Argumentation in the View of Modal Logic

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Abstract. The paper presents a study of abstract argumentation theory from the point of view of modal logic. The key thesis upon which the paper builds is that argumentation frameworks can be studied as Kripke frames. This simple observation allows us to import a number of techniques and results from modal logic to argumentation theory, and opens up new interesting avenues for further research. The paper gives a glimpse of the sort of techniques that can be imported, discussing complete calculi for argumentation, adequate model-checking and bisimulation games, and sketches an agenda for future research at the interface of modal logic and argumentation theory.

Keywords: Argumentation theory, modal logic.

1 Introduction

The paper advocates a perspective on abstract argumentation theory based on techniques and results borrowed from the field of formal logic and, in particular, of modal logic [3]. First steps in this line of research have been moved in [12] and [13]. The present paper recapitulates some of the results presented in those works and sketches a number of theoretical problems arising at the interface of logic and argumentation which constitute, in our view, an interesting and challanging agenda for future research in both disciplines.

The key point of the paper is that standard results in argumentation theory obtain elegant reformulations within well-investigated modal logics. Once this link is established a number of techniques (e.g., calculi, logical games), as well as results related to those techniques (e.g. completeness, adequacy), can be easily imported from modal logic to argumentation theory.

The paper presupposes some familiarity with both modal logic and abstract argumentation theory. Proofs are omitted for space reasons. The interested reader is referred to [12,13].

Outline of the paper. Section 2 starts off by applying a well-known modal logic to study a first set of notions of argumentation theory. This enables the possibility of using calculi to derive argumentation-theoretic results such as the Fundamental Lemma [7]. Along the same line, Section 3 tackles the formalization of the notion of grounded extension within the modal μ-calculus. In Section 4 semantic games are studied for the logic introduced in Section 2 which provide a version of

P. McBurney, I. Rahwan, and S. Parsons (Eds.): ArgMAS 2010, LNAI 6614, pp. 190–208, 2011.

Table 1. Basic notions of argumentation theory (X denotes a set of arguments)

$\mathsf{c}_{\mathcal{A}}$ is the characteristic function of $\mathcal{A}$	iff $\mathsf{c}_{\mathcal{A}} : 2^A \longrightarrow 2^A$ s.t.
	$\mathsf{c}_{\mathcal{A}}(X) = \{a \mid \forall b : [b \rightarrow a \Rightarrow \exists c \in X : c \rightarrow b]\}$
X is acceptable w.r.t. Y in $\mathcal{A}$	iff $X \subseteq \mathsf{c}_{\mathcal{A}}(Y)$
X is conflict-free in $\mathcal{A}$	iff $\nexists a, b \in X$ s.t. $a \rightarrow b$
X is admissible in $\mathcal{A}$	iff X is conflict-free and $X \subseteq \mathsf{c}_{\mathcal{A}}(X)$
	iff X is a post-fixpoint of $\mathsf{c}_{\mathcal{A}}$
X is a complete extension of $\mathcal{A}$	iff X is conflict-free and $X = \mathsf{c}_{\mathcal{A}}(X)$
	(X is a conflict-free fixpoint of $\mathsf{c}_{\mathcal{A}}$)
X is a stable extension of $\mathcal{A}$	iff X is a complete extension of $\mathcal{A}$
	and $\forall b \notin X, \exists a \in X : a \rightarrow b$
	iff $X = \{a \in A \mid \nexists b \in X : b \rightarrow a\}$
X is the grounded extension of $\mathcal{A}$	iff X is the minimal complete extension of $\mathcal{A}$
	iff X is the least fixpoint of $\mathsf{c}_{\mathcal{A}}$
X is a preferred extension of $\mathcal{A}$	iff X is a maximal complete extension of $\mathcal{A}$

games for argumentation by means of model-checking games. Section 5 tackles the question—not yet addressed in the literature—of when two arguments, or two argumentation frameworks, are equivalent from the point of view of argumentation theory. For this purpose the model-theoretic notion of bisimulation is introduced and bisimulation games are presented as a procedural method to check the 'behavioral equivalence' of two argumentation frameworks. Section 6 sketches some of the possible lines of research that we consider worth pursuing by applying logic-based methods to abstract argumentation. Section 7 briefly concludes.

2 Arguments in Modal Disguise

The section moves the first steps towards looking at argumentation frameworks as structures upon which to interpret modal languages.

2.1 Argumentation Frameworks

Let us start with the basic structures of argumentation theory [7].

Definition 1 (Argumentation frameworks). *An argumentation framework is a relational structure* $\mathcal{A} = (A, \rightarrow)$ *where* A *is a non-empty set of arguments,* *and* $\rightarrow \subseteq A^2$ *is a so-called 'attack' relation on* A*. A pointed argumentation framework is a pair* $(\mathcal{A}, a)$ *with* $a \in A$*. The set of all argumentation frameworks is called* $\mathfrak{A}$*.*

The intuitive reading of "$a \rightarrow b$" is that argument a attacks argument b. Doing abstract argumentation theory means, essentially, to study specific properties of subsets of the set of arguments A in a given $\mathcal{A}$. For space reasons the paper cannot introduce argumentation theory in an extensive way but, to make it as most self-contained as possible, the main argumentation-theoretic notions from [7] have been recapitulated in Table 1. As such notions are formalized along the paper, their intuitive reading will also be provided.

The paper is based on the simple idea of viewing argumentation frameworks as the structures known in modal logic as Kripke frames, that is, structures (S, R) where S is taken to be a non-empty set of states, and R a binary relation on elements of S [3] . In essence, the paper studies what modal logic can say about argumentation frameworks when S is set to be A, i.e., the modal states are taken to be arguments, and R is set to be the inverse of the attack relation, that is, relation $\rightarrow^{-1}$. The entire paper and all its results hinge on this simple assumptions.

The reader might ask himself why R is taken to be the inverse $\rightarrow^{-1}$ of the attack relation instead of the attack relation $\rightarrow$ itself. This will become clear as the paper develops. However, a simple inspection of Table 1 should already show that all the key argumentation-theoretic notions can be defined in terms of the characteristic function, and that the characteristic function is defined by taking, for any argument a in the given input X, the set of attackers b of a—that is, the set of arguments by which *a is attacked*—for which there always exists another attacker c—that is, an argument by which the attacker of *a is attacked*. So, the characteristic function looks, for any argument, at whether its attackers are attacked. To put it in modal logic terms, the characteristic function is defined by looking at the tree-unraveling of $\rightarrow^{-1}$ at each point a, and not at the tree-unraveling of $\rightarrow$. We will come back to this issue in Section 3.1, now we proceed to the use of argumentation frameworks in a modal logic setting.

2.2 Argumentation Models

If an argumentation framework can be viewed as a Kripke frame, then an argumentation framework plus a function assigning names from a set $\mathbf{P}$ to sets of arguments can be viewed as a Kripke model [3].

Definition 2 (Argumentation models). *Let* $\mathbf{P}$ *be a set of propositional atoms. An argumentation model* $\mathcal{M} = (\mathcal{A}, \mathcal{I})$ *is a structure such that:* $\mathcal{A} = (A, \rightarrow)$ *is an argumentation framework;* $\mathcal{I} : \mathbf{P} \longrightarrow 2^A$ *is an assignment from* $\mathbf{P}$ *to subsets of* A*. The set of all argumentation models is called* $\mathfrak{M}$*. A pointed argumentation model is a pair* $(\mathcal{M}, a)$ *where* $\mathcal{M}$ *is an argumentation model and* a *an argument from* A*.*

Argumentation models are nothing but argumentation frames together with a way of 'naming' sets of arguments or, to put it otherwise, of 'labeling' arguments.

The fact that an argument a belongs to $\mathcal{I}(p)$ in a given model $\mathcal{M}$, which in logical notation reads $(\mathcal{A}, \mathcal{I}), a \models p$, can be interpreted as stating that "argument a has property p" , or that "p is true of a". By substituting p with a Boolean compound φ (e.g., $\varphi := p \wedge q$) we can say that "a belongs to both the sets called p and q", and the same can be done for all other Boolean connectives.

This much as to Boolean properties of arguments. But what about statements of the sort: "argument a is attacked by an argument in a set φ"; "argument a is defended by the set φ", or, "φ attacks an attacker of argument a"? These are modal statements, and in order to express them, it suffices to introduce a dedicated modal operator $\langle\leftarrow\rangle$ whose intuitive reading is "there exists an attacking argument such that". To this we turn in the next section.

2.3 Argumentation and Logic $\mathsf{K}^{\forall}$

This section introduces logic $\mathsf{K}^{\forall}$, an extension of the minimal modal logic K with universal modality. The section shows how such a simple and standard modal logic is already able of capturing quite a few argumentation-theoretic notions.

Language. The language of $\mathsf{K}^{\forall}$ is a standard modal language with two modalities: $\langle\leftarrow\rangle$ and $\langle\forall\rangle$, i.e., the universal modality. It is built on the set of atoms $\mathbf{P}$ by the following BNF:

$$\mathcal{L}^{\mathsf{K}^{\forall}} : \varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle\leftarrow\rangle\varphi \mid \langle\forall\rangle\varphi$$

where p ranges over $\mathbf{P}$. The other standard boolean $\{\top, \vee, \rightarrow\}$ and modal $\{[\leftarrow], [\forall]\}$ connectives are defined as usual.

Semantics

Definition 3 (Satisfaction). *Let $\varphi \in \mathcal{L}^{\mathsf{K}^{\forall}}$. The satisfaction of φ by a pointed argumentation model $(\mathcal{M}, a)$ is inductively defined as follows (Boolean clauses are omitted):*

$$\mathcal{M}, a \models \langle\leftarrow\rangle\varphi \quad \textit{iff} \quad \exists b \in A : (a, b) \in \rightarrow^{-1} \ \text{AND}\ \mathcal{M}, b \models \varphi$$
$$\mathcal{M}, a \models \langle\forall\rangle\varphi \quad \textit{iff} \quad \exists b \in A : \mathcal{M}, b \models \varphi$$

As usual, φ is valid in an argumentation model $\mathcal{M}$ iff it is satisfied in all pointed models of $\mathcal{M}$, i.e., $\mathcal{M} \models \varphi$. The truth-set of a formula φ is denoted $|\varphi|_{\mathcal{M}}$.

Logic $\mathsf{K}^{\forall}$ is therefore endowed with modal operators of the type "there exists an argument attacking the current one such that", i.e., $\langle\leftarrow\rangle$, and "there exists an argument such that", i.e., $\langle\forall\rangle$, together with their duals. Given an argumentation model $\mathcal{M}$ we can thereby express statements such as the ones adverted to above: "a is attacked by an argument in a set called φ" corresponds to $\langle\leftarrow\rangle\varphi$ being true in the pointed model $(\mathcal{M}, a)$ and "a is defended by the set φ" corresponds to $\langle\leftarrow\rangle\langle\leftarrow\rangle\varphi$ being true in the pointed model $(\mathcal{M}, a)$.

On the ground of this semantics, it becomes already clear that logic $\mathsf{K}^{\forall}$ is expressive enough to capture several basic notions of argumentation theory such as: conflict freeness, acceptability, admissibility, complete extensions, stable extensions.

$$Acc(\varphi, \psi) := [\forall](\varphi \rightarrow [\leftarrow]\langle\leftarrow\rangle\psi) \quad (1)$$
$$CFree(\varphi) := [\forall](\varphi \rightarrow [\leftarrow]\neg\varphi) \quad (2)$$
$$Adm(\varphi) := [\forall](\varphi \rightarrow ([\leftarrow]\neg\varphi \wedge [\leftarrow]\langle\leftarrow\rangle\varphi)) \quad (3)$$
$$Compl(\varphi) := [\forall]((\varphi \rightarrow [\leftarrow]\neg\varphi) \wedge (\varphi \leftrightarrow [\leftarrow]\langle\leftarrow\rangle\varphi)) \quad (4)$$
$$Stable(\varphi) := [\forall](\varphi \leftrightarrow [\leftarrow]\neg\varphi) \quad (5)$$

Intuitively, a set of arguments φ is acceptable with respect to the set of arguments ψ if and only all φ-arguments are such that for all their attackers there exists a defender in ψ (Formula 1). A set of arguments φ is conflict free if and only if all φ-arguments are such that none of their attackers is in φ (Formula 2). A set of arguments φ is admissible if and only if it is conflict free and acceptable with respect to itself (Formula 3). A set φ is a complete extension if and only if it is conflict free and it is equivalent to the set of arguments all the attackers of which are attacked by some φ-argument (Formula 4). Finally, a set φ is a stable extension if and only if it is equivalent to the set of arguments whose attackers are not in φ (Formula 5). The adequacy of these definitions with respect to the ones in Table 1 is easily checked.

Axiomatics. Logic $\mathsf{K}^{\forall}$ is axiomatized as follows, where $i \in \{\leftarrow, \forall\}$:

(**Prop**)	propositional tautologies
(**K**)	$[i](\varphi_1 \rightarrow \varphi_2) \rightarrow ([i]\varphi_1 \rightarrow [i]\varphi_2)$
(**T**)	$[\forall]\varphi \rightarrow \varphi$
(**4**)	$[\forall]\varphi \rightarrow [\forall][\forall]\varphi$
(**5**)	$\neg[\forall]\varphi \rightarrow [\forall]\neg[\forall]\varphi$
(**Incl**)	$[\forall]\varphi \rightarrow [i]\varphi$
(**Dual**)	$\langle i\rangle\varphi \leftrightarrow \neg[i]\neg\varphi$

The axiom system combines the axioms of logic K for the $[\leftarrow]$ operator, the axioms of logic $\mathsf{S5}$ for the universal operator $[\forall]$, and the interaction axiom **Incl**. It can be proven that this axiomatics is sound and strongly complete for the class $\mathfrak{A}$ of argumentation frames [3, Ch. 7].

The fact that $\mathsf{K}^{\forall}$ is axiomatized by the axioms and rules above gives us a calculus by means of which we can prove theorems of abstract argumentation theory in a purely formal manner. A notable example is the following generalized version of the fundamental lemma from [7], which states that if φ is admissible and both ψ and ξ are acceptable with respect to it, then also $\psi \vee \xi$ is admissible and ξ is acceptable with respect to $\varphi \vee \psi$.

Theorem 1 (Fundamental Lemma [7]). *The following formula is a theorem of* $\mathsf{K}^{\forall}$*:*

$$Adm(\varphi) \wedge Acc(\psi \vee \xi, \varphi) \rightarrow Adm(\varphi \vee \psi) \wedge Acc(\xi, \varphi \vee \psi) \quad (6)$$

The theorem could be proven semantically by then calling in completeness. However, to give a detailed example of an application of the above axiomatics, a formal derivation of the theorem is provided in the appendix.

Other examples of theorems of [7] that could be casted in this logic are, for instance: $Stable(\varphi) \rightarrow Adm(\varphi)$ and $Stable(\varphi) \rightarrow Compl(\varphi)$.

3 Modal Fixpoints

The present section shows what kind of modal machinery is needed to capture the notion of grounded extension left aside in Section 2. In [7], the grounded extension is defined as the smallest fixpoint of the characteristic function of an argumentation framework (see Table 1).

3.1 Characteristic Functions in Modal Logic

Each argumentation framework $\mathcal{A} = (A, \rightarrow)$ determines a *characteristic function* $c_{\mathcal{A}} : 2^A \longrightarrow 2^A$ such that for any set of arguments X, $c_{\mathcal{A}}(X)$ yields the set of arguments in A which are acceptable with respect to X, i.e., $\{a \in A \mid \forall b \in A : [b \rightarrow a \Rightarrow \exists c \in X : c \rightarrow b]\}$. Does logic $\mathsf{K}^{\forall}$ have a syntactic counterpart of the characteristic function? The answer turns out to be yes.

Let $\mathcal{L}^{[\leftarrow]\langle\leftarrow\rangle}$ be the language defined by the following BNF:

$$\mathcal{L}^{[\leftarrow]\langle\leftarrow\rangle} : \varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid [\leftarrow]\langle\leftarrow\rangle\varphi$$

where p belongs to the set of atoms $\mathbf{P}$. Language $\mathcal{L}^{[\leftarrow]\langle\leftarrow\rangle}$ is the fragment of $\mathcal{L}^{\mathsf{K}^{\forall}}$ containing only the compounded modal operator $[\leftarrow]\langle\leftarrow\rangle$ or, also, simply the fragment of $\mathcal{L}^{\mathsf{K}}$ (i.e., f $\mathcal{L}^{\mathsf{K}^{\forall}}$ without universal modality) containing only the $[\leftarrow]\langle\leftarrow\rangle$-operator. Let $\mathcal{A}^{+} = (2^A, \cap, -, \emptyset, c_{\mathcal{A}})$ be the power set algebra on 2^A extended with operator $c_{\mathcal{A}}$, and consider the term algebra $\mathfrak{ter}_{\mathcal{L}^{[\leftarrow]\langle\leftarrow\rangle}} = (\mathcal{L}^{[\leftarrow]\langle\leftarrow\rangle}, \wedge, \neg, \bot, [\leftarrow]\langle\leftarrow\rangle)$. Finally, let $\mathcal{I}^{*} : \mathcal{L}^{[\leftarrow]\langle\leftarrow\rangle} \longrightarrow 2^A$ be the inductive extension of a valuation function $\mathcal{I} : \mathbf{P} \longrightarrow 2^A$ according to the semantics given in Definition 3. We can prove the following result.

Theorem 2 ($c_{\mathcal{A}}$ vs. $[\leftarrow]\langle\leftarrow\rangle$). *Let $\mathcal{M} = (\mathcal{A}, \mathcal{I})$ be an argumentation model. Function $\mathcal{I}^{*}$ is a homomorphism from $\mathfrak{ter}_{\mathcal{L}^{[\leftarrow]\langle\leftarrow\rangle}}$ to $\mathcal{A}^{+}$.*

In other words, Theorem 2 shows that the complex modal operator $[\leftarrow]\langle\leftarrow\rangle$, under the semantics provided in Definition 3, behaves exactly like the characteristic function of the argumentation frameworks on which the argumentation models are built. To put it yet otherwise, formulae of the form $[\leftarrow]\langle\leftarrow\rangle\varphi$ denote the value of the characteristic function applied to the set φ of arguments. Notice also that from Theorem 2 the adequacy of Formulae 1-5 with respect to the definitions in Table 1 follows straightforwardly.

Characteristic functions are known to be monotonic [7] hence, by Theorem 2, we get that $[\leftarrow]\langle\leftarrow\rangle$ denotes a monotonic function and therefore, by the Knaster-Tarski theorem[1] we have that there always exist a greatest and a least $[\leftarrow]\langle\leftarrow\rangle$-fixpoint. From a logical point of view this means that, in order to be able to

[1] We refer the interested reader to [5].

express the grounded extension, it suffices to add to the K fragment of $\mathsf{K}^{\forall}$ a least fixpoint operator. This takes us to the realm of μ-calculi.

3.2 Argumentation and the μ-Calculus

Language. To add the least fixpoint operator μ to logic K we first define language $\mathcal{L}^{\mathsf{K}^{\mu}}$ via the following BNF:

$$\mathcal{L}^{\mathsf{K}^{\mu}} : \varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle\leftarrow\rangle\varphi \mid \mu p.\varphi(p)$$

where p ranges over $\mathbf{P}$ and $\varphi(p)$ indicates that p occurs free in φ (i.e., it is not bounded by fixpoint operators) and under an even number of negations.[2] In general, the notation $\varphi(\psi)$ stands for ψ occurs in φ. The usual definitions for Boolean and modal operators can be applied. Intuitively, $\mu p.\varphi(p)$ denotes the smallest formula p such that $p \leftrightarrow \varphi(p)$. This intuition is made precise in the semantics of $\mathcal{L}^{\mathsf{K}^{\mu}}$.

Semantics

Definition 4 (Satisfaction). *Let $\varphi \in \mathcal{L}^{\mathsf{K}^{\mu}}$. The satisfaction of φ by a pointed model $(\mathcal{M}, a)$, with $\mathcal{M} = (\mathcal{A}, \mathcal{I})$, is inductively defined as follows (Boolean clauses, as well as the clause for $\langle\leftarrow\rangle$, are as in Definition 3):*

$$\mathcal{M}, a \models \mu p.\varphi(p) \;\; \textit{iff} \;\; a \in \bigcap\{X \in 2^{A} \mid |\varphi|_{\mathcal{M}[p:=X]} \subseteq X\}$$

where $|\varphi|_{\mathcal{M}[p:=X]}$ denotes the truth-set of φ once $\mathcal{I}(p)$ is set to be X. As usual, we say that: φ is valid in an argumentation model $\mathcal{M}$ iff it is satisfied in all pointed models of $\mathcal{M}$, i.e., $\mathcal{M} \models \varphi$; φ is valid in a class $\mathfrak{M}$ of argumentation models iff it is valid in all its models, i.e., $\mathfrak{M} \models \varphi$.

We have now all the logical machinery in place to express the notion of grounded extension. Set $\varphi(p) := [\leftarrow]\langle\leftarrow\rangle p$, that is, take $\varphi(p)$ to be the modal version $[\leftarrow]\langle\leftarrow\rangle$ of the characteristic function, and apply it to formula p. What we obtain is a modal formula expressing the least fixpoint of a characteristic function, that is, the grounded extension:

$$\mathit{Grounded} := \mu p.[\leftarrow]\langle\leftarrow\rangle p \tag{7}$$

Notice that, unlike the notions formalized in Formulae 1-5, the grounded extension of a framework is always unique and does not depend on the particular labeling of a given model.

We refrain here from providing a sound and complete axiomatization of μ-calculus. The interested reader is referred to [19]. However, just like we did for logic $\mathsf{K}^{\forall}$ we give now a couple of examples of the kind of argumentation-theoretic results formalizable in K^{μ}. Well-known theorems of argumentation theory are provable formulae of the μ-calculus.

[2] This syntactic restriction guarantees that every formula $\varphi(p)$ defines a monotonic set transformation.

Theorem 3 (The grounded extension is conflict-free). *The following formula is a theorem of* K^{μ}*:*

$$Grounded \to [\leftarrow]\neg Grounded \tag{8}$$

We can also study complexity results from this modal perspective and, unsurprisingly, the results are in accordance with complexity studies in argumentation theory [8], although the proofs take different routes.

Theorem 4 (Model-checking grounded). *Let* $\mathcal{A}$ *be an argumentation framework. It can be decided in polynomial time whether an argument* a *belongs to the grounded extension of* $\mathcal{A}$*, that is, whether* $\mathcal{A}, a \models Grounded$*.*

4 Dialogue Games and Logic Games

The proof-theory of abstract argumentation is commonly given in terms of dialogue games [16]. The present section introduces a new game-theoretic proof procedure for argumentation theory based on model-checking games. In model-checking games, a proponent or verifier (∃ve) tries to prove that a given formula φ holds in a point a of a model $\mathcal{M}$, while an opponent or falsifier (∀dam) tries to disprove it. The present section deals with the model-checking game for $\mathsf{K}^{\forall}$. For the K^{μ}-variant of this game we refer the reader to [18].

4.1 Model-Checking Game for $\mathsf{K}^{\forall}$

A model-checking game is a *graph game*, that is, a game played by two agents on a directed graph, where each node—called position—is labelled by the player that is supposed to move next. The structure of the graph determines which are the *admissible moves* at any given position. If a player has to move in a certain position but there are no available moves, then it loses and its opponent wins. In general, graph games might have infinite paths, but this is not the case in the game we are going to introduce. A match of a graph game is then just the set of positions visited during play, that is, a complete path through the graph.

Definition 5 ($\mathsf{K}^{\forall}$-model-checking game). *Let* $\varphi \in \mathcal{L}^{\mathsf{K}^{\forall}}$*, and* $\mathcal{M}$ *be an argumentation model. The model-checking game* $\mathcal{C}(\varphi, \mathcal{M})$ *is defined by the following items.* **Players:** *The set of players is* $\{\exists, \forall\}$*. An element from* $\{\exists, \forall\}$ *will be denoted* P *and its opponent* $\overline{P}$*.* **Game form:** *The* game form *of* $\mathcal{C}(\varphi, \mathcal{M})$ *is defined by the board game in Table 2.* **Winning conditions:** *Player* P *wins if and only if* $\overline{P}$ *has to play in a position with no available moves.* **Instantiation:** *The* instance *of* $\mathcal{C}(\varphi, \mathcal{M})$ *with starting point* (φ, a) *is denoted* $\mathcal{C}(\varphi, \mathcal{M})@(\varphi, a)$*.*

The important thing to notice is that positions of the game are pairs of a formula and an argument, and that the type of formula in the position determines which player has to play: ∃ if the formula is a disjunction, a diamond, a false atom or ⊥, and ∀ in the remaining cases.[3]

We now define what it means to have a winning strategy and to be in a winning position in this type of games.

[3] Notice that positions use formulae in positive normal form.

Table 2. Rules of the model-checking game for $\mathsf{K}^{\forall}$

Position	Turn	Available moves
$(\varphi_1 \vee \varphi_2, a)$	$\exists$	$\{(\varphi_1, a), (\varphi_2, a)\}$
$(\varphi_1 \wedge \varphi_2, a)$	$\forall$	$\{(\varphi_1, a), (\varphi_2, a)\}$
$(\langle \leftarrow \rangle \varphi, a)$	$\exists$	$\{(\varphi, b) \mid (a, b) \in \rightarrow^{-1}\}$
$([\leftarrow]\varphi, a)$	$\forall$	$\{(\varphi, b) \mid (a, b) \in \rightarrow^{-1}\}$
$(\langle \forall \rangle \varphi, a)$	$\exists$	$\{(\varphi, b) \mid b \in A\}$
$([\forall]\varphi, a)$	$\forall$	$\{(\varphi, b) \mid b \in A\}$
$(\bot, a)$	$\exists$	$\emptyset$
$(\top, a)$	$\forall$	$\emptyset$
(p, a) & $a \notin \mathcal{I}(p)$	$\exists$	$\emptyset$
(p, a) & $a \in \mathcal{I}(p)$	$\forall$	$\emptyset$
$(\neg p, a)$ & $a \in \mathcal{I}(p)$	$\exists$	$\emptyset$
$(\neg p, a)$ & $a \notin \mathcal{I}(p)$	$\forall$	$\emptyset$

Definition 6 (Winning strategies and positions). *A strategy for player P in $\mathcal{C}(\varphi, \mathcal{M})@(\varphi, a)$ is a function telling P what to do in any match played from position (φ, a). Such a strategy is* winning for *P if and only if, in any match played according to the strategy, P wins. A position (φ, a) in $\mathcal{C}(\varphi, \mathcal{M})$ is* winning for *P if and only if P has a winning strategy in $\mathcal{C}(\varphi, \mathcal{M})@(\varphi, a)$. The set of winning positions of $\mathcal{C}(\varphi, \mathcal{M})$ is denoted $Win_P(\mathcal{C}(\varphi, \mathcal{M}))$.*

By Definitions 4.2 and 6 it follows that the model-checking game is a two-players zero-sum game with perfect information. It is known that such games are determined, that is, each match has a winner [20].

These games can be proven adequate. This means that if ∃ve has a winning strategy then the formula defining the game is true at the point of instantiation and, vice versa, that if a formula is true at a point in a model, then ∃ve has a winning strategy in the corresponding game instantiated at that point.

Theorem 5 (Adequacy). *Let $\varphi \in \mathcal{L}^{\mathsf{K}^{\forall}}$, and let $\mathcal{M} = (\mathcal{A}, \mathcal{I})$ be an argumentation model. Then, for all $a \in A$:*

$$(\varphi, a) \in Win_{\exists}(\mathcal{C}(\varphi, \mathcal{M})) \Longleftrightarrow \mathcal{M}, a \models \varphi.$$

4.2 Games for Model-Checking Extensions

The next example illustrates a model-checking game for stable extensions run on the so-called Nixon diamond [16].

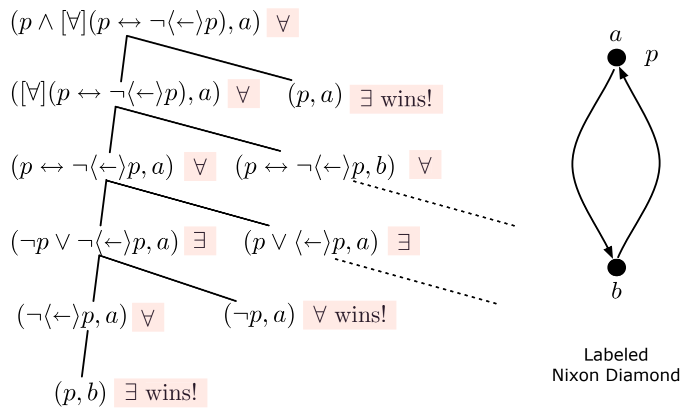

Fig. 1. Game for stable extensions in the 2-cycle with labeling (valuation) function

Example 1 (Model-checking the Nixon diamond). Let $\mathcal{A} = (\{a, b\}, \{(a, b), (b, a)\})$ be an argumentation framework consisting of two arguments a and b attacking each other (i.e., the Nixon diamond), and consider the labeling $\mathcal{I}$ assigning p to a and $\neg p$ to b (top right corner of Figure 1). We now want to run an evaluation game for checking whether a belongs to a stable extension corresponding to the truth-set of p. Such game is the game $\mathcal{C}(p \wedge \mathit{Stable}(p), (\mathcal{A}, \mathcal{I}))$ initialized at position $(p \wedge \mathit{Stable}(p), a)$. That is, spelling out the definition of $\mathit{Stable}(p)$: $\mathcal{C}(p \wedge [\forall](p \leftrightarrow \neg\langle\leftarrow\rangle p))@(p \wedge [\forall](p \leftrightarrow \neg\langle\leftarrow\rangle p), a)$. Such a game, played according to the rules in Definitions 4.2 and 6, gives rise to the tree in Figure 1.

In general, model-checking games provide a proof procedure for checking whether an argument belongs to a certain extension given an argumentation model. What must be noted is that the structure of such proof procedure is invariant, and the different games are obtained simply by choosing the right formula to be checked (Table 3).[4] This feature confers a high systematic flavor to this sort of games.

Now the natural question arises of what the precise relationship is between model-checking games and the sort of games studied in argumentation, sometimes called dialogue games [16,14]. The difference is as follows.

In model-checking games you are given a model $\mathcal{M} = (\mathcal{A}, \mathcal{I})$, a formula φ and an argument a, and ∃ve is asked to prove that $\mathcal{M}, a \models \varphi$. In dialogue games, the check appointed to ∃ve is inherently more complex since the input consists only of an argumentation framework $\mathcal{A}$, a formula φ and an argument a. ∃ve is then asked to prove one of the two following things:

– that there exists a labeling function $\mathcal{I}$ such that $(\mathcal{A}, \mathcal{I}), a \models \varphi$ (the so-called *credulous* semantics for φ);
– that for all the labeling functions $\mathcal{I}$, $(\mathcal{A}, \mathcal{I}), a \models \varphi$ (the so-called *skeptical* semantics for φ).

[4] Note that the game for checking grounded extensions is, obviously, the model-checking game for K^{μ} [18].

Table 3. Games for model-checking extensions in argumentation models

Adm :	$\mathcal{E}(\varphi \wedge Adm(\varphi), \mathcal{M})@(\varphi \wedge Adm(\varphi), a)$
$Complete$:	$\mathcal{E}(\varphi \wedge Complete(\varphi), \mathcal{M})@(\varphi \wedge Complete(\varphi), a)$
$Stable$:	$\mathcal{E}(\varphi \wedge Stable(\varphi)), \mathcal{M})@(\varphi \wedge Stable(\varphi), a)$
$Grounded$:	$\mathcal{E}(Grounded, \mathcal{M})@(Grounded, a)$

These are not a model-checking problems, but satisfiability problems in a pointed frame [3] which, in turn, are essentially model-checking problems in some fragment of monadic second-order logic. That is the problem of checking, given a frame $\mathcal{A}$ and an argument a, whether the following is the case:

$$\mathcal{A} \models \exists p_1, \dots, p_n ST_x(\varphi)[a]$$
$$\mathcal{A} \models \forall p_1, \dots, p_n ST_x(\varphi)[a]$$

where $p_1, \dots, p_n$ are the atoms occurring in φ and $ST_x(\varphi)[a]$ is the standard translation of φ realized in state a.[5]

To conclude, we might say that the games defined in Section 4.1 provide a proof procedure for a reasoning task which is computationally simpler than the one tackled by standard dialogue games. It should be noted, however, that this is no intrinsic limitation to the logic-based approach advocated in the present paper. Model-checking games for monadic second-order logic (or rather for appropriate fragments of it) would be able to perform the sort of tasks demanded in dialogue games and do that in the same systematic manner of modal model-checking games. We will come back to this issue in Section 6.

5 Equivalent Arguments

Since abstract argumentation neglects the internal structure of arguments, the natural question arises of when two arguments can be said to be equivalent. Such a notion of equivalence will necessarily be of a structural nature. The study of a notion of equivalence for argumentation has not received attention yet by the argumentation theory community, except for one recent notable exception [15], which defines a notion of strong equivalence for argumentation frameworks, borrowed from the analogous notion developed in logic programming.

Modal logic offers a readily available notion of structural equivalence, the notion of bisimulation (with all its variants) [3,11]. This section sketches the use of bisimulation for argumentation theoretic purposes. To illustrate the issue we use a simple motivating example depicted in Figure 2. We have two labelled argumentation frameworks which both contain an argument labeled p which is attacked by some arguments labelled q. Now the question would be: are the two p-arguments equivalent as far as abstract argumentation theory is concerned? The answer is yes, and the next sections explain why.

[5] For the definition of the standard translation we refer the reader to [3].

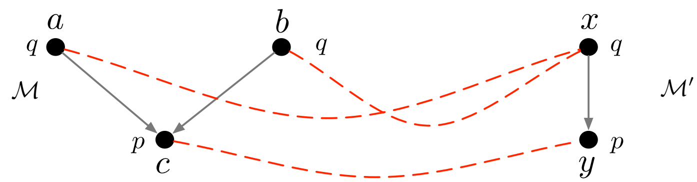

Fig. 2. Two (totally) bisimilar arguments (c and y) in two argumentation models

5.1 Bisimilar Arguments

t is well-known that logic K^{μ} is invariant under bisimulation [18]. In the present section we will focus on the specific notion of bisimulation which is tailored to $\mathsf{K}^{\forall}$, also called *total bisimulation.*

Definition 7 (Bisimulation). *Let $\mathcal{M} = (A, \rightarrow, \mathcal{I})$ and $\mathcal{M}' = (A', \rightarrow', \mathcal{I}')$ be two argumentation models. A bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ is a non-empty relation $Z \subseteq A \times A'$ such that for any a, a' s.t. aZa':* **Atom:** *a and a' are propositionally equivalent;* **Zig:** *if $a \rightarrow^{-1} b$ for some $b \in A$, then $a' \rightarrow^{-1} lb'$ for some $b' \in A'$ and bZb';* **Zag:** *if $a' \rightarrow^{-1} b'$ for some $b' \in A$ then $a \rightarrow^{-1} b$ for some $\underline{i}nA$ and aZa'. A* total bisimulation *is a bisimulation $Z \subseteq A \times A'$ such that its left projection covers A and its right projection covers A'. When a total bisimulation exists between $\mathcal{M}$ and $\mathcal{M}'$ we write $(\mathcal{M}, a) \Leftrightarrow (\mathcal{M}', a')$.*

Now, since logic $\mathsf{K}^{\forall}$ is invariant under total bisimulation [3] and logic K^{μ} under bisimulation [11], we obtain a natural notion of equivalence of arguments, which is weaker than the notion of isomorphism of argumentation frameworks. If two arguments are equivalent in this perspective, then they are equivalent from the point of view of argumentation theory, as far as the notions expressible in those logics are concerned. In particular, we obtain the following simple theorem.

Theorem 6 (Bisimilar arguments). *Let $(\mathcal{M}, a)$ and $(\mathcal{M}', a')$ be two pointed models, and let Z be a total bisimulation between $\mathcal{M}$ and $\mathcal{M}'$. It holds that:*

$$\mathcal{M}, a \models \mathit{Adm}(\varphi) \wedge \varphi \Longleftrightarrow \mathcal{M}', a' \models \mathit{Adm}(\varphi) \wedge \varphi$$
$$\mathcal{M}, a \models \mathit{CFree}(\varphi) \wedge \varphi \Longleftrightarrow \mathcal{M}', a' \models \mathit{CFree}(\varphi) \wedge \varphi$$
$$\mathcal{M}, a \models \mathit{Compl}(\varphi) \wedge \varphi \Longleftrightarrow \mathcal{M}', a' \models \mathit{Compl}(\varphi) \wedge \varphi$$
$$\mathcal{M}, a \models \mathit{Stable}(\varphi) \wedge \varphi \Longleftrightarrow \mathcal{M}', a' \models \mathit{Stable}(\varphi) \wedge \varphi$$
$$\mathcal{M}, a \models \mathit{Grounded} \Longleftrightarrow \mathcal{M}', a' \models \mathit{Grounded}$$

In other words, Theorem 6 states that if two arguments are totally bisimilar, then they are indistinguishable from the point of view of abstract argumentation in the sense that the first belongs to a given conflict-free, or admissible set φ if and only if also the second does, and the first belongs to a given stable, complete extension φ, or to the grounded extension, if and only if also the second does. Arguments c and y in Figure 2 are totally bisimilar arguments.

Table 4. Rules of the bisimulation game

Position	Available moves
$((\mathcal{M},a)(\mathcal{M}',a'))$	$\{((\mathcal{M},a)(\mathcal{M}',b')) \mid \exists b' \in A' : a' \leftarrow b'\}$
	$\cup\{((\mathcal{M},b)(\mathcal{M}',a')) \mid \exists b \in A : a \leftarrow b\}$
	$\cup\{((\mathcal{M},a)(\mathcal{M}',b')) \mid \exists b' \in A'\}$
	$\cup\{((\mathcal{M},b)(\mathcal{M}',a')) \mid \exists b \in A\}$

5.2 Total Bisimulation Games

We can associate a game to Definition 7. Such game checks whether two given pointed models $(\mathcal{M}, a)$ and $(\mathcal{M}', a')$ are bisimular or not. The game is played by two players: **S**poiler, which tries to show that the two given pointed models are not bisimilar, and **D**uplicator which pursues the opposite goal. A match is started by **S**, then **D** responds, and so on. If and only if **D** moves to a position where the two pointed models are not propositionally equivalent, or if it cannot move any more, **S** wins.

Definition 8 (Total bisimulation game). *Take two pointed models $\mathcal{M}$ and $\mathcal{M}'$. The* total bisimulation game $\mathcal{B}(\mathcal{M}, \mathcal{M}')$ *is defined by the following items.* **Players:** *The set of players is $\{\mathbf{D}, \mathbf{S}\}$. An element from $\{\mathbf{D}, \mathbf{S}\}$ will be denoted P and its opponent $\overline{P}$.* **Game form:** *The* game form *of $\mathcal{B}(\mathcal{M}, \mathcal{M}')$ is defined by Table 4.* **Turn function:** *If the round is even* **S** *plays, if it is odd* **D** *plays.* **Winning conditions:** **S** *wins if and only if either D has moved to a position $((\mathcal{M}, a)(\mathcal{M}', a'))$ where a and a' do not satisfy the same labels, or* **D** *has no available moves. Otherwise* **D** *wins.* **Instantiation:** *The* instance *of $\mathcal{B}(\mathcal{M}, \mathcal{M}')$ with starting position $((\mathcal{M}, a)(\mathcal{M}', a'))$ is denoted $\mathcal{B}(\mathcal{M}, \mathcal{M}')@(a, a')$.*

So, as we might expect, positions in a (total) bisimulation game are pairs of pointed models, that is, the pointed models that **D** tries to show are bisimilar. It might also be instructive to notice that such a game can have infinite matches, which, according to Definition 8, are won by **D**.

From Definition 8 we obtain the following notions of winning strategies and winning positions.

Definition 9 (Winning strategies and positions). *A strategy for player P in $\mathcal{B}(\mathcal{M}, \mathcal{M}')@(a, a')$ is a function telling P what to do in any match played from position (a, a'). Such a strategy is* winning for *P if and only if, in any match played according to the strategy, P wins. A position $((\mathcal{M}, a)(\mathcal{M}', a'))$ in $\mathcal{B}(\mathcal{M}, \mathcal{M}')$ is* winning for *P if and only if P has a winning strategy in $\mathcal{B}(\mathcal{M}, \mathcal{M}')@(a, a')$. The set of all winning positions of game $\mathcal{B}(\mathcal{M}, \mathcal{M}')$ for P is denoted by $Win_P(\mathcal{B}(\mathcal{M}, \mathcal{M}'))$.*

We have the following adequacy theorem. The proof is standard and the reader is referred to [11].

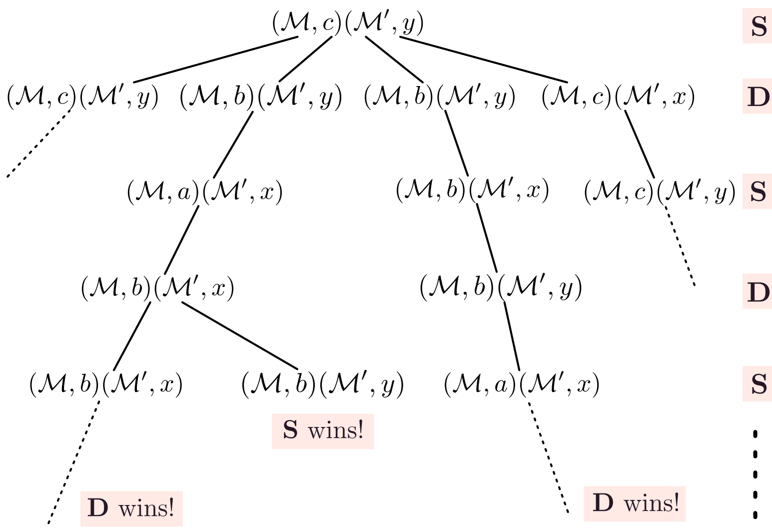

Fig. 3. Part of the total bisimulation game played on the models in Figure 2

Theorem 7 (Adequacy). *Take* $(\mathcal{M}, a)$ *and* $(\mathcal{M}', a')$ *to be two argumentation models. It holds that:*

$$((\mathcal{M}, a)(\mathcal{M}', a')) \in Win_{\mathbf{D}}(\mathcal{B}(\mathcal{M}, \mathcal{M}')) \Longleftrightarrow (\mathcal{M}, a) \leftrightarrows (\mathcal{M}', a').$$

In words, **D** has a winning strategy in the game $\mathcal{B}(\mathcal{M}, \mathcal{M}')@(a, a')$ if and only if $\mathcal{M}, a$ and $\mathcal{M}', a'$ are totally bisimilar. An example of such a game follows.

Example 2 (A total bisimulation game). Let us play a total bisimulation game on the two models $\mathcal{M}$ and $\mathcal{M}'$ given in Figure 2. A total bisimulation connects c with y, and a and b with x. Part of the extensive bisimulation game $\mathcal{B}(\mathcal{M}, \mathcal{M}')@(c, y)$ is depicted in Figure 3. Notice that **D** wins on those infinite paths where it can always duplicate **S**'s moves. On the other hand, it looses for instance when it replies to one of **S**'s moves $((\mathcal{M}, b)(\mathcal{M}', x))$ by moving in the second model to argument y, which is labelled p while b is not.

6 A Research Agenda between Logic and Argumentation

By recapitulating results presented in [12,13], the paper has given a glimpse of the sort of results that can be obtained about abstract argumentation theory by resorting to quite standard methods and techniques of modal logic. The present section proposes an agenda for this line of research which, in the author's view, is of definite interest for a deeper mathematical understanding of abstract argumentation theory.

6.1 Other Extension-Based Semantics in Modal Logic

The paper has left aside one key notion of argumentation: preferred extensions. In [7], preferred extensions are defined as maximal, with respect to set-inclusion, complete extensions. The natural question is whether the logics we have introduced are expressive enough to capture also this notion.

Technically, this means looking for a formula $\varphi(p)$ such that for any pointed model $\mathcal{M} = ((\mathcal{A}, \mathcal{I}), a)$ $\mathcal{M}, a \models \varphi(p)$ iff $a \in |p|_{\mathcal{M}}$ and $|p|_{\mathcal{M}}$ is a preferred extension of $\mathcal{A}$, where $p \in \mathbf{P}$. It is easy to see that such $\varphi(p)$ can be expressed in monadic second-order logic with a Π^1_1 quantification:

$$p \wedge ST_x(Compl(p)) \wedge \forall q(ST_x(Compl(q)) \rightarrow \neg(p \sqsubset q)) \quad (9)$$

where $ST_x(Compl(p))$ denotes the standard translation [3] of the $\mathsf{K}^{\forall}$ formula for complete extensions (Formula 4) and $q \sqsubseteq p$ means just that $|q|_{\mathcal{M}} \subseteq |p|_{\mathcal{M}}$, i.e., the truth set of q is included in the truth-set of p. The same question of representability within (possibly extended) modal languages can be posed for other types of extensions, such as the semi-stable one [4].

6.2 A Unified Game-Theoretic Proof-Theory for Argumentation

Section 4.2 has shown how model-checking games can be used to provide a form of game-theoretic proof theory to check the membership of a given argument to a given extension. Although Section 4.2 has then pointed out how these games differ from the standard dialogue games studied in argumentation theory, it is our thesis that a suitable extension of the expressivity of the modal languages used in this paper can offer a unified game-theoretic proof-theory for argumentation.

For instance, the question whether there exists, given a pointed frame $(\mathcal{A}, a)$, a stable extension of $\mathcal{A}$ containing a could be phrased as the model checking of a formula of the extension of $\mathsf{K}^{\forall}$ with second order quantification limited to alternation depth 1:[6]

$$\mathcal{A}, a \models \exists p.(Stable(p) \wedge p).$$

The prospect of an extension of this type is to provide each argumentation-theoretic notion with a game-theoretic proof-theory (both for its skepical and credulous versions) which would directly follow from the model-checking game of the underlying logic. We would thereby obtain also games for extensions which have not yet found a game-theoretic proof-theory in the literature on abstract argumentation theory such as, for instance, skeptical and credulous stable extensions, or skeptical preferred extensions.

6.3 Equivalence in Argumentation

Another original application of modal logic which could open up new venues for research is the study of invariance, or equivalence, in argumentation theory. Section 5 has shown how to tackle the question of when two labelled argumentation

[6] A well-known logical language for this purpose could be second order propositional modal logic [9].

frameworks can be considered equivalent, by looking at the existence of a (total) bisimulation relation between them.

The key observation in this case is that, depending on the features we consider relevant for the comparison of two argumentation frameworks, different modal languages can be chosen, which come with their characteristic notion of bisimulation, i.e., structural equivalence. For instance, if we were to compare two argumentation frameworks by considering, as a relevant property for the comparison, also the number of attackers, then the two arguments considered equivalent in Figure 2 would cease to be such, as the first one has two attackers, while the second has only one.

The modal language with the sort of expressivity necessary to 'count' the number of attackers of a given argument is called *graded modal logic* [10]. In such a language it becomes possible to say that:

$$\mathcal{A}, a \models \Diamond_2 \top$$

that is, a has at least two attackers. Going back to the example given in Figure 2, while argument c in the first framework satisfies $\Diamond_2 \top$, argument y in the second does not. In modal logic terms this implies that c and y are not bisimilar with respect to the language of graded modal logic or, put it otherwise, they are not *graded bisimilar* [17]. So, mapping all the relevant modal languages for argumentation theory would automatically provide a whole landscape of different equivalence notions which can be used to compare argumentation frameworks.

6.4 Argumentation Dynamics

The whole of abstract argumentation theory is built on structures—the argumentation frameworks—which are essentially static. To date, no theory has yet been systematically developed about how to modify argumentation frameworks by operations of addition and deletion of arguments and links.

The link with modal logic could offer again a wealth of techniques, stemming from dynamic logic [6,2], which might prove themselves useful for the development of such a theory of argumentation dynamics. The possibly simplest example in this line is sabotage modal logic [1], where formulae of the type:

$$(\mathcal{A}, \mathcal{I}), a \models \blacksquare \varphi$$

express, in an argumentation-theoretic reading, that after any possible removal of an attack relation, argument a still belongs to the truth-set of φ.

7 Conclusions

The paper has shown how rather standard modal logics—the extensions of K with universal modality and least fixpoint operator—can be applied to argumentation theory in an almost direct way. Both these logics come equipped with complete calculi, in which, therefore, theorems from argumentation theory can be formally derived, with model-checking games, which can be used to provide a game-theoretic proof-theory on argumentation models, and with characteristic notions of structural equivalence (bisimulation) which can be used to provide a formalization of notions of equivalence for argumentation frameworks.

We have concluded by pointing at several directions for future work, ranging from the problem of the formalization of preferred extensions, to second-order model checking games, to the study of argumentation equivalence via bisimulation, and to argumentation dynamics.

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Appendix: A Formal Proof of the *Fundamental Lemma*

1.	$\varphi \to \varphi \vee \psi$	**Prop**
2.	$\langle\leftarrow\rangle\varphi \to \langle\leftarrow\rangle(\varphi \vee \psi)$	1, K − derived rule
3.	$[\leftarrow]\langle\leftarrow\rangle\varphi \to [\leftarrow]\langle\leftarrow\rangle(\varphi \vee \psi)$	2, K − derived rule
4.	$(\alpha \vee \beta \to \gamma) \to (\beta \to \gamma)$	**Prop**
5.	$(\psi \vee \xi \to [\leftarrow]\langle\leftarrow\rangle\varphi) \to (\xi \to [\leftarrow]\langle\leftarrow\rangle\varphi)$	4, instance
6.	$(\psi \vee \xi \to [\leftarrow]\langle\leftarrow\rangle\varphi) \to (\xi \to [\leftarrow]\langle\leftarrow\rangle\varphi \vee \psi)$	5, 3, **Prop**, **MP**
7.	$[\forall](\psi \vee \xi \to [\leftarrow]\langle\leftarrow\rangle\varphi) \to [\forall](\xi \to [\leftarrow]\langle\leftarrow\rangle\varphi \vee \psi)$	6, K − derived rule
8.	$Acc(\psi \vee \xi, \varphi) \to Acc(\xi, \varphi \vee \psi)$	7, definition
9.	$(\psi \vee \xi \to [\leftarrow]\langle\leftarrow\rangle\varphi) \to (\psi \to [\leftarrow]\langle\leftarrow\rangle\varphi)$	4, instance
10.	$[\forall](\psi \vee \xi \to [\leftarrow]\langle\leftarrow\rangle\varphi) \to [\forall](\psi \to [\leftarrow]\langle\leftarrow\rangle\varphi)$	9, K − derived rule
11.	$Acc(\psi \vee \xi, \varphi) \to Acc(\psi, \varphi)$	10, definition
12.	$((\alpha \to \gamma) \wedge (\beta \to \gamma)) \to (\alpha \vee \beta \to \gamma)$	**Prop**
13.	$([\forall](\alpha \to \gamma) \wedge [\forall](\beta \to \gamma)) \to [\forall](\alpha \vee \beta \to \gamma)$	12, **N**, **K**, **MP**
14.	$([\forall](\varphi \to [\leftarrow]\langle\leftarrow\rangle\varphi) \wedge [\forall](\psi \to [\leftarrow]\langle\leftarrow\rangle\varphi)) \to [\forall](\varphi \vee \psi \to [\leftarrow]\langle\leftarrow\rangle\varphi)$	13, Instance
15.	$[\leftarrow]\langle\leftarrow\rangle\varphi \to [\leftarrow]\langle\leftarrow\rangle(\varphi \vee \psi)$	14, **Prop**, **K**, **N**
16.	$([\forall](\varphi \to [\leftarrow]\langle\leftarrow\rangle\varphi) \wedge [\forall](\psi \to [\leftarrow]\langle\leftarrow\rangle\varphi)) \to [\forall](\varphi \vee \psi \to [\leftarrow]\langle\leftarrow\rangle\varphi \vee \psi)$	15, **Prop**, **K**, **N**
17.	$Acc(\varphi, \varphi) \wedge Acc(\psi, \varphi) \to Acc(\varphi \vee \psi, \varphi \vee \psi)$	16, definition
18.	$Acc(\varphi, \varphi) \wedge Acc(\psi \vee \xi, \varphi) \to Acc(\varphi \vee \psi, \varphi \vee \psi)$	17, 9, **Prop**, **MP**
19.	$[\forall](\langle\leftarrow\rangle\varphi \to \neg\varphi) \to [\leftarrow](\langle\leftarrow\rangle\varphi \to \neg\varphi)$	**Incl**

20.	$[\forall]((\langle\leftarrow\rangle\varphi \to \neg\varphi) \to ([\leftarrow]\langle\leftarrow\rangle\varphi \to [\leftarrow]\neg\varphi)$	19, **Prop**, **MP**
21.	$[\forall][\forall]((\langle\leftarrow\rangle\varphi \to \neg\varphi) \to [\forall]([\leftarrow]\langle\leftarrow\rangle\varphi \to [\leftarrow]\neg\varphi)$	20, K − derived rule
22.	$[\forall]((\langle\leftarrow\rangle\varphi \to \neg\varphi) \to [\forall]([\leftarrow]\langle\leftarrow\rangle\varphi \to [\leftarrow]\neg\varphi)$	21, $\mathsf{S5}$ − derived rule
23.	$[\forall]((\langle\leftarrow\rangle\varphi \to \neg\varphi) \wedge [\forall](\varphi \vee \psi \to [\leftarrow]\langle\leftarrow\rangle\varphi)$	
	$\to [\forall](\varphi \vee \psi \to [\leftarrow]\langle\leftarrow\rangle\varphi) \wedge [\forall]([\leftarrow]\langle\leftarrow\rangle\varphi \to [\leftarrow]\neg\varphi)$	22, **Prop**, **MP**
24.	$[\forall]((\langle\leftarrow\rangle\varphi \to \neg\varphi) \wedge [\forall](\varphi \vee \psi \to [\leftarrow]\langle\leftarrow\rangle\varphi) \to [\forall](\varphi \vee \psi \to [\leftarrow]\neg\varphi)$	23, **Prop**, **MP**
25.	$[\forall]((\langle\leftarrow\rangle\varphi \to \neg\varphi \wedge \neg\psi) \to [\leftarrow]((\langle\leftarrow\rangle\varphi \to \neg\varphi \wedge \neg\psi)$	**Incl**
26.	$[\forall]((\langle\leftarrow\rangle\varphi \to \neg\varphi \wedge \neg\psi) \to ([\leftarrow]\langle\leftarrow\rangle\varphi \to [\leftarrow]\neg\varphi \wedge \neg\psi)$	25, **K**, **Prop**, **MP**
27.	$[\forall]((\langle\leftarrow\rangle\varphi \to \neg\varphi \wedge \neg\psi) \to [\forall]([\leftarrow]\langle\leftarrow\rangle\varphi \to [\leftarrow]\neg\varphi \wedge \neg\psi)$	26, $\mathsf{S5}$ − derived rule
28.	$[\forall]((\langle\leftarrow\rangle\varphi \to \neg\varphi) \wedge [\forall](\varphi \vee \psi \to [\leftarrow]\langle\leftarrow\rangle\varphi)$	
	$\to [\forall]([\leftarrow]\langle\leftarrow\rangle\varphi \to [\leftarrow]\neg\varphi \wedge \neg\psi)$	24, 27, **Prop**, **MP**
29.	$[\forall]((\langle\leftarrow\rangle\varphi \to \neg\varphi) \wedge [\forall](\varphi \vee \psi \to [\leftarrow]\langle\leftarrow\rangle\varphi)$	
	$\to [\forall](\varphi \vee \psi \to [\leftarrow]\langle\leftarrow\rangle\varphi) \wedge [\forall]([\leftarrow]\langle\leftarrow\rangle\varphi \to [\leftarrow]\neg\varphi \wedge \neg\psi)$	28, **Prop**, **MP**
30.	$[\forall](\alpha \to \beta) \wedge [\forall](\beta \to \gamma) \to [\forall](\alpha \to \gamma)$	$\mathsf{S5}$ − theorem
31.	$[\forall]([\leftarrow]\langle\leftarrow\rangle\varphi \to [\leftarrow](\neg\varphi \wedge \neg\psi)) \wedge [\forall](\varphi \vee \psi \to [\leftarrow]\langle\leftarrow\rangle\varphi)$	
	$\to [\forall](\varphi \vee \psi \to [\leftarrow](\neg\varphi \wedge \neg\psi))$	30, instance
32.	$[\forall]((\langle\leftarrow\rangle\varphi \to \neg\varphi) \wedge [\forall](\varphi \vee \psi \to [\leftarrow]\langle\leftarrow\rangle\varphi)$	
	$\to [\forall](\varphi \vee \psi \to [\leftarrow](\neg\varphi \wedge \neg\psi))$	29, 31, **Prop**, **MP**
33.	$\mathit{CFree}(\varphi) \wedge \mathit{Acc}(\varphi \vee \psi, \varphi) \to \mathit{CFree}(\varphi \vee \psi)$	32, definition
34.	$\mathit{Acc}(\varphi, \varphi) \wedge \mathit{Acc}(\psi, \varphi) \to \mathit{Acc}(\varphi \vee \psi, \varphi)$	14, definition
35.	$\mathit{CFree}(\varphi) \wedge \mathit{Acc}(\varphi, \varphi) \wedge \mathit{Acc}(\psi, \varphi) \to \mathit{CFree}(\varphi \vee \psi)$	33, 34, **Prop**, **MP**
36.	$\mathit{CFree}(\varphi) \wedge \mathit{Acc}(\varphi, \varphi) \wedge \mathit{Acc}(\psi \vee \xi, \varphi) \to \mathit{CFree}(\varphi \vee \psi)$	35, 9, **Prop**, **MP**
37.	$\mathit{CFree}(\varphi) \wedge \mathit{Acc}(\varphi, \varphi) \wedge \mathit{Acc}(\psi \vee \xi, \varphi)$	
	$\to \mathit{CFree}(\varphi \vee \psi) \wedge \mathit{Acc}(\varphi \vee \psi, \varphi \vee \psi)$	36, 18, **Prop**, **MP**
38.	$\mathit{CFree}(\varphi) \wedge \mathit{Acc}(\varphi, \varphi) \wedge \mathit{Acc}(\psi \vee \xi, \varphi)$	
	$\to \mathit{CFree}(\varphi \vee \psi) \wedge \mathit{Acc}(\varphi \vee \psi, \varphi \vee \psi) \wedge \mathit{Acc}(\xi, \varphi \vee \psi)$	37, 8, **Prop**, **MP**
39.	$\mathit{Adm}(\varphi) \wedge \mathit{Acc}(\psi \vee \xi, \varphi) \to \mathit{Adm}(\varphi \vee \psi)\mathit{Acc}(\xi, \varphi \vee \psi)$	38, definition