Solving a Generalized Version of the Exact Cover Problem with a Light-Based Device

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Abstract. We propose a light-based device that is capable of solving a generalized version of the exact cover problem.

1 Introduction

In recent times, significant amount of attention has been given to explore different unconventional computing techniques to solve various computational problems. Different problems that are hard to solve efficiently in conventional methods, have received significant attention in the literature from this point of view. Optical computing is an exciting research avenue in this regard. Very recently,a number of researchers have suggested light-based devices to solve combinatorially interesting problems. Most of the problems handled in this way are hard to solve in the conventional computing paradigm and are categorized as NP-Complete or NP-Hard problems. For example, a system which solves the Hamiltonian Path problem (an NP-Complete problem [1]) using light and its properties has been proposed in [6]. Similar system was devised in [7] and [8] to solve the Exact Cover Problem and the Subset Sum problem respectively, both of which are NP-Complete problems [1]. Following up the work of [8], very recently, a generalized version of the subset sum problem was solved using a light-based device in [2]. In the above-mentioned work, only the decision versions of the corresponding problems have been considered. Finally, a way to compute an actual solution of the subset sum problem was presented in [3].

This paper follows up the work of [7] and [2]. In particular, we revisit the Exact Cover problem from optical computing point of view. We combine the work of [7] and [2] to solve a generalized version of the exact cover problem. Notably, a number of unresolved issues were highlighted in [7] as future research directions. By devising a light-based device capable of solving a generalized version of the problem, we are able to resolve some of these open problems.

S. Dolev and M. Oltean (Eds.): OSC 2010, LNCS 6748, pp. 23–31, 2011.

2 Exact Cover Problem

We start this section with a formal definition of the exact cover problem.

Problem 1. Exact Cover Problem

Instance: We are given a set $U = \{u_1, u_2, ..., u_n\}$ of elements and a set C of subsets of U.

Question: Is there a subset S of C such that every element in U is contained in exactly one set in S.

In the paper [7], the authors presented a light based device to solve the Exact Cover problem. Notably, the device presented in [7] only can solve the problem as a decision problem; it doesn't have the capability to provide a solution. Indeed, the author pointed out in the conclusion that designing a device with such capability should be explored as a future research. In this paper, we consider a more general problem than the problem defined above. In particular, we solve the question whether there is a subset S of C such that $|S|$ is less than or equal to a given integer m. Formally, the problem we handle is defined below:

Problem 2. Generalized Exact Cover Problem

Instance: We are given a set $U = \{u_1, u_2, ..., u_n\}$ of elements, a set C of subsets of U and a positive number m.

Question: Is there a subset S of C such that every element in U is contained in exactly one set in S and $|S| \leq m$.

Clearly, the Exact Cover Problem (Problem 1) is a restriction of the Generalized Exact Cover Problem (Problem 2), since the latter reduces to the former when $m = |U|$; therefore, the latter is also NP-Complete.

2.1 Example

$U = \{u_1, u_2, u_3, u_4\}$
$C = \{C_1, C_2, C_3\}$
$C1 = \{u_1, u_3\}$
$C2 = \{u_2, u_3, u_4\}$
$C3 = \{u_2, u_4\}$

An exact cover S of U is, $S = \{C_1, C_3\}$ also there is no solution set S such that $|S| \leq 1$.

3 The Device

We design our device in a step by step manner. We start from the device proposed in [7] see figure 1. We first give brief review of the device proposed in [7].

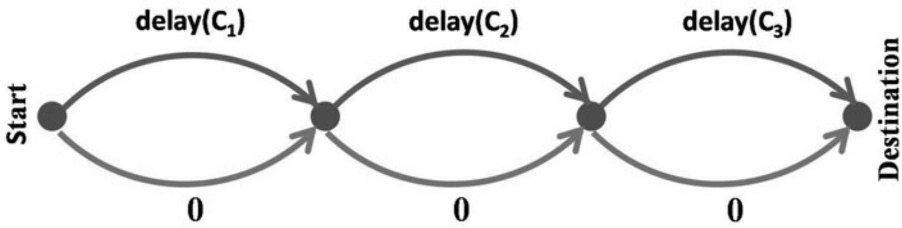

Fig. 1. previously proposed device

3.1 Previously Proposed Device

The previously proposed device in figure 1 is constructed for the problem definition given in 2.1. The device has a graph-like structure. The following two simple operations are performed by the device.

1. When passing through an arc the light ray is delayed by the amount of time assigned to the arc.
2. When passing through a node the light ray is divided into a number of rays equal to the external degree of that node.

Delay calculation. The $delay(C_i), 1 \leq i \leq |C|$ is computed as follows,

We assign each element of set $U = \{u_1, u_2, u_3, u_4\}$ a number $d_i(1 \leq i \leq n)$. These number have the following special property,

$$d_1 + d_2 + \ldots + d_n \neq a_1 \cdot d_1 + a_2 \cdot d_2 + \ldots + a_n \cdot d_n \quad (1)$$

where $a_i(1 \leq i \leq n)$ are natural numbers $(a \geq 0)$ and cannot be all 1 in the same time.

The integer number a_j tells us how many times a ray has passed through (covered) an element u_j. If value of a_j is equal to 1, we know that that particular element u_j has been covered (exactly once). Some example of numbers abiding the equation 1 is given in table 1. The numbers in this set are called Nialpdromes numbers (sequence A023758from The On-line Encyclopedia of Integer Numbers [9]).

These numbers have been used before to solve Hamiltonian path problem [6], Exact cover problem [7]. We can very easily generate these number for a given n by using the following formulae,

$$2^n - 2^{n-1},$$
$$2^n - 2^{n-2},$$
$$2^n - 2^{n-3},$$
$$\ldots$$
$$2^n - 2^0$$

Although the numbers are very easy to generate, for a given n they are of exponential order, and thus have a adverse effect on the complexity of the device.

For each element $u_i \in U(1 \leq i \leq n)$ we assign the value $d_i(1 \leq i \leq n)$. for the problem definition in 2.1 each delay is computed as follow,

$delay(C_1) = d_1 + d_2$

$delay(C_2) = d_2 + d_3 + d_4$

$delay(C_1) = d_2 + d_4$

Because of the values assigned to each element $u_i(1 \leq i \leq n)$ and the way $delay(C_i)(1 \leq i \leq |C|)$ is computed we can identify whether a exact cover exist or not. We do so by waiting for a light signal at moment $d_1 + d_2 + \ldots + d_n$. if a light signal is present then a solution exists otherwise there is no solution.

3.2 The Basic Idea of the System

The basic idea of our device is that $delay(C_i), (1 \leq i \leq |C|)$ plus some fraction value represent the delays induced to the signals(light) that passes through our device. For instance the, if the sets C_1, C_3 produce a exact cover, then the total delay of the signal should be $delay(C_1) + delay(C_1) + (fractionalValue)$. The fractional value will help us determine the maximum number of sets needed for exact cover. If using light we can easily induce some delays by forcing the ray to pass through an optical cable of given length.

This is why we have designed our device as a directed graph. Arcs, which are implemented by using optical cables, are labeled with $delay(C_i), (1 \leq i \leq |C|)$ plus a fractional value which depends on the size of the set C.

As can be seen in figure 2 that each number plus fractional value f is assigned to arcs with an alternative path of delay equal to zero.

Property of the fraction f:

1. The f is equal for all arcs, so that we can identify the number of sets participating in the solution.
2. Sum of all f is less then 1. The significance of this property will be realized when we prove the correctness of this device.

We can easily calculate such fraction by using the formulae, $f = \frac{1}{(|C|+1)}$. So for the problem instance shown in the figure 2 with $|C| = 3, f = \frac{1}{(3+1)} = 0.25$ the signal passing through the arcs containing C_1, C_3 will reach the destination at

Table 1. Delay that are used for a given set U of size n

Problem Size, n	Labels for $C_j, d(C_j)$
1	1
2	2, 3
3	4, 6, 7
4	8, 12, 14, 15
5	16, 24, 28, 30, 31
6	32, 48, 56, 60, 62, 63

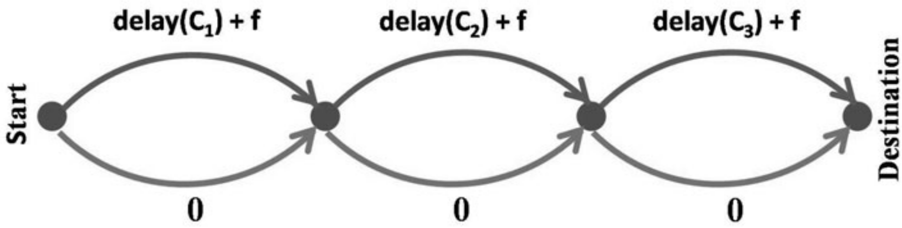

Fig. 2. First version of our device

time $delay(C_1) + delay(C_3) + 0.5$. So if we are interested in finding out whether there is a solution set S producing exact cover and $|S| = m$, then we wait for a signal at the moment $d_1 + d_2 + \ldots + d_n + \frac{m}{(|C|+1)}$.

Clearly, the above system is not capable to answer the question whether there is a solution subset S such that, $|S| \leq m$. In the rest of this section we present a novel system to achieve our goal.

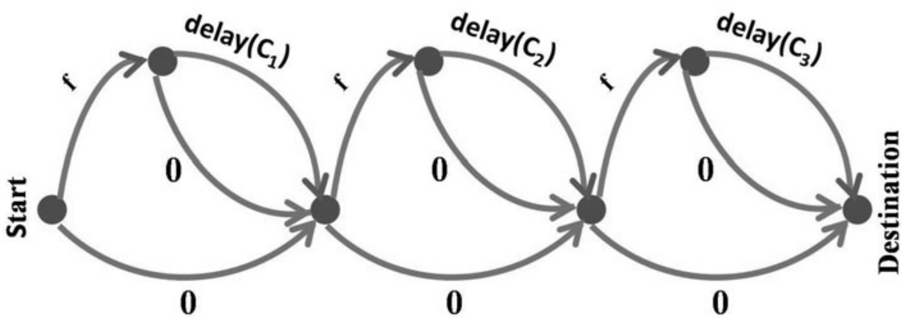

Fig. 3. Second version of our device

3.3 The Modified Device

We modify the device as shown in figure 3. This device ensures that if there is a light signal at moment $d_1 + d_2 + \ldots + d_n + \frac{\beta}{(|C|+1)}$, then there will be a light signal at moments $d_1 + d_2 + \ldots + d_n + \frac{\alpha}{(|C|+1)}$, where $\beta < \alpha \leq |C|$. For instance if $|C| = 3, f = 0.25$ the device in figure 3 will produce a signal which reaches destination at time $d_1 + d_2 + \ldots + d_4 + 0.5$ as before. But it will also produce a signal at time $d_1 + d_2 + \ldots + d_4 + 0.75$.

The final bit of modification needed is because of the presence of 0 arcs. Even if theoretically we could have arcs of length 0, we cannot have cables of length 0 in practice. We may use very short cables (let's say of length ϵ) for arcs which are supposed to have length 0. However, in that case we could obtain false signal because of the cumulative effect of cables with length ϵ. Even if there is no exact cover of the set U, still there will be possible to have a signal at moment $d_1 + d_2 + \ldots + d_n + (fractionalValue)$ due to the situation presented above. For avoiding this situation we have added a constant k to the length of each alternative path of the light signal. The schematic view of this device is depicted in figure 4.

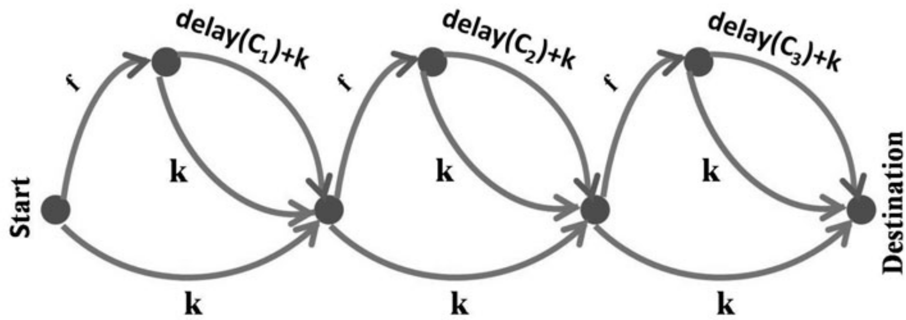

Fig. 4. Final version of our device considering implementation issues

We can see that each path from start to destination contains exactly $|C|$ time value k. Thus at the destination we will not wait anymore at moment $d_1 + d_2 + \ldots + d_n + \frac{m}{(|C|+1)}$, for a solution set S with $|S| \leq m$. Instead we will wait for a solution at moment $d_1 + d_2 + \ldots + d_n + \frac{m}{(|C|+1)} + |C| \cdot k$ Since all light signal will have the constant $|C| \cdot k$ added. Now, it is quite easy to realize that if there is a ray arriving to the destination (see figure 4) at moment $d_1 + d_2 + \ldots + d_n + \frac{m}{(|C|+1)} + |C| \cdot k$ then there is a subset S of C such that S contains every elements of U exactly once and $|S| \leq m$.

4 Physical Implementation

For implementing the proposed device the following components are required:

- A light source at the start node.
- Beam-splitters for splitting a light beam into two light rays. Half silvered mirror is a good example for beam-splitter
- Optical fibers of various length.
- Light sensor at the destination node, which converts optical pulses to electric signals. Photo diode is a good example for light sensor.
- Oscilloscope, a tool for detecting a fluctuation of power generated by photo diode.

5 Analysis of the Proposed Device

In this section, we analyze our proposed device following the framework for analysis presented in [6, 8].

5.1 Precision

A problem is that we cannot measure the moment $d_1+d_2+\ldots+d_n+\frac{m}{(|C|+1)}+|C|\cdot k$ exactly. We can do this measurement only with a given precision which depends on the tools involved in the experiments. Actually it will depend on the response time of the photodiode and the rise time of the oscilloscope. The rise-time of the best oscilloscope available on the market is in the range of picoseconds (10^{-12} seconds). This means that each signal arriving at the destination at distinct moment must maintain above mentioned time interval to be recognized correctly. This can be ensured by setting a lower bound on the length of the cable as follows. Since the speed of light is 3×10^8, the minimum cable length must be 0.0003 meter.

This value is the minimal delay that should be introduced by an arc. Also, note that, all lengths must be integer multiples of 0.0003. We cannot allow to have cables whose lengths can be written as $p \times 0.0003 + q$, where p is an integer and q is a positive real number less than 0.0003, since by combining this kind of numbers we can have a signal in the above mentioned interval and that signal may not contain valid information.

Once we have the length for the minimal delay it is quite easy to compute the length of the other cables that are used in order to induce a certain delay as follows. We assign $f = 0.0003$ meter, where f stands for the fractional value. Then we define $unit = (1 + |C|) \cdot f = (1 + |C|) \cdot 0.0003$ meter Now for each $delay(C_i), (1 \leq i \leq |C|)$, we use arc of length $delay(C_i) \cdot unit$ where $1 \leq i \leq |C|$.

5.2 Hardware Complexity

The maximum value of delay of a particular arc in the device, is $O(2^n)$. So the maximum cable length needed for physical representation of a element is, $2^n \cdot unit = (|C| + 1) \cdot 2^n \cdot \Delta = O(|C| \cdot 2^n)$. So the total length of the wire is $O(|C|^2 \cdot 2^n)$.

5.3 Runtime Complexity

The ray corresponding to solution takes $O((d_1 + d_2 + \ldots + d_n) \cdot |C| + |C| \cdot k)$ time to reach destination node. So the time complexity is $O(|C| \cdot n \cdot 2^n + |C| \cdot k)$ or $O(|C| \cdot n \cdot 2^n)$ assuming $2^n \gg k$.

5.4 Problem Size

We are also interested to find the size of the instances that can be solved by our device. let, L be the maximum length of available cable. And Δ be the minimum length of cable (see Section 5.1). Then, we have $2^n \cdot (1 + |C|) \cdot \Delta = L$. This implies $2^n \cdot (1+2^n) \cdot \Delta = L$, [because $C = O(2^n)$]. Finally, we get $n = \frac{1}{2} \cdot lg\frac{L}{\Delta}$ (approximately).

5.5 Power Decrease

Beam splitters are used in our approach for dividing a ray in two subrays. Because of that, the intensity of the signal is decreasing. In our proposed device there $2 \cdot |C|$ nodes(we ignore the destination node, because there is no split there). At each node the power of the signal is reduced to half. So for detection of the light signal at the destination the input signal must have $2^{2 \cdot |C|}$ power.

5.6 Technical Difficulties

There are many technical difficulties that must be solved to implement the proposed device. Some of them are described bellow:

- Cutting the optical fiber with precision. Failing to do so can generate a fluctuation of light pulse at the time when the signal representing the solution is scheduled to reach the destination.
- Finding high precision oscilloscope. This is essential to measure the power at the accurate time.

5.7 Improving the Device

In our proposed device the delay a light signal incurs as it passes through an optical fiber is used to encode the elements of the given set. So the maximum number representable in the device depends upon the maximum delay that is practically possible to produce with an optical fiber. We can either increase the length of the cable used or decrease the speed of light to do so. Increasing the length of the cables will also increase the device size. So decreasing light speed is a preferable option.

The speed of light in the optical fiber is much less than the speed of light in void space (60% of original speed of light). Other methods for reducing the speed of light is also available.A very interesting solution was proposed in [4] which is able to reduce the speed of light by 7 order. This could help our mechanism significantly. But using this idea in our device remains an open question because of the complex equipment involved in those experiments [5] [4].

6 Conclusion and Future Work

In this paper, we have developed a light-based device that is capable of solving a generalized version of the exact cover problem. Further research directions can be focused on the followings:

- implementation of the proposed device.
- the device proposed is mechanical in nature. i.e. we have to change the cable length and number of beam splitters in order to solve different instances of the exact cover problem. So we need to automate the whole process.

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