The Order Encoding: From Tractable CSP to Tractable SAT

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Many mathematical and practical problems can be expressed as *constraint satisfaction problems* (CSPs). The general CSP is known to be NP-complete, but many different conditions have been identified which are sufficient to ensure that classes of instances satisfying those conditions are tractable, that is, solvable in polynomial time [1,2,3,4,7]. The increasing efficiency of SAT-solvers has led to the development of SAT-based constraint solvers and various SAT encodings for CSPs [6]. However, most previous comparison between such encodings has been purely empirical. In a recent paper we showed that current SAT-solvers will decide the satisfiability of the *direct encoding* of any CSP instance with bounded width in expected polynomial time [5]. In this paper we give a theory-based argument to prefer the *order encoding* instead for certain other families of tractable constraint satisfaction problems. We consider problems of the form CSP(C), consisting of all CSP instances whose constraint relations belong to some fixed set of relations C, known as a *constraint language*. Schaefer's well-known dichotomy theorem [7] identifies all the tractable constraint languages over a Boolean domain, that is, all the tractable language classes for SAT.

A *sparse encoding* of a CSP instance introduces a new Boolean variable, $x^{=}_{v,a}$, for each possible variable assignment, $v = a$. The *log encoding* introduces a Boolean variable for each *bit* in the value of a CSP variable. It turns out that under such encodings tractable CSPs cannot be translated into tractable language classes of SAT. In particular, we have shown that:

Proposition 1. *No sparse encoding of a CSP instance with domain size > 2 belongs to a tractable language class of SAT. Moreover, the log encoding of any CSP instance with domain size > 7 containing certain unary constraints does not belong to any tractable language class of SAT.*

In the order encoding [8] each Boolean variable, $x^{\leq}_{v,c}$, represents a comparison, $v \leq c$. Under that encoding we have shown that certain tractable CSP classes are translated to tractable language classes of SAT, and hence efficiently solvable.

For example, a CSP instance is called *constant-closed* if every constraint in it allows some fixed constant value d to be assigned to all variables in its scope.

Theorem 1. *If all the constraints in a CSP instance are constant-closed for the lowest domain value, then its order encoding will be constant-closed for the value* True.

Hence, we have shown that using the order encoding to translate a CSP instance that is constant-closed for the lowest domain value gives a set of clauses satisfying

* The provision of an EPSRC Doctoral Training Award to Justyna Petke is gratefully acknowledged.

K.A. Sakallah and L. Simon (Eds.): SAT 2011, LNCS 6695, pp. 371–372, 2011.

the first condition of Schaefer's Dichotomy Theorem. Similarly, constraints that are constant-closed under the highest domain value translate under the order encoding to clauses that satisfy the second condition of that theorem.

A rather more interesting family of tractable constraint satisfaction problems is the class of CSPs whose constraints are all *max-closed.*

Lemma 1 ([4]). *If the domain of the variables is $\{True, False\}$, with $False < True$, then a constraint is min-closed if and only if it is logically equivalent to a conjunction of Horn clauses over literals representing comparisons.*

Theorem 2. *If a CSP instance P contains max-closed constraints only, then its order encoding will be min-closed.*

Hence, max-closed constraints translate using the order encoding to clauses satisfying the third condition of Schaefer's Dichotomy Theorem. By symmetry between min-closed and max-closed constraints, min-closed constraints translate to clauses satisfying the fourth condition of Schaefer's Dichotomy Theorem.

Connected-row-convex constraints were first defined in [3] using a standard matrix representation of binary relations. Here is an alternative characterisation:

Lemma 2 ([2]). *A constraint is connected-row-convex if and only if it is logically equivalent to a conjunction of 2-CNF clauses over literals representing comparisons.*

Connected-row-convex constraints translate to clauses satisfying the fifth condition of Schaefer's Dichotomy Theorem due to the following result:

Theorem 3. *If a CSP instance P contains only connected-row-convex constraints, then its order encoding will be connected-row-convex.*

The final, sixth, condition in Schaefer's Dichotomy Theorem can never be satisfied using the order encoding, since (for all domains with 3 or more elements) it is already broken by the consistency clauses, $\neg(x^{\leq}_{v,c-1}) \vee (x^{\leq}_{v,c})$. Hence we have given a complete list of all constraint languages which are encoded to tractable language classes for SAT using the order encoding.

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