatory solution of system (3), then each component of $x(t)$ is oscillatory.

2 Preliminary

Lemma 1 (Lemma 2 in [9]). If (6) is satisfied and there exists a constant vector $x^* = [x_1^*, x_2^*, \cdots, x_n^*]^T$ such that $Wg(x^*) + u = 0$, then given any $x_0 \in R^n$, system (3) has a unique solution $x = (t, x_0)$ defined on $[0, +\infty)$, where $W = (w_{ij})_{n\times n}$. Thus, under the conditions of Lemma 1 hold, if we set

$$\tilde{u}(t) = u(t) - u; \quad z(t) = [z_1(t), z_2(t), \cdots, z_n(t)]^T = x - x^*. \tag{7}$$

then system (3) can be transformed to the following equivalent system

$$z'(t) = Wf(z(t-\tau)) + \tilde{u}(t), \quad z(0) = x_0 - x^*. \tag{8}$$

where $f(z) = g(z + x^*) - g(x^*)$ satisfies that $f(0) = 0$.

Lemma 2. If the matrix W is a nonsingular matrix, then system (8) has a unique equilibrium point.

Proof. An equilibrium point $z^* = [z_1^*, z_2^*, \cdots, z_n^*]^T$ is a constant solution of the following algebraic equation

$$Wf(z^*) + \tilde{u}^* = 0. \tag{9}$$

Noting that (9) holds for any t, from (6) and the definition of $\tilde{u}(t) = u(t) - u$, if t is sufficiently large, it implies that $\tilde{u}^* = 0$. Meanwhile W is a nonsingular matrix, from (9) we get. $f(z^*) = 0$. Noting that $f(z)$ is a continuous monotone activation function, there exists only one z^* such that $f(z^*) = 0$. Since $f(0) = 0$, so the unique equilibrium point z^* is exactly 0. Since system (8) is an equivalent system of (3), hence, the oscillatory behavior of system (8) implies that the oscillatory behavior of system (3). Therefore, in the following we only deal with the oscillatory behavior of the unique equilibrium point of system (8). Noting that $f(0) = 0$. and is a continuous monotone activation function, then in a sufficiently small neighborhood of zero point, there exist $0 < \beta_j \le k_j \ (j = 1, 2, \cdots, n)$, a sufficiently small constant $\varepsilon (0 < \varepsilon \ll 1)$ such that

$$(\beta_j - \varepsilon) z_j(t - \tau_j) \le f_j(z_j(t - \tau_j)) \le (\beta_j + \varepsilon) z_j(t - \tau_j) \ (j = 1, 2, \cdots, n)$$

hold. According to [1, 6], the oscillatory behavior of system (8) about equilibrium point equals to the oscillation of the following system about zero point:

$$z_i'(t) = \sum_{j=1}^{n} (\beta_j \pm \varepsilon) w_{ij} z_j(t - \tau_j) + \tilde{u}_i(t), \quad i = 1, 2, \cdots, n. \tag{10}$$

3 Oscillation Analysis

Theorem 1. Suppose that the matrix W is a nonsingular matrix and system (8) has a unique equilibrium point. Let $\rho_1, \rho_2, \cdots, \rho_n$ represent the characteristic values of matrix $B = (\beta_j w_{ij})_{n\times n}$. Assume that there exists at least an ρ_j such that $\rho_j > 0,\ j \in \{1, 2, \cdots, n\}$. Then the unique equilibrium point of system (8) is unstable, and system (8) has a oscillatory solution.

Proof. Since ε is a sufficiently small constant, the oscillatory behavior of system (10) is the same as the following

$$z_i'(t) = \sum_{j=1}^{n} \beta_j w_{ij} z_j(t-\tau_j) + \tilde{u}_i(t), \quad i = 1, 2, \cdots, n. \tag{11}$$

From condition (6), when t is sufficiently large, $u_i(t) = u_i \pm \delta_i$, where δ_i are sufficiently small positive constants. This means that when t is sufficiently large, $\tilde{u}_i(t) = u_i \pm \delta_i - u_i = \pm\delta_i$. Therefore, if we neglect $\tilde{u}_i(t)$ in system (11), and get the following system:

$$z_i'(t) = \sum_{j=1}^{n} \beta_j w_{ij} z_j(t-\tau_j), \quad i = 1, 2, \cdots, n. \tag{12}$$

We claim that system (12) has an oscillatory solution implies that system (11) has an oscillatory solution. Because the unique equilibrium point $z^* = [z_1^*, z_2^*, \cdots, z_n^*]^T$ in system (12) is unstable. Then for the sufficiently small positive constant ε, $z^* \pm \varepsilon$ is still neither eventually positive nor eventually negative. Since $\rho_1, \rho_2, \cdots, \rho_n$ are characteristic values of *B*, the characteristic equation of (12) can be written as $\prod_{i=1}^{n} (\lambda - \rho_i e^{-\lambda\tau_i}) = 0$. This means that for each *i* we get

$$\lambda - \rho_i e^{-\lambda\tau_i} = 0, i = 1, 2, \cdots, n. \tag{13}$$

There is at least one characteristic value of (13) larger than zero. Noting that the characteristic equation (13) is a transcendental equation, the characteristic values may be complex numbers. However, there still exists a real positive root under our assumptions. Since for some $j \in \{1, 2, \cdots, n\}$, $\rho_j > 0$, we consider function of λ for this *j* as follows:

$$h(\lambda) = \lambda - \rho_j e^{-\lambda\tau_j}. \tag{14}$$

Obviously, $h(\lambda)$ is a continuous function of λ, and $h(0) = -\rho_j < 0$. Noting that delay $\tau_j > 0$. Then $e^{-\lambda\tau_j}$ tends to zero as λ tends to positive infinity. Therefore, there exists a suitable large $\lambda^* > 0$ such that $h(\lambda^*) = \lambda^* - \rho_j e^{-\lambda^* \tau_j} > 0$. By means of the mean value theorem, there exists an $\bar{\lambda} \in (0, \lambda^*)$ such that $h(\bar{\lambda}) = 0$. In other words, $\bar{\lambda}$ is a positive characteristic value of system (14). This means that equation (13) has a positive characteristic value. Therefore, the unique equilibrium point of (12) is unstable, which implies that system (8) has an oscillatory solution.

Theorem 2. If the matrix W is a nonsingular matrix and system (8) has a unique equilibrium point. Suppose that the following conditions hold:

$$\beta_i w_{ii} < 0, \quad \max_{1\le i\le n}(\beta_i w_{ii} + \sum_{j=1,\, j\ne i} |\beta_j w_{ij}|) = -\delta < 0, \quad \delta\tau e > 1 \tag{15}$$

where $\tau = \min\{\tau_1, \tau_2, \cdots \tau_n\}$. Then system (8) has an oscillatory solution.

Proof. From system (12), assume, for the sake of contradiction, there exists a t^* such that for $t > t^*$, we always have $|z_i(t)| > 0$ $(i = 1, 2, \cdots, n)$. Noting that $\beta_i w_{ii} < 0$, then for
$t > t^*$ and $i = 1, 2, \cdots, n$, we get

$$|z_i'(t)| \le -\beta_i w_{ii} |z_i(t-\tau_i)| + \sum_{j=1,\, j\ne i} |\beta_j w_{ij}| \cdot |z_j(t-\tau_j)| \tag{16}$$

therefore,

$$\sum_{i=1}^{n} |z_i'(t)| \le -\delta \sum_{i=1}^{n} |z_i(t-\tau_i)| \tag{17}$$

It is easily to see from (17) that $\lim_{t\to\infty} z_i(t) = 0$. Otherwise, suppose that $\lim_{t\to\infty} z_i(t) = c > 0$. Then there exists a sufficiently large $t_0 (> \tau_i)$ such that when $t > t_0 - \tau_i$ we have $z_i(t) > 0.5c$. By integrating both sides of (17) from t_0 to t, we get

$$\sum_{i=1}^{n} (|z_i(t)| - |z_i(t_0)|) \le -\delta \int_{t_0}^{t} \sum_{i=1}^{n} |z_i(s-\tau_i)| ds = -\delta \int_{t_0-\tau_i}^{t-\tau_i} \sum_{i=1}^{n} |z_i(s)| ds \tag{18}$$

namely,

$$\sum_{i=1}^{n} |z_i(t)| + \delta \int_{t_0-\tau_i}^{t-\tau_i} n \cdot \frac{c}{2} ds \le \sum_{i=1}^{n} |z_i(t_0)| \tag{19}$$

Noting that $\delta\int_{t_0-\tau_i}^{t-\tau_i} n\cdot\frac{c}{2}ds\to\infty$ as $t\to\infty$, and the right hand of (19) is a constant. This contradiction implies that $\lim_{t\to\infty} z_i(t)=0.$ Again integrating both sides of (17) from $t_0(>\tau_i)$ to $+\infty$, we get

$$0-\sum_{i=1}^{n}|z_i(t)|\le -\delta\int_{t}^{+\infty}\sum_{i=1}^{n}|z_i(s-\tau_i)|ds \tag{20}$$

Namely, $\sum_{i=1}^{n}|z_i(t)|\ge\delta\int_{t}^{+\infty}\sum_{i=1}^{n}|z_i(s-\tau_i)|ds.$ Set $\tau=\min\{\tau_1,\tau_2,\cdots\tau_n\}$, then for each τ_i we have $t-\tau\ge t-\tau_i$, hence

$$\sum_{i=1}^{n}|z_i(t)|\ge\delta\int_{t}^{+\infty}\sum_{i=1}^{n}|z_i(s-\tau_i)|ds\ge\delta\int_{t-\tau}^{+\infty}\sum_{i=1}^{n}|z_i(s)|ds \tag{21}$$

Set $y(t)=\sum_{i=1}^{n}|z_i(t)|$, then $y(t)\ge 0$ and from (21) we get $y(t)\ge\delta\int_{t-\tau}^{+\infty}y(s)ds.$ Define a sequence as follows:

$$\xi_0(t)=y(t),\quad \xi_{k+1}(t)=\begin{cases} y(t)-y(T)+\delta\int_{t-\tau}^{+\infty}\xi_k(s)ds, & t\le T,\\ \delta\int_{t-\tau}^{+\infty}\xi_k(s)ds, & t>T.\end{cases} \tag{22}$$

For $t>T$ by induction we can easily to see that $\xi_0(t)\ge\xi_1(t)\ge\cdots\ge\xi_k(t)\ge\cdots\ge 0.$ Therefore, $\lim_{k\to\infty}\xi_k(t)=\xi$ exists, and ξ is an eventually positive bounded solution of the following equation:

$$y'(t)=-\delta y(t-\tau),\ t>T. \tag{23}$$

Thus, the characteristic equation of (23) has a real root which is negative. From $\lambda=-\delta e^{-\lambda\tau}$ has a negative root, so that $-\lambda>0$, then using formula of $e^x\ge ex$, $(x>0)$, we obtain that $1=\frac{\delta\tau e^{-\lambda\tau}}{-\lambda\tau}\ge\frac{\delta\tau e(-\lambda\tau)}{-\lambda\tau}=\delta\tau e$, which contradicts assumption (15). The result follows.

4 Simulation Results

Example 1. Consider the following three-neuron system

$$\begin{cases} x_1' = -18g(x_1) + 0.2g(x_2) - 1.2g(x_3) + 0.5t/(1+t), \\ x_2' = -2g(x_1) - 6.4g(x_2) + 0.1g(x_3) + 1.5t/(1+t), \\ x_3' = -0.24g(x_1) + 6g(x_2) - 12g(x_3) + 2.5t/(1+t). \end{cases} \quad (24)$$

where $g(x_i) = g(x_i(t-\tau_i)), i = 1,2,3.$ Let $g(x) = 0.5\times(|x+1| - |x-1|).$ For function $g(x)$, we can select $\beta_1 = \beta_2 = \beta_3 = 1,$ then $\beta_1 w_{11} = -18,$ $\beta_2 w_{22} = -6.4,$ $\beta_3 w_{33} = -12.$ $\max\limits_{1\le i\le 3}(\beta_i w_{ii} + \sum_{j=1,j\ne i} |\beta_j w_{ij}|) = -\delta = -4.3.$ From (24), $u = (0.5,\ 1.5,\ 2.5)$. When we select $\tau_1 = 1,$ $\tau_2 = 2,$ $\tau_3 = 4,$ and $\tau_1 = 3,$ $\tau_2 = 6,$ $\tau_3 = 12,$ respectively, the conditions of Theorem 2 are satisfied. System (24) generates oscillations (Fig.1A and Fig.1B).

Example 2. Consider the following four-neuron system

$$\begin{cases} x_1' = -6.8g(x_1) + 0.2g(x_2) - 0.2f(x_3) + 0.5f(x_4) + u_1(t) \\ x_2' = 2.4g(x_1) - 8.5g(x_2) + 0.2f(x_3) - 0.1f(x_4) + u_2(t) \\ x_3' = 0.25g(x_1) + 0.3g(x_2) - 3.8f(x_3) - 1.2f(x_4) + u_3(t) \\ x_4' = 1.6g(x_1) + 0.2g(x_2) - 0.2f(x_3) - 4.2f(x_4) + u_4(t) \end{cases} \quad (25)$$

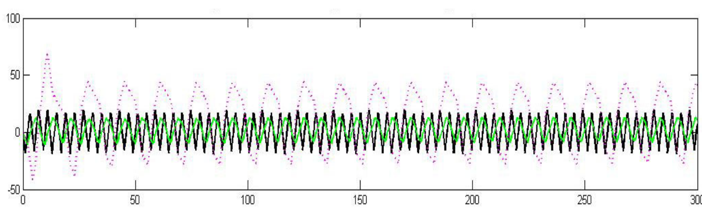

Fig. 1A. Oscillation of the equilibrium point, u=(0.5, 1.5, 2.5), Solid line X_1(t), dashed line: X_2(t), dotted line: X_3(t), delays: 1, 2, 4.

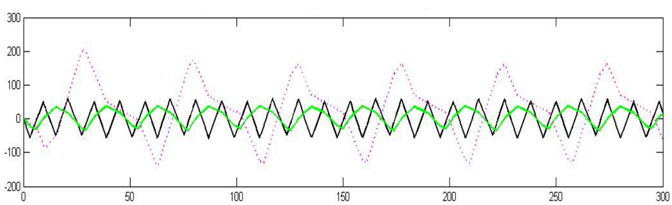

Fig. 1B. Oscillation of the equilibrium point, u=(0.5, 1.5, 2.5), Solid line X_1(t), dashed line: X_2(t), dotted line: X_3(t), delays: 3, 6, 12.

where $g(x) = \arctan(x), f(x) = \tanh(x),$ we can select $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1,$ then the characteristic values of $B = (\beta_j w_{ij})_{4\times4}$ are 0.2020, -3.9043, -6.8447, -8.7530. It is known that a characteristic value 0.2020>0. When $u_1(t), u_2(t), u_3(t), u_4(t)$ are taken the values and $(2t^2/(1+t^2), 4t^2/(1+t^2), 6t^2/(1+t^2), 4t^2/(1+t^2))$ respectively, we select $\tau_1 = 1, \tau_2 = 2, \tau_3 = 2, \tau_4 = 1$, the conditions of Theorem 1 are satisfied. System (25) generates oscillations (Fig.2A and Fig.2B).

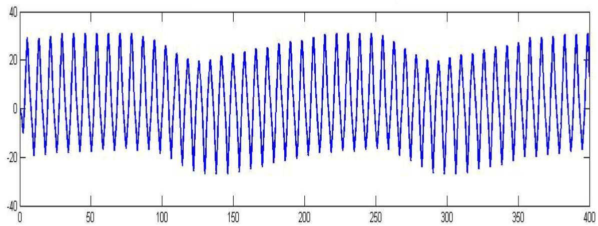

Fig. 2A. Oscillation of the second neuron output, u= (2, 0, 2, 0), delays: 1, 2, 2, 1.

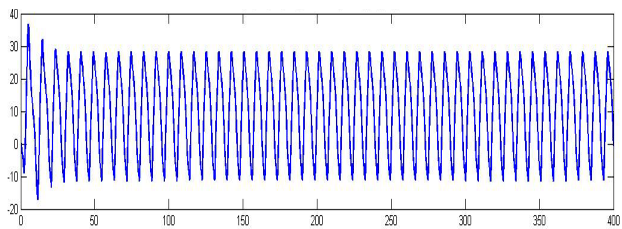

Fig. 2B. Oscillation of the second neuron output, u= (2, 4, 6, 5), delays: 1, 2, 2, 1.

5 Conclusion

This paper discusses the oscillatory behavior for a recurrent neural network with time delays between neural interconnections and time-varying input. By using continuous and monotone increasing activation function and continuous time-varying inputs, two criteria to guarantee oscillations of global output have been proposed.

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