A Branching Time Logic with Two Types of Probability Operators

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Abstract. We introduce a propositional logic whose formulas are built using the language of CTL^*, enriched by two types of probability operators: one speaking about probabilities on branches, and one speaking about probabilities of sets of branches with the same initial state. An infinitary axiomatization for the logic, which is shown to be sound and strongly complete with respect to the corresponding class of models, is proposed.

1 Introduction

Interest in temporal reasoning came from theoretical and practical points of view. Logicians [5,6,30] investigated consequences of different assumptions about the structure of time, while temporal formalisms can be used in computer science to reason about properties of programs [11,29]. In both cases discrete linear and branching time logics have been extensively studied. Linear temporal logics are suitable for specification and verification of universal properties of all executions of programs. On the other hand, the branching time approach is appropriate to analyze nondeterministic computations described in the form of execution trees. In the later framework a state (a node) may have many successors. Then, it is natural to attach probabilities to the corresponding transitions and to analyze the corresponding discrete time Markov chains as the underlying structures. All this led to probabilistic branching temporal logic [2,3,17,18,21,35]. The mentioned papers mainly investigate semantical properties of the logics and do not offer any axiomatic system. The only exception is [35], where the logic with a very restricted language is presented. A more detailed overview on the topic is presented in Section 5, when we will be able to precisely formulate relevant notions and connections between them using the formalism from Section 2.

In this paper we consider a propositional discrete probabilistic branching temporal logic (denoted pBTL). We use a logical language which allows us to formulate statements that combine temporal and qualitative probabilistic features. Thus, the statements as "in at least half of paths α holds in at least a third of states" and "if α holds in the next moment, then the probability of α is positive" are expressible in our logic. To the best of our knowledge, the former

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sentence is not expressible in any of existing logics. The language for pBTL is obtained by adding temporal operators $\bigcirc$ ("next"), A (universal path operator) and U ("until"), as well as the two types of probability operators, $P^p_{\geqslant r}$ and $P^s_{\geqslant r}$ ($r \in \mathbb{Q} \cap [0,1]$), to the classical propositional language. The temporal operators are well known from other formalizations of branching time logics, while the intended meaning of $P^s_{\geqslant r}\alpha$ ($P^p_{\geqslant r}\alpha$) is "the probability that α is true on a randomly chosen branch is at least r" ("the probability that α holds on a particular branch is at least r"). The superscript s in $P^s_{\geqslant r}$ (p in $P^p_{\geqslant r}$) indicates that the probability depends only on a time instant - state (on a chosen branch - path).

We present a class of suitable models for the pBTL-language and an infinitary axiomatization, for which we prove strong completeness theorem ("every consistent set of formulas is satisfiable", in contrast to weak completeness: "every consistent formula is satisfiable"). Up to our knowledge it is the first such result reported in literature. The corresponding proof uses ideas (the Henkin construction) presented in [7,8,23,24,25,26,27,31].

The rest of the paper is organized as follows. In Section 2 we define syntax and semantics for pBTL. Section 3 introduces an infinitary axiomatization for the logic, which is proved to be strongly complete in Section 4. Comparison with the related work is discussed in Section 5. Section 6 contains concluding remarks and directions for further work.

2 Syntax and Semantics

Let $\mathcal{P}$ be at most countable set of propositional letters. The set of formulas For of the logic pBTL is the smallest set which satisfies the following conditions:

- $\mathcal{P} \subseteq For$,
- if $\alpha, \beta \in For$, then $\alpha \wedge \beta, \neg\alpha \in For$,
- if $\alpha, \beta \in For$, then $\bigcirc\alpha, \alpha U \beta, A\alpha \in For$,
- if $\alpha \in For$ and $r \in \mathbb{Q} \cap [0,1]$, then $P^p_{\geqslant r}\alpha, P^s_{\geqslant r}\alpha \in For$.

Intuitively, the operators mean:

- $\bigcirc\alpha$: α holds in the next time instant on a particular branch,
- $\alpha U \beta$: α holds in every time instant (on a particular branch) until β becomes true,
- $A\alpha$: α holds on every branch which passes through the current state,
- $P^p_{\geqslant r}\alpha$: the probability that α holds at a randomly chosen time instant on a particular branch is at least r, and
- $P^s_{\geqslant r}\alpha$: the probability of branches (with a particular initial time instant) on which α holds is at least r".

A formula is a *state formula* if it is a boolean combination of propositional letters, formulas of the form $P^s_{\geqslant r}\alpha$ and formulas of the form $A\alpha$. We denote the set of all state formulas by St. For $n \in \omega$, we define $\bigcirc^{n+1}\alpha$ as $\bigcirc(\bigcirc^n\alpha)$. If T is a set of formulas, then $\bigcirc T$ denotes $\{\bigcirc\alpha | \alpha \in T\}$, and AT denotes $\{A\alpha | \alpha \in T\}$. The temporal operators F (sometime), G (always) and E (existential path quantifier) are defined as follows:

- $F\alpha$ is $\top U\alpha$,
- $G\alpha$ is $\neg F\neg\alpha$,
- $E\alpha$ is $\neg A\neg\alpha$.

Also, in order to simplify notation, we introduce the following convention:

- $P^p_{<r}\alpha$ is $\neg P^p_{\geqslant r}\alpha$, $P^p_{\leqslant r}\alpha$ is $P^p_{\geqslant 1-r}\neg\alpha$, $P^p_{>r}\alpha$ is $\neg P^p_{\leqslant r}\alpha$ and $P^p_{=r}\alpha$ is $P^p_{\geqslant r}\alpha\wedge P^p_{\leqslant r}\alpha$,
- $P^s_{<r}\alpha$, $P^s_{\leqslant r}\alpha$, $P^s_{>r}\alpha$ and $P^s_{=r}\alpha$ are defined in a similar way.

An example of a formula is

$$EG\alpha \rightarrow P^s_{\geqslant\frac{1}{2}}P^p_{\geqslant\frac{1}{3}}\alpha,$$

which can be read as: "if there exists a path on which the formula α always holds, then on at least a half of paths α holds in at least a third of time instants".

Definition 1. *A* model $\mathcal{M}$ *is any tuple* $\langle S, v, R, \Sigma, Prob^{state}, Prob^{path}\rangle$ *such that:*

- *S is a non-empty set of states (time instants),*
- *$v : S \times \mathcal{P} \longrightarrow \{0,1\}$ assigns a truth labelling to every state.*
- *R is a binary relation on S, which is total (for every $s \in S$ there is $t \in S$ such that sRt),*
- *Σ is a set of ω-sequences $\sigma = s_0, s_1, s_2, \ldots$ of states from S, such that s_iRs_{i+1}, for all $i \in \omega$. A* path *is an element of Σ. We assume that Σ is suffix-closed, i.e., if $\sigma = s_0, s_1, s_2, \ldots$ is a path and $i \in \omega$, the sequence $s_i, s_{i+1}, s_{i+2}, \ldots$ is also a path.*
- *$Prob^{state}$ associates to every $s \in S$, a probability space $Prob_s = \langle H_s, \mu_s\rangle$ such that:*
  - *H_s is an algebra of subsets of $\Sigma_s = \{\sigma \in \Sigma \mid \sigma_0 = s\}$, i.e., it contains Σ_s and it is closed under complements and finite union,*
  - *$\mu_s : H_s \longrightarrow [0,1]$ is a finitely additive probability measure, i.e.,*
    - *$\mu_s(H_s) = 1$, and*
    - *$\mu_s(X \cup Y) = \mu_s(X) + \mu_s(Y)$, whenever X and Y are disjoint.*
- *$Prob^{path}$ associates to every $\sigma \in \Sigma$, $\sigma = s_0, s_1, \ldots, s_i, s_{i+1}, s_{i+2}, \ldots$, a probability space $Prob_\sigma = \langle A_\sigma, \mu_\sigma\rangle$ such that:*
  - *A_σ is an algebra of subsets of $S_\sigma = \{\pi \in \Sigma \mid \pi = s_i, s_{i+1}, s_{i+2}, \ldots, \text{for } i \in \omega\}$,*
  - *$\mu_\sigma : A_\sigma \longrightarrow [0,1]$ is a finitely additive probability measure.*

Let $\sigma = s_0, s_1, s_2, \ldots$ In the rest of the paper, we will use the following abbreviations:

- $\sigma_{\geq i}$ is the path $s_i, s_{i+1}, s_{i+2}, \ldots$
- σ_i is the state s_i.

Definition 2. *Let* $\mathcal{M} = \langle S, v, R, \Sigma, Prob^{state}, Prob^{path}\rangle$ *be any model. The* satisfiability relation $\models$ *(we denote the fact that a formula* α *is satisfied at a path* σ *in a model* $\mathcal{M}$ *by* $\mathcal{M}, \sigma \models \alpha$*) is defined recursively as follows:*

- *if* $p \in \mathcal{P}$*, then* $\mathcal{M}, \sigma \models p$ *iff* $v(s_0, p) = 1$,
- $\mathcal{M}, \sigma \models \neg\alpha$ *iff* $\mathcal{M}, \sigma \not\models \alpha$,
- $\mathcal{M}, \sigma \models \alpha \wedge \beta$ *iff* $\mathcal{M}, \sigma \models \alpha$ *and* $\mathcal{M}, \sigma \models \beta$,
- $\mathcal{M}, \sigma \models \bigcirc\alpha$ *iff* $\mathcal{M}, \sigma_{\geq 1} \models \alpha$,
- $\mathcal{M}, \sigma \models A\alpha$ *iff for every path* π*, if* $\sigma_0 = \pi_0$ *then* $\mathcal{M}, \pi \models \alpha$.
- $\mathcal{M}, \sigma \models \alpha U \beta$ *iff there is some* $i \in \omega$ *such that* $\mathcal{M}, \sigma_{\geq i} \models \beta$ *and for each* $j \in \omega$*, if* $0 \leq j < i$ *then* $\mathcal{M}, \sigma_{\geq j} \models \alpha$,
- $\mathcal{M}, \sigma \models P^s_{\geqslant r}\alpha$ *iff* $\mu_{\sigma_0}\{\pi \in \Sigma_{\sigma_0} \mid \mathcal{M}, \pi \models \alpha\} \geqslant r$,
- $\mathcal{M}, \sigma \models P^p_{\geqslant r}\alpha$ *iff* $\mu_{\sigma}\{\pi \in S_{\sigma} \mid \mathcal{M}, \pi \models \alpha\} \geqslant r$.

Note that the satisfiability of any state formula (for example $P^s_{\geqslant r}\alpha$) depends only on the initial state of the path, while the other formulas are path-dependent.

If $\mathcal{M} = \langle S, v, R, \Sigma, Prob^{state}, Prob^{path}\rangle$ is a model and $\sigma \in \Sigma$, we will denote:

- $[\alpha]^{path}_{\mathcal{M},\sigma} = \{\pi \in S_\sigma \mid \mathcal{M}, \pi \models \alpha\}$, and
- $[\alpha]^{state}_{\mathcal{M},s} = \{\pi \in \Sigma_s \mid \mathcal{M}, \pi \models \alpha\}$.

The possible problems in Definition 2 are that for an α the sets $[\alpha]^{path}_{\mathcal{M},\sigma}$ and $[\alpha]^{state}_{\mathcal{M},s}$ might not be in A_σ and in H_s, respectively. To overcome this, in the rest of the paper we will consider only so-called measurable models.

Definition 3. *A model* $\mathcal{M} = \langle S, v, R, \Sigma, Prob^{state}, Prob^{path}\rangle$ *is* measurable *if the following conditions are satisfied:*

- $[\alpha]^{path}_{\mathcal{M},\sigma} \in A_\sigma$*, for every* $\alpha \in For$,
- $[\alpha]^{state}_{\mathcal{M},s} \in H_s$*, for every* $\alpha \in For$.

We will denote the probabilistic branching-time temporal logic characterized by the class of all measurable models by pBTL_{Meas}.

The expression $\mathcal{M}, \sigma \models T$ denotes the fact that $\mathcal{M}, \sigma \models \alpha$, for every $\alpha \in T$. A formula α is satisfiable if there is a path σ in a model $\mathcal{M}$ such that $\mathcal{M}, \sigma \models \alpha$. A formula is valid if $\mathcal{M}, \sigma \models \alpha$ for every model $\mathcal{M}$ and every path σ of $\mathcal{M}$. We write $T \models \alpha$ ("α is a semantical consequence of T"), if for every model $\mathcal{M}$ and every σ in $\mathcal{M}$, if $\mathcal{M}, \sigma \models T$, then $\mathcal{M}, \sigma \models \alpha$.

3 Axiomatization

Propositional axioms

A1. all the tautologies of the classical propositional logic

Temporal axioms

A2. $\bigcirc(\alpha \rightarrow \beta) \rightarrow (\bigcirc\alpha \rightarrow \bigcirc\beta)$
A3. $\neg \bigcirc \alpha \leftrightarrow \bigcirc\neg\alpha$
A4. $\alpha U\beta \leftrightarrow \beta \vee (\alpha \wedge \bigcirc(\alpha U\beta))$
A5. $p \rightarrow Ap,\ \ p \in \mathcal{P}$
A6. $Ep \rightarrow p,\ \ p \in \mathcal{P}$
A7. $A\alpha \rightarrow \alpha$
A8. $A(\alpha \rightarrow \beta) \rightarrow (A\alpha \rightarrow A\beta)$
A9. $A\alpha \rightarrow AA\alpha$
A10. $E\alpha \rightarrow AE\alpha$

Probabilistic axioms ($x \in \{p, s\}$)

A11. $P^x_{\geq 0}\alpha$
A12. $P^{\overline{x}}_{\leq s}\alpha \rightarrow P^x_{<t}\alpha,\ t > s$
A13. $P^{\overline{x}}_{<s}\alpha \rightarrow P^x_{\leq s}\alpha$
A14. $(P^x_{\geq s}\alpha \wedge P^{\overline{x}}_{\geq r}\beta \wedge P^x_{\geq 1}(\neg\alpha \vee \neg\beta)) \rightarrow P^x_{\geq \min(1,s+r)}(\alpha \vee \beta)$
A15. $(P^{\overline{x}}_{\leq s}\alpha \wedge P^{\overline{x}}_{<r}\beta) \rightarrow P^x_{<s+r}(\alpha \vee \beta),\ s + r \leq 1$

Axioms about probability and temporality

A16. $G\alpha \rightarrow P^p_{\geq 1}\alpha$
A17. $A\alpha \rightarrow P^{\overline{s}}_{\geq 1}\alpha$
A18. $P^s_{\geq r}\alpha \rightarrow AP^s_{\geq r}\alpha$
A19. $EP^s_{\geq r}\alpha \rightarrow P^{\overline{s}}_{\geq r}\alpha$

Inference rules

R1. from $\{\alpha, \alpha \rightarrow \beta\}$ infer β
R2. from α infer $\bigcirc\alpha$
R3. from α infer $A\alpha$
R4. from the set of premises

$$\{\gamma \rightarrow \neg((\wedge^i_{k=0} \bigcirc^k \alpha) \wedge \bigcirc^{i+1}\beta) \mid i \in \omega\}$$

infer $\gamma \rightarrow \neg(\alpha U\beta)$
R5. from the set of premises

$$\{\beta \rightarrow \bigcirc^m P^x_{\geq r-\frac{1}{k}}\alpha \mid k \in \omega, k \geq \frac{1}{r}\}$$

infer $\beta \rightarrow \bigcirc^m P^x_{\geq r}\alpha$ (for any $m \in \omega$ and $x \in \{p, s\}$)

Let us briefly discuss some of the above axioms and rules. By the axiom A1 and the inference rule R1 (Modus ponens), pBTL extends the classical propositional logic. The axioms A2–A4 are standard axioms of discrete linear-time temporal logic, while the axioms A5–A10 concern the non-linear aspect of the temporal logic [34]. Probabilistic axioms captures the basic properties of probability: non-negativity and finite additivity. The last group of axioms concerns mixing of probabilistic and temporal reasoning.

The inference rules R2 and R3 are the variants of modal Necessitation. They can be applied only to theorems. The rules R4 and R5 are infinitary inference rules. The former one characterizes the until operator, while the later one intuitively says that if the probability is arbitrarily close to r, then it is at least r.

We say that a formula α is deducible from a set T of formulas, and write $T \vdash \alpha$, if there is an at most countable sequence of formulas $\alpha_0, \alpha_1, \ldots, \alpha$, such that every α_i is an axiom or a formula from T, or it is derived from the preceding formulas by an inference rule (with the exception that R2 and R3 can be applied to theorems only). That sequence is called the proof of α from T. The formula α is a theorem, denoted by $\vdash \alpha$, if it is deducible from the empty set. A set T of formulas is consistent if there is at least one formula which is not deducible from T; otherwise it is inconsistent. A consistent set T of sentences is said to be maximally consistent if for every $\alpha \in For$, either $\alpha \in T$ or $\neg\alpha \in T$.

It is easy to prove soundness of the proposed axiomatic system (with respect to the considered class of models), using a straightforward induction on the length of the inference.

4 Completeness

In this section, some straightforward parts of the proof are omitted because of limited space.

Theorem 1 (Deduction theorem). *If T is a set of formulas, φ is a formula, and $T, \varphi \vdash \psi$, then $T \vdash \varphi \to \psi$.*

Proof. The proof is on the the transfinite induction on the length of the inference. We will only consider the case when we apply the inference rule R4.

If $T, \varphi \vdash \gamma \to \neg(\alpha U \beta)$ is obtained by the inference rule R4, then $T, \varphi \vdash \gamma \to \neg((\wedge_{k=0}^{i} \bigcirc^k \alpha) \wedge \bigcirc^{i+1}\beta)$, for all $i \in \omega$. By the induction hypothesis, we have $T \vdash \varphi \to (\gamma \to \neg((\wedge_{k=0}^{i} \bigcirc^k \alpha) \wedge \bigcirc^{i+1}\beta))$ (for all $i \in \omega$). From A1 we obtain $T \vdash (\varphi \wedge \gamma) \to (\neg((\wedge_{k=0}^{i} \bigcirc^k \alpha) \wedge \bigcirc^{i+1}\beta)))$, for all $i \in \omega$. Applying the inference rule R4 we conclude $T \vdash (\varphi \wedge \gamma) \to (\neg(\alpha U \beta))$. Finally, by A1 we obtain $T \vdash \varphi \to (\gamma \to \neg(\alpha U \beta))$.

The cases when ψ is a theorem and when we apply Modus ponens are standard, while the cases when we apply the inference rules R2 and R3 are trivial, since they can be applied to theorems only. In the case when we apply R5, the proof is similar to the considered case (R4). □

Lemma 1. *Let α, β, γ be formulas.*

1. *the following inference rule is derivable: from the set of formulas*

$$\{\gamma \to \bigcirc^i \beta \mid i \in \omega\}$$

   infer $\gamma \to G\beta$,
2. *if $\vdash \alpha$, then $\vdash G\alpha$,*
3. $\vdash G \bigcirc \alpha \leftrightarrow \bigcirc G\alpha$,

4. $\vdash (\bigcirc\alpha \to \bigcirc\beta) \to \bigcirc(\alpha \to \beta)$,
5. $\vdash \bigcirc(\alpha \wedge \beta) \leftrightarrow (\bigcirc\alpha \wedge \bigcirc\beta)$,
6. $\vdash \bigcirc(\alpha \vee \beta) \leftrightarrow (\bigcirc\alpha \vee \bigcirc\beta)$,
7. $G\alpha \vdash \bigcirc^i\alpha$ *for every* $i \geq 0$,
8. *if* $T \vdash \alpha$, *where* T *is a set of formulae, then* $\bigcirc T \vdash \bigcirc\alpha$.
9. *for* $j \geq 0$, $\bigcirc^j\beta, \bigcirc^0\alpha, \ldots, \bigcirc^{j-1}\alpha \vdash \alpha U\beta$,
10. *if* T *is a set of formulas and* $T \vdash \alpha$, *then* $AT \vdash A\alpha$.
11. $\vdash G\alpha \leftrightarrow \alpha \wedge \bigcirc G\alpha$,
12. $\vdash G(\alpha \to \beta) \to (G\alpha \to G\beta)$,
13. $\vdash G(\alpha \to \bigcirc\alpha) \to (\alpha \to G\alpha)$,
14. $\vdash (G(\alpha \to \alpha_1) \wedge (\alpha U\beta)) \to (\alpha_1 U\beta)$,
15. $\vdash (G(\beta \to \beta_1) \wedge (\alpha U\beta)) \to (\alpha U\beta_1)$,
16. $\vdash F\alpha \leftrightarrow F\neg\neg\alpha$
17. $\vdash \alpha U\beta \to F\beta$.

Proof. (1) is an immediate consequence of R4, obtained by replacing α and β with $\top$ and $\neg\beta$, respectively. (2) follows from (1) and R2.

For the proof of (3), (8) and (9) we refer the reader to [26], while the proof of (10) can be found in [7].
(14) Note that by (9) we have:

– $G(\alpha \to \alpha_1) \vdash \neg(\alpha_1 U\beta) \to \neg((\wedge_{k=0}^{i-1} \bigcirc^k \alpha_1) \wedge \bigcirc^i\beta)$, for every $i \geq 0$
– $G(\alpha \to \alpha_1) \vdash \neg(\alpha_1 U\beta) \to ((\wedge_{k=0}^{i-1} \bigcirc^k \alpha_1) \to \neg \bigcirc^i \beta)$, for every $i \geq 0$
– $G(\alpha \to \alpha_1) \vdash \neg(\alpha_1 U\beta) \to ((\wedge_{k=0}^{i-1} \bigcirc^k \alpha) \to \neg \bigcirc^i \beta)$, for every $i \geq 0$
– $G(\alpha \to \alpha_1) \vdash \neg(\alpha_1 U\beta) \to \neg((\wedge_{k=0}^{i-1} \bigcirc^k \alpha) \wedge \bigcirc^i\beta)$, for every $i \geq 0$
– $G(\alpha \to \alpha_1) \vdash \neg(\alpha_1 U\beta) \to \neg((\alpha U\beta))$, by R4

Thus, the statement holds. The statement (15) can be proved in a similar way, while (16) follows from the definition of $F\alpha = \top U\alpha$ and the previous steps. (17) follows directly from (14), taking $\alpha_1 = \top$. The remaining statements are easy consequences of the temporal part of the above axiomatization. □

Note that Lemma 1 states that some of the formulas and inference rules, proposed as the part of some (weakly) complete axiomatic systems [4,32,34] for temporal reasoning, hold in our logic. Thus, the temporal part of our axiomatization is sufficient to capture the semantical properties of the operators $\bigcirc$, A and U.

Theorem 2. *Every consistent set* T *of formulas can be extended to a maximal consistent set* T^*.

Proof. Let us assume that $For = \{\alpha_i \mid i \in \omega\}$. The maximally consistent set T^* is defined recursively, as follows:

1. $T_0 = T$.
2. If α_i is consistent with T_i, then $T_{i+1} = T_i \cup \{\alpha_i\}$.
3. If α_i is not consistent with T_i, then:

(a) Otherwise, if α_i has the form $\gamma \to \neg(\alpha U \beta)$, then

$$T_{i+1} = T_i \cup \{\gamma \to ((\wedge_{k=0}^{n_0} \bigcirc^k \alpha) \wedge \bigcirc^{n_0+1}\beta)\},$$

where n_0 is a positive integer such that T_{i+1} is consistent.

(b) Otherwise, if α_i is of the form $\gamma \to \bigcirc^m P^x_{\geq r}\beta$, for $x \in \{p, s\}$, then

$$T_{i+1} = T_i \cup \{\gamma \to \neg \bigcirc^m P^x_{\geq r-\frac{1}{n_1}}\beta\}$$

where n_1 is a positive integer such that T_{i+1} is consistent.

(c) Otherwise, $T_{i+1} = T_i$.

4. $T^* = \bigcup_{n\in\omega} T_n$.

Let us prove the existence of the number n_0 in 3(a). If we suppose that $\gamma \to ((\wedge_{k=0}^{n} \bigcirc^k \alpha) \wedge \bigcirc^{n+1}\beta)$ is not consistent with T_i, for every $n \in \omega$, then, by Theorem 1, $T_i \vdash \neg(\gamma \to ((\wedge_{k=0}^{n} \bigcirc^k \alpha) \wedge \bigcirc^{n+1}\beta))$, for every $n \in \omega$. By A1 we obtain $T_i \vdash \gamma \to \neg((\wedge_{k=0}^{n} \bigcirc^k \alpha) \wedge \bigcirc^{n+1}\beta)$, for every $n \in \omega$. By R4 we have $T_i \vdash \gamma \to \neg(\alpha U \beta)$, which contradicts the assumption. The proof of the existence of the number n_1 in 3(b) is similar.

It is easy to show that T_i is consistent for every i, and that for each $\alpha \in For$, either $\alpha \in T^*$ or $\neg\alpha \in T^*$.

Note that deductive closeness of T^* would imply its consistency: $T^* \vdash \bot$ would imply $\bot \in T^*$, thus there would exist i such that $\bot \in T_i$, which is impossible. In order to prove that T^* is deductively closed, it is sufficient to prove that it is closed under the inference rules, since all instances of axioms are obviously in T^*. We will only prove closeness under the inference rule R4, since the case when we consider R5 is similar, while the other cases are trivial.

Suppose that $\gamma \to \neg(\alpha U \beta) \notin T^*$, while $\gamma \to \neg((\wedge_{k=0}^{i} \bigcirc^k \alpha) \wedge \bigcirc^{i+1}\beta) \in T^*$ for every $i \in \omega$. By maximality of T^*, $\neg(\gamma \to \neg(\alpha U \beta)) \in T^*$, or, equivalently, $\gamma \wedge (\alpha U \beta) \in T^*$. Consequently, $\gamma \in T^*$ and $\alpha U \beta \in T^*$, so there are $m, n \in \omega$ such that $\gamma \in T_m$ and $\alpha U \beta \in T_n$. If $\gamma \to \neg(\alpha U \beta) = \alpha_l$, then, by the construction of T^*, there is n_0 such that $\gamma \to ((\wedge_{k=0}^{n_0} \bigcirc^k \alpha) \wedge \bigcirc^{n_0+1}\beta) \in T_l$. By Lemma 1(9), $T_l \vdash \alpha U \beta$. Consequently, $T_{\max\{l,m,n\}}$, which is in contradiction with consistency of $T_{\max\{l,m,n\}}$. □

We define the equivalence relation $\sim$ on the set of maximally consistent sets of formulas as follows:

$T_1^* \sim T_2^*$ iff $T_1^* \cap St = T_2^* \cap St$.

The equivalence class of T^* is $[T^*] = \{T_1^* \mid T_1^* \sim T^*\}$.

A canonical model $\mathcal{M}^* = \langle S, v, R, \Sigma, Prob^{state}, Prob^{path}\rangle$ is defined in the following way:

- $S = \{[T^*] \mid T^*$ is maximally consistent set of formulas$\}$,
- $v([T^*], p) = 1$ iff $T^* \vdash p,\ \ p \in \mathcal{P}$,
- $[T_1^*]R[T_2^*]$ if there exist $T_3^* \sim T_1^*$, $T_4^* \sim T_2^*$ such that $T_4^* = \{\alpha \mid \bigcirc \alpha \in T_3^*\}$,

– Σ is the set of paths $[T_0^*]$, $[T_1^*]$, $[T_2^*]$,... such that $T_{i+1}^* = \{\alpha | \bigcirc \alpha \in T_i^*\}$, for all $i \in \omega$. If the sequence $\{T_i^*\}_{i\in\omega}$ determines a path σ, we will write $\sigma(i)$ for T_i^*,
– $Prob^{path}$ is defined as follows: for every $\sigma = [T_0^*]$, $[T_1^*]$, $[T_2^*]$,..., $Prob_\sigma = \langle A_\sigma, \mu_\sigma \rangle$ is a probability space such that:
  - $A_\sigma = \{[\alpha]_\sigma \mid \alpha \in For\}$, where $[\alpha]_\sigma = \{\sigma_{\geqslant} i \mid T_i^* \vdash \alpha, i \in \omega\}$,
  - $\mu_\sigma([\alpha]_\sigma) = \sup\{r \in \mathbb{Q} \cap [0,1] \mid T_0^* \vdash P^p_{\geqslant r}\alpha\}$,
– $Prob^{state}$ is defined as follows: for every $\sigma = [T_0^*]$, $[T_1^*]$, $[T_2^*]$,..., the probability space $Prob_\sigma = \langle A_\sigma, \mu_\sigma \rangle$ is determined by the following conditions: that:
  - $H_s = \{[\alpha]_s \mid \alpha \in For\}$, where $[\alpha]_s = \{\pi \mid \pi(0) \sim T_0^*,\ \pi(0) \vdash \alpha\}$,
  - $\mu_s([\alpha]_s) = \sup\{r \in \mathbb{Q} \cap [0,1] \mid T_0^* \vdash P^s_{\geqslant r}\alpha\}$.

Theorem 3. *$\mathcal{M}^*$ is a* pBTL*-model.*

Proof. Note that definitions of v and μ_s depend on the chosen element of equivalence class. We will show that the definition of $\mathcal{M}^*$ is correct:

– v is well defined, since $\mathcal{P} \subseteq St$, so $T_1^* \vdash p$ iff $T_2^* \vdash p$, whenever $T_1^* \sim T_2^*$, $p \in \mathcal{P}$.
– The definition of R is correct. Namely, using Temporal axioms, one can show that the properties of consistency and maximality transfer from T^* to $\{\alpha | \bigcirc \alpha \in T^*\}$. Moreover, R is obviously a total relation.
– A_σ is an algebra of sets. It is easy to show that $S_\sigma = [\top]_\sigma$, $[\alpha]_\sigma^c = [\neg\alpha]_\sigma$ and $[\alpha]_\sigma \cup [\beta]_\sigma = [\alpha \vee \beta]_\sigma$. Similarly, H_s is an algebra of sets.
– The function μ_s is well defined, since any formula of the form $P^s_{\geqslant r}\alpha$ is a state formula, so it belongs to a maximally consistent set T_1^* if and only if it belongs to any other maximally consistent set $T_2^* \in [T_1^*]$. Consequently, $\sup\{r \in \mathbb{Q} \cap [0,1] \mid T_1^* \vdash P^s_{\geqslant r}\alpha\} = \sup\{r \in \mathbb{Q} \cap [0,1] \mid T_2^* \vdash P^s_{\geqslant r}\alpha\}$.

By the axiom A11, $\mu_s(\alpha) \geqslant 0$, for every $\alpha \in For$. By R3, $\vdash A\top$, so, by A17, $T^* \vdash P^s_{\geqslant 1}\top$, for every maximally consistent set T^*. Since $H_s = [\top]_s$, we obtain $\mu_s(H_s) = 1$. Similarly, $\mu_\sigma(A_\sigma) = 1$ (by Lemma 1(2) and A16).

For the proof of finite additivity of μ_s and μ_σ, we refer the reader to [26], where a similar result is proved. □

Note that, since each $[T^*]$ may contain many maximally consistent sets, it is possible that one state belongs to several paths.

Theorem 4 (Strong completeness theorem). *Every consistent set of formulas is satisfiable.*

Proof. Let T be a consistent set of formulas, and let $\mathcal{M}^*$ be the model constructed above. We will prove that for every $\alpha \in For$, $\mathcal{M}^*, \sigma \models \alpha$ iff $\alpha \in \sigma(0)$.

If α is a propositional letter, this is immediate consequence of the definition of v. The proof in the cases when α is a negation or a conjunction is standard. For the proof in the cases when α is of the form $\bigcirc\beta$ or $\beta U \gamma$, we refer the reader to [26], where the similar proofs are presented.

Let $\alpha = A\beta$. If $\mathcal{M}^*, \sigma \not\models A\beta$, then there exists $\pi \in \Sigma_{\sigma_0}$ such that $\mathcal{M}^*, \pi \models \neg\beta$. By the induction hypothesis we obtain $\neg\beta \in \pi(0)$, so $\beta \notin \pi(0)$. By Axiom A7, $A\beta \notin \pi(0)$. From $\pi(0) \sim \sigma(0)$ and $A\beta \in St$, we conclude $A\beta \notin \sigma(0)$. For the other direction, suppose that $\mathcal{M}^*, \sigma \models A\beta$. Then for all $\pi \in \Sigma_{\sigma_0}$, $\mathcal{M}^*, \sigma \models \beta$. Consequently, by the induction hypothesis, for all $\pi \in \Sigma_{\sigma_0}$, $\beta \in \pi(0)$. If $A\beta \notin \sigma(0)$, using Temporal axioms one can show that there exists $\rho \in \Sigma_{\sigma_0}$ such that $\beta \notin \rho(0)$, which contradicts the assumption.

Let $\alpha = P^s_{\geq r}\beta$ (in the case when $\alpha = P^p_{\geq r}\beta$ the proof is similar). Suppose that $\mathcal{M}^*, \sigma \models P^s_{\geq r}\beta$. If $\sup\{t \in \mathbb{Q} \cap [0,1] \mid P^s_{\geq t}\beta \in \sigma(0)\} = r$, then $P^s_{\geq r}\beta \in \sigma(0)$, by the maximality of $\sigma(0)$ and the rule R3. If $\sup\{t \in \mathbb{Q} \cap [0,1] \mid P^s_{\geq t}\beta \in \sigma(0)\} > r$, then there exists $q \in \mathbb{Q} \cap (r, \sup\{t \in \mathbb{Q} \cap [0,1] \mid P^s_{\geq t}\beta \in \sigma(0)\}]$ such that $P^s_{\geq q}\beta \notin \sigma(0)$. By deductively closeness of $\sigma(0)$, $P^s_{\geq r}\beta \in \sigma(0)$. On the other hand, if $P^s_{\geq r}\beta \in \sigma(0)$, then $\mu_s(\{\pi \mid \pi(0) \sim \sigma(0),\ \pi(0) \vdash \beta\}) = \sup\{t \in \mathbb{Q} \cap [0,1] \mid P^s_{\geq t}\beta \in \sigma(0)\} \geq r$. By the induction hypothesis, $\{\pi \mid \pi(0) \sim \sigma(0),\ \pi(0) \vdash \beta\} = \{\pi \mid \pi(0) \sim \sigma(0),\ \mathcal{M}^*, \pi \models \beta\}$, so $\mathcal{M}^*, \sigma \models P^s_{\geq r}\beta$.

Let T^* be a maximally consistent set such that $T \subseteq T^*$. If $\sigma = [T^*], [\{\alpha | \bigcirc\alpha \in T^*\}], [\{\alpha | \bigcirc^2 \alpha \in T^*\}] \ldots$, then $\mathcal{M}^*, \sigma \models T$. □

Note that, by the proof of the previous theorem, $[\alpha]_\sigma = \{\sigma_{\geqslant} i \mid T^*_i \vdash \alpha, i \in \omega\} = \{\pi \in S_\sigma \mid \mathcal{M}^*, \pi \models \alpha\} = [\alpha]^{path}_{\mathcal{M},\sigma}$. Similarly, $[\alpha]_s = [\alpha]^{state}_{\mathcal{M}^*,\sigma}$, so $\mathcal{M}^*$ is a measurable model.

Corollary 1. *If α is a formula and T is a set of formulas, then $T \models \alpha$ implies $T \vdash \alpha$.*

Proof. Let $T \models \alpha$. Then $T \cup \{\neg\alpha\}$ is not satisfiable. By Theorem 4, $T \cup \{\neg\alpha\} \vdash \bot$, and, by Theorem 1, $T \vdash \alpha$.

5 Related Work

The branching-time logic PCTL for reasoning about time and probability is described in [17]. The underlying temporal logic is Computational Tree Logic CTL (Emerson, Clark, Sistla [10]). The statements of the form: "after a request for service there is at least a 98% probability that the service will be carried out within 2 seconds" are expressible in the language of PCTL. Formulas are interpreted over discrete time Markov chains and algorithms for checking satisfiability of formulas by a given Markov chain are described. No axiomatization is presented. The logic follows the division of CTL into state formulas and path formulas. The classical propositional language is enriched in the following way:

- $\alpha U^{\leq t}\beta$ and $\alpha \mathcal{U}^{\leq t}\beta$ are path formulas, if α and β are state formulas, and $t \in \omega \cup \{\infty\}$. The intuitive meaning of $\alpha U^{\leq t}\beta$ is similar to the meaning of $\alpha U\beta$, with the exception that β has to become true within t time instances (for $t = \infty$, $U^{\leq t}$ and U coincide). The relation of $\alpha\mathcal{U}^{\leq t}\beta$ to $\alpha\mathcal{U}\beta \equiv \alpha U\beta \vee G\alpha$ is analogous.

– $\alpha U^{\leq t}_{>r}\beta$ and $\alpha \mathcal{U}^{\leq t}_{>r}\beta$ are state formulas, if α and β are path formulas, and $t \in \omega \cup \{\infty\}$. The meaning of those formulas is given by the satisfiability relation (formulation is adopted according to our terminology):
$\mathcal{M}, \sigma \models \alpha U^{\leq t}_{>r}\beta$ iff $\mu_{\sigma_0}(\{\pi \mid \sigma_0 = \pi_0,\ \mathcal{M}, \pi \models \alpha U^{\leq t}\beta\}) > r$,

The formulas of PCTL are expressible in our language. For example:

– $\alpha U^{\leq n}\beta$ may be written as $\beta \vee \bigvee_{i=1}^{n}((\wedge_{k=0}^{i-1} \bigcirc^k \alpha) \wedge \bigcirc^i \beta)$,
– $\alpha U^{\leq n}_{>r}\beta$ may be written as $P^s_{>r}(\beta \vee \bigvee_{i=1}^{n}((\wedge_{k=0}^{i-1} \bigcirc^k \alpha) \wedge \bigcirc^i \beta))$.

On the other hand, our operator $P^p_{\geqslant r}$ is not expressible in PCTL. Also, boolean combinations of state and path formulas are not PCTL-formulas.

A more expressive branching-time logic denoted PCTL* is described in [2]. The underlying temporal logic is CTL* with path quantifiers replaced by probabilities ($P_{=1}$, $P_{>0}$). Thus, the propositional language is extended with:

– state formulas: $P_{\geq r}\alpha$ (α is a path formula),
– path formulas: $\bigcirc\alpha$, $\alpha U \beta$ (α and β are state formulas).

According to definition of satisfiability, their probability operator $P_{\geqslant r}$ corresponds to our operator $P^s_{\geqslant r}$, while our operator $P^p_{\geqslant r}$ is not expressible in PCTL*. Similarly as in PCTL, the conjunction of a state formula and a path formula is not a formula. No axiomatization for PCTL* is given.

The paper [3] presents model-checking algorithms for extensions of PCTL and PCTL* that involve non-determinism.

A probabilistic modal logic PPL is introduced in [35]. It allows applying probabilities to sequences of formulas (giving so called path expressiveness). A Gentzen-style axiom system is presented and proved to be sound and complete. Probabilities are expressed using terms (similarly as in [12]). The language allows linear combinations of terms of the form $P(\alpha_1, \ldots, \alpha_n)$ which means "the probability of the sequence of formulas." Iteration of probabilities in a term is allowed. The formula $P(\alpha_1, \ldots, \alpha_n) \geq r$ is expressible in our logic as $P^s_{\geqslant r}(\bigwedge_{i=1}^{n} \bigcirc^i \alpha_i)$. On the other hand, formulas of PPL can not express probability within a path ($P^p_{\geqslant r}$). Also, the temporal operators are not definable in PPL. Although our language does not allow linear combinations of probabilities, combining the techniques from [8,9,28], where arithmetical operations are built into the syntax of probabilistic logic, with the ideas presented in this paper, would lead to a logic in which formulas of PPL are expressible.

In [18] and [21] propositional logics that use the languages of CTL and CTL* are presented. The probabilities are not expressible in syntax, but the formulas are interpreted over Markov systems which can simulate the execution of probabilistic programs.

The papers [15,19,22] introduce real-time interval logics that can be used in design of an embedded real-time systems. The infinite intervals are considered in [16].

The language of the logic presented in [13] is based on the propositional dynamic logic, and the main objects are programs. Probabilistic operators can be applied on a limited class of formulas, and the completeness problem is not solved. A fragment of [13]) is considered in [20]. A dynamic generalization of the logic of qualitative probabilities from [33] is presented in [14]. Completeness is proved using an infinitary rule, similarly as in our approach.

6 Conclusion and Future Work

We have introduced the propositional probabilistic branching time logic pBTL that enables us to formulate (and combine) both purely temporal statements and the expressions such as: "in at least half of paths α holds in at least a third of states". The formulas are interpreted over models that involve a class of probability measures assigned to states, and a class of probability measures assigned to paths. We have proved that the infinitary axiomatic system for pBTL is sound and strongly complete.

One of the main axiomatization issues for temporal logics with the operators $\bigcirc$ and G, and for real valued probability logics is the non-compactness phenomena. The set of formulas $\{P^s_{>0}\alpha\} \cup \{P^s_{\leqslant \frac{1}{n}}\alpha \mid n \in \omega\}$ and $\{G\alpha\} \cup \{\bigcirc^n \neg\alpha \mid n \in \omega\}$ are finitely satisfiable but they are not satisfiable. It is well known that, in the absence of compactness, any finitary axiomatization would be incomplete. Thus, infinitary axiomatic systems are the only way to establish strong completeness.

The temporal fragment of pBTL uses the language of CTL^*. The restricted class of models (without probabilities) corresponds to the class of models of so-called $\forall LT$ logic from [34] (compare Lemma 1 and the axiomatic system from [34]). The paper [32] solved the problem of (weak) completeness of Full Computation Tree Logic (with the class of models satisfying the desirable properties FC (Fusion closed) and LC (Limit closed)), extending the axiomatization of $\forall LT$. Thus, the question of extending the temporal part of our axiomatization, with the aim to obtain completeness of probabilistic Full Computation Tree Logic, naturally arise.

Also, we believe that there are several other promising ways to extend the results presented here, along the lines of our previous research:

- Combining the techniques from this paper and [7] may lead to the first-order extension of pBTL. That logic would be not only of theoretical interest, since the set of all valid formulas is not recursively enumerable [1], and no complete finitary axiomatization is possible in that undecidable framework. In this situation, a complete (even if infinitary) axiomatization would be of great practical significance.
- A branching time logic in which linear combinations of probabilities are expressible could be developed combining the ideas presented here with the ideas from [8,9,28]. The formulas of the logic presented in [35] would be expressible in the resulting language (see Section 5).
- It is well known that CTL and CTL^* are decidable [11]. We expect that, similarly as it is done in [26] for probabilistic linear time logic, it is possible to adapt the corresponding procedures to prove decidability of the logic presented here.

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