Coloring and Maximum Independent Set of Rectangles

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Abstract. In this paper, we consider two geometric optimization problems: Rectangle Coloring problem (RCOL) and Maximum Independent Set of Rectangles (MISR). In RCOL, we are given a collection of n rectangles in the plane where overlapping rectangles need to be colored differently, and the goal is to find a coloring using minimum number of colors. Let q be the maximum clique size of the instance, i.e. the maximum number of rectangles containing the same point. We are interested in bounding the ratio $\sigma(q)$ between the total number of colors used and the clique size. This problem was first raised by graph theory community in 1960 when the ratio of $\sigma(q) \leq O(q)$ was proved. Over decades, except for special cases, only the constant in front of q has been improved. In this paper, we present a new bound for $\sigma(q)$ that significantly improves the known bounds for a broad class of instances.

The bound $\sigma(q)$ has a strong connection with the integrality gap of natural LP relaxation for MISR, in which the input is a collection of rectangles where each rectangle is additionally associated with non-negative weight, and our objective is to find a maximum-weight independent set of rectangles. MISR has been studied extensively and has applications in various areas of computer science. Our new bounds for RCOL imply new approximation algorithms for a broad class of MISR, including (i) $O(\log \log n)$ approximation algorithm for unweighted MISR, matching the result by Chalermsook and Chuzhoy, and (ii) $O(\log \log n)$-approximation algorithm for the MISR instances arising in the Unsplittable Flow Problem on paths. Our technique builds on and generalizes past works.

1 Introduction

In this paper, we devise algorithms for two geometric optimization problems: Rectangle Coloring problem (RCOL) and Maximum Independent Set of Rectangles (MISR). In RCOL, we are given a collection $\mathcal{R}$ of n rectangles. Our objective is to find a valid coloring of rectangles such that no two overlapping rectangles get the same color, while minimizing the number of colors. Clearly, this problem is a special case of Graph Coloring problem: Define graph $G = (V, E)$ where the vertex set corresponds to rectangles, and there is an edge connecting two vertices that correspond to overlapping rectangles. We denote by $\omega(\mathcal{R})$ the size of maximum clique of collection $\mathcal{R}$ and $\chi(\mathcal{R})$ its chromatic number. When $\mathcal{R}$ is clear from context, we will often use q to denote $\omega(\mathcal{R})$. Note that $\chi(\mathcal{R}) \geq \omega(\mathcal{R})$, so an interesting question to ask is how large the ratio $\chi(\mathcal{R})/\omega(\mathcal{R})$ can be.

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We denote such ratio by $\sigma(\mathcal{R})$ and define $\sigma_{\mathsf{rect,q}} = \sup_{\mathcal{R}:\omega(\mathcal{R})=q} \sigma(\mathcal{R})$. We are interested in bounding $\sigma_{\mathsf{rect,q}}$ as a function that depends on q but not on the input size. Notice that obtaining such bounds in general graphs is impossible as Erdös observed that there are family of graphs with maximum clique size 2 and arbitrarily large chromatic number [14]. However, for several interesting family of geometric intersection graphs, such as rectangles, segments and circular arc graphs, this ratio is well defined and has been studied; please refer to the survey by Kostochka [21] for more detail. For rectangle intersection graphs, not much progress has been made. In 1960, Asplund and Grünbaum show that any collection $\mathcal{R}$ of rectangles with clique size q can be colored by at most $O(q^2)$ colors, implying the ratio $\sigma_{\mathsf{rect,q}} \leq O(q)$, where they also prove the lower bound of 3. This bound remained asymptotically best known. In this paper, we show a new bound of $\sigma(\mathcal{R}) \leq O(\gamma \log q)$ where γ is a parameter we will define later. Since our γ is at most q, in the worst case, we still have the bound of $\tilde{O}(q)$, and we get an improvement when $\gamma = o(\frac{q}{\log q})$. For a broad class of instances, γ is expected to be constant.

It turns out that this bound is enough for us to get an improved approximation factor for a large class of MISR instances. In the MISR problem, the input is the set $\mathcal{R}$ of rectangles in the plane where each rectangle $R \in \mathcal{R}$ is associated with weight w_R, and the goal is to find a maximum weight subset of non-overlapping rectangles. Being one of the most fundamental problem in computational geometry, MISR comes up in various areas of computer science, e.g. in data mining [20,17,22], map labeling [1,13], channel admission control [23], and pricing [12]. MISR is NP-hard [16,19], and there has been a long line of attack on the problem, proposing approximation algorithms for both general cases [20,1,24,5,23] and special cases [15,10,8]. Currently the best known approximation ratio is $O(\log n / \log\log n)$ by Chan and Har-peled [11]. Through the connection between RCOL and MISR, our bounds for $\sigma(\mathcal{R})$ immediately give $O(\gamma \log\log n)$ approximation algorithms for MISR, where $\gamma \leq O(\log n)$.

Here we discuss some consequences of our results. For unweighted setting of MISR, the value of γ is one, so our result would give $O(\log\log n)$ approximation algorithm, matching the bound of [8]. For general (weighted) MISR, if γ is constant, our algorithm would give $O(\log\log n)$ approximation factor. An evidence that our result could be useful is when Bonsma, Schulz, and Wiese [7] recently showed a constant factor approximation algorithm for UNSPLITTABLE FLOW problem, and their main ingredient is an algorithm that solves a very restricted instance of MISR. Despite being very special cases, no approximation algorithms for MISR existing in the literature could directly apply to give $O(\log^{1-\epsilon} n)$ bound for it. However, it is easy to show that the instances they solve have $\gamma \leq 2$ (details are in the full version), so we obtain an $O(\log\log n)$ approximation algorithm for solving this instance; note, however, that the algorithm of [7] solves this instance exactly in polynomial time.

Our contributions: The main technical contribution of this paper is a new bound for $\sigma(\mathcal{R})$ improving upon the bound in [18] when $\gamma = o(\frac{q}{\log q})$. We now discuss this quantity precisely. For each rectangle $R \in \mathcal{R}$, we define the *containment* *depth* $\mathsf{d}(R)$ as the number of rectangles $R' \neq R$ that completely contains R, and $\mathsf{d}(\mathcal{R}) = \max_{R \in \mathcal{R}} \mathsf{d}(R)$. Notice that $\mathsf{d}(\mathcal{R}) \leq q$. The set $H(R)$ is defined as $H(R) = \{R' : R' \subseteq R\}$. Then we let $h(R)$ be the size of the maximum independent set $\mathcal{S} \subseteq H(R)$ such that all rectangles in $\mathcal{S}$ can be pierced by one horizontal line (in other words, the projection of rectangles in $\mathcal{S}$ onto vertical line share some common point). Then define $h(\mathcal{R}) = \max_{R \in \mathcal{R}} h(R)$.

Theorem 1. *For any collection $\mathcal{R}$ of rectangles with $\gamma = \min\{\mathsf{d}(\mathcal{R}), h(\mathcal{R})\} + 1$, there is a polynomial time algorithm that finds $O(\gamma q \log q)$-coloring of $\mathcal{R}$. In particular, $\sigma(\mathcal{R}) \leq O(\gamma \log q)$, which is $o(q)$ when $\gamma = o(\frac{q}{\log q})$.*

For purpose of solving the unweighted MISR problem, we may assume without loss of generality that there is no containment of any two rectangles in $\mathcal{R}$ (so $\mathsf{d}(\mathcal{R}) = 0$): Assume there are two rectangles R, R' such that R' contains R. We simply remove rectangle R' from the collection without affecting the optimal solution. Therefore, it is interesting and natural to study the ratio $\sigma_{\mathsf{rect-nc,q}}$ defined as the maximum $\sigma(\mathcal{R})$ where $\mathcal{R}$ is a collection of rectangles with $\mathsf{d}(\mathcal{R}) = 0$.

Corollary 1. *For collection $\mathcal{R}$ in which for any two rectangles R and R', R does not contain R', there is an $O(q \log q)$-coloring algorithm that runs in polynomial time. In particular, $\sigma_{\mathsf{rect-nc,q}} \leq O(\log q)$.*

Through the connection between MISR and RCOL (presented in Section 3), we get the following approximation bound for MISR.

Theorem 2. *For any collection $\mathcal{R}$ of rectangles, let $\gamma = \min\{\mathsf{d}(\mathcal{R}), \mathsf{h}(\mathcal{R})\} + 1$. There is an $O(\gamma \log \log n)$ approximation algorithm for MISR.*

We observe that our work relies a lot on dealing with the "corner information" of the intersecting rectangles. In fact, many recent works that solve optimization problems on rectangle intersection graphs have exploited this information in one way or another [4,11,8,23]. In Section 5, we investigate special cases of RCOL and MISR by restricting the intersection types and show additional results.

Related work: The study of the ratio $\sigma_{\mathsf{rect,q}}$ for rectangle intersection graphs started in 1948, when Bielecki [6] asked whether the value of $\sigma_{\mathsf{rect,q}}$ is independent of the instance size n. This question was answered positively by Asplund and Grünbaum in 1960 [3], when they show that $\chi(\mathcal{R}) \leq 4q^2 - 3q$, which implies that $\sigma_{\mathsf{rect,q}} \leq 4q - 3$. The bound was later improved to $\sigma_{\mathsf{rect,q}} \leq 3q - 2$ by Hendler [18], while the best lower bound remains $\sigma_{\mathsf{rect,q}} \geq 3$ by constructing a set of rectangles with clique size 2 and chromatic number 6 [3]; in fact, their result implies the exact bound $\sigma_{\mathsf{rect,2}} = 6$. Better bounds are known for special cases. For squares, a better bound of $\sigma_{\mathsf{squares,q}} \leq 4$ was shown by Ahlswede and Karapetyan, and independently by Perepelitsa (see [2]). Lewin-Eytan et al. show that $\sigma_{\mathsf{rect-non-corner,q}} = 1$ where the collections of interest do not have any rectangle that contains corners of other rectangles [23]. All these upper bounds imply polynomial time algorithms for finding the coloring. We refer to the survey by Kostochka for more related work [21].

For more related works on MISR, we refer the readers to [8,11] and references therein.

Organization: In Section 2, we recall the standard terms in graph theory, stated in the context of rectangles, and define the notations. We discuss the connection between MISR and RCOL in Section 3. Then we show the coloring algorithms in section 4.

2 Preliminaries

A rectangle R is given by a quadruple $(x^l(R), x^r(R), y^t(R), y^b(R))$ of real numbers, corresponding to the x-coordinates of its left and right boundaries and the y-coordinates of its top and bottom boundaries respectively. Furthermore, we assume that each rectangle is closed, i.e. each $R \in \mathcal{R}$ is defined as follows: $R = \{(x, y) : x^l(R) \leq x \leq x^r(R) \text{ and } y^b(R) \leq y \leq y^t(R)\}$. We say that rectangles R and R' intersect iff $R \cap R' \neq \emptyset$. In both RCOL and MISR, we are given a collection $\mathcal{R}$ of n-axis parallel rectangles. For MISR, each rectangle R is associated with weight w_R. The goal of the RCOL is to find a minimum coloring, while the goal of MISR is to find maximum-weight independent set.

We will distinguish among the three types of intersections: corner, crossing, and containment (see Figure 1) whose formal definitions are as follows. For two overlapping rectangles R, R', we let $j(R, R')$ denote the number of corners of R contained in R', and let $c(R, R') = \max\{j(R, R'), j(R', R)\}$. We say that the intersection between R and R' is a *corner* intersection iff $c(R, R') \in \{1, 2\}$. It is called a *crossing* iff $c(R, R') = 0$. Otherwise $c(R, R') = 4$, and we say that two rectangles have *containment* intersection.

2.1 Polynomially Bounded Weights

We argue that we can assume, by losing a constant factor in the approximation ratio, that all weights w_R are positive integers of values at most $2n$. We first scale the weights of rectangles so that the minimum weight is at least 1. Let $W_{\max}$ be the weight of the maximum weight rectangle. For each rectangle $R \in \mathcal{R}$, we assign a new weight

$$w'_R = \left\lfloor w_R \cdot \frac{2n}{W_{\max}} \right\rfloor$$

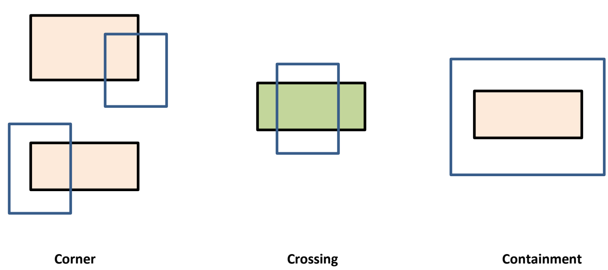

Fig. 1. Three possible intersection types

In the new instance, the weight of maximum weight rectangle becomes $2n$. It is easy to see that any γ-approximate solution to the new instance gives a 2γ-approximate solution to the original instance.

2.2 Rectangle Coloring and Degenerate Instances

Here we recall some standard terms in graph theory and state them in our context of rectangles, which is more convenient for our purpose. First, a set of rectangles $\mathcal{Q} \subseteq \mathcal{R}$ forms a *clique* if the intersection of all the rectangles in $\mathcal{Q}$ is non-empty. Let $\mathcal{R}'$ be a collection of rectangles. We say that $\mathcal{R}'$ admits a c-coloring if there exists an assignment $b : \mathcal{R}' \to [c]$ such that no two overlapping rectangles in $\mathcal{R}'$ get the same number. A collection $\mathcal{R}'$ of rectangles is k-*degenerate* if for every sub-collection $\mathcal{R}'' \subseteq \mathcal{R}'$, there exists a rectangle $R \in \mathcal{R}''$ such that R intersects with at most k other rectangles in $\mathcal{R}''$. It is a standard fact that any k-degenerate collection $\mathcal{R}'$ is $(k+1)$-colorable, and such coloring can be computed efficiently: Choose a rectangle $R \in \mathcal{R}'$ such that the size of neighbors of R, i.e. $|\{S \in \mathcal{R}' : S \cap R \neq \emptyset\}|$, is at most k. Recursively color the collection $\mathcal{R}' \setminus \{R\}$. Assign any color to R that does not conflict with any of R's neighbors.

2.3 Sparse Instances

We say that a collection of rectangles $\mathcal{R}$ is s-*sparse* if, for any rectangle $R \in \mathcal{R}$, there exist s points $p_1^R, p_2^R, \ldots, p_s^R \in \mathbb{R}^2$ associated with rectangle R (to be called *representative points* of R) such that the following holds. For any overlapping rectangles $R, R' \in \mathcal{R}$, either $p_i^{R'} \in R$ for some i or $p_j^R \in R'$ for some j. We note that Chan [9] uses similar ideas to define β-fat objects. For example, any collection of rectangles with only corner and containment intersections is 4-sparse: for each rectangle R, we can define $\left\{p_1^R, p_2^R, p_3^R, p_4^R\right\}$ to be the set of the four corners of R, so whenever two rectangles overlap, one rectangle contains some representative point of another.

Now we generalize the lemma in [23], again restated in our terms. The proof follows along the same lines and is omitted from this extended abstract.

Lemma 1. *Let $\mathcal{R}'$ be an s-sparse instance with maximum clique size q. Then $\mathcal{R}'$ is $(2sq)$-degenerate, and therefore is $(2sq+1)$-colorable.*

3 Independent Set and Coloring

In this section, we discuss the connections between RCOL and MISR. We remark that any c-coloring algorithm for $\mathcal{R}$ trivially implies an algorithm that finds an independent set of size $|\mathcal{R}|/c$. However, this bound is too loose for our purpose. The following theorem, whose analogous unweighted version was used implicitly in the prior work of the author with Chuzhoy [8], summarizes the connection between the two problems:

Theorem 3. *The integrality gap of natural LP relaxation of* MISR *is at most* $\sigma_{\mathsf{rect},\mathsf{O}(\log \mathsf{n})}$. *Moreover, if there is a polynomial time algorithm that finds a valid coloring of rectangles of* $\mathcal{R}$ *using at most* $qf(q)$ *colors, then we have a* $c_1 f(\log n)$ *randomized approximation algorithm for* MISR *where* c_1 *is a constant that does not depend on the input instance.*

In particular, if the right answer for $\sigma_{\mathsf{rect},\mathsf{q}}$ is constant independent of q, the integrality gap for MISR would be constant as well. Or, if we can bound $\sigma(\mathcal{R})$ for a particular instance $\mathcal{R}$, we would get $\sigma(\mathcal{R})$ approximate solution for MISR problem on $\mathcal{R}$. For unweighted MISR, Theorem 3 translates to the following:

Corollary 2. *The integrality gap of natural LP relaxation of unweighted* MISR *is at most* $\sigma_{\mathsf{rect-nc},\mathsf{O}(\log \mathsf{n})}$. *Moreover, if there is a polynomial time algorithm that finds a valid coloring of rectangles of* $\mathcal{R}$ *using at most* $qf(q)$ *colors, then we have a* $c_2 f(\log n)$ *randomized approximation algorithm for unweighted* MISR *where* c_2 *is a constant that does not depend on* q.

Now we prove the theorem.

Proof. (Of Lemma 3) We first consider a natural LP relaxation of the problem. We have, for each rectangle R, an indicator variable x_R of whether R is included in the intended independent set. Let $X = \left\{x^l(R), x^r(R) : R \in \mathcal{R}\right\}$ and $Y = \left\{y^t(R), y^b(R) : R \in \mathcal{R}\right\}$ be the set of all x and y coordinates of the boundaries of input rectangles respectively. We define $\mathcal{P}$ to be the set of all "interesting points" of the plane: $\mathcal{P} = \{(x, y) : x \in X, y \in Y\}$. Notice that $|\mathcal{P}| \leq (2n)^2$. The LP relaxation is as follows.

$$\begin{aligned} \text{(LP)} \max & \sum_{R \in \mathcal{R}} w_R x_R \\ \text{s.t.} & \sum_{R : p \in R} x_R \leq 1 \text{ for all } p \in \mathcal{P} \end{aligned}$$

To avoid confusion, we will be using the term LP-value to refer to the value of specific LP variable x_R. We say LP-cost of collection $\mathcal{R}'$ to mean the quantity $\sum_{R \in \mathcal{R}'} w_R x_R$.

Let z be an optimal LP solution with associated LP-cost OPT. Observe that if $\mathsf{OPT} \leq O(n)$, getting a constant approximation is trivial: simply output the maximum-weight rectangle, whose weight is always $2n$. Therefore, we assume that $\mathsf{OPT} \geq 32n$. Let $M = 64 \log n$. The next lemma states that we can convert z into solution z' that is $(\frac{1}{M})$-integral having roughly the same LP-value with high probability. The proof only uses standard randomized rounding techniques and is deferred to the full version

Lemma 2. *There is an efficient randomized algorithm that, given an optimal LP-solution of value* $\mathsf{OPT} \geq 32n$ *for* $\mathcal{R}$, *produces with high probability, a feasible solution* z' *for (LP) that is* $(\frac{1}{M})$*-integral whose LP-value is* $\Omega(\mathsf{OPT})$.

Given an LP solution z', we create a multi-subset $\mathcal{R}'$ of $\mathcal{R}$ as follows: for each $R \in \mathcal{R}$, add $c_R = Mz'_R$ copies of R to $\mathcal{R}'$. Notice that we can associate each copy in $\mathcal{R}'$ with an LP-weight of $1/M$, so the maximum clique size is at most M. Moreover the total LP-cost in $\mathcal{R}'$ is $\sum_{R \in \mathcal{R}'} w_R x_R = \Omega(\mathsf{OPT})$. Now assume that we have $qf(q)$-coloring algorithm for any collection of rectangles with clique size q. By invoking this algorithm on $\mathcal{R}'$, we divide rectangles in $\mathcal{R}'$ into sets $\mathcal{R}'_1, \ldots, \mathcal{R}'_{Mf(M)}$ according to their colors. Let $\mathcal{R}'_j$ be the color class having maximum total LP-cost among the sets $\{\mathcal{R}'_{j'}\}$. We have that $\sum_{R \in \mathcal{R}'_j} \frac{w_R}{M} \geq \frac{\mathsf{OPT}}{Mf(M)}$.

Therefore, the total weight of rectangles in $\mathcal{R}'_j$ is $\sum_{R \in \mathcal{R}'_j} w_R \geq \Omega(\mathsf{OPT}/f(64 \log n))$, as desired. If we are satisfied with non-constructive bound, we can invoke $(\sigma_{\mathsf{rect,M}} M)$-coloring of $\mathcal{R}'$, and we would get the integrality gap bound of $O(\sigma_{\mathsf{rect,\log n}})$.

4 Coloring Algorithms

In this extended abstract, we prove a weaker result which gives $O(q^{3/2})$-coloring for a special case. This case captures most of the key challenges of the problem.

4.1 $O(q^2)$-Coloring

We first show how to color $\mathcal{R}$ using $O(q^2)$ colors. This coloring algorithm will be used later as a subroutine of our main result. For each rectangle $R \in \mathcal{R}$, we denote by $V(R)$ the set of all rectangles $R' \in \mathcal{R}$ such that R and R' cross each other and the width of R' is smaller than the width of R. Let $v(R)$ be the size of the maximum clique formed by the rectangles in $V(R)$. Notice that since the maximum clique size of $\mathcal{R}$ is q, we have that $0 \leq v(R) \leq q - 1$ for all rectangle $R \in \mathcal{R}$. It is easy to see that if $v(R) = v(R')$ for a pair of rectangles R and R', then it is impossible for R to cross R': Assume for contradiction that R crosses R', and the width of R is smaller than the width of R'. Let $\mathcal{Q} \subseteq V(R)$ be a clique such that $|\mathcal{Q}| = v(R)$. Then $\{R\} \cup \mathcal{Q} \subseteq V(R')$ is a clique, and therefore $v(R') \geq v(R) + 1$.

Claim. Any collection of rectangles with clique size q is $O(q^2)$-colorable.

Proof. We compute the values $v(R)$ for all rectangles $R \in \mathcal{R}$. Partition $\mathcal{R}$ into q subsets $S_1, \ldots, S_q$ where $R \in S_i$ iff $v(R) = i - 1$. Since each set S_i does not have crossing, it is 4-sparse (representative points are just the four corners), and so by Lemma 1 each such collection is $O(q)$-colorable. This implies that the set $\mathcal{R}$ is $O(q^2)$-colorable.

4.2 An $O(q^{3/2})$ Coloring for Restricted Setting

The coloring result in the previous section uses the values $v(R)$ to define a "grouping" of rectangles such that the intersection patterns of rectangles in the same set S_i are limited to only crossing and containment, which is 4-sparse. Now we try to push this idea further. We would like to say such things as "if $v(R)$ and $v(R')$ are close, the intersection patterns of R and R' are limited", and we would expect that if we group rectangles with roughly the same values of $v(R)$ together, such collection should be "almost" sparse.

We start with the following lemma about the combinatorial structures of sets of intersecting rectangles. This lemma was used implicitly in [8].

Lemma 3 (Structure Lemma). *Let $\mathcal{C}$ be a clique, and R be any rectangle such that $\mathcal{C} \subseteq V(R)$. Then we have that*

$$v(R) \geq \min_{R' \in \mathcal{C}} \{v(R')\} + \lfloor |\mathcal{C}|/2 \rfloor$$

In other words, if we have a sub-collection of rectangles $\mathcal{R}'$ such that $|v(R) - v(R')| \leq \delta$ for all $R, R' \in \mathcal{R}'$, then any clique $\mathcal{C} \subseteq \mathcal{R}'$ of size larger than 2δ is not a subset of $V(R)$.

Proof. Let $p = (x, y)$ be any point contained in the intersection of rectangles in $\mathcal{C} \cup \{R\}$. Consider now vertical line L passing through p. Let $\mathcal{Q} \subseteq \mathcal{C}$ be the set of $\lfloor |\mathcal{C}|/2 \rfloor$ rectangles whose left boundary is closest to L in $\mathcal{C}$, and let $P \in \mathcal{Q}$ be the rectangle whose right boundary is closest to L among the rectangles in $\mathcal{Q}$. Notice that all rectangles in $\mathcal{C} \setminus \mathcal{Q}$ intersect the left boundary of P, and all rectangles in $\mathcal{Q} \setminus \{P\}$ intersect the right boundary of P. Let $\mathcal{C}'$ be a clique of size $v(P)$ in $V(P)$. This is the clique whose rectangles contribute to the value $v(P)$. Observe that each rectangle in $\mathcal{C}'$ belongs to $V(R)$, and that $\mathcal{C}$ is disjoint with $\mathcal{C}'$ since rectangles in $\mathcal{C}$ intersect the left or the right boundary of P while rectangles in $\mathcal{C}'$ do not. Let $p' = (x', y)$ be any point in the intersection of rectangles in $\mathcal{C}' \cup \{P\}$ (the intersection region is shown as a black stripe in Figure 2) that is horizontally aligned with point p. Assume first that $x' > x$. Then every rectangle in $\mathcal{Q}$ contains p' because each rectangle in $\mathcal{Q}$ contains p (so its left boundary must lie on the left side of p) and intersects the right boundary of P (so its right boundary must be on the right of p'). Therefore $\mathcal{C}' \cup \mathcal{Q} \subseteq V(R)$ form a clique of size at least $v(P) + \lfloor |\mathcal{C}|/2 \rfloor$. Similarly, if $x' \leq x$, then every rectangle in $\mathcal{C} \setminus \mathcal{Q}$ contains p', and we have that the set $\mathcal{C}' \cup (\mathcal{C} \setminus \mathcal{Q}) \cup \{P\}$ forms a clique of size at least $v(P) + \lfloor |\mathcal{C}|/2 \rfloor$.

We show how to use the above lemma to get a better coloring result. We introduce the key definition, similar to the one used in [8].

Definition 1. *Let $\mathcal{R}'$ be a sub-collection of rectangles. Consider rectangle R, and let $X_1, X_2 \subseteq \mathcal{R}'$ be collections of rectangles such that $|X_1| = |X_2| = \alpha$. We say that they form an α-covering of R with respect to $\mathcal{R}'$ iff:*

- *Each rectangle in X_1 (resp. X_2) intersects the top (resp. bottom) boundary of R.*
- *$X_1 \cup X_2 \cup \{R\}$ forms a clique.*

We denote by $\alpha_{\mathcal{R}'}(R)$ the maximum integer α such that there exist $X_1, X_2 \subseteq \mathcal{R}'$ that form an α-covering of R. When the choice of $\mathcal{R}'$ is clear from context, we write $\alpha(R)$ instead of $\alpha_{\mathcal{R}'}(R)$.

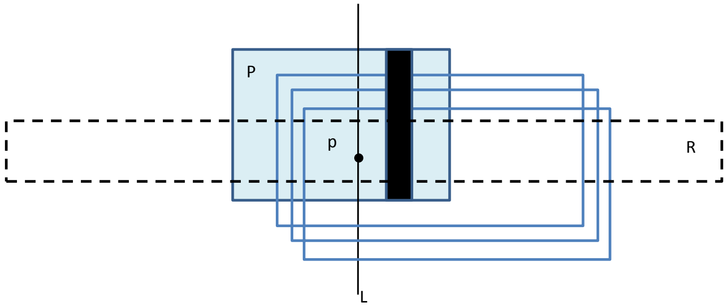

Fig. 2. Proof of Lemma 2 when $x' > x$. The black stripe shows the intersection region of $V(P)$

We will often call the set X_1 and X_2 the top and bottom α-coverage of R respectively. It is easy to see that $\alpha_{\mathcal{R}'}(R)$ can be computed in polynomial time: Fix rectangle R. For each interesting point $p \in R$, we compute the set of rectangles containing p and intersecting the top boundary of R. Denote this set by X_1^p. Set X_2^p is defined and computed similarly. Then we have that $\alpha_{\mathcal{R}'}(R) = \max_{p \in R} \min\{|X_1^p|, |X_2^p|\}$.

Claim. Consider a collection of rectangles $\mathcal{R}'$, and assume that $\mathcal{R}'$ contains a clique of size q'. Then there is at least one rectangle $R \in \mathcal{R}'$ such that $\alpha_{\mathcal{R}'}(R) \geq q'/2 - 1$.

Proof. Let $\mathcal{C} \subseteq \mathcal{R}'$ be the clique of size q', and let $X_1 \subseteq \mathcal{C}$ denote the set of $q'/2 - 1$ rectangles with highest top boundaries (breaking ties arbitrarily). Then define $X_2 \subseteq \mathcal{C}$ as the set of $q'/2 - 1$ rectangles with lowest bottom boundaries in $\mathcal{C} \setminus X_1$ (breaking ties arbitrarily). Consider any rectangle R in $\mathcal{C} \setminus (X_1 \cup X_2)$. It is easy to see that every rectangle in X_1 (resp. X_2) intersects the upper (resp. lower) boundary of R. Therefore, X_1, X_2 is a $(q'/2 - 1)$-covering of R.

Corollary 3. *For any collection $\mathcal{R}'$ of rectangles, let $\mathcal{R}'' \subseteq \mathcal{R}'$ be a set of rectangles with $\alpha_{\mathcal{R}'}(R) \geq \nu$ for some $\nu > 2$. Then $\omega(\mathcal{R}' \setminus \mathcal{R}'') \leq 3\nu$.*

Proof. Let $\tilde{\mathcal{R}} = \mathcal{R}' \setminus \mathcal{R}''$ be the set of remaining rectangles. Suppose a large clique of size 3ν remains in $\tilde{\mathcal{R}}$. Then by Claim 4.2, we would have a rectangle $R \in \tilde{\mathcal{R}}$ with $\alpha_{\tilde{\mathcal{R}}}(R) \geq 3\nu/2 - 1 > \nu$. And so we have $\alpha_{\mathcal{R}'}(R) > \nu$, a contradiction.

We are ready to describe an $O(q^{3/2})$-coloring algorithm. For simplicity of presentation, let us for now restrict the intersection types and assume that we do not have an intersection of rectangles R and R' such that R contains at least two corners of R'. So now there are only two restricted types of intersection: (i) crossing and (ii) corner intersection where one rectangle contains exactly one corner of another. We will show in the next section how this assumption can be removed.

1. Compute the values $v(R)$ for rectangles $R \in \mathcal{R}$ in the beginning. This value is used throughout the algorithm.
2. Partition the rectangles into $\sqrt{q}$ sets $\{S_i\}_{i=1}^{\sqrt{q}}$ where rectangle R belongs to S_i iff $(i-1)\sqrt{q} \leq v(R) < i\sqrt{q}$. So $|v(R) - v(R')| \leq \sqrt{q}$ for all R, R' in the same set.
3. For each $i : 1 \leq i \leq \sqrt{q}$,
   (a) Define $T_i = \left\{R \in S_i : \alpha_i(R) \geq 10\sqrt{q}\right\}$, where $\alpha_i(R)$ denotes the α-covering of R with respect to set S_i.
   (b) $S_i' \leftarrow S_i \setminus T_i$.
   (c) Color T_i and S_i' using $O(q)$ colors.

Assuming that step 3(c) can be implemented, it is clear that the total number of colors used is $O(q^{3/2})$. It is therefore sufficient to show that sets T_i and S_i' are $O(q)$-colorable. For each set S_i', notice that the clique size in S_i' is at most $O(\sqrt{q})$ after the removal of T_i from S_i, due to Corollary 3. Using the $O(\omega(S_i')^2)$-coloring algorithm from the previous section, we can get $O(q)$-coloring for each set S_i'. The following claim shows that we can also color T_i.

Claim. Each set T_i is 5-sparse. Therefore, it is $O(q)$-colorable.

Proof. Recall that each $R \in T_i$ has $(i-1)\sqrt{q} \leq v(R) < i\sqrt{q}$. We need to define, for each $R \in T_i$, five representative points $p_1^R, \ldots, p_5^R$. Now we fix R. Define $p_1^R, \ldots, p_4^R$ to be the four corners of R. Let (X_1, X_2) be a $10\sqrt{q}$-coverage of R in S_i (these rectangles may not be in T_i), and $\mathcal{C} = X_1 \cup X_2$. Define p_5^R to be any common point of rectangles in $\mathcal{C}$.

Now we proceed to prove that the collection T_i is sparse. Consider two intersecting rectangles R and R' in T_i. If it is a corner intersection, we would be done. Otherwise, it is a crossing, and assume that the width of R' is larger than the width of R, i.e. $R \in V(R')$. We claim that $p_5^R \in R'$: If not, assume without loss of generality that p_5^R is below the bottom boundary of R'. Consider the "top α-coverage" X_1 of R. Recall that all rectangles in X_1 contain p_5^R and intersect the top boundary of R. Therefore, the only possible layout is that R' crosses every rectangle in X_1 (because of our initial assumption), or in other words, $X_1 \subseteq V(R')$. Applying Lemma 3, we have that $v(R') \geq (i-1)\sqrt{q} + 2\sqrt{q} = (i+1)\sqrt{q}$, which is impossible, thus concluding the proof.

Notice that the proof of this claim would fail if we do not restrict the intersection types of rectangles. To deal with the general case, we need to deal with another notion of covering. Our $O(\gamma q \log q)$-coloring result is obtained through an iterative application of the ideas used in this section. Please see the full version for more detail.

5 Special Cases

In this section, we discuss the special cases of MISR and RCOL categorized by the types of intersections allowed. Table 1 summarizes known results on the special

Table 1. Summary of best known upper bounds . New results are marked by (*)

	no corner	no containment	no crossing	all
$\sigma(\mathcal{R})$	1^*	$O(\log q)^*$	5 [23]	$3q-2$ [18]
MISR	1^*	$O(\log\log n)$ [8]	4 [23]	$O(\log n/\log\log n)$ [11]

cases. The bound of $O(\log q)$ was implied by Theorem 1 due to the fact that, without containment intersection, we have $\gamma = 1$.

We study the case when corner intersection is not allowed and prove the following theorem. Due to lack of space, the proof appears in the full version.

Theorem 4. *Let $\mathcal{R}$ be a collection of rectangles with clique size q, in which the intersection types are only containment and crossing. Then $\chi(\mathcal{R}) = \omega(\mathcal{R})$. This implies that $\sigma(\mathcal{R}) = 1$ and maximum independent set of $\mathcal{R}$ can be found in polynomial time.*

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