On the Stability of Fully-Explicit Finite-Difference Scheme for Two-Dimensional Parabolic Equation with Nonlocal Conditions

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Abstract. We construct and analyse a fully-explicit finite-difference scheme for a two-dimensional parabolic equation with nonlocal integral conditions. The main attention is paid to the stability of the scheme. We apply the stability analysis technique which is based on the investigation of the spectral structure of the transition matrix of a finite-difference scheme and demonstrate that depending on the parameters of nonlocal conditions the proposed method can be stable or unstable. The results of numerical experiment with one test problem are also presented and they validate theoretical results.

Keywords: parabolic equation, nonlocal integral conditions, fully-explicit finite-difference scheme, stability

1 Introduction

We consider the two-dimensional parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y, t), \quad 0 < x < 1, \quad 0 < y < 1, \quad 0 < t \leqslant T, \tag{1}$$

subject to nonlocal integral conditions

$$u(0, y, t) = \gamma_1 \int_0^1 u(x, y, t)\mathrm{d}x + \mu_1(y, t), \tag{2}$$

$$u(1, y, t) = \gamma_2 \int_0^1 u(x, y, t)\mathrm{d}x + \mu_2(y, t), \quad 0 < y < 1, \quad 0 < t \leqslant T, \tag{3}$$

boundary conditions

$$u(x, 0, t) = \mu_3(x, t), \quad u(x, 1, t) = \mu_4(x, t), \quad 0 < x < 1, \quad 0 < t \leqslant T, \tag{4}$$

and initial condition

$$u(x, y, 0) = \varphi(x, y), \quad 0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 1, \tag{5}$$

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where $f(x,y,t)$, $\mu_1(y,t)$, $\mu_2(y,t)$, $\mu_3(x,t)$, $\mu_4(x,t)$, $\varphi(x,y)$ are given functions, γ_1, γ_2 are given parameters, and function $u(x,y,t)$ is unknown. We assume that for all t, $0 < t \leqslant T$, nonlocal integral conditions (2), (3) and boundary conditions (4) are compatible, i.e., the following compatibility conditions are satisfied:

$$\gamma_1 \int_0^1 \mu_3(x,t)\mathrm{d}x + \mu_1(0,t) = \mu_3(0,t),$$

$$\gamma_1 \int_0^1 \mu_4(x,t)\mathrm{d}x + \mu_1(1,t) = \mu_4(0,t),$$

$$\gamma_2 \int_0^1 \mu_3(x,t)\mathrm{d}x + \mu_2(0,t) = \mu_3(1,t),$$

$$\gamma_2 \int_0^1 \mu_4(x,t)\mathrm{d}x + \mu_2(1,t) = \mu_4(1,t).$$

The present paper is devoted to the numerical solution of the two-dimensional differential problem (1)–(5). We construct the fully-explicit finite-difference scheme and analyse its stability. In order to efficiently apply the stability analysis technique which has been applied for other types of finite-difference schemes, we formulate the proposed numerical method as splitting finite-difference scheme.

The stability of finite-difference schemes for the corresponding one-dimensional parabolic problems with nonlocal integral conditions similar to conditions (2), (3) or with more general integral conditions is investigated by M. Sapagovas [12,13], Ž. Jesevičiūtė and M. Sapagovas [4] and other authors. The alternating direction implicit (ADI) and locally one-dimensional (LOD) methods for the two-dimensional differential problem (1)–(5) has been proposed and the stability of these methods has been analysed by S. Sajavičius [5,6,7]. The LOD technique for two-dimensional parabolic problems with nonlocal integral condition (the specification of mass/energy) has been investigated by M. Dehghan [3]. The paper [14] deals with the ADI method for the two-dimensional parabolic equation (1) with Bitsadze-Samarskii type nonlocal boundary condition. We use the similar technique and argument in order to construct the fully-explicit finite-difference scheme for the differential problem (1)–(5) and to investigate the stability of that method.

The paper is organized as follows. In Sect. 2, the details of the fully-explicit finite-difference scheme are described. Section 3 reviews the well-known stability analysis technique which is based on the spectral structure of the transition matrix of a finite-difference scheme, and discusses possibilities to use such technique in order to analyse the stability of the proposed finite-difference scheme. The results of numerical experiment with a particular test problem are presented in Sect. 4. Some remarks in Sect. 5 conclude the paper.

2 The Fully-Explicit Finite-Difference Scheme

To solve the two-dimensional differential problem (1)–(5) numerically, we apply the finite-difference technique [11]. Let us define discrete grids with uniform steps,

$$\omega_{h_1} = \{x_i = ih_1, i = 1, 2, \dots, N_1 - 1, N_1 h_1 = 1\},$$
$$\overline{\omega}_{h_1} = \omega_{h_1} \cup \{x_0 = 0, x_{N_1} = 1\},$$
$$\omega_{h_2} = \{y_j = jh_2, j = 1, 2, \dots, N_2 - 1, N_2 h_2 = 1\},$$
$$\overline{\omega}_{h_2} = \omega_{h_2} \cup \{y_0 = 0, y_{N_2} = 1\},$$
$$\omega = \omega_{h_1} \times \omega_{h_2}, \quad \overline{\omega} = \overline{\omega}_{h_1} \times \overline{\omega}_{h_2},$$
$$\omega^\tau = \{t^k = k\tau, k = 1, 2, \dots, M, M\tau = T\}, \quad \overline{\omega}^\tau = \omega^\tau \cup \{t^0 = 0\}.$$

We use the notation $U_{ij}^k = U(x_i, y_j, t^k)$ for functions defined on the grid $\overline{\omega} \times \overline{\omega}^\tau$ or its parts, and the notation $U_{ij}^{k+1/2} = U(x_i, y_j, t^k + 0.5\tau)$ (some of the indices can be omitted). We define one-dimensional discrete operators

$$\Lambda_1 U_{ij} = \frac{U_{i-1,j} - 2U_{ij} + U_{i+1,j}}{h_1^2}, \quad \Lambda_2 U_{ij} = \frac{U_{i,j-1} - 2U_{ij} + U_{i,j+1}}{h_2^2}.$$

Now we explain the main steps of the fully-explicit (splitting) finite-difference scheme for the numerical solution of problem (1)–(5). First of all, we replace the initial condition (5) by equations

$$U_{ij}^0 = \varphi_{ij}, \quad (x_i, y_j) \in \overline{\omega}. \tag{6}$$

Then, for any k, $0 \leqslant k < M - 1$, the transition from the kth layer of time to the $(k+1)$th layer can be carried out by splitting it into two stages and solving one-dimensional finite-difference subproblems in each of them. The both of subproblems are fully-explicit. The values $U_{ij}^{k+1/2}$, $x_i \in \omega_{h_1}$, can be computed from the first subproblem:

$$\frac{U_{ij}^{k+1/2} - U_{ij}^k}{\tau} = \Lambda_1 U_{ij}^k, \quad y_j \in \omega_{h_2}, \tag{7}$$
$$U_{i0}^{k+1/2} = (\overline{\mu}_3)_i, \tag{8}$$
$$U_{iN_2}^{k+1/2} = (\overline{\mu}_4)_i, \tag{9}$$

where

$$\overline{\mu}_3 = (\mu_3)^k + \tau \Lambda_1 (\mu_3)^k, \quad \overline{\mu}_4 = (\mu_4)^k + \tau \Lambda_1 (\mu_4)^k.$$

In the second subproblem, nonlocal integral conditions (2), (3) are approximated by the trapezoidal rule and from this subproblem we can compute the values U_{ij}^{k+1}, $y_j \in \omega_{h_2}$:

$$\frac{U_{ij}^{k+1} - U_{ij}^{k+1/2}}{\tau} = \Lambda_2 U_{ij}^{k+1/2} + f_{ij}^{k+1}, \quad x_i \in \omega_{h_1}, \tag{10}$$
$$U_{0j}^{k+1} = \gamma_1(1, U)_j^{k+1} + (\mu_1)_j^{k+1}, \tag{11}$$
$$U_{N_1 j}^{k+1} = \gamma_2(1, U)_j^{k+1} + (\mu_2)_j^{k+1}, \tag{12}$$

where

$$(1,U)_j^{k+1} = h_1\left(\frac{U_{0j}^{k+1} + U_{N_1 j}^{k+1}}{2} + \sum_{i=1}^{N_1-1} U_{ij}^{k+1}\right).$$

Every transition is finished by computing

$$U_{i0}^{k+1} = (\mu_3)_i^{k+1}, \quad U_{iN_2}^{k+1} = (\mu_4)_i^{k+1}, \quad x_i \in \overline{\omega}_{h_1}. \tag{13}$$

Thus, the procedure of numerical solution can be stated as follows:

procedure `The Fully-Explicit Finite-Difference Scheme`
begin
  Compute U_{ij}^0, $(x_i, y_j) \in \overline{\omega}$, from (6);
  for $k = 0, 1, \ldots, M-1$
    for each $x_i \in \omega_{h_1}$
      Compute $U_{ij}^{k+1/2}$, $y_j \in \overline{\omega}_{h_2}$, from (7)–(9);
    end for
    for each $y_j \in \omega_{h_2}$
      Compute U_{ij}^{k+1}, $x_i \in \overline{\omega}_{h_1}$, from (10)–(12);
    end for
    Compute U_{i0}^{k+1} and $U_{iN_2}^{k+1}$, $x_i \in \overline{\omega}_{h_1}$, from (13);
  end for
end

It is known [11] that the finite-difference scheme (7)–(12) approximates the two-dimensional differential problem (1)–(5) with error $\mathrm{O}(\tau + h_1 + h_2)$.

Now let us transform the finite-difference scheme (7)–(12) to the matrix form. From (11) and (12) we obtain

$$U_{0j}^{k+1} = \overline{\alpha} \sum_{i=1}^{N_1-1} U_{ij}^{k+1} + (\overline{\mu}_1)_j^{k+1},$$

$$U_{N_1 j}^{k+1} = \overline{\beta} \sum_{i=1}^{N_1-1} U_{ij}^{k+1} + (\overline{\mu}_2)_j^{k+1},$$

where

$$\overline{\alpha} = \frac{\gamma_1 h_1}{D}, \quad \overline{\beta} = \frac{\gamma_2 h_1}{D},$$

$$(\overline{\mu}_1)_j^{k+1} = \left(\frac{1}{D} - \frac{\overline{\beta}}{2}\right)(\mu_1)_j^{k+1} + \frac{\overline{\alpha}}{2}(\mu_2)_j^{k+1},$$

$$(\overline{\mu}_2)_j^{k+1} = \frac{\overline{\beta}}{2}(\mu_1)_j^{k+1} + \left(\frac{1}{D} - \frac{\overline{\alpha}}{2}\right)(\mu_2)_j^{k+1},$$

$$D = 1 - \frac{h_1}{2}(\gamma_1 + \gamma_2).$$

If $M_1 = \max\{|\gamma_1|, |\gamma_2|\} < \infty$ and the grid step $h_1 < 1/M_1$, then $D > 0$.

Let us introduce $(N_1-1)\times(N_1-1)$ and $(N_2-1)\times(N_2-1)$ matrices

$$\widetilde{\Lambda}_1 = h_1^{-2}\begin{pmatrix} -2+\overline{\alpha} & 1+\overline{\alpha} & \overline{\alpha} & \cdots & \overline{\alpha} & \overline{\alpha} & \overline{\alpha} \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ \overline{\beta} & \overline{\beta} & \overline{\beta} & \cdots & \overline{\beta} & 1+\overline{\beta} & -2+\overline{\beta} \end{pmatrix}$$

and

$$\widetilde{\Lambda}_2 = h_2^{-2}\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & -2 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

Now we can define matrices of order $(N_1-1)\cdot(N_2-1)$,

$$A_1 = -E_{N_2-1}\otimes\widetilde{\Lambda}_1, \quad A_2 = -\widetilde{\Lambda}_2\otimes E_{N_1-1},$$

where E_N is the identity matrix of order N and $A\otimes B$ denotes the Kronecker (tensor) product of matrices A and B. We can directly verify that A_1 and A_2 are commutative matrices, i.e.,

$$A_1A_2 = A_2A_1 = \widetilde{\Lambda}_2\otimes\widetilde{\Lambda}_1.$$

Introducing the matrices A_1 and A_2 allow us to rewrite the finite-difference scheme (7)–(12) in the following form:

$$U^{k+1/2} = \big(E-\tau A_1\big)U^k, \tag{14}$$

$$U^{k+1} = \big(E-\tau A_2\big)U^{k+1/2} + \tau F^{k+1}, \tag{15}$$

where E is the identity matrix of order $(N_1-1)\cdot(N_2-1)$,

$$U = \big(\widetilde{U}_1, \widetilde{U}_2, \dots, \widetilde{U}_j, \dots, \widetilde{U}_{N_2-1}\big)^T, \quad \widetilde{U}_j = \big(U_{1j}, U_{2j}, \dots, U_{ij}, \dots, U_{N_1-1,j}\big)^T,$$

and

$$F^{k+1} = \big(F_1^{k+1}, F_2^{k+1}, \dots, F_j^{k+1}, \dots, F_{N_2-1}^{k+1}\big)^T,$$

$$F_j^{k+1} = \left(\frac{(\overline{\mu}_1)_j^{k+1}}{h_1^2} + f_{1j}^{k+1}, f_{2j}^{k+1}, \dots, f_{N_1-2,j}^{k+1}, \frac{(\overline{\mu}_2)_j^{k+1}}{h_1^2} + f_{N_1-1,j}^{k+1}\right)^T.$$

From (14) and (15) it follows that

$$U^{k+1} = SU^k + \overline{F}^k, \tag{16}$$

where

$$S = \big(E - \tau A_2\big)\big(E - \tau A_1\big), \quad \overline{F}^k = \tau F^{k+1}.$$

3 Analysis of the Stability

Let us recall some facts related with the stability of the finite-difference schemes. The finite-difference scheme (16) is called stepwise stable [2] if for all fixed τ and h_1, h_2 there exists a constant $C = C(\tau, h_1, h_2)$ such that $|U_{ij}^k| \leqslant C$, $(x_i, y_j) \in \overline{\omega}$, $k = 0, 1, \ldots$. We know [11] that a sufficient stability condition for the finite-difference scheme (16) can be written in the form

$$\|S\| \leqslant 1 + c_0\tau,$$

where a non-negative constant c_0 is independent on τ and h_1, h_2. The necessary and sufficient condition to define a matrix norm $\|\cdot\|_*$ such that $\|S\|_* < 1$ is the inequality [1]

$$\rho(S) = \max_{\lambda(S)} |\lambda(S)| < 1,$$

where $\lambda(S)$ is the eigenvalues of S and $\rho(S)$ is the spectral radius of S. If S is a simple-structured matrix, i.e., the number of linearly independent eigenvectors is equal to the order of the matrix, then it is possible to define the norm [12]

$$\|S\|_* = \|P^{-1}SP\|_\infty = \rho(S),$$

which is compatible with the vector norm

$$\|U\|_* = \|P^{-1}U\|_\infty,$$

where columns of the matrix P are linearly independent eigenvectors of S, and the norms

$$\|B\|_\infty = \max_{1\leqslant i\leqslant m} \sum_{j=1}^{m} |b_{ij}|, \quad \|V\|_\infty = \max_{1\leqslant i\leqslant m} |v_i|,$$

m is the order of matrix $B = (b_{ij})_{i,j=1}^m$ and vector $V = (v_1, v_2, \ldots, v_m)^T$. Therefore, we will use the stability condition $\rho(S) < 1$ in the analysis of the stability of the finite-difference scheme (16).

The spectral structure of finite-difference and differential operators with nonlocal conditions are investigated by many authors (see, e.g., [8,9,10] and references therein). The eigenvalue problem for the matrix $(-\widetilde{\Lambda}_1)$ has been investigated in the paper [13]. When $\gamma_1 + \gamma_2 \leqslant 2$, then all the eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ are non-negative and algebraically simple real numbers: $\lambda_i(-\widetilde{\Lambda}_1) \geqslant 0$, $i = 1, 2, \ldots, N_1 - 1$. If $\gamma_1 + \gamma_2 > 2$, then there exists one and only one negative eigenvalue of the matrix $(-\widetilde{\Lambda}_1)$. It is well-known [11] that all the eigenvalues of the matrix $(-\widetilde{\Lambda}_2)$ are real, positive and algebraically simple:

$$\lambda_j(-\widetilde{\Lambda}_2) = \frac{4}{h_2^2}\sin^2\frac{j\pi h_2}{2}, \quad j = 1, 2, \dots, N_2 - 1.$$

Let us denote

$$\Delta_2 = \max_{\lambda(A_2)} \lambda(A_2) = \max_{1 \leqslant j \leqslant N_2 - 1} \lambda_j(-\widetilde{\Lambda}_2) = \lambda_{N_2-1}(-\widetilde{\Lambda}_2) = \frac{4}{h_2^2}\cos^2\frac{\pi h_2}{2}.$$

Since A_1 and A_2 are simple-structured matrices as Kornecker products of two simple-structured matrices, then S is a simple-structured matrix too, and the eigenvalues of the matrix S can be expressed as follows:

$$\lambda(S) = \big(1 - \tau\lambda(A_1)\big) \cdot \big(1 - \tau\lambda(A_2)\big). \tag{17}$$

If all the eigenvalues of the matrix A_1 are real and non-negative numbers, then the finite-difference scheme (16) is stable under condition

$$\tau < \tau^* = 2\min\left\{\frac{1}{\rho(A_1)}, \frac{1}{\Delta_2}\right\}. \tag{18}$$

Indeed, from (17) it follows that

$$|\lambda(S)| = |1 - \tau\lambda(A_1)| \cdot |1 - \tau\lambda(A_2)|.$$

Thus, we conclude that $\rho(S) < 1$, if condition (18) are fulfilled.

The eigenvalues of the matrix A_1 coincide with the eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ and they are multiple. Thus, the fully-explicit finite-difference scheme is stable if all the eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ are non-negative, i.e., if $\gamma_1 + \gamma_2 \leqslant 2$. The non-negativity of the eigenvalues of the matrix $(-\widetilde{\Lambda}_1)$ ensures the stability of the finite-difference scheme (16), but it is notable [14] that the scheme can be stable even if there exists a negative eigenvalue of the matrix $(-\widetilde{\Lambda}_1)$.

4 Numerical Results

In order to demonstrate the efficiency of the considered fully-explicit finite-difference scheme and practically justify the stability analysis technique, we solved a particular test problem. In our test problem, functions $f(x, y, t)$, $\mu_1(y, t)$, $\mu_2(y, t)$, $\mu_3(x, t)$, $\mu_4(x, t)$ and $\varphi(x, y)$ were chosen so that the function

$$u(x, y, t) = x^3 + y^3 + t^3$$

would be the solution to the differential problem (1)–(5), i.e.,

$$f(x,y,t) = -3(2x+2y-t^2),$$
$$\mu_1(y,t) = y^3+t^3-\gamma_1(0.25+y^3+t^3),$$
$$\mu_2(y,t) = 1+y^3+t^3-\gamma_2(0.25+y^3+t^3),$$
$$\mu_3(x,t) = x^3+t^3,$$
$$\mu_4(x,t) = x^3+1+t^3,$$
$$\varphi(x,y) = x^3+y^3.$$

The finite-difference scheme was implemented in a stand-alone C application. Numerical experiment with $h_1 = h_2 = 10^{-2}$, $T = 2.0$, $\gamma_1 = -\gamma_2 = 1$ and with different values of τ was performed using the technologies of grid computing. To estimate the accuracy of the numerical solution, we calculated the maximum norm of computational error,

$$\|\varepsilon\|_{C_h} = \max_{0\leqslant k\leqslant M} \max_{\substack{0\leqslant i\leqslant N_1\\ 0\leqslant j\leqslant N_2}} |U_{ij}^k - u(x_i,y_j,t^k)|.$$

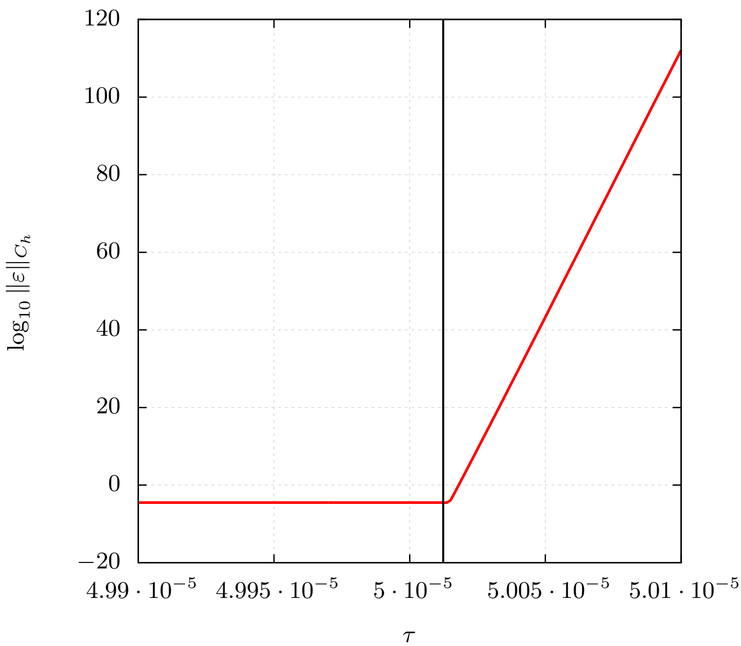

Fig. 1. The dependence of $\log_{10}\|\varepsilon\|_{C_h}$ for various values of τ ($\gamma_1 = -\gamma_2 = 1$). The vertical straight line denotes the line $\tau = \tau^*$.

Note that

$$\min_{0\leqslant t\leqslant T}\min_{\substack{0\leqslant x\leqslant 1\\0\leqslant y\leqslant 1}} u(x,y,t)=u(0,0,0)=0,$$

$$\max_{0\leqslant t\leqslant T}\max_{\substack{0\leqslant x\leqslant 1\\0\leqslant y\leqslant 1}} u(x,y,t)=u(1,1,T)=10.$$

For the numerical analysis of the spectrum of the matrix $(-\widetilde{\Lambda}_1)$, MATLAB (The MathWorks, Inc.) software package [15] was used. The results of the numerical analysis of the spectrum of the matrix $(-\widetilde{\Lambda}_1)$ shown that $\rho(A_1) = 3.999013170072345\cdot 10^4$. Since $\Delta_2 = 3.999013120731463\cdot 10^4 < \rho(A_1)$, then $\tau^* = 2/\rho(A_1) = 5.001233841807574\cdot 10^{-5}$.

The influence of condition (18) for the stability of the finite-difference scheme (16) was investigated. From Fig. 1 we see that the norm $\|\varepsilon\|_{C_h}$ is small enough when $\tau < \tau^*$.

5 Concluding Remarks

We developed the fully-explicit finite-difference scheme for the two-dimensional parabolic equation with two nonlocal integral conditions. Applying quite a simple technique allow us to investigate the stability of this method. The stability analysis technique is based on the analysis of the spectrum of the transition matrix of a finite-difference scheme. We demonstrate that the proposed scheme can be stable or unstable depending on the parameters of nonlocal conditions. The results of numerical experiment with a particular test problem justify theoretical results.

The fully-explicit finite-difference scheme can be generalized for the corresponding two-dimensional differential problem with nonlocal integral conditions

$$u(0,y,t)=\gamma_1\int_0^1 \alpha(x)u(x,y,t)\mathrm{d}x+\mu_1(y,t),$$

$$u(1,y,t)=\gamma_2\int_0^1 \beta(x)u(x,y,t)\mathrm{d}x+\mu_2(y,t),\quad 0<y<1,\quad 0<t\leqslant T,$$

where $\alpha(x)$ and $\beta(x)$ are given functions.

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