Orders on Partial Partitions and Maximal Partitioning of Sets*

Christian Ronse

LSIIT UMR 7005 CNRS-UdS
Parc d'Innovation, Boulevard Sébastien Brant
BP 10413, 67412 ILLKIRCH CEDEX, France
cronse@unistra.fr
http://lsiit-cnrs.unistra.fr/miv

Abstract. The segmentation of a function on a set can be considered as the construction of a maximal partial partition of that set with blocks satisfying some criterion for the function. Several order relations on partial partitions are considered in association with types of operators and criteria involved in the segmentation process. We investigate orders for which this maximality of the segmentation partial partition is preserved in compound segmentation with two successive criteria. Finally we consider valuations on partial partitions, that is, strictly isotone functions with positive real values; this gives an alternative approach where the valuation, not the partial partition, should be maximized.

1 Introduction

The purpose of this communication is to discuss order-theoretic issues involved in image segmentation. We consider images as functions $E \to T$, where E is the space of points and T is the set of image values. A *partition* of E is a family of nonvoid mutually disjoint subsets of E, called *blocks*, whose union is E. Soille [15] summarizes conventional requirements of image segmentation as follows:

1. The segmentation method relies on a criterion that determines, for every function F and every subset A of E, whether F is homogeneous on A or not.
2. Given a function F, its segmentation is a partition of E into connected blocks on which F is homogeneous; these blocks are called segmentation classes.
3. Merging two or more adjacent segmentation classes, F is not homogeneous on the resulting set; in other words F cannot be homogeneous on a connected union of two or more segmentation classes.

Let us formalize these principles. We first recall some terminology. Write $\mathit{\Pi}(E)$ for the set of all partitions of E. Now $\mathit{\Pi}(E)$ is ordered by *refinement*: for $\pi_1, \pi_2 \in \mathit{\Pi}(E)$, we say that π_1 is *finer* than π_2, or that π_2 is *coarser* than π_1, and write $\pi_1 \leq \pi_2$ (or $\pi_2 \geq \pi_1$), iff every block of π_1 is included in a block of π_2, that

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is, every block of π_2 is a union of blocks of π_1. Then $(\Pi(E), \leq)$ is a complete lattice. For any family $\mathcal{C} \subseteq \mathcal{P}(E)$, let $\Pi(E,\mathcal{C}) = \Pi(E) \cap \mathcal{P}\bigl(\mathcal{C} \setminus \{\emptyset\}\bigr)$ be the family of all partitions whose blocks belong to $\mathcal{C}$ (in fact, to $\mathcal{C} \setminus \{\emptyset\}$).

The connectivity of sets is given by a *connection* $\mathcal{C}$ on $\mathcal{P}(E)$ [10,4,9]. By item 1, the segmentation method is based on a *criterion* [12,9], that is a map $\mathsf{cr} : T^E \times \mathcal{P}(E) \to \{0,1\}$, where for $F : E \to T$ and $A \in \mathcal{P}(E)$, we have $\mathsf{cr}[F,A] = 1$ if A is connected (i.e., $A \in \mathcal{C}$) *and* F is homogeneous on A. For any $F : E \to T$, we obtain the family $\mathcal{C}^F_{\mathsf{cr}}$ of all connected sets on which F is homogeneous, that is,

$$\mathcal{C}^F_{\mathsf{cr}} = \{A \in \mathcal{P}(E) \mid \mathsf{cr}[F,A] = 1\} \ . \tag{1}$$

By item 2, the segmentation of F is a partition π^F_{cr} of E with blocks in $\mathcal{C}^F_{\mathsf{cr}}$, that is, $\pi^F_{\mathsf{cr}} \in \Pi(E, \mathcal{C}^F_{\mathsf{cr}})$, and by item 3, for any partition π strictly coarser than π^F_{cr}, the blocks of π cannot belong to $\mathcal{C}^F_{\mathsf{cr}}$, that is, $\pi^F_{\mathsf{cr}} < \pi \;\Rightarrow\; \pi \notin \Pi(E, \mathcal{C}^F_{\mathsf{cr}})$. Thus the segmentation of F is a maximal element of $\Pi(E, \mathcal{C}^F_{\mathsf{cr}})$ for the refinement order.

As remarked by Serra [12,9], it is necessary to consider the segmentation of a function F not only on E, but on any subset A of E. Thus in the above conditions, we have to consider the segmentation of a function F on a subset A of E, which is a partition of A. Here the segmentation method based on criterion cr associates to every function $F : E \to T$ and subset A of E the segmentation $\sigma^F_{\mathsf{cr}}(A)$, which is a partition of A; we require then that $\sigma^F_{\mathsf{cr}}(A)$ is a maximal element of $\Pi(A, \mathcal{C}^F_{\mathsf{cr}})$ for the refinement ordering. We call this requirement the *maximality principle.*

This wider approach has the advantage of allowing to consider the segmentation process as an operator acting on the lattice of partitions [7]. Indeed, we have introduced here a *set splitting operator* σ^F_{cr} that maps each $A \in \mathcal{P}(E)$ on a partition of A. From σ^F_{cr} we derive the *block splitting operator* $\beta(\sigma^F_{\mathsf{cr}}) : \Pi(E) \to \Pi(E)$ that acts on a partition π by applying σ^F_{cr} to each block of π, in other words $\beta(\sigma^F_{\mathsf{cr}})(\pi) = \bigcup_{B \in \pi} \sigma^F_{\mathsf{cr}}(B)$, see [7]. Then the maximality principle means that $\beta(\sigma^F_{\mathsf{cr}})(\pi)$ is a maximal element of the set of all $\pi' \in \Pi(E, \mathcal{C}^F_{\mathsf{cr}})$ such that $\pi' \leq \pi$.

Now this order-theoretic approach to image segmentation has been generalized to partial partitions [5,9]. A *partial partition* of E is a partition of any subset of E; in other words, it is a family of nonvoid mutually disjoint subsets of E, called *blocks*, but here we do no more assume that their union covers E. In parallel, the theory of connections has been generalized to that of *partial connections* [5]. Indeed, apart from the obvious fact that some segmentation algorithms produce a partial partition rather than a partition (the points of E not covered by it constitute the borders between regions), the "partial" framework is more versatile, allowing to represent individual or multiple set markers, as well as the progressive steps in the construction of a segmentation; also the construction of new connections [5] or new operators on partitions [6] is easier with the use of partial connections and partial partitions.

Write $\Pi^*(E)$ for the set of all partial partitions of E. Write $\boldsymbol{\varnothing}$ for the empty partial partition (with no block), and for any $A \in \mathcal{P}(E) \setminus \{\emptyset\}$, let $\mathbf{1}_A = \{A\}$ and $\mathbf{0}_A = \bigl\{\{p\} \mid p \in A\bigr\}$, while $\mathbf{1}_\emptyset = \mathbf{0}_\emptyset = \boldsymbol{\varnothing}$ [5]. For $\pi \in \Pi^*(E)$, the *support* of π, written $\mathsf{supp}(\pi)$, is the union of its blocks: $\mathsf{supp}(\pi) = \bigcup \pi$; the complement $E \setminus \mathsf{supp}(\pi)$ of the support is the *background* of π. For $\mathcal{C} \subseteq \mathcal{P}(E)$, let $\Pi^*(E,\mathcal{C}) = \Pi^*(E) \cap \mathcal{P}\big(\mathcal{C} \setminus \{\emptyset\}\big)$. The refinement order on $\Pi(E)$ extends naturally to $\Pi^*(E)$: for $\pi_1, \pi_2 \in \Pi^*(E)$, we write $\pi_1 \leq \pi_2$ (or $\pi_2 \geq \pi_1$), iff every block of π_1 is included in a block of π_2. Now contrarily to $\Pi(E)$, for $\pi_2 \geq \pi_1$ a block of π_2 is not necessarily a union of blocks of π_1, it can also contain points outside the support of π_1, it can even contain no block of π_1; thus a "coarsening" of a partial partition results not only from merging blocks, but also from inflating individual blocks or creating new blocks. Hence this order relation on $\Pi^*(E)$ should not be called refinement, a more appropriate denomination could be *extended refinement*; we will simply call it the *standard order*. Its main advantages are that (1) it naturally constitutes $\Pi^*(E)$ into a complete lattice, (2) its restriction to $\Pi(E)$ coincides with the refinement order, and the non-empty supremum and infimum operations in $\Pi(E)$ are inherited from those in $\Pi^*(E)$, and (3) we easily obtain a nice theory about idempotent block splitting operators [7,8].

Let us recall here that each order relation R is identified with the set of ordered pairs (a,b) such that $a \; R \; b$, so if we say that the order S is included in the order R, or that R contains S, this means that $a \; S \; b \Rightarrow a \; R \; b$.

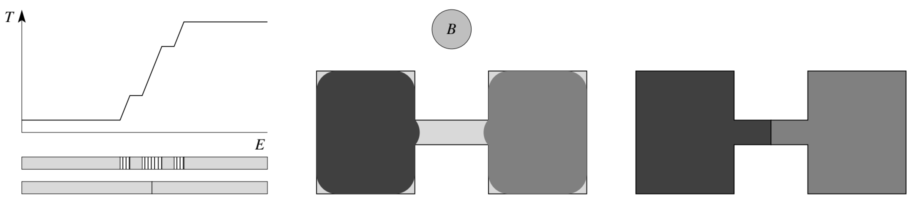

Fig. 1. Left: the graph of a one-dimensional grey-level edge; below we show its segmentation into connected classes with bounded slope (light grey rectangles for non-singleton classes, vertically hatched ones for groups of singleton classes); notice the large number of small classes on the edge; eliminating them, the final segmentation (bottom) consists of the influence zones of the two large classes. Middle: a subset of the plane is segmented into two connected zones open by a disk B, while the remaining points form singletons; right: the desired segmentation is obtained by the the influence zones of the two open zones.

Serra [13,14] noticed that in many image segmentation algorithms, "small parasitic" segmentation classes appear along contours and transitions, where the region homogeneity criterion fails; they can be eliminated and then one can take as final partition the influence zones of the significant segmentation classes corresponding to objects. See Figure 1. In order to analyse this process, he defined the *building order* $\Subset$ on $\Pi^*(E)$ as follows[1]: $\pi_1 \Subset \pi_2$ iff every block of π_2 contains at least one block of π_1. Then $\Subset$ is a partial order relation, and it is generally unrelated to the standard order $\leq$, except when the partial partitions

[1] In fact, Serra wrote $\pi_1 \preceq \pi_2$, but we will use the symbol $\prec$ for the covering relation.

have the same support: if $\pi_1 \leq \pi_2$ and $\mathsf{supp}(\pi_1) = \mathsf{supp}(\pi_2)$, then $\pi_1 \Subset \pi_2$; in particular for partitions, the building order $\Subset$ contains the refinement order $\leq$. However the building order does not constitute a lattice, and it is not easy to define operators with given order-theoretic properties (for example, isotony). Serra constructs extensive operators for the building order in two steps; starting from a partial partition π_0:

1. Remove "small parasitic" blocks from π_0 (through some "parasitism" and size criterion); the resulting partial partition π_1 satisfies $\pi_0 \supseteq \pi_1$, thus $\pi_0 \geq \pi_1$, but $\pi_0 \Subset \pi_1$.
2. Inflate the blocks of π_1 (for example by a SKIZ), without creating any new block; the merging of blocks is not excluded, but it is not used in practice; the resulting partial partition π_2 satisfies both $\pi_1 \leq \pi_2$ and $\pi_1 \Subset \pi_2$.

Then the partial partition π_2, having fewer but bigger blocks than π_0, is "better", a quality that is certified by the order $\pi_0 \Subset \pi_2$.

We remark that the construction of π_2 from π_0 involves two operations using two distinct criteria, and two distinct orders included in $\Subset$, which are at the same time included in $\geq$ and $\leq$ respectively. Furthermore, although $\pi_0 \Subset \pi_1$, π_1 is not considered as a "good" result; in all practical examples, the block growth of step 2 must be repeated until the blocks removed in step 1 are fully covered, in other words $\mathsf{supp}(\pi_2) = \sup(\pi_0)$ (in fact, Serra considers that π_0 and π_2 are partitions of E).

We propose that a meaningful order relation on partial partitions should be viewed through the family of operations that "enlarge" a partial partition, and these operations should effectively be involved in image segmentation techniques; if possible, they should be linked with segmentation criteria. We will indeed obtain several relevant orders included in the standard order $\leq$. Following the *maximality principle*, namely that the segmentation of $F : E \to T$ on $A \subseteq E$ following a criterion cr, is a maximal element of $\Pi^*(A, \mathcal{C}^F_{\mathsf{cr}})$, we show that for some of these orders, this principle is preserved in the compound segmentation paradigm [11,9], where after a first segmentation, a second one (with another criterion) is applied to the residue. Finally we investigate *valuations* on partial partitions, that is, strictly isotone maps $\Pi^*(E) \to \mathbb{R}^+$, and the possibility to replace "maximal" by "having maximal valuation" in the maximality principle.

2 Partial Order Relations on $\Pi^*(E)$

We will investigate several partial order relations on $\Pi^*(E)$. Each order is denoted by a variant of $\leq$, such as $\trianglelefteq$, with the associated notation $\blacktriangleleft$ for the corresponding strict order, and the mirror notation $\trianglerighteq$ for the inverse order and $\blacktriangleright$ for the inverse strict order. We first propose three *primary* partial order relations, from which other ones will be built:

1. The *inclusion* order $\subseteq$: $\pi_1 \subseteq \pi_2$ iff every block of π_1 is a block of π_2. This order is involved in the elimination of "parasitic" segmentation classes, cf.

Serra's step 1 above, but also in the compound segmentation paradigm, where we add to the blocks of a first segmentation those of a second segmentation of the residue.

2. The *pure refinement* order $\sqsubseteq$: $\pi_1 \sqsubseteq \pi_2$ iff $\pi_1 \leq \pi_2$ and $\mathsf{supp}(\pi_1) = \mathsf{supp}(\pi_2)$; in other words, every block of π_1 is included in a block of π_2, and every block of π_2 is a union of blocks of π_1; we say then that π_1 is *purely finer* than π_2, or that π_2 is *purely coarser* than π_1. This order is involved in split-and-merge operations in segmentation.
3. The *inflating* order $\trianglelefteq$: $\pi_1 \trianglelefteq \pi_2$ iff $\pi_1 \leq \pi_2$ and every block of π_2 contains exactly one block of π_1, in other words the inclusion relation between blocks of π_1 and those of π_2 is a bijection; we say then that π_1 is a *deflation* of π_2, or that π_2 is an *inflation* of π_1. This order is involved in parallel region growing (SKIZ, watershed), cf. Serra's step 2 above.

Next we give two *secondary* partial order relations; as their names (and notation) suggest, they can be constructed by composing two of the three primary orders, see Proposition 1:

4. The *inclusion-inflating* order $\mathrel{\subseteq\!\!\!\!\lhd}$ is defined by $\pi_1 \mathrel{\subseteq\!\!\!\!\lhd} \pi_2$ iff $\pi_1 \leq \pi_2$ and every block of π_2 contains at most one block of π_1.
5. The *refinement-inflating* order $\mathrel{\sqsubseteq\!\!\!\!\lhd}$ is the intersection of the building and standard orders, it is defined by $\pi_1 \mathrel{\sqsubseteq\!\!\!\!\lhd} \pi_2$ iff $\pi_1 \leq \pi_2$ and every block of π_2 contains at least one block of π_1.

The above five order relations are included in the standard order $\leq$, in other words, each such relation on (π_1, π_2) implies that $\pi_1 \leq \pi_2$. More precisely:

Proposition 1. *Here π_1, π_2, π, π' designate arbitrary partial partitions of E.*

1. *The standard order contains the inclusion, pure refinement and inflating orders: each of $\pi_1 \subseteq \pi_2$, $\pi_1 \sqsubseteq \pi_2$ and $\pi_1 \trianglelefteq \pi_2$ implies $\pi_1 \leq \pi_2$. It is generated by inclusion followed by pure refinement: $\pi_1 \leq \pi_2 \iff \exists\, \pi,\ \pi_1 \subseteq \pi \sqsubseteq \pi_2$.*
2. *The inclusion-inflating order contains the inclusion and inflating orders: each of $\pi_1 \subseteq \pi_2$ and $\pi_1 \trianglelefteq \pi_2$ implies $\pi_1 \mathrel{\subseteq\!\!\!\!\lhd} \pi_2$. It it is generated by composing them in any order:*

$$\pi_1 \mathrel{\subseteq\!\!\!\!\lhd} \pi_2 \iff \left(\exists\, \pi,\ \pi_1 \subseteq \pi \trianglelefteq \pi_2\right) \iff \left(\exists\, \pi',\ \pi_1 \trianglelefteq \pi' \subseteq \pi_2\right) .$$

3. *The refinement-inflating order contains the pure refinement and inflating orders: each of $\pi_1 \sqsubseteq \pi_2$ and $\pi_1 \trianglelefteq \pi_2$ implies $\pi_1 \mathrel{\sqsubseteq\!\!\!\!\lhd} \pi_2$. It is generated by composing them in any order:*

$$\pi_1 \mathrel{\sqsubseteq\!\!\!\!\lhd} \pi_2 \iff \left(\exists\, \pi,\ \pi_1 \sqsubseteq \pi \trianglelefteq \pi_2\right) \iff \left(\exists\, \pi',\ \pi_1 \trianglelefteq \pi' \sqsubseteq \pi_2\right) .$$

4. *The building order contains the inverse inclusion, pure refinement and inflating orders: each of $\pi_1 \supseteq \pi_2$, $\pi_1 \sqsubseteq \pi_2$ and $\pi_1 \trianglelefteq \pi_2$ implies $\pi_1 \Subset \pi_2$. It is generated by inverse inclusion followed by inflating: $\pi_1 \Subset \pi_2 \iff \exists\, \pi,\ \pi_1 \supseteq \pi \trianglelefteq \pi_2$.*

Each primary order on $\Pi^*(E)$ corresponds to the operations on blocks used for enlarging a partial partition; thus the inclusion $\subseteq$, inverse inclusion $\supseteq$, pure refinement $\sqsubseteq$ and inflating $\trianglelefteq$ orders correspond respectively to: creating, removing, merging and inflating blocks. We show in Figure 2 the sequence of two such operations involved in the orders $\leq$, $\trianglelefteqq$, $\sqtrianglelefteq$ and $\Subset$.

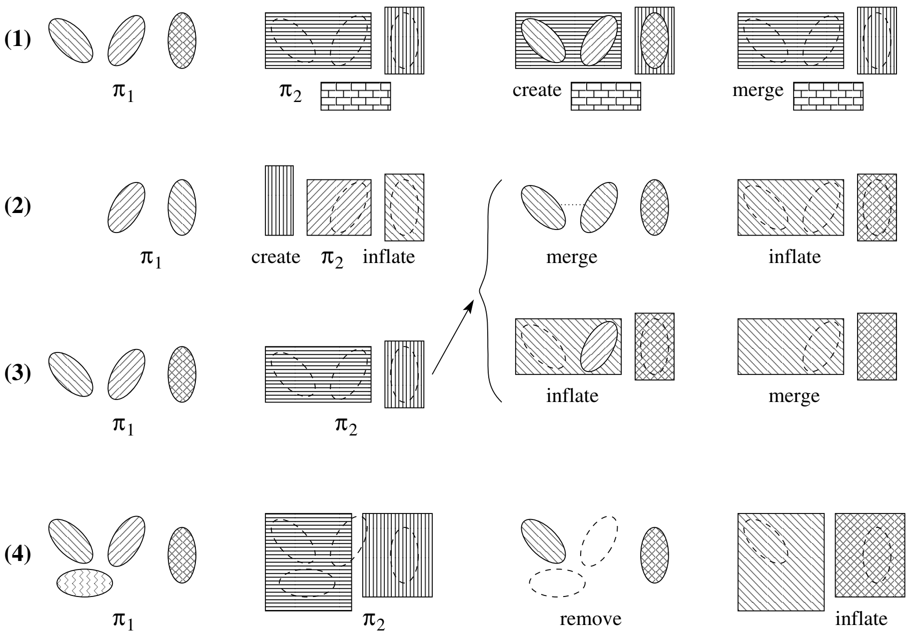

Fig. 2. In each partial partition, blocks are distinguished by their hatching. (1) $\pi_1 \leq \pi_2$; π_2 is obtained from π_1 by creating, then merging blocks. (2) $\pi_1 \trianglelefteqq \pi_2$; π_2 is obtained from π_1 by creating blocks outside $\mathsf{supp}(\pi_1)$ and at the same time inflating blocks of π_1. (3) $\pi_1 \sqtrianglelefteq \pi_2$; π_2 is obtained from π_1 by merging, then inflating blocks, or vice versa. (4) $\pi_1 \Subset \pi_2$; π_2 is obtained from π_1 by removing, then inflating blocks.

In watershed segmentation, the *saliency* of a watershed line between two basins is the flooding level at which these two basins merge [2]. More generally, given a *hierarchy of segmentations*, that is, an increasing sequence $\mathbf{0}_E = \pi_0 \leq \ldots \leq \pi_n = \mathbf{1}_E$ of segmentation partitions (with connected blocks), given two neighbouring blocks $B, C \in \bigcup_{i=0}^{n} \pi_i$, the *saliency* of the edge separating B and C is the level i at which this edge disappears, in other words the least level i such that B and C are included in the same block of π_i [1]. For partial partitions, things are more complicated, because the growth of the supports can lead to the creation of new edges. Between two neighbouring digital points p, q lies an edge element $e(p, q)$ (line in 2D, surface in 3D). In a partial partition π, any edge element belongs to one of the following 4 categories:

1. *background*: lying between two points of the background of π;
2. *outer*: lying between a point of the support and one of the background of π;
3. *separating*: lying between two points belonging to two distinct blocks of π;
4. *inner*: lying between two points belonging to the same block of π.

In a hierarchy $\boldsymbol{\emptyset} = \pi_0 \leq \ldots \leq \pi_n = \mathbf{1}_E$ of segmentation partial partitions (for the standard order), as the growth adds points to the support and merges blocks, each edge element can change category only in increasing order: *background* < *outer* < *separating* < *inner*. Thus the saliency associated to an edge element on a block boundary is given by two numbers, the level where it appears (transition *background* → *outer* or *separating*), and the higher level where it disappears (transition *outer* or *separating* → *inner*); for edge elements not belonging to block boundaries, we have the transition *background* → *inner*.

Alternately, the saliency $s(p)$ of a point $p \in E$ is the least level i such that $p \in \mathsf{supp}(\pi_i)$, and the saliency $s(p,q)$ of an edge element $e(p,q)$ is the least level j such that p and q belong to the same block of π_j. For two neighbouring pixels p, q, the 3 salencies $s(p)$, $s(q)$ and $s(p,q)$ satisfy $s(p,q) \geq \max(s(p), s(q))$, and they determine for each level the category of $e(p,q)$: for $i < \min(s(p), s(q))$: *background*; for $\min(s(p), s(q)) \leq i < \max(s(p), s(q))$: *outer*; for $\max(s(p), s(q)) \leq i < s(p,q)$: *separating*; for $s(p,q) \leq i$: *inner*.

3 Maximality in Compound Segmentation

A *set splitting operator on* $\mathcal{P}(E)$ is a map $\sigma : \mathcal{P}(E) \to \Pi^*(E)$ such that for every $X \in \mathcal{P}(E)$, $\sigma(X) \in \Pi^*(X)$; the corresponding *block splitting operator on* $\Pi^*(E)$ is the map $\beta(\sigma) : \Pi^*(E) \to \Pi^*(E) : \pi \mapsto \bigcup_{B \in \pi} \sigma(B)$ (i.e., it applies σ to each block of a partial partition) [7]. Given a partial order relation $\trianglelefteq$ on $\Pi^*(E)$, a family $\mathcal{C} \subseteq \mathcal{P}(E)$, and a set splitting operator σ on $\mathcal{P}(E)$, we say that σ *is* $\mathcal{C}$*-maximal for* $\trianglelefteq$ if for every $A \in \mathcal{P}(E)$, $\sigma(A)$ is a maximal element of $\Pi^*(A, \mathcal{C})$ according to the order $\trianglelefteq$.

For a segmentation method based on a criterion cr, to any function $F : E \to T$ is associated the family $\mathcal{C}^F_{\mathsf{cr}}$ of "homogeneous sets", see (1), and the set splitting operator σ^F_{cr}, where for $A \in \mathcal{P}(E)$, $\sigma^F_{\mathsf{cr}}(A)$ is the segmentation of F on A. Now the maximality principle states that for every function $F : E \to T$, σ^F_{cr} is $\mathcal{C}^F_{\mathsf{cr}}$-maximal for the chosen order (usually the standard order $\leq$ [7], but possibly the building order $\Subset$ [13]).

In *connective segmentation* [12,5,9], each $\mathcal{C}^F_{\mathsf{cr}}$ is a partial connection, equivalently, for every $A \in \mathcal{P}(E)$, $\Pi^*(A, \mathcal{C}^F_{\mathsf{cr}})$ has a greatest element (for the standard order $\leq$), namely the partial partion of A into its $\mathcal{C}^F_{\mathsf{cr}}$-components. Thus by the maximality principle, the segmentation $\sigma^F_{\mathsf{cr}}(A)$ must necessarily be that greatest element. In fact, this corresponds to the case where $\beta(\sigma^F_{\mathsf{cr}})$ is an opening [7]. Since partial connections constitute a complete lattice for the inclusion order, such connective segmentations also form a complete lattice [12,5,9]; for a family cr_i $(i \in I)$ of connective criteria, the infimum of the connective segmentation operators $\sigma^F_{\mathsf{cr}_i}$ is $\sigma^F_{\inf_{i \in I} \mathsf{cr}_i}$, and their supremum is $\bigvee_{i \in I} \sigma^F_{\mathsf{cr}_i}$.

There is a priori no such lattice structure in the non-connective case. However the maximality principle is consistent with Serra's compound segmentation paradigm [11,9], where after a first segmentation producing a partial partition, the residue (the set of points not covered by that partial partition) is partitioned by another segmentation method, and the union of the two partial partitions forms the compound segmentation. This operation has been formalized as follows [8]. Let σ_1, σ_2 be two set splitting operators on $\mathcal{P}(E)$; the *residual combination of* σ_1 *followed by* σ_2 is the set splitting operator $\sigma_1 \ltimes \sigma_2$ on $\mathcal{P}(E)$ defined by $(\sigma_1 \ltimes \sigma_2)(X) = \sigma_1(X) \cup \sigma_2\big(X \setminus \mathsf{supp}[\sigma_1(X)]\big)$ for any $X \in \mathcal{P}(E)$. Note that in [8] we wrote it $\mathsf{rc}[\sigma_1, \sigma_2]$.

Let us define the binary operation $\curlywedge$ on subsets of $\mathcal{P}(E)$ by setting for any $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{P}(E)$:

$$\mathcal{H}_1 \curlywedge \mathcal{H}_2 = \mathcal{H}_1 \cup \big\{X \in \mathcal{H}_2 \mid \forall Y \in \mathcal{P}(X) \setminus \{\emptyset\},\ Y \notin \mathcal{H}_1\big\} \ .$$

In other words $\mathcal{H}_1 \curlywedge \mathcal{H}_2 = \mathcal{H}_1 \cup \big\{X \in \mathcal{H}_2 \mid \mathcal{P}(X) \cap \mathcal{H}_1 \subseteq \{\emptyset\}\big\}$. Let us say that a partial order relation $\trianglelefteq$ on $\Pi^*(E)$ is *well-composed* if for any $\pi_0, \pi_1, \pi_2 \in \Pi^*(E)$ we have:

$$\pi_0 \subseteq \pi_1 \implies \pi_0 \trianglelefteq \pi_1 \implies \pi_0 \leq \pi_1,$$
$$\big(\pi_0 \leq \pi_1 \subseteq \pi_2 \text{ and } \pi_0 \trianglelefteq \pi_2\big) \implies \pi_0 \trianglelefteq \pi_1 \ .$$

For example the inclusion $\subseteq$, inclusion-inflating $\trianglelefteq$ and standard $\leq$ orders are well-composed.

Theorem 2. *Let* $\trianglelefteq$ *be a well-composed partial order relation on* $\Pi^*(E)$*, let* $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathcal{P}(E)$*, and let* σ_1, σ_2 *be set splitting operators on* $\mathcal{P}(E)$*, respectively* $\mathcal{C}_1$*-maximal and* $\mathcal{C}_2$*-maximal for* $\trianglelefteq$*. Then* $\sigma_1 \ltimes \sigma_2$ *is* $\mathcal{C}_1 \curlywedge \mathcal{C}_2$*-maximal for* $\trianglelefteq$*.*

The residual combination $\ltimes$ is associative [8], thus we can define $\sigma_1 \ltimes \cdots \ltimes \sigma_n$ for any $n \geq 2$, and we have for any $X \in \mathcal{P}(E)$:

$$(\sigma_1 \ltimes \cdots \ltimes \sigma_n)(X) = \bigcup_{i=1}^{n} \sigma_i(X_i), \quad \text{where}$$
$$X_1 = X \quad \text{and} \quad \text{for } i = 2, \ldots, n,\ X_i = X \setminus \bigcup_{j=1}^{i-1} \mathsf{supp}\big[\sigma_j(X_j)\big] \ .$$

We can easily show that the operation $\curlywedge$ is also associative, and we have

$$\mathcal{H}_1 \curlywedge \cdots \curlywedge \mathcal{H}_n = \mathcal{H}'_1 \cup \cdots \cup \mathcal{H}'_n, \quad \text{where} \quad \mathcal{H}'_1 = \mathcal{H}_1 \quad \text{and}$$
$$\text{for } i = 2, \ldots, n,\ \mathcal{H}'_i = \Big\{X \in \mathcal{H}_i \mid \forall Y \in \mathcal{P}(X) \setminus \{\emptyset\},\ Y \notin \bigcup_{j=1}^{i-1} \mathcal{H}_j\Big\} \ .$$

Corollary 3. *Let* $\trianglelefteq$ *be a well-composed partial order relation on* $\Pi^*(E)$*, let* $\mathcal{C}_1, \ldots, \mathcal{C}_n \subseteq \mathcal{P}(E)$ *(*$n \geq 2$*), and let* $\sigma_1, \ldots, \sigma_n$ *be set splitting operators on* $\mathcal{P}(E)$ *such that for each* $i = 1, \ldots, n$*,* σ_i *is* $\mathcal{C}_i$*-maximal for* $\trianglelefteq$*. Then* $\sigma_1 \ltimes \cdots \ltimes \sigma_n$ *is* $\mathcal{C}_1 \curlywedge \cdots \curlywedge \mathcal{C}_n$*-maximal for* $\trianglelefteq$*.*

4 Block Splitting Thinnings

We will consider now the possible idempotence of $\beta(\sigma)$ for a $\mathcal{C}$-maximal set splitting operator σ on $\mathcal{P}(E)$. But let us first briefly introduce some general terminology. Consider a poset $(P, \leq)$; for any operator π on P, let $\mathsf{Inv}(\psi) = \{x \in P \mid \psi(x) = x\}$ be the *invariance domain* of ψ. A *thinning* is an anti-extensive idempotent operator. A *max-thinning* is an operator ψ such that for all $x \in P$, $\psi(x)$ is a maximal element of $\{y \in \mathsf{Inv}(\psi) \mid y \leq x\}$. A *C-thinning* [8] (in [3] we said an *open-condensation*) is an anti-extensive operator ψ such that for any $x, y \in P$, $\left[\psi(x) \leq y \leq x\right] \Rightarrow \psi(y) = \psi(x)$. Then every C-thinning is a max-thinning, and every max-thinning is a thinning.

Proposition 4. *Let $\trianglelefteq$ be a well-composed partial order relation on $\Pi^*(E)$ and let $\mathcal{C} \subseteq \mathcal{P}(E)$. A set splitting operator σ on $\mathcal{P}(E)$ is $\mathcal{C}$-maximal for $\trianglelefteq$ iff for any $\pi \in \Pi^*(E)$, $\beta(\sigma)(\pi)$ is a maximal element (for $\trianglelefteq$) of $\{\pi' \in \Pi^*(E, \mathcal{C}) \mid \pi' \leq \pi\}$.*

In fact, $\beta(\sigma)$ will be idempotent iff for every $A \in \mathcal{C} \setminus \{\emptyset\}$, $\mathbf{1}_A$ is the greatest element (for $\trianglelefteq$) of $\Pi^*(A, \mathcal{C})$.

Let us now consider the particular case where $\trianglelefteq$ is the standard order $\leq$. Following Proposition 4, we see that a set splitting operator σ on $\mathcal{P}(E)$ is $\mathcal{C}$-maximal for $\leq$ iff $\beta(\sigma)$ is a max-thinning with $\mathsf{Inv}(\beta(\sigma)) = \Pi^*(E, \mathcal{C})$.

In [8] we showed that given two set splitting operators σ_1, σ_2 such that $\beta(\sigma_1)$ and $\beta(\sigma_2)$ are C-thinnings on $\Pi^*(E, \mathcal{C})$, then $\beta(\sigma_1 \ltimes \sigma_2)$ will be a C-thinning on $\Pi^*(E, \mathcal{C})$. This result was in fact a particular case of a general construction on posets, of which another special case was a result of [3]: given two C-thinnings θ_1, θ_2 on $\mathcal{P}(E)$, the operator on $\mathcal{P}(E) : X \mapsto \theta_1(X) \cup \theta_2\big(X \setminus \theta_1(X)\big)$ will be a C-thinning. However Theorem 2 is specific to $\Pi^*(E)$, the analogous result on $\mathcal{P}(E)$ is not valid. For example, let $E = \mathbb{Z}^2$ and define the operator θ on $\mathcal{P}(E)$ that extracts from any $X \in \mathcal{P}(E)$ a rectangle $\theta(X) \subseteq X$ by applying rules (a,b,c,d) with decreasing priority: (1°) if $width(X) < height(X)$, $\theta(X)$ has (a) greatest height, then (b) greatest width, and is located (c) topmost, then (d) leftmost; (2°)

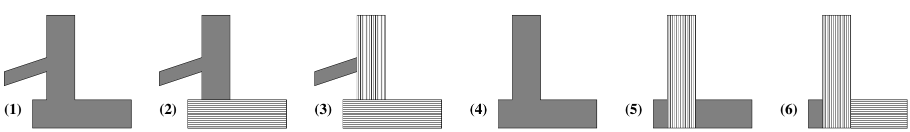

Fig. 3. $E = \mathbb{Z}^2$. (1) The set X, with $width(X) > height(X)$. (2) The rectangle $A = \theta(X)$ (shown with horizontal hatching) is selected with priority to width. (3) Now $width(X \setminus A) < height(X \setminus A) < \infty$, so the rectangle $B = \theta(X \setminus A)$ (shown with vertical hatching) is selected with priority to height; then $(\mathbf{1}_\theta \ltimes \mathbf{1}_\theta)(X) = \{A, B\}$ is invariant under further application of $\beta(\mathbf{1}_\theta \ltimes \mathbf{1}_\theta)$. (4) $Y = A \cup B$, with $width(Y) < height(Y) < \infty$. (5) The rectangle $C = \theta(Y)$ (shown with vertical hatching) is selected with priority to height. (6) Now $width(Y \setminus C) > height(Y \setminus C)$, so the rectangle $D = \theta(Y \setminus C)$ (shown with horizontal hatching) is selected with priority to width; then $(\mathbf{1}_\theta \ltimes \mathbf{1}_\theta)(Y) = \{C, D\}$, but $C \cup D \subset Y = A \cup B$.

if $width(X) \geq height(X)$, $\theta(X)$ has (a) greatest width, then (b) greatest height, and is located (c) leftmost, then (d) topmost. Consider now the *set shrinking* operator [7] $\mathbf{1}_\theta : X \mapsto \mathbf{1}_{\theta(X)}$. Then $\mathbf{1}_\theta$ is is $\mathcal{H}$-maximal for the set $\mathcal{H}$ of all rectangles; however the operator on $\psi : \mathcal{P}(E) \to \mathcal{P}(E) : X \mapsto \theta(X) \cup \theta\big(X \setminus \theta(X)\big)$ is not idempotent, see Figure 3.

In [8] we showed also that given a C-thinning κ and an adjunction (ε, δ), $\delta\kappa\varepsilon$ will be a C-thinning. This does not hold for max-thinnings. Indeed, the map $\mathsf{supp} : \Pi^*(E) \to \mathcal{P}(E) : \pi \mapsto \mathsf{supp}(\pi)$ is the lower adjoint of the map $\mathbf{1}_\bullet : \mathcal{P}(E) \to \Pi^*(E) : A \mapsto \mathbf{1}_A$ [5], and $\beta(\mathbf{1}_\theta \ltimes \mathbf{1}_\theta)$ is a max-thinning, but $\psi = \mathsf{supp} \cdot \beta(\mathbf{1}_\theta \ltimes \mathbf{1}_\theta) \cdot \mathbf{1}_\bullet$ is not idempotent.

5 Valuations on Partial Partitions

Let the space E be finite, so that $\Pi^*(E)$ is finite, and let $\trianglelefteq$ be a partial order relation on $\Pi^*(E)$ having $\emptyset$ as least element. For $\pi_1, \pi_2 \in \Pi^*(E)$, we say that π_2 *covers* π_1 if $\pi_1 \triangleleft \pi_2$ but there is no $\pi \in \Pi^*(E)$ with $\pi_1 \triangleleft \pi \triangleleft \pi_2$. The covering relation is written $\prec$, with possible variants like $\overset{x}{\prec}$. We call a *valuation* on $(\Pi^*(E), \trianglelefteq)$ a map $f : \Pi^*(E) \to \mathbb{R}$ such that $f(\emptyset) = 0$ and for all $\pi_1, \pi_2 \in \Pi^*(E)$, $\pi_1 \triangleleft \pi_2 \;\Rightarrow\; f(\pi_1) < f(\pi_2)$; thus f is in fact $\Pi^*(E) \to \mathbb{R}^+$. Since $\Pi^*(E)$ is finite, the order $\trianglelefteq$ is determined by the covering relation $\prec$, and $f : \Pi^*(E) \to \mathbb{R}$ is a valuation iff $f(\emptyset) = 0$ and for $\pi_1, \pi_2 \in \Pi^*(E)$, $\pi_1 \prec \pi_2 \;\Rightarrow\; f(\pi_1) < f(\pi_2)$. The valuation f is said to be *linear* if for any $\pi \in \Pi^*(E)$, $f(\pi) = \sum_{B \in \pi} f(\mathbf{1}_B)$, in other words if there is a map $g : \mathcal{P}(E) \setminus \{\emptyset\} \to \mathbb{R}$ such that for any $\pi \in \Pi^*(E)$, $f(\pi) = \sum_{B \in \pi} g(B)$. It is said to be *homogeneous* if it is invariant under any permutation of the points of E; equivalently, for any $\pi \in \Pi^*(E)$, $f(\pi)$ depends only on the histogram of sizes of all blocks of π, that is, the function $\mathbb{N} \to \mathbb{N} : n \mapsto \big|\{B \in \pi \mid |B| = n\}\big|$ (here $|X|$ denotes the size of a set X). It is both homogeneous and linear if there is a map $h : \mathbb{N} \setminus \{0\} \to \mathbb{R}$ such that for any $\pi \in \Pi^*(E)$, $f(\pi) = \sum_{B \in \pi} h(|B|)$.

Let us briefly describe the covering relation and linear valuations for the standard order $\leq$ on $\Pi^*(E)$. A similar analysis can easily be done for the well-composed order relations $\subseteq$ and $\trianglelefteq$ (that admit also $\emptyset$ as least element). For $\pi_1, \pi_2 \in \Pi^*(E)$, we say that π_2 *m-covers* (resp., *c-covers*) π_1 and write $\pi_1 \overset{m}{\prec} \pi_2$ (resp., $\pi_1 \overset{c}{\prec} \pi_2$) if π_2 is obtained from π_1 by merging two blocks (resp., by creating a new singleton block):

$$\pi_1 \overset{m}{\prec} \pi_2 \iff \exists\, B, C \in \pi_1,\ B \neq C,\ \pi_2 = \big(\pi_1 \setminus \{B, C\}\big) \cup \{B \cup C\} \ ,$$
$$\pi_1 \overset{c}{\prec} \pi_2 \iff \exists\, p \in E \setminus \mathsf{supp}(\pi_1),\ \pi_2 = \pi_1 \cup \big\{\{p\}\big\} \ .$$

We have $\pi_1 \prec \pi_2$ iff $\pi_1 \overset{m}{\prec} \pi_2$ or $\pi_1 \overset{c}{\prec} \pi_2$ [6]. Thus a linear valuation takes the form $f(\pi) = \sum_{B \in \pi} g(B)$, where $g : \mathcal{P}(E) \setminus \{\emptyset\} \to \mathbb{R}$ satisfies $g(\{p\}) > 0$ for all $p \in E$, and $g(B \cup C) > g(B) + g(C)$ for all disjoint $B, C \in \mathcal{P}(E) \setminus \{\emptyset\}$. A homogeneous linear valuation takes the form $f(\pi) = \sum_{B \in \pi} h(|B|)$, where $h : \mathbb{N} \setminus \{0\} \to \mathbb{R}$ satisfies $h(1) > 0$ and $h(m + n) > h(m) + h(n)$ for all $m, n > 0$. In a covering chain $\emptyset \prec \cdots \prec \pi$, we always have $|\mathsf{supp}(\pi)| - |\pi|$ m-coverings $\overset{m}{\prec}$ and $|\mathsf{supp}(\pi)|$ c-coverings $\overset{c}{\prec}$; assigning them respective costs $\mu, \gamma > 0$, we get the homogeneous linear valuation

$$f(\pi) = \mu\big(|\mathsf{supp}(\pi)| - |\pi|\big) + \gamma|\mathsf{supp}(\pi)| = (\mu + \gamma)|\mathsf{supp}(\pi)| - \mu|\pi| \ ; \qquad (2)$$

here $h(n) = (\mu + \gamma)n - \mu$ for all $n > 0$. For example, $\mu = \gamma = 1$ gives the *height* $2|\mathsf{supp}(\pi)| - |\pi|$ of π [6], with $h(n) = 2n - 1$.

Let us now apply valuations in image segmentation methodology. According to the maximality principle, $\sigma^F_{\mathsf{cr}}(A)$, the segmentation of $F : E \to T$ on $A \subseteq E$ following a criterion cr, is a maximal element of $\Pi^*(A, \mathcal{C}^F_{\mathsf{cr}})$ for the chosen order $\trianglelefteq$ on $\Pi^*(E)$. We can modify that principle by requiring that $\sigma^F_{\mathsf{cr}}(A)$ is an element of $\Pi^*(A, \mathcal{C}^F_{\mathsf{cr}})$ having maximum valuation; for the valuation f, we have:

$$\forall\, F \in T^E,\ \forall\, A \in \mathcal{P}(E), \quad f\big(\sigma^F_{\mathsf{cr}}(A)\big) = \max\big\{f(\pi) \mid \pi \in \Pi^*(A, \mathcal{C}^F_{\mathsf{cr}})\big\}$$

This represents in fact a strengthening of the maximality principle; indeed, since $\pi_1 \blacktriangleleft \pi_2 \ \Rightarrow\ f(\pi_1) < f(\pi_2)$, an element of $\Pi^*(A, \mathcal{C}^F_{\mathsf{cr}})$ with maximum valuation must necessarily be maximal.

Let now $\trianglelefteq$ be the standard order $\leq$. In connective segmentation, for each $F : E \to T$, $\mathcal{C}^F_{\mathsf{cr}}$ is a partial connection, that is, for every $A \in \mathcal{P}(E)$, $\Pi^*(A, \mathcal{C}^F_{\mathsf{cr}})$ has a greatest element; then this greatest element has necessarily maximum valuation. Hence maximum valuation does not modify connective segmentation.

In Serra's method for "eliminating parasitic segmentation classes" [14,13], we start from an initial segmentation $\pi_0 \in \Pi^*(A, \mathcal{C}^F_{\mathsf{cr}})$, eliminate "small parasitic" blocks, getting $\pi_1 \in \Pi^*(A, \mathcal{C}^F_{\mathsf{cr}})$, then inflate the blocks of π_1, getting $\pi_2 \in \Pi^*(A, \mathcal{C}^F_{\mathsf{cr}})$; ideally $\mathsf{supp}(\pi_0) \subseteq \mathsf{supp}(\pi_2) \subseteq A$. Clearly $\pi_0 \Subset \pi_1 \Subset \pi_2$, so from Serra's point of view, the modifications $\pi_0 \mapsto \pi_1 \mapsto \pi_2$ are two successive improvements. In the light of a valuation f, since $\pi_0 > \pi_1 < \pi_2$, we have $f(\pi_0) > f(\pi_1) < f(\pi_2)$, which is consistent with our view that the intermediate result π_1 is not satisfactory; now taking f as in (2), since π_2 has fewer blocks than π_0, and a larger or equal support, we get $f(\pi_2) > f(\pi_0)$, confirming that π_2 improves on π_1.

6 Conclusion and Perspectives

We have analysed image segmentation as the construction of an "optimal" partial partition with "homogeneous" blocks, where the optimality can mean either maximality for an order, or maximality of a valuation; the latter is generally a stronger requirement, except for connective segmentation, where the two are equivalent. Maximality for a well-composed order is preserved in compound segmentation, leading to max-thinnings on partial partitions. It seems that maximal valuation can lead to a segmentation eliminating "small parasitic" classes.

We have analysed several orders included in the standard order. There are other orders included in the building order, for example: $\pi_1 \trianglelefteq \pi_2$ iff each block of π_2 is a union of blocks of π_1, i.e., $\exists\, \pi,\, \pi_1 \supseteq \pi \sqsubseteq \pi_2$. Serra's work [13,14] suggests a new order relation on $\Pi^*(E)$, the *apportioning* order $\overline{\sqsubseteq}$, where $\pi_1 \mathrel{\overline{\sqsubseteq}} \pi_2$ iff $\pi_1 \Subset \pi_2$ and $\mathsf{supp}(\pi_1) = \mathsf{supp}(\pi_2)$. Consider further the *apportioning-inflating* order $\overline{\trianglelefteq}$, with $\pi_1 \mathrel{\overline{\trianglelefteq}} \pi_2$ iff $\pi_1 \Subset \pi_2$ and $\mathsf{supp}(\pi_1) \subseteq \mathsf{supp}(\pi_2)$ (briefly considered and written $\sqsubseteq$ in [13]), and the *extended* order $\overline{\leq}$, with $\pi_1 \mathrel{\overline{\leq}} \pi_2$ iff $\pi_1 \Subset \pi_2 \cap \mathcal{P}(\mathsf{supp}(\pi_1))$ and $\mathsf{supp}(\pi_1) \subseteq \mathsf{supp}(\pi_2)$. We then get the analogue of items 1 and 3 of Proposition 1 with $\overline{\sqsubseteq}, \overline{\trianglelefteq}, \overline{\leq}$ instead of $\sqsubseteq, \trianglelefteq, \leq$.

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