5

Counting Quantifiers on Automatic Structures

In chapter 2 we asked how first-order logic can be extended and analyzed using infinitary logic, which lead us to the regular game quantifier and to clarifying the connection to games.

In this chapter we consider extensions of first-order logic in another direction, by generalized unary quantifiers. As usual, we require these extensions to preserve regularity on automatic structures. It turns out that the only generalized unary quantifiers with this property are the counting quantifiers:

- the *modulo counting quantifiers* "there exist k mod m many",
- the *infinity quantifier* "there exist infinitely many", and
- the *uncountability quantifier* "there exist uncountably many".

While it is known that all counting quantifiers indeed preserve regularity over finite-word automatic structures, and even over injectively presented ω-automatic structures, this was open for general ω-automatic structures. Our proof [10] uses ω-semigroups and leads to an additional corollary that all countable ω-automatic structures have injective presentations. It follows that countable ω-automatic structures have automatic presentations over finite words, which answers a question of Blumensath [11].

5.1 Generalized Quantifiers Preserving Regularity

To extend first-order logic with additional quantifiers it is useful to have an abstract definition of a *generalized quantifier*. We borrow the definition given by Lindström [60].

Definition 5.1. *A generalized quantifier Q over a relational signature $\tau = \{R_1, \dots, R_k\}$ is a class of structures with signature τ that is closed under isomorphism. Let $\mathfrak{A}$ be a structure and $\varphi_1(\overline{x}_1, \overline{z}), \dots, \varphi_k(\overline{x}_k, \overline{z})$ formulas over the signature $\sigma(\mathfrak{A})$ possibly different from τ, such that $|\overline{x}_i| = r_i$, i.e. the length of the vector $\overline{x}_i$ is the same as the arity of R_i. In first-order logic extended with the quantifier Q we allow to write formulas of the form $Q\overline{x}_1 \dots \overline{x}_k(\varphi_1, \dots, \varphi_k)$*

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and define their semantics in the following way. If θ maps $\overline{z}$ to the tuple $\overline{a}$ of elements of $\mathfrak{A}$ then

$$\mathfrak{A}, \theta \models Q\overline{x}_1 \dots \overline{x}_k(\varphi_1, \dots, \varphi_k) \Leftrightarrow (A, \varphi_1^{\mathfrak{A}}(-, \overline{a}), \dots, \varphi_k^{\mathfrak{A}}(-, \overline{a})) \in Q,$$

where by $\varphi_i^{\mathfrak{A}}(-, \overline{a})$ we denote the relation satisfied by exactly those tuples $\overline{b}$ for which $\varphi_i^{\mathfrak{A}}(\overline{b}, \overline{a})$ holds. The arity of the quantifier Q is the maximum of the lengths $|\overline{x}_i|$, so a unary quantifier *is one where each of the vectors $\overline{x}_i$ is just a single variable.*

To illustrate this definition observe that the classical quantifier $\exists$ is given by $\{(A, X) : X \neq \emptyset\}$ and $\forall$ is given by $\{(A, X) : X = A\}$. The quantifiers "there exist infinitely many" or "there exist k mod m many" can be represented in a similar way, but we give the standard definition.

The extension of first-order logic with counting quantifiers, denoted $\mathsf{FO}[\mathsf{C}]$, allows to write all quantifiers of the following form:

- $\exists^{(r \bmod m)} x\ \varphi$ meaning that the number of x satisfying φ is finite and is congruent to r mod m,
- $\exists^{\infty} x\ \varphi$ meaning that there are infinitely many x satisfying φ,
- $\exists^{\leq \aleph_0} x\ \varphi$ and $\exists^{> \aleph_0} x\ \varphi$ meaning that the cardinality of the set of all x satisfying φ is countable, or uncountable, respectively.

The logic $\mathsf{FO}[\mathsf{C}]$ has intimate relation to quantifiers that preserve regularity. To define this relation we first need to say that a generalized quantifier Q preserves (ω-)regularity if for every (ω-)automatic presentation $\mathfrak{d}, f$ of a structure $\mathfrak{A}$ every formula

$$\psi(\overline{z}) = Q\overline{x}_1 \dots \overline{x}_k\big(\varphi_1(\overline{x}_1, \overline{z}) \dots \varphi_k(\overline{x}_1, \overline{z})\big)$$

defines a relation $\psi^{\mathfrak{A}}$ that is (ω-)regular in the presentation $\mathfrak{d}, f$, i.e. such that $f^{-1}(\psi^{\mathfrak{A}})$ is (ω-)regular.

Moreover, we say that a quantifier Q over a signature τ is definable in $\mathsf{FO}[\mathsf{C}]$ (or any other extension of FO) if there exists a formula $\varphi_Q \in \mathsf{FO}[\mathsf{C}]$ over the same signature τ such that $Q = \{\mathfrak{A} : \mathfrak{A} \models \varphi\}$. We can now state the result that shows the relationship between $\mathsf{FO}[\mathsf{C}]$ and regularity-preserving quantifiers: every unary quantifier that preserves (ω-)regularity is definable in $\mathsf{FO}[\mathsf{C}]$. This result is proved in [79] for regularity preserving quantifiers and the proof extends to ω-regularity preserving ones.

The remaining question is whether every quantifier in $\mathsf{FO}[\mathsf{C}]$ preserves regularity, and whether it does so in an effective way. For finite-word automatic structures the basic Theorem 1.6 can be extended to $\mathsf{FO}[\mathsf{C}]$ as follows.

Theorem 5.2 (Cf. [45, 53, 14]).

- *There is an effective procedure that given an automatic presentation $\mathfrak{d}, f$ of a structure $\mathfrak{A}$, and given an $\mathsf{FO}[\mathsf{C}]$ formula $\varphi(\overline{x})$ defining a k-ary relation R over $\mathfrak{A}$, constructs a k-tape synchronous automaton recognizing $f^{-1}(R)$.*

- *The* $\mathsf{FO[C]}$*-theory of every automatic structure is decidable.*
- *The class of automatic structures is closed under* $\mathsf{FO[C]}$*-interpretations.*

It has been observed that Theorem 5.2 can be extended to *injective* ω-automatic presentations [55, 58]. Moreover, Kuske and Lohrey show that the cardinality of any set definable in $\mathsf{FO[C]}$ is either countable or equal to that of the continuum. In the next section we work to extend this result to all, not necessarily injective automatic structures.

5.2 Defining Uncountability Using Equal Ends

We characterize when there exist countably many words x satisfying a given formula *with parameters* $\varphi(x, \overline{z})$ in some ω-automatic structure $\mathfrak{A}$. The characterization is first-order expressible in an ω-automatic extension of $\mathfrak{A}$ by the equal ends relation $\sim_{\mathrm{e}}$ and the quantifier rank of the resulting formula depends on a constant C, which itself depends on φ and on the given presentation of $\mathfrak{A}$.

Let us fix an ω-automatic presentation $\mathfrak{d}$ of a structure $\mathfrak{A}$ with congruence $\approx$, and a first-order formula $\varphi(x, \overline{z})$ in the language of $\mathfrak{A}$ with x and $\overline{z}$ as free variables.

Proposition 5.3. *There is a constant C, computable from the presentation $\mathfrak{d}$, so that for all tuples $\overline{z}$ of infinite words the following are equivalent:*

(i) $\varphi(-, \overline{z})$ is satisfiable and $\approx$ restricted to the domain $\varphi(-, \overline{z})$ has countably many equivalence classes,
(ii) there exist C-many words $x_1, \ldots, x_C$, each satisfying $\varphi(-, \overline{z})$, so that every x satisfying $\varphi(-, \overline{z})$ is $\approx$-equivalent to some $y \sim_e x_i$; formally, the structure $(\mathfrak{A}, \approx, \sim_e)$ models the sentence

$$\forall \overline{z} \left(\exists^{\leq \aleph_0} w \; \varphi(w, \overline{z}) \longleftrightarrow \exists x_1 \ldots x_C \left(\bigwedge_i \varphi(x_i, \overline{z}) \wedge \forall x \left(\varphi(x, \overline{z}) \rightarrow \exists y \; (y \approx x \wedge \bigvee_i y \sim_e x_i) \right) \right) \right).$$

Proof. Suppose $\mathfrak{d}$, $\mathfrak{A}$, and φ are given. Define C to be c^2, where c is the size of the largest ω-semigroup corresponding to any of the given automata from the presentation $\mathfrak{d}$ or corresponding to φ. We fix the parameters $\overline{z}$ and let $\approx$ denote the equivalence relation $\approx$ restricted to the domain of $\varphi(-, \overline{z})$.

(ii) $\Rightarrow$ *(i)*: Condition *(ii)* and the fact that every $\sim_{\mathrm{e}}$-class is countable imply that all words satisfying $\varphi(-, \overline{z})$ are contained in a countable number of $\approx$-classes.

(i) ⇒ (ii): The negation of Condition *(ii)* says that given $D < C$ many words $x_1, \dots, x_D$, each satisfying $\varphi(-, \overline{z})$, there exists a word x_{D+1} also satisfying $\varphi(-, \overline{z})$ whose $\approx$-class does not meet any of the $\sim_{\mathrm{e}}$-classes of the x_i for $i \leq D$.

Thus we can inductively define words $x_1, \dots, x_C$, each satisfying the formula $\varphi(-, \overline{z})$, and such that for $1 \leq i < j \leq C$ the $\approx$-class of x_j does not meet the $\sim_{\mathrm{e}}$-class of x_i. In particular, the x_is are pairwise non-equivalent with respect to $\sim_{\mathrm{e}}$.

The plan is to produce uncountably many pairwise non-$\approx$ words that satisfy $\varphi(-, \overline{z})$. In the first 'Ramsey step', similar to what is done in [58], we find two words from the given C many, say $x_1, x_2 \in \Sigma^*$, and a factorization $H \subset \mathbb{N}$ so that both words behave the same way along the factored sub-words with respect to the $\approx$- and φ-semigroups. In the second 'Coarsening step' we identify a technical property of finite semigroups recognizing transitive relations. This allows us to produce an altered factorization G and new, well-behaving words y_1, y_2. In the final step, the new words are 'shuffled along G' to produce continuum many pairwise non-$\approx$ words, each satisfying $\varphi(-, \overline{z})$.

5.2.1 Ramsey Step

This step effectively allows us to discard the parameters $\overline{z}$. Before we use Ramsey's theorem, we introduce a convenient notation to talk about factorizations of words.

Definition 5.4. *Let $A = a_1 < a_2 < \dots$ be any infinite subset of $\mathbb{N}$ and $h : \Sigma^* \to S$ be a morphism into a finite semigroup S. For an ω-word $\alpha \in \Sigma^\omega$, and element $e \in S$, say that A* is an h, e-homogeneous factorization of α *if for all $n \in \mathbb{N}^+$, $h\big(\alpha[a_n, a_{n+1})\big) = e$.*

Observe the following facts.

- If A is an h, s-homogeneous factorization of α and $k \in \mathbb{N}^+$ then the set $\{a_{k \cdot i}\}_{i \in \mathbb{N}^+}$ is an h, s^k-homogeneous factorization of α.
- If A is an h, e-homogeneous factorization of α and e is idempotent, then every infinite $B \subset A$ is also an h, e-homogeneous factorization of α.

In the following we write w^φ and $w^\approx$ to denote the image of w under the semigroup morphism into the finite semigroup associated to φ and $\approx$, respectively, as determined by the presentation. Accordingly, we will speak of e.g. φ, s_i-homogeneous factorizations.

Let us now color every $\{n, m\} \in [\mathbb{N}]^2$ with $n < m$ by the tuple of ω-semigroup elements

$$\Big(\big(\otimes (x_i, \overline{z})[n, m)^\varphi \big)_{0 \leq i \leq C} \, , \, \big(\otimes (x_i, x_j)[n, m)^\approx \big)_{0 \leq i \leq j \leq C} \Big) .$$

By Ramsey's theorem there exists an infinite $H \subset \mathbb{N}$ and a tuple of ω-semigroup elements

$$\big((s_i)_{1\le i\le C},(t_{(i,j)})_{1\le i\le j\le C}\big)$$

so that for all $0 \le i \le j \le C$,

- H is a φ, s_i-homogeneous factorization of the word $\otimes(x_i, \overline{z})$,
- H is a $\approx, t_{(i,j)}$-homogeneous factorization of the word $\otimes(x_i, x_j)$.

Note that by virtue of additivity of our coloring and Ramsey's theorem each of the s_i and $t_{(i,j)}$ above are idempotents. Since there are at most c-many s_is and c-many $t_{(i,i)}$s, there are at most c^2 many pairs $(s_i, t_{(i,i)})$ and so there must be two indices, we may suppose 1 and 2, with $s_1 = s_2$ and $t_{(1,1)} = t_{(2,2)}$.

5.2.2 Coarsening Step

For technical reasons we now refine H and alter x_1, x_2 so that the semigroup elements have certain additional properties.

To start with, using the fact that $x_1 \not\sim_{\mathrm{e}} x_2$ and the facts we observed on homogeneous factorizations, we assume without loss of generality that H is coarse enough so that $x_1[h_n, h_{n+1}) \ne x_2[h_n, h_{n+1})$ for all $n \in \mathbb{N}$.

Lemma 5.5. *There exists a subset $G \subset H$, listed as $g_1 < g_2 < \ldots$, and ω-words y_1, y_2 with the following properties:*

(1) The words y_1 and y_2 are neither $\approx$-equivalent nor $\sim_e$-equivalent, and each satisfies $\varphi(-, \overline{z})$.
(2) There exists an idempotent φ-semigroup element s such that G is a φ, s-homogeneous factorization for each of $\otimes(y_1, \overline{z})$ and $\otimes(y_2, \overline{z})$.
(3) There exist idempotent $\approx$-semigroup elements $t, t^{\uparrow}, t^{\downarrow}$ so that for $y_j \in \{y_1, y_2\}$
   - *both $t^{\uparrow}$ and $t^{\downarrow}$ absorb t*
   - *$\otimes(y_j, y_j)[0, g_1)^{\approx}$ absorbs t*
   - *G is an $\approx, t$-homogeneous factorization of $\otimes(y_j, y_j)$*
   - *G is an $\approx, t^{\uparrow}$-homogeneous factorization of $\otimes(y_1, y_2)$*
   - *G is an $\approx, t^{\downarrow}$-homogeneous factorization of $\otimes(y_2, y_1)$.*

Proof. Define ω-words $y_1 := x_2[0, h_2)x_1[h_2, \infty)$, and y_2 by

$$y_2[0, h_2) := x_2[0, h_2) \text{ and}$$
$$y_2[h_{2n}, h_{2n+2}) := x_2[h_{2n}, h_{2n+1})x_1[h_{2n+1}, h_{2n+2}) \text{ for } n > 0.$$

Item 1. Clearly, $y_1 \not\sim_{\mathrm{e}} y_2$ and each $y_j \in \{y_1, y_2\}$ satisfies $\varphi(y_j, \overline{z})$ since by homogeneity and $s_1 = s_2$

$$\begin{aligned}\otimes(y_1, \overline{z})^{\varphi} &= \otimes(x_2, \overline{z})[0, h_2)^{\varphi} s_1^{\omega} \\ &= \otimes(x_2, \overline{z})[0, h_2)^{\varphi} s_2^{\omega} \\ &= \otimes(x_2, \overline{z})^{\varphi},\end{aligned}$$

and similarly

$$\begin{aligned}\otimes(y_2,\overline{z})^{\varphi} &= \otimes(x_2,\overline{z})[0,h_2)^{\varphi}(s_2s_1)^{\omega}\\ &= \otimes(x_2,\overline{z})[0,h_2)^{\varphi}s_2^{\omega}\\ &= \otimes(x_2,\overline{z})^{\varphi}.\end{aligned}$$

Next we check that $y_1 \not\approx y_2$.

$$\begin{aligned}\otimes(y_1,y_2)^{\approx} &= \pi_{\approx}\Big(\otimes(x_2,x_2)[0,h_2)^{\approx},\\ &\qquad\big(\otimes(x_1,x_2)[h_{2n},h_{2n+1})^{\approx},\\ &\qquad\ \otimes(x_1,x_1)[h_{2n+1},h_{2n+2})^{\approx}\big)_{n\in\mathbb{N}^+}\Big)\\ &= \otimes(x_2,x_2)[0,h_1)^{\approx}\,t_{(2,2)}\ (t_{(1,2)}t_{(1,1)})^{\omega}\\ &= \otimes(x_2,x_2)[0,h_1)^{\approx}\,t_{(2,2)}t_{(2,2)}\ (t_{(1,2)}t_{(1,1)})^{\omega}\\ &= \otimes(x_2,x_2)[0,h_1)^{\approx}\,t_{(2,2)}t_{(2,2)}\ (t_{(1,2)}t_{(2,2)})^{\omega}\\ &= \otimes(x_2,x_2)[0,h_1)^{\approx}\,t_{(2,2)}\ (t_{(2,2)}t_{(1,2)})^{\omega}\\ &= \pi_{\approx}\Big(\otimes(x_2,x_2)[0,h_2)^{\approx},\\ &\qquad\big(\otimes(x_2,x_2)[h_{2n},h_{2n+1})^{\approx},\\ &\qquad\ \otimes(x_1,x_2)[h_{2n+1},h_{2n+2})^{\approx}\big)_{n\in\mathbb{N}^+}\Big)\\ &= \otimes(y_2,x_2)^{\approx}\end{aligned}$$

Thus, if $y_1 \approx y_2$ then also $y_2 \approx x_2$ and so *by transitivity* $y_1 \approx x_2$. But since $y_1 \sim_{\mathrm{e}} x_1$, the $\approx$-class of x_2 meets the $\sim_{\mathrm{e}}$-class of x_1, contradicting the initial choice of the x_is.

Items 2 and 3. Define intermediate semigroup elements $q := s_1$, $r := t_{(1,1)}$, $r^{\uparrow} := t_{(1,2)}t_{(1,1)}$ and $r^{\downarrow} := t_{(2,1)}t_{(1,1)}$. Then

1. both $r^{\uparrow}$ and $r^{\downarrow}$ absorb r, since $t_{(1,1)}$ is idempotent,
2. $\otimes(y_j,y_j)[0,h_2)^{\approx} = \otimes(y_j,y_j)[0,h_1)^{\approx}t_{(2,2)}$ and thus absorbs r (for $y_j \in \{y_1,y_2\}$).

In this notation, for all $i \in \mathbb{N}^+$ and $y_j \in \{y_1,y_2\}$,

- $\otimes(y_j,\overline{z})[h_{2i},h_{2i+2})^{\varphi}$ is $qq = q$,
- $\otimes(y_j,y_j)[h_{2i},h_{2i+2})^{\approx}$ is $rr = r$,
- $\otimes(y_1,y_2)[h_{2i},h_{2i+2})^{\approx}$ is $t_{(1,2)}t_{(1,1)} = r^{\uparrow}$,
- $\otimes(y_2,y_1)[h_{2i},h_{2i+2})^{\approx}$ is $t_{(2,1)}t_{(1,1)} = r^{\downarrow}$.

Finally, define the set $G := \{h_{2ki}\}_{i>1}$, i.e. $g_i = h_{2k(i+1)}$, and the semigroup elements $t := r^k$, $t^{\uparrow} := (r^{\uparrow})^k$, $t^{\downarrow} := (r^{\downarrow})^k$ and $s := q^k$. The extra multiple of k (defined as the product of the exponents of the semigroups for $\sim_{\mathrm{e}}$ and $\approx$) ensures all these semigroup elements (in particular $t^{\uparrow}$ and $t^{\downarrow}$) are idempotent. We now verify the absorption properties:

$$t^{\uparrow}t = r^{\uparrow k}r^k = r^{\uparrow k} = t^{\uparrow} \quad \text{because } r^{\uparrow} \text{ absorbs } r.$$

Similarly, $t^{\downarrow}t$ absorbs t. Further, since $g_1 = h_{4k}$, we have

$$\begin{aligned}\otimes(y_j, y_j)[0, g_1)^{\approx} &= \otimes(y_j, y_j)[0, h_2)^{\approx} \otimes (y_j, y_j)[h_2, h_{4k})^{\approx} \\ &= \otimes(y_j, y_j)[0, h_2)^{\approx} r^{4k-2} \\ &= \otimes(y_j, y_j)[0, h_2)^{\approx} r^{3k-2} t\end{aligned}$$

and thus absorbs t.

Finally, we verify the homogeneity properties. Observe that G is an $\approx, t^{\downarrow}$-homogeneous factorization of $\otimes(y_2, y_1)$ since for $i \in \mathbb{N}^+$

$$\begin{aligned}\otimes(y_2, y_1)[g_i, g_{i+1})^{\approx} &= \otimes(y_2, y_1)[h_{2k(i+1)}, h_{2k(i+2)})^{\approx} \\ &= (r^{\downarrow})^k \;=\; t^{\downarrow}.\end{aligned}$$

The other cases are similar. □

5.2.3 Shuffling Step

We continue the proof of Proposition 5.3 by shuffling the words y_1 and y_2 along G resulting in continuum many pairwise distinct words that are pairwise not $\approx$-equivalent, each satisfying $\varphi(-, \overline{z})$. To this end, we define for $S \subset \mathbb{N}^+$ the characteristic word χ_S by

$$\chi_S[0, g_1) := y_2[0, g_1) \text{ , and}$$

$$\chi_S[g_n, g_{n+1}) := \begin{cases} y_2[g_n, g_{n+1}) & \text{if } n \in S \\ y_1[g_n, g_{n+1}) & \text{otherwise} \end{cases}$$

First observe that $\mathfrak{A} \models \varphi(\chi_S, \overline{z})$. Indeed, by item (2) of Lemma 5.5

$$\begin{aligned}\otimes(\chi_S, \overline{z})^{\varphi} &= \otimes(y_2, \overline{z})[0, g_1)^{\varphi} s^{\omega} \\ &= \otimes(y_2, \overline{z})^{\varphi}\end{aligned}$$

and $\mathfrak{A} \models \varphi(y_2, \overline{z})$ by item (1) of Lemma 5.5. Moreover, for $S \not\sim_{\mathrm{e}} T$ the construction gives that $\chi_S \not\sim_{\mathrm{e}} \chi_T$. This is due to our initial choice of $x_1 \not\sim_{\mathrm{e}} x_2$ and the assumption that the factorization $(h_n)_{n \in \mathbb{N}}$ is coarse enough so that $x_1[h_n, h_{n+1}) \neq x_2[h_n, h_{n+1})$ and thus also $y_1[g_n, g_{n+1}) \neq y_2[g_n, g_{n+1})$ for all n.

The following two lemmas establish that if both $S \setminus T$ and $T \setminus S$ are infinite then $\chi_S \not\approx \chi_T$. We denote by $x_{\circ\bullet}$ the word $\chi_{2\mathbb{N}^+}$ and by $x_{\bullet\circ}$ the word $\chi_{2\mathbb{N}^+-1}$, and we write p for $\otimes(y_2, y_2)[0, g_1)^{\approx}$.

Lemma 5.6. *For all S, T such that both $S \setminus T$ and $T \setminus S$ are infinite*

$$\otimes(\chi_S, \chi_T)^{\approx} = \begin{cases} \otimes(x_{\circ\bullet}, x_{\bullet\circ})^{\approx} & \textit{or} \\ \otimes(x_{\bullet\circ}, x_{\circ\bullet})^{\approx} & \end{cases}$$

Proof. Define semigroup-elements p_n for $n \in \mathbb{N}$ by

$$p_n := \begin{cases} t^{\downarrow} & \text{if } n \in S \setminus T \\ t^{\uparrow} & \text{if } n \in T \setminus S \\ t & \text{otherwise} \end{cases}$$

Let m be the smallest number in $S \triangle T$. Suppose that $m \in S \setminus T$. Because both $t^{\uparrow}$ and $t^{\downarrow}$ are idempotent and since t is absorbed by both p, $t^{\uparrow}$ and $t^{\downarrow}$, and both $t^{\uparrow}$ and $t^{\downarrow}$ appear infinitely often (as both $S \setminus T$ and $T \setminus S$ are infinite), we have

$$\begin{aligned} \otimes(\chi_S, \chi_T)^{\approx} &= \pi_{\approx}\left(p,\ (p_n)_{n \in \mathbb{N}}\right) \ = \ p(t^{\downarrow} t^{\uparrow})^{\omega} \\ &= \otimes(x_{\bullet\circ}, x_{\circ\bullet})^{\approx}. \end{aligned}$$

The case that $m \in T \setminus S$ similarly results in $\otimes(x_{\circ\bullet}, x_{\bullet\circ})^{\approx}$. □

Lemma 5.7. $x_{\circ\bullet} \not\approx x_{\bullet\circ}$.

Proof. Define an intermediate word $x_{\circ\bullet\circ\circ} := \chi_{4\mathbb{N}^+ - 2}$. By computations similar to the above we find that

$$\begin{aligned} \otimes(x_{\bullet\circ}, x_{\circ\bullet\circ\circ})^{\approx} &= p(t^{\downarrow} t^{\uparrow} t^{\downarrow} t)^{\omega} \ = \ p(t^{\downarrow} t^{\uparrow} t^{\downarrow})^{\omega} \ = \ p(t^{\downarrow} t^{\uparrow})^{\omega} \\ &= \otimes(x_{\bullet\circ}, x_{\circ\bullet})^{\approx} \end{aligned}$$

and

$$\begin{aligned} \otimes(x_{\circ\bullet}, x_{\circ\bullet\circ\circ})^{\approx} &= p(tttt^{\downarrow})^{\omega} \ = \ p(t^{\downarrow})^{\omega} \\ &= \otimes(y_2, y_1)^{\approx}. \end{aligned}$$

Therefore, if $x_{\bullet\circ} \approx x_{\circ\bullet}$ then also $x_{\bullet\circ} \approx x_{\circ\bullet\circ\circ}$ and so *by symmetry and by transitivity* $x_{\circ\bullet} \approx x_{\circ\bullet\circ\circ}$. But in this case also $y_2 \approx y_1$, contradicting item (1) of Lemma 5.5. □

We are now able to complete the proof of Proposition 5.3. There are continuum many classes in $\mathcal{P}(\mathbb{N})/\sim_{\mathrm{e}}$, thus there is a continuum of pairwise non-$\sim_{\mathrm{e}}$-equivalent sets S. To construct sets with pairwise infinite differences, we define for a set $S \subseteq \mathbb{N}$ the swap set

$$\widehat{S} = \{2n+1 \ : \ n \in S\} \cup \{2n+2 \ : \ n \notin S\}.$$

Observe that if $S \not\sim_{\mathrm{e}} T$ then both $\widehat{S} \setminus \widehat{T}$ and $\widehat{T} \setminus \widehat{S}$ are infinite. Therefore taking the words $\chi_{\widehat{S}}$ for the continuum of pairwise non-$\sim_{\mathrm{e}}$-equivalent sets S yields a continuum of non-$\approx$-equivalent words, each satisfying $\varphi(-, \overline{z})$. □

5.3 FO[C] over ω-Automatic Structures

Using the results about countability of the previous section, we are finally able to extend Theorem 5.2 to ω-automatic structures.

Theorem 5.8. *The statements of Theorem 5.2 hold true for* FO[C] *over all (not necessarily injective) ω-automatic presentations.*

Proof. We prove the first item, i.e. give the procedure for constructing automata for formulas, from which the rest of the theorem follows immediately. We inductively eliminate occurrences of cardinality and modulo quantifiers in the following way.

The countability quantifier $\exists^{\leq\aleph_0}$ and uncountability quantifier $\exists^{>\aleph_0}$ can be eliminated (in an extension of the presentation by $\sim_{\mathrm{e}}$) by the formula given in Proposition 5.3.

For the remaining quantifiers we further expand the presentation with the ω-regular relations

- $\pi(a,b,c)$ saying that $a \sim_{\mathrm{e}} b \sim_{\mathrm{e}} c$ and the last position where a differs from c is no larger than the last position where b differs from c, and
- $\lambda(a,b,c)$ saying that $\pi(a,b,c)$ and $\pi(b,a,c)$ and that the word $a[0,k]$ is lexicographically smaller than the word $b[0,k]$, where k is the common last position where a and b differ from c.

Now $\exists^{<\infty} x\ \varphi(x,\overline{z})$ is equivalent to

$$\exists x_1 \dots x_C\ \Psi(x_1,\dots,x_C,\overline{z})$$

where Ψ expresses that $x_1,\dots x_C$ satisfy $\varphi(-,\overline{z})$ and there exists a position, say $k \in \mathbb{N}$, so that every $\approx$-class contains a word satisfying $\varphi(-,\overline{z})$ that coincides with one of the x_i from position k onwards. This additional condition can be expressed by

$$\exists y_1 \dots y_C \forall x \exists y \left(\varphi(x,\overline{z}) \rightarrow x \approx y \wedge \bigvee_i \pi(y,y_i,x_i) \right).$$

Consequently, $\exists^{(r \bmod m)} x\,.\,\varphi(x,\overline{z})$ can be eliminated since we can pick out unique representatives of the $\approx$-classes. We write $i(w)$ for the smallest index i for which $w \sim_{\mathrm{e}} x_i$. The representatives are those x that satisfy the following properties for every $y \neq x$ in the same $\approx$-class as x.

- Either the index $i(x) < i(y)$, or
- the index $i(x) = i(y)$ and $\lambda(x,y,x_{i(x)})$ holds.

Now we can apply the construction of [58] or [55] for elimination of the $\exists^{(r \bmod m)}$ quantifier. □

5.4 Presentations of Countable ω-Automatic Structures

As a corollary of Proposition 5.3 we obtain that for every ω-regular equivalence relation with countably many classes a set of unique representatives is definable.

Corollary 5.9. *Let $\approx$ be an ω-automatic equivalence relation on Σ^ω. There is a constant C, depending on the presentation, so that the following are equivalent:*

(1) $\approx$ has countably many equivalence classes,
(2) there exist C many $\sim_e$-classes so that every $\approx$-class has a non-empty intersection with at least one of these $\sim_e$-classes.

If one of these conditions holds, then there exists an ω-regular set of representatives of $\approx$. Moreover, an automaton for this set can be effectively constructed given an automaton for $\approx$.

Proof. The first two items are simply a specialization of Proposition 5.3. We construct the ω-regular set of representatives as follows.

Write A for the domain of $\approx$ and consider the formula $\psi(x_1, \ldots, x_C)$ with free variables $x_1, \ldots, x_C$:

$$\bigwedge_i x_i \in A \ \wedge \ \forall x \in A \ \exists y \ (y \approx x \wedge \bigvee_i y \sim_{\mathrm{e}} x_i)$$

The relation defined by ψ is ω-regular since it is a first order formula over ω-regular relations. By assumption it is non-empty, and therefore it contains an ultimately periodic word of the form $\otimes(a_1, \ldots, a_C)$. Each of these a_is is thus ultimately periodic, and we write $a_i = v_i(u_i)^\omega$.

By definition of ψ, every word has now an $\approx$-representative in $B = \bigcup_i \Sigma^*(u_i)^\omega$. It remains to prune B to select unique representatives for each $\approx$-class.

It is easy to construct an ω-regular well-founded linear order on B. For every $w \in B$, let $p(w) \in \Sigma^*$ be the length-lexicographically smallest word such that w has period $p(w)$. Also let $t(w) \in \Sigma^*$ be the length-lexicographically smallest word so that $w = t(w) \cdot p(w)^\omega$. Define an order $\prec$ on B by $w \prec w'$ if $p(w)$ is length-lexicographically smaller than $p(w')$, or otherwise if $p(w) = p(w')$ and $t(w)$ is length-lexicographically smaller than $t(w')$. The ordering $\prec$ is ω-regular since it is FO-definable in terms of ω-regular relations. Finally, the required set of representatives may be defined as the set of $\prec$-minimal elements of every $\approx$-class. An automaton for this set can be constructed from an automaton for $\approx$ as all the steps we made used definable relations. □

Corollary 5.9 immediately yields an *injective* ω-automatic presentation from a given ω-automatic presentation. This is especially interesting together with the following proposition by which countable injective ω-automatic presentations can be transformed to automatic ones.

Proposition 5.10. *([11, Theorem 5.32]) Let $\mathfrak{d}$ be an injective ω-automatic presentation of a countable structure $\mathcal{A}$. Then, an (injective) automatic presentation $\mathfrak{d}'$ of $\mathcal{A}$ can be effectively constructed.*

Combining Proposition 5.10 and Corollary 5.9 we are able to answer affirmatively a question of Blumensath [11] and conclude that every countable ω-automatic structure is already automatic.

Corollary 5.11. *A countable structure is ω-automatic if and only if it is automatic. Transforming a presentation of one type into the other can be done effectively.*

The techniques used in this chapter not only give insight into the cardinality of the ω-automatic equivalence relations, but can also be used to study cliques built from an arbitrary binary ω-automatic relation. This was exploited recently in [57] to investigate which Ramsey-like theorems hold for ω-automatic structures.

Remarkably, the existence of injective presentations can not be extended from countable to arbitrary ω-automatic structures. Consider a disjoint sum of the Boolean algebra of sets of natural numbers $\mathfrak{B} = (\mathcal{P}(\mathbb{N}), \cup, \cap, ^C)$ and the uncountable atomless Boolean algebra $\mathfrak{B}/\sim_e$, where sets with finite symmetric difference are identified. Let us define the structure $\mathfrak{A} = (\mathfrak{B} \sqcup (\mathfrak{B}/\sim_e), B_1, B_2, f)$ which extends the disjoint sum $\mathfrak{B} \sqcup (\mathfrak{B}/\sim_e)$ with two predicates denoting the universes of the two components of the sum and a function that takes elements of $\mathfrak{B}$ to their corresponding classes in the other component of the disjoint sum, i.e. $f(B) = [B]_{\sim_e}$ for $B \in \mathfrak{B}$.

Observe that there is an ω-automatic presentation of $\mathfrak{A}$. Elements of $\mathfrak{A}$ are represented as ω-words over $\{0, 1\}$ with the first bit indicating whether the word represents an element of $\mathfrak{B}$ or of $\mathfrak{B}/\sim_e$ and the other bits listing which numbers belong to the represented subset. Equality is defined as equality of words for words staring with 1, i.e. representing elements of $\mathfrak{B}$, and as $\sim_e$ for words starting with 0. Boolean operations can be represented by automata in the standard way and the function f must only check that the $\sim_e$-classes of the components coincide, which can be done by the $\sim_e$ automaton ignoring the first bit.

The fact that there is no injective ω-automatic presentation of the structure $\mathfrak{A}$ was recently shown by Hjörth, Khoussainov, Montalban and Nies [44]. The proof is based on the topological observation that certain morphism between $\mathfrak{B}/\sim_e$ and $\mathfrak{B}$ can not be Borel, which would be contradicted by an injective presentation of $\mathfrak{A}$. It follows that decidability of $\mathsf{FO}[\mathsf{C}]$ on the structure $\mathfrak{A}$, which is a consequence of Theorem 5.8, can not be deduced from the previous Theorem 5.2, and so it shows that Theorem 5.8 is a strong generalization of the previous result.