Pairwise-Interaction Games

Martin Dyer and Velumailum Mohanaraj

School of Computing, University of Leeds
Leeds LS2 9AS, United Kingdom
http://www.comp.leeds.ac.uk

Abstract. We study the complexity of computing Nash equilibria in games where players arranged as the vertices of a graph play a *symmetric 2-player game* against their neighbours. We call this a *pairwise-interaction game*. We analyse this game for n players with a fixed number of actions and show that (1) a mixed Nash equilibrium can be computed in constant time for any game, (2) a pure Nash equilibrium can be computed through Nash dynamics in polynomial time for games with a symmetrisable payoff matrix, (3) determining whether a pure Nash equilibrium exists for zero-sum games is NP-complete, and (4) counting pure Nash equilibria is #P-complete even for 2-strategy games. In proving (3), we define a new *defective* graph colouring problem called *Nash colouring*, which is of independent interest, and prove that its decision version is NP-complete. Finally, we show that pairwise-interaction games form a proper subclass of the usual graphical games.

Keywords: Nash equilibrium, graphical game, computational complexity, pairwise interaction.

1 Introduction

1.1 Overview

The *Nash equilibrium* [21] is the central solution concept in game theory. Plausibility of an equilibrium concept like the Nash equilibrium is partly determined by the complexity of computing equilibria [14]. As a result, many recent studies have focused on the complexity of finding Nash equilibria (e.g. [1,4,5,7,11,12,13]). For the complexity problem to be meaningful, however, the game, particularly its payoffs, should allow a compact representation [23].

Many succinctly representable games have been studied in the literature of which *graphical games*, proposed by Kearns *et al.* [18], have received much attention (see [5] and the references therein). In these games, players are arranged as the vertices of a graph and can play the game only with their immediate neighbours. In effect, a vertex k of degree d_k plays a $(d_k + 1)$-player game. If the number of pure strategies available to k is r, payoffs for k can be specified using r^{d_k+1} numbers. Thus, an n-player game with r strategies can be represented by an n-vertex graph and $nr^{\Delta+1}$ numbers, where Δ is the maximum vertex degree.

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The representation can be further simplified using symmetries in games. A game is *symmetric* if every player has the same payoff matrix and a player's payoff depends only on the player's strategy and the number of other players playing each pure strategy available. Hence, a symmetric graphical game (see [5] for a formal definition) with the neighbourhood size of d can be described by a d-regular graph with n vertices and $r\binom{d+r-1}{r-1}$ numbers. It must be noted that a symmetric graphical game cannot be considered symmetric by definition unless the graph is highly regular like the complete graph. In these games, each player plays the game with a different group of players and largely ignores what happens outside the group, whereas symmetry is defined globally.

The idea of playing games on graphs predates the idea of graphical games. Nearly a decade before graphical games were introduced by [18], Nowak and May [22] empirically studied the impact on the emergence of cooperation of placing players at the vertices of a grid graph. Their investigations stimulated research in this area, and a spate of new work followed, studying the impact of many other types of graph (see [24] and the references therein). In this setting, players are arranged as the vertices of a graph. Each vertex chooses a single pure or mixed strategy from a common strategy space and plays an identical, but independent, symmetric 2-player game with its immediate neighbours using this strategy. The payoff for a vertex is the sum of the payoffs it receives from playing the 2-player game with all its neighbours. This captures the natural tendency of players to treat each interaction as a separate 2-player game when the interactions are pairwise. This game, which we call a *pairwise-interaction game*, is the subject of this paper. Pairwise-interaction games are not necessarily symmetric, due to the reason stated above for symmetric graphical games, even though the 2-player games are assumed to be symmetric. Moreover, in pairwise interaction games, neighbourhoods of players can even be of different sizes.

Predictably, pairwise-interaction games can be represented even more succinctly than symmetric graphical games. More precisely, an n-player pairwise-interaction game with r strategies can be described by an n-vertex graph and a single $r \times r$ matrix. Strategic interactions in political, social, biological and economic situations are often pairwise [3]. Hence, pairwise-interaction games have many applications while admitting an extremely compact representation.

Surprisingly though, to the best of our knowledge, a systematic study of pairwise-interaction games has not been carried out from the computational game theory perspective. That is the main purpose of this paper. Perhaps unsurprisingly, it turns out that the set of pairwise-interaction games is in fact a *proper* subclass of graphical games. What makes our study even more interesting, but surely disappointing from a game-theoretic point of view, is the fact that even for this simple subclass, the problem of deciding whether a pure Nash equilibrium exists is hard for zero-sum games with more than two strategies.

1.2 Our Results

In this paper, we study n-player pairwise-interaction games with a fixed number of strategies r. Clearly, Nash's theorem [21] that there exists a mixed Nash equilibrium in all finite games holds for pairwise-interaction games. Thus, we have the following easy theorem about mixed strategies. Many detailed proofs including that of Theorem 1 are omitted for brevity.

Theorem 1. *For any pairwise-interaction game with a fixed number of pure strategies, a symmetric mixed Nash equilibrium can be computed in constant time. This strategy corresponds to all players playing the symmetric mixed Nash equilibrium strategy for the 2-player game.*

Although mixed Nash equilibria exist in any game, there is no convincing justification for players deliberately randomising their actions. Hence, the pure Nash equilibrium is considered a better solution concept for games where one exists. This gives rise to two computational problems: (1) does a given game have any pure Nash equilibrium? (2) if it has, can that be computed in polynomial time? We address these questions for pairwise-interaction games. We first prove the following theorem that the *Nash dynamics* converges for games with symmetric matrices. This is the simple dynamics in which, at every iteration, some player switches to the best response to the current strategies of their neighbours. Its convergence implies the existence of a pure Nash equilibrium.

Theorem 2. *For any pairwise-interaction game with r strategies and a symmetric payoff matrix, the Nash dynamics converges in at most $n2^{K/2}(2\Delta + 1)^{K+1}$ steps, where $K = r(r + 1)/2 - 2$ and Δ is the maximum vertex degree.*

We show further that adding a constant to any column of the payoff matrix does not affect the Nash equilibria. So, the above result applies to games with payoff matrices that can be symmetrised using this operation. This, in particular, means that the above result applies to all 2-strategy games.

Perhaps more significantly, in Section 4 we prove the following theorem for *zero-sum* games.

Theorem 3. *For all $r \geq 3$, and all antisymmetric $r \times r$ payoff matrices $\mathbf{A}$ such that the 2-player game has a unique mixed strategy which is not a pure strategy, deciding whether there is a pure Nash equilibrium in the pairwise-interaction game with payoff matrix $\mathbf{A}$ on a Δ-regular graph is NP-complete.*

That the mixed strategy cannot be a pure strategy is clear, since otherwise all players playing this strategy would give a pure Nash equilibrium by Theorem 1. The condition of having a unique mixed strategy is made for technical reasons, and we believe the theorem to be true without this assumption. However, we note that having a unique mixed strategy is generic, and we will show this in Lemma 5.

In Section 5 we show that even for some 2-strategy pairwise-interaction games for which the problem of finding a pure Nash equilibrium is in FP, the problem of exactly counting them is #P-hard. Surprisingly, it turns out that even approximately counting them in polynomial time is not possible unless NP=RP. Finally, we have the following theorem that pairwise-interaction games form a (small) proper subset of symmetric graphical games. Thus our hardness results are much stronger than those previously known for graphical games.

Theorem 4. *For given $r > 2$ and $\Delta = \Omega(r)$, pairwise-interaction games form an exponentially small fraction of symmetric graphical games on Δ-regular graphs.*

1.3 Related Work

Two-strategy games. The *parity affiliation game* [12] with all edge weights -1 and the *cut game* [4] with all edge weights $+1$ correspond to the pairwise-interaction game with payoff matrix $\mathbf{P} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Any pure Nash equilibrium in these games is a STABLE-CONFIGURATION and a MAX-CUT, in the sense of Schäffer and Yannakakis [25]. Thus, finding a pure Nash equilibrium for these games is P-complete [25].

We also note that some 2-strategy games considered here are essentially equivalent to the *defective* 2*-colouring* problem [10]. Similar results to those we give are known for the defective 2-colouring problem.

Our convergence proof in Theorem 2 employs a potential function similar to that used in the convergence of Hopfield's neural networks [16] and other related problems. However, our proof applies to more than two strategies. In addition, due to the simplified nature of pairwise-interaction games, we are able to show that a pure Nash equilibrium can be computed in polynomial time.

Complexity of games. The problem of computing mixed Nash equilibria of n-player normal-form games is PPAD-complete for all $n \geq 2$ [7,11]. When $n \geq 4$, this problem is equivalent to the problem of computing Nash equilibria in graphical games of maximum degree $\Delta \geq 3$, with two strategies per player. Hence, the latter is also PPAD-complete [11]. Some positive results are known when these games are symmetric. A mixed Nash equilibrium of a symmetric n-player normal-form game with r strategies can be computed in polynomial time if $r = O(\log n / \log\log n)$ [23]. Using this result, it is shown in [5] that for the symmetric graphical games with degree Δ, an equilibrium can be computed in polynomial time if $r = O(\log \Delta / \log\log \Delta)$.

For symmetric n-player normal-form games with a constant number of strategies, r, the problem of determining the existence of a pure Nash equilibrium is in AC^0 [2,6]. It is known that these games are guaranteed to have a pure Nash equilibrium when $r = 2$ [8]. For graphical games, the problem of determining whether there exist a pure Nash equilibrium is NP-complete, in general, even if all players have only two strategies and neighbourhoods of size 2 [13].

The problem of counting Nash equilibria is generally hard. Counting the number of (mixed) Nash equilibria is #P-hard even for symmetric 2-player games [9]. For graphical games, counting the number of pure Nash equilibria is #P-hard even for symmetric games with neighbourhood size of 2 [5].

As we will see later, pairwise-interaction games with symmetrisable payoff matrices are in fact *general potential games* [12]. By definition, every potential game has a pure Nash equilibrium. *Congestion games* [12] are a special class of potential games. Computing a pure Nash equilibrium for these games is PLS-complete [12].

2 Preliminaries

2.1 Notations

If all elements of a matrix $\mathbf{A}$ or a vector $\mathbf{n}$ are positive, we write $\mathbf{A} \geq \mathbf{0}$ or $\mathbf{n} \geq \mathbf{0}$ respectively. Here and elsewhere, matrices and column vectors with all 1's and 0's are denoted by $\mathbf{1}$ and $\mathbf{0}$ respectively. By $\mathbf{A}^T$ and $\mathbf{n}^T$, we denote the transpose of $\mathbf{A}$ and $\mathbf{n}$ respectively. We write column vectors as row vectors with the transpose operation, e.g. $(n_0, \ldots, n_{r-1})^T$. An integer set $\{0, \ldots, x-1\}$ is denoted by $[x]$. Interchangeably, we refer to a participant in a pairwise-interaction game as a player or vertex, since each participant is represented by a graph vertex.

2.2 Strategic Games

Definition 1. *A* normal-form game *is given by a set of players $\mathcal{Q}$, and for each player $k \in \mathcal{Q}$ a finite set of pure strategies $\mathcal{S}_k$ and a payoff function $u_k : \times_{k \in \mathcal{Q}} \mathcal{S}_k \to \mathbb{R}$.*

A *pure strategy* of player k is an element of $\mathcal{S}_k$. A *mixed strategy* for player k is a probability distribution Σ_k over $\mathcal{S}_k$, so is a nonnegative vector of length $|\mathcal{S}_k|$. A set of strategies $s = (s_1, \ldots, s_k, \ldots, s_n)$ $(s_k \in \mathcal{S}_k)$ is called a *pure strategy profile*, and $\sigma = (\sigma_1, \ldots, \sigma_k, \ldots, \sigma_n)$ $(\sigma_k \in \Sigma_k)$ is called a *mixed strategy profile.*

A 2-player normal-form game can be conveniently represented by two real matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$. The game is *symmetric* if $\mathbf{B} = \mathbf{A}^T$, and is *zero-sum* if $\mathbf{A} + \mathbf{B} = \mathbf{0}$. Hence for a symmetric zero-sum game, $\mathbf{A}$ is *antisymmetric.* Thus, one payoff matrix $\mathbf{A}$ is sufficient to describe any symmetric 2-player game.

Let $G = (V, E)$ be a graph. Let $\mathcal{N}(k) = \{v \in V \mid (k, v) \in E\}$, and $d_k = |\mathcal{N}(k)|$. By s_{-k} and σ_{-k} we denote the pure and mixed strategies of all neighbours of k respectively. Then, the pairwise-interaction game is defined as follows.

Definition 2. *A* pairwise-interaction game $\mathcal{G}$ *is defined by:*

- *An undirected graph $G = (V, E)$, where the vertices $V = [n]$ represent players. Without loss of generality, we may assume G is connected.*
- *A symmetric 2-player game $\langle \mathcal{S}, \mathbf{A} \rangle$, where $\mathcal{S} = [r]$ is the set of pure strategies available to each vertex and $\mathbf{A} = (a_{ij})$ $(i, j \in [r])$ is the payoff matrix.*
- *The payoff for any player k $(k \in [n])$ is*

$$u(\sigma_k; \sigma_{-k}) = \sum_{p \in \mathcal{N}(k)} \sigma_k^T \mathbf{A} \sigma_p \ . \tag{1}$$

We will regard r as being a constant. We show later that the numbers in $\mathbf{A}$ can be taken as being polynomially bounded in n, so the size of these numbers does not need be included in the input size. Thus, for complexity purposes, the input size is measured only by n.

We shall denote the set of mixed strategies over $\mathcal{S}$ by Σ. To avoid trivialities, we will always assume $r \geq 2$. If the 2-player game is zero-sum, we refer to the pairwise-interaction game as zero-sum.

Let $\mathcal{B}(\sigma_{-k})$ be the set of mixed strategy best responses of vertex k to the neighbour strategies σ_{-k}. Then we have

$$\mathcal{B}(\sigma_{-k}) \;=\; \{\sigma_k \in \Sigma \mid u(\sigma_k;\sigma_{-k}) \geq u(\sigma'_k;\sigma_{-k})\ \forall \sigma'_k \in \Sigma\ \}\ . \qquad (2)$$

Definition 3. *A strategy profile* $\sigma^* = \left\{\sigma^*_0, \ldots, \sigma^*_k, \ldots, \sigma^*_{n-1}\right\}$ *(*$\sigma^*_k \in \Sigma$*) is a* mixed Nash equilibrium *if* $\sigma^*_k \in \mathcal{B}(\sigma_{-k})\ \forall k \in V$*. We say* σ^* *is a* pure Nash equilibrium *if it is a pure strategy profile.*

Let $n_j^{(k)}$ denote the number of neighbours of k playing strategy $j \in \mathcal{S}$. We shall call a combination of neighbour strategies a *neighbourhood*. Then, instead of using s_{-k} to denote it, for symmetric games, it is convenient to use a column vector of $n_j^{(k)}$ with one entry for each $j \in [r]$, e.g. $\mathbf{n_k} = \left(n_0^{(k)}, \ldots, n_{r-1}^{(k)}\right)^T$ where $\sum_{j=0}^{r-1} n_j^{(k)} = d_k$. Using this notation, for pure strategies, (1) could be rewritten as

$$u\big(s_k; n_0^{(k)}, \ldots, n_{r-1}^{(k)}\big) \;=\; \sum_{j\in\mathcal{S}} n_j^{(k)} a_{s_k j}\ . \qquad (3)$$

Similarly, (2) could be written as $\mathcal{B}(n_0^{(k)}, \ldots, n_{r-1}^{(k)})$. We will use this notation in the analysis of pure Nash equilibria, and (1) and (2) in the analysis of mixed Nash equilibria.

Now the following proposition is easy to prove.

Proposition 1. *Adding an arbitrary constant to all entries of any column of* $\mathbf{A}$ *does not affect the Nash equilibria of pairwise-interaction games.*

We next define Nash dynamics and provide a proposition that links its convergence and the existence of a pure Nash equilibrium (e.g. [12]).

Definition 4. Nash Dynamics*: In this dynamics, at every step, some player playing a suboptimal strategy improves their payoff by switching to the best response to the current strategies of their neighbours.*

Proposition 2. *If the Nash dynamics converges, then there is a pure Nash equilibrium.*

We use the following notion of equivalence of games throughout the paper.

Definition 5. *Two games are* equivalent *if they have identical best responses to every combination of opponents' strategies.*

3 Symmetric Payoff Matrices

In this section we prove Theorem 2 about pairwise interaction games with symmetrisable payoff matrix $\mathbf{A}$. The following lemma shows that there always exists a pure Nash equilibrium for these games.

Lemma 1. *An r-strategy pairwise-interaction game with symmetric payoff matrix* $\mathbf{A}$ *has a pure Nash equilibrium.*

Proof. We prove this using a potential function $\psi : \mathcal{S}^n \to \mathbb{R}$. Let $s = (s_1, \ldots, s_k, \ldots, s_n) \in \mathcal{S}^n$ be a pure strategy profile. Then $\psi(s)$ is defined as

$$\psi(s) \;=\; \sum_{k \in V} u\big(s_k; n_0^{(k)}, \ldots, n_{r-1}^{(k)}\big) \;=\; \sum_{k \in V} \sum_{j \in S} n_j^{(k)} a_{s_k j} \;. \tag{4}$$

Now suppose, at some point in the Nash dynamics, vertex k switches from its current strategy s_k to its best response $\bar{s_k}$, taking the strategy profile from s to $\bar{s} = (s_1, \ldots, \bar{s_k}, \ldots, s_n)$. Then the payoff of k increases by $\theta_k = \sum_{j \in S} a_{\bar{s_k} j} n_j^{(k)} - \sum_{j \in S} a_{s_k j} n_j^{(k)} > 0$, while the total payoff of the neighbours of k increases by $\theta_{\mathrm{N}} = \sum_{j \in S} a_{j \bar{s_k}} n_j^{(k)} - \sum_{j \in S} a_{j s_k} n_j^{(k)}$. By symmetry of $\mathbf{A}$, we have $\theta_k = \theta_{\mathrm{N}}$. Hence we get $\psi(\bar{s}) - \psi(s) \;=\; \theta_k + \theta_{\mathrm{N}} = 2\theta_k \; > 0$. So, at every step of the Nash dynamics, the potential function increases by a positive value. Thus, the Nash dynamics converges. □

Remark 1. The above proof shows that pairwise-interaction games with a symmetric payoff matrix belong to the class of *weighted potential games* [20]. The weight for each player is 1/2: when a player improves his payoff by x, the potential function increases by $2x$.

The Nash dynamics converges for these games, but how long does this take? Theorem 2 states that the convergence is fast for the games considered in Lemma 1. To prove this, we need the following lemma.

Lemma 2. *Any $r \times r$ payoff matrix $\mathbf{A}$ can be rescaled such that the minimum difference between any two payoffs is one, and at least one payoff is zero. This can be done without affecting the Nash equilibria or the symmetry of $\mathbf{A}$. The rescaling requires only constant time for fixed r.*

Proof sketch of Theorem 2: We first rescale the payoff matrix using Lemma 2. Then, there is at least one payoff 0 and another 1. We consider the remaining payoffs as variables. Let $\mathbf{v} = (v_1, \ldots, v_K)^T$ denote the vector containing these variables, where K is the total number of variables. As $\mathbf{A}$ is symmetric, we have $K = r(r+1)/2 - 2$. Consider a vertex k with degree d. Let $P(r,d) \;=\; \binom{d+r-1}{r-1}$ be the number different neighbourhood configurations for k. Let $\mathbf{a}_i$ $(i \in [r])$ denote the rows of the payoff matrix. For each configuration of the neighbour strategies $\mathbf{n}_p$ $(p \in [P])$, the best response is determined by the ordering of $\mathbf{a}_i \mathbf{n}_p$ $(i \in [r])$. Now, for each neighbourhood $\mathbf{n}_p$ $(p \in [P])$, and each distinct pair of strategies i and j $(0 \le i, j \le r-1)$, we add the inequality $(\mathbf{a}_i - \mathbf{a}_j)^T \mathbf{n}_p \ge 1$ if i yields a higher payoff than j, and we add $(\mathbf{a}_i - \mathbf{a}_j)^T \mathbf{n}_p = 0$ if both strategies yield the same payoff. These inequalities form a convex nonempty polyhedron in K-dimensional space. It is nonempty because the original payoffs satisfy all these inequalities. This polyhedron defines a class of games that are equivalent to $\mathbf{A}$ and have the property that every best response move improves the player's payoff by at least 1. Let $\mathbf{N}\mathbf{v} = \mathbf{b}$ be the set of K inequalities that are tight at a vertex of the polyhedron. Applying Cramer's rule on this, we can find the coordinates of this vertex in terms of the elements of $\mathbf{n}_p$'s. Then, Hadamard's inequality can be used to bound these coordinates in terms of Δ and r, which actually are the payoffs of an equivalent game. In this context, we may allow exponential dependence on r, since this is assumed to be constant. For the new, equivalent game, $\psi(s)$ is polynomially bounded and its value increases by at least 2 at every step. It can then be shown that the Nash dynamics converges as claimed. □

Note that the computation of an equivalent game in the above proof is valid even if $\mathbf{A}$ is not symmetric, except that K will now be $r^2 - 2$. Hence, the following holds by a similar proof to the one above.

Corollary 1. *For any pairwise-interaction game with a fixed number of strategies, there is an equivalent game with a payoff matrix whose entries are polynomially bounded in n.*

As mentioned before, the payoff matrix of any 2-strategy game can be symmetrised. Let $\mathbf{A} = \left(\begin{smallmatrix} P & T \\ S & R \end{smallmatrix}\right)$ be the original payoff matrix. This can be symmetrised to give $\mathbf{P} = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$, where $\alpha = P - S$ and $\beta = R - T$ (see Proposition 1). So, Theorem 2 applies to these games. However, in the following theorem, we get tighter results than that of Theorem 2 for regular graphs by exploiting the unique properties of the game, albeit using essentially similar techniques.

Theorem 5. *For any 2-strategy pairwise-interaction game on a Δ-regular graph $G = (V, E)$ with n vertices and m edges, starting from an arbitrary initial state, the Nash dynamics converges in at most $3n/2$ steps if $\alpha + \beta$ and β are of opposite signs and m steps if they are of the same sign.*

It might be possible to extend the above result to non-regular graphs. For example, consider the unweighted *cut game* where $\alpha = \beta = -1$. In this game, we have $0 \geq \psi(s) \geq \sum_{k \in V} -d_k = -2m$, so it takes only m steps for convergence on any graph. However, we have an alternative proof for general graphs.

Theorem 6. *For any 2-strategy pairwise-interaction game with payoff matrix $\mathbf{A} = \left(\begin{smallmatrix} P & T \\ S & R \end{smallmatrix}\right)$ on a graph with n vertices and m edges, starting from an arbitrary initial state, the Nash dynamics converges in at most (i) $3m - n$ steps if $T > R$ and $S > P$, (ii) $3m$ steps if $T < R$ and $S < P$, (iii) n steps, otherwise.*

Proof (Sketch). The proof is similar to the proofs of Lemma 1 and Theorem 5, but uses an algorithm similar to the graph partitioning algorithm of [15].

4 Zero-Sum Games

In this section, we study pairwise interaction zero-sum games with $r \geq 3$ strategies and prove Theorem 3. The main tool of the proof is the following proposition.

Proposition 3. *In a zero-sum pairwise-interaction game, the best response to any neighbourhood configuration yields a nonnegative payoff. Furthermore, in Nash equilibrium, every player earns zero payoff.*

The neighbourhoods in a Nash equilibrium can be characterised using the proposition above.

Definition 6. *In a zero-sum pairwise-interaction game, a neighbourhood will be called a Nash Equilibrium neighbourhood (NE neighbourhood) if the best response to the neighbourhood yields zero payoff.*

Corollary 2. *If* $\mathbf{n} = (n_0, \ldots, n_{r-1})^T$ *is a NE neighbourhood, then* $\mathbf{An} \leq \mathbf{0}$.

We now show that a highly nontrivial elimination of strategies is possible for zero-sum games, which, in a sense, is much stronger than the usual iterated elimination of dominated strategies. That is, if a strategy earns a negative payoff in any NE neighbourhood, it can be eliminated, implying that *any* surviving strategy is a best response to *any* NE neighbourhood.

Lemma 3. *If a strategy earns a negative payoff when played against a NE neighbourhood, no player will play it in any pure Nash equilibrium.*

We now consider the question of the existence of NE neighbourhoods for rational payoff matrices. But, as we shall see, this does not imply that a pure Nash equilibrium exists in a d-regular graph.

Lemma 4. *If* $\mathbf{A}$ *has rational entries then, for some integer* d*, there exists a NE neighbourhood for a vertex of degree* d.

The proof, which is omitted here, reveals a remarkable connection between a NE neighbourhood in a zero-sum pairwise interaction game and the optimal mixed strategy of the 2-player game. This suggests a heuristic approach to pairwise-interaction games: from each player's point of view, their neighbourhood can be viewed as a single opponent playing a mixed strategy. For a general pairwise-interaction game, this approach has no real validity. Since individual players are not able to play mixed strategies, the above view is asymmetric. But, surprisingly, for zero-sum games it is actually valid.

In this context, Lemma 4 can be linked to some well-known mixed strategy results. Consider games with a *unique mixed strategy*, by which we mean the games with a unique NE neighbourhood. These are games for which the surviving strategies are *completely mixed* [17]. (A completely mixed strategy is one for which every pure strategy has a positive probability.) Kaplansky [17] showed that a symmetric two-player zero-sum game can be completely mixed only if the number of strategies is odd. Thus, for games with a unique NE neighbourhood, the number of surviving strategies must be odd. For the remainder of this section we consider only payoff matrices $\mathbf{A}$ which have unique mixed strategies. But we note that there is always a matrix arbitrarily close to $\mathbf{A}$ for which this is true.

Lemma 5. *Let* $\mathbf{A} = (a_{ij})$ *be an antisymmetric payoff matrix. Then, there always exists an antisymmetric payoff matrix* $\mathbf{B} = (b_{ij})$ *that has a unique mixed strategy and satisfies* $|a_{ij} - b_{ij}| \leq 1/M$ $(i, j \in [r])$*, for any* $M > 0$.

Next, let us define an interesting computational problem related to improper vertex colouring that will be used in the proof of Theorem 3.

Definition 7. *NASH-COLOURABLE is a decision problem whose instance is a graph $G = (V, E)$, a set of colours $[r]$ and, for each vertex degree d in G, a set of r nonnegative integers $(c_0^d, c_1^d \dots, c_{r-1}^d)$ such that $d = \sum_{i=0}^{r-1} c_i^d$. The question is whether there is an improper vertex colouring of G with r colours such that a vertex with degree d has exactly c_i^d neighbours with colour $i \in [r]$.*

If the answer is positive, the graph G will be said to be Nash colourable *and the particular assignment of colours will be called a* Nash colouring *of the graph.*

Definition 8. *$(c_0, c_1, \dots, c_{r-1})_\Delta$-NASH-COLOURABLE will mean the Nash colouring problem for Δ-regular graphs. In this case we will write $(c_0^\Delta, c_1^\Delta, \dots, c_{r-1}^\Delta) = (c_0, c_1, \dots, c_{r-1})$, so $\Delta = \sum_{i=0}^{r-1} c_r$. Since there is only one vertex degree, we will specify $(c_0, c_1, \dots, c_{r-1})$ in the prefix.*

Proof sketch of Theorem 3: A Nash equilibrium in the zero-sum pairwise-interaction game corresponds to a Nash colouring of the graph. The result then following from Theorem 7 that NASH-COLOURABLE is NP-complete. □

Theorem 7. *If $r \geq 3$, and $c_0, c_1, \dots, c_{r-1}$ are any positive integers such that $\sum_{i=0}^{r-1} c_i = \Delta$, then $(c_0, c_1, \dots, c_{r-1})_\Delta$-NASH-COLOURABLE is NP-complete.*

Proof (Sketch). The problem is clearly in NP. To prove that it is NP-hard, we use a gadget-based reduction from CHROMATIC-INDEX of r-regular graphs to $(c_0, c_1, \dots, c_{r-1})_\Delta$-NASH-COLOURABLE. The hardness then follows from the result of [19] that CHROMATIC-INDEX is NP-complete for Δ-regular graphs with degree $\Delta \geq 3$. □

5 Some Further Results

The following two theorems consider some games whose Nash equilibria correspond to maximal independent sets in the corresponding graphs. For these games, Theorem 8 shows that exactly counting the Nash equilibria is hard while Theorem 9 proves that even counting them approximately is hard.

Theorem 8. *Suppose a 2-strategy pairwise-interaction game with payoff matrix $\mathbf{A} = \left(\begin{smallmatrix} P & T \\ S & R \end{smallmatrix}\right)$ is played on a graph of maximum degree 4. Then, if the payoffs are such that $T > R$, $S > P$ and $0 < \gamma < 1/4$ or $3/4 < \gamma < 1$, where $\gamma = (S - P)/(T + S - R - P)$, the problem of counting the pure Nash equilibria is #P-complete.*

Theorem 9. *For the same game considered in Theorem 8 except that the game is played on a graph of maximum degree 7 and the payoffs are such that $0 < \gamma < 1/7$ or $6/7 < \gamma < 1$, there does not exist a fully polynomial time approximation scheme (FPTAS) to count the pure Nash equilibria unless RP=NP.*

For some 2-strategy pairwise interaction games, computation of a pure Nash equilibrium is inherently sequential, and the following theorem holds.

Theorem 10. *The problem of computing pure Nash equilibria is P-complete for some* 2*-strategy pairwise-interaction games.*

Thus finding a pure Nash equilibrium almost certainly cannot be done in constant time, in sharp contrast to Theorem 1 for mixed Nash equilibria.

6 Open Problems

We have initiated a systematic study of pairwise-interaction games and presented results for the games with symmetric or antisymmetric payoff matrix. A natural extension of our work is to investigate the remaining case, i.e. the games with asymmetric payoff matrices that are not antisymmetric. We have examples of matrices of this kind for which it is easy to compute a pure Nash equilibrium. Hence, we know that there are easy cases left to study, and we believe there are also hard cases. But, we conjecture that there is a *dichotomy*: the problem of deciding whether a pure Nash equilibrium exists is either in P or is NP-complete. Similarly, we believe that for the problem of counting Nash equilibria there is a dichotomy, thus the problem is in FP or is #P-complete. We leave finding the dichotomy conditions as open problems. For the approximate counting problem, we showed that there does not exist an FPTAS even for some 2-strategy games. Classifying the complexity of approximately counting Nash equilibria for these games we leave as another open question. In addition, considering our hardness results, another topic of interest would be to explore approximate Nash equilibria for these games.

Two-strategy pairwise-interaction games on the complete graph can be modelled as congestion games [8]. In [1], the combinatorial structures in congestion games that ensure that the Nash dynamics converges in polynomial time are studied. It would be interesting to explore if there is any connection between the congestion games with a polynomial time convergence and other pairwise-interaction games with a polynomial time convergence.

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