Connective Segmentation Generalized to Arbitrary Complete Lattices

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Abstract. We begin by defining the setup and the framework of connective segmentation. Then we start from a theorem based on connective criteria, established for the power set of an arbitrary set. As the power set is an example of a complete lattice, we formulate and prove an analogue of the theorem for general complete lattices.

Secondly, we consider partial partitions and partial connections. We recall the definitions, and quote a result that gives a characterization of (partial) connections. As a continuation of the work in the first part, we generalize this characterization to complete lattices as well.

Finally we link these two approaches by means of a commutative diagram, in two manners.

Keywords: Connective segmentation, complete lattice, partial partition, block-splitting opening, commutative diagram.

1 Introduction

The theory of connective segmentation on sets, developed by, among others, Serra (see e.g. [8]) and Ronse (see e.g. [3]), has proved fruitful in image segmentation. In this article, we generalize this theory and concider connective segmentation on arbitrary complete lattices, rather than on the power set lattice $\mathcal{P}(E)$ of subsets of a set E. Apart from the theoretical value of such a generalization, it is also relevant in practice, as a number of important lattices are not of type $\mathcal{P}(E)$. Two major examples are

– viscous lattices, as described in [7]. The elements of a viscous lattice are the images of the subsets of a set E under a given dilation, ordered by inclusion. The smallest and greatest elements are $\emptyset \subseteq E$ and E, respectively, and for a non-empty family of subsets, the supremum is the union, whereas the infimum is the opening (corresponding to the dilation) of the intersection. Viscous lattices are atomistic, the atoms being not the points of E, as is true for $\mathcal{P}(E)$, but rather the images of such points under the dilation. For example, if $E = \mathbb{R}^n$, the structure element of the dilation may be a ball of fixed radius r. This gives a model of the physical world in the sense that atoms indeed are not singular points.

P. Soille, M. Pesaresi, and G.K. Ouzounis (Eds.): ISMM 2011, LNCS 6671, pp. 61–72, 2011.

– the lattice of functions defined on an arbitrary set, and taking values on the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. While this lattice is indeed important, it is not even atomistic, and thus there is a motivation to extend the theory beyond atomisticity as well.

Section 1.2 gives the definitions and one of the main results to be generalized.

1.1 Notation

Throughout this paper, all sets considered that are not obtained from other sets as subsets (and that are usually denoted by E) are assumed to be non-empty, unless otherwise stated. Analogously, in all complete lattices that do not arise from a construction on other complete lattices (and that are usually denoted by $\mathcal{L}$) we assume that the smallest element, usually denoted by 0, and the greatest element, usually denoted by 1, do not coincide. Finally, the symbols $<$ and $\subset$ always denote strict inequality and strict inclusion, respectively.

1.2 Fundamentals of Connective Segmentation

First, we recall the definition of connections and connective criteria.

Definition 1. *[8] Let E and T be sets, and $\mathcal{F}$ a family of functions $f : E \to T$.*

1. *A* criterion *on $\mathcal{F}$ is a map $\sigma : \mathcal{F} \times \mathcal{P}(E) \to \{0, 1\}$ satisfying $\sigma(f, \emptyset) = 1$. The criterion σ is said to be* validated *on $(f, A) \in \mathcal{F} \times \mathcal{P}(E)$ whenever $\sigma(f, A) = 1$; otherwise it is* refuted *on (f, A).*
2. *A subset $\mathcal{C}$ of $\mathcal{P}(E)$ is a* connection *if*
   (a) $\emptyset \in \mathcal{C}$,
   (b) $\forall x \in E; \{x\} \in \mathcal{C}$, and
   (c) if $\{C_i\}_{i\in I} \subseteq \mathcal{C}$ for some index set I, and $\bigcap_{i\in I} C_i \neq \emptyset$, then $\bigcup_{i\in I} C_i \in \mathcal{C}$.
3. *A criterion σ on $\mathcal{F}$ is* connective *if for each $f \in \mathcal{F}$, the set $\sigma_f = \{A \in \mathcal{P}(E); \sigma(f, A) = 1\}$ is a connection.*

The elements of $\mathcal{C}$ are said to be *$\mathcal{C}$-connected*, or *connected* if the connection is clear from the context. A *connected component* of a set $A \subseteq E$ is an element $C \in \mathcal{C} \cap \mathcal{P}(A)$, $C \neq \emptyset$, such that there exists no $D \in \mathcal{C}$ that satisfies $C \subset D \subseteq A$; if $A = E$, such sets will simply be called connected components.

Next we define partitions and segmentation.

Definition 2. *A* partition *of E is a map $\pi : E \to \mathcal{P}(E)$ that satisfies*

1. *$\forall x \in E; x \in \pi(x)$, and*
2. *$\forall x, y \in E$ such that $\pi(x) \cap \pi(y) \neq \emptyset$ it holds that $\pi(x) = \pi(y)$.*

When confusion is improbable, we will denote the image of a partition π, i.e the collection of all its classes, by π as well. Thus the notation $C \in \pi$ means that C is a class of π.

The collection of all partitions of a set E, denoted $\Pi(E)$, is known to be a complete lattice under the *refinement order*, defined as follows. For $\pi, \pi' \in \Pi(E)$, $\pi \leq \pi'$ if and only if $\pi(x) \subseteq \pi'(x)$ for each $x \in E$, i.e. if and only if each class of π is contained in a class of π'. The smallest element (or empty supremum) in this lattice is the partition $I; I(x) = \{x\}$ for all $x \in E$, while the greatest partition (or empty infimum) is $1; 1(x) = E$ for all $x \in E$. The infimum π of a non-empty family $\{\pi_i\}_{i \in I}$ of partitions, where I is an index set, is the partition whose classes are the intersections

$$\pi(x) = \bigcap_{i \in I} \pi_i(x) \tag{1}$$

while the supremum π' of $\{\pi_i\}_{i \in I}$ is the smallest partition such that for each $x \in E$, $\pi_i(x) \subseteq \pi'(x)$ for all $i \in I$.[1] Partitions are the corner stone of segmentation.

Definition 3. *[8] Given $f \in \mathcal{F}$ where $\mathcal{F}$ is a family of functions from a set E to a set T, and given $A \in \mathcal{P}(E)$ and a criterion σ on $\mathcal{F}$, let $\Pi(A, \sigma_f)$ be the family of all partitions π on A such that σ is validated on each class of π. The criterion σ is said to* segment *f (over A) if*

1. *$\forall x \in E, \sigma(f, \{x\}) = 1$, and*
2. *the family $\Pi(A, \sigma_f)$ is closed under the supremum of partitions.*

In that case, the supremum of $\Pi(A, \sigma_f)$ is the segmentation of f on A by σ.

Remark 1. Closure under the supremum in item 2 is equivalently expressed by saying that $\Pi(A, \sigma_f)$ is a dual Moore family in $\Pi(A)$. When the set A is not specified, the segmentation is understood to be over E (as is the case in the theorem below). Note that item 1 is the special case of item 2 when taking the empty supremum of partitions, thus making item 1 redundant. Nevertheless, we use this formulation to prepare for the generalized version of the definition.

We aim at generalizing the following result by Serra.

Theorem 1. *[8] Given two sets E and T, let $\mathcal{F}$ be a family of functions $f : E \to T$, and σ a criterion on $\mathcal{F}$. Then σ is connective if and only if σ segments all $f \in \mathcal{F}$.*

From [8] we know that in this case, for each $f \in \mathcal{F}$ the segmentation obtained is the partition of E into its σ_f-connected components.[2]

2 Generalization to Arbitrary Complete Lattices

In this section, we will consider arbitrary complete lattices $\mathcal{L} = (\mathcal{L}, \leq)$ sup-generated by a set $S \subseteq \mathcal{L}$. (This indeed holds for all complete lattices $\mathcal{L}$ by

[1] This is a formal definition. A way to construct the supremum is by *chaining*; see [2].

[2] This will indeed be the case in every generalization of the theorem, as the reader will see in the coming sections.

choosing $S = \mathcal{L} \setminus \{0\}$, but often more interesting choices can be made. If, for example, $\mathcal{L}$ is atomistic, then S can be chosen to be the set of all atoms.) We always assume $0 \notin S$. An important property of a sup-generating set is that for any $x, y \in \mathcal{L}$, it holds that[3]

$$(\forall s \in S; s \leq x \Rightarrow s \leq y,) \Longleftrightarrow x \leq y \ . \tag{2}$$

2.1 Adapting the Setting

We begin by generalizing the definitions.

Definition 4. *Let $\mathcal{L}$ be a complete lattice sup-generated by a subset $S \subseteq \mathcal{L}$, T an arbitrary set, and $\mathcal{F}$ a family of functions $f : S \to T$.*

1. *A* criterion *on $\mathcal{F}$ is a map $\sigma : \mathcal{F} \times \mathcal{L} \to \{0, 1\}$ satisfying $\sigma(f, 0) = 1$. The criterion σ is said to be* validated *on $(f, l) \in \mathcal{F} \times \mathcal{L}$ whenever $\sigma(f, l) = 1$; otherwise it is* refuted *on (f, l).*
2. *[5] A subset $\mathcal{C}$ of $\mathcal{L}$ is an S-*connection *if it satisfies*
   (a) $0 \in \mathcal{C}$,
   (b) $S \subseteq \mathcal{C}$, and
   (c) if $\{c_i\}_{i \in I} \subseteq \mathcal{C}$ for some index set I, and $\bigwedge_{i \in I} c_i \neq 0$, then $\bigvee_{i \in I} c_i \in \mathcal{C}$.
3. *A criterion σ on $\mathcal{F}$ is S-*connective *if for each $f \in \mathcal{F}$, the set $\sigma_f = \{l \in \mathcal{L}; \sigma(f, l) = 1\}$ is an S-connection.*

Note that the domain of the functions is S and not $\mathcal{L}$; indeed S generalizes the set E of Definition 1, and $\mathcal{L}$ generalizes $\mathcal{P}(E)$. (In [5], an S-connection is just called a *connection*.)

For simplicity, when the function f is clear from context, we will say that σ is validated on $l \in \mathcal{L}$, meaning that $\sigma(f, l) = 1$. An S-connected component of an element $l \in \mathcal{L}$ greater than or equal to a given $s \in S$ is an element $c \in \mathcal{C}$ such that $s \leq c \leq l$ and there exists no $d \in \mathcal{C}$ such that $c < d \leq l$; due to Definition 4.2.(c), this is precisely $\bigvee \{c \in \mathcal{C}; s \leq c \leq l\}$.

Next is the definition of partitions on complete lattices.

Definition 5. *[1] Let $\mathcal{L}$ be a complete lattice sup-generated by a subset $S \subseteq \mathcal{L}$. An S-*partition *on $\mathcal{L}$ is a map $\pi : S \to \mathcal{L}$ that satisfies*

1. *$\forall s \in S; s \leq \pi(s)$, and*
2. *$\forall s, t \in S$ such that $\pi(s) \wedge \pi(t) \neq 0$ it holds that $\pi(s) = \pi(t)$.*

Ordered by refinement, in analogy with the case of $\Pi(E)$, the set $\Pi_S(\mathcal{L})$ of S-partitions on $\mathcal{L}$ is a complete lattice. As before, we will use the same notation for a partition and its image (which we continue to call the set of its classes, even though the term no longer bears its original meaning), and the prefix S will sometimes be dropped when the set S is clear from context. Segmentation is defined as follows.

[3] Thus this holds for $S = \mathcal{L} \setminus \{0\}$ as a special case; indeed, the strength of the notion of a sup-generating set is that any such set S satisfies this property.

Definition 6. *Let $\mathcal{L}$ be a complete lattice sup-generated by a subset $S \subseteq \mathcal{L}$. Given $f \in \mathcal{F}$ where $\mathcal{F}$ is a family of functions from S to a set T, and a criterion σ on $\mathcal{F}$, let $\Pi_S(\mathcal{L}, \sigma_f)$ be the family of all S-partitions π of $\mathcal{L}$ such that σ is validated on each class of π. The criterion σ is said to S-*segment *f if*

1. *$\forall s \in S, \sigma(f, s) = 1$, and*
2. *the family $\Pi_S(\mathcal{L}, \sigma_f)$ is closed under the supremum of S-partitions.*

*In that case the supremum of $\Pi_S(\mathcal{L}, \sigma_f)$ is the S-*segmentation of *f* by *σ.*

Remark 2. For a set E, the lattice $\mathcal{P}(E)$ is sup-generated by the set of singleton sets, i.e. the points of E. Setting $\mathcal{L} = \mathcal{P}(E)$ and $S = \{\{x\}; x \in E\}$ we retrieve the original definitions from the generalized ones upon identifying $E = \{\{x\}; x \in E\}$.

Remark 3. In general lattices, the empty supremum of partitions is not the map $\pi : S \to \mathcal{L}, \pi(s) = s$, as this is not necessarily a partition (see, for instance, the lattice in Example 1). Thus item 2 does not imply item 1. Lemma 1 below will describe the empty supremum from a connectivity point of view.

2.2 Main Result

We now aim to generalize Theorem 1, for which we need the following

Lemma 1. *Let $\mathcal{L}$ be a complete lattice sup-generated by a subset $S \subseteq \mathcal{L}$. Let $\Pi_S(\mathcal{L})$ be the complete lattice of all S-partitions on $\mathcal{L}$, and denote by π its smallest element. Then for any S-connection $\mathcal{C}$ on $\mathcal{L}$, $\pi(s) \in \mathcal{C}$ for all $s \in S$.*

Proof. Given an S-connection $\mathcal{C}$, consider the map $\pi_C : S \to \mathcal{L}$ defined by[4]

$$\pi_C(s) = \bigvee\{c \in \mathcal{C}; s \leq c \leq \pi(s)\} \tag{3}$$

for all $s \in S$. Then as π is an S-partition, it follows that π_C is an S-partition. Moreover, $\pi_C(s) \in \mathcal{C}$ for all $s \in S$ by Definition 4.2.(c), and in the lattice $\Pi_S(\mathcal{L})$ we have $\pi_C \leq \pi$, thus $\pi_C = \pi$ since π is the smallest S-partition on $\mathcal{L}$. □

Theorem 2. *Let $\mathcal{L}$ be a complete lattice sup-generated by a subset $S \subseteq \mathcal{L}$. Let $\mathcal{F}$ be a family of functions $f : S \to T$ where T is an arbitrary set, and let σ be a criterion on $\mathcal{F}$.*

1. *If σ is S-connective, then σ S-segments all $f \in \mathcal{F}$.*
2. *If $\mathcal{L}$ is atomistic, and S is the set of all atoms of $\mathcal{L}$, it conversely holds that if σ S-segments all $f \in \mathcal{F}$, then σ is S-connective.*

Proof. Throughout the proof, let f be any element of $\mathcal{F}$. Let $\sigma_f = \{l \in \mathcal{L}; \sigma(f, l) = 1\}$, and let $\Pi_S(\mathcal{L}, \sigma_f)$ be the collection of all S-partitions all of whose classes belong to σ_f. We start by proving item 1.

[4] This definition is equivalent to saying that π_C assigns to each $s \in S$ the connected component of $\pi(s)$ greater than or equal to it.

Assume that σ is S-connective. By Definition 4.2.(b), the first item of Definition 6 is satisfied. Definition 6.2. is satisfied for the empty supremum (i.e. smallest S-partition) of $\Pi_S(\mathcal{L}, \sigma_f)$ since by Lemma 1 the empty supremum of $\Pi_S(\mathcal{L})$ is in $\Pi_S(\mathcal{L}, \sigma_f)$. Next, let $\{\pi_i\}_{i\in I}$, for some index set I, be a non-empty family in $\Pi_S(\mathcal{L}, \sigma_f)$ with $\pi = \bigvee_{i\in I} \pi_i$. Let $\pi_C : S \to \sigma_f$ be defined by

$$\pi_C(s) = \bigvee \{c \in \sigma_f ; s \le c \le \pi(s)\} \tag{4}$$

for all $s \in S$. It is readily checked that π_C is a partition. Now, $\pi_C \le \pi$, and for all $i \in I$ and $s \in S$, $\pi_i(s) \le \pi_C(S)$ since $\pi_i(s) \in \sigma_f$ and $s \le \pi_i(s) \le \pi(s)$. Hence π_C is an upper bound of $\{\pi_i\}_{i\in I}$, and π being the least such, we have $\pi \le \pi_C$ and altogether $\pi = \pi_C$. Thus $\Pi_S(\mathcal{L}, \sigma_f)$ is also closed under non-empty suprema. This proves item 1.

As for the converse, by Definitions 4.1 and 6.1, $\sigma(f, l) = 1$ for all $l \in \{0\} \cup S$. Take any $\{c_i\}_{i\in I} \subseteq \sigma_f$ satisfying $\bigwedge_{i\in I} c_i \ne 0$. Since S is sup-generating, this implies that $\exists s_0 \in S; s_0 \le \bigwedge_{i\in I} c_i$. Each c_i induces a map $\pi_i : S \to \mathcal{L}$ defined by

$$\begin{cases} \pi_i(s) = c_i & \text{if } s \le c_i \text{ and} \\ \pi_i(s) = s & \text{otherwise .} \end{cases} \tag{5}$$

If each s is an atom, then indeed each π_i is a partition, and clearly we have $\pi_i \in \Pi_S(\mathcal{L}, \sigma_f)$ for all $i \in I$. Since σ S-segments all $f \in \mathcal{F}$, it follows that $\pi = \bigvee_{i\in I} \pi_i \in \Pi_S(\mathcal{L}, \sigma_f)$. Define $x = \bigvee_{i\in I} c_i$. We need show that $x \in \sigma_f$. For any $i \in I$, the definition of π_i implies that $\pi_i(s_0) = c_i$ and hence $\pi_i(s_0) \le x$ for all $i \in I$. Now define the partition π' by

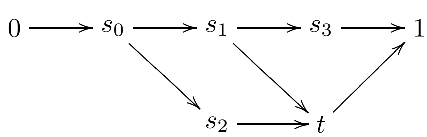
$$\begin{cases} \pi'(s) = x & \text{if } s \le x \text{ and} \\ \pi'(s) = s & \text{otherwise} \end{cases} \tag{6}$$

for all $s \in S$. For all $i \in I$ it holds that $\pi_i \le \pi'$, and as $\pi = \bigvee_{i\in I} \pi_i$, we have $\pi \le \pi'$, and thus $\pi(s_0) \le \pi'(s_0) = x$.

Since $\pi_i \le \pi$, we have $c_i = \pi_i(s_0) \le \pi(s_0)$ for all $i \in I$. By the definition of the supremum this implies that $x = \bigvee_{i\in I} c_i \le \pi(s_0)$. Altogether $x = \pi(s_0)$, whereby $x \in \sigma_f$ since $\Pi_S(\mathcal{L}, \sigma_f)$ is closed under suprema. Thus σ_f is an S-connection and the proof is complete. □

The following example demonstrates that item 2 of the theorem is in general false when atomisticity is not required.

Example 1. Let $\mathcal{L}$ be the set $\{0, 1, s_0, s_1, s_2, s_3, t\}$ and define the partial order $\le$ on $\mathcal{L}$ by



(7)

where for $l \in \mathcal{L}$ and $m \in \mathcal{L}$, $l \le m$ if and only if there exists a directed path from l to m. It is easily checked that $\mathcal{L}$ is a lattice, hence a complete lattice by finiteness, and that it is sup-generated by the set $S = \{s_i, i = 0, 1, 2, 3\}$. It is however not atomistic as e.g. s_0 and s_1 necessarily belong to each sup-generating set. Let T be any set, and $\mathcal{F}$ any family of functions from S to T. Define the criterion σ on $\mathcal{F}$ by

$$\begin{cases} \sigma(f, l) = 0 & \text{if } l = t \text{ and} \\ \sigma(f, l) = 1 & \text{otherwise} \end{cases} \tag{8}$$

for all $f \in \mathcal{F}$. Since $t = s_1 \vee s_2$, and $s_1 \wedge s_2 \neq 0$, this implies that σ is not connective. However, by the structure of $\mathcal{L}$ the only possible S-partition on $\mathcal{L}$ is the one whose unique class is 1, and as $\sigma(f, 1) = 1$, we get that σ S-segments f.

Thus we conclude that Theorem 1 generalizes completely to the atomistic setting, but in general not farther. The approach of Section 3 will, as it turns out, in fact mend this issue.

3 Partial Connections and Partial Partitions

We now extend this study in another direction, by looking at operators that act on partitions. More specifically, we study a certain type of openings on partial partitions, and their relation to partial connections (concepts to be defined). This follows the work of Ronse, who has established this theory for the case of the power set lattice in [4]. We begin by the definitions for that case.

Definition 7. *[3] Let E be a set.*

1. *A subset $\mathcal{C}$ of $\mathcal{P}(E)$ is a* partial connection *on E if it satisfies*
   (a) $\emptyset \in \mathcal{C}$, and
   (b) if $\{C_i\}_{i \in I} \subseteq \mathcal{C}$ for some index set I, and $\bigcap_{i \in I} C_i \neq \emptyset$, then $\bigcup_{i \in I} C_i \in \mathcal{C}$.
2. *A* partial partition *of E is a map $\pi : E \to \mathcal{P}(E)$ that satisfies*
   (a) $\forall x \in E; \pi(x) = \emptyset$ or $x \in \pi(x)$, and
   (b) $\forall x, y \in E; x \in \pi(y) \Rightarrow \pi(x) = \pi(y)$.

It is seen that every connection is a partial connection, and by [3] every partition is a partial partition. As in the case of partitions, the notation $C \in \pi$ means that C is a class of π. Definition 7 implies that singleton sets are not generally assumed to be connected, and that the connected components of a partial connection and the classes of a partial partition do not necessarily cover E. In [4], one motivation of using partiality is that in computations, considering singleton sets to be a priori connected becomes impractical. In general, this offers a great deal of flexibility to the theory, as we will see.

The collection of partial connections on E, denoted by $\Gamma^*(E)$, is a complete lattice under the inclusion order: for $\mathcal{C}, \mathcal{D} \in \Gamma^*(E)$, $\mathcal{C} \leq \mathcal{D} \iff \forall C \in \mathcal{C}, C \in \mathcal{D}$; the set of partial partitions on E is denoted $\Pi^*(E)$; it is a complete lattice under the refinement order previously defined for partitions. [3] The greatest element in this lattice is the partition $1; 1(x) = E$ for all $x \in E$, while the smallest partial partition is $0; 0(x) = \emptyset$ for all $x \in E$.

The notions of *partially connective criteria* and *partial segmentation*, in which a criterion *segments* a function *partially*, are obtained by replacing connections and partitions in Definitions 1.3 and 3.2 by partial connections and partial partitions, and disregarding Definition 3.1. (about singleton sets). A detailed study of partial connections and partial partitions is conducted in [3]. We move on to a class of operators on $\Pi^*(E)$, defined in two steps.

Definition 8. *[4] Let E be a set.*

1. *A map $\psi : \mathcal{P}(E) \to \Pi^*(E)$ is* set splitting *if $\psi(A) \in \Pi^*(A)$, i.e. if $\psi(A)(x) \subseteq A$ for all $x \in E$.*
2. *Given a set splitting map ψ, the* class splitting operator induced by ψ *is the operator $\psi^* : \Pi^*(E) \to \Pi^*(E)$ such that for each partial partition $\pi \in \Pi^*(E)$, $\psi^*(\pi) = \bigvee_{C \in \pi} \psi(C)$.*

Note that the supremum in item 2 of the definition is in fact the partial partition whose classes are the images under ψ of the classes of π.

From [4] one learns that ψ is set splitting if and only if for all $A \subseteq E$ and $x \in E \setminus A$, $\psi(A)(x) = \emptyset$. In [4] class splitting operators are called *block splitting.*

3.1 Openings on Partial Partitions

An opening (say, ω) on a complete lattice is an anti-extensive ($\omega(x) \leq x$), idempotent and order-preserving operator on it. The set of openings on a complete lattice $\mathcal{L}$ is a subset of the complete lattice of operators on $\mathcal{L}$. It exhibits additional structure by the following well-known fact.

Lemma 2. *[6] Let $\mathcal{L}$ be a complete lattice, and $\mathcal{L}^{\mathcal{L}}$ the complete lattice of operators on $\mathcal{L}$. The subset of $\mathcal{L}^{\mathcal{L}}$ consisting of all openings on $\mathcal{L}$ is a dual Moore family[5] in $\mathcal{L}^{\mathcal{L}}$. It is thus a complete lattice under the order induced by the order on $\mathcal{L}^{\mathcal{L}}$.*

In this lattice, the supremum resp. the infimum of a non-empty family P of openings is the supremum of P in $\mathcal{L}^{\mathcal{L}}$ resp. the greatest opening smaller than or equal to the infimum of P in $\mathcal{L}^{\mathcal{L}}$; the smallest element in the lattice of openings is the map $l \mapsto 0$ for all $l \in \mathcal{L}$, and the greatest element is the identity map $l \mapsto l$ for all $l \in \mathcal{L}$.

There is indeed a greatest opening smaller than or equal to a given element in $\mathcal{L}^{\mathcal{L}}$, by definition of a dual Moore family; it is for the same reason that the supremum in $\mathcal{L}^{\mathcal{L}}$ of a family of openings is an opening.

The following result by Ronse is central in the characterization of partial connections in terms of class splitting openings.

Proposition 1. *[4] The set $\Omega(E)$ of class splitting openings on $\Pi^*(E)$ for a set E is a complete sublattice of the lattice of openings on $\Pi^*(E)$. It is isomorphic to the lattice $\Gamma^*(E)$ of partial connections. The isomorphism is given by the map*

$$\lambda : \Gamma^*(E) \to \Omega(E); \mathcal{C} \mapsto \mathcal{C}^*_\bullet \tag{9}$$

where for all $\pi \in \Pi^*(E)$, $\mathcal{C}^*_\bullet(\pi)(x) = \bigvee\{C \in \mathcal{C}; x \in C \subseteq \pi(x)\}$ for all $x \in E$.

[5] See Remark 1.

Put simply, λ maps each partial connection $\mathcal{C}$ to the (class splitting) opening that splits the classes of a partial partition into their $\mathcal{C}$-connected components.

3.2 Generalizing Ronse's Result

Following the lines of our treatment of partitions and connections on complete lattices, Definition 7 generalizes readily. Note that, in contrast to the generalization of the concept of a connection (Definition 4.2), the partial counterpart is independent of any sup-generating set. Indeed, the dependence of Definition 4.2 on a sup-generating set lies in its second item, which does not appear here, generalizing the aforementioned fact that in the power set lattice, singleton sets are now not a priori assumed connected.

Definition 9. *1. Let $\mathcal{L}$ be a complete lattice. A subset $\mathcal{C}$ of $\mathcal{L}$ is a* partial connection *if it satisfies*
  (a) $0 \in \mathcal{C}$, and
  (b) if $\{c_i\}_{i \in I} \subseteq C$ for some index set I, and $\bigwedge_{i \in I} c_i \neq 0$, then $\bigvee_{i \in I} c_i \in \mathcal{C}$.
2. Let $\mathcal{L}$ be a complete lattice sup-generated by a subset $S \subseteq \mathcal{L}$. A partial S-partition *on $\mathcal{L}$ is a map $\pi : S \to \mathcal{L}$ that satisfies*
  (a) $\forall s \in S; \pi(s) = 0$ or $s \leq \pi(s)$, and
  (b) $\forall s, t \in S$ such that $s \leq \pi(t)$ it holds that $\pi(s) = \pi(t)$.

The sets of partial connections and partial S-partitions on $\mathcal{L}$, henceforth denoted by $\Gamma^*(\mathcal{L})$ and $\Pi^*_S(\mathcal{L})$, respectively, are complete lattices by a straight-forward proof. *Partial S-segmentation*, where a criterion *S-segments* a function *partially*, and *partially connective criteria* are defined in an analogous manner to that described above Definition 8 for the power set lattice.

Example 2. If $\mathcal{L}$ is a complete lattice sup-generated by $S \subseteq \mathcal{L}$, then for each $m \in \mathcal{L}$, the function $1_m : S \to \mathcal{L}$ defined by $1_m(s) = m$ if $s \leq m$, and $1_m(s) = 0$ otherwise, is a partial S-partition with m as unique non-zero class.

The class splitting operators become as follows in the general setting.

Definition 10. *Let $\mathcal{L}$ be a complete lattice sup-generated by a subset $S \subseteq \mathcal{L}$.*

1. *A map $\psi : \mathcal{L} \to \Pi^*_S(\mathcal{L})$ is* element splitting *if $\psi(l) \leq 1_l$ for all $l \in \mathcal{L}$; equivalently, if $\forall l \in \mathcal{L}; \forall s \in S; \psi(l)(s) \leq l$.*
2. *Given an element splitting map ψ, the* class splitting operator induced by ψ *is the operator $\psi^* : \Pi^*_S(\mathcal{L}) \to \Pi^*_S(\mathcal{L})$ such that for each partial S-partition $\pi \in \Pi^*_S(\mathcal{L})$, $\psi^*(\pi) = \bigvee_{c \in \pi} \psi(c)$.*

Now we are ready to generalize Proposition 1.

Theorem 3. *Let $\mathcal{L}$ be a complete lattice sup-generated by a subset $S \subseteq \mathcal{L}$. Let $\Omega_S(\mathcal{L})$ be the set of all class splitting openings on $\Pi^*_S(\mathcal{L})$. Then $\Omega_S(\mathcal{L})$ is a complete sublattice of the lattice of openings on $\Pi^*_S(\mathcal{L})$. As a complete lattice, it is isomorphic to the lattice $\Gamma^*(\mathcal{L})$ of partial connections on $\mathcal{L}$, via the isomorphism*

$$\lambda : \Gamma^*(\mathcal{L}) \to \Omega_S(\mathcal{L}); \mathcal{C} \mapsto \mathcal{C}^*_\bullet \tag{10}$$

where for all $\pi \in \Pi^*_S(\mathcal{L})$, $\mathcal{C}^*_\bullet(\pi)(s) = \bigvee\{c \in \mathcal{C}; s \leq c \leq \pi(s)\}$ for all $s \in S$.

The long, technical proof leading to this result follows the lines of the proof of Proposition 1, given in [4], and we here give a brief sketch due to space limits.

Proof. The smallest opening on $\Pi^*_S(\mathcal{L})$ is $\mathbf{0} : \pi \mapsto 0$, which is class splitting. The greatest opening on $\Pi^*_S(\mathcal{L})$ is the identity map, which is class splitting as induced by the element splitting map $m \mapsto 1_m$. Consider next a non-empty family of class splitting openings $\{\psi^*_i\}_{i\in I}$ for an index set I. Each ψ^*_i is shown to be induced by a unique, order-preserving element splitting map ψ_i. By Lemma 2, the supremum $\bigvee_{i\in I}(\psi^*_i)$ is an opening, which is in fact class splitting. The infimum in the lattice of openings is described in Lemma 2. One deduces that it is class splitting from the ψ_i being order preserving. This proves the first part. Given $\mathcal{C} \in \Gamma^*(\mathcal{L})$, the operator $\mathcal{C}^*_\bullet : \Pi^*_S(\mathcal{L}) \to \Pi^*_S(\mathcal{L})$ is a class splitting opening, induced by the map that splits an element of $\mathcal{L}$ into its $\mathcal{C}$-connected components. Thus λ is well-defined, and an isomorphism by technical considerations. □

We moreover give the following result, which is the analogue of Theorem 2 in the partial setup. The special case for the power set lattice was proven in [3].

Lemma 3. *Let $\mathcal{L}$ be a complete lattice sup-generated by a subset $S \subseteq \mathcal{L}$, and let $\mathcal{C} \subseteq \mathcal{L}$ be a subset. Then $\mathcal{C}$ is a partial connection if and only if the set $\Pi^*_S(\mathcal{L}, \mathcal{C})$ of partial S-partitions all whose classes are in $\mathcal{C}$ is closed under arbitrary suprema.*

Corollary 1. *Let $\mathcal{L}$ be a complete lattice sup-generated by a subset $S \subseteq \mathcal{L}$, $\mathcal{F}$ a family of functions $f : S \to T$ for a set T, and σ a criterion on $\mathcal{F}$. Then σ is partially connective if and only if σ S-segments all $f \in \mathcal{F}$ partially.*

Note how partiality circumvents the need of atomisticity in order for the result to be fully valid as no longer all $s \in S$ are required to be connected.

4 Relating Ronse's Approach to Serra's

In this section we will state and prove a relation between the characterization of (partially) segmenting criteria of Corollary 1 (Theorem 2), and that of class splitting openings of Theorem 3; this will be done by means of a commutative diagram. We give the link in both the partial and non-partial setup, and we state it directly for general lattices; of course, it then also holds for the special case of the power set lattice. To the authors' knowledge, such a link has not been presented before in any setting.

For connective segmentation in the non-partial sense, we have the following

Theorem 4. *Let $\mathcal{L}$ be a complete lattice, sup-generated by a subset $S \subseteq \mathcal{L}$, T a set, and $\mathcal{F}$ a family of functions $f : S \to T$. Define $\Sigma_S(\mathcal{F}, \mathcal{L})$ to be the set of all S-connective criteria on $\mathcal{F}$, $\Pi_S(\mathcal{L})$ the complete lattice of S-partitions on $\mathcal{L}$, $\Pi^*_S(\mathcal{L})$ the complete lattice of partial S-partitions on $\mathcal{L}$, $\Gamma^*(\mathcal{L})$ the complete lattice of partial connections on $\mathcal{L}$, and $\Omega_S(\mathcal{L})$ the complete lattice of class splitting openings on $\Pi^*_S(\mathcal{L})$. Let moreover*

– *for each* $f \in \mathcal{F}, \phi_f : \Sigma_S(\mathcal{F}, \mathcal{L}) \to \Pi_S(\mathcal{L})$ *be the map that assigns to each* S*-connective criterion* σ *its* S*-segmentation of* f*, and*
– $\lambda : \Gamma^*(\mathcal{L}) \to \Omega_S(\mathcal{L})$ *be the lattice isomorphism that assigns to each partial connection* $\mathcal{C}$ *the class splitting opening* $\mathcal{C}^*_\bullet$*.*

Then for each $f \in \mathcal{F}$ *there exist maps* μ_f *and* κ *such that following diagram commutes*

$$\begin{array}{ccc} \Sigma_S(\mathcal{F}, \mathcal{L}) & \xrightarrow{\phi_f} & \Pi_S(\mathcal{L}) \\ \big\downarrow{\scriptstyle \mu_f} & & \big\uparrow{\scriptstyle \kappa} \\ \Gamma^*(\mathcal{L}) & \xrightarrow[\lambda]{} & \Omega_S(\mathcal{L}) \end{array} \quad . \tag{11}$$

In other words we have $\phi_f(\sigma) = \kappa\lambda\mu_f(\sigma)$ *for all* $f \in \mathcal{F}$ *and* $\sigma \in \Sigma_S(\mathcal{F}, \mathcal{L})$*. Specifically,*

– μ_f *is the map that assigns to each* S*-connective criterion* σ *the* S*-connection* $\{c \in \mathcal{L}; \sigma(f, c) = 1\}$*, and*
– κ *is the map that assigns to each class splitting opening* γ^* *the partial* S*-partition* $\gamma^*(1)$*, where* 1 *is the greatest* S*-partition on* $\mathcal{L}$*.*

Proof. Given $f \in \mathcal{F}$, take any criterion $\sigma \in \Sigma_S(\mathcal{F}, \mathcal{L})$. Then $\mu_f(\sigma) = \{c \in \mathcal{L}; \sigma(f, c) = 1\}$, and composing μ_f with λ and κ gives $\kappa\lambda\mu_f(\sigma) = (\mu_f(\sigma))^*_\bullet(1)$, where for all $s \in S$,

$$(\mu_f(\sigma))^*_\bullet(1)(s) = \bigvee\{c \in \mu_f(\sigma); s \le c \le 1\} = \bigvee\{c \in \mu_f(\sigma); s \le c\} \ , \tag{12}$$

i.e. $(\mu_f(\sigma))^*_\bullet(1)(s)$ is equal to the $\mu_f(\sigma)$-connected component greater than or equal to s, if it exists, and otherwise it is equal to 0. Since $\mu_f(\sigma)$ is a partial connection, $(\mu_f(\sigma))^*_\bullet(1)$ is a partial S-partition all of whose classes belong to $\mu_f(\sigma)$. It is moreover an S-partition, since $\mu_f(\sigma)$ is an S-connection. This is the largest S-partition all of whose classes belong to $\mu_f(\sigma)$. Hence it is the S-segmentation of f by σ, which completes the proof. □

Next, we replace connections and partitions by partial connections and partial partitions, respectively, to get the following

Theorem 5. *Let* $\mathcal{L}$ *be a complete lattice, sup-generated by a subset* $S \subseteq \mathcal{L}$*,* T *a set, and* $\mathcal{F}$ *a family of functions* $f : S \to T$*. Define* $\Sigma^*(\mathcal{F}, \mathcal{L})$ *to be the set of all partially connective criteria on* $\mathcal{F}$*,* $\Pi^*_S(\mathcal{L})$ *the complete lattice of partial* S*-partitions on* $\mathcal{L}$*,* $\Gamma^*(\mathcal{L})$ *the complete lattice of partial connections on* $\mathcal{L}$*, and* $\Omega_S(\mathcal{L})$ *the complete lattice of class splitting openings on* $\Pi^*_S(\mathcal{L})$*. Let moreover*

– *for each* $f \in \mathcal{F}, \phi'_f : \Sigma^*(\mathcal{F}, \mathcal{L}) \to \Pi^*_S(\mathcal{L})$ *be the map that assigns to each partially connective criterion* σ *its partial* S*-segmentation of* f*, and*
– $\lambda : \Gamma^*(\mathcal{L}) \to \Omega_S(\mathcal{L})$ *be the lattice isomorphism that assigns to each partial connection* $\mathcal{C}$ *the class splitting opening* $\mathcal{C}^*_\bullet$*.*

Then for each $f \in \mathcal{F}$ *there exist maps* μ'_f *and* κ *such that following diagram commutes*

$$\begin{array}{ccc} \Sigma^*(\mathcal{F},\mathcal{L}) & \xrightarrow{\phi'_f} & \Pi^*_S(\mathcal{L}) \\ \Big\downarrow{\scriptstyle \mu'_f} & & \Big\uparrow{\scriptstyle \kappa} \\ \Gamma^*(\mathcal{L}) & \xrightarrow[\lambda]{} & \Omega_S(\mathcal{L}) \end{array} \tag{13}$$

where μ'_f *is the map that assigns to each partially connective criterion* σ *the partial connection* $\{c \in \mathcal{L}; \sigma(f,c) = 1\}$*, and* κ *is the map that assigns to each class splitting opening* γ^* *the partial* S*-partition* $\gamma^*(1)$ *(where* 1 *is the greatest* S*-partition on* $\mathcal{L}$*).*

The proof is analogous to that of Theorem 4. Theorems 4 and 5 thus relate the two approaches to connective segmentation discussed in Sections 1–3.

5 Conclusion

As we have seen, the theory of connective segmentation generalizes to atomistic lattices in a natural way. When dealing with non-atomistic lattices, partiality becomes necessary in order to maintain the full strength of the theory. We have also seen that the two approaches to connective segmentation are linked on all levels. In short, the apparatus of connective segmentation lends itself to investigations of a large number of complete lattices that arise naturally in different theoretical and practical contexts of mathematical morphology.

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