The Rationality Properties of Three Preference-Based Fuzzy Choice Functions*

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Abstract. In this paper, three $PFCFs$ are selected and whether they satisfies the rationality properties such as $F\alpha$, $F\beta$, $F\gamma$, $WFCA$ is investigated, then the satisfaction of preference-based fuzzy choice functions is obtained as a consequence.

Keywords: Fuzzy Preference Relation, Preference-based Fuzzy Choice Function, Rationality Property.

1 Introduction

In the research on choice functions, there is lots of literature in which the relationships between rationality conditions and the characterization of the rationalization of choice functions are discussed whereas little attention is paid to the rationality of some specific choice functions. The rationality of some specific choice functions include: Barrett [1] proposed nine preference-based non-fuzzy choice functions and assess whether those satisfy $RPWD$、 $RPSD$、 $SREJ$、 $WREJ$、 UF、 LF; Roubens [2] defined three preference-based fuzzy choice functions and analyzed some properties of these choice functions. Based on them, in this paper we select three preference-based fuzzy choice functions and assess these preference-based fuzzy choice functions selected in terms of some usual rationality conditions in fuzzy case.

2 Preliminaries

Let X denote a finite set of alternatives, $P(X)$ the set of all non-empty crisp subsets of X, and $F(X)$ the set of all non-empty fuzzy subsets of X.

Definition 1 [1]: A fuzzy binary preference relation ($FBPR$) is a function

$$R : X \times X \rightarrow (0,1).$$

* The project is supported by the Natural Science Foundation of Shanxi Province (2009011006) and Yong Foundation of TUT.
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H. Deng et al. (Eds.): AICI 2011, CCIS 237, pp. 113–119, 2011.

Remark 1. An $FBPR\ R$ satisfies transitivity iff

$$\text{for all } x, y, z \in X, R(x,z) \geq R(x,y) \wedge R(y,z).$$

Remark 2. G denotes the non-empty set of all $FBPRs$.

Definition 2 [1]: A fuzzy choice function is a function

$$C : P(X) \to F(X) \text{ such that } \forall S \in P(X), \sup pC(S) \subseteq S,$$

where $\sup pS$ denotes the support of S for all $S \in F(X)$. We denote it as $C(\cdot)$.

Remark 3. A fuzzy choice function $C(\cdot)$ is called a preference-based fuzzy choice function($PFCF$) iff $C(\cdot)$ is determined by a fuzzy relation Q. We denote it as $C(\cdot, Q)$.

The choice set $C(S)$ is normal iff $\forall S \in P(X)$, $\exists x \in S$, such that $C(S) = 1$.

Definition 3 [3]: Let $C(\cdot)$ be a fuzzy choice function, the fuzzy preference relation revealed by $C(\cdot)$ is defined by

$$\forall x, y \in X, \forall S \in P(X), \ R(x,y) = \bigvee_{\{S | x, y \in S\}} C(S)(x).$$

Remark 4. It is clear that $\forall x \in X, R(x,y) \geq C(S)(x)$ $(\forall y \in S)$.

In this section we select three $PFCFs$ as follows:

Let Q is a $FBPR$, $\forall x, y \in X$, $\forall S \in P(X)$, then we denote

(1) $C_1(S,Q)(x) = \bigwedge_{y \in S} Q(x,y)$.

(2) $C_2(S,Q)(x) = 1 - \bigvee_{y \in S} P(y,x)$ such that $P(y,x) = (Q(y,x) - Q(x,y)) \vee 0$.

Obviously $P(x,x) = 0$.

(3) $C_3(S,Q)(x) = \bigvee_{y \in S} P(x,y)$.

Remark 5. In general, the three functions above may not be non-empty, so we cannot assure those are choice functions. The following results are based on the hypothesis that those three $PFCFs$ are all choice functions.

Remark 6. $C_1(S,Q)(x)$ is the same as $D^Q(x)$ [2], it means the degree to which x is preferred to any element of S in terms of Q; $C_2(S,Q)(x)$ is the same as $N^Q(x)$ [2], it means the degree to which any element of S is not preferred to x in terms of Q; $C_3(S,Q)(x)$ is different from $SD^Q(x)$ [2], it means the degree to which x is strictly preferred to any element of S in terms of Q.

3 Some Rationality Properties of Preference-Based Choice Functions

In this section we consider some restrictions on $PFCFs$, which may be viewed as representing different aspects of rationality. Some of these properties have very well known counterparts in the vast literature [3,4,5] and we examine, in terms of these properties, the performance of the different $PFCFs$ defined in section 2. Now we list these properties as follows:

$F\alpha: \forall S_1, S_2 \in P(X),\ \forall x \in S_1, S_1 \subseteq S_2 \Rightarrow C(S_2)(x) \le C(S_1)(x)$.

$F\beta: \forall S_1, S_2 \in P(X), \forall x, y \in S_1$,

$S_1 \subseteq S_2 \Rightarrow C(S_1)(x) \wedge C(S_1)(y) \wedge C(S_2)(x) \le C(S_2)(y)$.

$F\gamma: \forall S_1, S_2 \in P(X), \forall x \in S_1 \cap S_2,\ C(S_1)(x) \wedge C(S_2)(x) \le C(S_1 \cup S_2)(x)$.

$F\beta^+ : \forall S_1, S_2 \in P(X), \forall x, y \in S_1,\ S_1 \subseteq S_2 \Rightarrow C(S_1)(x) \wedge C(S_2)(y) \le C(S_2)(x)$.

$FA1: \forall A, B \in P(X)$, $x \in A \cap B$, then $C(A)(x) \wedge C(B)(x) = C(A \cup B)(x)$,

[5] point out : $FA1 \Leftrightarrow F\alpha + F\gamma$.

$FA2$: $\forall A, B \in P(X)$, if $B \subseteq A - \sup pC(A)$, then $C(A-B) \subseteq C(A)$.

Obviously , $FA2$ is satisfied iff $\forall x \in B, C(A)(x) = 0$ for $\forall A, B \in P(X)$,

then $C(A-B)(x) \le C(A)(x)$ *for* $\forall x \in A - B$.

$WFCA: \forall S \in P(X), \forall x, y \in S,\ R(x,y) \wedge C(S)(y) \le C(S)(x)$.

$FC1: \forall S \in P(X), \forall x, y \in S$, $\{y$ is dominant in S$\} \Rightarrow C(S)(x) = R(x,y)$.

$FC2: \forall S \in P(X), \forall x, y \in S$,

$\{y$ is dominant in S and $R(x,y) \ge R(y,x)\} \Rightarrow x$ is dominant in S .

$FC3$: $\forall S \in P(X)$, $\forall x, y \in S$ and all real numbers k such that $0 < k \le 1$,

$\{C(S)(y) \ge k$ *and* $R(x,y) \ge k\} \Rightarrow C(S)(x) \ge k$.

4 Result

Proposition 1. (1) $C_1(\cdot, Q)$ and $C_2(\cdot, Q)$ satisfy $F\alpha$.

(2) $C_3(\cdot, Q)$ violates $F\alpha$.

Proof

(1) Let $\forall S_1, S_2 \in P(X)$, $S_1 \subseteq S_2$, $S_1 \subseteq S_2$, $\forall Q \in G$, $\forall x \in S_1$

$$C_1(S_2, Q)(x) = \bigwedge_{z \in S_2} Q(x,z) \le \bigwedge_{z \in S_1} Q(x,z) = C_1(S_1, Q)(x)$$

$$C_2(S_2, Q)(x) = 1 - \bigvee_{z \in S_2} P(z,x) \le 1 - \bigvee_{z \in S_1} P(z,x) = C_2(S_1, Q)(x)$$

so $C_1(\cdot, Q)$ and $C_2(\cdot, Q)$ satisfy $F\alpha$.

(2) The proof consists of counterexamples.

Let $X=\{x,y,z\}, S_1=\{x,y\}, S_2=X$, $Q=\begin{pmatrix}1 & 0.6 & 0.6\\ 0.2 & 1 & 0.2\\ 0.1 & 0.3 & 1\end{pmatrix}$.

Then $C_3(S_2,Q)(x)>C_3(S_1,Q)(x)$,

But $S_1\subseteq S_2$, so $C_3(\cdot,Q)$ violates $F\alpha$. □

Proposition 2. **(1)** $C_3(\cdot,Q)$ **satisfies** $F\beta^+$.

(2) $C_1(\cdot,Q)$ and $C_2(\cdot,Q)$ violate $F\beta$.

Proof

(1) Let $\forall S_1,S_2\in P(X),\forall x,y\in S_1,S_1\subseteq S_2,\forall Q\in G$,

$C_3(S_1,Q)(x)\wedge C_3(S_2,Q)(y)=[\vee_{z\in S_1}P(x,z)]\wedge[\vee_{z\in S_2}P(y,z)]$

$\le \vee_{z\in S_2}P(x,z)=C_3(S_2,Q)(x)$,

so $C_3(\cdot,Q)$ satisfies $F\beta^+$.

(2) Let $X=\{x,y,z\}, S_1=\{x,y\}, S_2=X$,

Consider Q_1,Q_2 such that $Q_1=\begin{pmatrix}1 & 0.6 & 0.4\\ 0.8 & 1 & 0.3\\ 0.6 & 0.3 & 1\end{pmatrix}, Q=\begin{pmatrix}1 & 0.8 & 0.2\\ 0.6 & 1 & 0.3\\ 0.4 & 0.6 & 1\end{pmatrix}$.

By computing , we have

$C_1(\cdot,Q_1), C_2(\cdot,Q_2)$ violates $F\beta$. □

Remark 7. (1)By $F\beta^+\Rightarrow F\beta$, $C_3(\cdot,Q)$ satisfies $F\beta$.

(2) $F\beta^+$ is stronger than $F\beta$ so $C_1(\cdot,Q)$ and $C_2(\cdot,Q)$ violate $F\beta^+$.

Proposition 3. $C_1(\cdot,Q)$, $C_2(\cdot,Q)$ and $C_3(\cdot,Q)$ all satisfy $F\gamma$.

Proof Let $\forall S_1,S_2\in P(X),\forall x\in S_1\cap S_2, \forall Q\in G$,

(1) $C_1(S_1,Q)(x)\wedge C_1(S_2,Q)(x)=[\wedge_{z\in S_1}Q(x,z)]\wedge[\wedge_{z\in S_2}Q(x,z)]$

$=\wedge_{z\in S_1\cup S_2}Q(x,z)=C_1(S_1\cup S_2,Q)(x)$,

so $C_1(\cdot,Q)$ satisfies $F\gamma$.

(2) $\vee_{z\in S_1\cup S_2}P(z,x)=[\vee_{z\in S_1}P(z,x)]\wedge[\vee_{z\in S_2}P(z,x)]$

suppose $\vee_{z\in S_1}P(z,x)\ge \vee_{z\in S_2}P(z,x)$

then $C_2(S_1,Q)(x)\wedge C_2(S_2,Q)(x)=[1-\vee_{z\in S_1}P(z,x)]\wedge[1-\vee_{z\in S_2}P(z,x)]$

$$=[1-\vee_{z\in S_1} P(z,x)]=[1-\vee_{z\in S_1\cup S_2} P(z,x)]= C_2(S_1\cup S_2,Q)(x)$$

so $C_2(\cdot,Q)$ satisfies $F\gamma$.

(3) $C_3(S_1,Q)(x)\wedge C_3(S_2,Q)(x) =[\vee_{z\in S_1} P(x,z)]\wedge[\vee_{z\in S_2} P(x,z)]$

$\le \vee_{z\in S_1} P(x,z)]\le \vee_{z\in S_1\cup S_2} P(x,z) = C_3(S_1\cup S_2,Q)(x)$

so $C_3(\cdot,Q)$ satisfies $F\gamma$. □

By [5], $FA1 \Leftrightarrow F\alpha+F\gamma$, then the following proposition is satisfied.

Proposition 4. $C_1(\cdot,Q)$ and $C_2(\cdot,Q)$ satisfy $FA1$.

Remark 8. By $C_3(\cdot,Q)$ violates $F\alpha$, so it violates $FA1$ (See the proof of proposition(2)).

Proposition 5. (1) $C_3(\cdot,Q)$ satisfies $FA2$.

(2) $C_1(\cdot,Q)$ and $C_2(\cdot,Q)$ violate $FA2$.

Proof

(1) Let $\forall A,B\in P(X),B\subseteq A-\sup pC(A)$, $\forall x\in A-B$, $\forall Q\in G$,

then $C_3(A-B,Q)(x)=\vee_{z\in A-B} P(x,z)\le \vee_{z\in A} P(x,z)=C_3(A,Q)(x)$,

so $FA2$ is satisfied.

(2) We only proof $C_2(\cdot,Q)$ violate $FA2$, the proof of $C_1(\cdot,Q)$ is similar.

Let $X=\{x,y,z\},A=\{x,y,z\},B=\{y\}$, $Q=\begin{pmatrix}1 & 1 & 0.8\\ 0 & 1 & 0.9\\ 0.2 & 0.2 & 1\end{pmatrix}$.

then $A-\sup pC_2(A,Q)=\{y\}$, $B\subseteq A-\sup pC_2(A,Q)$,

But $C_2(A-B,Q)(z)>C_2(A,Q)(z)$,

so $C_2(\cdot,Q)$ violates $FA2$. □

Proposition 6. (1) If Q satisfies transitivity then $C_1(\cdot,Q)$ satisfies $WFCA$.

(2) $C_2(\cdot,Q)$, $C_3(\cdot,Q)$ violate $WFCA$.

Proof

(1) Let $\forall S\in P(X)$, $\forall x,y\in S$, $\forall Q\in G$ and Q satisfies transitivity ,

$$R(x,y)\wedge C_1(S,Q)(y)=Q(x,y)\wedge[\wedge_{z\in S} Q(y,z)] = \wedge_{z\in S}[Q(x,y)\wedge Q(y,z)]$$

$$\le \wedge_{z\in S} Q(x,z)=C_1(S,Q)(x)$$

then $WFCA$ is satisfied.

(2) We only proof $C_2(\cdot,Q)$ violate $WFCA$, the proof of $C_3(\cdot,Q)$ is similar.

Let $X = \{x, y, z\}$, $Q = \begin{pmatrix} 1 & 0.3 & 0.1 \\ 0.8 & 1 & 0.3 \\ 0.8 & 0.7 & 1 \end{pmatrix}$, then $C_2(\cdot, Q)$ violates $WFCA$. □

In the literature [6], Guo proposed $WFCA \Leftrightarrow FC3 \Leftrightarrow F\alpha + F\beta$ in the hypothesis that $T = \wedge$ and the choice set is normal; by [7], we have $FC3 \Rightarrow FC2$, so the following notation is hold

Remark 9. If the choice set is normal and Q satisfies transitivity then $C_1(\cdot, Q)$ satisfies $WFCA$, $FC3$ and $FC2$.

5 Conclusions

In this paper, we have investigated the rationality properties of three preference-based choice functions, our main results are summarized in the following table.

Table 1. The rationality of preference-based fuzzy choice functions

	$C_1(\cdot, Q)$	$C_2(\cdot, Q)$	$C_3(\cdot, Q)$
$F\alpha$	*	*	
$F\beta$	*"		*
$F\beta^+$			*
$F\gamma$	*	*	*
$FA1$	*	*	
$FA2$			*
$WFCA$	*'		

In the table, * indicates that the $PFCF$ satisfies the property under consideration; *′ indicates that when Q satisfies transitivity the $PFCF$ satisfies the property; *″ indicates that the $PFCF$ satisfies the property under the condition that the choice set is normal and Q satisfies transitivity.

Remark 10. In addition, $C_1(\cdot, Q)$ satisfies $FC3$ and $FC2$ under the condition that the choice set is normal and Q satisfies transitivity.

As is clear from the table, $C_1(\cdot, Q)$ performs better than $C_2(\cdot, Q)$ and $C_3(\cdot, Q)$ if the fuzzy preferences are constrained; $C_2(\cdot, Q)$ and $C_3(\cdot, Q)$ perform better than $C_1(\cdot, Q)$ if the fuzzy preferences are not constrained. Through our comparative study, the rationality properties of some specific preference-based fuzzy choice functions are further understood and thus the research is of significance without doubt to the selection of choice functions in practice.

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