The Planet-4D Model: An Original Hypersymmetric Music Space Based on Graph Theory

Gilles Baroin

Laboratoire LLA Creatis, Université de Toulouse
Gilles@Baroin.org

Abstract. Beside a geometrical part that has been calculated with the help of the graph theory, the Planet-4D model includes twelve ideograms that can either symbolize notes, chords or scales depending on the context. Based on symmetry principles, it presents the following innovations:

1. the hyper spherical environment grants each symbol an equivalent physical position, and involves more symmetries than any 3D model,
2. the concept of bi-dimensional ideograms provides an intuitive understanding of pitch relationships,
3. it contains implicitly the chromatic and fourth circles as well as the original Tonnetz.

NB: the pertinence of this model is effective when demonstrated in motion with colored CGI animations of the 4D Space including sound examples. Videos shown during this conference are available on the web at www.planetes.info.

Keywords: Symmetry, Hypersphere, Pitch Space, Tonnetz, Spectral Projections, Graph Theory, Animated GCI, Quaternions.

1 Symmetry

Nöthers theorem specifies that "to every local symmetry there corresponds a conservation law" [1,2] Interpreting symmetry as invariance to the transposition leads to a model where each pitch class plays the same role. That is actually the case for the Tonnetze describing the equal tempered space.

We will also use symmetry as invariance between musical interval and geometrical distance. The physical distance within the geometrical model for a defined interval will be constant i.e., each fifth will be drawn in the model with the same length. That condition is not present in a 3D model but can be respected on a 2D circle.

We apply finally the invariance principle to the ideographic system: since our ideograms are two-dimensional, moving into one direction will conserve one of the two parameters: color or form! That condition on the symbols is an enhancement to the merely mathematical decomposition.

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2 Mathematical Background

Using different schemes, mathematicians and music theorists have demonstrated that the tempered twelve tone pitch space [3,4] can be considered as a combination of minor and major thirds [5,6,7]. We use the Cartesian product of two circular graphs C3 $\square$ C4 to build the 12 edge graph called the Planet graph [8] (see Fig. 1).

The demonstration involves a step by step composition of 12 vertex graphs as a combination of simple sub elements. The Planet graph is an Abelian Cayley graph on Z_{12} with neighborhood of 0 generated by (±3,±4).

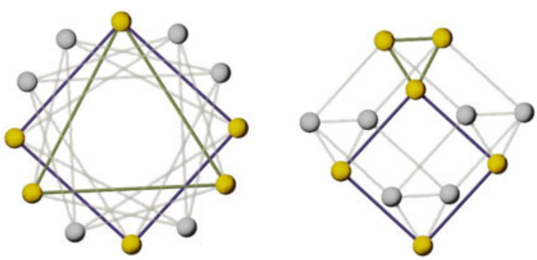

Fig. 1. Two isomorphic representations of the Planet graph: each Triangle graph (C3) and Square graph (C4) always have one unique and common node

A graph being a combinatoric object, in order to build a geometrical model, we perform a spectral analysis to determine its Eigen spaces and obtain geometrical coordinates [8]. The spectral projection gives 12 points equally dispatched on a Hypersphere. Since the 4D space is built up with two orthogonal spaces, each point is represented by a quaternion defined by a couple

$$(\tfrac{1}{\sqrt{3}}e^{i\frac{2\pi k}{3}}, \tfrac{1}{\sqrt{2}}e^{i\frac{2k'\pi}{4}}),$$

with $k = 0$ to 2 and $k' = 0$ to 3, where the couple (k, k') determines the belonging to triangles and squares. If we consider $n = 0$ to 11 as the pitch index, the quaternions ensemble is defined by

$$(\tfrac{1}{\sqrt{3}}e^{i\frac{2n\pi}{3}}, \tfrac{1}{\sqrt{2}}e^{i\frac{2n\pi}{4}}).$$

We call "physical distance" the Euclidian distance in the geometric space; the "logical distance" is the number of rotations to perform to reach the next point, (distance on the graph). The acoustic distance is calculated in semi tones.

The table of distances (Table 1) shows three different physical distances whose values are noteworthy numbers. The distance between neighbor pitch classes is 1. For notes that are neighbor on the chromatic circle or on the circle of fourth or have a triton interval, we obtain $\sqrt{2}$. Finally, the biggest possible distance between two nodes: $\sqrt{3}$ (diagonal of the unity cube) applies for the whole tone.

Table 1. Table of distances

Physical	Logical	Acoustic	Interval
1	1	3;9;4;8	M3;m6;m3;M6
$\sqrt{2}$	2	1;11;5;7;6	m2;M7;P4;P5;TT
$\sqrt{3}$	3	2;10	M2;m7

3 Vizualisation

In conventional Tonnetz, pitch classes are represented with letters or numbers. Since the decomposition involves two sets and each pitch class being a unique combination of these two sub-groups, we use bi-dimensional ideogram. Each dimension of the pictogram is linked to a subgroup. The two dimensions (parameters) for the ideograms have been arbitrary chosen: form and color [9].

In order to see the fourth dimension and to feel its symmetry, we project the model into our 3D space [10] and let it rotate about its two main symmetry axes (see Fig. 2).

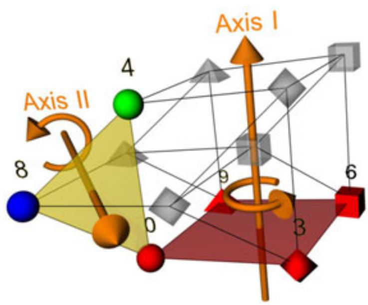

Fig. 2. Major axes of the Model

We must imagine that the first axis passes through the center of every square and the second axis through each triangle center. To feel that the model resides in true 4D space, it will be set in motion in our 3D space (see Fig. 3). In the computer animated model, the nearest node to the spectator represents the current position within the model. We control the camera focal distance in order to blur the distant nodes and focus the attention to the nearest nodes (one major or minor third).

As the rotation of a 3D cube on a screen looks like a 2D deformation within the plane, the rotation of the 4D model will be perceived as a deformation within our 3D space without affecting the hull. The feeling is analog to manipulating a Rubik^{TM} cube. As for the torus, chromatic scale and circle of fourth are included within the Planet-4D model. They are obtained by rotating the Hypersphere in both directions simultaneously.

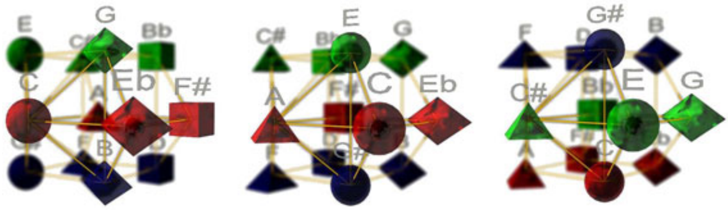

Fig. 3. 4D Rotations from start position C (center image)

4 Applications and Perspectives

Since the experiment raised curiosity of both mathematicians and musicians, we have applied the same mathematical principles (spectral graph projections) to other musical representation in order to let appear the inner structure of pitch space decomposition. The same principles of visualization in hyperspaces are experimented for a representation of the dual space of the Tonnetz on the Hypersphere. In this case, we illustrate visually and musically, some well known harmonic path, made of P,L,R relations [11,12]. Beside existing methods, these new techniques enable the layman and the music student to better understand proposals that are evident to experimented musicians.

References

1. Byers, N.: E.Noether's Discovery of the Deep Connection Between Symmetries and Conservation Laws. IMC 12 (1999)
2. Lederman, L., Hill, C.: Symmetry and the Beautiful Universe. Prometheus Books (2004)
3. Andreatta, M., Agon, C.: Formalisation algébrique des structures musicales aide de la Set-Theory: aspects théoriques et analytiques. Actes des Journées d'Informatique Musicale (2003)
4. Forte, A.: The Structure of Atonal Music. Yale University Press, New Haven (1973)
5. Balzano, G.: The group-theoretic description of 12-fold and microtonal pitch systems. Computer Music Journal, 66–84 (1980)
6. Mazzola, G.: The Topos of Music. Birkhuser Verlag, Basel (2003)
7. Noll, T.: Geometry of Chords. In: Puebla, E.L. (ed.) Electronic Bulletin of the Sociedad Matematica Mexicana, vol. 1 (2001)
8. Baroin, G., Ferré, L.: Representation of musical pitch space using graphs, spectrum and hyperspaces. In: Proceedings of 8FCC Combinatorical Conference, Paris (2010)
9. Baroin, G.: De Newton à Riemann, Graphes et Graphisme: Interactions Mathématico-Musico-Plastiques. Litter@incognita (3), 41–57 (2010)
10. Leys, J., Ghys, E., Alvarez, A.: Dimensions: A walk through mathematics! (CGI movie) (2008), `http://www.dimensions-math.org`
11. Cohn, R.: Dramatization of hypermetric conflicts in the Scherzo of Beethoven's Ninth Symphony. 19th-Century Music 15, 22–40 (1992)
12. Crans, A., Fiore, T., Satyendra, R.: Musical Actions of Dihedral Groups. Amer. Math. Monthly 116(6), 479–495 (2009)