

Chapter 6

Universal Relations for Fermions with Large Scattering Length

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6.1 Introduction

Particles with short-range interactions that produce a large scattering length have universal properties that depend only on the scattering length [1]. A system consisting of such particles is strongly interacting in the sense that there are effects of the interactions that must be treated nonperturbatively. These strong interactions give rise to strong correlations among the particles. Many theoretical methods, even if they are nonperturbative, are inadequate for dealing with such strong correlations. However, such a system is also governed by *universal relations* that follow from the short-distance and short-time dynamics associated with the large scattering length. These universal relations provide powerful constraints on the behavior of the system. They hold for any state of the system: few-body or many-body, ground state or nonzero temperature, homogeneous or in a trapping potential, normal state or superfluid, balanced in the two spin states or imbalanced. They connect various properties of the system, ranging from thermodynamic variables to large-momentum and high-frequency tails of correlation functions.

The systems for which the universal relations have been most extensively studied are those consisting of fermions with two spin states. The universal relations that have been derived thus far all involve a property of the system called the *contact*, which measures the number of pairs of fermions in the two spin states with small separations. Many of these relations were first derived by Shina Tan, and they are known as the *Tan relations* [2–4]. Tan derived these relations by exploiting the fact that the large scattering length can be taken into account through boundary conditions on the many-body Schrödinger wavefunction for otherwise noninteracting particles. The universal relations can also be derived concisely within a quantum field theory framework [5], where they follow from renormalization and from the operator

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product expansion. Such a framework facilitates the derivation of additional universal relations and the systematic inclusion of corrections associated with the nonzero range of the interactions.

In this review, we summarize the current theoretical and experimental status of universal relations for systems consisting of fermions with two spin states and a large scattering length. We begin in Sect. 6.2 by presenting the Tan relations. In Sect. 6.3, we discuss the physical interpretation of the contact and we provide some illustrative examples. In Sect. 6.4, we present other universal relations that have been derived more recently. In Sect. 6.5, we describe exciting recent developments in the field of ultracold atoms involving measurements of the contact and experimental tests of the universal relations. In Sect. 6.6, we discuss the derivation of the universal relations, with an emphasis on quantum field theory methods.

6.2 The Tan Relations

The Tan relations were derived by Shina Tan in a series of three papers [2–4]. The first two of these papers were written in 2005, but they were not published until 2008, when all three papers were published back-to-back in *Annals of Physics*.

The Tan relations apply to systems consisting of fermions with two spin states whose scattering length a is large compared to the range r_0 of their interactions. We will refer to the fermions as atoms and label the two spin states by an index σ with values 1 and 2. In a many-body system, the number densities n_σ and the temperature T must also be small enough that the corresponding length scales are large compared to the range: $n_\sigma^{-1/3} \gg r_0$ and $\lambda_T \gg r_0$, where $\lambda_T = (2\pi\hbar^2/mk_B T)^{1/2}$. If the system is in an external trapping potential $V(\mathbf{r}) = \frac{1}{2}m\omega^2 r^2$, the length scale associated with the trap should also be large compared to the range: $(\hbar/m\omega)^{1/2} \gg r_0$.

The Tan relations involve an extensive quantity, the *contact* C , which is the integral over space of a local quantity, the *contact density* $\mathcal{C}(\mathbf{R})$:

$$C = \int d^3R \mathcal{C}(\mathbf{R}). \quad (6.1)$$

We proceed to present the Tan relations in chronological order.

6.2.1 Tails of Distributions

In the first of Tan's two 2005 papers, he derived three universal relations [2]. The first was for the tails of the momentum distributions $n_\sigma(\mathbf{k})$ for the two spin states $\sigma = 1, 2$:

Tail of the momentum distribution. The distributions of the wavevector \mathbf{k} have power-law tails at large k :

$$n_\sigma(\mathbf{k}) \longrightarrow \frac{C^4}{k}. \quad (6.2)$$

The coefficient C is the contact and it is the same for both spin states.

The asymptotic behavior in (6.2) actually applies only in the scaling region $|a|^{-1} \ll k \ll r_0^{-1}$. The wavenumber must also be larger than the scales associated with the system, such as $n^{1/3}$, λ_T^{-1} , and $(m\omega/\hbar)^{1/2}$. The momentum distributions in Eq. 6.2 have been normalized so that the total number of atoms in the spin state σ is

$$N_\sigma = \int \frac{d^3k}{(2\pi)^3} n_\sigma(\mathbf{k}). \quad (6.3)$$

The universal relation in Eq. 6.2 implies that the contact is positive definite and has dimension $(\text{length})^{-1}$. Thus the contact density has dimensions $(\text{length})^{-4}$.

The total energy E of the system is the sum of the kinetic energy T , the interaction energy U , and the energy V associated with an external potential:

$$E = T + U + V. \quad (6.4)$$

The kinetic energy T (which should not be confused with the temperature) can be expressed as an integral over the momentum distribution:

$$T \equiv \sum_\sigma \int \frac{d^3k}{(2\pi)^3} \left(\frac{\hbar^2 k^2}{2m} \right) n_\sigma(\mathbf{k}). \quad (6.5)$$

The asymptotic behavior of the momentum distribution in Eq. 6.2 implies that T is ultraviolet divergent. This divergence actually occurs only in the zero-range limit $r_0 \rightarrow 0$. For interactions with a finite range, the integral in Eq. 6.5 is cut off by the range and therefore has a contribution that behaves like $1/r_0$ as $r_0 \rightarrow 0$. Thus the physical interpretation of the ultraviolet divergence is that T is sensitive to the range. The second Tan relation in Ref. [2] implies that the sensitivity of the kinetic energy to the range is cancelled by the interaction energy:

Energy relation. The sum of the kinetic and interaction energies is ultraviolet finite and it is completely determined by the momentum distributions $n_\sigma(\mathbf{k})$ and the contact C :

$$T + U = \sum_\sigma \int \frac{d^3k}{(2\pi)^3} \frac{\hbar^2 k^2}{2m} \left(n_\sigma(\mathbf{k}) - \frac{C}{k^4} \right) + \frac{\hbar^2}{4\pi m a} C. \quad (6.6)$$

In the integral on the right side, the subtraction term cancels the tail of the momentum distribution and makes the integral convergent in the ultraviolet. The sum of the two terms in Eq. 6.6 proportional to the contact C is the interaction energy. The first of those two terms is ultraviolet divergent. Thus the interaction energy is sensitive to the range, but that sensitivity is exactly cancelled by the kinetic energy. The last

term in Eq. 6.6 is the interaction energy that remains after subtracting the divergent term. Remarkably, it is also proportional to the contact.

The third Tan relation in Ref. [2] gives the asymptotic behavior of the correlation function for the densities of the two spin states at short distances:

Density–density correlator at short distances. The correlation between the number densities for the two spin states at points separated by a small distance r diverges as $1/r^2$ and the coefficient of the divergence is proportional to the contact density:

$$\left\langle n_1 \left(\mathbf{R} + \frac{1}{2} \mathbf{r} \right) n_2 \left(\mathbf{R} - \frac{1}{2} \mathbf{r} \right) \right\rangle \longrightarrow \frac{1}{16\pi^2} \left(\frac{1}{r^2} - \frac{2}{ar} \right) \mathcal{C}(\mathbf{R}). \quad (6.7)$$

Tan also pointed out that the contact density appears in the short-distance expansion for the correlator of the quantum field operators that create and annihilate the atoms. This expansion will be discussed in Sect. 6.6.4.4.

6.2.2 Changes in the Scattering Length

From the three universal relations described above, one might conclude that the contact is an esoteric property of the system that has only to do with tails of distributions. In the second of Tan’s 2005 papers [3], he derived another universal relation that makes it clear that the contact is an absolutely central property of the system:

Adiabatic relation. The rate of change of the energy due to a small change in the inverse scattering length is proportional to the contact:

$$\left(\frac{dE}{da^{-1}} \right)_S = -\frac{\hbar^2}{4\pi m} C. \quad (6.8)$$

The derivative is evaluated with the entropy S held fixed. The particle numbers N_1 and N_2 are also implicitly held fixed.

In the simplest case, E is just an energy eigenvalue. The adiabatic relation also holds for any statistical mixture of eigenstates if the derivative is evaluated with the occupation numbers held fixed. By the adiabatic theorem of quantum mechanics, if the scattering length changes sufficiently slowly with time, the occupation numbers remain constant. Thus if the contact C is known as a function of a , the adiabatic relation in Eq. 6.8 can be integrated to obtain the accumulated change in E .

The adiabatic relation can also be expressed in terms of the derivative of the free energy $F = E - TS$ with the temperature T held fixed:

$$\left(\frac{dF}{da^{-1}} \right)_T = -\frac{\hbar^2}{4\pi m} C. \quad (6.9)$$

As pointed out by Tan, this implies that the contact determines the thermodynamics of the system. Given the contact of a system as a function of the scattering length a and other variables, such as N_1 , N_2 , and T , the free energy F can be obtained by

integrating Eq. 6.9 with respect to a . A convenient boundary condition is provided by the limit $a \rightarrow 0^-$, in which the atoms are noninteracting. From F , one can determine all the other thermodynamic functions.

If one uses Eq. 6.2 to define the contact in terms of the tail of the momentum distribution, this appears to be a case of the tail wagging the dog. The thermodynamic behavior of the system seems to be determined by the tail of the momentum distribution. However the proper interpretation is that the contact is a central property of the system that determines both the thermodynamics and the tail of the momentum distribution.

The adiabatic relation in Eq. 6.8 determines the change in the total energy when the scattering length changes very slowly. Tan also considered the opposite limit in which the scattering length changes very rapidly [3]:

Sudden change in the scattering length. If the scattering length is changed suddenly from a to a' , the change in the total energy is proportional to the contact:

$$\Delta E = -\frac{\hbar^2}{4\pi m} \left(\frac{1}{a'} - \frac{1}{a} \right) C, \quad (6.10)$$

where C is the initial value of the contact.

This result requires the time scale for the sudden change in scattering length to be much slower than the time scale mr_0^2/\hbar associated with the range. Tan also presented a more general result for the change in the energy due to a rapid change in the scattering length, which will be described in Sect. 6.4.5.

6.2.3 Additional Tan Relations

In Tan's 2008 paper, he derived two additional universal relations that apply for specific forms of the external potential [4].

Virial theorem. For a system in a harmonic trapping potential, the components of the energy E in Eq. 6.4 satisfy

$$T + U - V = -\frac{\hbar^2}{8\pi ma} C. \quad (6.11)$$

The virial theorem for the unitary limit $a = \pm\infty$ was first derived and also verified experimentally by Thomas, Kinast, and Turlapov [6]. The virial theorem in Eq. 6.11 is the generalization to finite scattering length.

Pressure relation. For a homogeneous system, the pressure and the energy density are related by

$$\mathcal{P} = \frac{2}{3}\mathcal{E} + \frac{\hbar^2}{12\pi ma} C. \quad (6.12)$$

The pressure relation was actually first derived in Ref. [3] for the special case of a balanced gas in which the two spin states have equal populations. The derivation was extended to the general case in Ref. [4].

If there are inelastic two-body scattering processes with a large energy release, they will result in a decrease in the number of low-energy atoms. Tan realized that the rate at which the number density of low-energy atoms decreases is proportional to the contact density \mathcal{C} (Tan S, private communication). The proportionality constant was first given in Ref. [5]. If there are inelastic two-body scattering channels, the scattering length a has a negative imaginary part. The proportionality constant in the universal relation can be expressed in terms of that complex scattering length:

Inelastic two-body losses. If there are inelastic two-body scattering processes with a large energy release, the number density of low-energy atoms decreases at a rate that is proportional to the contact density:

$$\frac{d}{dt}n_{\sigma}(\mathbf{R}) = -\frac{\hbar(-\text{Im } a)}{2\pi m|a|^2} \mathcal{C}(\mathbf{R}). \quad (6.13)$$

6.3 What is the Contact?

Given the Tan relations described in Sect. 6.2, it is evident that the contact is a central property of the system. But what is it? In this section, we provide an intuitive interpretation of the contact. We also provide additional insights into the contact by giving analytic expressions in some simple cases.

6.3.1 Intuitive Interpretation

An intuitive interpretation of the contact density can be derived from the universal relation for the density–density correlator in Eq. 6.7. That relation can be expressed in the form

$$\langle n_1(\mathbf{R} + \mathbf{r}_1) n_2(\mathbf{R} + \mathbf{r}_2) \rangle \longrightarrow \frac{1}{16\pi^2|\mathbf{r}_1 - \mathbf{r}_2|^2} \mathcal{C}(\mathbf{R}). \quad (6.14)$$

If we integrate both \mathbf{r}_1 and \mathbf{r}_2 over a ball of radius s , we obtain

$$N_{\text{pair}}(\mathbf{R}, s) \longrightarrow \frac{s^4}{4} \mathcal{C}(\mathbf{R}). \quad (6.15)$$

The left side simply counts the number of pairs inside that ball, which is the product $N_1 N_2$ of the number of atoms in the two spin states. The volume of that ball is $V = \frac{4}{3}\pi s^3$. One might naively expect the number of pairs to scale as V^2 as $V \rightarrow 0$. However, according to Eq. 6.15, it scales instead as $V^{4/3}$. That scaling

behavior applies only for s smaller than $|a|$ and also smaller than the length scales associated with the system, such as $n^{-1/3}$, λ_T , and $(\hbar/m\omega)^{1/2}$. The scaling behavior extends down to s of order the range r_0 .

A naive definition of the density of pairs is the limit as $V \rightarrow 0$ of $N_1 N_2 / V^2$, where N_1 and N_2 are the numbers of atoms in the volume V . This quantity has dimensions $(\text{length})^{-6}$. The result in Eq. 6.15 implies that the combination with a nontrivial small-volume limit is $N_1 N_2 / V^{4/3}$, which has dimensions $(\text{length})^{-4}$. Thus a more appropriate definition of the local pair density is the small-volume limit of $N_1 N_2 / V^{4/3}$, up to a normalization constant that can be chosen by convention. The unusual dimensions of $(\text{length})^{-4}$ for this local pair density can be expressed concisely by saying that this quantity has *scaling dimension* 4. The difference -2 between the scaling dimension and the naive dimension 6 is called the *anomalous dimension*. This anomalous dimension comes from the strong correlations associated with the large scattering length. This anomalous scaling behavior implies that the number of pairs in a very small volume is much larger than one would naively expect by extrapolating from larger volumes. The contact density is a measure of the local pair density that takes into account this anomalous scaling behavior.

Further intuition for the contact can be gleaned from the universal relation for the tail of the momentum distribution in Eq. 6.2. It implies that the number of atoms in either spin state whose wavenumber \mathbf{k} is larger than K is

$$N_\sigma(|\mathbf{k}| > K) = \frac{1}{2\pi^2 K} C, \quad (6.16)$$

provided K is in the scaling region $1/|a| \ll K \ll 1/r_0$ and is larger than the wavenumber scales set by the system. Thus the contact is a measure of the number of atoms with large momentum.

When interpreting the contact density as a measure of the local pair density, one should not confuse those pairs with Cooper pairs, which are pairs with a specific momentum correlation. Under conditions in which Cooper pairs are well defined, the typical separation of the atoms in a Cooper pair is much larger than the interparticle spacing. The number of Cooper pairs in a volume V that is comparable to or smaller than the volume per particle is not well defined. It certainly does not have the anomalous scaling behavior $V^{-4/3}$ of the total number of pairs.

6.3.2 Few-Body Systems

The adiabatic relation in Eq. 6.8 can be used as an operational definition of the contact. If the energy of a system is known as a function of the scattering length, we can simply differentiate to get an expression for the contact. A simple example in the case $a > 0$ is the weakly-bound diatomic molecule, or *shallow dimer*, consisting of two atoms with spins 1 and 2. The universal result for its binding energy is \hbar^2/ma^2 . Its energy is therefore

$$E_{\text{dimer}} = -\frac{\hbar^2}{ma^2}. \quad (6.17)$$

Using the adiabatic relation in Eq. 6.8, we find that the contact for the dimer is

$$C_{\text{dimer}} = \frac{8\pi}{a}. \quad (6.18)$$

Blume and Daily have calculated the contact numerically for the ground state of 4 fermions, 2 in each spin state, trapped in a harmonic potential and interacting through a potential with a large adjustable scattering length a [7]. They determined the contact as a function of a using four different universal relations: the tail of the momentum distribution in Eq. 6.2, the density-density correlator at short distances in Eq. 6.7, the adiabatic relation in Eq. 6.8, and the virial theorem in Eq. 6.11. The small differences between the four determinations of the contact were compatible with systematic errors associated with the nonzero range of the potential.

6.3.3 *Balanced Homogeneous Gas*

The contact density \mathcal{C} for the homogeneous gas can be obtained by using the adiabatic relation in Eq. 6.8 as an operational definition. Dividing both sides of the equation by the volume, the relation can be expressed in the form

$$\mathcal{C} = \frac{4\pi ma^2}{\hbar^2} \frac{d\mathcal{E}}{da}. \quad (6.19)$$

Analytic results for the energy density \mathcal{E} are available in various limits, and they can be used to obtain analytic expressions for the contact density.

We first consider the case of a balanced gas, in which the two spin states are equally populated, at zero temperature. The total number density $n = 2n_1 = 2n_2$ determines the Fermi momentum: $k_F = (3\pi^2 n)^{1/3}$. The ground state is determined by the dimensionless interaction variable $1/k_F a$. As this variable changes from $-\infty$ to 0 to $+\infty$, the ground state changes smoothly from a mixture of two weakly-interacting Fermi gases to a Bose-Einstein condensate of diatomic molecules. The ground state is always a superfluid. The mechanism for superfluidity evolves smoothly from the BCS mechanism, which is the Cooper pairing of atoms in the two spin states, to the BEC mechanism, which is the Bose-Einstein condensation of dimers.

In the *BCS limit* $a \rightarrow 0^-$, the energy density can be expanded in powers of $k_F a$:

$$\mathcal{E} = \frac{\hbar^2 k_F^5}{10\pi^2 m} \left(1 + \frac{10}{9\pi} k_F a + \dots \right). \quad (6.20)$$

Using Eq. 6.19, we find that the contact density in the BCS limit is

$$\mathcal{C} \longrightarrow 4\pi^2 n^2 a^2. \quad (6.21)$$

This is proportional to the low-energy cross section $4\pi a^2$ and to the product $(n/2)^2$ of the number densities of the two spin states. The contact density decreases to 0 as $a \rightarrow 0^-$, but it decreases only as a^2 . This result emphasizes that the contact density is not closely related to the density of Cooper pairs, which decreases exponentially as $\exp(-\pi/k_F|a|)$ as $a \rightarrow 0^-$.

In the *unitary limit* $a \rightarrow \pm\infty$, the energy density can be expanded in powers of $1/k_F a$:

$$\mathcal{E} = \frac{\hbar^2 k_F^5}{10\pi^2 m} \left(\xi - \frac{\zeta}{k_F a} + \dots \right), \quad (6.22)$$

where ξ and ζ are numerical constants. Using Eq. 6.19, we find that the contact density in the unitary limit is

$$\mathcal{C} \longrightarrow \frac{2\zeta}{5\pi} (3\pi^2 n)^{4/3}. \quad (6.23)$$

Since the interaction provides no length scale in the unitary limit, the contact density must be proportional to k_F^4 by dimensional analysis. An estimate $\zeta \approx 1$ for the numerical constant in Eq. 6.23 can be obtained from numerical calculations of the energy density near the unitary limit using quantum Monte Carlo methods [8, 9]. A more precise value can be obtained from numerical calculations of the density-density correlator in the unitary limit using the fixed-node diffusion Monte Carlo method [10]. Using the Tan relation for the density-density correlator in Eq. 6.7, we obtain the value

$$\zeta \approx 0.95. \quad (6.24)$$

In the *BEC limit* $a \rightarrow 0^+$, the energy density can be expanded in powers of $(k_F a)^{3/2}$:

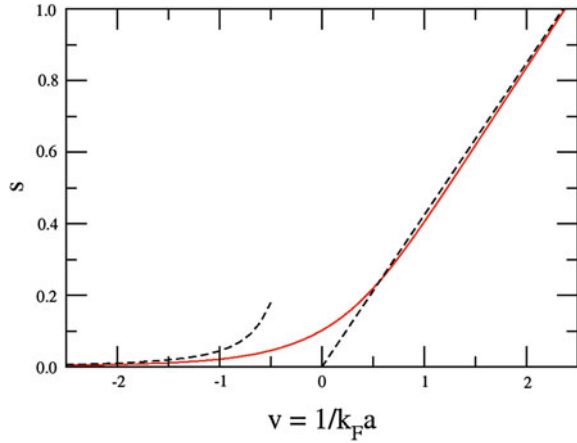
$$\mathcal{E} = -\frac{\hbar^2 n}{2ma^2} + \frac{\pi \hbar^2 n^2 a_{dd}}{4m} \left(1 + \frac{128}{15} \sqrt{na_{dd}^3/2\pi} + \dots \right), \quad (6.25)$$

where $a_{dd} \approx 0.60a$ is the dimer-dimer scattering length [11]. The leading term is the total binding energy density for dimers with number density $n/2$ and binding energy given by Eq. 6.17. The second term is the mean-field energy of a Bose-Einstein condensate of dimers with dimer-dimer scattering length a_{dd} . Using Eq. 6.19, we find that the contact density in the BEC limit is

$$\mathcal{C} \longrightarrow \frac{4\pi n}{a}. \quad (6.26)$$

This is equal to the contact $8\pi/a$ for a dimer, which is given in Eq. 6.18, multiplied by the dimer number density $n/2$. The contact density diverges as $1/a$ as $a \rightarrow 0^+$. The first correction to the leading term in Eq. 6.26 is suppressed by a factor of $(k_F a)^3$.

Fig. 6.1 The dimensionless contact density $s = \mathcal{C}/k_F^4$ for the balanced homogeneous gas as a function of the dimensionless coupling strength $1/k_F a$, from Ref. [12]. The left *dashed line* is the leading contribution in the BCS limit given by Eq. 6.21. The right *dashed line* is the leading contribution in the BEC limit given by Eq. 6.26. The contact density in the unitary limit may be underpredicted by about 16%



The contact density \mathcal{C} for the ground state of the balanced homogeneous gas can be expressed as k_F^4 multiplied by a monotonically-increasing dimensionless function of $1/k_F a$. Haussmann, Punk, and Zwerger have calculated this function numerically [12]. They used the Luttinger–Ward self-consistent formalism to calculate the single particle spectral functions. The contact density was determined using the Tan relation in Eq. 6.2 for the tail of the momentum distribution. Their result for the contact density as a function of $1/k_F a$ is shown in Fig. 6.1. It interpolates smoothly between the BCS limit in Eq. 6.21 and the BEC limit in Eq. 6.26. Their result in the unitary limit corresponds to a value $\zeta \approx 0.80$ for the constant in Eq. 6.23. This is about 16% smaller than the value in Eq. 6.24 that was obtained from a fixed-node diffusion Monte Carlo calculation. This difference reflects a systematic theoretical error in the calculational method of either Ref. [10] or Ref. [12] or both.

The dependence of the contact density on the temperature has been determined analytically in various limits by Yu, Bruun, and Baym [13]. In the low-temperature limit, the leading thermal contribution to the contact density comes from phonons. The leading thermal correction at small T increases like T^4 [13]. It becomes significant when T is comparable to the Fermi temperature: $k_B T_F = \hbar^2 k_F^2 / 2m$. In the BCS limit and in the unitary limit, the thermal contribution differs from the contact at $T = 0$ by a factor of $(T/T_F)^4$ multiplied by a numerical constant. In the BEC limit, the thermal contribution is suppressed by a factor of $(T/T_F)^4$ and by a further factor of $(k_F a)^{1/2}$.

The contact density at high temperature can be calculated using the virial expansion. The high-temperature limit¹ was determined by Yu, Bruun, and Baym [13]:

¹ In Ref. [13], the contact density was denoted by $16\pi^2 \hbar^2 C$ and the total number density was denoted by $2n$ instead of n .

$$\mathcal{C} \longrightarrow \frac{8\pi^2 \hbar^2 n^2}{mk_B T}. \quad (6.27)$$

Since the contact density increases as T^4 at small T and decreases as $1/T$ at sufficiently large T , it must achieve a maximum somewhere in between. The maximum occurs for T of order T_F . The maximum is pronounced only when a is near the unitary limit [13].

Palestini, Perali, Pieri, and Strinati have calculated the contact density for the balanced homogeneous gas at nonzero temperature using a diagrammatic t-matrix approximation [14]. They determined the contact from the high-momentum tail of the momentum distribution in Eq. 6.2 and from the high-frequency tail of the radio-frequency transition rate (which is discussed in Sect. 6.4.1).

Thus far, the calculational methods that have been used to calculate the contact density numerically involve uncontrolled approximations. While they may be accurate in certain limits, there may also be regions of $k_F a$ and T/T_F in which the systematic theoretical errors are not negligible.

6.3.4 Strongly-Imbalanced Homogeneous Gas

We now consider the strongly-imbalanced gas, in which a tiny population of minority atoms in state 2 is immersed in a system of atoms in state 1. The minority atoms can be considered as a dilute gas of impurities in the Fermi sea of majority atoms. In the homogeneous gas with number densities n_1 and n_2 , the ground state is determined by the dimensionless interaction variable $1/k_{F1}a$, where $k_{F1} = (6\pi^2 n_1)^{1/3}$ is the Fermi wavenumber for the majority atoms. In the BCS limit $a \rightarrow 0^-$, the impurity particle is an atom in state 2. In the BEC limit $a \rightarrow 0^+$, the impurity particle is the dimer whose binding energy is given by Eq. 6.17. Using a diagrammatic Monte Carlo method, Prokof'ev and Svistunov have shown that there is a phase transition at a critical value a_c of the scattering length given by $1/k_{F1}a_c = 0.90 \pm 0.02$ [15]. As $1/k_{F1}a$ increases through this critical value, the impurity changes from a quasiparticle associated with the atom in state 2, which is called a *polaron*, to a dimer quasiparticle.

Analytic expressions for the contact density for the ground state of the strongly-imbalanced homogeneous gas can be obtained from the energy density using Eq. 6.19. In the BCS limit $a \rightarrow 0^-$, the contact density is

$$\mathcal{C} \longrightarrow 16\pi^2 n_1 n_2 a^2. \quad (6.28)$$

This is proportional to the low-energy cross section $4\pi a^2$ and to the product $n_1 n_2$ of the two number densities. The contact density is an increasing function of $1/a$ that is smooth except for a discontinuity at the phase transition at $a = a_c$. In the BEC limit $a \rightarrow 0^+$, the contact density is

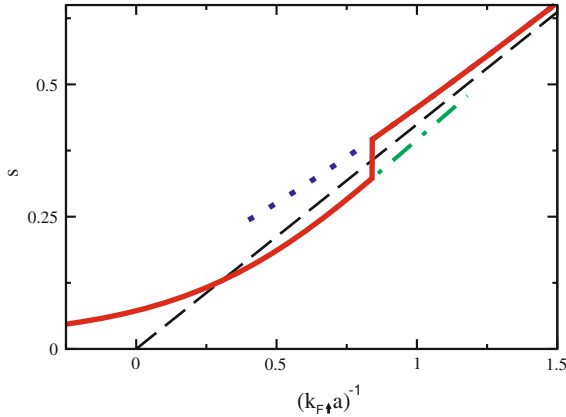


Fig. 6.2 The dimensionless contact density $s = \mathcal{C}/(6\pi^2 k_{F1} n_2)$ for the strongly-imbalanced homogeneous gas as a function of the dimensionless coupling strength $1/k_{F1}a$, from Ref. [16]. The *dashed line* is the leading contribution in the BEC limit given by Eq. 6.29. The *dotted* and *dash-dotted lines* are continuations of the *solid line* past the phase transition into metastable regions. The position of the phase transition may be underpredicted by about 7%

$$\mathcal{C} \longrightarrow \frac{8\pi n_2}{a}. \quad (6.29)$$

This is equal to the contact $8\pi/a$ for the dimer, which is given in Eq. 6.18, multiplied by the dimer number density n_2 .

The contact density \mathcal{C} for the ground state of the strongly-imbalanced homogeneous gas can be expressed as $k_{F1}n_2$ multiplied by a monotonically-increasing dimensionless function of $1/k_{F1}a$. This function has been calculated numerically by Punk, Dumitrescu, and Zwerger [16]. They used the adiabatic relation in Eq. 6.19 to obtain the contact from the ground state energy density, which they calculated using a variational method that gives a fairly good approximation to the results from the diagrammatic Monte Carlo method [15]. The results of Ref. [16] for the contact density as a function of $1/k_{F1}a$ are shown in Fig. 6.2. For large negative values of $1/k_{F1}a$, the contact density can be approximated by the BCS limit in Eq. 6.28. It is predicted to increase to about $5k_{F1}n_2$ in the unitary limit $ato \pm \infty$ and then to about $20k_{F1}n_2$ at the phase transition a_c , where it is predicted to have a discontinuity of about 20%. The prediction of the variational method for the position of the phase transition is $(k_{F1}a_c)^{-1} \approx 0.84$, which is about 7% smaller than the diagrammatic Monte Carlo result 0.90 ± 0.02 . This difference reflects a systematic theoretical error in the variational method of Ref. [12]. For large positive values of $1/k_{F1}a$, the contact density can be approximated by the BEC limit in Eq. 6.29.

6.4 Other Universal Relations

Many new universal relations involving the contact have been discovered in recent years. They reveal that the contact plays a central role in many of the most important probes for ultracold atoms.

6.4.1 RF Spectroscopy

Given a system of atoms in spin states 1 and 2, a radio-frequency (rf) signal that is tuned to near the transition frequency between an atom in state 2 and an atom in a third spin state 3 can transform the atoms in state 2 coherently into linear combinations of atoms in states 2 and 3. These atoms can be subsequently transformed by decohering processes into a mixture of atoms in states 2 and 3. The net effect is a transfer of atoms from the state 2 to the state 3. The transition rate $\Gamma(\omega)$ for this process depends on the frequency ω of the rf signal. It is convenient to choose the offset for ω to be the transition frequency for a single atom. The transition rate for an extremely dilute sample of N_2 atoms is then a delta function at $\omega = 0$:

$$\Gamma(\omega) \longrightarrow \pi \Omega^2 \delta(\omega) N_2, \quad (6.30)$$

where Ω is the Rabi frequency associated with the transition. In a many-body system consisting of atoms in states 1 and 2, $\Gamma(\omega)$ can be modified by initial-state interactions between atoms in states 1 and 2 and by final-state interactions between atoms in state 3 and atoms in states 1 or 2. However the effects of these interactions are constrained by a sum rule [17, 18]:

$$\int_{-\infty}^{\infty} d\omega \Gamma(\omega) = \pi \Omega^2 N_2. \quad (6.31)$$

If the atoms interact through large pair scattering lengths $a_{12} \equiv a$, a_{13} , and a_{23} , there are universal relations that govern the rf transition rate $\Gamma(\omega)$. One of these universal relations is a sum rule derived by Punk and Zwirger [19] and by Baym, Pethick, Yu, and Zwierlein [20]:

$$\int_{-\infty}^{\infty} d\omega \omega \Gamma(\omega) = \frac{\hbar \Omega^2}{4m} \left(\frac{1}{a_{12}} - \frac{1}{a_{13}} \right) C_{12}, \quad (6.32)$$

where $C_{12} \equiv C$ is the contact for atoms in states 1 and 2. The term proportional to $1/a_{13}$ comes from final-state interactions between atoms in states 1 and 3. If we divide the sum rule in Eq. 6.32 by the sum rule in Eq. 6.31, we get an expression for the frequency shift $\langle \omega \rangle$ averaged over the system. This frequency shift is called the

clock shift. The universal relation for the clock shift has several interesting features. If $a_{13} = a_{12}$, the clock shift vanishes because of a symmetry relating atoms 2 and 3. The clock shift has smooth behavior in the unitary limit $a_{12} \rightarrow \pm\infty$. This behavior was first observed in experiments on rf spectroscopy in ${}^6\text{Li}$ atoms [21]. If we take the limit $a_{13} \rightarrow 0$ in Eq. 6.32, the sum rule diverges. This implies that if the scattering length a_{13} is not large, the clock shift is sensitive to the range.

Another universal relation for rf spectroscopy is that the high-frequency tail of $\Gamma(\omega)$ is proportional to the contact. The general result for large scattering lengths a_{12} and a_{13} was derived in Ref. [22]:

$$\Gamma(\omega) \longrightarrow \frac{\Omega^2(a_{13}^{-1} - a_{12}^{-1})^2}{4\pi\omega(m\omega/\hbar)^{1/2}(a_{13}^{-2} + m\omega/\hbar)} C_{12}. \quad (6.33)$$

The asymptotic behavior in Eq. 6.33 holds when ω is much larger than the many-body frequency scales $\hbar k_F^2/m$ and $k_B T/\hbar$, but still much smaller than the frequency scale \hbar/mr_0^2 associated with the range. If $\omega \gg \hbar/mr_0^2$, the high-frequency tail decreases as $\omega^{-5/2}$. The result if the scattering length a_{13} is not large can be obtained by taking the limit $a_{13} \rightarrow 0$ in Eq. 6.33:

$$\Gamma(\omega) \longrightarrow \frac{\Omega^2}{4\pi\omega(m\omega/\hbar)^{1/2}} C_{12}. \quad (6.34)$$

In this case, the high-frequency tail decreases as $\omega^{-3/2}$. This scaling behavior was derived in Ref. [23] and the coefficient was first calculated correctly by Schneider and Randeria [24]. The scaling behavior was also pointed out in Ref. [25]. If $\Gamma(\omega)$ decreases asymptotically as $\omega^{-3/2}$, the sum rule in Eq. 6.32 diverges. Thus it is the high-frequency tail in Eq. 6.34 that makes this sum rule sensitive to the range in the case of a_{13} that is not large.

6.4.2 Photoassociation

Photoassociation uses a laser to transfer pairs of low-energy atoms into an excited molecular state with very high energy. The wavefunction of the molecule has support only over very short distances much smaller than the range r_0 , so the pair of atoms must be very close together to have a reasonable probability of making the transition. If there is a closed-channel molecule near the two-atom threshold that can be excited by the laser, it can dominate the photoassociation rate. The rate is then proportional to the number N_{mol} of closed-channel molecules:

$$\Gamma = \frac{\Omega^2}{\gamma} N_{\text{mol}}, \quad (6.35)$$

where Ω is the Rabi frequency of the laser and γ is the line width of the excited molecule.

Werner, Tarruel, and Castin [27] and Zhang and Leggett [28] pointed out that if the large scattering length a comes from the tuning of the magnetic field B to near a Feshbach resonance associated with this closed-channel molecule, then N_{mol} is proportional to the contact. If the zero of energy is chosen to coincide with the threshold for the atoms, the rate of change of the energy of the system with respect to the magnetic field can be expressed as

$$\frac{dE}{dB} = -\mu_{\text{mol}}N_{\text{mol}}, \quad (6.36)$$

where μ_{mol} is the difference between the magnetic moment of the closed-channel molecule and twice the magnetic moment of an atom in the open channel. The scattering length $a(B)$ near a Feshbach resonance at B_0 can be parametrized by

$$a(B) = a_{\text{bg}} \left(1 - \frac{\Delta}{B - B_0} \right). \quad (6.37)$$

Combining Eqs. 6.36 and 6.37 and using the adiabatic relation in Eq. 6.8, we obtain an expression for the number of closed-channel molecules that is proportional to the contact [27]:

$$N_{\text{mol}} = \frac{R_* \Delta^2}{4\pi[\Delta - (B - B_0)]^2} C, \quad (6.38)$$

where R_* is a positive length that characterizes the width of the Feshbach resonance:

$$R_* = -\frac{\hbar^2}{m\mu_{\text{mol}}a_{\text{bg}}\Delta}. \quad (6.39)$$

That length can also be expressed as $R_* = -\frac{1}{2}r_s$, where r_s is the effective range at the center of the resonance $B = B_0$.

The universal relation between the number of closed-channel molecules and the contact was previously derived formally by Braaten, Kang and Platter [29], but they did not make the connection to photoassociation.

6.4.3 Structure Factors

Structure factors encode information about density-density correlations in a system. For a many-body system of fermions with two spin states, the correlations between the densities of the two spin states are particularly important. The corresponding static structure factor $S_{12}(q)$ for a homogeneous system is the Fourier transform in the separation vector $\mathbf{r}_1 - \mathbf{r}_2$ of the correlator $\langle n_1(\mathbf{r}_1)n_2(\mathbf{r}_2) \rangle$ of the two densities. The dynamic structure factor $S_{12}(q, \omega)$ is the Fourier transform in the separation

vector and the time interval between the two densities. It encodes information about the degrees of freedom that can be excited by density fluctuations. The static structure factor can be obtained by integrating $S_{12}(q, \omega)$ over ω .

If the scattering length is large, the static structure factor $S_{12}(q)$ has a high-momentum tail that decreases like $1/q$ [30]. Hu, Liu, and Drummond have pointed out that this tail is proportional to the contact density [31]:

$$S_{12}(q) \longrightarrow \frac{1}{8} \left(\frac{1}{q} - \frac{4}{\pi a q^2} \right) \mathcal{C}. \quad (6.40)$$

The normalization of $S_{12}(q)$ in Ref. [31] differs from that in Eq. 6.40 by a factor of $2/N$, where $N = N_1 + N_2$ is the total number of atoms. The universal relation in Eq. 6.40 follows simply by taking the Fourier transform of the Tan relation in Eq. 6.7 for the density-density correlation at short distances.

Son and Thompson have studied the dynamic structure factor $S_{12}(q, \omega)$ in the unitary limit [32]. They showed that the leading contribution in the scaling limit $\omega \rightarrow \infty$ and $q \rightarrow \infty$ with $x = \hbar q^2/2m\omega$ fixed is proportional to the contact density. The coefficient of \mathcal{C} is $(m\omega^3/\hbar)^{-1/2}$ multiplied by a complicated function of the dimensionless scaling variable x , which they calculated analytically. For small x , their result reduces to

$$S_{12}(q, \omega) \longrightarrow \frac{4q^4}{45\pi^2\omega(m\omega/\hbar)^{5/2}} \mathcal{C}. \quad (6.41)$$

Taylor and Randeria have also determined the high-frequency tail of the dynamic structure factor [33]. Their result for the limits $q \rightarrow 0$ followed by $\omega \rightarrow \infty$ differs from that in Eq. 6.41 by a factor of $3/2$.

6.4.4 Viscosity Spectral Functions

Taylor and Randeria have derived universal relations for the viscosity spectral functions of a homogeneous gas [33]. They found that the shear viscosity spectral function $\eta(\omega)$ has a high-frequency tail that is proportional to the contact density. An error in the coefficient was corrected by Enss, Haussmann, and Zwerger [34]:

$$\eta(\omega) \longrightarrow \frac{\hbar^2}{15\pi(m\omega/\hbar)^{1/2}} \mathcal{C}. \quad (6.42)$$

Taylor and Randeria also derived a sum rule for $\eta(\omega)$:

$$\int_0^\infty d\omega \left(\eta(\omega) - \frac{\hbar^2 \mathcal{C}}{15\pi(m\omega/\hbar)^{1/2}} \right) = \frac{\pi \hbar}{3} \left(\mathcal{E} - \frac{3\hbar^2}{15\pi m a} \mathcal{C} \right), \quad (6.43)$$

where \mathcal{E} is the energy density. Given the high-frequency tail of $\eta(\omega)$ in Eq. 6.42, the subtraction term on the left side of Eq. 6.43 is necessary to make the integral convergent.

Taylor and Randeria have also derived a sum rule for the bulk viscosity spectral function $\zeta(\omega)$ [33]:

$$\int_0^{\infty} d\omega \zeta(\omega) = \frac{\hbar^3}{72ma^2} \left(\frac{dC}{da^{-1}} \right)_{S/N}, \quad (6.44)$$

where the derivative is taken with the entropy per particle held fixed. Since the spectral function $\zeta(\omega)$ is positive definite, the sum rule in Eq. 6.44 implies that the contact density C is a strictly increasing function of a^{-1} .

6.4.5 Rapid Change in the Scattering Length

In Ref. [3], Tan showed that if the scattering length $a(t)$ is time dependent, the total energy of the system changes at a rate that is proportional to the contact:

$$\frac{d}{dt} E(t) = -\frac{\hbar^2}{4\pi m} C(t) \dot{b}(t), \quad (6.45)$$

where $b(t) = 1/a(t)$ and $C(t)$ is the instantaneous contact at time t . If the external potential $V(\mathbf{r})$ is also changing with time, there is an additional term proportional to $\dot{V}(\mathbf{r})$ on the right side of Eq. 6.45. Tan referred to that equation as the *dynamic sweep theorem*.

The simplest case of a time-dependent scattering length is a sudden change in a , for which the change in the energy is given in Eq. 6.10. Tan also presented a more general result for a scattering length $a(t)$ that changes rapidly enough that the contact does not have time to evolve significantly from its original value. If the scattering length $a(t)$ changes over a short time interval T from an initial value $a(0)$ to a final value $a(T)$, the change in the total energy is

$$\Delta E = -\frac{\hbar^2}{4\pi m} \left(\frac{1}{a(T)} - \frac{1}{a(0)} + \sqrt{\frac{8\hbar}{\pi m}} \int_0^T dt \int_0^t dt' \sqrt{t-t'} \dot{b}(t) \dot{b}(t') \right) C, \quad (6.46)$$

where C is the contact at the initial time $t = 0$. The time interval T must be short compared to the time scales for the evolution of the system. It must also be long compared to the time scale mr_0^2/\hbar set by the range.

Son and Thompson have also considered rapid changes in the scattering length for a system that is initially in the unitary limit [32]. The simplest case is a small-amplitude oscillation of the inverse scattering length about the unitary limit: $a(t)^{-1} =$

$\gamma_0 \cos(\omega t)$. If ω is large compared to the Fermi frequency $\hbar k_F^2/2m$, the energy density increases at a rate that is proportional to the contact density:

$$\frac{d}{dt} \mathcal{E} = \frac{\gamma_0^2 \hbar^2}{8\pi m} \sqrt{\frac{\hbar\omega}{m}} C. \quad (6.47)$$

The frequency ω must also be small compared to the frequency scale \hbar/mr_0^2 set by the range.

Another case of a rapidly varying scattering length considered in Ref. [32] is a pulse $a(t)^{-1}$ that differs from zero only during a short time interval. The change in the total energy is proportional to the contact:

$$\Delta E = \frac{\hbar^{5/2}}{4\pi^2 m^{3/2}} \left(\int_0^\infty d\omega \sqrt{\omega} |\tilde{\gamma}(\omega)|^2 \right) C, \quad (6.48)$$

where $\tilde{\gamma}(\omega)$ is the Fourier transform of the pulse: $\tilde{\gamma}(\omega) = \int_0^T dt \exp(i\omega t) a(t)^{-1}$. This is consistent with the more general result in Eq. 6.46, which does not require the unitary limit before and after the pulse.

6.5 Making Contact with Experiment

There are some exciting recent developments in the study of the universal relations using experiments with ultracold atoms. They involve various measurements of the contact and the experimental verification of some of the universal relations.

6.5.1 Photoassociation

The Hulet group at Rice University in Houston has measured the photoassociation rate of a balanced mixture of ${}^6\text{Li}$ atoms in the lowest two hyperfine spin states [35]. The scattering length was controlled by using the Feshbach resonance near 834 G. They measured the photoassociation rate at various values of the scattering length, with $1/k_F a$ ranging from about -1.4 to about $+5.4$. The photoassociation laser can excite the closed-channel molecule that is responsible for the Feshbach resonance. The photoassociation rate can therefore be interpreted as a measurement of the number of closed-channel molecules. According to the universal relation in Eq. 6.38, this is proportional to the contact. However, at the time of the experiment, the concept of the contact was still unfamiliar. The number of closed-channel molecules was expected to be proportional to the square of the order parameter $|\Delta|^2$ [35]. In the BEC limit, $|\Delta|^2 \propto k_F/a$, which has the same dependence on a as the contact in Eq. 6.26. In the BCS limit, the order parameter decreases exponentially with

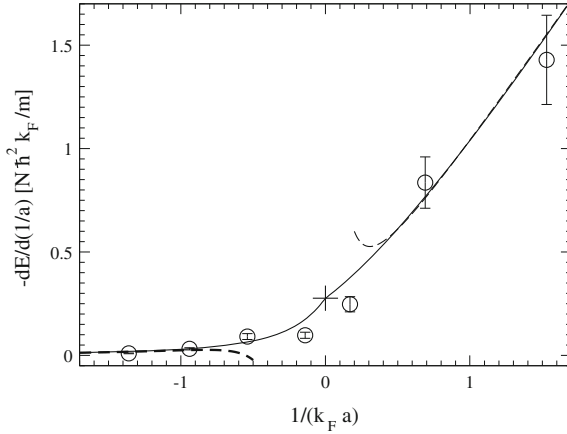


Fig. 6.3 The derivative of the energy E with respect to $1/a$ determined from measurements of the photoassociation rate of a trapped gas of ${}^6\text{Li}$ atoms, from Ref. [27]. The data come from measurements of the number of closed-channel molecules in Ref. [35]. The *solid line* is a theoretical prediction using the local density approximation, with the contact density for the homogeneous system obtained by interpolating between the BCS, unitary, and BEC limits. The symbol $+$ indicates the prediction for the contact in the unitary limit. The *dashed lines* are extrapolations from the BCS and BEC limits

$1/|a| : |\Delta|^2 \propto k_F^2 \exp(-\pi/(k_F|a|))$. This behavior is dramatically different from the contact in Eq. 6.21, which decreases like a^2 as $a \rightarrow 0^-$. Nevertheless, the measurements of the photoassociation rate were compatible with the assumption that it was proportional to $|\Delta|^2$, even for negative values of $1/k_F a$ as large as -1.4 .

The first analysis of the data from Ref. [35] in terms of the contact was carried out by Werner, Tarruell, and Castin [27]. Their results are shown in Fig. 6.3. The contact extracted from the measured number of closed-channel molecules was in reasonable agreement with a theoretical prediction using the local density approximation, with the contact density for the homogeneous system obtained by interpolating between the BCS, unitary, and BEC limits.

6.5.2 Static Structure Factor

The static and dynamic structure factors for systems consisting of ultracold atoms can be probed by using Bragg spectroscopy. Bragg scattering is a two-photon process in which an atom absorbs a photon from one laser beam and emits a photon into a second laser beam. The net effect is the transfer of a selected momentum $\hbar\mathbf{k}$ and a selected energy $\hbar\omega$ to the atom, where \mathbf{k} is the difference between the wavevectors of the two lasers and ω is the difference between their frequencies.

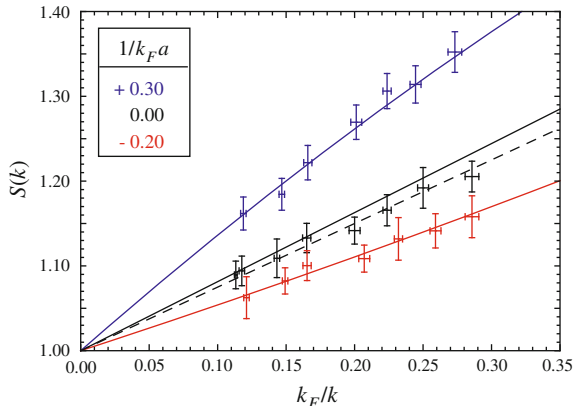


Fig. 6.4 The structure factor $S(k)$ for a trapped gas of ${}^6\text{Li}$ atoms as a function of k_F/k for three values of $1/k_F a$, from Ref. [37]. The *solid lines* are the universal predictions from Eq. 6.40 using the contact obtained from a theoretical calculation. The *dashed line* is a linear fit to the data for $1/k_F a = 0$

The Vale group at Swinburne University of Technology in Melbourne has used Bragg spectroscopy to study the static structure factor $S_{12}(k)$ [36]. They used a balanced mixture of ${}^6\text{Li}$ atoms in the lowest two hyperfine spin states. The scattering length was controlled by using the Feshbach resonance near 834 G. In Ref. [37], they reported measurements of the static structure factor $S_{12}(k)$ as a function of a for $k = 4.8k_F$. These measurements are in good agreement with the universal relation for the large-momentum tail in Eq. 6.40, with the contact density for the homogeneous system obtained by interpolating between the BCS, unitary, and BEC limits. In Ref. [37], they also reported measurements of $S_{12}(k)$ as a function of k for $1/k_F a = -0.2, 0$, and $+0.3$, which are shown in Fig. 6.4. The measurements are linear in k_F/k , as predicted by the universal relation in Eq. 6.40. The slope is predicted to be proportional to the contact C . The contact for their trapped system at zero temperature was calculated in Ref. [37] using the local density approximation and a below-threshold Gaussian fluctuation theory for the homogeneous system. For $1/k_F a = -0.2$ and $+0.3$, the slope agrees well the universal relation. In the unitary limit $1/k_F a = 0$, the slope is smaller than predicted. The discrepancy could be attributed to the effects of nonzero temperature.

6.5.3 Comparing Measurements of the Contact

The Jin group at JILA in Boulder has measured the contact C for a trapped gas of atoms using three independent methods [38]. They used a balanced mixture of ${}^{40}\text{K}$ atoms in the two lowest hyperfine spin states at a temperature of about $0.1T_F$. The scattering length was controlled by using the Feshbach resonance near 201 G.

The first method for measuring C used the Tan relation in Eq. 6.2 for the high-momentum tail of the momentum distribution $n_2(k)$. The interactions between the trapped atoms were turned off by changing the magnetic field to 209 G where the scattering length vanishes. The trapping potential was then turned off, and the momentum distribution was measured from the ballistic expansion of the cloud of atoms. The contact C is the large-momentum limit of $k^4 n_2(k)$. It was measured for values of $1/k_F a$ ranging from about -2.7 to about $+0.2$.

The second method for measuring the contact used the universal relation in Eq. 6.34 for the high-frequency tail of the radio-frequency (rf) transition rate. The rf signal was used to transfer atoms from state 2 into a third spin state 3 for which the pair scattering length a_{13} is not large. The rf transition rate $\Gamma(\omega)$ was determined by measuring the number of atoms transferred. The contact was then determined from the behavior of $\Gamma(\omega)$ at large ω . It was measured for values of $1/k_F a$ ranging from about -1.7 to about $+0.2$.

The third method for measuring C also used the Tan relation in Eq. 6.2, but the tail of the momentum distribution was determined by photoemission spectroscopy (PES). This involves using momentum-resolved rf spectroscopy to measure the distribution $n_2(k, \omega)$ of the momentum and energy of atoms in state 2, and then integrating over ω to determine the momentum distribution $n_2(k)$. The contact C is the large-momentum limit of $k^4 n_2(k)$. It was measured for the same values of $1/k_F a$ as the second method.

The three sets of measurements of the contact by the Jin group [38] are shown in Fig. 6.5. The results from the three methods are all consistent. They lie close to the theoretical prediction of Ref. [27], which was based on the local density approximation, with the contact density for the homogeneous system obtained by interpolating between the BEC, unitary, and BCS limits. These results provide direct experimental verification of the role of the contact in large-momentum and high-frequency tails for a many-body system of fermions with a large scattering length.

6.5.4 Tests of the Thermodynamic Tan Relations

The adiabatic relation in Eq. 6.8 and the virial theorem in Eq. 6.11 relate different contributions to the total energy E to the contact C . The adiabatic relation expresses a derivative of $(T + U) + V$ in terms of the contact. The virial theorem expresses the combination $(T + U) - V$ in terms of the contact. Thus measurements of $T + U$, V , and C provide two nontrivial tests of the Tan relations for the thermodynamic properties of the system.

The Jin group at JILA in Boulder has tested these Tan relations by measuring $T + U$ and V for the same system of ^{40}K atoms for which they measured the contact C [38], as described in Sect. 6.5.3. They measured the external potential energy V by imaging the spatial distribution of the cloud of atoms, which was trapped in a harmonic potential. They measured the combination $T + U$, which can be called

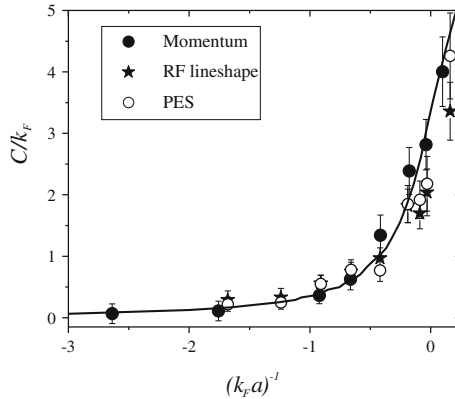


Fig. 6.5 Three measurements of the dimensionless contact C/k_F for a trapped gas of ^{40}K atoms as a function of $1/k_F a$, from Ref. [38]. The Fermi wavenumber k_F is defined by the Fermi energy for the trapped system: $E_F = \hbar^2 k_F^2 / 2m$. Two of the data sets are from the tail of the momentum distribution measured directly by ballistic expansion (*solid dots*) and indirectly by photoemission spectrometry (*open dots*). The third data set is from the high-frequency tail of the rf lineshape (*stars*)

the release energy, by turning off the trapping potential and observing the resulting expansion of the cloud. They measured $T + U$ and V at values of $1/k_F a$ ranging from about -3 to about $+0.3$. For the contact C , they used their measurements from photoemission spectroscopy described in Sect. 6.5.3. They found good agreement between the two sides of the adiabatic relation in Eq. 6.8 as shown in Fig. 6.6. They also found that the two sides of the virial theorem in Eq. 6.11 agreed to within the errors, which were roughly 1% of the Fermi energy. These results provide direct experimental verification of the role of the contact in the thermodynamics of a many-body system of fermions with a large scattering length.

6.5.5 Contact Density Near Unitarity

The contact density for the homogeneous gas has been determined by the Salomon group at École Normale Supérieure in Paris [39]. They used an imbalanced mixture of ^6Li atoms in the lowest two hyperfine spin states at a magnetic field near 834 G. They determined the equation of state for the homogeneous gas by measuring the number densities of the two spin states in a harmonic trapping potential integrated over the two transverse dimensions. Their result for the numerical constant in Eq. 6.23 was $\zeta = 0.93(5)$. This is consistent with the value in Eq. 6.24 obtained from diffusion Monte Carlo calculations of the density–density correlator.

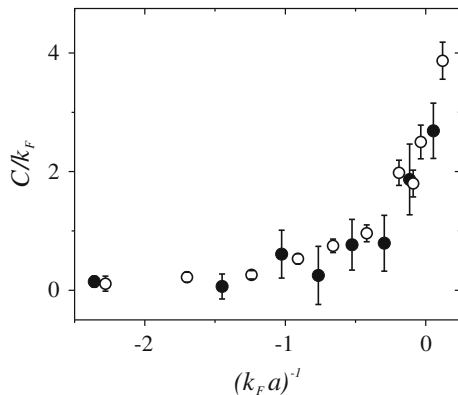


Fig. 6.6 Test of the adiabatic relation in a trapped gas of ^{40}K atoms, from Ref. [38]. The dimensionless contact C/k_F as a function of $1/k_F a$ determined from the derivative of the energy E with respect to $1/a$ (*solid dots*) is compared to the measurements using photoemission spectrometry (*open dots*). The Fermi wavenumber k_F is defined by the Fermi energy for the trapped system: $E_F = \hbar^2 k_F^2 / 2m$

6.6 Derivations of Universal Relations

In this section, we give an overview of various derivations of the universal relations. We begin by describing briefly the novel methods used in the original derivations of the Tan relations. We then describe briefly various other approaches that have been used to rederive the Tan relations. Finally, we describe in more detail how universal relations can be derived using quantum field theory methods involving renormalization and the operator product expansion.

6.6.1 Preliminaries

The scattering amplitude for S-wave atom–atom scattering can be written as

$$f(k) = \frac{1}{k \cot \delta_0(k) - ik}, \quad (6.49)$$

where k is the relative wavenumber and $\delta_0(k)$ is the S-wave phase shift. If the interactions have a finite range, the low-energy expansion of the phase shift can be expressed as a power series in k^2 :

$$k \cot \delta_0(k) = -1/a + \frac{1}{2} r_s k^2 + \dots, \quad (6.50)$$

where r_s is the effective range. The coefficient of $(k^2)^n$ has dimensions $(\text{length})^{2n-1}$. Generically, all these coefficients are comparable to the range raised to the appropriate power. By fine tuning the interactions, the scattering length can be made much larger than the range. This fine-tuning leads to universal properties that depend on the interactions only through the scattering length.

The universality for large scattering length reflects a well-behaved *zero-range limit*, in which all the terms on the right side of Eq. 6.50 go to zero except the leading term. The phase shift reduces in this limit to

$$k \cot \delta_0(k) = -1/a. \quad (6.51)$$

The model in which the phase shift has this simple form up to arbitrarily large momentum is called the *Zero-Range Model*. The universal properties of a general model with large scattering length are realized in the Zero-Range Model in a particularly simple form, because a is the only length scale that arises from interactions. The price that must be paid is that the zero-range limit leads to divergences in some observables. It also leads to singularities in intermediate steps of the derivations of universal relations. An illustration is provided by the energy relation in Eq. 6.6. The kinetic energy T and the interaction energy U are separately ultraviolet divergent, but the divergences cancel in the sum $T + U$. The singularities associated with the zero-range limit can be regularized by backing off from the zero-range limit or by some equivalent device.

One way to represent the Zero-Range Model is in terms of the Schrödinger equation for noninteracting particles with peculiar boundary conditions. The Schrödinger wavefunction for N_1 fermions in state 1 and N_2 fermions in state 2 is a complex function $\Psi(\mathbf{r}_1, \dots, \mathbf{r}_{N_1}; \mathbf{r}'_1, \dots, \mathbf{r}'_{N_1})$ that is totally antisymmetric in the first N_1 positions and totally antisymmetric in the last N_2 positions. The proper normalization of the wavefunction is

$$\frac{1}{N_1!N_2!} \int d^3r_1 \dots \int d^3r_{N_1} \int d^3r'_1 \dots \int d^3r'_{N_2} |\Psi(\mathbf{r}_1, \dots, \mathbf{r}_{N_1}; \mathbf{r}'_1, \dots, \mathbf{r}'_{N_2})|^2 = 1. \quad (6.52)$$

In the zero-range limit, this wavefunction diverges when the positions \mathbf{r}_i and \mathbf{r}'_j of any pair of fermions with different spins coincide. Its behavior when \mathbf{r}_1 and \mathbf{r}'_1 are nearly equal is

$$\begin{aligned} & \Psi(\mathbf{R} + \frac{1}{2}\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_{N_1}; \mathbf{R} - \frac{1}{2}\mathbf{r}, \mathbf{r}'_2, \dots, \mathbf{r}'_{N_2}) \\ & \longrightarrow \phi(r)\Phi(\mathbf{r}_2, \dots, \mathbf{r}_{N_1}; \mathbf{r}'_2, \dots, \mathbf{r}'_{N_2}; \mathbf{R}), \end{aligned} \quad (6.53)$$

where Φ is a smooth function of \mathbf{R} and $\phi(r)$ is the zero-energy scattering wavefunction for two particles interacting through a large scattering length:

$$\phi(r) = \frac{1}{r} - \frac{1}{a}. \quad (6.54)$$

The Fourier transform of this wavefunction is

$$\tilde{\phi}(\mathbf{k}) = \frac{4\pi}{k^2} - \frac{(2\pi)^3}{a} \delta^3(\mathbf{k}). \quad (6.55)$$

The Schrödinger equation for interacting particles with a large scattering length reduces in the zero-range limit to the Schrödinger equation for non-interacting particles with the wavefunction constrained to satisfy the Bethe–Peierls boundary conditions in Eq. 6.53.

6.6.2 Tan’s Derivations

Tan derived many of his universal relations by using generalized functions, or distributions, to deal with the singularities associated with the zero-range limit [2, 3]. He introduced distributions $\Lambda(\mathbf{k})$ and $L(\mathbf{k})$ whose values at finite \mathbf{k} are

$$\Lambda(\mathbf{k}) = 1, \quad |\mathbf{k}| < \infty, \quad (6.56a)$$

$$L(\mathbf{k}) = 0, \quad |\mathbf{k}| < \infty, \quad (6.56b)$$

and which have the following integrals over all \mathbf{k} :

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} \Lambda(\mathbf{k}) = 0 \quad (6.57a)$$

$$\int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2} L(\mathbf{k}) = 1. \quad (6.57b)$$

Using these properties, it is easy to verify that the Fourier transform of the zero-energy scattering wavefunction in Eq. 6.55 satisfies

$$\int \frac{d^3k}{(2\pi)^3} \left[\Lambda(\mathbf{k}) + \frac{1}{4\pi a} L(\mathbf{k}) \right] \tilde{\phi}(\mathbf{k}) = 0. \quad (6.58)$$

This property allows the Bethe–Peierls boundary condition in Eq. 6.53 to be expressed as an equality rather than as a limit:

$$\begin{aligned} & \int \frac{d^3k}{(2\pi)^3} \left[\Lambda(\mathbf{k}) + \frac{1}{4\pi a} L(\mathbf{k}) \right] \int d^3r e^{-i\mathbf{k}\cdot\mathbf{r}} \\ & \times \Psi \left(\mathbf{R} + \frac{1}{2}\mathbf{r}, \mathbf{r}_2, \dots, \mathbf{r}_{N_1}; \mathbf{R} - \frac{1}{2}\mathbf{r}, \mathbf{r}'_2, \dots, \mathbf{r}'_{N_2} \right) = 0. \end{aligned} \quad (6.59)$$

In addition to using $\Lambda(\mathbf{k})$ and $L(\mathbf{k})$ to impose the Bethe–Peierls boundary conditions, Tan used the distribution $\Lambda(\mathbf{k})$ to regularize the interaction term in the

Hamiltonian. He derived many of his universal relations by simple manipulations involving these distributions. They lead to an expression for the contact of the form

$$C = \frac{1}{2} \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3} L(\mathbf{k}) k^2 n_{\sigma}(\mathbf{k}). \quad (6.60)$$

By the properties of the distribution $L(\mathbf{k})$ in Eqs. 6.56b and 6.57b, the integral extracts the coefficient of $1/k^4$ in the high-momentum tail of $n_{\sigma}(\mathbf{k})$.

6.6.3 Other Derivations

6.6.3.1 Braaten, Kang, and Platter

Braaten and Platter [5] used quantum field theory methods to rederive all the Tan relations in Sect. 6.2 except for the relation in Eq. 6.10 for the sudden change in the scattering length. They used the formulation of the Zero-Range Model as a local quantum field theory, as described later in Sect. 6.6.4.1. The singularities associated with the zero-range limit were regularized by imposing an ultraviolet cutoff on the momenta of virtual particles. The contact density was identified as the expectation value of a local operator. The universal relations were derived using renormalization, as described in Sect. 6.6.4.2, and the operator product expansion, as described in Sects. 6.6.4.4 and 6.6.4.5.

Braaten, Kang, and Platter [29] used quantum field theory methods to derive universal relations for the *Resonance Model*, in which the S-wave scattering phase shift is given by

$$k \cot \delta_0(k) = - \left(\lambda + \frac{g^2}{k^2 - \nu} \right)^{-1}. \quad (6.61)$$

The Resonance Model is a two-channel model, in which the states in the two-atom sector consist of a point-like molecule as well as the usual two-atom scattering states. It provides a natural model for a Feshbach resonance. The scattering length $a = \lambda - g^2/\nu$ agrees with the expression in Eq. 6.37 if we set $\lambda = a_{\text{bg}}$, $\nu = -m\mu_{\text{mol}}(B - B_0)/\hbar^2$, and $g^2 = 1/R_*$, where R_* is given in Eq. 6.39. Braaten, Kang, and Platter found that in the various universal relations that correspond to the Tan relations, the contact density is replaced by various linear combinations of the expectation values of three local operators. If the expression for $k \cot \delta_0(k)$ in Eq. 6.61 is well-approximated by $-1/a$ for all wavenumbers from 0 up to the scale set by the system, the expectation values of the three local operators must coincide in order for the universal relations to reduce to those of the Zero-Range Model. For a broad Feshbach resonance, which is defined by $|a_{\text{bg}}| \gg R_*$, this requires only that $|a| \gg r_0$ and $kr_0 \ll 1$, where r_0 is the range of interactions in the absence of the Feshbach

resonance. For a narrow Feshbach resonance, which is defined by $|a_{\text{bg}}| \ll R_*$, this requires also that $kR_* \ll 1$.

6.6.3.2 Werner, Tarruell, and Castin

Werner [26] derived the virial theorem in Eq. 6.11 independently. The right side of Eq. 6.11 was expressed not in terms of the contact C but in terms of the derivative of the energy that appears in the adiabatic relation in Eq. 6.8.

Werner, Tarruell, and Castin [27] rederived the tail of the momentum distribution in Eq. 6.2, the adiabatic relation, and the density-density correlator at short distances in Eq. 6.7. The singularities associated with the zero-range limit were regularized by using a lattice model in which the fermions occupy the sites of a three-dimensional cubic lattice whose spacing b approaches 0. Werner, Tarruell and Castin also used a two-channel model to derive the universal relation for the number of closed-channel molecules in Eq. 6.38.

Werner and Castin [40] subsequently presented a much more thorough treatment of the universal relations for the Zero-Range Model and for the lattice model. In addition to the tail of the momentum distribution, the adiabatic relation, and the density-density correlator at short distances, they rederived the energy relation in Eq. 6.6 and the adiabatic sweep theorem, which is the generalization of Eq. 6.45 that allows for a time-dependent external potential. They showed that for a system in thermal equilibrium (which includes the ground state as a limiting case), the contact is an increasing function of $1/a$:

$$\frac{dC}{da^{-1}} > 0. \quad (6.62)$$

This inequality holds whether the derivative is evaluated at fixed entropy or at fixed temperature. The monotonic increase of the contact density with $1/a$ is illustrated in Figs. 6.1 and 6.2 for the cases of a homogeneous gas at zero temperature that is balanced and strongly imbalanced, respectively. Because the inequality in Eq. 6.62 does not hold in general, it should not be regarded as a universal relation.

Werner and Castin also generalized the universal relations to various other systems [40]. They considered the effects of a nonzero range for the interaction potential. They derived the generalizations of the universal relations to two spatial dimensions, which had been considered previously by Tan [41]. They also derived universal relations for two types of fermions with unequal masses $m_1 < m_2$, for identical bosons, and for mixtures of fermions and bosons with various masses. In some cases, including identical bosons and two types of fermions whose mass ratio m_2/m_1 exceeds the critical value 13.7, the universal relations are complicated by the Efimov effect [40].

6.6.3.3 Zhang and Leggett

Zhang and Leggett [28] rederived the adiabatic relation in Eq. 6.8 and the pressure relation in Eq. 6.12. They used a nonlocal quantum field theory with field operators $\psi_1(\mathbf{r})$ and $\psi_2(\mathbf{r})$ and with a short-range interaction potential $U(\mathbf{r})$. They restricted their attention to a homogeneous many-body system in equilibrium. The equal-time two-particle correlation function was expressed as a sum over eigenstates of the operator $\psi_2(\mathbf{r}_2)\psi_1(\mathbf{r}_1)$:

$$\langle \psi_1^\dagger(\mathbf{r}_1)\psi_2^\dagger(\mathbf{r}_2)\psi_2(\mathbf{r}_2)\psi_1(\mathbf{r}_1) \rangle = \sum_i v_i \phi^{(i)}(\mathbf{r}_1, \mathbf{r}_2)^* \phi^{(i)}(\mathbf{r}_1, \mathbf{r}_2), \quad (6.63)$$

where v_i is the average number of pairs with different spins in the eigenstate i and the eigenfunctions $\phi^{(i)}(\mathbf{r}_1, \mathbf{r}_2)$ are normalized accordingly in a large volume V . The Bethe–Peierls boundary conditions together with translation invariance imply that the limiting behavior of these eigenfunctions as the separation $r = |\mathbf{r}_1 - \mathbf{r}_2|$ goes to zero is

$$\phi^{(i)}\left(\mathbf{R} + \frac{1}{2}\mathbf{r}, \mathbf{R} - \frac{1}{2}\mathbf{r}\right) \longrightarrow C^{(i)} e^{i\mathbf{P}^{(i)} \cdot \mathbf{R}} \phi(r), \quad (6.64)$$

where the normalization constant $C^{(i)}$ and the momentum vector $\mathbf{P}^{(i)}$ depend on the eigenstate i and $\phi(r)$ is the zero-energy scattering wavefunction defined in Eq. 6.54. The integral of the correlation function weighted by an arbitrary short-distance function $s(|\mathbf{r}_1 - \mathbf{r}_2|)$ therefore reduces to:

$$\begin{aligned} & \int d^3r_1 \int d^3r_2 s(|\mathbf{r}_1 - \mathbf{r}_2|) \langle \psi_1^\dagger(\mathbf{r}_1)\psi_2^\dagger(\mathbf{r}_2)\psi_2(\mathbf{r}_2)\psi_1(\mathbf{r}_1) \rangle \\ &= \frac{1}{4\pi} V \mathcal{C} \int_0^\infty dr r^2 s(r) |\phi(r)|^2, \end{aligned} \quad (6.65)$$

where \mathcal{C} is the contact density, which is given by

$$\mathcal{C} = 16\pi^2 \sum_i v_i |C^{(i)}|^2. \quad (6.66)$$

Zhang and Leggett emphasized that the thermodynamics is universal and is completely determined by the contact density \mathcal{C} . Zhang and Leggett also used a two-channel model to derive a universal relation for the number of closed-channel molecules that can be reduced to Eq. 6.38 [28].

Zhang and Leggett derived a factorization formula for the interaction energy density [28] that separates the dependence on a and thermodynamic variables, such as the temperature and number densities, from the dependence on the short-distance parameters that determine the shape of the interaction potential $U(r)$. The interaction

energy is the special case of Eq. 6.65 in which the short-range function $s(r)$ is the interaction potential $U(r)$. The interaction energy density can be expressed as

$$\mathcal{U} = \frac{1}{4\pi} \mathcal{C} \int_0^\infty dr r^2 U(r) |\phi(r)|^2. \quad (6.67)$$

All the dependence on the thermodynamic variables and on the large scattering length a resides in the contact density \mathcal{C} .

6.6.3.4 Combescot, Alzetto, and Leyronas

Combescot, Alzetto, and Leyronas [42] rederived the tail of the momentum distribution in Eq. 6.2 and the energy relation in Eq. 6.6. They used the Schrödinger formalism in the coordinate representation to describe a system consisting of $N_1 + N_2$ fermions. The singularities associated with the zero-range limit were regularized by imposing a cutoff $|\mathbf{r}_i - \mathbf{r}'_j| > r_0$ on the separations of the two types of fermions. They expressed the contact in terms of the Fourier transform of the function Φ defined by the Bethe-Peierls boundary conditions in Eq. 6.53. By reverting to the coordinate representation, the contact density can be expressed as

$$\begin{aligned} \mathcal{C}(\mathbf{R}) = & \frac{16\pi^2}{(N_1 - 1)!(N_2 - 1)!} \int d^3r_2 \dots \int d^3r_{N_1} \int d^3r'_2 \dots \int d^3r'_{N_2} \\ & \times |\Phi(\mathbf{r}_2, \dots, \mathbf{r}_{N_1}; \mathbf{r}'_2, \dots, \mathbf{r}'_{N_2}; \mathbf{R})|^2, \end{aligned} \quad (6.68)$$

provided the wavefunction Ψ is properly normalized as in Eq. 6.52. This expression for the contact density was originally derived by Tan [2].

Combescot, Alzetto, and Leyronas generalized the tail of the momentum distribution and the energy relation to various other systems [42]. They generalized them to two spatial dimensions, which had been considered previously by Tan [41]. They generalized them to two types of fermions with different masses $m_1 < m_2$. They did not however consider the complications associated with the Efimov effect when the mass ratio m_2/m_1 exceeds the critical value 13.7 [40].

6.6.4 Quantum Field Theory Derivations

We proceed to explain how universal relations can be derived concisely using the methods of quantum field theory.

6.6.4.1 Zero-Range Model

A quantum field theory that describe atoms with two spin states must have fundamental quantum fields $\psi_\sigma(\mathbf{r})$, $\sigma = 1, 2$. The Hamiltonian operator for a local quantum field theory can be expressed as the integral over space of a Hamiltonian density operator: $H = \int d^3R \mathcal{H}$. If the atoms are in an external potential $V(\mathbf{r})$, the Hamiltonian density operator is the sum of a kinetic term \mathcal{T} , an interaction term \mathcal{U} , and an external potential term \mathcal{V} :

$$\mathcal{H} = \mathcal{T} + \mathcal{U} + \mathcal{V}. \quad (6.69)$$

In the quantum field theory formulation of the Zero-Range Model, the three terms in the Hamiltonian density operator are

$$\mathcal{T} = \sum_{\sigma} \frac{1}{2m} \nabla \psi_{\sigma}^{\dagger} \cdot \nabla \psi_{\sigma}^{(\Lambda)}(\mathbf{R}), \quad (6.70a)$$

$$\mathcal{U} = \frac{g_0(\Lambda)}{m} \psi_1^{\dagger} \psi_2^{\dagger} \psi_2 \psi_1^{(\Lambda)}(\mathbf{R}), \quad (6.70b)$$

$$\mathcal{V} = V(\mathbf{R}) \sum_{\sigma} \psi_{\sigma}^{\dagger} \psi_{\sigma}(\mathbf{R}). \quad (6.70c)$$

For simplicity, we have set $\hbar = 1$. The superscripts (Λ) on the operators in Eqs. 6.70a and 6.70b indicate that their matrix elements are ultraviolet divergent and an ultraviolet cutoff is required to make them well defined. For the ultraviolet cutoff, we impose an upper limit $|\mathbf{k}| < \Lambda$ on the momenta of virtual particles. In the limit $\Lambda \rightarrow \infty$, the Hamiltonian density in Eq. 6.69 describes atoms with the phase shift given by Eq. 6.51 provided we take the bare coupling constant to be

$$g_0(\Lambda) = \frac{4\pi a}{1 - 2a\Lambda/\pi}. \quad (6.71)$$

In Ref. [5], Braaten and Platter identified the operator that measures the contact density in the quantum field theory formulation of the Zero-Range Model. It is convenient to introduce the diatom field operator Φ defined by

$$\Phi(\mathbf{R}) = g_0(\Lambda) \psi_2 \psi_1^{(\Lambda)}(\mathbf{R}). \quad (6.72)$$

This operator annihilates a pair of atoms at the point \mathbf{R} . The superscript (Λ) on the operator $\psi_1 \psi_2$ indicates that its matrix elements are ultraviolet divergent. Their dependence on Λ is exactly compensated by the prefactor $g_0(\Lambda)$, so Φ is an ultraviolet finite operator. The contact density operator is $\Phi^{\dagger} \Phi$ [5]. This is just the interaction energy density operator multiplied by a constant that depends on the

ultraviolet cutoff: $\Phi^\dagger \Phi = mg_0 \mathcal{U}$. The contact is obtained by taking the expectation value of the contact density operator and integrating over space:

$$C = \int d^3 R \langle \Phi^\dagger \Phi(\mathbf{R}) \rangle. \quad (6.73)$$

6.6.4.2 Renormalization

Several of the Tan relations follow very simply from the renormalization of the Zero-Range Model [5]. The renormalization condition in Eq. 6.71 implies that the bare coupling constant $g_0(\Lambda)$ satisfies

$$g_0(\Lambda) = \left(\frac{1}{4\pi a} - \frac{\Lambda}{2\pi^2} \right) g_0^2(\Lambda). \quad (6.74)$$

Its derivative with respect to a is

$$\frac{d}{da} g_0(\Lambda) = \frac{1}{4\pi a^2} g_0^2(\Lambda). \quad (6.75)$$

The energy relation in Eq. 6.6 and the adiabatic relation in Eq. 6.8 follow simply from these properties of the bare coupling constant:

Energy relation. The kinetic and interaction terms \mathcal{T} and \mathcal{U} in the Hamiltonian density operator are given in Eqs. 6.70a and 6.70b. After inserting the expression in Eq. 6.74 for the bare coupling constant g_0 into \mathcal{U} , the sum of \mathcal{T} and \mathcal{U} can be expressed as the sum of two finite operators:

$$\mathcal{T} + \mathcal{U} = \left(\sum_{\sigma} \frac{1}{2m} \nabla \psi_{\sigma}^{\dagger} \cdot \nabla \psi_{\sigma}^{(\Lambda)} - \frac{\Lambda}{2\pi^2 m} \Phi^{\dagger} \Phi \right) + \frac{1}{4\pi m a} \Phi^{\dagger} \Phi. \quad (6.76)$$

By taking the expectation value of both sides of Eq. 6.76, integrating over space, and using the expression for C in Eq. 6.73, we obtain the energy relation in Eq. 6.6.

Adiabatic relation. According to the Feynman-Hellman theorem, the rate of change in the energy due to a change in the scattering length can be expressed in the form

$$\left(\frac{dE}{da} \right)_S = \int d^3 R \left\langle \frac{\partial \mathcal{H}}{\partial a} \right\rangle. \quad (6.77)$$

The Hamiltonian density \mathcal{H} depends on a only through the factor of g_0 in \mathcal{U} . Using the derivative of the bare coupling constant in Eq. 6.75, we obtain the derivative of the Hamiltonian density:

$$\frac{\partial \mathcal{H}}{\partial a} = \frac{1}{4\pi m a^2} \Phi^{\dagger} \Phi. \quad (6.78)$$

By inserting this into Eq. 6.77 and using the expression for C in Eq. 6.73, we obtain the adiabatic relation in Eq. 6.8.

The virial theorem in Eq. 6.73 and the pressure relation in Eq. 6.73 can be derived by combining renormalization with dimensional analysis. For the purposes of dimensional analysis, we can regard $\hbar = 1$ and m simply as conversion constants that allow any dimensionful quantity to be expressed as a length raised to an appropriate power.

Virial theorem. For a system in a harmonic trapping potential, the only parameters that an energy eigenvalue can depend on are the scattering length a and the angular frequency ω . The combinations with dimensions of length are a and $(m\omega)^{-1/2}$. Since an energy eigenvalue E has dimensions $1/(m \text{ length}^2)$, the constraint of dimensional analysis can be expressed as the requirement that a differential operator that counts the factors of length gives -2 when acting on $E = \int d^3R \langle \mathcal{H} \rangle$:

$$\left(a \frac{\partial}{\partial a} - 2\omega \frac{\partial}{\partial \omega} \right) \int d^3R \langle \mathcal{H} \rangle = -2E. \quad (6.79)$$

Using the Feynman-Hellman theorem, this equation can be written

$$\int d^3R \left(\frac{1}{4\pi m a} \langle \Phi^\dagger \Phi \rangle - 4 \langle \mathcal{V} \rangle \right) = -2E. \quad (6.80)$$

Using the expression for C in Eq. 6.73, we obtain the virial theorem in Eq. 6.11.

Pressure relation. For a homogeneous system, the only variables that the free energy density $\mathcal{F} = \mathcal{E} - TS$ can depend on are the scattering length a , the temperature T , and the number densities n_i . The combinations with dimensions of length are a , $(mk_B T)^{-1/2}$, and $n_i^{-1/3}$. Since \mathcal{F} has dimensions $1/(m \text{ length}^5)$, the constraint of dimensional analysis can be expressed as the requirement that a differential operator that counts the factors of length gives -5 when acting on \mathcal{F} :

$$\left(a \frac{\partial}{\partial a} - 2T \frac{\partial}{\partial T} - 3n_1 \frac{\partial}{\partial n_1} - 3n_2 \frac{\partial}{\partial n_2} \right) \mathcal{F} = -5\mathcal{F}. \quad (6.81)$$

Using the adiabatic relation in Eq. 6.8, this can be written

$$\frac{1}{4\pi m a} C + 2TS - 3\mu_1 n_1 - 3\mu_2 n_2 = -5\mathcal{F}, \quad (6.82)$$

where μ_i is the chemical potential for the spin state i . The pressure relation in Eq. 6.12 then follows from the thermodynamic identity $\mathcal{F} = -\mathcal{P} + \mu_1 n_1 + \mu_2 n_2$.

6.6.4.3 The Operator Product Expansion

Many universal relations are most concisely derived using the *operator product expansion* (OPE) of quantum field theory. The OPE was invented independently

in 1969 by three giants of theoretical physics: Leo Kadanoff [43], Alexander Polyakov [44], and Ken Wilson [45]. It is an expansion for the product of local operators at nearby points in terms of local operators at the same point:

$$O_A \left(\mathbf{R} + \frac{1}{2} \mathbf{r} \right) O_B \left(\mathbf{R} - \frac{1}{2} \mathbf{r} \right) = \sum_C f_{A,B}^C(\mathbf{r}) O_C(\mathbf{R}). \quad (6.83)$$

The *Wilson coefficients* $f_{A,B}^C(\mathbf{r})$ are ordinary functions of the separation vector \mathbf{r} . The local operators $O_C(\mathbf{R})$ include some that can be obtained by Taylor-expanding the operators on the left side in powers of \mathbf{r} , but they also include additional operators that take into account effects from quantum fluctuations. The Wilson coefficients for these operators are not necessarily analytic functions of the vector \mathbf{r} , and they can even diverge as $\mathbf{r} \rightarrow 0$. One particularly simple local operator is the unit operator I , whose expectation value in any state is 1. The sum over C in Eq. 6.83 can be extended to include all local operators if we allow Wilson coefficients that are 0.

The OPE is a natural tool for generating universal relations, because it is an operator identity. By taking the expectation value of both sides of the OPE in Eq. 6.83 in some state of the system, we get an expression for the correlator of the operators O_A and O_B in terms of the expectation values of local operators in that state. Since this expansion holds for any state of the system, it is a universal relation.

A local operator $O_C(\mathbf{R})$ is assigned the *scaling dimension* d_C if the correlation function of O_C and its hermitian conjugate at points separated by r decreases asymptotically as $1/r^{2d_C}$ at small r . The unit operator I is assigned scaling dimension 0. In a weakly-interacting theory, the scaling dimensions can be obtained simply by dimensional analysis. In a strongly-interacting theory, they can be significantly different. The difference between the scaling dimension and its value in the corresponding non-interacting theory is called the *anomalous dimension*. At very small \mathbf{r} , the leading behavior of a nontrivial Wilson coefficient is determined by the scaling dimensions of the operators:

$$f_{A,B}^C(\mathbf{r}) \sim r^{d_C - d_A - d_B}. \quad (6.84)$$

In the OPE in Eq. 6.83, the Wilson coefficients of higher dimension operators go to 0 more rapidly as $\mathbf{r} \rightarrow 0$. A Wilson coefficient can be suppressed by a further power of r if there is an explicitly broken symmetry which, if exact, would require $f_{A,B}^C(\mathbf{r})$ to vanish. The extra suppression factor is the dimensionless combination of r and the symmetry breaking parameter.

The technical assumptions required to prove the OPE have been discussed by Wilson and Zimmerman [46]. The OPE can be expressed more precisely as an asymptotic expansion in the separation $r = |\mathbf{r}|$. The OPE in Eq. 6.83 can be organized into an expansion in powers of r by expanding the Wilson coefficients as Laurent series in r . The OPE is an asymptotic expansion if for any power p , there are only finitely many terms that go to zero more slowly than r^p . The scaling behavior of the Wilson coefficients in Eq. 6.84 guarantees that the OPE is an asymptotic expansion provided

every local operator $O_C(\mathbf{R})$ has a positive scaling dimension $d_C > 0$ and there are only finitely many local operators with scaling dimension $d_C < d$ for any positive number d . These conditions are satisfied in the Zero-Range Model, the Resonance Model, and other renormalizable local quantum field theories that are relevant to cold atoms.

An illustration of the operator product expansion with anomalous dimensions is provided by the Ising Model in 2 dimensions. The exact solution by Lars Onsager in 1944 [47] implies that correlation functions in the continuum limit have scaling behavior with anomalous dimensions. For example, the leading term in the correlation function for two spin operators σ as their separation \mathbf{r} goes to 0 has the form

$$\left\langle \sigma\left(\mathbf{R} + \frac{1}{2}\mathbf{r}\right)\sigma\left(\mathbf{R} - \frac{1}{2}\mathbf{r}\right) \right\rangle \longrightarrow \frac{A}{|\mathbf{r}|^{1/4}}, \quad (6.85)$$

where A is a constant that does not depend on the state of the system. This correlator is singular as $r \rightarrow 0$. The power law behavior suggests that the system becomes scale invariant at short distances. The fractional power of r indicates that the spin operator has an anomalous dimension.

Kadanoff generalized the result for the correlator in Eq. 6.85 to an operator relation [43]:

$$\sigma\left(\mathbf{R} + \frac{1}{2}\mathbf{r}\right)\sigma\left(\mathbf{R} - \frac{1}{2}\mathbf{r}\right) = \frac{A}{|\mathbf{r}|^{1/4}}I + B|\mathbf{r}|^{3/4}\varepsilon(\mathbf{R}) + \dots, \quad (6.86)$$

where I is the identity operator, $\varepsilon(\mathbf{R})$ is the energy fluctuation operator, and B is another constant. The infinitely many terms that are not shown explicitly in Eq. 6.86 go to 0 more rapidly than $r^{3/4}$ as $r \rightarrow 0$. The short-distance tail of the correlator in Eq. 6.85 can be obtained simply by taking the expectation value of Eq. 6.86. Kadanoff showed that the critical exponents of the Ising model, such as the exponent $\frac{1}{4}$ of $1/|\mathbf{r}|$ in Eq. 6.85, could be deduced simply from the knowledge of which operator products have singular Wilson coefficients. From the OPE in Eq. 6.86, we can deduce that the spin operator has scaling dimension $\frac{1}{8}$ and the energy fluctuation operator has scaling dimension 1.

The OPE in Eq. 6.83 can be described more precisely as a *short-distance operator product expansion*, because it involves operators at the same time with small spatial separation. It can be generalized to a *short-time operator product expansion*, in which the operators also have a small separation in time:

$$\mathcal{O}_A\left(\mathbf{R} + \frac{1}{2}\mathbf{r}, T + \frac{1}{2}t\right)\mathcal{O}_B\left(\mathbf{R} - \frac{1}{2}\mathbf{r}, T - \frac{1}{2}t\right) = \sum_C f_{A,B}^C(\mathbf{r}, t)\mathcal{O}_C(\mathbf{R}, T). \quad (6.87)$$

The Wilson coefficients $f_{A,B}^C(\mathbf{r}, t)$ are functions of the separation vector \mathbf{r} and the time interval t . The short-time OPE is more subtle than the short-distance OPE because of the possibility that a correlator can have oscillatory behavior in t as $t \rightarrow 0$ [46]. The possibility of oscillatory behavior is avoided in the Euclidean version of the quantum field theory that corresponds to analytic continuation of

the time t to Euclidean time: $t \rightarrow -i\tau$. Thus the short-time OPE in Eq. 6.87 can be expressed most rigorously as an asymptotic expansion in $(\mathbf{r}, -i\tau)$, where τ is the Euclidean time separation obtained by the analytic continuation $t \rightarrow -i\tau$. In a Galilean-invariant theory, the appropriate scaling of $(\mathbf{r}, -i\tau)$ is such that τ scales in the same way as $|\mathbf{r}|^2$.

6.6.4.4 Short-Distance Operator Product Expansion

Universal relations for fermions with large scattering length can be derived from operator product expansions in the Zero-Range Model defined in Sect. 6.6.4.1. In this model, the scattering length a is the only length scale that arises from interactions. At distances much smaller than $|a|$, the model is scale invariant with nontrivial scaling dimensions. In the unitary limit $a \rightarrow \pm\infty$, the model is not only scale invariant at all distances but also conformally invariant [48]. If a is finite, we can regard $1/a$ as the symmetry-breaking parameter associated with the broken conformal symmetry.

The fundamental field operators ψ_1 and ψ_2 of the Zero-Range Model have the same scaling dimensions as in a noninteracting theory. However there are composite operators with anomalous scaling dimensions. The scaling dimension of an operator \mathcal{O}_C can be deduced from its propagator at large momentum \mathbf{k} , which in a Galilean-invariant theory scales as k^{2dc-5} . Since the propagator for ψ_1 or ψ_2 is simply $(\omega - k^2/2m)^{-1}$, these fields have scaling dimensions $\frac{3}{2}$. The scaling dimension of the number density operator $\psi_\sigma^\dagger \psi_\sigma$ is twice that of ψ_σ , which is 3. If there were no interactions, the scaling dimension of the composite operator $\psi_1 \psi_2$, or equivalently the diatom field operator Φ defined in Eq. 6.72, would also be 3. However the propagator for Φ is

$$\int dt e^{i\omega t} \int d^3 r e^{-i\mathbf{k}\cdot\mathbf{r}} \langle \Phi(\mathbf{r}, t) \Phi^\dagger(\mathbf{0}, 0) \rangle = \frac{-i4\pi m}{-1/a + \sqrt{-m(\omega - k^2/4m)}}. \quad (6.88)$$

We have dropped an additive constant that is independent of ω and \mathbf{k} , which could be removed by renormalization. Since the propagator in Eq. 6.88 scales as k^{-1} at large k , Φ has scaling dimension 2 and therefore anomalous dimension -1 . The scaling dimension of the contact density operator $\Phi^\dagger \Phi$ is twice that of Φ , which is 4.

The short-distance OPE can be used to derive the Tan relation for the tail of the momentum distribution in Eq. 6.2. The momentum distribution $n_\sigma(\mathbf{k})$ for atoms in the spin state σ can be expressed as

$$n_\sigma(\mathbf{k}) = \int d^3 R \int d^3 r e^{-i\mathbf{k}\cdot\mathbf{r}} \left\langle \psi_\sigma^\dagger \left(\mathbf{R} - \frac{1}{2}\mathbf{r} \right) \psi_\sigma \left(\mathbf{R} + \frac{1}{2}\mathbf{r} \right) \right\rangle. \quad (6.89)$$

The behavior at large \mathbf{k} is dominated by the small- \mathbf{r} region of the integral. We can therefore apply the OPE to the product of the operators ψ_σ^\dagger and ψ_σ . As shown in Ref. [5], the leading terms in the OPE are

$$\begin{aligned} & \psi_\sigma^\dagger \left(\mathbf{R} - \frac{1}{2} \mathbf{r} \right) \psi_\sigma \left(\mathbf{R} + \frac{1}{2} \mathbf{r} \right) \\ &= \psi_\sigma^\dagger \psi_\sigma(\mathbf{R}) + \frac{1}{2} \mathbf{r} \cdot \left[\psi_\sigma^\dagger \nabla \psi_\sigma(\mathbf{R}) - \nabla \psi_\sigma^\dagger \psi_\sigma(\mathbf{R}) \right] - \frac{r}{8\pi} \Phi^\dagger \Phi(\mathbf{R}) + \dots \quad (6.90) \end{aligned}$$

We have written explicitly all terms whose Wilson coefficients go to zero more slowly than r^2 as $r \rightarrow 0$. The first two terms on the right side of the OPE in Eq. 6.90 can be obtained by multiplying the Taylor expansions of the two operators. The third term arises from quantum fluctuations involving pairs of atoms with small separations. That its Wilson coefficient is proportional to r can be predicted from the scaling dimensions of the operators using Eq. 6.84. The coefficient of r can be calculated using diagrammatic methods described in Ref. [29]. Note that this Wilson coefficient is not an analytic function of the vector $\mathbf{r} = (x, y, z)$ at $\mathbf{r} = 0$, because it is proportional to $r = \sqrt{x^2 + y^2 + z^2}$. The expectation value of the OPE in Eq. 6.90 can be expressed as

$$\left\langle \psi_\sigma^\dagger \left(\mathbf{R} - \frac{1}{2} \mathbf{r} \right) \psi_\sigma \left(\mathbf{R} + \frac{1}{2} \mathbf{r} \right) \right\rangle = n_\sigma(\mathbf{R}) + i \mathbf{r} \cdot \mathcal{P}_\sigma(\mathbf{R}) - \frac{r}{8\pi} \mathcal{C}(\mathbf{R}) + \dots, \quad (6.91)$$

where \mathcal{P}_σ is the momentum density of atoms in the state σ . This form of the OPE was first written down by Tan [2].

Upon inserting the OPE in Eq. 6.90 into Eq. 6.89, the first two terms give a delta function in \mathbf{k} and the gradient of such a delta function. They correspond to contributions to $n_\sigma(\mathbf{k})$ that decrease at large \mathbf{k} faster than any power of k . In the third term, the Fourier transform of the Wilson coefficient at nonzero values of \mathbf{k} can be obtained from the identity

$$\int d^3 r e^{-i\mathbf{k} \cdot \mathbf{r}} r = -\frac{8\pi}{k^4}, \quad (6.92)$$

which can be derived by differentiating the Fourier transform of a $1/r$ potential. This term gives a power-law tail in the momentum distribution:

$$n_\sigma(\mathbf{k}) \longrightarrow \frac{1}{k^4} \int d^3 R \langle \Phi^\dagger \Phi(\mathbf{R}) \rangle. \quad (6.93)$$

Comparing with Eq. 6.2, we obtain the expression in Eq. 6.73 for the contact C in the Zero-Range Model. This verifies that the contact density operator is $\Phi^\dagger \Phi$, where Φ is the diatom field operator defined in Eq. 6.72.

The short-distance OPE can be used to derive the Tan relation in Eq. 6.7 for the density-density correlator at short distances [5]. The OPE for the number density operators $\psi_1^\dagger \psi_1$ and $\psi_2^\dagger \psi_2$ includes a term whose Wilson coefficient is singular as $\mathbf{r} \rightarrow 0$:

$$\psi_1^\dagger \psi_1 \left(\mathbf{R} + \frac{1}{2} \mathbf{r} \right) \psi_2^\dagger \psi_2 \left(\mathbf{R} - \frac{1}{2} \mathbf{r} \right) = \frac{1}{16\pi^2} \left(\frac{1}{r^2} - \frac{2}{ar} \right) \Phi^\dagger \Phi(\mathbf{R}) + \dots \quad (6.94)$$

All the other terms in the OPE are regular at $\mathbf{r} = 0$. Taking the expectation value of both sides, we get the Tan relation in Eq. 6.7.

The OPE in Eq. 6.94 can also be used to derive the universal relation for the static structure factor $S_{12}(q)$ in Eq. 6.40. That structure factor can be expressed as

$$S_{12}(q) = \int d^3 R \int d^3 r e^{-iq \cdot r} \langle \psi_1^\dagger \psi_1 \left(\mathbf{R} - \frac{1}{2} \mathbf{r} \right) \psi_2^\dagger \psi_2 \left(\mathbf{R} + \frac{1}{2} \mathbf{r} \right) \rangle. \quad (6.95)$$

According to the universal relation in Eq. 6.40, the high momentum tail has terms proportional to $1/q$ and $1/q^2$. They come from the singular terms proportional to $1/r^2$ and $1/r$ in the OPE in Eq. 6.94.

6.6.4.5 Short-Time operator product expansion

Other universal relations can be derived using the short-time operator product expansion in Eq. 6.87. We will illustrate the use of the short-time OPE by showing how it can be used to derive the universal relations for radio-frequency (rf) spectroscopy that were presented in Sect. 6.4.1.

The rf signal that causes a transition of an atom in spin state 2 into an atom in spin state 3 corresponds to the action of an operator $\psi_3^\dagger \psi_2(\mathbf{r}, t)$. The inclusive rate $\Gamma(\omega)$ for the production of atoms in state 3 can be expressed as

$$\Gamma(\omega) = \Omega^2 \text{Im} i \int dt e^{i(\omega+i\epsilon)t} \int d^3 R \int d^3 r \left\langle T \psi_2^\dagger \psi_3 \left(\mathbf{R} + \frac{1}{2} \mathbf{r}, t \right) \psi_3^\dagger \psi_2 \left(\mathbf{R} - \frac{1}{2} \mathbf{r}, 0 \right) \right\rangle. \quad (6.96)$$

The symbol T in the matrix element implies that the product of operators is time ordered. For a large frequency ω , the integrals are dominated by small Euclidean time intervals $t = -i\tau$ and by small separations \mathbf{r} . We can therefore apply the short-time OPE in Eq. 6.87 to the product of operators $\psi_2^\dagger \psi_3$ and $\psi_3^\dagger \psi_2$. The Wilson coefficients of the leading one-body operator $\psi_2^\dagger \psi_2$ and the leading two-body operator $\Phi^\dagger \Phi$ were determined in Ref. [22]:

$$\begin{aligned} & \int dt e^{i\omega t} \int d^3 r \psi_2^\dagger \psi_3 \left(\mathbf{R} + \frac{1}{2} \mathbf{r}, t \right) \psi_3^\dagger \psi_2 \left(\mathbf{R} - \frac{1}{2} \mathbf{r}, 0 \right) \\ &= \frac{i}{\omega} \psi_2^\dagger \psi_2(\mathbf{R}) + \frac{i(a_{12}^{-1} - a_{13}^{-1})[a_{12}^{-1} - \sqrt{-m\omega}]}{4\pi m\omega^2 [a_{13}^{-1} - \sqrt{-m\omega}]} \Phi^\dagger \Phi(\mathbf{R}) + \dots \end{aligned} \quad (6.97)$$

Upon inserting the OPE in Eq. 6.97 into Eq. 6.96, the $\psi_2^\dagger \psi_2$ term gives a delta function in ω . This corresponds to a contribution that decreases at large ω faster than any power. The leading contribution to the high-frequency tail comes from the $\Phi^\dagger \Phi$ term:

$$\Gamma(\omega) \longrightarrow \frac{\Omega^2 (a_{13}^{-1} - a_{12}^{-1})^2}{4\pi m\omega^2} \text{Im} \left(\frac{-1}{a_{13}^{-1} - \sqrt{-m(\omega + i\epsilon)}} \right) \int d^3 R \langle \Phi^\dagger \Phi(\mathbf{R}) \rangle. \quad (6.98)$$

This reduces to the universal relation for the high-frequency tail in Eq. 6.33.

Sum rules for the rf transition rate $\Gamma(\omega)$, such as those in Eqs. 6.31 and 6.32, can be derived by expressing the integral along the real ω axis as a contour integral in the complex ω plane that wraps around the real axis. If we allow for a general weighting function $f(\omega)$, the sum rule becomes

$$\int_{-\infty}^{\infty} d\omega f(\omega) \Gamma(\omega) = \Omega^2 \oint \frac{d\omega}{2\pi} f(\omega) \int dt e^{i\omega t} \int d^3 R \int d^3 r \times \left\langle T \psi_2^\dagger \psi_3 \left(\mathbf{R} + \frac{1}{2} \mathbf{r}, t \right) \psi_3^\dagger \psi_2 \left(\mathbf{R} - \frac{1}{2} \mathbf{r}, 0 \right) \right\rangle. \quad (6.99)$$

If that contour is deformed into a circle whose radius approaches infinity, then ω has a large imaginary part along most of the contour. This justifies the use of the short-time OPE in Eq. 6.97. The sum rule can then be derived by evaluating the contour integral along the circle at infinity for each of the Wilson coefficients. For $f(\omega) = 1$, the only nonzero contribution is from the $\psi_2^\dagger \psi_2$ term and the sum rule reduces to Eq. 6.31. For $f(\omega) = \omega$, the only nonzero contribution is from the $\Phi^\dagger \Phi$ term and the sum rule reduces to Eq. 6.32. The weighting function $f(\omega)$ does not need to be a polynomial in ω . The Lorentzian function $f(\omega) = [(\omega - \omega_0)^2 + \gamma^2]^{-1}$ gives a family of sum rules with two adjustable parameters that is less sensitive to range corrections than that in Eq. 6.32 [22].

There are other universal relations that can be derived using the short-time OPE. One example is the high-frequency behavior of the structure factor $S_{12}(\omega, q)$, such as the universal relation in Eq. 6.41. It can be derived from the short-time OPE for the number density operators $\psi_1^\dagger \psi_1$ and $\psi_2^\dagger \psi_2$.

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References

1. Braaten, E., Hammer, H.W.: Phys. Rept. **428**, 259 (2006). arXiv:cond-mat/0410417
2. Tan, S.: Ann. Phys. **323**, 2952 (2008). arXiv:cond-mat/0505200
3. Tan, S.: Ann. Phys. **323**, 2971 (2008). arXiv:cond-mat/0508320
4. Tan, S.: Ann. Phys. **323**, 2987 (2008). arXiv:0803.0841
5. Braaten, E., Platter, L.: Phys. Rev. Lett. **100**, 205301 (2008). arXiv:0803.1125
6. Thomas, J.E., Kinast, J., Turlapov, A.: Rev. Lett. **95**, 120402 (2005). arXiv:cond-mat/0503620
7. Blume, D., Daily, K.M.: Phys. Rev. A **80**, 053626 (2009). arXiv:0909.2701
8. Chang, S.Y., Pandharipande, V.R., Carlson, J., Schmidt, K.E.: Phys. Rev. A **70**, 043602 (2004). arXiv:physics/0404115
9. Astrakharchik, G.E., Boronat, J., Casulleras, J., Giorgini, S.: Phys. Rev. Lett. **93**, 200404 (2004). arXiv:cond-mat/0406113

10. Lobo, C., Carusotto, I., Giorgini, S., Recati, A., Stringari, S.: Phys. Rev. Lett. **97**, 100405 (2006). arXiv:cond-mat/0604282
11. Petrov, D.S., Salomon, C., Shlyapnikov, G.V.: Phys. Rev. Lett. **93**, 090404 (2004). arXiv:cond-mat/0309010
12. Haussmann, R., Punk, M., Zwerger, W.: Phys. Rev. A **80**, 063612 (2009). arXiv:0904.1333
13. Yu, Z., Bruun, G.M., Baym, G.: Phys. Rev. A **80**, 023615 (2009). arXiv:0905.1836
14. Palestini, F., Perali, A., Pieri, P., Strinati, G.C.: Phys. Rev. A **82**, 021605(R) (2010). arXiv:1005.1158
15. Prokof'ev, N., Svistunov, B.: Phys. Rev. B **77**, 020408(R) (2008). arXiv:0707.4259
16. Punk, M., Dumitrescu, P.T., Zwerger, W.: Phys. Rev. A **80**, 053605 (2009). arXiv:0908.1343
17. Yu, Z., Baym, G.: Phys. Rev. A **73**, 063601 (2006). arXiv:cond-mat/0510675
18. Zhang, S., Leggett, A.J.: Phys. Rev. A **77**, 033614 (2008)
19. Punk, M., Zwerger, W.: Phys. Rev. Lett. **99**, 170404 (2007). arXiv:0707.0792
20. Baym, G., Pethick, C.J., Yu, Z., Zwierlein, M.W.: Phys. Rev. Lett. **99**, 190407 (2007). arXiv:0707.0859
21. Zwierlein, M.W., Hadzibabic, Z., Gupta, S., Ketterle, W.: Phys. Rev. Lett. **91**, 250404 (2003). arXiv:cond-mat/0306627
22. Braaten, E., Kang, D., Platter, L.: Phys. Rev. Lett. **104**, 223004 (2010). arXiv:1001.4518
23. Schneider, W., Shenoy, V.B., Randeria, M.: Theory of radio frequency spectroscopy of polarized Fermi gases. arXiv:0903.3006
24. Schneider, W., Randeria, M.: Phys. Rev. A **81**, 021601(R) (2010). arXiv:0910.2693
25. Pieri, P., Perali, A., Strinati, G.C.: Nat. Phys. **5**, 736 (2009). arXiv:0811.0770
26. Werner, F.: Phys. Rev. A **78**, 025601 (2008). arXiv:0803.3277
27. Werner, F., Tarruell, L., Castin, Y.: Phys. J. B **68**, 401 (2009). arXiv:0807.0078
28. Zhang, S., Leggett, A.J.: Phys. Rev. A **79**, 023601 (2009). arXiv:0809.1892
29. Braaten, E., Kang, D., Platter, L.: Phys. Rev. A **78**, 053606 (2008). arXiv:0806.2277
30. Combescot, R., Giorgini, S., Stringari, S.: Europhys. Lett. **75**, 695 (2006). arXiv:cond-mat/0512048
31. Hu, H., Liu, X.-J., Drummond, P.D.: Europhys. Lett. **91**, 20005 (2010). arXiv:1003.3511
32. Son, D.T., Thompson, E.G.: Phys. Rev. A **81**, 063634 (2010). arXiv:1002.0922
33. Taylor, E., Randeria, M.: Phys. Rev. A **81**, 053610 (2010). arXiv:1002.0869
34. Enss, T., Haussmann, R., Zwerger, W.: Ann. Phys. **326**, 770 (2011). arXiv:1008.0007
35. Partridge, G.B., Strecker, K.E., Kamar, R.I., Jack, M.W., Hulet, R.G.: Phys. Rev. Lett. **95**, 020404 (2005). arXiv:cond-mat/0505353
36. Kuhnle, E., Veeravalli, G., Dyke, P., Vale, C.J.: Phys. Rev. Lett. **101**, 250403 (2008). arXiv:0809.2145
37. Hu, H., Kuhnle, E.D., Liu, X.-J., Dyke, P., Mark, M., Drummond, P.D., Hannaford, P., Vale, C.J.: Phys. Rev. Lett. **105**, 070402 (2010). arXiv:1001.3200
38. Stewart, J.T., Gaebler, J.P., Drake, T.E., Jin, D.S.: Phys. Rev. Lett. **104**, 235301 (2010). arXiv:1002.1987
39. Navon, N., Nascimbène, S., Chevy, F., Salomon, C.: Science **328**, 729 (2010). arXiv:1004.1465
40. Werner, F., Castin, Y.: Exact relations for quantum-mechanical few-body and many-body problems with short-range interactions in two and three dimensions. arXiv:1001.0774
41. Tan, S.: S-wave contact interaction problem: a simple description. arXiv:cond-mat/0505615
42. Combescot, R., Alzetto, F., Leyronas, X.: Phys. Rev. A **79**, 053640 (2009). arXiv:0901.4303
43. Kadanoff, L.P.: Phys. Rev. Lett. **23** 1430 (1969)
44. Wilson, K.G., Zimmermann, W.: Comm. Math. Phys. **24**, 87 (1972)
45. Onsager, L.: Phys. Rev. **65**, 117 (1944)
46. Son, D.T., Wingate, M.: Ann. Phys. **321**, 197 (2006). arXiv:cond-mat/0509786