Recursion Formulas in Determining Isochronicity of a Cubic Reversible System

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Abstract. In this paper, we present a new method to determining the isochronicity of a reversible cubic system based on the recursion formulas and mathematical induction. Unlike the method, which proves the isochronicity of the same system with the integrability, our new method gives the recursion formulas among the period coefficients firstly. Then, based on these recursion formulas and using mathematical induction, the isochronicity is proved.

Keywords: reversible cubic system, recursion formula, weak center, isochronous center, mathematical induction.

1 Introduction

The qualitative theory of differential equations is a very important branch of dynamic system since it can provide qualitative information for the dynamic system.

In 1989, Chicone and Jacobs put forward the concept, weak center, which can answer how many critical periods bifurcate from the center [1]. However, because of the computation complexity, only quadratic systems and some special cubic systems were discussed.

Recently, some efforts for reversible cubic systems were done based on computer algebra [2]-[6]. The main motivation for these methods is that computer algebra can help compute period coefficients of the systems.

Unfortunately, determining isochronicity is a challenge for most of systems since it has to show all period coefficients are equal to zeros [7]-[12]. One feasible alternative for this difficulty is studying the recursion relations among coefficients of the system and proving the isochronicity with the induction algorithm based on the relations.

In order to show the importance of our method, one origin of a specified system proposed by W Zhang etc, whose isochronicity is proved in [6], is considered.

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Unlike the method proposed by W Zhang [6], our new method presents recursion formulas among the period coefficients firstly. Then, based on these recursion formulas and using mathematical induction, the isochronicity is proved.

The remainder of this paper is as follows. Section 2 is devoted to the theory of weak centers. In section 3 some conclusions for weak centers of a reversible cubic system are given. In section 4 we prove the origin of the reversible cubic system is an isochronous center. We also give conclusions finally.

2 The Theory of Weak Centers

Let $V(x, y, \lambda)$ be a family of planar analytic vector fields parameterized by $\lambda \in R^n$ with a nondegenerate center at the origin, i. e., the vector field does not have an eigenvalue zero at the origin. $P(r, \lambda)$ denotes the minimum period of the closed orbit passing $(r, 0)$, a point in a sufficiently small open interval $J = (-\alpha, \alpha)$ on x axis.

Definition 1. *Let $F(r, \lambda_*) = P(r, \lambda_*) - P(0, \lambda_*)$. The origin is called a* weak center of finite order k *if*

$$F(0, \lambda_*) = F'(0, \lambda_*) = \cdots = F^{(2k+1)}(0, \lambda_*) = 0 \quad \textit{and} \quad F^{(2k+2)}(0, \lambda_*) \neq 0 \quad (1)$$

where the derivatives indicated are taken with respect to the first argument of the function F.

Definition 2. *The origin is called an* isochronous center*, i. e., all closed orbits surrounding the origin have the same period, that is, $F^{(k)}(0, \lambda_*) = 0, \forall k \geq 0$.*

Definition 3. Local critical period *is a period corresponding to a critical point of the period function which bifurcates from a weak center.*

Lemma 1 (Period Coefficient Lemma). *[1] If $P(0, \lambda) = 2\pi, \forall \lambda \in R^n$, then for any given $\lambda_* \in R^n$,*

$$P(r, \lambda) = 2\pi + \sum_{k=2}^{\infty} p_k(\lambda) r^k \quad (2)$$

which is analytic for $|r|$ and $|\lambda - \lambda_*|$ sufficiently small. Moreover, $p_k \in R[\lambda_1, \cdots, \lambda_n]$, the noetherian ring of polynomials; for $k \geq 1, p_{2k+1} \in (p_2, p_4, \cdots, p_{2k})$, the ideal generated by $p_{2i}, i = 1, \cdots, k$ over $R[\lambda_1, \cdots, \lambda_n]$; the first $k > 1$ such that $p_k(\lambda) \neq 0$ is even.

3 Reversible Cubic System

We consider the differential system

$$\begin{cases} \dot{x} = -y + \phi(x, y, \lambda) \\ \dot{y} = \quad x + \psi(x, y, \lambda) \end{cases} \quad (3)$$

where $\phi(0,0,\lambda) = \psi(0,0,\lambda) = 0, \forall\lambda \in R$.

Especially, for C^3, functions $\phi(x,y,\lambda)$ and $\psi(x,y,\lambda)$ are cubic systems

$$\begin{cases} \dot{x} = -y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 \\ \dot{y} = x + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 \end{cases} \tag{4}$$

Definition 4. *A planar vector field is said to be reversible if it is symmetric with respect to a line.*

Consider the cubic system Eq. 4 which is symmetric with respect to y axis. Clearly, such a reversible cubic system is

$$\begin{cases} \dot{x} = -y + a_1x^2 + a_2y^2 + a_3x^2y + a_4y^3 \\ \dot{y} = x + b_1xy + b_2x^3 + b_3xy^2 \end{cases} \tag{5}$$

with parameter $\lambda = (a_1, a_2, a_3, a_4, b_1, b_2, b_3) \in R^7$. This symmetry ensures that Eq. 5 has a center at the origin.

By Lemma 1 and Definition 1, we have

Theorem 1. *[1] If for a certain $\lambda_* \in R^7$, there is an integer $k \geq 1$ such that*

$$p_2(\lambda_*) = p_3(\lambda_*) = \cdots = p_{2k+1}(\lambda_*) = 0 \quad \textit{and} \quad p_{2k+2}(\lambda_*) \neq 0 \tag{6}$$

The origin is a weak center of order k. Otherwise, the origin is an isochronous center.

Deciding if an origin is an isochronous center using Theorem 1 or Definition 1, is a difficult problem since it needs compute all $F(r,\lambda_*)$s or p_ks. In this paper, we proposed a new method to determine if an origin is an isochronous center by presenting recursion formulas firstly, and then proving the origin is isochronous using these formulas.

Taking polar coordinate

$$x = r\cos\theta, \quad y = r\sin\theta \tag{7}$$

we have

$$\dot{r} = \dot{x}\cos\theta + \dot{y}\sin\theta = r^2G_2(\theta) + r^3G_3(\theta) \tag{8}$$

$$\dot{\theta} = (\dot{y}\cos\theta - \dot{x}\sin\theta)/r = 1 + rH_1(\theta) + r^2H_2(\theta) \tag{9}$$

where

$$\begin{aligned} G_2(\theta) &= a_1\cos^3\theta + (a_2 + b_1)\sin^2\theta\cos\theta \\ G_3(\theta) &= (a_3 + b_2)\cos^3\theta\sin\theta + (a_4 + b_3)\cos\theta\sin^3\theta \\ H_1(\theta) &= (b_1 - a_1)\cos^2\theta\sin\theta - a_2\sin^3\theta \\ H_2(\theta) &= (b_3 - a_3)\cos^2\theta\sin^2\theta + b_2\cos^4\theta - a_4\sin^4\theta \end{aligned} \tag{10}$$

Thus

$$\frac{dr}{d\theta} = \frac{r^2G_2(\theta) + r^3G_3(\theta)}{1 + rH_1(\theta) + r^2H_2(\theta)} \tag{11}$$

Lemma 2. *[6] The vector field defined by Eq. 11 is analytic and*

$$\frac{dr}{d\theta} = r^2 G_2 + \sum_{k=3}^{\infty} r^k (G_2 A_{k-2} + G_3 A_{k-3}) \tag{12}$$

in a sufficiently small neighborhood of $r = 0$*, where*

$$A_0 = 1, A_1 = -H_1, A_k = -H_2 A_{k-2} - H_1 A_{k-1}, \forall k \geq 3 \tag{13}$$

Consider the solution of Eq. 11 with $r(0, \lambda) = r_0 > 0$ in the form

$$r(\theta, \lambda) = \sum_{k=1}^{\infty} u_k(\theta, \lambda) r_0^k \tag{14}$$

The initial condition implies

$$u_1(0, \lambda) = 1, u_k(0, \lambda) = 0, \forall k > 1, \lambda \in R^n \tag{15}$$

Replacing r in Eq. 12 with the series Eq. 14 and comparing coefficients of $r_0^k, k = 1, 2, \cdots$, we get the following differential equations

$$\begin{array}{lcl} u_1' & = & 0 \\ u_2' & = & G_2 u_1^2 \\ u_3' & = & (G_2 A_1 + G_3) u_1^3 + 2 u_1 u_2 G_2 \\ \cdots\cdots\cdots\cdots\cdots \end{array} \tag{16}$$

where u_k' denotes $\frac{d}{d\theta} u_k(\theta, \lambda)$. Under the initial conditions in Eq. 15, we can obtain their solutions.

$$\begin{array}{lcl} u_1(\theta) & = & 1 \\ u_2(\theta) & = & \int_0^\theta G_2(\xi) d\xi \\ u_3(\theta) & = & \int_0^\theta (G_3 + G_2(2u_2 - H_1)) d\xi \\ \cdots\cdots\cdots\cdots\cdots \end{array} \tag{17}$$

Finally, we compute the period $P(r_0, \lambda)$ of the closed orbit $C(r_0)$ through $(r_0, 0)$ from Eq. 9 and

$$\frac{1}{1 + rH_1(\theta) + r^2(\theta)} = \sum_{k=3}^{\infty} r^k A_k \tag{18}$$

then

$$P(r_0, \lambda) = \int_{C(r_0)} dt$$

$$\begin{aligned}
&= \int_0^{2\pi} \frac{1}{1 + rH_1(\theta) + r^2 H_2(\theta)} d\theta \\
&= \int_0^{2\pi} (1 + \sum_{k=1}^{\infty} r^k A_k) d\theta \qquad (19) \\
&= 2\pi + \int_0^{2\pi} \sum_{k=1}^{\infty} r^k A_k d\theta
\end{aligned}$$

Meanwhile, from Eq. 14, we obtain the following power series expansion

$$\begin{aligned}
\textstyle\sum_{k=1}^{\infty} r^k A_k &= \textstyle\sum_{k=1}^{\infty} (\sum_{t=1}^{\infty} u_k r_0^t) A_k \\
&= A_1 u_1 r_0 + (A_1 u_2 + A_2 u_1^2) r_0^2 + (A_1 u_3 + 2A_2 u_1 u_2 + A_3 u_1^3) r_0^3 \qquad (20) \\
&\quad + (A_1 u_4 + A_2 (u_2^2 + 2u_1 u_3) + 3A_3 u_1^2 u_2 + A_4 u_1^4) r_0^4 + \cdots \\
&= p_1' r_0 + p_2' r_0^2 + p_3' r_0^3 + p_4' r_0^4 + \cdots
\end{aligned}$$

therefore

$$P(r_0, \lambda) = 2\pi + \sum_{k=1}^{\infty} P_k(\lambda) r_0^k \qquad (21)$$

where

$$\begin{aligned}
p_1(\lambda) &= \textstyle\int_0^{2\pi} p_1' d\theta = \int_0^{2\pi} A_1 u_1 d\theta = -\int_0^{2\pi} H_1(\theta) d\theta = 0 \\
p_2(\lambda) &= \textstyle\int_0^{2\pi} p_2' d\theta = \int_0^{2\pi} (A_1 u_2 + A_2 u_1^2) d\theta \\
p_3(\lambda) &= \textstyle\int_0^{2\pi} p_3' d\theta = \int_0^{2\pi} (A_1 u_3 + 2A_2 u_1 u_2 + A_3 u_1^3) d\theta \qquad (22) \\
p_4(\lambda) &= \textstyle\int_0^{2\pi} p_4' d\theta = \int_0^{2\pi} (A_1 u_4 + A_2 (u_2^2 + 2u_1 u_3) + 3A_3 u_1^2 u_2 + A_4 u_1^4) d\theta \\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

and A_k, u_k, $k = 1, 2, \cdots$ are determined by Eq. 13 and Eq. 17.

3.1 The Algorithm of the Reversible Cubic System

Summary the above procedures in this section, we have

1. Compute $G_2(\theta)$, $G_3(\theta)$, $H_1(\theta)$ and $H_2(\theta)$ using Eq. 10.
2. Compute $A_k, k = 1, 2, \cdots$ using Eq. 13.
3. Compute $\frac{dr}{d\theta}$ using Eq. 11.

4. Compute $u'_k(\theta), k = 1, 2, \cdots$ using Eq. 16 and $u_k(\theta), k = 1, 2, \cdots$ using Eq. 17 one by one.

5. Compute $p_k(\lambda), k = 1, 2, \cdots$ one by one using Eq. 22.

According to above procedures, $p_k(\lambda), k = 1, 2, \cdots$ can be computed one by one. Thus, if all p_ks are equal to zeros, the origin is an isochronous center. That is what we often do for weak focuses.

4 The New Method for Isochronicity

In this section, we analyze the weak center of one reversible cubic system. Consider the system in the following form

$$\begin{cases} \dot{x} = -y - ax^2 + ay^2 + cx^2y \\ \dot{y} = x - 2axy + cxy^2 \end{cases} \tag{23}$$

The isochronicity of the system has been proved by W Zhang in [6] with the integrability. Different from the method of W Zhang, we proposed a new method to prove the isochronicity based on recursion formulas and mathematical induction. In order to the consistency of this paper, the procedures of our method are explained one by one even for the W Zhang's overlapped parts.

4.1 The General Algorithm

Firstly, we compute coefficients about the system using the algorithm presented in Section 3.1.

1. Compute $G_2(\theta)$, $G_3(\theta)$, $H_1(\theta)$ and $H_2(\theta)$ using Eq. 10.

$$\begin{aligned} &G_2 = -a\text{cos}^3\theta + (a - 2a) = -a\text{cos}\theta \\ &H_1 = (-2a + a)\text{cos}^2\theta\text{sin}\theta - a\text{sin}^3\theta = -a\text{sin}\theta \\ &H_2 = 0 \\ &G_3 = (c + 0)\text{cos}^3\theta\text{sin}\theta + (0 + c)\text{cos}\theta\text{sin}^3\theta = c\text{cos}\theta\text{sin}\theta \end{aligned} \tag{24}$$

2. Compute $A_k, k = 1, 2, \cdots$ using Eq. 13.

$$A_0 = 1, A_k = -H_1 A_{k-1} \tag{25}$$

So

$$A_k = a^k\text{sin}^k\theta, \quad k = 0, 1, 2\cdots \tag{26}$$

3. Compute $\frac{dr}{d\theta}$ using Eq. 11.

$$\frac{dr}{d\theta} = \frac{G_2 A_{k-2} + G_3 A_{k-3}}{G_2 A_{k-3} + G_3 A_{k-4}} = a\text{sin}\theta \tag{27}$$

4. Compute $u'_k(\theta), k = 1, 2, \cdots$ using Eq. 16 and $u_k(\theta), k = 1, 2, \cdots$ using Eq. 17 one by one.

$$\begin{aligned}
u'_1 &= 0, & u_1 &= 1 \\
u'_2 &= G_2 u_1^2 = -a\cos\theta, & u_2 &= -a\sin\theta \\
u'_3 &= (G_2 A_1 + G_3)u_1^3 + 2u_1 u_2 G_2, & u_3 &= (\tfrac{1}{2}a^2 + \tfrac{1}{2}c)\sin^2\theta \\
&= (a^2 + c)\sin\theta\cos\theta \\
&\cdots\cdots\cdots\cdots & &\cdots\cdots\cdots\cdots
\end{aligned} \tag{28}$$

5. Compute $p_k(\lambda), k = 1, 2, \cdots$ using Eq. 22.

$$\begin{aligned}
p'_1 &= A_1(\theta)u_1(\theta) = a\sin\theta \\
p_1 &= \int_0^{2\pi} A_1 u_1 d\theta = 0 \\
p'_2 &= A_1(\theta)u_2(\theta) + A_2(\theta)u_1^2(\theta) = -a^2\sin\theta + a^2\theta = 0 \\
p_2 &= \int_0^{2\pi} (A_1(\theta)u_2(\theta) + A_2(\theta)u_1^2(\theta))d\theta = 0 \\
p'_3 &= A_1(\theta)u_3(\theta) + 2A_2(\theta)u_1(\theta)u_2(\theta) + A_3(\theta)u_1^3(\theta) \\
&= a\sin\theta(\frac{1}{2}a^2 + \frac{1}{2}c)\sin^2\theta + 2a^2\sin^2\theta(-a\sin\theta) + a^3\sin^3\theta \\
&= (\frac{c}{2} - \frac{a^2}{2})a\sin^3\theta \\
p_3 &= \int_0^{2\pi} (A_1(\theta)u_3(\theta) + 2A_2(\theta)u_1(\theta)u_2(\theta) + A_3(\theta)u_1^3(\theta))d\theta = 0 \\
&\cdots\cdots\cdots\cdots
\end{aligned} \tag{29}$$

By Theorem 1, we know that we only need to show the coefficients of $p'_{2k}, k = 1, 2, \cdots$, equal to zeros if we want to prove the origin is an isochronous center. Therefore, we only consider the coefficients of p_k, p'_k, u_k and $u'_k, k = 1, 2, \cdots$. The coefficients of $u_k, p_k, u'_k, p'_k, k = 1, 2, \cdots$ are written as u_k^*, p_k^*, du_k^*, $dp_k^*, k = 1, 2, \cdots$ respectively in the remainder of this paper.

4.2 Recursion Formulas

In this subsection, we give some preparations for the proof of the isochronous center. Firstly, we give some recursion formulas for the system.

Lemma 3. *The u_k^*s and p_k^*s of the cubic system Eq. 23 have following relation*

$$[(k-2)a^2 + c]u_k^* = \frac{c - a^2}{a}p_k^* - ac(\sum_{i=1}^{k-1} u_i^* u_{i-k}^*) \tag{30}$$

Proof. Because of

$$u'_k = (\frac{p'_k}{a^2\sin^2\theta} - \frac{u_k}{a\sin\theta})G_2 + (\frac{p'_k}{a^3\sin^3\theta} - \frac{u_k}{a^2\sin^2\theta} - \frac{\sum_{i=1}^{k-1} u_i u_{k-i}}{a\sin\theta})G_3$$

we integer both sides of the equation

$$u_k^* = -\frac{1}{k-1}(\frac{dp_k^*}{a} - u_k^*) + \frac{c}{k-1}(\frac{dp_k^*}{a^3} - \frac{u_k^*}{a^2} - \frac{\sum_{i=1}^{k-1} u_i^* u_{k-i}^*}{a}) \quad (31)$$

then

$$(\frac{k-2}{k-1} + \frac{c}{(k-1)a})u_k^* = \frac{dp_k^*}{(k-1)a}(\frac{c}{a^2} - 1) - \frac{c}{(k-1)a}\sum_{i=1}^{k-1} u_i^* u_{k-i}^*$$

times $(k-1)a^2$ both sides

$$[(k-2)a^2 + c]u_k^* = \frac{c-a^2}{a}dp_k^* - ac(\sum_{i=1}^{k-1} u_i^* u_{i-k}^*)$$

Lemma 4. *p'_n can be computed using*

$$p'_n = (p'_{n-1}u_1 + p'_{n-2}u_2 + \cdots + p'_1 u_{n-1} + u_n)A_1 \quad (32)$$

Proof. Since

$$\begin{aligned}&(u_1 + u_2 + \cdots + u_m)^{k+1} = (u_1 + u_2 + \cdots + u_m)^k(u_1 + u_2 + \cdots + u_m)\\ &= (u_1 + u_2 + \cdots + u_m)^k u_1 + (u_1 + u_2 + \cdots + u_m)^k u_2 + \cdots\\ &+(u_1 + u_2 + \cdots + u_m)^k u_m\end{aligned} \quad (33)$$

combing with Eq. 20, we have

$$p'_n = (p'_{n-1}u_1 + p'_{n-2}u_2 + \cdots + p'_1 u_{n-1} + u_n)A_1$$

Corollary 1. *The dp_k^* and $u_k^*, k = 1, 2, \cdots$ of the cubic system Eq. 23 can be represented as:*

$$dp_k^* = \frac{k-2}{k-1}a\sum_{i=1}^{k-1} dp_i^* u_{k-i}^* + \frac{c}{(k-1)a}(\sum_{i=1}^{k-1} dp_i^* u_{k-i}^* - a\sum_{i=1}^{k-1} u_i^* u_{k-i}^*) \quad (34)$$

$$u_k^* = \frac{-1}{k-1}\sum_{i=1}^{k-1} dp_i^* u_{k-i}^* + \frac{c}{(k-1)a^2}(\sum_{i=1}^{k-1} dp_i^* u_{k-i}^* - a\sum_{i=1}^{k-1} u_i^* u_{k-i}^*) \quad (35)$$

$$(k-1)a^2 u_k^* = (c-a^2)\sum_{i=1}^{k-1} dp_i^* u_{k-i}^* - ac\sum_{i=1}^{k-1} u_i^* u_{k-i}^* \quad (36)$$

Proof. According to Eq. 31

$$u_k^* = -\frac{1}{k-1}(\frac{dp_k^*}{a} - u_k^*) + \frac{c}{k-1}(\frac{dp_k^*}{a^3} - \frac{u_k^*}{a^2} - \frac{\sum_{i=1}^{k-1} u_i^* u_{k-i}^*}{a})$$

combining with Eq. 32, we have

$$u_k^* = \frac{-1}{k-1}\sum_{i=1}^{k-1} dp_i^* u_{k-i}^* + \frac{c}{(k-1)a^2}(\sum_{i=1}^{k-1} dp_i^* u_{k-i}^* - a\sum_{i=1}^{k-1} u_i^* u_{k-i}^*)$$

so

$$(k-1)a^2 u_k^* = (c-a^2)\sum_{i=1}^{k-1} dp_i^* u_{k-i}^* - ac\sum_{i=1}^{k-1} u_i^* u_{k-i}^*$$

From Eq. 32

$$p_n' = (p_{n-1}' u_1 + p_{n-2}' u_2 + \cdots + p_1' u_{n-1} + u_n)A_1$$

we have

$$dp_n^* = (dp_{n-1}^* u_1^* + dp_{n-2}^* u_2^* + \cdots + dp_1^* u_{n-1}^* + u_n^*)a \tag{37}$$

Replacing u_k^* in Eq. 37 with the right side of Eq. 35, the dp_k^* becomes

$$dp_k^* = \frac{k-2}{k-1}a\sum_{i=1}^{k-1} dp_i^* u_{k-i}^* + \frac{c}{(k-1)a}(\sum_{i=1}^{k-1} dp_i^* u_{k-i}^* - a\sum_{i=1}^{k-1} u_i^* u_{k-i}^*)$$

4.3 The New Method for Determining the Isochronicity

In this subsection, we will give a new method for determining the isochronicity of the same system proposed by W Zhang [6]. Although W Zhang has proved the isochronicity of the system in [6], in order to preserve the consistence of our statement, the isochronicity still is presented as the most important theorem in this subsection.

Lemma 5. *For the system defined by Eq. 23, if*

$$dp_{2i}^* = 0, i = 1, 2, \cdots, k \tag{38}$$

$$u_{2i}^* = -ac^{i-1}, i = 1, 2, \cdots, k \tag{39}$$

then

$$2ku_{2k+1}^* = [(2k-1)c - (2k-3)a^2]u_{2k-1}^* - ac\sum_{i=1}^{2k-2} u_i^* u_{2k-1-i}^*. \tag{40}$$

Proof. For $m = 1$,

$$2u_3^* = (c + a^2).$$

If for all $m \leq 2k$, Eq. 40 hold, we will prove the Equation will hold on $m = 2k+1$. According to Eq. 36, we have

$$2ka^2u^*_{2k+1} = (c-a^2)\sum_{i=1}^{2k-2} dp^*_i u^*_{2k-1-i} - ac\sum_{i=1}^{2k-2} u^*_i u^*_{2k-1-i}.$$

Since $dp^*_{2i} = 0, i = 1, 2, \cdots k$ and $u^*_{2j} \times u^*_{2i} = a^2c^{i+j-2}$ so we only consider the terms of u^*_{2m+1} and dp^*_{2m+1}, where m is an integer.

$$\begin{aligned}
2ka^2u^*_{2k+1} = (c-a^2) & \sum_{i=1,\text{ i is an odd number}}^{2k} dp^*_i u^*_{2k+1-i} \\
-ac & \sum_{i=1,\text{ i is an odd}}^{2k} u^*_i u^*_{2k+1-i} \\
= (c-a^2) & \sum_{i=1,\text{ i is an odd}}^{2k-2} dp^*_i u^*_{2k-1-i}c + dp^*_{2k-1}u^*_2(c-a^2) \\
-ac & \sum_{i=1,\text{ i is an odd}}^{2k-2} u^*_i u^*_{2k-1-i} - 2acu^*_{2k-1}u^*_2 \\
= (2k-2)u^*_{2k-1}ca^2 & + (c-a^2)dp^*_{2k-1}u^*_2 - 2acu^*_{2k-1}u^*_2
\end{aligned}$$

Dividing a^2 on both sides, above equation becomes

$$\begin{aligned}
2ku^*_{2k+1} &= (2k-2)u^*_{2k-1}c - [(2k-3)a^2 + c]u^*_{2k-1} \\
&\quad -ac \sum_{i=1,\text{ i is an odd}}^{2k-2} u^*_i u^*_{2k-1-i} \\
&= [(2k-1)c - (2k-3)a^2]u^*_{2k-1} \\
&\quad -ac \sum_{i=1,\text{ i is an odd}}^{2k-2} u^*_i u^*_{2k-1-i}
\end{aligned}$$

Corollary 2. *For the system defined by Eq. 23, if*

$$\begin{aligned}
dp^*_{2i} &= 0, i = 1, 2, \cdots, k \\
u^*_{2i} &= -ac^{i-1}, i = 1, 2, \cdots, k
\end{aligned}$$

then

$$2ku^*_{2k+1} = [(4k-3)c-(2k-3)a^2]u^*_{2k-1} + [-(2k-3)c+(2k-3)a^2]cu^*_{2k-3}. \quad (41)$$

Proof. Since

$$\sum_{i=1,\text{ i is an odd}}^{2k-2} u^*_i u^*_{2k-1-i} = c \sum_{i=1,\text{ i is an odd}}^{2k-4} u^*_i u^*_{2k-3-i} + 2u^*_{2k-3}$$

by Eq. 40, we have

$$(2k-2)u^*_{2k-1} - [(2k-3)c - (2k-5)a^2]u^*_{2k-3} = -ac \sum_{i=1,\text{ i is an odd}}^{2k-4} u^*_i u^*_{2k-3-i} \tag{42}$$

Because of

$$\begin{aligned}
2ku^*_{2k+1} &= [(2k-1)c - (2k-3)a^2]u^*_{2k-1} \\
&\quad -ac^2 \sum_{i=1,\text{ i is an odd}}^{2k-4} u^*_i u^*_{2k-3-i} - 2ca^2 u^*_{2k-3} \\
&= [(2k-1)c - (2k-3)a^2]u^*_{2k-1} + (2k-2)cu^*_{2k-1} \\
&\quad -[(2k-3)c - (2k-5)a^2]cu^*_{2k-3} - 2ca^2 u^*_{2k-3} \\
&= [(4k-3)c - (2k-3)a^2]u^*_{2k-1} + [-(2k-3)c + (2k-3)a^2]cu^*_{2k-3}
\end{aligned}$$

Theorem 2. *For the system defined by Eq. 23, if*

$$\begin{aligned}
dp^*_{2i} &= 0, i = 1, 2, \cdots, k \\
u^*_{2j} &= -ac^{j-1}, i = 1, 2, \cdots, k
\end{aligned}$$

for $k \geq 2$

$$\sum_{i=1,\ i\ is\ an\ odd}^{2k-1} u^*_i u^*_{2k-i} = [(k-1)a^2 + c]c^{k-2}. \tag{43}$$

Proof. For $m = 2$, we have

$$2u^*_3 = (a^2 + c)$$

We assume that Eq. 43 are correct for all $2 \leq m \leq 2k$, thus we will prove the equation is correct for $m = 2k+1$. According to Eq. 43, we have

$$\begin{aligned}
2u^*_3 &= (c + a^2) && (44) \\
4u^*_5 &= (5c - a^2)u^*_3 + (-c + a^2)c && (45) \\
&\cdots\cdots\cdots\cdots\cdots \\
2ku^*_{2k+1} &= [(4k-3)c - (2k-3)a^2]u^*_{2k-1} + [-(2k-3)c + (2k-3)a^2]cu^*_{2k-3} && (46)
\end{aligned}$$

then we times Eq. 44, Eq. 45, Eq. 46 with u^*_{2k-1}, u^*_{2k-3} and u^*_1 respectively and sum these equations both sides, we have

$$k \sum_{i=1,\text{ i is an odd number}}^{2k+1} u^*_i u^*_{2k+2-i}$$

$$= [(2k-1)c-(k-2)a^2] \sum_{i=1,\text{ i is an odd}}^{2k-1} u_i^* u_{2k-i}^*$$
$$+[-(k-1)c+(k-1)a^2]c \sum_{i=1,\text{ i is an odd}}^{2k-3} u_i^* u_{2k-2-i}^*$$
$$= [(2k-1)c-(k-2)a^2][(k-1)a^2+c]c^{k-2}+c[-(k-1)c+(k-1)a^2][(k-2)a^2+c]c^{k-3}$$
$$= k(ka^2+c)c^{k-1}$$

so

$$k \sum_{i=1,\text{ i is an odd}}^{2k+1} u_i^* u_{2k+2-i}^* = k(ka^2+c)c^{k-1}$$

That is

$$\sum_{i=1,\text{ i is an odd}}^{2k+1} u_i^* u_{2k+2-i}^* = (ka^2+c)c^{k-1}$$

Theorem 3. *For the system defined by Eq. 23, if*

$$dp_{2i}^* = 0, i = 1, 2, \cdots, k$$
$$u_{2i}^* = -ac^{i-1}, i = 1, 2, \cdots, k$$

we have

$$\sum_{i=1}^{2k-1} dp_i^* u_{2k-i}^* = ac^{k-1}. \tag{47}$$

Proof. For $m = 1$, we have

$$dp_1^* u_1^* = a = ac^{1-1}$$

We assume that Eq. 47 are correct for all $m \leq 2k+1$, thus we will prove the equation is correct for $m = 2k+2$. Since

$$dp_{2s}^* = 0, s = 1, 2, \cdots, k,$$

for all $m \leq 2k+1$, we have

$$\sum_{i=1,\text{ i is an odd}}^{2k-1} dp_i^* u_{2k-i}^* = ac^{k-1}$$

According to Eq. 47, we have

$$dp_1^* = a \tag{48}$$
$$dp_3^* = au_3^* + adp_1^* u_2^* \tag{49}$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$dp_{2k+1}^* = au_{2k+1}^* + adp_1^* u_{2k}^* + \cdots + adp_{2k-1}^* u_2^* \tag{50}$$

then we times Eq. 48, Eq. 49, Eq. 50 with u^*_{2k+1}, u^*_{2k-1} and u^*_1 respectively and sum these equations both sides, we have

$$\begin{aligned}
&\sum_{i=1,\text{ i is an odd}}^{2k+1} dp^*_i u^*_{2k-i} \\
&= a\sum_{i=1}^{2k+1} u^*_i u^*_{2k+2-i} \\
&\quad -a^2[\sum_{i=1}^{2k-1} dp^*_{2k-i}u^*_i + c\sum_{i=1}^{2k-3} dp^*_{2k-2-i}u^*_i + \cdots] \\
&= a(ka^2+c)c^{k-1} - ka^3c^{k-1} \\
&= ac^k
\end{aligned}$$

Theorem 4. *For the system defined by Eq. 23, we have*

$$\begin{aligned}
dp^*_{2i} &= 0, i = 1, 2, \cdots \\
u^*_{2i} &= -ac^{i-1}, i = 1, 2, \cdots.
\end{aligned}$$

Proof. For m=1, we have

$$dp^*_2 = 0$$

$$u^*_2 = -a = -ac^{1-1}$$

We assume that Eq. 38 and Eq. 39 are correct for all $m \leq 2k$, thus we will prove the equation is correct for $m = 2k+2$. According to corollary 1, dp^*_{2k+2} can be represented by

$$\begin{aligned}
(2k+1)adp^*_{2k+2} &= 2ka^2\sum_{i=1}^{2k+1} dp^*_i u^*_{2k+2-i} + c(\sum_{i=1}^{2k+1} dp^*_i u^*_{2k+2-i} - a\sum_{i=1}^{2k+1} u^*_i u^*_{2k+2-i}) \\
&= (2ka^2+c)\sum_{i=1}^{2k+1} dp^*_i u^*_{2k+2-i} - ac\sum_{i=1}^{2k+1} u^*_i u^*_{2k+2-i}
\end{aligned}$$

By theorem 2 and 3, we have

$$\begin{aligned}
(2k+1)adp^*_{2k+2} &= (2ka^2+c)\sum_{i=1}^{2k+1} dp^*_i u^*_{2k+2-i} - ac\sum_{i=1}^{2k+1} u^*_i u^*_{2k+2-i} \\
&= (2ka^2+c)ac^k - ac(2ka^2+c)c^{k-1} \\
&= 0
\end{aligned}$$

Thus, for all $i \geq 1$, $dp^*_{2i} = 0$.

Theorem 5. *The center of the system defined by Eq. 23 is an isochronous center.*

By theorem 4 and 1, the proof of theorem 5 is very clear.

5 Conclusions

In this paper, we have analyzed the coefficients of the reversible cubic system defined by Eq. 23 and give some recursion formulas for its coefficients. Based on these formulas, the origin of this system is an isochronous center is proved. This proof gives a new method to determining the isochronicity of the same system proposed by W Zhang [6].

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