A Frequency-Robust Solver for the Time-Harmonic Eddy Current Problem

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Abstract This work is devoted to fast and parameter-robust iterative solvers for frequency domain finite element equations, approximating the eddy current problem with harmonic excitation. We construct a preconditioned MinRes solver for the frequency domain equations, that is robust (= parameter-independent) in both the discretization parameter h and the frequency ω.

1 Introduction

In many practical applications, the excitation is time-harmonic. Switching from the time domain to the frequency domain allows us to replace expensive time-integration procedures by the solution of a system of partial differential equations for the amplitudes belonging to the sine- and to the cosine-excitation. Following this strategy Copeland et al. [7, 8] and Bachinger et al. [5, 6] applied harmonic and multiharmonic approaches to parabolic initial-boundary value problems and the eddy current problem, respectively. Indeed, in [7] a MinRes solver for the solution of parabolic initial-boundary value problems is constructed, that is robust with respect to both the discretization parameter h and the frequency ω. The aim of this work is

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B. Michielsen and J.-R. Poirier (eds.), *Scientific Computing in Electrical Engineering SCEE 2010*, Mathematics in Industry 16, DOI 10.1007/978-3-642-22453-9_11,

to generalize these ideas to the eddy current problem. Due to the non-trivial kernel of the **curl**-operator, the generalization of this solver is not straight forward. In order to achieve a positive definite reformulation of the frequency domain equations, we perform a regularization in terms of an additional gauging term. The regularized problem can be solved in a MinRes setting, applying a preconditioning technique proposed by Schöberl and Zulehner [19].

2 Frequency Domain FEM

As a model problem we consider the eddy current problem with homogeneous Dirichlet boundary condition and an inhomogeneous initial condition.

$$\begin{cases} \sigma \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{curl}\,(\nu\, \mathbf{curl}\, \mathbf{u}) = \mathbf{f} & \text{in } \Omega \times (0, T] \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{0} & \text{in } \bar{\Omega} \\ \mathbf{u}(\mathbf{x}, t) = \mathbf{0} & \text{on } \partial\Omega \times [0, T] \end{cases} \tag{1}$$

We assume, that $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain. The reluctivity $\nu = \nu(\mathbf{x})$ is supposed to be independent of $|\mathbf{curl}\ \mathbf{u}|$, i.e. we assume that the eddy current problem (1) is linear. The conductivity σ is piecewise constant and zero in non-conducting regions. We assume that the source $\mathbf{f}$ is weakly divergence free. Bachinger et al. [5] provide existence und uniqueness results for linear and non-linear eddy current problems in appropriate gauged spaces.

Furthermore we assume that $\mathbf{f}$ is given by a time-harmonic excitation with frequency $\omega > 0$ and amplitudes $\mathbf{f}^{\mathbf{c}}$ and $\mathbf{f}^{\mathbf{s}}$, i.e. $\mathbf{f}(\mathbf{x}, t) = \mathbf{f}^{\mathbf{c}}(\mathbf{x}) \cos(\omega t) + \mathbf{f}^{\mathbf{s}}(\mathbf{x}) \sin(\omega t)$ Therefore the solution $\mathbf{u}$ is time-harmonic as well, with the same base frequency ω:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}^{\mathbf{c}}(\mathbf{x}) \cos(\omega t) + \mathbf{u}^{\mathbf{s}}(\mathbf{x}) \sin(\omega t). \tag{2}$$

In fact, (2) is the real reformulation of a complex time-harmonic approach $\mathbf{u}(\mathbf{x}, t) = \hat{\mathbf{u}}(\mathbf{x}) e^{i\omega t}$ with the complex-valued amplitude $\hat{\mathbf{u}} = \mathbf{u}^{\mathbf{c}} - i\mathbf{u}^{\mathbf{s}}$. Using the real-valued time-harmonic representation of the solution (2), we can state the eddy current problem (1) in the frequency domain as follows:

$$\text{Find } \mathbf{u} = (\mathbf{u}^{\mathbf{c}}, \mathbf{u}^{\mathbf{s}}) : \quad \begin{cases} \mathbf{curl}\,(\nu\, \mathbf{curl}\, \mathbf{u}^{\mathbf{c}}) + \omega\, \sigma\, \mathbf{u}^{\mathbf{s}} = \mathbf{f}^{\mathbf{c}} \\ \mathbf{curl}\,(\nu\, \mathbf{curl}\, \mathbf{u}^{\mathbf{s}}) - \omega\, \sigma\, \mathbf{u}^{\mathbf{c}} = \mathbf{f}^{\mathbf{s}}, \end{cases} \tag{3}$$

with the corresponding boundary conditions from (1).

Remark 1. Having in mind applications to problems with non-linear reluctivity ν, we prefer to use the real reformulation (3) instead of a complex approach (see [3, Sect. 3.4]).

The finite element discretization of the variational formulation of (3) with lowest order edge elements, introduced by Nédélec in [13], yields the following system of linear equations

$$\begin{pmatrix} \mathbf{A_h} & \omega \mathbf{M}_{\sigma,\mathbf{h}} \\ -\omega \mathbf{M}_{\sigma,\mathbf{h}} & \mathbf{A_h} \end{pmatrix} \begin{pmatrix} \mathbf{u_h^c} \\ \mathbf{u_h^s} \end{pmatrix} = \begin{pmatrix} \mathbf{f_h^c} \\ \mathbf{f_h^s} \end{pmatrix} \tag{4}$$

with stiffness matrix $\mathbf{A_h}$ and mass matrix $\mathbf{M}_{\sigma,\mathbf{h}}$.

3 Exact Regularization

Eddy current problems are essentially different for conducting ($\sigma > 0$) and non-conducting regions ($\sigma = 0$). In order to gain uniqueness in the non-conducting regions, we persue an exact regularization strategy.

Due to the non-trivial kernel of the **curl**-operator, the resulting stiffness matrix $\mathbf{A_h}$ is only positive semi-definite. However, for later preconditioning purposes, we require that the sum of certain blocks of the system matrix (4) is positive definite. In order to achieve that, we follow a gauging strategy proposed by Kuhn [12]. The regularized variational problem reads as

$$\text{Find } \mathbf{u} = (\mathbf{u^c}, \mathbf{u^s}) \in H_0(\mathbf{curl})^2 : \quad a_Q(\mathbf{u}, \mathbf{v}) = \langle F, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in H_0(\mathbf{curl})^2 \tag{5}$$

with the regularized bilinear form

$$a_Q(\mathbf{u}, \mathbf{v}) := \sum_{j \in \{c,s\}} \int_\Omega \nu \, \mathbf{curl}\, \mathbf{u^j} \mathbf{curl}\, \mathbf{v^j} + \omega \nabla P_D \mathbf{u^j} \nabla P_D \mathbf{v^j} \mathrm{d}\mathbf{x} + \omega \int_\Omega \sigma \, [\mathbf{u^c v^s} - \mathbf{u^s v^c}] \, \mathrm{d}\mathbf{x}. \tag{6}$$

Here $P_D : H_0(\mathbf{curl}) \to H_0^1(\Omega)$ is the Helmholtz projection (see e.g. [12]). For any $\mathbf{v} \in H_0(\mathbf{curl})$, $P_D \mathbf{v} := p$ is defined by the unique solution of the variational problem

$$(\nabla p, \nabla q)_{L_2(\Omega)} = (\mathbf{v}, \nabla q)_{L_2(\Omega)}, \quad \forall q \in H_0^1(\Omega). \tag{7}$$

Hence we replace $\mathbf{A_h}$ by the sum of $\mathbf{A_h}$ and a regularization term $\omega \mathbf{Q_h}$, i.e. $\mathbf{A_h} + \omega \mathbf{Q_h}$. Here $\mathbf{Q_h}$ is the discretization of the operator $\mathbf{Q}$, defined by $(\mathbf{Q}\,\mathbf{u}, \mathbf{v})_{L_2(\Omega)} := \int_\Omega \nabla P_D \mathbf{u} \nabla P_D \mathbf{v} \mathrm{d}\mathbf{x}$, by Nédélec finite elements of lowest order.

$$\begin{pmatrix} \mathbf{A_h} + \omega \mathbf{Q_h} & \omega \mathbf{M}_{\sigma,\mathbf{h}} \\ -\omega \mathbf{M}_{\sigma,\mathbf{h}} & \mathbf{A_h} + \omega \mathbf{Q_h} \end{pmatrix} \begin{pmatrix} \mathbf{u_h^c} \\ \mathbf{u_h^s} \end{pmatrix} = \begin{pmatrix} \mathbf{f_h^c} \\ \mathbf{f_h^s} \end{pmatrix}. \tag{8}$$

The operator P_D and hence the matrix $\mathbf{Q_h}$ are chosen in such a way, that on the one hand it ensures the positive definiteness of the block $\mathbf{A_h} + \omega \mathbf{Q_h}$ and on the other hand $\mathbf{Q_h u_h^{c/s}}$ vanishes at the solution, i.e. the regularized system (8) and the original system (4) have one and the same solution. The proof of the equivalence of the original and exact regularized problem (5) follows the same steps as in [12].

4 MinRes Preconditioner

For preconditioning purpose we have to reformulate the system (8) with a positive definite but block skew-symmetric system matrix, as a symmetric but indefinite one. This system can be solved by a preconditioned MinRes method [14]. The key points for the construction of a parameter robust preconditioner are the introduction of a non-standard norm in $H_0(\mathbf{curl})$ and the theorem of Babuška-Aziz [2].

The symmetric and indefinite reformulation of the variational formulation with a positive definite but skew-symmetric bilinear form (5) is given by:

$$\text{Find } (\mathbf{x},\mathbf{y}) \in H_0(\mathbf{curl})^2 : \quad \mathscr{A}_M((\mathbf{x},\mathbf{y}),(\mathbf{v},\mathbf{w})) = \int_\Omega \left[\frac{1}{\omega}\mathbf{f}^c\mathbf{v} + \mathbf{f}^s\mathbf{w}\right] d\mathbf{x} \tag{9}$$

for all $(\mathbf{v},\mathbf{w}) \in H_0(\mathbf{curl})^2$, with the scaled vectors $(\mathbf{x},\mathbf{y}) = (\mathbf{u}^s, \frac{1}{\omega}\mathbf{u}^c)$ and $(\mathbf{v},\mathbf{w}) = (\omega\mathbf{v}^c, \mathbf{v}^s)$ and the symmetrised bilinear form $\mathscr{A}_M(\cdot,\cdot)$, given by

$$\begin{aligned}\mathscr{A}_M((\mathbf{x},\mathbf{y}),(\mathbf{v},\mathbf{w})) = {} & (\sigma\ \mathbf{x},\mathbf{v})_{L_2(\Omega)} - \omega^2(\sigma\ \mathbf{y},\mathbf{w})_{L_2(\Omega)} \\ & + (\nu\ \mathbf{curl}\ \mathbf{y}, \mathbf{curl}\ \mathbf{v})_{L_2(\Omega)} + \omega(\nabla P_D\mathbf{y}, \nabla P_D\mathbf{v})_{L_2(\Omega)} \\ & + (\nu\ \mathbf{curl}\ \mathbf{x}, \mathbf{curl}\ \mathbf{w})_{L_2(\Omega)} + \omega(\nabla P_D\mathbf{x}, \nabla P_D\mathbf{w})_{L_2(\Omega)}.\end{aligned}$$

Hence we can reformulate the block skew-symmetric and positive definite system (8) as a symmetric but indefinite system (10) with system matrix $\mathbf{D_h}$:

$$\begin{pmatrix} \mathbf{M}_{\sigma,\mathbf{h}} & \mathbf{A_h} + \omega\mathbf{Q_h} \\ \mathbf{A_h} + \omega\mathbf{Q_h} & -\omega^2\mathbf{M}_{\sigma,\mathbf{h}} \end{pmatrix} \begin{pmatrix} \mathbf{u_h^s} \\ \frac{1}{\omega}\mathbf{u_h^c} \end{pmatrix} = \begin{pmatrix} \frac{1}{\omega}\mathbf{f_h^c} \\ \mathbf{f_h^s} \end{pmatrix}. \tag{10}$$

Next we construct a block-diagonal preconditioner according to the preconditioning technique proposed by Schöberl and Zulehner [19]. We introduce the non-standard norm $\|\cdot\|_{V_M}$ in $H_0(\mathbf{curl})$

$$\|\mathbf{y}\|_{V_M}^2 = \frac{1}{\omega}\left[(\nu\ \mathbf{curl}\ \mathbf{y}, \mathbf{curl}\ \mathbf{y})_{L_2(\Omega)} + \omega\|\nabla P_D\mathbf{y}\|_{L_2(\Omega)}^2 + \omega(\sigma\ \mathbf{y},\mathbf{y})_{L_2(\Omega)}\right]. \tag{11}$$

Note, that the regularization term ensures, that this norm is well defined even in non-conducting regions. This definition gives rise to a non-standard norm $\|\cdot\|_{Q_M}$ in the product space $H_0(\mathbf{curl})^2$

$$\|(\mathbf{x},\mathbf{y})\|_{Q_M}^2 = \|\mathbf{x}\|_{V_M}^2 + \omega^2\|\mathbf{y}\|_{V_M}^2. \tag{12}$$

Lemma 1. *We have*

$$\frac{1}{\sqrt{2}}\|(\mathbf{x},\mathbf{y})\|_{Q_M} \leq \sup_{0\neq(\mathbf{v},\mathbf{w})\in H_0(\mathbf{curl})^2} \frac{\mathscr{A}_M((\mathbf{x},\mathbf{y}),(\mathbf{v},\mathbf{w}))}{\|(\mathbf{v},\mathbf{w})\|_{Q_M}} \leq \|(\mathbf{x},\mathbf{y})\|_{Q_M}. \tag{13}$$

Proof. Boundedness follows from reapplication of Cauchy's inequality. The lower estimate can be attained by choosing $\mathbf{v} = \omega\mathbf{y} + \mathbf{x}$ and $\mathbf{w} = \frac{1}{\omega}\mathbf{x} - \mathbf{y}$. □

Since we are dealing with conforming finite elements, the estimate (13) is also valid in the Nédélec finite element subspace. Hence, it follows by the theorem of Babuška-Aziz, that there exists a unique solution of the corresponding variational problem (9), and that the solution continuously depends on the data, uniformly on ω and σ. Hence we conclude that the block-diagonal preconditioner

$$\mathbf{C_h} = \frac{1}{\omega}\begin{pmatrix} \tilde{\mathbf{C}}_\mathbf{h} & \mathbf{0} \\ \mathbf{0} & \omega^2\tilde{\mathbf{C}}_\mathbf{h} \end{pmatrix}, \tag{14}$$

with $\tilde{\mathbf{C}}_\mathbf{h} = \omega(\mathbf{M}_\sigma,\mathbf{h} + \mathbf{Q_h}) + \mathbf{A_h}$, is robust with respect to both the discretization parameter h and the parameters ω and σ. Thus the spectral condition number (16) of the preconditioned system

$$\mathbf{C_h}^{-1}\mathbf{D_h}\mathbf{u_h} = \mathbf{C_h}^{-1}\mathbf{f_h} \tag{15}$$

can be estimated by a constant c that is independent of h, ω and σ i.e.

$$\kappa(\mathbf{C_h}^{-1}\mathbf{D_h}) := \|\mathbf{C_h}^{-1}\mathbf{D_h}\|_{\mathbf{C_h}}\|\mathbf{D_h}^{-1}\mathbf{C_h}\|_{\mathbf{C_h}} \leq c \neq c(\omega, h, \sigma). \tag{16}$$

Therefore the number of MinRes iterations required for reducing the initial error by some fixed factor $\varepsilon \in (0,1)$ is independent of the discretization parameter h and the frequency ω.

In practice, the diagonal blocks $\tilde{\mathbf{C}}_\mathbf{h}$ in (14) are replaced by some appropriate preconditioners, e.g. by robust multigrid preconditioners as proposed in [1].

Theorem 1 (Entire robust and optimal solver). *The MinRes method applied to the preconditioned system (15) converges. At the m-th iteration, the preconditioned residual* $\mathbf{r}_\mathbf{h}^\mathbf{m} = \mathbf{C_h}^{-1}\mathbf{f_h} - \mathbf{C_h}^{-1}\mathbf{D_h}\mathbf{u}_\mathbf{h}^\mathbf{m}$ *is bounded as*

$$\left\|\mathbf{r}_\mathbf{h}^\mathbf{2m}\right\|_{\mathbf{C_h}} \leq \frac{2q^m}{1+q^{2m}}\left\|\mathbf{r}_\mathbf{h}^\mathbf{0}\right\|_{\mathbf{C_h}} \quad where \quad q = \frac{\kappa(\mathbf{C_h}^{-1}\mathbf{D_h}) - 1}{\kappa(\mathbf{C_h}^{-1}\mathbf{D_h}) + 1}. \tag{17}$$

If we additionally apply the Arnold/Falk/Winther multigrid preconditioner [1] to the diagonal blocks, the whole convergence rate q is independent of ω and h.

Proof. The convergencerate of the MinRes method [14] can be found in [10]. Combining this result with the estimate of the condition number (16) and the multigrid convergence [1], yields the desired result. □

Table 1 Number of MinRes iterations for reducing the initial residual by 10^{-6}

DOF	$\log_{10}\omega$	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	CPU time
1,208	$h = 0.25$	3	3	3	5	7	14	15	16	14	8	6	4	4	< 0.48 s
8,368	$h = 0.125$	3	3	3	5	7	13	15	16	16	12	6	4	4	< 2.48 s
62,048	$h = 0.0625$	3	3	3	5	7	13	15	16	16	14	8	6	4	< 29.79 s
477,376	$h = 0.03125$	3	3	3	5	7	8	16	16	16	13	12		4	< 477.55 s
Skin depth $\sqrt{2\nu/(\omega\sigma)}$		141.4	44.6	14.1	4.5	1.4	0.4	0.14	0.044	< 0.03125					

Table 2 Number of MinRes iterations for reducing the initial residual by 10^{-6}

DOF	$\log_{10}\sigma_2$	−4	−3	−2	−1	1	2	3	4	5	6	7	8
196	$h = 0.5$	7	7	7	7	13	15	14	8	8	8	7	7
1,208	$h = 0.25$	6	6	6	7	11	15	16	12	8	8	7	7
8,368	$h = 0.125$	5	5	6	6	11	15	16	16	10	8	7	7
62,048	$h = 0.0625$	5	5	5	6	9	15	17	18	14	8	8	7

Since σ is not constant in general, we loose robustness with respect to σ in the multigrid procedure. Note that for constant σ, we additionally get robustness with respect to σ.

5 Numerical Results

Finally, we report two numerical tests for an academic three dimensional eddy current problem. The numerical results presented in this section were attained using ParMax [16]. First, we demonstrate the robustness of the block-diagonal preconditioner with respect to the frequency ω. Therefore, for the inversion of the diagonal blocks we use the exact solver PARDISO [17, 18]. Table 1 provides the number of MinRes iterations needed for reducing the initial residual by a factor of 10^{-6} for different ω and h. These numerical experiment was performed for a three-dimensional linear problem on the unit-cube, discretized by tetrahedra for the case $\nu = \sigma = 1$. These experiment demonstrates the independence of the frequency and the meshsize as the number of iterations is bounded by 16. Next we repeat the numerical experiment for piecewise constant conductivity σ, i.e.

$$\sigma = \begin{cases} \sigma_1 \text{ in } \Omega_1 = \{(x,y,z)^T \in [0,1]^3 : z > 0.5\} \\ \sigma_2 \text{ in } \Omega_2 = \{(x,y,z)^T \in [0,1]^3 : z \leq 0.5\} \end{cases}. \tag{18}$$

In Table 2 we give the number of iterations for fixed $\omega = 1$ and $\sigma_1 = 1$ and various σ_2. We observe, that the number of iterations is bounded by 18. Both experiments demonstrate the robustness of the block-diagonal preconditioner with respect to the involved parameters. Moreover, this theory-based parameter-robust block-diagonal preconditioner is appropriate to be incorporated in a Newton-based multiharmonic solver for solving (nonlinear) shielding and welding problems (see [5, 6]).

6 Further Applications

The presented preconditioning technique provides a robust tool for solving linear eddy current problems with time-harmonic excitation. The theory can be extended to multiharmonic excitations and even to problems with non-harmonic excitation of the right-hand side. The theory in this paper is presented for exact regularized problems. Furthermore we can develop this preconditioning technique also for inexact regularized problems.

6.1 Non-harmonic Excitation

By approximating any non-harmonic right-hand side by a multiharmonic excitation in terms of a truncated Fourier series, it follows, that the solution $\mathbf{u_N}$ has the structure:

$$\mathbf{u_N}(\mathbf{x}, t) = \sum_{k=0}^{N} \mathbf{u_k^c}(\mathbf{x}) \cos(k\omega t) + \mathbf{u_k^s}(\mathbf{x}) \sin(k\omega t). \tag{19}$$

Using the truncated Fourier approximation (19), the corresponding system matrix in the frequency domain decouples into a block-diagonal matrix of the form

$$\text{diag}\left\{ \begin{pmatrix} \mathbf{A_h} & k\omega \mathbf{M_{\sigma,h}} \\ -k\omega \mathbf{M_{\sigma,h}} & \mathbf{A_h} \end{pmatrix} \right\}_{k=0,\ldots,N}, \tag{20}$$

where each block has almost the same structure as the two-by-two system matrix in (4). Hence we can apply either the exact or the inexact regularization technique and precondition each block robustly with respect to the frequency ω, the mode k and the meshsize h. By approximating a general right-hand side $\mathbf{f}$ by a finite Fourier series with N summands, we introduce an additional truncation error of order N^{-1}.

$$\|\mathbf{u} - \mathbf{u_N}\|_{L_2((0,T),H_0(\mathbf{curl}))} = \mathscr{O}(N^{-1}). \tag{21}$$

6.2 Inexact Regularization (Conductivity Regularization)

Instead of the exact regularization an inexact regularization, as for example in [5], can also be applied by introducing a regularized conductivity σ_ε, defined as $\max\{\sigma, \varepsilon\}$ with the regularization parameter $\varepsilon > 0$. In this case the same strategy can be used to construct a block diagonal preconditioner, that is robust with respect to ω, h and σ_ε, leading to the system matrix $\mathbf{D_h}$ and the preconditioner $\mathbf{C_h}$.

$$\mathbf{D_h} = \begin{pmatrix} \mathbf{M}_{\sigma_\varepsilon,\mathbf{h}} & \mathbf{A_h} \\ \mathbf{A_h} & -\omega^2 \mathbf{M}_{\sigma_\varepsilon,\mathbf{h}} \end{pmatrix} \quad \mathbf{C_h} = \frac{1}{\omega} \begin{pmatrix} \omega \mathbf{M}_{\sigma_\varepsilon,\mathbf{h}} + \mathbf{A_h} & \mathbf{0} \\ \mathbf{0} & \omega^2(\omega \mathbf{M}_{\sigma_\varepsilon,\mathbf{h}} + \mathbf{A_h}) \end{pmatrix} \tag{22}$$

In contrast to the exact regularization, where no additional regularization error is introduced, in the case of inexact regularization, we have to deal with an additional error of order $\mathscr{O}(\varepsilon)$ (see [5]).

7 Conclusion and Outlook

The method developed in this work shows great potential for solving both, time-harmonic and non harmonic eddy current problems in a very efficient and robust way, in the linear case. Up to now, theory only guarantees robustness in the case of constant coefficients ω and σ, but currently we are working on the extension also to the piecewise constant case. Indeed, based on the results in [11], we are working on a domain decomposition preconditioner for the inversion of the diagonal blocks, that guarantees robustness also for piecewise constant conductivity σ.

In the non-linear case, i.e. $\nu = \nu(\mathbf{x}, |\mathbf{curl\ u}|)$, it turns out, that even for harmonic excitation of the right-hand side, we have to take all frequencies $k\omega$ into account. For earlier works see e.g. [4, 9, 15]. Additionally, due to the nonlinearity, we lose the advantageous block-diagonal structure and therefore have to deal with a fully-coupled system of non-linear equations in the Fourier coefficients. Since the Fréchet derivative of the non-linear frequency domain equations is explicitly computable, the nonlinearity can easily be overcome by applying Newton's method. Anyhow, at each step of Newton's iteration, a huge and fully block-coupled Jacobi system with sparse blocks has to be solved. The applicableness of the parameter-robust MinRes solver to the Jacobi system is not clear at the first glance.

Acknowledgements The authors gratefully acknowledge the financial support of the Austrian Science Fund (FWF) research project P19255 and DK W1214.

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