Hyperconnections and Openings on Complete Lattices

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Abstract. In this paper the notion of hyperconnectivity, which is an extension of connectivity is explored in the lattice theoretical framework. It is shown that a fourth axiom is needed when moving from connections to hyperconnections, in order to define hyperconnected components meaningfully, which is important for the definition of, e.g., viscous levellings. New hyperconnectivity openings, which are the hyperconnected equivalents of connectivity openings are then defined. It then shown that *all* algebraic openings which are translation and grey-scale invariant can be described as hyperconnected attribute filters. This means that hyperconnectivity lies at the heart of a vast range of morphological filters.

1 Introduction

Hyperconnectivity was first put forward by Serra [8] as a generalization of connectivity. As discussed in [10, 11], hyperconnectivity offers the prospect of generalizing a large number of morphological filters into a single framework, and the existence of a continuum of filters with properties intermediate between connected and structural filters was proposed. Fig. 1 shows an example comparing structural open-close with connected and viscous hyperconnected levellings. Though viscous *reconstruction* can readily be described in terms of a connection on a viscous lattice [9], this is awkward for a levelling, because the viscous lattice, consisting of the set of dilates of members of some lattice $\mathcal{L}$, is not complemented. This means we cannot define auto-dual filters because the complement of a member of a viscous lattice is not necessarily a member of that lattice. Because viscous hyperconnections are defined in terms of the original lattice in [10], we can define auto-dual filters such as levellings without difficulty. This is one reason why we need to define hyperconnections for complete lattices.

A shortcoming of [10], was that only the finite, set-theoretical case was discussed. In this paper, the theory is extended to the compete lattice framework, and the infinite case. The new theory provides deeper understanding of hyperconnectivity, and provides a theoretical basis for grey-scale hyperconnected filters such as those developed in [5, 4]. I will first define some basic notions and notation, and then recall the theory of connectivity classes or connections, as developed by Serra in the complete lattice case [8]. In the same paper Serra put forward the notion of hyperconnections, by generalizing the third axiom for connections. I will show that this generalization is insufficient to define hyperconnected components properly, and that a fourth axiom is required. This extra axiom allows definition of new hyperconnectivity openings, which generalize the operators developed in the finite, set theoretical case in [10]. In that paper, and in [11],

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(a) (b) (c) (d)

Fig. 1. Self-dual filters: (a) original image f; (b) point-wise average of structural open-close, and close-open of f; (c) levelling of f using (b); (c) equivalent viscous hyperconnected levelling

it was shown that structural openings, viscous filters, and path openings all were special cases of hyperconnected filters. In this paper I will show that all openings which when applied to any canonical sup-generator (e.g., impulse function) returns the zero element of the lattice is a hyperconnected attribute filter. This means that hyperconnectivity lies at the heart of a vast number of morphological filters.

2 Theory

In the following $\mathcal{L}$ denotes a complete lattice, with global infimum and supremum $\mathbf{0}$ and $\mathbf{1}$, and $\mathcal{P}(\mathcal{L})$ the power lattice of $\mathcal{L}$. Elements of $\mathcal{L}$ are denoted as lower-case a, b, c; elements of $\mathcal{P}(\mathcal{L})$ as upper-case A, B, C, etc. An algebraic opening is any operator γ which is idempotent ($\gamma(\gamma(a)) = \gamma(a)$), increasing ($a \leq b \Rightarrow \gamma(a) \leq \gamma(b)$) and anti-extensive ($a \leq \gamma(a)$). A cover C of any $a \in \mathcal{L}$ is an element of $\mathcal{P}(\mathcal{L})$ such that $\bigvee C = a$. A partition $P \in \mathcal{P}(\mathcal{L})$ of any $a \in \mathcal{L}$, is a cover such that any two $x_1, x_2 \in P$ are either equal or disjoint. A chain is any totally ordered set.

2.1 Redundancy Operators

Any non-empty $A \in \mathcal{P}(\mathcal{L})$ is the union of a set of maximal chains C_i, because any chain $C \subseteq A$ can be extended to a maximal chain according to the Hausdorff maximal principle. For every element $a \in A$, $\{a\}$ is a chain, which can be extended to a maximal chain C_i with $\{a\} \subseteq C_i \subseteq A$. Thus the union $\bigcup_i C_i \subseteq A$, because all maximal chains $C_i \subseteq A$. However, because all $a \in A$ are contained in at least one chain C_i, $A \subseteq \bigcup_i C_i$, and thus $A = \bigcup_i C_i$. This leads to the definition of *non-redundancy*:

Definition 1 (Non-redundancy). *Any element $A \in \mathcal{P}(\mathcal{L})$ is said to be* non-redundant *if*

$$a_i \leq a_j \quad \Rightarrow \quad a_i = a_j, \quad \forall a_i, a_j \in A \tag{1}$$

or, equivalently, that all maximal chains $C_i \subseteq A$ have cardinality $\#(C_i) = 1$.

It is readily seen that all partitions are non-redundant covers, but not all non-redundant covers are partitions.

Any set $A \in \mathcal{P}(\mathcal{L})$ is said to be *chain-sup-complete* if for every non-empty chain $C \subseteq A$ we have $\bigvee C \in A$. All non-redundant sets, and all finite A are chain-sup-complete. Note that if the supremum of the empty chain ($= \mathbf{0}$) is also member of A it is *chain complete* or *ω-complete* as in [3]. It can readily be shown that the union of two chain-sup-complete sets is itself chain-sup-complete. The set of all chain-sup-complete elements of $\mathcal{P}(\mathcal{L})$ is denoted as $\mathcal{P}_{csc}(\mathcal{L})$.

We define $\mathcal{N}(\mathcal{L}) \subseteq \mathcal{P}_{csc}(\mathcal{L}) \cup \{\emptyset\}$ as the family of all non-redundant elements of $\mathcal{P}_{csc}(\mathcal{L}) \cup \{\emptyset\}$. On $\mathcal{N}(\mathcal{L})$ we define the *refinement order* $\sqsubseteq$ as used in partitions

$$A \sqsubseteq B \quad \equiv \quad \forall a_i \in A \ \ \exists b_j \in B : a_i \leq b_j \tag{2}$$

If $A_1 \sqsubseteq A_2$ for two partitions or covers we state that A_1 is finer than A_2, or, equivalently, A_2 is coarser than A_1. Relation $\sqsubseteq$ is a partial order $\mathcal{N}(\mathcal{L})$, and a partial preorder on $\mathcal{P}(\mathcal{L})$, i.e., it is reflexive and transitive, but not antisymmetric. We can now define the *reduction operator* which is crucial to the selection of hyperconnected components:

Definition 2 (Reduction Operator). *The reduction operator $\psi_{\mathcal{N}} : \mathcal{P}(\mathcal{L}) \to \mathcal{N}(\mathcal{L})$ is defined as*

$$\psi_{\mathcal{N}}(A) = \{a \in A \mid \nexists b \in A : a < b\} \tag{3}$$

Because $\psi_{\mathcal{N}}(A) \subseteq A$, we can readily see from the definition that no two elements $a, b \in \psi_{\mathcal{N}}(A)$ exist such that $a \leq b$, and $\psi_{\mathcal{N}}(A) \in \mathcal{N}(\mathcal{L})$.

Proposition 1. *For any set $A \in \mathcal{P}_{csc}(\mathcal{L})$*

1. *$\psi_{\mathcal{N}}(A)$ is the least element in $\mathcal{N}(\mathcal{L})$ with $A \sqsubseteq \psi_{\mathcal{N}}(A)$*
2. *$\bigvee A = \bigvee \psi_{\mathcal{N}}(A)$*

Proof. If $A \in \mathcal{P}_{csc}(\mathcal{L})$, $\psi_{\mathcal{N}}(A)$ contains all maximal elements of A, and nothing else. Therefore, $A \sqsubseteq \psi_{\mathcal{N}}(A)$. If we obtain some $B \in \mathcal{N}(\mathcal{L})$, with $B \sqsubset \psi_{\mathcal{N}}(A)$, by either removing any element $a \in \psi_{\mathcal{N}}(A)$, or replacing it by some $b < a$, this means there exists no $c \in B$ such that $a \leq c$. Because $a \in A$ we have $A \not\sqsubseteq B$, proving property 1.

To prove the second point, we can write

$$A = \bigcup_i \{C_i\} \quad \Rightarrow \quad \bigvee A = \bigvee_i \left\{ \bigvee C_i \right\} \tag{4}$$

with $\{C_i\}$ the set of all maximal chains $C_i \subseteq A$. Because A is chain sup-complete, it contains the maximal elements of all its non-empty chains C_i, and we may write

$$\psi_{\mathcal{N}}(A) = \left\{ \bigvee C_i \right\} \quad \Rightarrow \quad \bigvee \psi_{\mathcal{N}}(A) = \bigvee_i \left\{ \bigvee C_i \right\} = \bigvee A \tag{5}$$

which proves Proposition 1.

If $A \notin \mathcal{P}_{csc}(\mathcal{L})$, these properties do not necessarily hold. For example, let A be the set of open balls b_r of radius r defined as $A = \{b_{(1-1/n)}, n \in \mathbb{Z}^+\}$. The supremum of this set is the open ball $b_1 \notin A$, and thus $A \notin \mathcal{P}_{csc}(\mathcal{L})$. Because for any element $a \in A$ there exist infinitely many $b \in A$ such that $a < b$, we obtain $\psi_{\mathcal{N}}(A) = \emptyset$. This violates both properties in Proposition 1.

2.2 Connectivity

Connectivity such as is used in morphological filtering is defined through the notion of connectivity classes [8,6] defined as follows.

Definition 3. *A connectivity class $\mathcal{C} \subseteq \mathcal{P}(\mathcal{L})$ is a set of elements of $\mathcal{L}$ with the following three properties:*

1. $\mathbf{0} \in \mathcal{C}$
2. *$\mathcal{C}$ is sup-generating,*
3. *for each family $\{c_i\} \subseteq \mathcal{C}$, $\bigwedge c_i \neq \mathbf{0}$ implies $\bigvee c_i \in \mathcal{C}$.*

Any element $c \in \mathcal{C}$ is said to be connected. Any element $a \in \mathcal{L}$ can be partitioned into connected components. These are the elements of $c \in \mathcal{C}$ such that $c \leq a$ of maximal extent, i.e., if $c \leq a$, $c \in \mathcal{C}$, and there exists no $d \in \mathcal{C}$ such that $c < d \leq a$, then c is a connected component of a. Let $\mathcal{C}_a$ be defined as follows:

$$\mathcal{C}_a = \{c \in \mathcal{C} \mid \mathbf{0} < c \leq a\}, \tag{6}$$

in other words $\mathcal{C}_a$ is the set of all elements of $\mathcal{C} \setminus \{\mathbf{0}\}$ majorated by a. The set of all connected components $\mathcal{C}_a^*$ of a is simply

$$\mathcal{C}_a^* = \psi_{\mathcal{N}}(\mathcal{C}_a). \tag{7}$$

We can do this because of the following property.

Proposition 2 (Chain completeness and connections). *Any connection $\mathcal{C}$ on any complete lattice $\mathcal{L}$ is chain complete.*

Proof. The supremum of the empty chain is $\mathbf{0} \in \mathcal{C}$. Let $B \subseteq \mathcal{C}$ be a non-empty chain. If $\bigwedge B \neq \mathbf{0}$, then $\bigvee B \in \mathcal{C}$ by definition. If $\bigwedge B = \mathbf{0}$, and $\bigvee B \neq \mathbf{0}$ we pick an arbitrary $m \in B$ such that $\mathbf{0} < m \leq \bigvee B$ and define the set

$$B^+ = \{b_i \in B \mid m \leq b_i\}. \tag{8}$$

Thus, $\mathbf{0} < m \leq \bigwedge B^+$ and therefore $\bigvee B^+ \in \mathcal{C}$. Obviously

$$\bigvee B = \bigvee B^+ \quad \Rightarrow \quad \bigvee B \in \mathcal{C}. \tag{9}$$

Finally, if $\bigvee B = \mathbf{0}$ we have $\bigvee B \in \mathcal{C}$ as well, proving Proposition 2.

The connected components of any image can be accessed through *connectivity openings* [8,1]:

Definition 4. *The connectivity opening γ_x of $a \in \mathcal{L}$ marked by some $x \in \mathcal{C} \setminus \{\mathbf{0}\}$, is given by*

$$\gamma_x(a) = \begin{cases} \bigvee\{c_i \in \mathcal{C} \mid x \leq c_i \leq a\} & \text{if } x \leq a \\ \mathbf{0} & \text{otherwise}. \end{cases} \tag{10}$$

In this definition the notion of maximum extent is derived by taking the supremum of all connected subsets of a but larger than or equal to x. It can readily be shown that this is equivalent to

$$\gamma_x(a) = \begin{cases} c_i \in \mathcal{C}_a^* : x \leq c_i & \text{if } x \leq a \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{11}$$

This equivalence stems from the fact that c_i in (10) for which of $x \leq c_i \leq a$ have infimum $x \neq \mathbf{0}$, and that their supremum is therefore connected. Serra [8] notes that in many important cases it is sufficient to use some canonical set of sup-generators $\mathcal{S}$ to obtain all connected components of any $a \in \mathcal{L}$, through a family of connectivity openings $\{\gamma_s, s \in \mathcal{S}\}$. Proposition 2 shows that if the smallest connection possible of $\mathcal{L}$ is $\{\mathbf{0}\} \cup \mathcal{S}$, as asserted in [8, 6], this is only true if $\mathcal{S}$ is chain-sup-complete.

2.3 Hyperconnectivity

Hyperconnectivity extends connectivity by generalizing the third condition of Definition 3 [8]. Instead of using a non-empty intersection, we can use any *overlap criterion* $\perp$ which is defined as follows.

Definition 5. *An overlap criterion in $\mathcal{L}$ is a mapping $\perp : \mathcal{P}(\mathcal{L}) \rightarrow \{\mathit{false}, \mathit{true}\}$ such that $\perp$ is decreasing, i.e., for any $A, B \in \mathcal{P}(\mathcal{L})$*

$$A \subseteq B \quad \Rightarrow \quad \perp(B) \leq \perp(A). \tag{12}$$

Note that $\mathit{false} \leq \mathit{true}$. Any $A \in \mathcal{P}(\mathcal{L})$ for which $\perp(A) = \mathit{true}$ is said to be *overlapping*. We can now define a *hyperconnectivity class* as follows.

Definition 6. *A hyperconnectivity class $\mathcal{H} \subseteq \mathcal{L}$ is a class with the following properties:*

1. $\mathbf{0} \in \mathcal{H}$
2. *$\mathcal{H}$ is sup-generating*
3. *$\mathcal{H}$ is chain-sup complete,*
4. *for each family $\{h_i\} \subset \mathcal{H}$, $\perp(\{h_i\})$ implies $\bigvee_i h_i \in \mathcal{H}$,*

with $\perp$ an overlap criterion, for which $\perp(A) \Rightarrow \bigwedge A \neq \mathbf{0}$.

The above definition of hyperconnections differs from that given by Serra [8], in that it has four axioms, rather than just three. The additional axiom (number 3) is necessary in the infinite case, as will be discussed shortly. If a canonical family of sup-generators $\mathcal{S}$ exists, axiom 2 can be rewritten to $s \in \mathcal{H}$ for all $s \in \mathcal{S}$ as in the connected case [8, 6].

Serra [8] showed that connectivity is an extension of hyperconnectivity, in which the third axiom in Def. 3 is replaced by a stricter requirement. For example we might require that there exist a ball b_r of some diameter r for which $b_r \leq \bigwedge_i h_i$, leading to viscous hyperconnections [10]. Axioms 1 and 3 of Def. 6 mean hyperconnections are chain complete, just like connections.

Like the notion of *connected* components for connectivity classes, we need to define the notion of *hyperconnected components* of $a \in \mathcal{L}$, which are members of the hyperconnection $h_i \in \mathcal{H}$, such that $h_i \leq a$ has maximal extent. In complete analogy with connected components we can first define the set $\mathcal{H}_a$ of all elements of $\mathcal{H}$ such that $\mathbf{0} < h_i \leq a$:

$$\mathcal{H}_a = \{h \in \mathcal{H} \mid \mathbf{0} < h \leq a\}. \tag{13}$$

$\mathcal{H}_a$ is either chain-sup-complete or empty. The latter is the case if $a = \mathbf{0}$. If $a \neq \mathbf{0}$, there exists at least one $h \in \mathcal{H}$ such $\mathbf{0} < h \leq a$, because $\mathcal{H}$ is sup-generating. In that case, consider any non-empty chain $\{h_i\} \subset \mathcal{H}_a$. Obviously,

$$\bigvee_i h_i \in \mathcal{H}, \tag{14}$$

through axiom 3 in Def. 6. Because all $h_i \leq a$ it follows that

$$\bigvee_i h_i \leq a, \quad \Rightarrow \quad \bigvee_i h_i \in \mathcal{H}_a. \tag{15}$$

This proves that $\mathcal{H}_a$ is chain-sup-complete if $a \neq \mathbf{0}$. The set of hyperconnected components $\mathcal{H}_a^*$ of a is defined equivalently

$$\mathcal{H}_a^* = \psi_{\mathcal{N}}(\mathcal{H}_a). \tag{16}$$

Unlike $\mathcal{C}_a^*$, $\mathcal{H}_a^*$ is not necessarily a partition of a, because two hyperconnected components h_j, h_k, with $h_j \neq h_k$, may have a non-zero infimum, but $h_j \vee h_k$ is not a member of $\mathcal{H}_a$ because $\perp(\{h_j, h_k\}) = \mathit{false}$. This is easily understood, because if $\perp(\{h_j, h_k\}) = \mathit{true}$, then $h_j \vee h_k \in \mathcal{H}_a$. However, because $h_j, h_k \in \mathcal{H}_a^*$, we have $h_j \not\leq h_k$ and $h_k \not\leq h_j$ because of the definition of $\psi_{\mathcal{N}}$. Thus $h_j < h_j \vee h_k$ and $h_k < h_j \vee h_k$. This means $h_j \vee h_k$ is a larger element of $\mathcal{H}_a$ than either h_j or h_k which contradicts their membership of $\mathcal{H}_a^*$. Note that due to Prop. 1, $\mathcal{H}_a^*$ only is a cover of a if $\mathcal{H}$ is chain-sup-complete. This is why we need the third axiom in Def. 6 in the infinite case.

2.4 Hyperconnectivity Openings

We now introduce the families of hyperconnectivity openings $\Upsilon_h : \mathcal{N}(\mathcal{L}) \to \mathcal{N}(\mathcal{L})$, with $h \in \mathcal{H}$, which return sets of hyperconnected components. If the lattice is supplied with a canonical set of sup-generators $\mathcal{S}$, we may restrict the family $\{\Upsilon_h, h \in \mathcal{H}\}$ to $\{\Upsilon_s, s \in \mathcal{S}\}$, as before. As in [10] we need the reduction operator.

Definition 7. *The hyperconnectivity opening $\Upsilon_x : \mathcal{N}(\mathcal{L}) \to \mathcal{N}(\mathcal{L})$, with $x \in \mathcal{H} \setminus \{\mathbf{0}\}$, associated with hyperconnectivity class $\mathcal{H}$ is defined as*

$$\Upsilon_x(A) = \psi_{\mathcal{N}}\left(\bigcup_{a \in A} \{h \in \mathcal{H}_a \mid x \leq h\}\right) \tag{17}$$

If the parameter A is just a singleton $\{a\}$, Υ_x extracts the set of hyperconnected components of a containing x. Note that $\Upsilon_x(\emptyset) = \Upsilon_x(\{\mathbf{0}\}) = \emptyset$, and more generally that if there exists no $a \in A$ such that $x \leq a$ we have $\Upsilon_x(A) = \emptyset$.

Theorem 1 **(Υ_x are algebraic openings).** *For any hyperconnection $\mathcal{H}$ on any complete lattice $\mathcal{L}$, $\{\Upsilon_x, x \in \mathcal{H} \setminus \{\mathbf{0}\}\}$ is a family of algebraic openings on $(\mathcal{N}(\mathcal{L}), \sqsubseteq)$, for which*

$$\{\mathbf{0}\} \cup \left(\bigcup_{x \in \mathcal{H} \setminus \{\mathbf{0}\}} \bigcup_{A \in \mathrm{Inv}(\Upsilon_x)} A \right) = \mathcal{H}. \tag{18}$$

$\mathrm{Inv}(\gamma) \subseteq \mathcal{L}$ denotes the *invariance domain* of γ, which is the set $\{a \in \mathcal{L} \mid a = \gamma(a)\}$.

Proof. By definition, the output of $\Upsilon_h(A)$ is a non-redundant subset of $\mathcal{H}$ such that for each element $h_i \in \Upsilon_x(A) \subseteq \mathcal{H}$ we have $s \leq h_i$. Furthermore, $\Upsilon_x(A) \sqsubseteq A$, because we can see from the definition that for every $h_i \in \Upsilon_x(A)$ there exists an $a \in A$ such that $h_i \leq a$, which proves anti-extensiveness in terms of $\sqsubseteq$ as defined in (2).

It is readily verified that $\Upsilon_x(\emptyset) = \emptyset$ for all $x \in \mathcal{H} \setminus \{\mathbf{0}\}$, so idempotence holds whenever $\Upsilon_x(A) = \emptyset$. Let $\{h_i\} = \Upsilon_x(A) \neq \emptyset$, with i from some index set. Applying Υ_x to $\{h_i\}$ is equivalent to applying the reduction operator to the union of all sets

$$\mathcal{H}^s_{h_i} = \{h_j \in \mathcal{H} \mid s \leq h_j \leq h_i\}. \tag{19}$$

All $\mathcal{H}^s_{h_i}$ contain h_i because $s \leq h_i$ and $h_i \in \mathcal{H}$. Thus each $\mathcal{H}^s_{h_i}$ contains h_i as maximal element. Therefore we have

$$\Upsilon_x(\Upsilon_x(A)) = \Upsilon_x(\{h_i\}) = \psi_{\mathcal{N}}\left(\bigcup_i \mathcal{H}^s_{h_i} \right) = \{h_i\} \tag{20}$$

because the reduction operator removes all elements h_j of every $\mathcal{H}^s_{h_i}$, such that $h_j < h_i$, for each $h_i \in \Upsilon_x(A)$. All elements $h_i \in \Upsilon_x(A)$ are preserved because it is a non-redundant set, which cannot be reduced any further. Thus

$$\Upsilon_x(\Upsilon_x(A)) = \Upsilon_x(A) \tag{21}$$

and $\Upsilon_x(A)$ is idempotent. The above arguments also show that all elements of the invariance domain of any Υ_x consist of subsets of $\mathcal{H}$. Furthermore, any non-zero $h_i \in \mathcal{H}$ is marked by at least one sup-generator, so that every non-zero element of $\mathcal{H}$ is contained in the union of the elements of the invariance domains of all $\Upsilon_x, x \in \mathcal{H} \setminus \{\mathbf{0}\}$, simply because $\Upsilon_x(\{h_i\}) = \{h_i\}$, for any $x \leq h_i$. Thus, the union of all elements of the invariance domains of all $\Upsilon_x, x \in \mathcal{H} \setminus \{\mathbf{0}\}$, augmented with **0** is equal to $\mathcal{H}$.

Finally, we must show that Υ_x is increasing in terms of $\sqsubseteq$. Let $A, B \in \mathcal{N}(\mathcal{L})$ with $A \sqsubseteq B$. This means that for every $a \in A$ there exists a $b \in B$ such that $a \leq b$. We already know that for every $h_i \in \Upsilon_x(A)$ there exists an $a \in A$ such that $h_i \leq a$. Therefore, there exists a $b \in B$ such that $h_i \leq b$. This means that either $h_i \in \Upsilon_x(B)$, or there exists an $h_j \in \Upsilon_x(B)$ such that $h_i < h_j$. Thus, for every $h_i \in \Upsilon_x(A)$ there exists an $h_j \in \Upsilon_x(B)$ such that $h_i \leq h_j$, and thus

$$A \sqsubseteq B \quad \Rightarrow \quad \Upsilon_x(A) \sqsubseteq \Upsilon_x(B), \tag{22}$$

proving increasingness, and all $\Upsilon_x, x \in \mathcal{H} \setminus \{\mathbf{0}\}$ are algebraic openings, proving Theorem 1, and that the family $\{\Upsilon_x, x \in \mathcal{H} \setminus \{\mathbf{0}\}\}$ retrieves $\mathcal{H}$, through (18).

What needs to be done is to assess which properties a family of mappings $\Upsilon_x : \mathcal{N}(\mathcal{L}) \to \mathcal{N}(\mathcal{L})$ requires for it to define a hyperconnectivity class, in the same way a family of connectivity openings defines a connectivity.

Theorem 2 (Hyperconnectivity Openings). *On any lattice $\mathcal{L}$, every hyperconnection $\mathcal{H}$ associated to an overlap criterion $\perp$ is equivalent to a family of algebraic openings $\{\Upsilon_x, x \in \mathcal{H} \setminus \{\mathbf{0}\}\}$ on $(\mathcal{N}(\mathcal{L}), \sqsubseteq)$ with the following properties*

1. *Υ_x is an algebraic opening indexed by $x \in \mathcal{H} \setminus \{\mathbf{0}\}$.*
2. *for all $x \in \mathcal{H} \setminus \{\mathbf{0}\}$ we have $\Upsilon_x(\{x\}) = \{x\}$*
3. *for all $A \in \mathcal{N}(\mathcal{L})$, and all $x \in \mathcal{H} \setminus \{\mathbf{0}\}$ we have $s \not\leq a, \forall a \in A \;\Rightarrow\; \Upsilon_x(A) = \emptyset$;*
4. *for any $x, y \in \mathcal{H} \setminus \{\mathbf{0}\}$ and any $A \in \mathcal{P}(\mathcal{L})$, $h_i \in \Upsilon_x(A)$ and $y \leq h_i \;\Rightarrow\; h_i \in \Upsilon_y(A)$.*
5. *for all $x \in \mathcal{H} \setminus \{\mathbf{0}\}$ and all $a \in \mathcal{L}$, and any $\{h_i\} \subseteq \Upsilon_x(\{a\})$ we have $\#\{h_i\} \neq 1 \;\Rightarrow\; \perp(\{h_i\}) = false$.*

Proof. The first property follows directly from Theorem 1, and its proof. The second property follows directly from Def. 7 and the fact that

The third property derives directly from Def. 7, because (17) implies that if $x \not\leq a$ for all $a \in A$, $\Upsilon_x(A) = \emptyset$. Together with the first two this guarantees $\bigvee \Upsilon(A) = \bigvee A$.

The fourth property can be derived directly from (17): Because $h_i \in \Upsilon_x(A)$, there exists no hyperconnected set $h_j \in \Upsilon_x(A)$, such that $h_i < h_j$, because $\Upsilon_x(A)$ is non-redundant. If $t \leq h_i$ but $h_i \notin \Upsilon_t(A)$, this would imply that there is some $h_j \in \Upsilon_x(A)$, such that $h_i \leq h_j$, leading to contradiction.

The fifth property states that no set of two or more sets $\{h_i\} \in \Upsilon_x(\{a\})$ can overlap in the sense of $\perp$. If they did, $\bigvee_i h_i \in \mathcal{H}$ and $s \leq \bigvee_i h_i \in \Upsilon_x(\{a\}$, and $\{h_i\} \not\subset \Upsilon_x(\{a\})$.

We now show that any family of operators $\{\Upsilon_x, x \in \mathcal{H} \setminus \{\mathbf{0}\}\}$ with the above properties for a given overlap criterion $\perp$ is associated with a hyperconnection. Suppose we have some family $\{\Upsilon_x, x \in \mathcal{H} \setminus \{\mathbf{0}\}\}$ of algebraic openings marked by $x \in \mathcal{H} \setminus \{\mathbf{0}\}$. Let $\mathcal{I}$ be defined as

$$\mathcal{I} = \{\mathbf{0}\} \cup \left(\bigcup_{x \in \mathcal{H}} \bigcup_{A \in \mathrm{Inv}(\Upsilon_x)} A \right), \tag{23}$$

for our family $\{\Upsilon_x, x \in \mathcal{H} \setminus \{\mathbf{0}\}\}$. We must now show that $\mathcal{I}$ is a hyperconnection.

The second property implies $\mathcal{H} \subset \mathcal{I}$, proving $\mathcal{I}$, is sup-generating, conforming to property 2 of Def. 6. The definition of $\mathcal{I}$ states that $\mathbf{0} \in \mathcal{I}$, meeting property 1 of Def. 6. Furthermore, let $C \subset \mathcal{I}$ be a chain. This implies $\Upsilon_x(\{\bigvee C\}) = \{\bigvee C\}$. If not, we would have to represent the hyperconnected components $\bigvee C$ as a chain with cardinality larger than one, which contradicts the fact that $\Upsilon_x(\{\bigvee C\}) \in \mathcal{N}(\mathcal{L})$. Thus $\mathcal{I}$ is chain sup complete.

Similarly, let $\perp(H) = \mathit{true}$ for some $H \subseteq \mathcal{I}$. According to property 5, and the fact that $\bigvee \Upsilon(A) = \bigvee A$ for any $A \in \mathcal{N}(\mathcal{L})$, we have $\Upsilon_x(\{\bigvee H\}) = \bigvee H$ and thus $\perp(H) = \mathit{true}$ guarantees $\bigvee H \in \mathcal{I}$, and $\mathcal{I}$ is a hyperconnection.

Finally, we show that applying (17) to the hyperconnection $\mathcal{I}$, we find the same family of openings we started out with, note that $\mathcal{H} \subseteq \mathcal{I}$, due to property 2 in Theorem 2.

Let the family $\{\Upsilon_x^{\mathcal{I}}, x \in \mathcal{H} \setminus \{\mathbf{0}\}\}$ be the new family of operators, with each $\Upsilon_x^{\mathcal{I}}$ defined as

$$\Upsilon_x^{\mathcal{I}}(A) = \psi_{\mathcal{N}}\left(\bigcup_{a \in A} \{h \in \mathcal{I}_a \mid x \leq h\}\right) \tag{24}$$

which is just rewriting (17), by replacing $\mathcal{H}$ with $\mathcal{I}$. It is readily verified that the invariance domain $\mathrm{Inv}(\Upsilon_x^{\mathcal{I}})$ of each $\Upsilon_x^{\mathcal{I}}$ is given by

$$\mathrm{Inv}(\Upsilon_x^{\mathcal{I}}) = \{A \in \mathcal{N}(\mathcal{I}) \mid \forall a \in A : x \leq a\} \tag{25}$$

with $\mathcal{N}(\mathcal{I})$ the family of non-redundant subsets of $\mathcal{I}$. This is true because the input of $\Upsilon_x^{\mathcal{I}}$ must be non-redundant by definition, and if and only if the input A of $\Upsilon_x^{\mathcal{I}}$ is a subset of $\mathcal{I}$ and all elements of A are larger than or equal to x is A left unaffected by $\Upsilon_x^{\mathcal{I}}$. For the original family $\{\Upsilon_x, x \in \mathcal{H} \setminus \{\mathbf{0}\}\}$, we have that all members of $\mathrm{Inv}(\Upsilon_x)$, must also be non-redundant, and a subset of $\mathcal{I}$, each element of which must be larger than or equal to x, due to properties 2 and 3 of Theorem 2, and increasingness of openings. Thus,

$$\mathrm{Inv}(\Upsilon_x) = \{A \in \mathcal{N}(\mathcal{I}) \mid \forall a \in A : x \leq a\} = \mathrm{Inv}(\Upsilon_x^{\mathcal{I}}) \tag{26}$$

for all $x \in \mathcal{H} \setminus \{\mathbf{0}\}$. Because any opening γ maps the input A to the largest element $B \in \mathrm{Inv}(\gamma)$ such that $B \leq A$, the invariance domain uniquely defines an opening. This means $\Upsilon_x = \Upsilon_x^{\mathcal{I}}$ for all $x \in \mathcal{H} \setminus \{\mathbf{0}\}$, proving Theorem 2.

3 Openings and Hyperconnections

We now turn to the question of which openings form part of the gamut of hyperconnected filters. In this discussion we restrict ourselves to the case in which a *chain-sup-complete*, canonical set $\mathcal{S}$ of sup-generators exists, i.e. $\mathcal{S} \cup \{\mathbf{0}\}$ is a minimal connection. In the binary case we have $\mathcal{S} = \{\{x\} \mid x \in E\}$. In the lattice of functions from E to $\mathbb{R} \cup \{-\infty, +\infty\}$ we obtain $\mathcal{S} = \{\delta_x^a, x \in E, a \in \mathbb{R} \cup \{+\infty\}\}$, with

$$\delta_x^a(y) = \begin{cases} a & \text{if } x = y \\ -\infty & \text{otherwise,} \end{cases} \tag{27}$$

i.e., we need to include both finite and infinite impulses.

In the following we need the property that $\mathrm{Inv}(\gamma)$ is closed under supremum [7] for any algebraic opening. In analogy to connected filters we arrive at the following definition.

Definition 8 (Hyperconnected openings). *An opening $\gamma : \mathcal{L} \to \mathcal{L}$ is* hyperconnected *if there exists a hyperconnection $\mathcal{H}$ such that*

$$\mathcal{H}^*(\gamma(a)) \subseteq \mathcal{H}^*(a) \quad \forall a \in \mathcal{L} \tag{28}$$

The meaning of this is that a hyperconnected opening only removes hyperconnected components. Due their anti-extensive nature, no opening can extend an existing hyperconnected component. In analogy with connected filters, no new components may arise. Thus, any hyperconnected component present in the opening must be present in the original.

Hyperconnected attribute filters can be defined in much the same way as connected attribute filters, using trivial filters. A *trivial filter* $\Psi_\Lambda(h)$ based on criterion $\Lambda : \mathcal{H} \to \{\mathit{false}, \mathit{true}\}$ returns h if $\Lambda(h) = 1$, and **0** otherwise, for any $h \in \mathcal{H}$. Let $\Psi_\Lambda(\mathcal{H}_a^*)$ be shorthand for the set of all $h_j \in \mathcal{H}_a^*$ such that $\Lambda(h_j) = 1$. A hyperconnected attribute filter $\Psi^\Lambda : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ based on criterion $\Lambda : \mathcal{H} \to \{\mathit{false}, \mathit{true}\}$ is defined as

$$\Psi^\Lambda(a) = \bigvee_{s \leq X} \bigvee_{h_i \in \Upsilon_x(a)} \Psi_\Lambda(h_i) = \bigvee_{h_j \in \mathcal{H}_a^*} \Psi_\Lambda(h_j) = \bigvee_{h_k \in \Psi_\Lambda(\mathcal{H}_a^*)} h_k, \tag{29}$$

The following theorem links openings with hyperconnections.

Theorem 3 (Hyperconnections and openings). *Every algebraic opening γ on any complete, sup-generated lattice $\mathcal{L}$ with a* chain-sup-complete *set of sup-generators $\mathcal{S}$ is associated with a hyperconnection $\mathcal{H}_\gamma$ given by*

$$\mathcal{H}_\gamma = \mathrm{Inv}(\gamma) \cup \mathcal{S}. \tag{30}$$

with overlap criterion

$$\perp_\gamma(A) = \bigwedge A \neq \mathbf{0} \quad \wedge \quad A \subseteq \mathrm{Inv}(\gamma) \tag{31}$$

Proof. Because $\mathbf{0} = \gamma(\mathbf{0})$ by anti-extensiveness, and inclusion of the sup-generators, we adhere to the first two properties of hyperconnections. It is trivial to show that $\perp_\gamma$ is decreasing because both terms are decreasing, and all must be met. The first term of (31) guarantees that the infimum of A is not **0**, which meets the requirement that $\perp_\gamma$ must imply a non-zero infimum. The second term states that $A \subseteq \mathrm{Inv}(\gamma)$, which guarantees $\bigvee A \in \mathcal{H}_\gamma$, because $\mathrm{Inv}(\gamma)$ is closed under supremum.

Finally, $\mathcal{H}_\gamma$ is the union of of two chain-sup-complete sets, and is therefore chain-sup-complete. Therefore $\mathcal{H}_\gamma$ is a hyperconnection, proving Theorem 3.

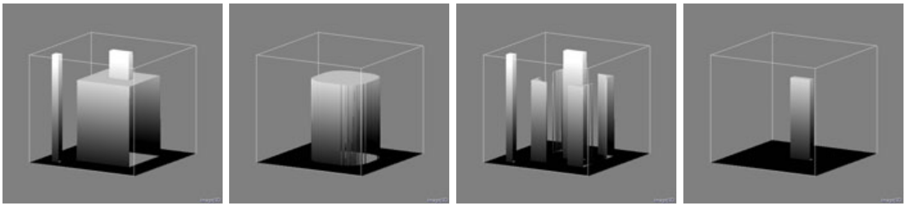
We have now shown any opening on any complete lattice, sup-generated by a chain-sup-complete family $\mathcal{S}$ to be associated with a hyperconnection, but that does not prove they are hyperconnected openings. For that we need to see whether the result of any opening contains only hyperconnected components of the input.

The hyperconnected components of any $a \in \mathcal{L}$ are simply $\gamma(a)$ augmented with such sup-generators as are required to "fill the gaps" between a and $\gamma(a)$, or

$$\mathcal{H}_\gamma^*(a) = \psi_\mathcal{N}(\{\gamma(a)\} \cup \{s \in \mathcal{S} | s \leq a\}). \tag{32}$$

This yields a non-redundant cover of a because $\mathcal{H}_\gamma$ is a hyperconnection if $\mathcal{S}$ is chain-sup complete. For example, take the image in Fig. 2(a). The main hyperconnected component $\gamma(f)$ is shown in Fig. 2(b). The remaining hyperconnected components are impulse functions at all points (x, y) where $\gamma(f)(x, y) \neq f(x, y)$. The height of each impulse function at (x, y) is given by $f(x, y)$, and their supremum is shown in Fig. 2(c). Fig. 2(d) shows that the infimum of (b) and(c) is not zero. We now consider the following opening γ_s with $s \in \mathcal{S}$, defined as

$$\gamma_s(a) = s \wedge a. \tag{33}$$



(a) (b) (c) (d)

Fig. 2. Hyperconnection derived from structural opening: (a) surface plot of grey-scale image f; (b) structural opening $\gamma(f)$; (c) union of sup-generators $s \in \mathcal{S}$ such that $s \leq f$ but $s \nleq \gamma(f)$; (d) the non-zero infimum of (b) and (c), typical for a hyperconnection, rather than a connection

This is an opening because it is idempotent, anti-extensive and increasing. Now let $t \in \mathcal{S}$ such that $s \leq t$. Obviously $\mathcal{H}^*_{\gamma_s}(t) = \{t\}$ because t is the only hyperconnected component of t. The only hyperconnected component of $\gamma_s(t)$ is s and

$$\mathcal{H}^*_{\gamma_s}(\gamma_s(t)) = \{s\} \nsubseteq \mathcal{H}^*_{\gamma_s}(t), \tag{34}$$

and therefore γ_s is not a hyperconnected filter according to Def. 8.

However, consider the subset of all openings on $\mathcal{L}$ which treats all sup-generators equally, in the sense that either

$$\mathcal{S} \subset \mathrm{Inv}(\gamma) \quad \text{or} \quad \gamma(s) = \mathbf{0} \quad \forall s \in \mathcal{S}. \tag{35}$$

The first case only holds for the identity operator, because if all sup-generators are in $\mathrm{Inv}(\gamma)$, so are all elements of $\mathcal{L}$. The identity operator is obviously hyperconnected. The second case holds in all openings that only preserve substructures in an object larger than any single sup-generator (remember $\mathcal{S} \in \mathcal{P}_{csc}(\mathcal{L})$). In particular, in the lattice of grey-scale images it holds for all translation and grey-scale invariant openings not equal to the identity operator. We will call such openings $\mathcal{S}$-*rejecting* openings.

Theorem 4 ($\mathcal{S}$-rejecting openings). *All $\mathcal{S}$-rejecting openings γ on any complete lattice $\mathcal{L}$, sup-generated by $\mathcal{S} \in \mathcal{P}_{csc}(\mathcal{L})$ are hyperconnected.*

Proof. If $x \in \mathcal{S}$, we have $\gamma(x) = \mathbf{0}$, because γ is $\mathcal{S}$-rejecting, and $\mathcal{H}^*_\gamma(\gamma(x)) = \emptyset$.

For any $x \in \mathcal{L} \setminus \mathcal{S}$, we have either $\gamma(x) = \mathbf{0}$, or $\gamma(x) \in \mathrm{Inv}(\gamma) \setminus \{\mathbf{0}\}$. In the first case we have the same situation as above, and in the second case $h_0 = \gamma(x)$ is the sole hyperconnected component of $\gamma(x)$. Because $h_0 \in \mathcal{H}^*_\gamma(x)$, γ is hyperconnected.

The above hyperconnected openings can be described as attribute filters in which the criterion $\Lambda : \mathcal{H}_\gamma \rightarrow \{\mathit{false}, \mathit{true}\}$ is

$$\Lambda(h) = (h \in \mathrm{Inv}(\gamma)). \tag{36}$$

4 Conclusion

In this paper I have shown how to extend the theory of hyperconnections to complete lattices. It was shown that we must require hyperconnections to be chain complete in the sense of [3], in the infinite case, in order to guarantee the existence of hyperconnected components.

Furthermore, a large family of openings, well beyond the cases discussed in [10,5,11] can be shown to be hyperconnected attribute filters. However, though we can write the operator in this way, it is not particularly useful, because the attribute criterion is defined in terms of $\mathrm{Inv}(\gamma)$, which is most easily checked by computing $\gamma(h)$, which is rather circular. Also, the generation of a cover consisting of a single huge entity plus a few details filled in by sup-generators might also be thought of as unsatisfactory.

However, given that there exists a hyperconnection describing any given $\mathcal{S}$-rejecting opening, it should in principle be possible to find a smallest hyperconnection $\mathcal{H}_\gamma^0 \leq \mathcal{H}_\gamma$ which describes the opening under study most efficiently. The smallest possible one would be $\{\mathbf{0}\} \cup \mathcal{S}$, which is impractical, because the acceptance or rejection of the hyperconnected components by the filter would not just rely on the components themselves, but on their context as well. An attribute filter using this description would not be adjacency stable [2]. Somewhere between $\{\mathbf{0}\} \cup \mathcal{S}$ and $\mathcal{H}_\gamma$ lies an optimum hyperconnection hyperconnection $\mathcal{H}_\gamma^0$ which is the smallest one which allows description of γ as an adjacency stable, hyperconnected, attribute filter. This hyperconnection would be the *characteristic hyperconnection* of γ. For structural openings, path openings, and viscous openings, these characteristic hyperconnections have been found in [10, 11]. For others, we must still obtain them.

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