Fuzzy Bipolar Mathematical Morphology: A General Algebraic Setting

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Abstract. Bipolar information is an important component in information processing, to handle both positive information (e.g. preferences) and negative information (e.g. constraints) in an asymmetric way. In this paper, a general algebraic framework is proposed to handle such information using mathematical morphology operators, leading to results that apply to any partial ordering.

Keywords: Bipolar information, fuzzy bipolar dilation and erosion, bipolar connectives.

1 Introduction

A recent trend in contemporary information processing focuses on bipolar information, both from a knowledge representation point of view, and from a processing and reasoning one. Bipolarity is important to distinguish between (i) positive information, which represents what is guaranteed to be possible, for instance because it has already been observed or experienced, and (ii) negative information, which represents what is impossible or forbidden, or surely false [1]. In this paper, we propose to handle such bipolar information using mathematical morphology operators. Mathematical morphology on bipolar fuzzy sets was proposed for the first time in [2], by considering the complete lattice defined from the Pareto ordering. Then it was further developed, with additional properties, geometric aspects and applications to spatial reasoning, in [3,4]. The lexicographic ordering was considered too in [5]. Here we propose a more general algebraic setting and we show that the usual properties considered in mathematical morphology hold in any complete lattice representing bipolar information, whatever the choice of the partial ordering. Recently, mathematical morphology on interval-valued fuzzy sets and intuitionistic fuzzy sets was addressed, independently, in [6], but without considering the algebraic framework of adjunctions, thus leading to weaker properties. This group then extended its approach with more properties in [7]. Pareto ordering was used in this work. Again this paper proposes a more general and powerful setting.

Mathematical morphology [8] usually relies on the algebraic framework of complete lattices [9]. Although it has also been extended to complete semi-lattices and general posets [10], based on the notion of adjunction [11], in this

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paper we only consider the case of complete lattices. Let us assume that bipolar information is represented by a pair (μ, ν), where μ represents the positive information and ν the negative information, under a consistency constraint [1]. Let us denote by $\mathcal{B}$ the set of all (μ, ν). We assume that it is possible to define a spatial ordering $\preceq$ on $\mathcal{B}$ such that $(\mathcal{B}, \preceq)$ is a complete lattice. We denote by $\bigvee$ and $\bigwedge$ the supremum and infimum, respectively. Once we have a complete lattice, it is easy to define algebraic dilations δ and erosions ε on this lattice, as classically done in mathematical morphology [11,12], as operations that commute with the supremum and with the infimum, respectively:

$$\forall (\mu_i, \nu_i) \in \mathcal{B},\ \delta(\bigvee_i (\mu_i, \nu_i)) = \bigvee_i \delta((\mu_i, \nu_i)), \tag{1}$$

$$\forall (\mu_i, \nu_i) \in \mathcal{B},\ \varepsilon(\bigwedge_i (\mu_i, \nu_i)) = \bigwedge_i \varepsilon((\mu_i, \nu_i)), \tag{2}$$

where (μ_i, ν_i) is any family (finite or not) of elements of $\mathcal{B}$.

Classical results derived from the properties of complete lattices and adjunctions [11,12] hold in the bipolar case too.

Bipolar information can be represented in different frameworks, leading to different forms of μ and ν, for instance:

- positive and negative information are subsets P and N of some set, and the consistency constraint is expressed as $P \cap N = \emptyset$, expressing that what is possible or preferred (positive information) should be included in what is not forbidden (negative information) [1];
- μ and ν are membership functions to fuzzy sets, defined over a space $\mathcal{S}$, and the consistency constraint is expressed as $\forall x \in \mathcal{S}, \mu(x) + \nu(x) \leq 1$ [2]. The pair (μ, ν) is then called a bipolar fuzzy set;
- positive and negative information are represented by logical formulas φ and ψ, generated by a set of propositional symbols and connectives, and the consistency constraint is then expressed as $\varphi \wedge \psi \models \bot$ (ψ represents what is forbidden or impossible).
- Other examples include functions such as utility functions or capacities [13], preference functions [14], four-valued logics [15], possibility distributions [16].

One of the main issues in the proposed extensions of mathematical morphology to bipolar information is to handle the two components (i.e. positive and negative information) and to define an adequate and relevant ordering. Two extreme cases are Pareto ordering (also called marginal ordering) and lexicographic ordering. The Pareto ordering handles both components in a symmetric way, while the lexicographic ordering on the contrary gives a strong priority to one component, and the other one is then seldom considered. This issue has been addressed in other types of work, where different partial orderings have been discussed, such as color image processing (see e.g. [17]) and social choice (see e.g. [18]). The works in these domains, and the various partial orderings proposed, can guide the choice of an ordering adapted to bipolar information.

In the following, we will detail the case of bipolar fuzzy sets, extending our previous work in [2,3] to any partial ordering. This includes the other examples described above: the case of sets corresponds to the case where only bipolarity should be taken into account, without fuzziness (hence the membership functions take only values 0 and 1). In the case of logical formulas, we consider the models $[\![\varphi]\!]$ and $[\![\psi]\!]$ as sets or fuzzy sets. Hence the case of bipolar fuzzy sets is general enough to cover several other mathematical settings.

The lattice structure is described in Section 2. Then bipolar connectives and their properties are detailed in Section 3. They are then used to define general forms of morphological dilations and erosions in Section 4, based on bipolar degrees of intersection and inclusion. Proofs are omitted here, and can be found in [19].

2 Partial Ordering and Lattice of Bipolar Fuzzy Sets

Let $\mathcal{S}$ be the underlying space (the spatial domain for spatial information processing). A bipolar fuzzy set on $\mathcal{S}$ is defined by an ordered pair of functions (μ, ν) such that $\forall x \in \mathcal{S}, \mu(x) + \nu(x) \leq 1$. Note that bipolar fuzzy sets are formally linked to intuitionistic fuzzy sets [20], interval-valued fuzzy sets [21] and vague sets, where the interval at each point x is $[\mu(x), 1-\nu(x)]$, or to clouds when boundary constraints are added [22], as shown by several authors [23]. However their respective semantics are very different, and we keep here the terminology of bipolarity, for handling asymmetric bipolar information [16].

For each point x, $\mu(x)$ defines the degree to which x belongs to the bipolar fuzzy set (positive information) and $\nu(x)$ the non-membership degree (negative information). This formalism allows representing both bipolarity and fuzziness. The set of bipolar fuzzy sets defined on $\mathcal{S}$ is denoted by $\mathcal{B}$.

Let us denote by $\mathcal{L}$ the set of ordered pairs of numbers (a, b) in $[0, 1]$ such that $a + b \leq 1$ (hence $(\mu, \nu) \in \mathcal{B} \Leftrightarrow \forall x \in \mathcal{S}, (\mu(x), \nu(x)) \in \mathcal{L}$). Let $\preceq$ be a partial ordering on $\mathcal{L}$ such that $(\mathcal{L}, \preceq)$ is a complete lattice. We denote by $\bigvee$ and $\bigwedge$ the supremum and infimum, respectively. The smallest element is denoted by $0_\mathcal{L}$ and the largest element by $1_\mathcal{L}$.

The partial ordering on $\mathcal{L}$ induces a partial ordering on $\mathcal{B}$, also denoted by $\preceq$ for the sake of simplicity:

$$(\mu_1, \nu_1) \preceq (\mu_2, \nu_2) \text{ iff } \forall x \in \mathcal{S}, (\mu_1(x), \nu_1(x)) \preceq (\mu_2(x), \nu_2(x)). \quad (3)$$

Then $(\mathcal{B}, \preceq)$ is a complete lattice, for which the supremum and infimum are also denoted by $\bigvee$ and $\bigwedge$. The smallest element is the bipolar fuzzy set (μ_0, ν_0) taking value $0_\mathcal{L}$ at each point, and the largest element is the bipolar fuzzy set $(\mu_\mathbb{I}, \nu_\mathbb{I})$ always equal to $1_\mathcal{L}$. Note that the supremum and the infimum do not necessarily provide one of the input bipolar numbers or bipolar fuzzy sets (in particular if they are not comparable according to $\preceq$). However, they do in case $\preceq$ is a total ordering.

3 Bipolar Connectives

Let us now introduce some connectives, that will be useful in the following and that extend to the bipolar case the connectives classically used in fuzzy set theory. In all what follows, increasingness or decreasingness is intended according to the partial ordering $\preceq$.

A **bipolar negation, or complementation**, on $\mathcal{L}$ is a decreasing operator N such that $N(0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $N(1_{\mathcal{L}}) = 0_{\mathcal{L}}$. In this paper, we restrict ourselves to involutive negations, such that $\forall (a,b) \in \mathcal{L}, N(N((a,b))) = (a,b)$ (these are the most interesting ones for mathematical morphology).

A **bipolar conjunction** is an operator C from $\mathcal{L} \times \mathcal{L}$ into $\mathcal{L}$ such that $C(0_{\mathcal{L}}, 0_{\mathcal{L}}) = C(0_{\mathcal{L}}, 1_{\mathcal{L}}) = C(1_{\mathcal{L}}, 0_{\mathcal{L}}) = 0_{\mathcal{L}}$, $C(1_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}$, and that is increasing in both arguments, i.e.: $\forall ((a_1,b_1),(a_2,b_2),(a'_1,b'_1),(a'_2,b'_2)) \in \mathcal{L}^4, (a_1,b_1) \preceq (a'_1,b'_1)$ and $(a_2,b_2) \preceq (a'_2,b'_2) \Rightarrow C((a_1,b_1),(a_2,b_2)) \preceq C((a'_1,b'_1),(a'_2,b'_2))$.

A **bipolar t-norm** is a commutative and associative bipolar conjunction such that $\forall (a,b) \in \mathcal{L}, C((a,b),1_{\mathcal{L}}) = C(1_{\mathcal{L}},(a,b)) = (a,b)$ (i.e. the largest element of $\mathcal{L}$ is the unit element of C). If only the property on the unit element holds, then C is called a **bipolar semi-norm**.

A **bipolar disjunction** is an operator D from $\mathcal{L} \times \mathcal{L}$ into $\mathcal{L}$ such that $D(1_{\mathcal{L}}, 1_{\mathcal{L}}) = D(0_{\mathcal{L}}, 1_{\mathcal{L}}) = D(1_{\mathcal{L}}, 0_{\mathcal{L}}) = 1_{\mathcal{L}}$, $D(0_{\mathcal{L}}, 0_{\mathcal{L}}) = 0_{\mathcal{L}}$, and that is increasing in both arguments.

A **bipolar t-conorm** is a commutative and associative bipolar disjunction such that $\forall (a,b) \in \mathcal{L}, D((a,b),0_{\mathcal{L}}) = D(0_{\mathcal{L}},(a,b)) = (a,b)$ (i.e. the smallest element of $\mathcal{L}$ is the unit element of D).

A **bipolar implication** is an operator I from $\mathcal{L} \times \mathcal{L}$ into $\mathcal{L}$ such that $I(0_{\mathcal{L}}, 0_{\mathcal{L}}) = I(0_{\mathcal{L}}, 1_{\mathcal{L}}) = I(1_{\mathcal{L}}, 1_{\mathcal{L}}) = 1_{\mathcal{L}}$, $I(1_{\mathcal{L}}, 0_{\mathcal{L}}) = 0_{\mathcal{L}}$ and that is decreasing in the first argument and increasing in the second argument.

Proposition 1. *Any bipolar conjunction C has a null element, which is the smallest element of $\mathcal{L}$: $\forall (a,b) \in \mathcal{L}, C((a,b),0_{\mathcal{L}}) = C(0_{\mathcal{L}},(a,b)) = 0_{\mathcal{L}}$. Similarly, any bipolar disjunction has a null element, which is the largest element of $\mathcal{L}$: $\forall (a,b) \in \mathcal{L}, D((a,b),1_{\mathcal{L}}) = D(1_{\mathcal{L}},(a,b)) = 1_{\mathcal{L}}$. For implications, we have $\forall (a,b) \in \mathcal{L}, I(0_{\mathcal{L}},(a,b)) = I((a,b),1_{\mathcal{L}}) = 1_{\mathcal{L}}$.*

As in the fuzzy case, conjunctions and implications may be related to each other based on the residuation principle, which corresponds to a notion of adjunction. This principle is expressed as follows in the bipolar case: a pair of bipolar connectives (I,C) forms an adjunction if, $\forall (a_i,b_i) \in \mathcal{L}, i = 1...3$,

$$C((a_1,b_1),(a_3,b_3)) \preceq (a_2,b_2) \Leftrightarrow (a_3,b_3) \preceq I((a_1,b_1),(a_2,b_2)). \tag{4}$$

These connectives can be linked to each other in different ways (again this is similar to the fuzzy case).

Proposition 2. *The following properties hold:*

- *Given a bipolar t-norm C and a bipolar negation N, the following operator D defines a bipolar t-conorm: $\forall ((a_1,b_1),(a_2,b_2)) \in \mathcal{L}^2$,*

$$D((a_1,b_1),(a_2,b_2)) = N(C(N((a_1,b_1)),N((a_2,b_2)))). \tag{5}$$

- *A bipolar implication I induces a bipolar negation N defined as:*

$$\forall (a,b) \in \mathcal{L}, N((a,b)) = I((a,b), 0_{\mathcal{L}}). \tag{6}$$

- *The following operator I_N, derived from a bipolar negation N and a bipolar conjunction C, defines a bipolar implication:* $\forall ((a_1,b_1),(a_2,b_2)) \in \mathcal{L}^2$,

$$I_N((a_1,b_1),(a_2,b_2)) = N(C((a_1,b_1), N((a_2,b_2)))). \tag{7}$$

- *Conversely, a bipolar conjunction C can be defined from a bipolar negation N and a bipolar implication I:* $\forall ((a_1,b_1),(a_2,b_2)) \in \mathcal{L}^2$,

$$C((a_1,b_1),(a_2,b_2)) = N(I((a_1,b_1), N((a_2,b_2)))). \tag{8}$$

- *Similarly, a bipolar implication can be defined from a negation N and a bipolar disjunction D as:* $\forall ((a_1,b_1),(a_2,b_2)) \in \mathcal{L}^2$,

$$I_N((a_1,b_1),(a_2,b_2)) = D(N((a_1,b_1)), (a_2,b_2)). \tag{9}$$

- *A bipolar implication can also be defined by residuation from a bipolar conjunction C such that* $\forall (a,b) \in \mathcal{L} \backslash 0_{\mathcal{L}}, C(1_{\mathcal{L}}, (a,b)) \neq 0_{\mathcal{L}}$*:* $\forall ((a_1,b_1),(a_2,b_2)) \in \mathcal{L}^2$,

$$I_R((a_1,b_1),(a_2,b_2)) = \bigvee \{(a_3,b_3) \in \mathcal{L} \mid C((a_1,b_1),(a_3,b_3)) \preceq (a_2,b_2)\}.$$

  The operators C and I_R are then said to be adjoint (see the definition in Equation 4).
- *Conversely, from a bipolar implication I_R such that* $\forall (a,b) \in \mathcal{L} \setminus 1_{\mathcal{L}}, I_R(1_{\mathcal{L}}, (a,b)) \neq 1_{\mathcal{L}}$*, the conjunction C such that (C, I_R) forms an adjunction is given by:* $\forall ((a_1,b_1),(a_2,b_2)) \in \mathcal{L}^2$,

$$C((a_1,b_1),(a_2,b_2)) = \bigwedge \{(a_3,b_3) \in \mathcal{L} \mid (a_2,b_2) \preceq I_R((a_1,b_1),(a_3,b_3))\}.$$

Proposition 3. *If C and I are bipolar connectives such that (I, C) forms an adjunction (i.e. verifies Equation 4), then C distributes over the supremum and I over the infimum on the right, i.e.:* $\forall (a_i,b_i) \in \mathcal{L}, \forall (a,b) \in \mathcal{L}$,

$$\bigvee_i C((a,b),(a_i,b_i)) = C((a,b), \bigvee_i (a_i,b_i)), \quad \bigwedge_i I((a,b),(a_i,b_i)) = I((a,b), \bigwedge_i (a_i,b_i)).$$

Note that the distributivity on the left requires C to be commutative.

The following properties of adjunctions will also be useful for deriving mathematical morphology operators.

Proposition 4. *Let (I, C) be an adjunction. Then the following properties hold:*

- *C is increasing in the second argument and I in the second one. If furthermore C is commutative, then it is also increasing in the first one.*
- *$0_{\mathcal{L}}$ is the null element of C on the right and $1_{\mathcal{L}}$ is the null element of I on the right, i.e.* $\forall (a,b) \in \mathcal{L}, C((a,b), 0_{\mathcal{L}}) = 0_{\mathcal{L}},\ I((a,b), 1_{\mathcal{L}}) = 1_{\mathcal{L}}$.

4 Morphological Dilations and Erosions of Bipolar Fuzzy Sets

We now assume that $\mathcal{S}$ is an affine space (or at least a space on which translations can be defined), and we use the notion of structuring element, which defines a spatial neighborhood of each point in $\mathcal{S}$ (or a binary relation between worlds in a logical framework). Here we consider fuzzy bipolar structuring elements. More generally, without any assumption on the underlying domain $\mathcal{S}$, a structuring element is defined as a binary relation between two elements of $\mathcal{S}$ (i.e. y is in relation with x if and only if $y \in B_x$). This allows on the one hand dealing with spatially varying structuring elements (when $\mathcal{S}$ is the spatial domain), or with graph structures, and on the other hand establishing interesting links with several other domains, such as rough sets, formal logics, and, in the more general case where the morphological operations are defined from one set to another one, with Galois connections and formal concept analysis. The general principle underlying morphological erosions consists in translating the structuring element at every position in space and check if this translated structuring element is included in the original set [8]. This principle has also been used in the main extensions of mathematical morphology to fuzzy sets or to logics. Similarly, defining morphological erosions of bipolar fuzzy sets, using bipolar fuzzy structuring elements, requires to define a degree of inclusion between bipolar fuzzy sets. Such inclusion degrees have been proposed in the context of intuitionistic fuzzy sets [24]. With our notations, a degree of inclusion of a bipolar fuzzy set (μ', ν') in another bipolar fuzzy set (μ, ν) is defined as [2]:

$$\bigwedge_{x \in \mathcal{S}} I((\mu'(x), \nu'(x)), (\mu(x), \nu(x))) \tag{10}$$

where I is a bipolar implication, and a degree of intersection is defined as:

$$\bigvee_{x \in \mathcal{S}} C((\mu'(x), \nu'(x)), (\mu(x), \nu(x))) \tag{11}$$

where C is a bipolar conjunction. Note that both inclusion and intersection degrees are elements of $\mathcal{L}$, i.e. they are defined as bipolar degrees.

Based on these concepts, we can now propose a general definition for morphological erosions and dilations, thus extending our previous work in [2,3,5].

Definition 1. *Let (μ_B, ν_B) be a bipolar fuzzy structuring element (in $\mathcal{B}$). The erosion of any (μ, ν) in $\mathcal{B}$ by (μ_B, ν_B) is defined from a bipolar implication I as:*

$$\forall x \in \mathcal{S},\ \varepsilon_{(\mu_B,\nu_B)}((\mu, \nu))(x) = \bigwedge_{y \in \mathcal{S}} I((\mu_B(y-x), \nu_B(y-x)), (\mu(y), \nu(y))). \tag{12}$$

In this equation, $\mu_B(y-x)$ (respectively $\nu_B(y-x)$) represents the value at point y of the translation of μ_B (respectively ν_B) at point x.

Definition 2. *Let (μ_B, ν_B) be a bipolar fuzzy structuring element (in $\mathcal{B}$). The dilation of any (μ, ν) in $\mathcal{B}$ by (μ_B, ν_B) is defined from a bipolar conjunction C as:*

$$\delta_{(\mu_B,\nu_B)}((\mu,\nu))(x) = \bigvee_{y\in\mathcal{S}} C((\mu_B(x-y), \nu_B(x-y)), (\mu(y), \nu(y))). \tag{13}$$

Proposition 5. *Definitions 1 and 2 are consistent: they actually provide bipolar fuzzy sets of $\mathcal{B}$, i.e. $\forall(\mu,\nu) \in \mathcal{B}, \forall(\mu_B,\nu_B) \in \mathcal{B}, \delta_{(\mu_B,\nu_B)}((\mu,\nu)) \in \mathcal{B}$ and $\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu)) \in \mathcal{B}$.*

Proposition 6. *In case the bipolar fuzzy sets are usual fuzzy sets (i.e. $\nu = 1-\mu$ and $\nu_B = 1-\mu_B$), the definitions lead to the usual definitions of fuzzy dilations and erosions. Hence they are also compatible with classical morphology in case μ and μ_B are crisp.*

Proposition 7. *Definitions 1 and 2 provide an adjunction (ε, δ) if and only if (I, C) is an adjunction.*

Proposition 8. *If I and C are bipolar connectives such that (I, C) is an adjunction, then the operator ε defined from I by Equation 12 commutes with the infimum and the operator δ defined from C by Equation 13 commutes with the supremum, i.e. they are algebraic erosion and dilation. Moreover they are increasing with respect to (μ, ν).*

Proposition 9. *If (I, C) is an adjunction such that C is increasing in the first argument and I is decreasing in the first argument (typically if they are a bipolar conjunction and a bipolar implication), then the operator ε defined from I by Equation 12 is decreasing with respect to the bipolar fuzzy structuring element and the operator δ defined from C by Equation 13 is increasing with respect to the bipolar fuzzy structuring element.*

Proposition 10. *C distributes over the supremum and I over the infimum on the right if and only if ε and δ defined by Equations 12 and 13 are algebraic erosion and dilation, respectively.*

Proposition 11. *Let δ and ε be a dilation and an erosion defined by Equations 13 and 12. Then, for all $(\mu_B, \nu_B), (\mu, \nu), (\mu', \nu')$ in $\mathcal{B}$, we have:*

$$\delta_{(\mu_B,\nu_B)}((\mu,\nu) \wedge (\mu',\nu')) \preceq \delta_{(\mu_B,\nu_B)}((\mu,\nu)) \wedge \delta_{(\mu_B,\nu_B)}((\mu',\nu')), \tag{14}$$

$$\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu)) \vee \varepsilon_{(\mu_B,\nu_B)}((\mu',\nu')) \preceq \varepsilon_{(\mu_B,\nu_B)}((\mu,\nu) \vee (\mu',\nu')). \tag{15}$$

Proposition 12. *A dilation δ defined by Equation 13 is increasing with respect to the bipolar fuzzy structuring element, while an erosion ε defined by Equation 12 is decreasing with respect to the bipolar fuzzy structuring element.*

These results fit well with the intuitive meaning behind the morphological operators. Indeed, a dilation is interpreted as a degree of intersection, which is easier to achieve with a larger structuring element, while an erosion is interpreted as a degree of inclusion, which means a stronger constraint if the structuring element is larger.

Proposition 13. *Let δ and ε be a dilation and an erosion defined by Equations 13 and 12. Then, for all $(\mu_B, \nu_B), (\mu'_B, \nu'_B), (\mu, \nu)$ in $\mathcal{B}$, we have:*

$$\delta_{(\mu_B,\nu_B)\wedge(\mu'_B,\nu'_B)}((\mu,\nu) \preceq \delta_{(\mu_B,\nu_B)}((\mu,\nu)) \wedge \delta_{(\mu'_B,\nu'_B)}((\mu,\nu)), \tag{16}$$

$$\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu)) \vee \varepsilon_{(\mu'_B,\nu'_B)}((\mu,\nu)) \preceq \varepsilon_{(\mu_B,\nu_B)\wedge(\mu'_B,\nu'_B)}((\mu,\nu)). \tag{17}$$

Proposition 14. *Let δ be a dilation defined by Equation 13 from a bipolar conjunction C. The dilation satisfies $\delta_{(\mu_B,\nu_B)}((\mu,\nu)) = \delta_{(\mu,\nu)}((\mu_B,\nu_B))$ if and only if C is commutative.*

This result is quite intuitive. When interpreting the dilation as a degree of intersection, it is natural to expect this degree to be symmetrical in both arguments. Hence the commutativity of C has to be satisfied.

Proposition 15. *Let δ be a dilation defined by Equation 13 from a bipolar conjunction C. It satisfies the iterativity property, i.e.:*

$$\delta_{(\mu_B,\nu_B)}(\delta_{(\mu'_B,\nu'_B)}(\mu,\nu)) = \delta_{\delta_{(\mu_B,\nu_B)}((\mu'_B,\nu'_B))}((\mu,\nu)),$$

if and only if C is associative.

Proposition 16. *Let δ be a dilation defined by Equation 13 from a bipolar conjunction C. If C is a bipolar conjunction that admits $1_\mathcal{L}$ as unit element on the left (i.e. $\forall(a,b) \in \mathcal{L}, C(1_\mathcal{L},(a,b)) = (a,b)$) and $C((a,b),1_\mathcal{L}) \neq 1_\mathcal{L}$ for $(a,b) \neq 1_\mathcal{L}$, then the dilation is extensive, i.e. $\delta_{(\mu_B,\nu_B)}((\mu,\nu)) \succeq (\mu,\nu)$, if and only if $(\mu_B,\nu_B)(0) = 1_\mathcal{L}$, where 0 denotes the origin of space $\mathcal{S}$.*

A similar property holds for erosion and if I is a bipolar implication that admits $1_\mathcal{L}$ as unit element to the left (i.e. $\forall(a,b) \in \mathcal{L}, I(1_\mathcal{L},(a,b)) = (a,b)$) and $I((a,b),0_\mathcal{L}) \neq 0_\mathcal{L}$ for $(a,b) \neq 1_\mathcal{L}$, then the erosion is anti-extensive, i.e. $\varepsilon_{(\mu_B,\nu_B)}((\mu,\nu)) \preceq (\mu,\nu)$, if and only if $(\mu_B,\nu_B)(0) = 1_\mathcal{L}$.

The second condition on C holds in particular if $1_\mathcal{L}$ is also unit element on the right. This holds in specific cases in which C is a bipolar t-norm, which are the most interesting ones from a morphological point of view, as shown below.

Note that the condition $(\mu_B,\nu_B)(0) = 1_\mathcal{L}$ (i.e. the origin of space completely belongs to the bipolar fuzzy set, without any indetermination) is equivalent to the conditions on the structuring element found in classical [8] and fuzzy [25] morphology to have extensive dilations and anti-extensive erosions.

Proposition 17. *If I is derived from C and a negation N, then δ and ε are dual operators, i.e.: $\delta_{(\mu_B,\nu_B)}(N(\mu,\nu)) = N(\varepsilon_{(\check{\mu}_B,\check{\nu}_B)}((\mu,\nu)))$, where $(\check{\mu}_B,\check{\nu}_B)$ denotes the symmetrical of (μ_B,ν_B) with respect to the origin of $\mathcal{S}$.*

Duality with respect to complementation, which was advocated in the first developments of mathematical morphology [8], is important to handle in an consistent way an object and its complement for many applications (for instance in image processing and spatial reasoning). Therefore it is useful to know exactly under which conditions this property may hold, so as to choose the appropriate operators if it is needed for a specific problem. On the other hand, adjunction is a major feature of the "modern" view of mathematical morphology, with strong algebraic bases in the framework of complete lattices [9]. This framework is now widely considered as the most interesting one, since it provides consistent definitions with sound properties in different settings (continuous and discrete ones) and extending mathematical morphology to bipolar fuzzy sets in this framework inherits a set of powerful and important properties. Due to the interesting features of these two properties of duality and adjunction, in several applications both are required.

From all these results, we can derive the following theorem, which shows that the proposed forms are the most general ones for C being a bipolar t-norm.

Theorem 1. *Definition 2 defines a dilation with all properties of classical mathematical morphology if and only if C is a bipolar t-norm. The adjoint erosion is then defined by Equation 1 from the residual implication I_R derived from C. If the duality property is additionally required, then C and I have also to be dual operators with respect to a negation N.*

This important result shows that taking any conjunction may not lead to dilations that have nice properties. For instance the iterativity of dilation is of prime importance in concrete applications, and it requires associative conjunctions. This is actually a main contribution of our work, which differs from [6], where some morphological operators are suggested on intuitionistic fuzzy sets and for the Pareto ordering, but without referring to the algebraic framework, and leading to weaker properties (for instance the erosion defined in this work does not commute with the infimum and is then not an algebraic erosion). This group has then proposed some extensions in [7], still for the specific case of Pareto ordering, which closely follow our previous results in [2,3,5]. Moreover the result expressed in Theorem 1 is stronger and more general since it applies for any partial ordering leading to a complete lattice on $\mathcal{B}$. Note that pairs of adjoint operators are not necessarily dual. Therefore requiring both adjunction and duality properties may drastically reduce the choice for C and I. Note that this strong constraint is similar to the one proved for fuzzy sets in [26]. Although the choice of C and I is limited by the results expressed in Theorem 1 if sufficiently strong properties are required for the morphological operators, some choice may remain. The following property expresses a monotony property with respect to this choice.

Proposition 18. *Dilations and erosions are monotonous with respect to the choice of C and I:*

$$C \preceq C' \Rightarrow \delta^C \preceq \delta^{C'}$$

where δ^C *is the dilation defined by Equation 13 using the bipolar conjunction or t-norm* C*, and*

$$I \preceq I' \Rightarrow \varepsilon^I \preceq \varepsilon^{I'}$$

where ε^I *is the erosion defined by Equation 12 using the bipolar implication* I*.*

Examples of connectives and derived morphological operators, along with their properties, can be found for the Pareto ordering and for the lexicographic ordering in our previous work [2,3,5].

5 Conclusion

A general algebraic framework for fuzzy bipolar mathematical morphology was proposed, along with a set of properties. This general formulation is an original contribution, leading to new theoretical results. More properties on the compositions $\delta\varepsilon$ and $\varepsilon\delta$ can also be derived [19]. This framework can now be instantiated for various partial orderings. The case of Pareto ordering and lexicographic ordering have been detailed in [2,3,5], showing different properties, behaviors and interpretations.

From the basic morphological operators, other ones can be derived, as classically done in mathematical morphology, thus endowing the complete toolbox of operations with a bipolarity layer. Some examples of such operators (opening, closing, conditional operators, gradient...), along with geometrical measures and distances on bipolar fuzzy sets have been proposed in [4].

Let us now briefly comment on the applicability of these new tools for image processing and understanding. When dealing with spatial information, both fuzziness and bipolarity occur. Fuzziness may be related to the observed phenomenon itself, to the image acquisition process, to the image processing steps, to the knowledge used for image understanding and recognition, etc. This is now taken into account in a number of works. As for bipolarity, which has not been much addressed until now in this domain, several situations could benefit from its modeling. For instance, when assessing the position of an object in space, we may have positive information expressed as a set of possible places, and negative information expressed as a set of impossible or forbidden places (for instance because they are occupied by other objects). As another example, let us consider spatial relations. Human beings consider "left" and "right" as opposite relations. But this does not mean that one of them is the negation of the other one. The semantics of "opposite" captures a notion of symmetry (with respect to some axis or plane) rather than a strict complementation. In particular, there may be positions which are considered neither to the right nor to the left of some reference object, thus leaving room for some indetermination, neutrality or indifference.

As an illustrative example, a typical scenario showing the interest of bipolar representations of spatial relations and of morphological operations on these representations for spatial reasoning has been described in [3,4], for recognizing brain structures in medical images. The recognition was guided by anatomical knowledge, expressing its bipolarity. For instance, the putamen is exterior (i.e. to the right in the right hemisphere and to the left in the left one) of the union of lateral ventricles and thalamus (positive information) and cannot be interior (negative information); the putamen is quite close to the union of lateral ventricles and thalamus (positive information) and cannot be very far (negative information). Merging this information allows reducing the search area for the putamen, by dilating reference objects (lateral ventricles and thalamus in this example) with bipolar fuzzy sets representing these spatial constraints, thus focusing on the only regions of space where the spatial relations are satisfied, while avoiding forbidden regions.

Developing these preliminary examples, future work aims at applying this framework in the domain of spatial reasoning, in particular for knowledge-based object recognition in images. Another line of research is its application in the domain of preference modeling, for fusion, mediation and argumentation.

References

1. Dubois, D., Kaci, S., Prade, H.: Bipolarity in Reasoning and Decision, an Introduction. In: International Conference on Information Processing and Management of Uncertainty, IPMU 2004, Perugia, Italy, pp. 959–966 (2004)
2. Bloch, I.: Dilation and erosion of spatial bipolar fuzzy sets. In: Masulli, F., Mitra, S., Pasi, G. (eds.) WILF 2007. LNCS (LNAI), vol. 4578, pp. 385–393. Springer, Heidelberg (2007)
3. Bloch, I.: Bipolar fuzzy mathematical morphology for spatial reasoning. In: Wilkinson, M.H.F., Roerdink, J.B.T.M. (eds.) ISMM 2009. LNCS, vol. 5720, pp. 24–34. Springer, Heidelberg (2009)
4. Bloch, I.: Bipolar Fuzzy Spatial Information: Geometry, Morphology, Spatial Reasoning. In: Jeansoulin, R., Papini, O., Prade, H., Schockaert, S. (eds.) Methods for Handling Imperfect Spatial Information, pp. 75–102. Springer, Heidelberg (2010)
5. Bloch, I.: Lattices of fuzzy sets and bipolar fuzzy sets, and mathematical morphology. Information Sciences 181, 2002–2015 (2011)
6. Nachtegael, M., Sussner, P., Mélange, T., Kerre, E.: Some Aspects of Interval-Valued and Intuitionistic Fuzzy Mathematical Morphology. In: IPCV 2008 (2008)
7. Mélange, T., Nachtegael, M., Sussner, P., Kerre, E.: Basic Properties of the Interval-Valued Fuzzy Morphological Operators. In: IEEE World Congress on Computational Intelligence, WCCI 2010, Barcelona, Spain, pp. 822–829 (2010)
8. Serra, J.: Image Analysis and Mathematical Morphology. Academic Press, London (1982)
9. Ronse, C.: Why Mathematical Morphology Needs Complete Lattices. Signal Processing 21(2), 129–154 (1990)
10. Keshet, R.: Mathematical Morphology on Complete Semilattices and its Applications to Image Processing. Fundamenta Informaticae 41, 33–56 (2000)
11. Heijmans, H.J.A.M., Ronse, C.: The Algebraic Basis of Mathematical Morphology – Part I: Dilations and Erosions. Computer Vision, Graphics and Image Processing 50, 245–295 (1990)
12. Heijmans, H.: Morphological Image Operators. Academic Press, Boston (1994)
13. Grabisch, M., Greco, S., Pirlot, M.: Bipolar and bivariate models in multicriteria decision analysis: Descriptive and constructive approaches. International Journal of Intelligent Systems 23(9), 930–969 (2008)

14. Öztürk, M., Tsoukias, A.: Bipolar preference modeling and aggregation in decision support. International Journal of Intelligent Systems 23(9), 970–984 (2008)
15. Konieczny, S., Marquis, P., Besnard, P.: Bipolarity in bilattice logics. International Journal of Intelligent Systems 23(10), 1046–1061 (2008)
16. Dubois, D., Prade, H.: An Overview of the Asymmetric Bipolar Representation of Positive and Negative Information in Possibility Theory. Fuzzy Sets and Systems 160, 1355–1366 (2009)
17. Aptoula, E., Lefèvre, S.: A Comparative Study in Multivariate Mathematical Morphology. Pattern Recognition 40, 2914–2929 (2007)
18. Bouyssou, D., Dubois, D., Pirlot, M., Prade, H.: Concepts and Methods of Decision-Making. In: ISTE. Wiley, Chichester (2009)
19. Bloch, I.: Mathematical morphology on bipolar fuzzy sets: general algebraic framework. Technical Report 2010D024, Télécom ParisTech (November 2010)
20. Atanassov, K.T.: Intuitionistic Fuzzy Sets. Fuzzy Sets and Systems 20, 87–96 (1986)
21. Zadeh, L.A.: The Concept of a Linguistic Variable and its Application to Approximate Reasoning. Information Sciences 8, 199–249 (1975)
22. Neumaier, A.: Clouds, fuzzy sets, and probability intervals. Reliable Computing 10(4), 249–272 (2004)
23. Dubois, D., Gottwald, S., Hajek, P., Kacprzyk, J., Prade, H.: Terminology Difficulties in Fuzzy Set Theory – The Case of "Intuitionistic Fuzzy Sets". Fuzzy Sets and Systems 156, 485–491 (2005)
24. Deschrijver, G., Cornelis, C., Kerre, E.: On the Representation of Intuitionistic Fuzzy t-Norms and t-Conorms. IEEE Transactions on Fuzzy Systems 12(1), 45–61 (2004)
25. Bloch, I., Maître, H.: Fuzzy Mathematical Morphologies: A Comparative Study. Pattern Recognition 28(9), 1341–1387 (1995)
26. Bloch, I.: Duality vs. Adjunction for Fuzzy Mathematical Morphology and General Form of Fuzzy Erosions and Dilations. Fuzzy Sets and Systems 160, 1858–1867 (2009)