Vector Optimization

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Recent Developments in Vector Optimization

Springer

Vector Optimization

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Vector Optimization

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Editors

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This volume is dedicated to Prof. Franco Giannessi for his contribution to the theory of vector variational inequalities and optimization. We would also like to dedicate this volume to our families.

Preface

We always come cross several decision-making problems in our daily life. Such problems are always conflicting in which many different view points should be satisfied. In politics, business, industrial systems, management science, networks, etc. one often encounters such kind of problems. The most important and difficult part in such problems is the conflict between various objectives and goals. In these problems, one has to find the minimum (or maximum) for several objective functions. Such problems are called vector optimization problems (VOP), multi-criteria optimization problems or multi-objective optimization problems.

This volume deals with several different topics/aspects of vector optimization theory ranging from the very beginning to the most recent one. It contains fourteen chapters written by different experts in the field of vector optimization.

Chapter 1 deals with the solution concepts in vector optimization and set optimization, existence results and applications. The concepts of a pre-order and a partial order of a set which naturally induce the notion of minimal and maximal elements for vector and set optimization problems are presented. Some different optimality notions such as minimal, weakly minimal, strongly minimal and properly minimal elements in a pre-ordered linear space are presented and the relations among these notions are discussed. Two different approaches – the vector approach and the set approach – for defining optimal solutions of set optimization are presented. Several optimality concepts for set optimization as well as order relations for sets are discussed. The relevance of vector optimization in practice is illustrated with an application to a bicriterial problem in field design of a magnetic resonance system.

Chapter 2 is devoted to the existence of Lagrange multipliers, (strong) duality results, linear scalarizations of various classes of solutions to vector optimization problems by using theorems of the alternative. The chapter starts by recalling the Fan–Glicksberg–Hoffman alternative theorem (1957) for convex functions. Then, many equivalent formulations to a general Gordan-type alternative theorem valid for (not necessarily pointed) convex cones with possibly empty interior, are established. They are expressed in terms of quasi relative interior. Several classes of generalized convexity for sets and for vector-valued mappings, are revisited. Applications to linear characterizations of weakly efficient, (Benson) proper efficient solutions, and to characterize the Fritz-John type optimality condition in vector optimization, are discussed. Finally, some recent developments about proper efficiency are presented.

In Chap. 3, the generalized notions of infimum and supremum, called infimal and supremal sets, are studied. It is shown that a set-valued approach to vector optimization yields a complete lattice such that the infimum and supremum can be expressed as certain infimal and supremal sets, respectively. This approach is applied to establish duality results based on a consequent usage of infimum and supremum. Exemplary, Lagrange and conjugate duality results, being completely analogous to their scalar counterparts, are proven.

In Chap. 4, the vector optimization problems with a variable ordering structure are studied. A cone-valued map which associates to each element of the space a cone of dominated (or preferred) directions defines thereby the variable ordering structure. The importance of such vector optimization problems is illustrated with several applications for instance in medical image registration. The different optimality notions which are known so far in the literature are given and properties and relations among them are discussed. The linear and nonlinear scalarization functionals which allow a characterization of optimal elements are presented and a basic duality theory is provided. Special attention to variable ordering structures where the images of the cone-valued map are Bishop-Phelps cones is paid.

In Chap. 5, some interesting results on optimality conditions for vector optimization problems are presented. The VOP both in finite and infinite dimensions is studied. For example, the study of strong KKT conditions is emphasized and in the infinite dimensional setting the optimality conditions when the ordering cone has an empty interior, is focused. A scalar-valued gap function and its regularization is also studied and an error bound is presented for a strongly convex vector optimization problem. Some results on second order sufficient optimality conditions for vector optimization problems are presented.

Chapter 6 deals with the optimality conditions, which are expressed by means of vector variational inequalities where the operator is the Gâteaux derivative of the objective function. The image space analysis for VOP with cone constraints of the form $g(x) \in D$, where D is a closed and convex cone is presented. In particular, following a vector separation scheme in the image space, generalized vector Lagrangian functions associated with VOP are introduced, and scalar and vector saddle points conditions are derived. The Kuhn–Tucker type first order optimality conditions for Gâteaux differentiable VOP with cone constraints and for semidifferentiable VOP (in the sense of Giannessi) with inequality constraints are discussed.

In Chap. 7, some elementary concepts from nonlinear analysis, convex analysis and invex analysis are reviewed. The directional derivatives, Gâteaux derivative, Dini (lower and upper) directional derivative, Dini-Hadamard (lower and upper) directional derivative, Clarke directional derivative and their properties are presented. By treating these directional derivatives as a bifunction, different kinds of invexities and invariant monotonicities for such a bifunction are introduced. Several properties of these invexities and invariant monotonicities are discussed. The vector variational-like inequalities for bifunctions are presented in such a way that if we treat the Dini upper directional derivative of a function as a bifunction, then we get, so called, the nonsmooth vector variational-like inequalities involving Dini upper directional derivative. Some existence results for these kinds of vector variational-like inequalities are presented. The vector optimization problem by using our vector variational-like inequalities is studied. Several relationships among the weakly efficient solution and efficient solution of the VOP, and the solutions of our vector variational-like inequalities are given.

Chapter 8 is concerned with a convex vector optimization problem which is to minimize a vector objective function consisting of a finite number of real-valued convex functions with infinitely many convex constraint functions, and an abstract constraint set (called convex semi-infinite vector problem). For such a problem, two kinds of approximate efficient solutions, namely, ε-efficient solutions and weakly ε-efficient solutions are considered. Optimality conditions for these kinds of approximate solutions are established in terms of Fenchel conjugates and ε-subdifferentials of the data involved. Optimality conditions for efficient and weakly efficient solutions of such a problem are derived as a special case of the general results. These results are established by using a new version of Farkas lemma for systems of infinitely many convex inequalities and under new regularity conditions which give rise to new results even for problems with finite constraints. Several numerical examples are given to illustrate the meaning of the results.

Chapter 9 surveys some existing results on solution stability and connectedness of the solution sets of linear fractional vector optimization problems and of convex quadratic vector optimization problems. The main concern is the situation where the constraint set is unbounded. Some open problems are also mentioned.

In Chap. 10, various types of Levitin–Polyak (LP, in short) well-posedness for scalar and vector optimization problems with functional constraints are introduced. Various criteria and characterizations for these types of LP well-posedness are derived. Relationships among these types of LP well-posedness are presented. Applications of some types of LP well-posedness to the convergence analysis of augmented Lagrangian methods and penalty methods for constrained scalar or vector optimization problems are also given.

In Chap. 11, new minimal point theorems in product spaces and the corresponding vector variational principles for set-valued functions are presented. As special cases, many of the existing variational principles of Ekeland's type are derived. Moreover, a new approach is used to get extensions of Ekeland's variational principles (EVPs) of Isac-Tammer's and Ha's types, as well as extensions of EVPs for bi-functions. An important tool for deriving variational principles is a general nonlinear scalarization technique. Some useful properties of scalarizing functionals is studied. Several applications, especially necessary conditions for solutions of VOP are presented.

In Chap. 12, several versions of the Fermat rule and the Lagrange multiplier rule for various efficient solutions of set-valued optimization problems are presented. These rules are expressed in terms of coderivatives in the senses of Fréchet, Ioffe, Clarke and Mordukhovich and provide illustrating examples. The fuzzy and exact versions of the Fermat rule and the Lagrange multiplier rule containing necessary conditions for Pareto efficient solutions are given. The Fermat rule and the Lagrange multiplier rule containing both necessary and sufficient conditions (the sufficient conditions require additional convex assumptions) for strongly efficient solutions, weakly efficient solutions, positive properly efficient solutions, Hurwicz properly efficient solutions, Henig global properly efficient solutions, Henig properly efficient solutions, super efficient solutions and Benson properly efficient solutions are obtained in an unified scheme. It is based on the fact that these solutions can be characterized as solutions to scalar optimization problems with the objective functions being linear functionals or the Hiriart-Urruty signed distance function or a Minkowski-type function.

Chapter 13 contains new developments on necessary conditions for minimal points of sets and their applications to deriving refined necessary optimality conditions in general models of set-valued optimization with geometric, functional, and operator constraints in finite and infinite dimensions. The results obtained address the new notions of extended Pareto optimality with preference relations generated by ordering sets satisfying the local asymptotic closedness property instead of that generated by convex and closed cones. In this way, most of the known notions of efficiency/optimality in multiobjective models are unified and extended. Some optimality conditions that are new even in standard settings are obtained. The approach is based on advanced tools of variational analysis and generalized differentiation.

The last chapter is devoted the theory of cooperative games which is a very important and interesting topic in applied mathematics. An ordinary cooperative game is specified by a real-valued characteristic function defined on the set of coalitions of a finite number of players. These games can be extended to cooperative games with vector-valued, both in the finite-dimensional real space and in a more general partially ordered linear space, characteristic functions and also set-valued characteristic maps. Some solution concepts such as the core and the Shapley value are investigated. On the other hand, since several interesting and useful games are derived from optimization problems, there are some results concerning games derived from vector optimization problems such as linear production programming problems and minimum cost spanning problems. Thus, a review on some relationships between vector optimization and cooperative games is provided.

We would like to express our profound thanks and gratitude to Prof. Johannes Jahn, who encouraged us to edit this kind of volume for the Springer series on Vector Optimization. This volume could not be completed without the support of all the esteemed authors who have contributed to this volume. We are profoundly thankful to them for their painstaking efforts. We would like to take this opportunity to thank our friend Prof. S. Al-Homidan, Dhahran, Saudi Arabia, for his kind encouragement at different times during the preparation of this volume. Also, we would like to express our appreciation to Springer for publishing this volume in the series on Vector Optimization. We hope that this volume would be useful to the students, researchers, and those who are interested in this emerging field of Applied

Mathematics. Last, but not the least, we wish to thank our family for their infinite patience, encouragement and forbearance. This work would not have seen the light of day without their unflinching support during this project. We remain indebted to them.

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Acronyms

AVI(M,q,K)	Affine variational inequality defined by a matrix M, vector q and a polyhedral convex set K
AVVI(ω,K)	Affine vector variational inequality defined by a data set $\omega = (M_1,\ldots,M_m,q_1,\ldots,q_m)$ and a polyhedral convex set K
BP cone	Bishop-Phelps cone
(CP)	Constrained set-valued optimization problem
IMRT	Intensity modulated radiation therapy
LAC	Local asymptotic closedness
LFVOP	Linear fractional vector optimization problem
LP	Levitin-Polyak
MR	Magnetic resonance
Minty VVI or MVVI	Minty vector variational inequality
Minty VVIP or MVVIP	Minty vector variational inequality problem
Minty VVLI or MVVLI	Minty vector variational-like inequality
Minty VVIP or MVVIP	Minty vector variational-like inequality problem
NDC	Net demand qualification
NMVVLIP	Nonsmooth Minty vector variational-like inequality problem
NSVVLIP	Nonsmooth Stampacchia vector variational-like inequality problem
NWMVVLIP	Nonsmooth weak Minty vector variational-like inequality problem
NWSVVLIP	Nonsmooth weak Stampacchia vector variational-like inequality problem
PSNC	Partially sequential normal compact
PSVVLIP	Perturbed Stampacchia vector variational-like inequality problem
RF	Radio frequency
SAP	Specific absorption rate

SNC	Sequential normal compact
SOP	Set optimization problem
(SP)	Set-valued optimization problem
SVVI or (SVVI)	Stampacchia vector variational inequality
SVVIP or (SVVIP)	Stampacchia vector variational inequality problem
SVVLI or (SVVLI)	Stampacchia vector variational-like inequality
SVVLIP or (SVVLIP)	Stampacchia vector variational-like inequality problem
VI or (VI)	Variational inequality
VIP or (VIP)	Variational inequality problem
VOP or (VP)	Vector optimization problem
VVI or (VVI)	Vector variational inequality
VVIP or (VVIP)	Vector variational inequality problem
VVLI or (VVLI)	Vector variational-like inequality
VVLIP or (VVLIP)	Vector variational-like inequality problem
WSVVLIP	Weak Stampacchia vector variational-like inequality problem

List of Abbreviations

e.g.	For example
Fig.	Figure
i.e.	That is
u.H.c.	Upper Hausdorff semicontinuous
u.s.c.	Upper semicontinuous
v.m.p.	Vector minimum point
w. r. t.	With respect to
weak v.m.p	Weak vector minimum point

List of Notations and Symbols

$\langle \cdot, \cdot \rangle$	The duality pairing between Y and its topological dual Y^*
$\leq_K$	The partial ordering induced by the cone K
$\leq_1, \leq_2$	The relations defined by the cone-valued map $\mathscr{D}$
$\preccurlyeq_c$	The certainly less order relation
$\preccurlyeq_e$	The equitability relation
$\preccurlyeq_m$	The minmax less order relation
$\preccurlyeq_{mc}$	The minmax certainly less order relation
$\preccurlyeq_{mn}$	The minmax certainly nondominated order relation
$\preccurlyeq_p$	The possibly less order relation
$\preccurlyeq_s$	The set less or KNY order relation
$\top$	The transpose
$\\| \cdot \\|_*$	The norm in the dual space Y^*
$\\| \cdot \\|_p$	The l_p norm
δ_A	The indicator function of the set A
f^*	The conjugate of f
$\bigtriangledown f$	The gradient of f
$f'_+(\bar{x}; v)$	The right-sided directional derivative of f at $\bar{x} \in X$ in the direction v
$f'_-(\bar{x}; v)$	The left-sided directional derivative of f at $\bar{x} \in X$ in the direction v
$f'(\bar{x}; v)$	The directional derivative of f at $\bar{x} \in X$ in the direction v
$f^G(\bar{x}; v)$	The Gâteaux derivative of f at $\bar{x} \in X$ in the direction v
$f^D(\bar{x}; v)$	The Dini upper derivative of f at $\bar{x} \in X$ in the direction v
$f^D_-(\bar{x}; v)$	The Dini lower derivative of f at $\bar{x} \in X$ in the direction v

$f^C(\overline{x};v)$ or $f^0(\overline{x};v)$	The Clarke generalized derivative of f at $\overline{x}\in X$ in the direction v
$f^{DH}(\overline{x};\overline{v})$	The Dini-Hadamard upper directional derivative f at $\overline{x}\in X$ in the direction v
$f_{DH}(\overline{x};\overline{v})$	The Dini-Hadamard lower directional derivative f at $\overline{x}\in X$ in the direction v
$\underline{\mathscr{D}}_G f\left(\overline{x};\frac{x-\overline{x}}{\lVert x-\overline{x}\rVert}\right)$	The lower G-semiderivative of f at $\overline{x}$
$\overline{\mathscr{D}}_G f\left(\overline{x};\frac{x-\overline{x}}{\lVert x-\overline{x}\rVert}\right)$	The upper G-semiderivative of f at $\overline{x}$
$\partial f(\overline{x})$	The subdifferential of f at $\overline{x}\in X$
$\underline{\partial}_G f(\overline{x})=\partial\underline{\mathscr{D}}_G f(\overline{x};0)$	The generalized subdifferential of f at $\overline{x}$
$\overline{\mathscr{D}}_G f(\overline{x};0)$	The G-subdifferential of f at $\overline{x}$
$\overline{B}(u,\delta)$	Closed ball with center at u and radius δ
$\chi(X)$	The number of the connected components of X
$\Delta T(v)$	The dividend
∂	The subdifferential
∂_A	The subdifferential in the sense of Ioffe
∂_C	The subdifferential in the sense of Clarke
∂_F	The subdifferential in the sense of Fréchet
∂_M	The subdifferential in the sense of Mordukhovich
$\xrightarrow{w^*}$	weak* convergence
$x\xrightarrow{\Omega}\overline{x}$	$x\to\overline{x}$ and $x\in\Omega$
$x\xrightarrow{\varphi}\overline{x}$	$x\to\overline{x}$ and $\varphi(x)\to\varphi(\overline{x})$
$\mathrm{cl}\,F: X\rightrightarrows Z$	$\mathrm{gph}\,(\mathrm{cl}\,F)=\mathrm{cl}\,(\mathrm{gph}\,F)$
$\sigma_A(\cdot)=\sigma(\cdot\,\vert A)$	the support function of A
$\varphi_A(\cdot)=\varphi(\cdot\,\vert A)$	the scalarization funtional based on support function of A
$\mathrm{argmin}_K\, f$	The set of minima of f on K
A_y	The section of A
bd$A$ or∂A	The boundary of A
$\mathbb{C}$	The set of complex numbers
C^*	The dual cone of C
C°	The polar cone of C
cl_+A	The upper closure of A
cl$A$ or cl(A) or $\overline{A}$	The closure of A
$C(l,\gamma)$	The Bishop-Phelps cone with parameters l and γ
$\overline{\mathrm{co}}(A)$	The closed convex hull of A
cone$A$ or cone(A)	The conic hull of A
conv$A$ or co(A)	The convex hull of A
core A or cor$A$ or ri A	The algebraic interior of A
$\mathscr{D}(A)$	The image of the set A under $\mathscr{D}$
$d_A(x)$	The distance function from point x to set A

$D^*F(x,y)$	The coderivative of F at (x,y)
$D^*_A F(x,y)$	The coderivative of F at (x,y) in the sense of Ioffe
$D^*_C F(x,y)$	The coderivative of F at (x,y) in the sense of Clarke
$D^*_F F(x,y)$	The coderivative of F at (x,y) in the sense of Fréchet
$D^*_M F(x,y)$	The coderivative of F at (x,y) in the sense of Mordukhovich or the mixed coderivative of F at $(x,y) \in \text{gph } F$
$D^*_N F(x,y)$	The normal coderivative of F at $(x,y) \in \text{gph } F$
$DC(v)$	The dominance core
$\text{dom} f$	The domain of f
d_S	The distance map w. r. t. the set S
$\mathscr{E}_F$	The epigraphical multifunction of F
$\mathscr{E}_\Omega$	The epigraphical set of Ω
$e(S;x)$	The excess
epi F	The (generalized) epigraph of F
$\mathscr{F}_C(Y)$	The space of upper closed (w.r.t. C) subsets of Y
$F(S)$	The image set of S under F
$\mathscr{F}(X)$	The family of all closed subsets of a topological space X
$frtA$	The frontier of A
G^N	The set of cooperative games
gph F	The graph of F
haus(A,B) or $\mathscr{H}(A,B)$	The Hausdorff distance between sets A and B
icr A	The intrinsic core of A
$\mathscr{I}_C(Y)$	The space of self-infimal (w.r.t. C) subsets of $\overline{Y}$
$\inf A$	The infimum of A (in the sense of a complete lattice)
Inf A	The infimal set of A
int A or int$A$ or int(A) or $\overset{\circ}{A}$	The (topological) interior of A
$I(v), I(V)$	The imputation sets
$I^*(v), I^*(V)$	The preimputation sets
$K^\#$	The quasi-interior of the dual cone of the cone K
$K^\#_{Y^*}$	The quasi-interior of the topological dual cone
K^*	The topological dual cone of the cone K
$L_\infty(\Omega)$	Linear space of all (equivalent classes of) essentially bounded functions $f\colon \Omega \to \mathbb{R}$ with $\Omega \subset \mathbb{R}^n$, $\Omega \neq \emptyset$
$L_c(\cdot,\cdot)$	The Lagrangian
Lim sup	The sequential Painlevé-Kuratowski upper/outer limit
$\mathscr{L}(X,Y)$	The space of continuous linear maps from X to Y
m^π	The marginal contribution vector

$\max A$	The set of maximal elements of A
$\min A$	The set of minimal elements of A
$\mathbb{N}$	The set of natural numbers
$N_A(x)$	The normal cone to A at x
$N(x;\Omega)$	The normal cone to Ω at x
$N_A(x;\Omega)$	The normal cone to Ω at x in the sense of Ioffe
$N_F(x;\Omega)$ or $\widehat{N}(x;\Omega)$	The normal cone to Ω at x in the sense of Fréchet
$N_C(x;\Omega)$	The normal cone to Ω at x in the sense of Clarke
$N_M(x;\Omega)$	The normal cone to Ω at x in the sense of Mordukhovich
$(N;V)$	The multiobjective cooperative game
$\Pi(N)$	The set of permutations on N
$\mathscr{P}(Y)$ or 2^Y	The power set of Y
$\mathbb{Q}$	The set of rational numbers
$\operatorname{qi} A$	The quasi interior of A
$\operatorname{qri} A$	The quasi relative interior of A
$\mathbb{R}$	The set of real numbers
$\mathbb{R}^k$	The finite dimensional space of real vectors of dimension k
$\mathbb{R}^k_+$	The nonnegative orthant in $\mathbb{R}^k$
$\mathbb{R}^k_{++}$	The interior of $\mathbb{R}^k_+$
$\mathscr{S}^n_+$	The cone of real symmetric positive semidefinite matrices
sG^N	The set of set-valued cooperative games
$\operatorname{sgn}(\cdot)$	The signum function
$\operatorname{sqri} A$	The strong quasi relative interior of A
$\sup A$	The supremum of A (in the sense of a complete lattice)
$\operatorname{Sup} A$	The supremal set of A
$T(A;\bar{x})$	The contingent cone of A at $\bar{x} \in A$
vG^N	The set of vector-valued cooperative games
$\operatorname{wMax} A$	The set of weakly maximal elements of A
$\operatorname{wMin} A$	The set of weakly minimal elements of A
Y^*	The topological dual space of Y
Z_{co}	The elements of Z (conlinear space) satisfying second distributive law

Chapter 1
Vector Optimization Problems and Their Solution Concepts

Gabriele Eichfelder and Johannes Jahn

1.1 Introduction

In vector optimization one investigates optimal elements of a set in a pre-ordered space. The problem of determining these optimal elements, if they exist at all, is called a vector optimization problem. Problems of this type can be found not only in mathematics but also in engineering and economics. There, these problems are also called multiobjective (or multi criteria or Pareto) optimization problems or one speaks of multi criteria decision making. Vector optimization problems arise, for example, in functional analysis (the Hahn–Banach theorem, the lemma of Bishop–Phelps, Ekeland's variational principle), multiobjective programming, multi-criteria decision making, statistics (Bayes solutions, theory of tests, minimal covariance matrices), approximation theory (location theory, simultaneous approximation, solution of boundary value problems) and cooperative game theory (cooperative n player differential games and, as a special case, optimal control problems). In the last decades vector optimization has been extended to problems with set-valued maps. This field, called set optimization, has important applications to variational inequalities and optimization problems with multivalued data.

In the applied sciences Edgeworth [14] (1881) and Pareto [28] (1906) were probably the first who introduced an optimality concept for vector optimization problems. Both have given the standard optimality notion in multiobjective optimization. Therefore, optimal points are called Edgeworth–Pareto optimal points in the modern special literature.

We give a brief historical sketch of the early works of Edgeworth and Pareto.

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Q.H. Ansari and J.-C. Yao (eds.), *Recent Developments in Vector Optimization*, Vector Optimization 1, DOI 10.1007/978-3-642-21114-0_1,

Edgeworth introduces notions in his book [14] on page 20: “Let P, the utility of X, one party, $= F(xy)$, and Π, the utility of Y, the other party, $= \Phi(xy)$.” Then he writes on page 21: “It is required to find a point (xy) such that, *in whatever direction* we take an infinitely small step, P and Π do not increase together, but that, while one increases, the other decreases.” Hence, Edgeworth presents the definition of a minimal solution, compare Definition 1.8, for the special case of $Y = \mathbb{R}^2$ partially ordered by the natural ordering, i.e. for two objectives $f_1 : S \to \mathbb{R}$ and $f_2 : S \to \mathbb{R}$ and with $K = \mathbb{R}^2_+$.

In the English translation of Pareto’s book [28] one finds on page 261: “We will say that the members of a collectivity enjoy *maximum ophelimity* in a certain position when it is impossible to find a way of moving from that position very slightly in such a manner that the ophelimity enjoyed by each of the individuals of that collectivity increases or decreases. That is to say, any small displacement in departing from that position necessarily has the effect of increasing the ophelimity which certain individuals enjoy, and decreasing that which others enjoy, of being agreeable to some and disagreeable to others.” The concept of “ophelimity” used by Pareto, is explained on page 111: “In our Cours we proposed to designate economic *utility* by the word *ophelimity*, which some other authors have since adopted,” and it is written on page 112: “For an individual, the ophelimity of a certain quantity of a thing, added to another known quantity (it can be equal to zero) which he already possesses, is the pleasure which this quantity affords him.” In our modern terms “ophelimity” can be identified with an objective function and so, the definition of a minimal solution given in Definition 1.8 actually describes what Pareto explained.

These citations show that the works of Edgeworth and Pareto concerning vector optimization are very close together and, therefore, it makes sense to speak of *Edgeworth–Pareto optimality* as proposed by Stadler [34]. It is historically not correct that optimal points are called Pareto optimal points as it is done in various papers.

In mathematics this branch of optimization has started with a paper of Kuhn and Tucker [24]. Since about the end of the 1960s research is intensively made in vector optimization.

In the following sections we first recall the concepts of a pre-order and a partial order of a set which naturally induce the notion of minimal and maximal elements for vector and set optimization problems. In the third section, we present different optimality notions such as minimal, weakly minimal, strongly minimal and properly minimal elements in a pre-ordered linear space and discuss the relations among these notions. In Sect. 1.4, we discuss set optimization problems and different approaches for defining optimal elements in set optimization. Conditions guaranteeing the existence of minimal, weakly minimal and properly minimal elements in linear spaces are given in the fifth section. The chapter is concluded with an engineering application in magnetic resonance systems.

1.2 Pre-Orders and Partial Orders

Minimizing a scalar valued function $f\colon X \to \mathbb{R}$ on some set X, two objective function values are compared by saying $f(x)$ is better than $f(y)$ if $f(x) \leq f(y)$. In vector optimization problems, i.e. in optimization problems with a vector valued objective function, and even more general in set optimization problems, i.e. in optimization problems with a set-valued objective function, we need relations for comparing several vectors or even sets. For that, let us recall some concepts from the theory of ordered sets [31].

Definition 1.1. Let $\mathscr{Q}$ be an arbitrary nonempty set with a binary relation $\leq$. Let $A,\ B,\ D \in \mathscr{Q}$ be arbitrarily chosen. The binary relation $\leq$ is said to be

- *Reflexive* if $A \leq A$
- *Transitive* if $A \leq B$ and $B \leq D$ imply $A \leq D$
- *Symmetric* if $A \leq B$ implies $B \leq A$
- *Antisymmetric* if $A \leq B$ and $B \leq A$ imply $A = B$

Definition 1.2. The binary relation $\leq$ is said to be

- A *pre-order* if it is reflexive and transitive.
- A *partial order* if it is reflexive, transitive and antisymmetric or in other words, if it is a pre-order that is antisymmetric.
- An *equivalence relation* if it is reflexive, transitive and symmetric.

When the relation $\leq$ is a pre-order/a partial order, we say that $\mathscr{Q}$ is a pre-ordered/partially ordered set.

It is important to note that in a pre-ordered set two arbitrary elements cannot be compared, in general, in terms of the pre-order.

Throughout this section let Y be an arbitrary real linear space. For $\mathscr{Q} = Y$ we say that the pre-order is compatible with the linear structure of the space if it is compatible with addition, i.e. for $x,y,w,z \in Y$ and $x \leq y,\ w \leq z$ we obtain $x+w \leq y+z$, and compatible with multiplication with a nonnegative real number, i.e. for $x,y \in Y$, $\alpha \in \mathbb{R}_+$ and $x \leq y$ we obtain $\alpha x \leq \alpha y$. For introducing this definition in a more general setting we need the power set of Y,

$$\mathscr{P}(Y) := \{A \subseteq Y \mid A \text{ is nonempty}\}.$$

Notice that the power set $\mathscr{P}(Y)$ of Y is a conlinear space introduced by Hamel [16]. In a conlinear space addition and multiplication with a nonnegative real number are defined but in contrast to the properties of a linear space the second distributive law is not required.

Definition 1.3. Suppose that $\mathscr{Q}$ is a subset of the power set $\mathscr{P}(Y)$. We say that the binary relation $\leq$ is

- *Compatible with the addition* if $A \leq B$ and $D \leq E$ imply $A+D \leq B+E$ for all $A,B,D,E \in \mathscr{Q}$.

- *Compatible with the multiplication* with a nonnegative real number if $A \le B$ implies $\lambda A \le \lambda B$ for all scalars $\lambda \ge 0$ and all $A, B \in \mathscr{Q}$.
- *Compatible with the conlinear structure* of $\mathscr{P}(Y)$ if it is compatible with both the addition and the multiplication with a nonnegative real number.

By setting $Q := \{\{y\} \mid y \in Y\} \subset \mathscr{P}(Y)$, Definition 1.3 includes as a special case the compatibility of a pre-order with the linear structure of the space Y as discussed above.

The connection between a pre-order (a partial order) in a linear space and a (pointed) convex cone is given in the following theorem.

Theorem 1.1. *Let Y be a real linear space.*

(a) If $\le$ is a pre-order which is compatible with the addition and the multiplication with a nonnegative real number, then the set

$$K := \{y \in Y \mid 0_Y \le y\}$$

is a convex cone. If, in addition, $\le$ is antisymmetric, i.e $\le$ is a partial order, then K is pointed, i.e. $K \cap (-K) = \{0_Y\}$.

(b) If K is a convex cone, then the binary relation

$$\le_K := \{(x,y) \in Y \times Y \mid y - x \in K\}$$

is a pre-order on Y which is compatible with the addition and the multiplication with a nonnegative real number. If, in addition, K is pointed, then $\le_K$ is a partial order.

If the convex cone K introduces some pre-order we speak of an *ordering cone*.

Let us consider some examples illustrating the above concepts.

Example 1.1. (a) Let Y be the linear space of all $n \times n$ real symmetric matrices. Then the pointed convex cone $\mathscr{S}^n_+$ of all positive semidefinite matrices introduces a partial order on Y.

(b) Let $K \subset Y$ be an ordering cone. For $\mathscr{Q} = \mathscr{P}(Y)$ we define a binary relation by the following: Let $A, B \in \mathscr{P}(Y)$ be arbitrarily chosen sets. Then

$$A \preccurlyeq_s B \quad :\Longleftrightarrow \quad \big(\forall\, a \in A\ \exists\, b \in B : a \le_K b\big) \text{ and } \big(\forall\, b \in B\ \exists\, a \in A : a \le_K b\big).$$

This relation is called *set less* or *KNY order relation* $\preccurlyeq_s$ and has been independently introduced by Young [40] and Nishnianidze [27]. It has been presented by Kuroiwa [25] in a slightly modified form. This relation is a pre-order and compatible with the conlinear structure of the space.

Based on a pre-order we can define minimal and maximal elements of some set $\mathscr{Q}$.

Definition 1.4. Let $\mathscr{Q}$ be a pre-ordered set. Let $\mathscr{A}$ be a nonempty subset of $\mathscr{Q}$, $T \in \mathscr{Q}$ and $\overline{A} \in \mathscr{A}$. We say that:

- $\overline{A}$ is a *minimal element* of $\mathscr{A}$ if $A \leq \overline{A}$ for some $A \in \mathscr{A}$ implies $\overline{A} \leq A$.
- $\overline{A}$ is a *maximal element* of $\mathscr{A}$ if $\overline{A} \leq A$ for some $A \in \mathscr{A}$ implies $A \leq \overline{A}$.
- T is a *lower bound* of $\mathscr{A}$ if $T \leq A$ for all $A \in \mathscr{A}$.
- T is an *upper bound* of $\mathscr{A}$ if $A \leq T$ for all $A \in \mathscr{A}$.

When the binary relation $\leq$ is a partial order, $\overline{A}$ is a minimal element of $\mathscr{A}$ if $A \not\leq \overline{A}$ for all $A \in \mathscr{A}$, $A \neq \overline{A}$, and $\overline{A}$ is a maximal element of $\mathscr{A}$ if $\overline{A} \not\leq A$ for all $A \in \mathscr{A}$, $A \neq \overline{A}$. If $K \subset Y$ denotes a pointed convex cone that introduces a partial order in Y we thus have that some element $\bar{y} \in A$ is a minimal element of $A \subset Y$ if

$$(\{\bar{y}\} - K) \cap A = \{\bar{y}\}. \tag{1.1}$$

Minimal elements are also known as *Edgeworth–Pareto-minimal* or *efficient* elements and will be discussed more detailed in the following section together with variations of this definition. Similar, some element $\bar{y} \in A$ is a maximal element of $A \subset Y$ if

$$(\{\bar{y}\} + K) \cap A = \{\bar{y}\}. \tag{1.2}$$

Moreover, $\bar{y} \in Y$ is a lower bound of A if $A \subset \{\bar{y}\} + K$ and an upper bound if $A \subset \{\bar{y}\} - K$.

Let $\min A$ and $\max A$ denote the sets of minimal elements and maximal elements of A w.r.t. the partial order $\leq_K$, i.e.

$$\min A = \{\bar{a} \in A \mid A \cap (\bar{a} - K) = \{\bar{a}\}\},$$
$$\max A = \{\bar{a} \in A \mid A \cap (\bar{a} + K) = \{\bar{a}\}\}.$$

Using these sets we can define new order relations for comparing sets as introduced in [22].

Example 1.2. Let

$$\mathscr{M} := \{A \in \mathscr{P}(Y) \mid \min A \text{ and } \max A \text{ are nonempty}\}.$$

Note that for instance in a topological real linear space Y for every compact set in $\mathscr{P}(Y)$ minimal and maximal elements exist. For $A, B \in \mathscr{M}$ the *minmax less order relation* $\preccurlyeq_m$ is defined by

$$A \preccurlyeq_m B \quad :\Longleftrightarrow \quad \min A \preccurlyeq_s \min B \text{ and } \max A \preccurlyeq_s \max B$$

(the subscript m stands for minmax). This relation is a pre-order which is compatible with the multiplication with nonnegative real numbers. In general, this relation is not antisymmetric.

We end this section with the definition of a chain and the famous Zorn's lemma, which is the most important result which provides a sufficient condition for the existence of a minimal element of a set, see Sect. 1.5.

Definition 1.5. Let $\mathscr{Q}$ be a pre-ordered set.

- $A,B \in \mathscr{Q}$ are said to be *comparable* if either $A \leq B$ or $B \leq A$ holds.
- A nonempty subset $\mathscr{A}$ of $\mathscr{Q}$ is called a *chain* if any pair $A,\ B \in \mathscr{A}$ is comparable.

Lemma 1.1 (Zorn's Lemma). *Every pre-ordered set, in which every chain has an upper (lower) bound, contains at least one maximal (minimal) element.*

1.3 Optimality Concepts in Linear Spaces

In this section we discuss more detailed optimality notions in vector optimization. In the following, let Y denote a real linear space that is pre-ordered by some convex cone $K \subset Y$ and let A denote some nonempty subset of Y. In general, one is mainly interested in minimal and maximal elements of the set A, but in certain situations it also makes sense to study variants of these concepts. For instance weakly minimal elements are often of interest in theoretical examinations whereas properly minimal elements are sometimes more of interest for applications.

First and second part in the definition below coincide with Definition 1.4 (first and second part) in the case of $\mathscr{Q}$ a linear space and a pre-order given by the convex cone K.

Definition 1.6.

- An element $\bar{y} \in A$ is called a *minimal element* of the set A, if

$$(\{\bar{y}\} - K) \cap A \subset \{\bar{y}\} + K. \tag{1.3}$$

- An element $\bar{y} \in A$ is called a *maximal element* of the set A, if

$$(\{\bar{y}\} + K) \cap A \subset \{\bar{y}\} - K. \tag{1.4}$$

- An element $\bar{y} \in A$ is called a *strongly minimal element* of the set A, if

$$A \subset \{\bar{y}\} + K. \tag{1.5}$$

- Let K have a nonempty algebraic interior, i.e. $\mathrm{cor}K \neq \emptyset$. An element $\bar{y} \in A$ is called a *weakly minimal element* of the set A, if

$$(\{\bar{y}\} - \mathrm{cor}K) \cap A = \emptyset. \tag{1.6}$$

If the ordering cone K is pointed, then the inclusions (1.3) and (1.4) can be replaced by (1.1) and (1.2), respectively. Of course, corresponding concepts as *strongly maximal* and *weakly maximal* can be defined analogously. Since every maximal element of A is also minimal w.r.t the pre-order induced by the convex cone $-K$, without loss of generality it is sufficient to study the minimality notion. In terms of lattice theory a strongly minimal element of a set A is also called *zero* element of A. It is a lower bound of the considered set, compare Definition 1.4 (third part). As this notion is very restrictive it is often not applicable in practice. Notice that the notions "minimal" and "weakly minimal" are closely related. Take an arbitrary weakly minimal element $\bar{y} \in A$ of the set A, that is $(\{\bar{y}\} - \mathrm{cor}(K)) \cap A = \emptyset$. The set $\hat{K} := \mathrm{cor}(K) \cup \{0_Y\}$ is a convex cone and it induces another pre-order in Y. Consequently, $\bar{y}$ is also a minimal element of the set A with respect to the pre-order induced by $\hat{K}$. Figures 1.1 and 1.2a illustrate the different optimality notions.

Example 1.3. (a) Let Y be the real linear space of functionals defined on a real linear space X and pre-ordered by a pointwise order. Moreover, let A denote the subset of Y which consists of all sublinear functionals on X. Then the algebraic dual space X' is the set of all minimal elements of A. This is proved in [21, Lemma 3.7] and is a key for the proof of the basic version of the Hahn–Banach theorem.

(b) Let X and Y be pre-ordered linear spaces with the ordering cones K_X and K_Y, and let $T : X \to Y$ be a given linear map. We assume that there is a $q \in Y$ so that the set $A := \{x \in K_X \mid T(x) + q \in K_Y\}$ is nonempty. Then an *abstract complementary problem* leads to the problem of finding a minimal element of

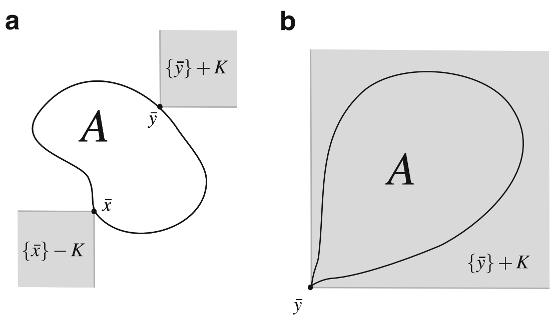

Fig. 1.1 (**a**) Minimal element $\bar{x}$ and maximal element $\bar{y}$ of a set A. (**b**) Strongly minimal element $\bar{y}$ of a set A

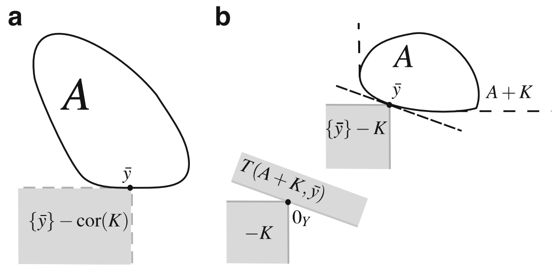

Fig. 1.2 (**a**) Weakly minimal element $\bar{y}$ of a set A. (**b**) Properly minimal element $\bar{y}$ of a set A

the set A. For further details we refer to [9, 12]. Obviously, if $q \in K_Y$, then 0_X is a strongly minimal element of the set A.

The next lemma gives relations between the different optimality concepts.

Lemma 1.2. *(a) Every strongly minimal element of the set A is also a minimal element of A.*

(b) Let K have a nonempty algebraic interior and $K \neq Y$. Then every minimal element of the set A is also a weakly minimal element of the set A.

Proof. (a) It holds $A \subset \{\bar{y}\} + K$ for any strongly minimal element $\bar{y}$ of A. Thus

$$(\{\bar{y}\} - K) \cap A \subset A \subset \{\bar{y}\} + K.$$

(b) The assumption $K \neq Y$ implies $(-\mathrm{cor}(K)) \cap K = \emptyset$. Therefore, for an arbitrary minimal element $\bar{y}$ of A it follows

$$\begin{aligned}\emptyset &= (\{\bar{y}\} - \mathrm{cor}(K)) \cap (\{\bar{y}\} + K) \\ &= (\{\bar{y}\} - \mathrm{cor}(K)) \cap (\{\bar{y}\} - K) \cap A \\ &= (\{\bar{y}\} - \mathrm{cor}(K)) \cap A\end{aligned}$$

which means that $\bar{y}$ is also a weakly minimal element of A. □

In general, the converse statement of Lemma 1.2 is not true. This fact is illustrated by

Example 1.4. Let $Y = \mathbb{R}^2$ and let a partial order be induced by the cone $K = \mathbb{R}^2_+$. Consider the set $A = [0,1] \times [0,1]$. The unique minimal element is $0_{\mathbb{R}^2}$ while all elements of the set $\{(y_1, y_2) \in A \mid y_1 = 0 \ \vee \ y_2 = 0\}$ are weakly minimal elements. Note that $0_{\mathbb{R}^2}$ is also a strongly minimal element.

The following lemma, compare [35], indicates that the minimal elements of a set A and the minimal elements of the set $A + K$ where K denotes the ordering cone are closely related. This result is of interest for further theoretical examinations, for instance regarding duality results. Especially if the set $A + K$ is convex while the set A is not convex, the consideration of $A + K$ instead of A is advantageous, e.g. when necessary linear scalarization results as given in [21, Theorem 5.11, 5.13] should be applied.

Lemma 1.3. *(a) If the ordering cone K is pointed, then every minimal element of the set $A + K$ is also a minimal element of the set A.*

(b) Every minimal element of the set A is also a minimal element of the set $A + K$.

Proof. (a) Let $\bar{y} \in A + K$ be an arbitrary minimal element of the set $A + K$. If we assume that $\bar{y} \notin A$, then there is an element $y \neq \bar{y}$ with $y \in A$ and $\bar{y} \in \{y\} + K$. Consequently, we get $y \in (\{\bar{y}\} - K) \cap (A + K)$ which contradicts the assumption that $\bar{y}$ is a minimal element of the set $A+K$. Hence, we obtain $\bar{y} \in A \subset A+K$ and, therefore, $\bar{y}$ is also a minimal element of the set A.

(b) Take an arbitrary minimal element $\bar{y} \in A$ of the set A, and choose any $y \in (\{\bar{y}\}-K) \cap (A+K)$. Then there are elements $a \in A$ and $k \in K$ so that $y = a+k$. Consequently, we obtain $a = y-k \in \{\bar{y}\}-K$, and since $\bar{y}$ is a minimal element of the set A, we conclude $a \in \{\bar{y}\}+K$. But then we get also $y \in \{\bar{y}\}+K$. This completes the proof. □

If the cone K has a nonempty algebraic interior, the statement of Lemma 1.3 is also true if we replace minimal by weakly minimal [21, Lemma 4.13].

Another refinement of the minimality notion is helpful from a theoretical point of view. These optima are called properly minimal. Until now there are various types of concepts of proper minimality. The notion of proper minimality (or proper efficiency) was first introduced by Kuhn–Tucker [24] and modified by Geoffrion [15], and later it was formulated in a more general framework (Benson–Morin [2], Borwein [6], Vogel [35], Wendell-Lee [36], Wierzbicki [38], Hartley [17], Benson [1], Borwein [7], Nieuwenhuis [26], Henig [18] and Zhuang [41]). We present here a definition introduced by Borwein [6] and Vogel [35]. For a collection of other definitions of proper minimality see for instance [21, p. 113f].

Recall that the *contingent cone* (or *Bouligand tangent cone*) $T(A,\bar{y})$ to a subset A of a real normed space $(Y,\|\cdot\|)$ in $\bar{y} \in \mathrm{cl}(A)$ is the set of all tangents h which are defined as follows: An element $h \in Y$ is called a tangent to A in $\bar{y}$, if there are a sequence $(y_n)_{n\in\mathbb{N}}$ of elements $y_n \in A$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive real numbers λ_n so that

$$\bar{y} = \lim_{n\to\infty} y_n \quad \text{and} \quad h = \lim_{n\to\infty} \lambda_n (y_n - \bar{y}).$$

Here, $\mathrm{cl}(A)$ denotes the closure of A.

Definition 1.7. Let $(Y,\|\cdot\|)$ be a real normed space. An element $\bar{y} \in A$ is called a *properly minimal element* of the set A, if $\bar{y}$ is a minimal element of the set A and the zero element 0_Y is a minimal element of the contingent cone $T(A+K,\bar{y})$ (see Fig. 1.2b).

It is evident that a properly minimal element of a set A is also a minimal element of A.

Example 1.5. Let Y be the Euclidean space $\mathbb{R}^2$ and let a partial order be induced by the cone $K = \mathbb{R}^2_+$. Consider $A = \left\{(y_1,y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 \le 1\right\}$. Then all elements of the set $\left\{(y_1,y_2) \in \mathbb{R}^2 \mid y_1 \in [-1,0],\ y_2 = -\sqrt{1-y_1^2}\right\}$ are minimal elements of A. The set of all properly elements of A reads as

$$\left\{(y_1,y_2) \in \mathbb{R}^2 \mid y_1 \in (-1,0),\ y_2 = -\sqrt{1-y_1^2}\right\}.$$

The optimality concepts for subsets of a real linear space naturally induce concepts of optimal solutions for *vector optimization problems*. Let X and Y be real linear spaces, and let K, as before, be a convex cone in Y. Furthermore, let S be a nonempty subset of X, and let $f : S \to Y$ be a given map. Then the *vector optimization problem*

$$\min_{x \in S} f(x) \tag{VOP}$$

is to be interpreted in the following way: Determine a (weakly, strongly, properly) minimal solution $\bar{x} \in S$ which is defined as the inverse image of a (weakly, strongly, properly) minimal element $f(\bar{x})$ of the image set $f(S) = \{f(x) \in Y \mid x \in S\}$.

Definition 1.8. An element $\bar{x} \in S$ is called a *(weakly, strongly, properly) minimal solution* of problem (VOP) w.r.t. the pre-order induced by K, if $f(\bar{x})$ is a (weakly, strongly, properly) minimal element of the image set $f(S)$ w.r.t. the pre-order induced by K.

For $Y = \mathbb{R}^m$ partially ordered by the natural ordering, i.e. $K = \mathbb{R}^m_+$, we call (VOP) also a *multiobjective optimization* problem, as the m objectives $f_i : S \to \mathbb{R}$, $i = 1, \ldots, m$, are minimized simultaneously. A minimal solution, then also called Edgeworth–Pareto optimal, compare page 2, is thus a point $\bar{x} \in S$ such that there exists no other $x \in S$ with

$$f_i(x) \leq f_i(\bar{x}) \text{ for all } i = 1, \ldots, m,$$

and

$$f_j(x) < f_j(\bar{x}) \text{ for at least one } j \in \{1, \ldots, m\}.$$

Example 1.6. Let $X = \mathbb{R}^2$ and Y be the Euclidean space $\mathbb{R}^2$ and let a partial order be induced by the cone $K = \mathbb{R}^2_+$. Consider the constraint set

$$S := \left\{(x_1, x_2) \in \mathbb{R}^2 \;\middle|\; x_1^2 - x_2 \leq 0,\; x_1 + 2x_2 - 3 \leq 0\right\}$$

and the vector function $f : S \to \mathbb{R}^2$ with

$$f(x_1, x_2) = \begin{pmatrix} -x_1 \\ x_1 + x_2^2 \end{pmatrix} \quad \text{for all } (x_1, x_2) \in S.$$

The point $\left(\frac{3}{2}, \frac{57}{16}\right)$ is the only maximal element of $T := f(S)$, and the set of all minimal elements of T reads

$$\left\{(y_1, y_2) \in \mathbb{R}^2 \;\middle|\; y_1 \in \left[-1, \frac{1}{2}\sqrt[3]{2}\right] \text{ and } y_2 = -y_1 + y_1^4\right\}.$$

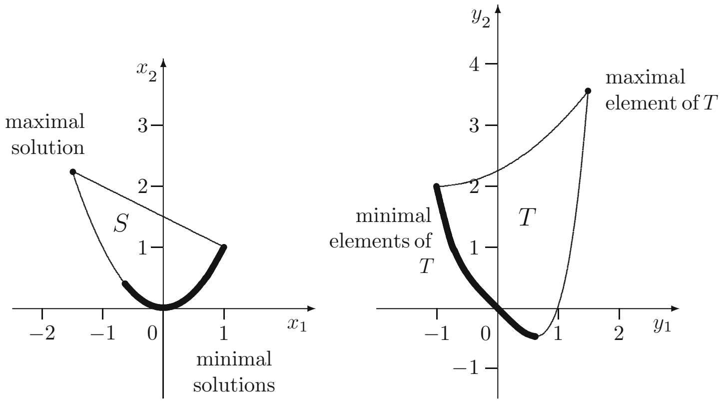

Fig. 1.3 Minimal and maximal elements of $T = f(S)$

The set of all minimal solutions of the vector optimization problem $\min_{x \in S} f(x)$ is given as

$$\left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in \left[-\frac{1}{2}\sqrt[3]{2}, 1\right] \text{ and } x_2 = x_1^2\right\}$$

(see Fig. 1.3).

1.4 Optimality Concepts in Set Optimization

Now we introduce *set optimization problems* as special vector optimization problems. Various optimality concepts are discussed for these problems.

In this section let S be a nonempty set, let Y be a real linear space, let $K \subset Y$ be a convex cone and let $F : S \rightrightarrows Y$ be a set-valued map. Then we consider the set optimization problem

$$\min_{x \in S} F(x). \tag{SOP}$$

Up to now many authors have used a vector approach for the formulation of optimality notions for this problem. First, we discuss this approach and then we present a more suitable set approach.

1.4.1 Vector Approach

In this subsection we give a short overview on some concepts of optimal solutions of problem (SOP) based on a vector approach. For simplicity we assume in this subsection that the convex cone K is pointed. Then, similar to minimal solutions of a vector optimization problem, see Definition 1.8, we can say that a pair $(\bar{x}, \bar{y})$ with

$\bar{x} \in S$ and $\bar{y} \in F(\bar{x})$ is a *minimizer* of (SOP) if

$$F(S) \cap (\{\bar{y}\} - (K \backslash \{0_Y\})) = \emptyset$$

for $F(S) := \bigcup_{x \in S} F(x)$, which means that $\bar{y} \in \min F(S)$. In general only one element does not imply that the whole set $F(\bar{x})$ is in a certain sense minimal with respect to a sets $F(x)$ with $x \in S$.

Another optimality notion has been recently introduced in [10, Definition 1.3]. An element $\bar{x} \in S$ is called a *feeble (multifunction) minimal point* of problem (SOP) if

$$\exists\, \bar{y} \in F(\bar{x}) : F(S \backslash \{\bar{x}\}) \cap (\{\bar{y}\} - (K \backslash \{0_Y\})) = \emptyset.$$

The equality means that $\bar{y}$ is not dominated by any arbitrary point in the set $F(S \backslash \{\bar{x}\})$. It is not required that the element $\bar{y}$ is a minimal element of the set $F(\bar{x})$. Obviously, this optimality notion is even weaker than the concept of a minimizer because the set $F(\bar{x})$ is not considered in the definition. The following simple example illustrates possible difficulties with this notion.

Example 1.7. For $S := \{1, 2, 3\}$ consider $F(1) = \{1\}, F(2) = \{2\}, F(3) = [1, 3]$ and $K := \mathbb{R}_+$. It is evident that $\bar{x} = 3$ (with $\bar{y} = 1$) is a feeble minimal point of problem (SOP), although $F(1)$ would be the "better" set because $F(2), F(3) \subset F(1) + K$.

A variation of this feeble notion is given in [10, Definition 1.3] in the following way: An element $\bar{x} \in S$ is called a *(multifunction) minimum point* of problem (SOP) if

$$F(S \backslash \{\bar{x}\}) \cap (\{y\} - (K \backslash \{0_Y\})) = \emptyset \text{ for all } y \in F(\bar{x}).$$

This condition is equivalent to the equality

$$F(S \backslash \{\bar{x}\}) \cap (F(\bar{x}) - (K \backslash \{0_Y\})) = \emptyset\ ,$$

which means that the set $F(\bar{x})$ is not dominated by any set $F(x)$ with $x \in S$, $x \neq \bar{x}$. The next example shows that this optimality notion is too strong in set optimization.

Example 1.8. For $S := \{1, 2\}$ consider for arbitrary real numbers a, b, c, d with $-\infty < a < b < \infty$ and $-\infty < c < d < \infty$ the intervals $F(1) = [a, b]$ and $F(2) = [c, d]$, and set $K := \mathbb{R}_+$. In this case $\bar{x} = 1$ is a minimum point of problem (SOP) if and only if

$$F(2) \cap (F(1) - (K \backslash \{0\})) = \emptyset,$$

which means that $[c, d] \cap (-\infty, b) = \emptyset$ or $b \leq c$. So, this is a very strong requirement.

Variants of the discussed notions have also been mentioned in [39, p. 10], where the nondomination concept is again used for different optimality notions (but notice that the utilized solution concept in [39] uses the set less relation of interval analysis, which is covered by the unified set approach).

1.4.2 Set Approach

Although the concept of a minimizer is of mathematical interest, it cannot be often used in practice. In order to avoid this drawback it is necessary to work with practically relevant order relations for sets. In Example 1.1.(b) the set less order relation $\preccurlyeq_s$ and in Example 1.2 the minmax less order relation $\preccurlyeq_m$ have been already defined for the comparison of sets. In interval analysis there are even more order relations in use, like the *certainly less* $\preccurlyeq_c$ or the *possibly less* $\preccurlyeq_p$ relations (see [11]), i.e. for arbitrary nonempty sets $A, B \subset Y$ one defines

$$A \preccurlyeq_c B \;\; :\Longleftrightarrow \;\; (A = B) \text{ or } \; (A \neq B, \forall\, a \in A\; \forall\, b \in B : a \leq b)$$

and

$$A \preccurlyeq_p B \;\; :\Longleftrightarrow \;\; (\exists\, a \in A\; \exists\, b \in B : a \leq b).$$

Let $\mathscr{M}$ be defined as in Example 1.2. Then for arbitrary $A, B \in \mathscr{M}$ the *minmax certainly less order relation* $\preccurlyeq_{mc}$ is defined by

$$A \preccurlyeq_{mc} B \; :\Longleftrightarrow (A = B) \text{ or } (A \neq B, \min A \preccurlyeq_c \min B \text{ and } \max A \preccurlyeq_c \max B)$$

and the *minmax certainly nondominated order relation* $\preccurlyeq_{mn}$ is defined by

$$A \preccurlyeq_{mn} B \;\; :\Longleftrightarrow \;\; (A = B) \text{ or } (A \neq B, \max A \preccurlyeq_s \min B),$$

see [22]. Figure 1.4 illustrates the order relation $\preccurlyeq_{mc}$.

The set less order relation $\preccurlyeq_s$ and the order relations $\preccurlyeq_m$, $\preccurlyeq_{mc}$ and $\preccurlyeq_{mn}$ are pre-orders. If $\preccurlyeq$ denotes one of these four order relations, then we can define optimal solutions w.r.t. the pre-order $\preccurlyeq$.

Definition 1.9. An element $\bar{x} \in S$ is called an *optimal solution* of problem (SOP) w.r.t. the pre-order $\preccurlyeq$ if

$$F(x) \preccurlyeq F(\bar{x}) \text{ for some } x \in S \Rightarrow \; F(\bar{x}) \preccurlyeq F(x).$$

So, we can use the same minimality concept as in vector optimization for the definition of optimal solutions in set optimization.

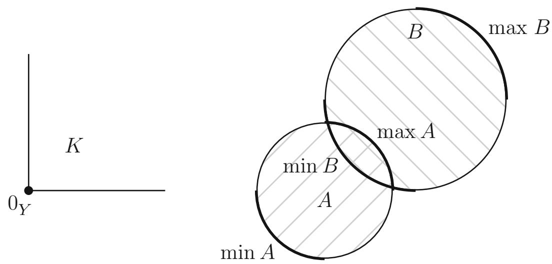

Fig. 1.4 Illustration of two sets $A, B \in \mathscr{M}$ with $A \preccurlyeq_{mc} B$

1.5 Existence Results in Vector Optimization

In this section we give assumptions which guarantee that at least one optimal element of a subset of a pre-ordered linear space exists. These investigations will be done for the minimality, the properly minimality and the weakly minimality notion. Strongly minimal elements are not considered because this optimality notion is too restrictive.

In order to get existence results under weak assumptions on a set we introduce the following

Definition 1.10. Let A be a nonempty subset of a pre-ordered linear space Y where the pre-order is introduced by a convex cone $K \subset Y$. If for some $y \in Y$ the set $A_y = (\{y\} - K) \cap A$ is nonempty, A_y is called a *section* of the set A (see Fig. 1.5).

The assertion of the following lemma is evident.

Lemma 1.4. *Let A be a nonempty subset of a pre-ordered linear space Y with an ordering cone K.*

(a) Every minimal element of a section of the set A is also a minimal element of the set A.
(b) If $cor(K) \neq \emptyset$, then every weakly minimal element of a section of the set A is also a weakly minimal element of the set A.

It is important to remark that for the notion of proper minimality a similar statement is not true in general. We begin now with a discussion of existence results for the notion of minimal elements. The following existence result is a consequence of Zorn's lemma (Lemma 1.1). Recall that an ordering cone in a real topological linear space is called *Daniell* if every decreasing net (i.e. $i \leq j \;\Rightarrow\; y_j \leq_K y_i$) which has a lower bound converges to its infimum. And a real topological linear space Y with an ordering cone K is called *boundedly order complete*, if every bounded decreasing net has an infimum.

Theorem 1.2. *Let Y be a topological linear space which is pre-ordered by a closed ordering cone K. Then we have:*

(a) If the set A has a closed section which has a lower bound and the ordering cone K is Daniell, then there is at least one minimal element of the set A.

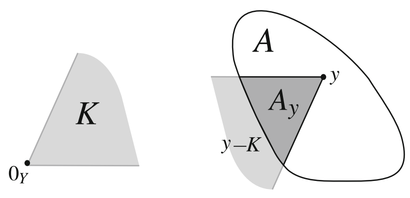

Fig. 1.5 Section A_y of a set A

(b) If the set A has a closed and bounded section and the ordering cone K is Daniell and boundedly order complete, then there is at least one minimal element of the set A.

(c) If the set A has a compact section, then there is at least one minimal element of the set A.

Proof. Let A_y (for some $y \in Y$) be an appropriate section of the set A. If we show that every chain in the section A_y has a lower bound, then by Zorn's lemma (Lemma 1.1) A_y has at least one minimal element which is, by Lemma 1.4.(a), also a minimal element of the set A.

Let $\{a_i\}_{i\in I}$ be any chain in the section A_y. Let $\mathscr{F}$ denote the set of all finite subsets of I which are pre-ordered with respect to the inclusion relation. Then for every $F \in \mathscr{F}$ the minimum

$$y_F := \min\{a_i \mid i \in F\}$$

exists and belongs to A_y. Consequently, $(y_F)_{F\in\mathscr{F}}$ is a decreasing net in A_y. Next, we consider several cases.

(a) A_y is assumed to have a lower bound so that $(y_F)_{F\in\mathscr{F}}$ has an infimum. Since A_y is closed and K is Daniell, $(y_F)_{F\in\mathscr{F}}$ converges to its infimum which belongs to A_y. This implies that any chain in A_y has a lower bound.

(b) Since A_y is bounded and K is boundedly order complete, the net $(y_F)_{F\in\mathscr{F}}$ has an infimum. The ordering cone K is Daniell and, therefore, $(y_F)_{F\in\mathscr{F}}$ converges to its infimum. And since A_y is closed, this infimum belongs to A_y. Hence, any chain in A_y has a lower bound.

(c) Now, A_y is assumed to be compact. The family of compact subsets A_{a_i} $(i \in I)$ has the finite intersection property, i.e., every finite subfamily has a nonempty intersection. Since A_y is compact, the family of subsets A_{a_i} $(i \in I)$ has a nonempty intersection (see Dunford–Schwartz [13, p. 17]), that is, there is an element

$$\hat{y} \in \bigcap_{i\in I} A_{a_i} = \bigcap_{i\in I} (\{a_i\} - K) \cap A_y.$$

Hence, $\hat{y}$ is a lower bound of the subset $\{a_i\}_{i\in I}$ and belongs to A_y. Consequently, any chain in A_y has a lower bound. □

Notice that the preceding theorem remains valid, if "section" is replaced by the set itself. Theorem 1.2 as well as the following example is due to Borwein [9], but Theorem 1.2 (c) was first proved by Vogel [35] and Theorem 1.2 (a) can essentially be found, without proof, in a survey article of Penot [29].

Example 1.9. We consider again the problem formulated in Example 1.3.(b). Let X and Y be pre-ordered topological linear spaces with the closed ordering cones K_X and K_Y where K_X is also assumed to be Daniell. Moreover, let $T : X \to Y$ be a continuous linear map and let $q \in Y$ be given so that the set $A := \{x \in K_X \mid T(x) + q \in K_Y\}$ is nonempty. Clearly the set A is closed and has a lower bound (namely 0_X). Then by Theorem 1.2 (a) the set A has at least one minimal element.

For the next existence result we need the so-called *James theorem* [23].

Theorem 1.3 (James Theorem). *Let A be a nonempty bounded and weakly closed subset of a real quasi-complete locally convex space Y. If every continuous linear functional $l \in Y^*$ attains its supremum on A, then A is weakly compact.*

Using this theorem together with Theorem 1.2 (c) we obtain the following result due to Borwein [9].

Theorem 1.4. *Let A be a nonempty subset of a real locally convex space Y.*

(a) If A is weakly compact, then for every closed convex cone K in Y the set A has at least one minimal element with respect to the pre-order induced by K.
(b) In addition, let Y be quasi-complete (for instance, let Y be a Banach space). If A is bounded and weakly closed and for every closed convex cone K in Y the set A has at least one minimal element with respect to the pre-order induced by K, then A is weakly compact.

Proof. (a) Every closed convex cone K is also weakly closed [21, Lemma 3.24]. Since A is weakly compact, there is a compact section of A. Then, by Theorem 1.2 (c), A has at least one minimal element with respect to the pre-order induced by K.

(b) It is evident that the functional 0_{Y^*} attains its supremum on the set A. Therefore, take an arbitrary continuous linear functional $l \in Y^* \backslash \{0_{Y^*}\}$ (if it exists) and define the set $K := \{y \in Y \mid l(y) \leq 0\}$ which is a closed convex cone. Let $\bar{y} \in A$ be a minimal element of the set A with respect to the pre-order induced by K, i.e.

$$(\{\bar{y}\} - K) \cap A \subset \{\bar{y}\} + K. \tag{1.7}$$

Since

$$\{\bar{y}\} - K = \{y \in Y \mid l(y) \geq l(\bar{y})\}$$

and

$$\{\bar{y}\} + K = \{y \in Y \mid l(y) \leq l(\bar{y})\},$$

the inclusion (1.7) is equivalent to the implication

$$y \in A,\ l(y) \geq l(\bar{y}) \implies l(y) = l(\bar{y}).$$

This implication can also be written as

$$l(\bar{y}) \geq l(y) \text{ for all } y \in A.$$

This means that the functional l attains its supremum on A at $\bar{y}$. Then by the James theorem (Theorem 1.3) the set A is weakly compact. □

The preceding theorem shows that the weak compactness assumption on a set plays an important role for the existence of minimal elements.

Next, we study existence theorems which follow from scalarization results, compare [20]. Recall that a nonempty subset A of a real normed space $(Y, \|\cdot\|)$ is called *proximinal*, if every $y \in Y$ has at least one best approximation from A, that is, for every $y \in Y$ there is an $\bar{y} \in A$ with

$$\|y - \bar{y}\| \leq \|y - a\| \text{ for all } a \in A.$$

Any nonempty weakly closed subset of a real reflexive Banach space is proximinal [21, Corollary 3.35]. A functional $f\colon A \to \mathbb{R}$ with A a nonempty subset of a linear space pre-ordered by K is called *strongly monotonically increasing* on A, if for every $\bar{y} \in A$

$$y \in (\{\bar{y}\} - K) \cap A,\ y \neq \bar{y} \implies f(y) < f(\bar{y}).$$

If $\operatorname{cor}(K) \neq \emptyset$, then f is called *strictly monotonically increasing*, if for every $\bar{y} \in A$

$$y \in (\{\bar{y}\} - \operatorname{cor}(K)) \cap A \implies f(y) < f(\bar{y}).$$

Theorem 1.5. *Assume that either assumption (a) or assumption (b) below holds:*

(a) Let A be a nonempty subset of a partially ordered normed space $(Y, \|\cdot\|_Y)$ with a pointed ordering cone K, and let Y be the topological dual space of a real normed space $(Z, \|\cdot\|_Z)$. Moreover, for some $y \in Y$ let a weak-closed section A_y be given.*

(b) Let A be a nonempty subset of a partially ordered reflexive Banach space $(Y, \|\cdot\|_Y)$ with a pointed ordering cone K. Furthermore, for some $y \in Y$ let a weakly closed section A_y be given.

If, in addition, the section A_y has a lower bound $\hat{y} \in Y$, i.e. $A_y \subset \{\hat{y}\} + K$, and the norm $\|\cdot\|_Y$ is strongly monotonically increasing on K, then the set A has at least one minimal element.

Proof. Let the assumptions of (a) be satisfied. Take any $z \in Z^* \setminus A_y = Y \setminus A_y$ and any $a \in A_y$. Since every closed ball in $Z^* = Y$ is weak*-compact, the set

$$A_y \cap \{w \in Y \mid \|w\|_Y \leq \|a\|_Y\}$$

is weak*-compact as well. Notice that the functional mapping from Y to $\mathbb{R}$ given by $w \mapsto \|z - w\|_Y$ is weakly* lower semicontinuous. Thus the section A_y is proximinal. On the other hand, if the assumption (b) is satisfied, then the section A_y is proximinal as well. Consequently, there is an $\bar{y} \in A_y$ with

$$\|\bar{y} - \hat{y}\|_Y \leq \|a - \hat{y}\|_Y \text{ for all } a \in A_y. \tag{1.8}$$

The norm $\|\cdot\|_Y$ is strongly monotonically increasing on K and because of $A_y - \{\hat{y}\} \subset K$ the functional $\|\cdot - \hat{y}\|_Y$ is strongly monotonically increasing on A_y, compare [21, Theorem 5.15(b)].

Next we show that $\bar{y}$ is a minimal element of A_y. Assume this is not the case. Then there is an element $a \in (\{\bar{y}\} - K) \cap A_y$ with $a \neq \bar{y}$. This implies $\|a - \hat{y}\|_Y < \|\bar{y} - \hat{y}\|_Y$ in contradiction to (1.8).

Finally, an application of Lemma 1.4.(a) completes the proof. □

Example 1.10. Let A be a nonempty subset of a pre-ordered Hilbert space $(Y, \langle .,. \rangle)$ with an ordering cone K_Y. Then the norm on Y is strongly monotonically increasing on K_Y if and only if $K_Y \subset K_Y^*$ with $K_Y^* = \{y^* \in Y^* \mid y^*(y) \geq 0 \text{ for all } y \in K_Y\}$ the dual cone of K_Y [30, 37]. Thus, if the ordering cone K_Y has the property $K_Y \subset K_Y^*$ and A has a weakly closed section bounded from below, then A has at least one minimal element.

For the minimality notion a scalarization result concerning positive linear functionals leads to an existence theorem which is contained in Theorem 1.4.(a). But for the proper minimality notion such a scalarization result is helpful. We recall the important *Krein–Rutman theorem*. For a proof see [8, p. 425] or [21, Theorem 3.38].

Theorem 1.6 (Krein–Rutman Theorem). *In a real separable normed space $(Y, \|\cdot\|)$ with a closed pointed convex cone $K \subset Y$ the quasi-interior*

$$K_{Y^*}^{\#} := \{y^* \in Y^* \mid y^*(y) > 0 \ \textit{for all}\ y \in K \setminus \{0_Y\}\}$$

of the topological dual cone is nonempty.

Theorem 1.7. *Let A be a weakly compact subset of a partially ordered separable normed space $(Y, \|\cdot\|)$ with a closed pointed ordering cone K. Then there exists at least one properly minimal element $\bar{y} \in A$.*

Proof. According to the Krein–Rutman Theorem 1.6, the quasi-interior of the topological dual cone is nonempty. Then every continuous linear functional which belongs to that quasi-interior attains its infimum on the weakly compact set A. So there exists some $\bar{y} \in A$ and some $l \in K_{Y^*}^{\#}$ with

$$l(\bar{y}) \leq l(y) \quad \text{for all} \quad y \in A. \tag{1.9}$$

As $l \in K_{Y^*}^{\#}$, l is strongly monotonically increasing on A. First we assume $\bar{y}$ is not a minimal element of A. Then there is an element $a \in (\{\bar{y}\} - K) \cap A$ with $a \neq \bar{y}$. This implies $l(a) < l(\bar{y})$ which is a contradiction to (1.9). Thus $\bar{y}$ is a minimal element of A and it remains to show that 0_Y is a minimal element of the contingent cone $T(A + K, \bar{y})$.

Take any tangent $h \in T(A + K, \bar{y})$. Then there are a sequence $(y_n)_{n \in \mathbb{N}}$ of elements in $A + K$ and a sequence $(\lambda_n)_{n \in \mathbb{N}}$ of positive real numbers with $\bar{y} = \lim_{n \to \infty} y_n$ and

$h = \lim_{n\to\infty} \lambda_n(y_n - \bar{y})$. The linear functional l is continuous and, therefore, we get $l(\bar{y}) = \lim_{n\to\infty} l(y_n)$. Since the functional l is also strongly monotonically increasing on Y, the inequality (1.9) implies

$$l(\bar{y}) \le l(y) \ \text{ for all } \ y \in A + K.$$

Then it follows

$$l(h) = \lim_{n\to\infty} l(\lambda_n(y_n - \bar{y})) = \lim_{n\to\infty} \lambda_n(l(y_n) - l(\bar{y})) \ge 0.$$

Hence we obtain

$$l(0_Y) = 0 \le l(h) \ \text{ for all } \ h \in T(A+K, \bar{y}).$$

With the same arguments as before we conclude that 0_Y is a minimal element of $T(A+K,\bar{y})$. This completes the proof. □

A further existence theorem for properly minimal elements is given by

Theorem 1.8. *Assume that either assumption (a) or assumption (b) below holds:*

(a) Let A be a nonempty subset of a partially ordered normed space $(Y, \|\cdot\|_Y)$ with a pointed ordering cone K which has a nonempty algebraic interior, and let Y be the topological dual space of a real normed space $(Z, \|\cdot\|_Z)$. Moreover, let the set A be weak-closed.*

(b) Let A be a nonempty subset of a partially ordered reflexive Banach space $(Y, \|\cdot\|_Y)$ with a pointed ordering cone K which has a nonempty algebraic interior. Furthermore, let the set A be weakly closed.

If, in addition, there is an $\hat{y} \in Y$ with $A \subset \{\hat{y}\} + cor(K)$ and the norm $\|\cdot\|_Y$ is strongly monotonically increasing on K, then the set A has at least one properly minimal element.

Proof. The proof is similar to that of Theorem 1.5. Since the norm $\|\cdot\|_Y$ is strongly monotonically increasing on K and $A - \{\hat{y}\} \subset \text{cor}(K)$ we get with the same arguments that there is some $\bar{y} \in A$ with

$$\|\bar{y} - \hat{y}\|_Y \le \|y - \hat{y}\|_Y \text{ for all } y \in A \tag{1.10}$$

and that $\bar{y}$ is a minimal element of A. It remains to show that 0_Y is a minimal element of $T(A+K,\bar{y})$.

Since the norm $\|\cdot\|_Y$ is assumed to be strongly monotonically increasing on K, we obtain from (1.10)

$$\|\bar{y} - \hat{y}\|_Y \le \|y - \hat{y}\|_Y \le \|y + k - \hat{y}\|_Y \ \text{ for all } y \in A \ \text{ and all } \ k \in K.$$

This results in

$$\|\bar{y}-\hat{y}\|_Y \le \|y-\hat{y}\|_Y \text{ for all } y \in A+K. \tag{1.11}$$

It is evident that the functional $\|\cdot - \hat{y}\|_Y$ is both convex and continuous in the topology generated by the norm $\|\cdot\|_Y$. Then, see for instance [21, Theorem 3.48], the inequality (1.11) implies

$$\|\bar{y}-\hat{y}\|_Y \le \|\bar{y}-\hat{y}+h\|_Y \text{ for all } h \in T(A+K,\bar{y}). \tag{1.12}$$

With $T := T(A+K,\bar{y}) \cap (\{\hat{y}-\bar{y}\}+K)$ the inequality (1.12) is also true for all $h \in T$, i.e.

$$\|0_Y - (\hat{y}-\bar{y})\|_Y \le \|h-(\hat{y}-\bar{y})\|_Y \text{ for all } h \in T$$

and $\|\cdot - (\hat{y}-\bar{y})\|_Y$ is because of $T-\{\hat{y}-\bar{y}\} \subset K$ strongly monotonically increasing on T. With the same arguments as in Theorem 1.5 0_Y is a minimal element of T.

Next we assume that 0_Y is not a minimal element of the contingent cone $T(A+K,\bar{y})$. Then there is an $y \in (-K) \cap T(A+K,\bar{y})$ with $y \neq 0_Y$. Since $A \subset \{\hat{y}\} + \mathrm{cor}(K)$ and $\bar{y} \in A$, there is a $\lambda > 0$ with $\bar{y}+\lambda y \in \{\hat{y}\}+K$ or $\lambda y \in \{\hat{y}-\bar{y}\}+K$. Consequently, we get

$$\lambda y \in (-K) \cap T(A+K,\bar{y}) \cap (\{\hat{y}-\bar{y}\}+K)$$

and therefore, we have $\lambda y \in (-K) \cap T$ which contradicts the fact that 0_Y is a minimal element of the set T. Hence, 0_Y is a minimal element of the contingent cone $T(A+K,\bar{y})$ and the assertion is obvious. □

Example 1.11. Let A be a nonempty subset of a partially ordered Hilbert space $(Y,\langle .,. \rangle)$ with an ordering cone K_Y which has a nonempty algebraic interior and for which $K_Y \subset K_Y^*$ (compare Example 1.10). If A is weakly closed and there is an $\hat{y} \in Y$ with $A \subset \{\hat{y}\} + \mathrm{cor}(K_Y)$, then the set A has at least one properly minimal element.

Finally, we turn our attention to the weak minimality notion. Using Lemma 1.2.(b) we can easily extend the existence theorems for minimal elements to weakly minimal elements, if we assume additionally that the ordering cone $K \subset Y$ does not equal Y and that it has a nonempty algebraic interior. This is one possibility in order to get existence results for the weak minimality notion. In the following theorems we use directly appropriate scalarization results for this optimality notion.

Theorem 1.9. *Let A be a nonempty subset of a pre-ordered locally convex space Y with a closed ordering cone $K_Y \neq Y$ which has a nonempty algebraic interior. If A has a weakly compact section, then the set A has at least one weakly minimal element.*

Proof. Applying a separation theorem we get that, since the ordering cone K_Y is closed and does not equal Y, there is at least one continuous linear functional $l \in K_Y^* \backslash \{0_{Y^*}\}$ with $K_Y^* = \{y^* \in Y^* \mid y^*(y) \ge 0 \text{ for all } y \in K_Y\}$. This functional attains its infimum on a weakly compact section of A, i.e. there is some $\bar{y} \in A$ and some $y \in Y$ with

$$l(\bar{y}) \leq l(a) \quad \text{for all} \quad a \in A_y. \tag{1.13}$$

Assume $\bar{y}$ is not a weakly minimal element of A_y. Then there is some $a \in (\{\bar{y}\} - \text{cor}(K)) \cap A_y$, and as l is strictly monotonically increasing on A_y due to $l \in K_Y^* \setminus \{0_Y\}$ we get $l(a) < l(\bar{y})$ in contradiction to (1.13). Thus $\bar{y}$ is a weakly minimal element of A_y and because of Lemma 1.4.(b) also of A. □

Notice that Theorem 1.9 could also be proved using Theorem 1.4.(a) and Lemma 1.2.(b).

Theorem 1.10. *Assume that either assumption (a) or assumption (b) below holds:*

(a) Let A be a nonempty subset of a pre-ordered normed space $(Y, \|\cdot\|_Y)$ with an ordering cone K which has a nonempty algebraic interior, and let Y be the topological dual space of a real normed space $(Y, \|\cdot\|_Y)$. Moreover, for some $y \in Y$ let a weak-closed section A_y be given.*

(b) Let A be a nonempty subset of a pre-ordered reflexive Banach space $(Y, \|\cdot\|_Y)$ with an ordering cone K which has a nonempty algebraic interior. Furthermore, for some $y \in Y$ let a weakly closed section A_y be given.

If, in addition, the section A_y has a lower bound $\hat{y} \in Y$, i.e. $A_y \subset \{\hat{y}\} + K$, and the norm $\|\cdot\|_Y$ is strictly monotonically increasing on K, then the set A has at least one weakly minimal element.

Proof. The proof is similar to that of Theorem 1.5. □

Example 1.12. Let A be a nonempty subset of $L_\infty(\Omega)$, the linear space of all (equivalence classes of) essentially bounded functions $f\colon \Omega \to \mathbb{R}$ ($\emptyset \neq \Omega \subset \mathbb{R}^n$) with the norm $\|\cdot\|_{L_\infty(\Omega)}$ given by

$$\|f\|_{L_\infty(\Omega)} := \text{ess sup}_{x\in\Omega}\{|f(x)|\} \quad \text{for all} \quad f \in L_\infty(\Omega).$$

The ordering cone $K_{L_\infty(\Omega)}$ is defined as

$$K_{L_\infty(\Omega)} := \{f \in L_\infty(\Omega) \mid f(x) \geq 0 \ \text{ almost everywhere on } \Omega\}.$$

It has a nonempty topological interior and it is weak* Daniell. We show that if the set A has a weak*-closed section bounded from below, then A has at least one weakly minimal element:

If we consider the linear space $L_\infty(\Omega)$ as the topological dual space of $L_1(\Omega)$, then the assertion follows from Theorem 1.10, if we show that the norm $\|\cdot\|_{L_\infty(\Omega)}$ is strictly monotonically increasing on the ordering cone $K_{L_\infty(\Omega)}$. It is evident that

$$\begin{aligned}\text{int}(K_{L_\infty(\Omega)}) = \{f \in L_\infty(\Omega) \mid\, & \text{there is an } \alpha > 0 \text{ with} \\ & f(x) \geq \alpha \text{ almost everywhere on } \Omega\} \neq \emptyset.\end{aligned}$$

As $K_{L_\infty(\Omega)}$ is convex with a nonempty topological interior, $\text{int}(K_{L_\infty(\Omega)})$ equals the algebraic interior of $K_{L_\infty(\Omega)}$. Take any functions $f, g \in K_{L_\infty(\Omega)}$ with $f \in \{g\} - \text{int}(K_{L_\infty(\Omega)})$. Then we have $g - f \in \text{int}(K_{L_\infty(\Omega)})$ which implies that there is an $\alpha > 0$ with

$$g(x) - f(x) \geq \alpha \text{ almost everywhere on } \Omega$$

and

$$g(x) \geq \alpha + f(x) \text{ almost everywhere on } \Omega.$$

Consequently, we get

$$\text{ess sup}_{x \in \Omega} \{g(x)\} \geq \alpha + \text{ess sup}_{x \in \Omega} \{f(x)\}$$

and

$$\|g\|_{L_\infty(\Omega)} > \|f\|_{L_\infty(\Omega)}.$$

Hence, the norm $\|\cdot\|_{L_\infty(\Omega)}$ is strictly monotonically increasing on $K_{L_\infty(\Omega)}$.

We conclude this section with the *Bishop–Phelps lemma* [5], which is a special type of an existence result for maximal elements. First we recall that in a real normed space $(Y, \|\cdot\|_Y)$ for an arbitrary continuous linear functional $l \in Y^*$ and an arbitrary $\gamma \in (0,1)$ the cone

$$C(l, \gamma) := \{y \in Y \mid \gamma \|y\|_Y \leq l(y)\}$$

is called *Bishop–Phelps cone*. Notice that this cone is convex and pointed and, therefore, it can be used as an ordering cone in the space Y.

Lemma 1.5 (Bishop–Phelps Lemma). *Let A be a nonempty closed subset of a real Banach space $(Y, \|\cdot\|_Y)$, and let a continuous linear functional $l \in Y^*$ be given with $\|l\|_{Y^*} = 1$ and $\sup_{y \in A} l(y) < \infty$. Then for every $y \in A$ and every $\gamma \in (0,1)$ there is a maximal element $\bar{y} \in \{y\} + C(l, \gamma)$ of the set A with respect to the Bishop–Phelps ordering cone $C(l, \gamma)$.*

For the proof we refer to [5] as well as to [19, p. 164].

1.6 Application: Field Design of a Magnetic Resonance System

In this section we discuss a vector optimization problem of the type (VOP) which is of importance in magnetic resonance systems in medical engineering. Magnetic resonance (MR) systems are significant devices in medical engineering which may produce images of soft tissue of the human body with high resolution and good contrast. Among others, it is a useful device for cancer diagnosis. The images are physically generated by the use of three types of magnetic fields: the main field, the gradient field and the radio frequency (RF) field, compare [32].

MR uses the spin of the atomic nuclei in a human body and it is the hydrogen proton whose magnetic characteristics are used to generate images. One does not consider only one spin but a collection of spins in a voxel being a small volume element. Without an external magnetic field the spins in this voxel are randomly oriented and because of their superposition their effects vanish (see Fig. 1.6a). By using the main field which is generated by super-conducting magnets, the spin magnets align in parallel or anti-parallel to the field (see Fig. 1.6b). There is a small majority of up spins in contrast to down spins and this difference leads to a very weak magnetization of the voxel. The spin magnet behaves like a magnetic top used by children; this is called the spin precession (see Fig. 1.7).

With an additional RF pulse the magnetization flips. This stimulation with an RF pulse leads to magnetic resonances in the body. In order to get the slices that give us the images, we use a so-called gradient field with the effect that outside the defined slice the nuclear spins are not affected by the RF pulse. The obtained voxel information in a slice can then be used for the construction of MR images via a 2-dimensional Fourier transform. A possible MR image of a human head is given in Fig. 1.8.

There are various optimization problems in the context of the improvement of the quality of MR images. We restrict ourselves to the description of the following bicriterial optimization problem, i.e. we consider a vector optimization problem as presented in (VOP) with $Y = \mathbb{R}^2$ the Euclidean space. This problem was already

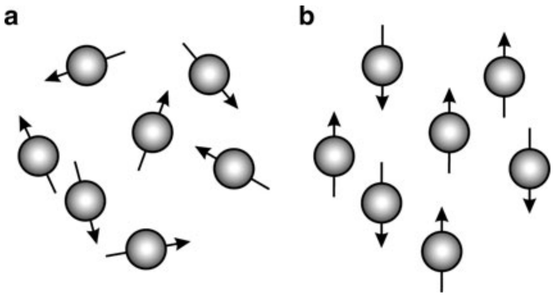

Fig. 1.6 (**a**) Arbitrary spins. (**b**) Parallel and anti-parallel aligned spins

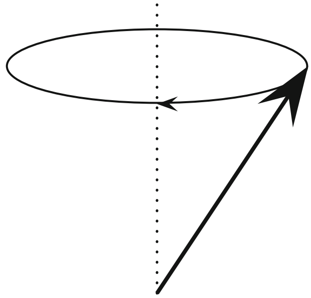

Fig. 1.7 Spin precession

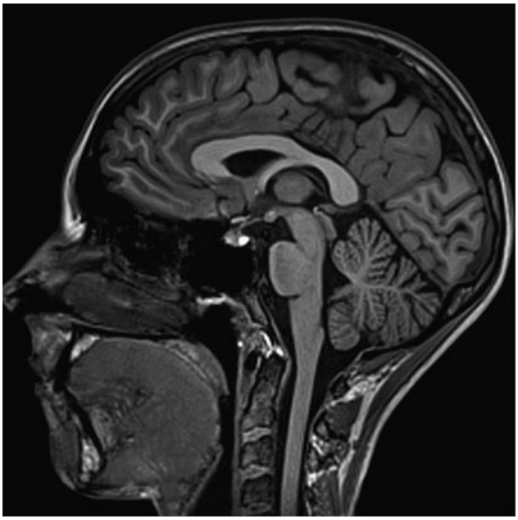

Fig. 1.8 A so-called sagittal T1 MP-RAGE image taken up by the 3 T system MAGNETOM Skyra produced by Siemens AG. With kind permission of Siemens AG Healthcare Sector

considered by Bijick (Schneider), Diehl and Renz [3, 4]. We assume that in the images space a partial order is introduced by the cone $K = \mathbb{R}^2_+$. For good MR images it is important to improve the homogeneity of the RF field for specific slices. Here we assume that the MR system uses $n \in \mathbb{N}$ antennas. The complex design variables $x_1, \ldots, x_n \in \mathbb{C}$ are the so-called scattering variables. Thus we choose $X = \mathbb{C}^n$. For a slice with $p \in \mathbb{N}$ voxels let $H^x_{k\ell}, H^y_{k\ell} \in \mathbb{C}$ (for $k \in \{1, \ldots, p\}$ and $\ell \in \{1, \ldots, n\}$) denote the cartesian components of the RF field of the k-th antenna in the ℓ-th voxel, if we work with a current of amplitude 1 A and phase 0. Then the objective function f_1 which is a standard deviation, reads as follows

$$f_1(x) := \frac{\sqrt{\frac{1}{p-1} \sum_{k=1}^{p} \left(H_k^-(x)\overline{H_k^-(x)} - \sum_{k=1}^{p} w_k H_k^-(x)\overline{H_k^-(x)} \right)^2}}{\sum_{k=1}^{p} w_k H_k^-(x)\overline{H_k^-(x)}}$$

for all $x \in \mathbb{C}^n$ with

$$H_k^-(x) := \frac{1}{2} \sum_{\ell=1}^{n} \overline{x}_\ell \overline{(H^x_{kl} - i H^y_{kl})} \text{ for all } x \in \mathbb{C}^n \text{ and } k \in \{1, \ldots, p\}$$

(here i denotes the imaginary unit and the overline means the conjugate complex number). Moreover, we would like to reduce the specific absorption rate (SAR) which is the RF energy absorbed per time unit and kilogram. Global energy absorption in the entire body is an important value for establishing safety thresholds.

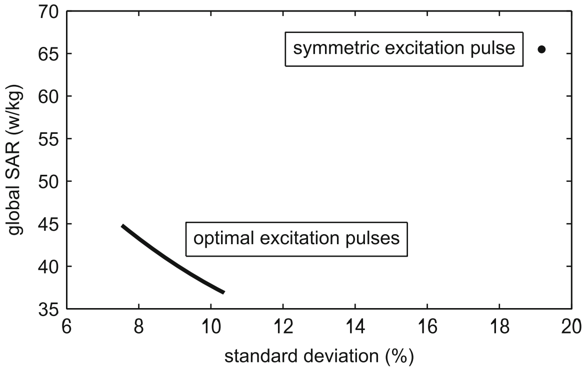

Fig. 1.9 Qualitative illustration of the image points of minimal solutions and the image point of the standard excitation pulse

If $m > 0$ denotes the mass of the patient and $M \in \mathbb{R}^{(n,n)}$ denotes the so-called scattering matrix, then the second objective function f_2 is given by

$$f_2(x) := \frac{1}{2m} x^\top (I - M^\top M)x \text{ for all } x \in \mathbb{C}^n$$

where I denotes the (n,n) identity matrix. f_2 describes the global SAR.

The constraints of this bicriterial problem describing the set S in (VOP) are given by upper bounds for the warming of the tissue within every voxel. The HUGO body model which is a typical human body model based on anatomical data of the Visible Human Project®, has more than 380,000 voxels which means that this bicriterial optimization problem has more than 380,000 constraints. A discussion of these constraints cannot be done in detail in this text. Using the so-called modified Polak method [21, Algorithm 12.1] one obtains an approximation of the image set of the set of minimal solutions of this large-scale bicriterial problem. The numerical results qualitatively illustrated in Fig. 1.9 are obtained by Bijick (Schneider, 2010, University of Erlangen-Nürnberg, Erlangen, Private communication). These results are better than the realized parameters in an ordinary MR system which uses a symmetric excitation pulse.

Notice in Fig. 1.9 that the global SAR measured in $\frac{w}{kg}$ is considered per time unit which may be very short because one considers only short RF pulses.

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Chapter 2
Gordan-Type Alternative Theorems and Vector Optimization Revisited

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2.1 Introduction and Formulation of the Problem

Alternative theorems have proved to be important in deriving key results in optimization theory like the existence of Lagrange multipliers, duality results, scalarization of vector functions, etc. Since the pioneering result due to Julius Farkas in 1902 concerning his alternative lemma which is well known in linear programming, or even the elder alternative result established by Paul Gordan in 1873, many mathematicians have made a lot of effort to generalize both results in a nonlinear setting. To these author's knowledge the first Gordan type result for convex functions is due to Fan et al. [13] and was established in 1957. Such a result says the following:

Let $K \subseteq \mathbb{R}^n$ be convex, and $f_i : K \to \mathbb{R}$, $i = 1, \ldots, m$, be convex functions. Then, exactly one of the following two systems has a solution:

(a) $f_i(x) < 0$, $i = 1, \ldots, m$, $x \in K$
(b) $p \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$, $\sum_{i=1}^m p_i f_i(x) \geq 0$ for all $x \in K$

After that, the problem without the convexity became an interesting challenge in mathematics.

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To be precise, let us consider a real locally convex topological vector space Y and a closed convex cone $P \subseteq Y$ such that $\operatorname{int} P \neq \emptyset$. We denote by Y^* the topological dual space of Y, and by P^* the (positive) polar cone of P. Given a nonempty set $A \subseteq Y$, a Gordan-type alternative theorem asserts the validity of exactly one of the following assertions:

$$\exists\, a \in A \text{ such that } a \in -\operatorname{int} P; \tag{2.1}$$

$$\exists\, p^* \in P^*,\ p^* \neq 0, \text{ such that } \langle p^*, a\rangle \geq 0 \ \ \forall\, a \in A. \tag{2.2}$$

Here $\langle\cdot,\cdot\rangle$ stands for the duality pairing between Y and Y^* and $\operatorname{int} P$ denotes the topological interior of P. We recall that P^* is defined by

$$P^* = \{p^* \in Y^* :\ \langle p^*, p\rangle \geq 0 \ \ \forall\, p \in P\}.$$

The closedness and convexity of the cone P is equivalent to $P = P^{**}$ by the bipolar theorem. In this case,

$$p \in P \iff \langle p^*, p\rangle \geq 0 \ \ \forall\, p^* \in P^*.$$

Moreover,

$$p \in \operatorname{int} P \iff \langle p^*, p\rangle > 0 \ \ \forall\, p^* \in P^* \setminus \{\mathbf{0}\}. \tag{2.3}$$

Via the last equation, we see that the inconsistency of assertions (2.1) and (2.2) is straightforward, whereas the validity of (2.2) by assuming that (2.1) does not hold, requires a careful analysis due to the lack of convexity of A.

In fact, because of many applications, one of our purposes in this chapter is to avoid convexity and to allow convex cones possibly with empty topological interior. The latter happens for instance if $(1 < p < +\infty)$

$$P = L^p_+ \doteq \{u \in L^p(\Omega) :\ u \geq 0 \quad \text{a.e. } x \in \Omega\},$$

or if P is of the form $P = Q \times \{\mathbf{0}\}$ with $\operatorname{int} Q \neq \emptyset$.

A good substitute for the interior is the quasi interior and even the quasi-relative interior. Borwein and Lewis in [5] introduced the quasi-relative interior of a convex set $A \subseteq Y$, although the concept of quasi interior was introduced earlier. We use both notions in order to deal with convex cones with possibly empty interior. In this situation, the convex hull arises naturally.

One of the main goals of the present chapter is to characterize those sets A for which the negation of (2.1) implies (2.2). The negation of (2.1) means

$$A \cap (-\operatorname{int} P) = \emptyset, \tag{2.4}$$

which is equivalent to

$$\overline{\operatorname{cone}}(A + P) \cap (-\operatorname{int} P) = \emptyset. \tag{2.5}$$

Therefore, by assuming the convexity of $\overline{\operatorname{cone}}(A+P)$, a standard separation theorem of convex sets provides the existence of p^* satisfying (2.2): this fact was proved in [42], see also [22, 32, 43] for additional sufficiency conditions of alternative theorems. In [14, Theorem 4.1] is established that such a convexity assumption is necessary and sufficient to get the implication (2.4) $\Longrightarrow$ (2.2) provided the space is two dimensional; whereas it is far to being necessary in dimension greater than or equal to three [14, Example 3.8]. We shall revise that alternative theorem in dimension two for convex cones having possibly empty interior, as well as various equivalences to the above convexity assumption.

This chapter is organized as follows. Section 2.2 gives the necessary basic definitions together with some elementary results about cones: in particular, when P is a halfspace, a complete answer to the validity of a Gordan-type alternative theorem is given, see Corollary 2.2. In Sect. 2.3, we establish several equivalent formulations to the Gordan-type alternative theorems valid for (not necessarily pointed or closed) convex cones with possibly empty (topological) interior, see Theorem 2.1 and Corollary 2.3. This is given in terms of quasi interior and quasi relative interior. We also compare various of the previously introduced notions of generalized convexity for sets and vector functions. As a consequence of these results, we are able to derive and strengthen several of the already known alternative theorems. Section 2.4 establishes an optimal alternative theorem in 2-dimension for a cone with possibly empty interior under a regularity assumption, which always holds if the interior is nonempty, see Theorem 2.4. Section 2.5 is devoted to applications. One of them is devoted to characterize those mappings $F : K \to \mathbb{R}^2$ for which a equivalence between

$$\bigcup_{p^*\in P^*\setminus\{\mathbf{0}\}} \operatorname{argmin}_K\langle p^*, F(\cdot)\rangle \;\; (\text{resp.} \bigcup_{p^*\in \operatorname{int} P^*} \operatorname{argmin}_K\langle p^*, F(\cdot)\rangle)$$

and E_W (resp. E_{pr}, the properly efficient set) holds, where E_W denotes the set of weakly efficient solutions to F on K. Such an equivalence is expected to be useful for developing a well-posedness theory in vector optimization as in [12]. In addition, as another application, we revise the Fritz–John optimality conditions for a class of nonconvex vector minimization problems. Finally, we also present some recent developments about proper efficiency.

2.2 Basic Definitions and Preliminaries

Throughout the chapter, X will be a vector space and Y a real locally convex topological vector space, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between Y and its topological dual space, Y^*. Given $x, y \in X$ we set $[x, y] = \{tx + (1-t)y : t \in [0,1]\}$. The segments $]x, y]$, $]x, y[$, etc., are defined analogously.

A set $P \subseteq Y$ is said to be a *cone* if $tP \subseteq P\ \forall\, t \geq 0$; given $A \subseteq Y$, $\operatorname{cone}(A)$ stands for the smallest cone containing A, that is,

$$\operatorname{cone}(A) = \bigcup_{t\geq 0} tA,$$

whereas $\overline{\text{cone}}(A)$ denotes the smallest closed cone containing A: obviously $\overline{\text{cone}}(A) = \overline{\text{cone}(\overline{A})}$, where $\overline{A}$ denotes the closure of A. Furthermore, we set

$$\text{cone}_+(A) \doteq \bigcup_{t>0} tA.$$

Evidently, $\text{cone}(A) = \text{cone}_+(A) \cup \{\mathbf{0}\}$ and therefore $\overline{\text{cone}(A)} = \overline{\text{cone}_+(A)}$. In [32, 33, 42, 43] the notation $\text{cone}(A)$ instead of $\text{cone}_+(A)$ is employed.

Given a convex set $A \subseteq Y$ and $x \in A$, $N_A(x)$ stands for the *normal cone* to A at x, defined by $N_A(x) = \{\xi \in Y^* : \langle \xi, a - x\rangle \le 0, \ \forall\, a \in A\}$.

Definition 2.1. We say that $x \in A$ is a (see for instance [7]):

- *Quasi interior point* of A, denoted by $x \in \text{qi}\,A$, if $\overline{\text{cone}}(A - x) = Y$, or equivalently, $N_A(x) = \{\mathbf{0}\}$;
- *Quasi relative interior point* of A, denoted by $x \in \text{qri}\,A$, if $\overline{\text{cone}}(A - x)$ is a linear subspace of Y, or equivalently, $N_A(x)$ is a linear subspace of Y^*.
- [31, 44] *core point* of A, denoted by $x \in \text{core}\,A$, if $\text{cone}(A - x) = Y$.
- [6, 18, 44] *intrinsic core point* of A, denoted by $x \in \text{icr}\,A$, if $\text{cone}(A - x)$ is a linear subspace of Y.
- [31] *strong-quasi relative interior point* of A, denoted by $x \in \text{sqri}\,A$, if $\text{cone}(A - x)$ is a closed linear subspace of Y.

For any convex set A, we have that [7, 25] $\text{qi}\,A \subseteq \text{qri}\,A$ and, $\text{int}\,A \ne \emptyset$ implies $\text{int}\,A = \text{qi}\,A$. Similarly, if $\text{qi}\,A \ne \emptyset$, then $\text{qi}\,A = \text{qri}\,A$. Moreover [5], if Y is a finite dimensional space, then $\text{qi}\,A = \text{int}\,A$ and $\text{qri}\,A = \text{ri}\,A$, where $\text{ri}\,A$ means the relative interior of A, which is the interior of A with respect to the affine hull of A. In addition,

$$\text{core}\,A \subseteq \text{sqri}\,A \subseteq \text{qri}\,A \ \text{ and } \ \text{core}\,A \subseteq \text{qi}\,A \subseteq \text{qri}\,A.$$

Let $B \subseteq Y$ another convex set. Then

$$\text{qri}\,A + \text{qri}\,B \subseteq \text{qri}(A + B);\ \text{qri}\,A \times \text{qri}\,B = \text{qri}(A \times B);\ \text{qri}(A - x) = \text{qri}\,A - x;$$

$$\text{qri}(tA) = t\text{qri}\,A \ \ \forall\, t \in \mathbb{R};\ \text{qri}\,A = A, \text{ provided } A \text{ is affine};\ \text{qri}(\text{qri}\,A) = \text{qri}\,A;$$
$$\overline{\text{qri}\,A} = \overline{A};\ \overline{\text{cone}}(\text{qri}\,A) = \overline{\text{cone}}\,A, \quad \text{if qri}\,A \ne \emptyset.$$

Thus, all results in this paper involving $\text{qi}\,A$ are also true for $\text{int}\,A$, provided the latter set is nonempty. On the other hand, the cone l^p_+ has nonempty quasi interior, but its interior (and even the relative algebraic interior) is empty for all $p \in [1, +\infty[$. Likewise, the core and even the strong quasi relative interior of L^p_+ is empty. Quasi relative interior points share some properties of the interior points; for instance, if $x \in \text{qri}\,A$ and $y \in A$ then $[x, y[\subseteq \text{qri}\,A$. In particular, $\text{qri}\,A$ is convex.

If P is a closed convex cone, then it is easy to check that $x \in \text{qi}\,P$ if and only if $\langle x^*, x\rangle > 0$ for all $x^* \in P^* \backslash \{\mathbf{0}\}$, or equivalently if the set $B = \{x^* \in P^* : \langle x^*, x\rangle = 1\}$

is a w*-closed base for P^* (we recall that a convex set B is called a base for P^* if $\mathbf{0}$ is not in the w*-closure of B and $P^* = \text{cone}(B)$). If $P \neq Y$, then $\mathbf{0} \notin \text{qi } P$. Note also that $\text{qi } P = \text{cone}_+(\text{qi } P)$ and $P + \text{qi } P = \text{qi } P$.

In the rest of the chapter, $\{\mathbf{0}\} \neq P \subsetneq Y$ will be a convex cone.

Some elementary properties of sets and cones are collected in the next proposition.

Proposition 2.1. *Let $A, K \subseteq Y$ be any nonempty sets.*

(a) $\overline{\text{co}}(A) = \overline{\text{co}}(\overline{A})$, $\overline{\text{cone}}(\overline{A}) = \overline{\text{cone}}(A)$.
(b) *if A is open then* $\text{cone}_+(A)$ *is open.*
(c) $\text{cone}(\text{co}(A)) = \text{co}(\text{cone}(A))$ $\text{cone}_+(\text{co}(A)) = \text{co}(\text{cone}_+(A))$.
(d) $\text{co}(A+K) = \text{co}(A) + K$ *provided K is convex.*
(e) $\text{cone}_+(A+K) = \text{cone}_+(A) + K$ *provided K is such that $tK \subseteq K \ \forall\, t > 0$.*
(f) $\overline{A+K} = \overline{A+\overline{K}}$.
(g) $K \subseteq \overline{\text{cone}}(A+K)$ *provided K is a cone.*
(h) $\text{cone}(A+K) \subseteq \text{cone}(A) + K \subseteq \overline{\text{cone}}(A+K)$ *provided K is a cone; if additionally $\mathbf{0} \in A$, then*

$$\text{cone}(A+K) = \text{cone}(A) + K;$$

In the following, K is a convex cone such that $\text{int } K \neq \emptyset$.
(i) $\overline{A + \text{int } K} = \overline{A+K}$, $\text{int}\overline{A+K} = A + \text{int} K = \text{int}(A+K)$.
(j) $\overline{\text{cone}}(A + \text{qri } P) = \overline{\text{cone}}(A+P)$, *provided P is a convex cone with* $\text{qri } P \neq \emptyset$.
(k) $\text{cone}_+(A + \text{int } K)$ *is convex* $\Longleftrightarrow$ $\text{cone}(A) + \text{int } K$ *is convex* $\Longleftrightarrow$ $\overline{\text{cone}}(A+K)$ *is convex.*

Proof. (a), (b), (c), (d), and (e) are straightforward.

(f) Since $K \subseteq \overline{K}$, we have $\overline{A+K} \subseteq \overline{A+\overline{K}}$. On the other hand, it is not difficult to obtain $A + \overline{K} \subseteq \overline{A+K}$, which completes both inclusions.
(g) For any fix $a \in A$, every $x \in K$ can be obtained as the limit of $\frac{1}{n}(a+nx)$. Hence $K \subseteq \overline{\text{cone}}(A+K)$.
(h) The first inclusion is obvious. According to (e), $\text{cone}_+(A)+K = \text{cone}_+(A+K) \subseteq \overline{\text{cone}}(A+K)$, which along with (g) prove the second inclusion. The remaining equality is trivial.
(i) The first part follows from (f), and the other is in [8, 36].
(j) $\overline{\text{cone}}(A + \text{qri } P) = \overline{\text{cone}}(\overline{A + \text{qri } P}) = \overline{\text{cone}}(\overline{A + \overline{\text{qri } P}}) = \overline{\text{cone}}(\overline{A+\overline{P}}) = \overline{\text{cone}}(A+P)$.
(k) By (e) and (i),

$$\begin{aligned}\text{cone}_+(A+\text{int } K) &= \text{cone}_+(A)+\text{int } K = \text{int}(\text{cone}_+(A)+K) = \text{int}(\overline{\text{cone}_+(A)+K})\\ &= \text{int}(\overline{\text{cone}(A)+K}) = \text{int}(\text{cone}(A)+K) = \text{cone}(A)+\text{int } K.\end{aligned}$$

This proves the first equivalence. We also obtain

$$\text{int}(\overline{\text{cone}}(A+K)) = \text{int}(\overline{\text{cone}_+(A)+K}) = \text{cone}_+(A)+\text{int } K = \text{cone}_+(A+\text{int } K),$$

proving the equivalence between the first and third sets. □

Remark 2.1. Proposition 2.1 (k) does not hold with qi P in the place of int P. Indeed, let $Y = l^1$ and $P = l^1_+$. Then qi $l^1_+ = \{(\alpha_i)_{i\in\mathbb{N}} : \alpha_i > 0\}$ while int $l^1_+ = \emptyset$. Set

$$A = l^1 \setminus \left(-\text{qi } l^1_+\right) = \left\{(\alpha_i)_{i\in\mathbb{N}} : \exists\, i \in \mathbb{N} \text{ with } \alpha_i \geq 0\right\}.$$

Each $(a_i)_{i\in\mathbb{N}} \in l^1$ can be written as a limit of a sequence of elements each of which has a finite number of nonzero coordinates. Thus $\overline{A} = l^1$ and $\overline{\text{cone}}(A + l^1_+) = l^1$ is convex. However, one can readily check that $\text{cone}_+(A + \text{qi } P) = A + \text{qi } P = \left\{(\alpha_i)_{i\in\mathbb{N}} : \exists\, i \in \mathbb{N} \text{ with } \alpha_i > 0\right\}$ is not convex.

Proposition 2.2. *Let $\emptyset \neq A \subseteq Y$. The following assertions hold:*

(a) $\alpha A + (1-\alpha)A \subseteq \overline{\text{cone}}(A)\ \forall\ \alpha \in\ ]0,1[\ \Longleftrightarrow \overline{\text{cone}}(A)$ is convex $\Longleftrightarrow \text{co}(A) \subseteq \overline{\text{cone}}(A)$
(b) $\alpha A + (1-\alpha)A \subseteq \text{cone}(A)\ \forall\ \alpha \in\ ]0,1[\ \Longleftrightarrow \text{cone}(A)$ *is convex* $\Longleftrightarrow \text{co}(A) \subseteq \text{cone}(A)$
(c) $\alpha A + (1-\alpha)A \subseteq \text{cone}_+(A)\ \forall\ \alpha \in\ ]0,1[\ \Longleftrightarrow \text{cone}_+(A)$ *is convex* $\Longleftrightarrow \text{co}(A) \subseteq \text{cone}_+(A)$

Proof. (a) Let x_i, $i = 1,2$, such that there are nets $\{t_i^\alpha\}_{\alpha\in\Lambda}$, $\{x_i^\alpha\}_{\alpha\in\Lambda}$ such that $t_i^\alpha \geq 0$, $x_i^\alpha \in A$ and $t_i^\alpha x_i^\alpha \to x_i$, $\alpha \in \Lambda$. We may assume $t_i^\alpha > 0$ for all α, $i = 1,2$. For any fixed $\lambda \in\ ]0,1[$, set $t^\alpha = \lambda t_1^\alpha + (1-\lambda)t_2^\alpha > 0$. Then

$$\begin{aligned}\lambda t_1^\alpha x_1^\alpha + (1-\lambda)t_2^\alpha x_2^\alpha &= t^n \left(\frac{\lambda t_1^\alpha}{t^\alpha} x_1^\alpha + \frac{(1-\lambda)t_2^\alpha}{t^\alpha} x_2^\alpha\right) \in t^\alpha \text{co}\, A \subseteq t^\alpha \overline{\text{cone}(A)} \\ &= \overline{\text{cone}(A)}.\end{aligned}$$

Hence, $\lambda x_1 + (1-\lambda)x_2 \in \overline{\text{cone}(A)}$. This proves the first implication; the next one results from the inclusion $A \subseteq \overline{\text{cone}}(A)$, and the remaining implication to close the circle is a consequence of $\alpha A + (1-\alpha)A \subseteq \text{co}(A)$.

(b) Let x_1, x_2 be in $\text{cone}(A)$ and $\lambda \in\ ]0,1[$. Then $x_i = t_i k_i$ for some $t_i > 0$ (if $t_i = 0$ for some i, there is nothing to prove), $k_i \in A$. Hence, setting $t = \lambda t_1 + (1-\lambda)t_2 > 0$, we have

$$\lambda x_1 + (1-\lambda)x_2 = \lambda t_1 k_1 + (1-\lambda)t_2 k_2 = t\left(\frac{\lambda t_1}{t} k_1 + \frac{(1-\lambda)t_2}{t} k_2\right) \in \text{cone}(A),$$

proving the first implication; the second one follows from the inclusion $A \subseteq \text{cone}(A)$, whereas the remaining implication is straightforward.

(c) The only implication to close the circle we have to check corresponds to the first one, and it is a consequence of (b) since $\text{cone}(A)$ is convex if and only if $\text{cone}_+(A)$ is convex. □

Corollary 2.1. *Let $\emptyset \neq A \subseteq Y$, P be a convex cone. The following assertions hold:*

(a) $\alpha A + (1-\alpha)A \subseteq \text{cone}_+(A+P)\ \ \forall\ \alpha \in\ ]0,1[\ \Leftrightarrow \text{cone}_+(A+P)$ *is convex* $\Leftrightarrow \text{co}(A) \subseteq \text{cone}_+(A+P)$

(b) $\alpha A + (1-\alpha)A + P \subseteq \operatorname{cone}(A+P) \;\; \forall\, \alpha \in \;]0,1[\;\; \Leftrightarrow \operatorname{cone}(A+P)$ *is convex* $\Leftrightarrow \operatorname{co}(A) + P \subseteq \operatorname{cone}(A+P)$

(c) $\alpha A + (1-\alpha)A + \operatorname{int} P \subseteq \operatorname{cone}_+(A+\operatorname{int} P) \;\forall\, \alpha \in \;]0,1[\;\Leftrightarrow \operatorname{cone}_+(A+\operatorname{int} P)$ *is convex* $\Leftrightarrow \operatorname{co}(A) + \operatorname{int} P \subseteq \operatorname{cone}_+(A+\operatorname{int} P)$

(d) $\alpha A + (1-\alpha)A \subseteq \operatorname{cone}_+(A+\operatorname{int} P) \;\; \forall\, \alpha \in \;]0,1[\;\Leftrightarrow \operatorname{cone}_+(A+\operatorname{int} P)$ *is convex* $\Leftrightarrow \operatorname{co}(A) \subseteq \operatorname{cone}_+(A+\operatorname{int} P)$

(e) $\alpha A + (1-\alpha)A \subseteq \operatorname{cone}(A)+P \;\forall\, \alpha \in \;]0,1[\;\Leftrightarrow \operatorname{cone}(A)+P$ *is convex* $\Leftrightarrow \operatorname{co}(A) \subseteq \operatorname{cone}(A)+P$

(f) $\alpha A + (1-\alpha)A \subseteq \overline{\operatorname{cone}}(A+P) \,\forall\, \alpha \in \;]0,1[\;\Leftrightarrow \overline{\operatorname{cone}}(A+P)$ is convex $\Leftrightarrow \operatorname{co}(A) \subseteq \overline{\operatorname{cone}}(A+P)$

Proof. (a), (b) follow from (c) and (b), respectively, of the previous proposition applied to $A+P$; (c) is a consequence of (c) by taking $A+\operatorname{int} P$.

(d) follows from (c) of the previous proposition. (e) One implication for the second equivalence results from the inclusions $A \subseteq \operatorname{cone}(A) \subseteq \operatorname{cone}(A)+P$; whereas the other follows from the following (use Proposition 2.1(c))

$$\begin{aligned}\operatorname{co}(\operatorname{cone}(A)+P) &= \operatorname{co}(\operatorname{cone}(A))+P = \operatorname{cone}(\operatorname{co}(A))+P \\ &\subseteq \operatorname{cone}(\operatorname{cone}(A)+P)+P \subseteq \operatorname{cone}(A)+P.\end{aligned}$$

The first equivalence is straightforward.

(f) It comes from (a) of the previous proposition applied to $A+P$ and the fact that $P+\overline{\operatorname{cone}}(A+P) \subseteq \overline{\operatorname{cone}}(A+P)$. □

Part (f) already appeared in [32].

Remark 2.2. From Proposition 2.1 (h), we obtain

$$\overline{\operatorname{cone}}(A+\operatorname{int} P) = \overline{\operatorname{cone}(A)+P} = \overline{\operatorname{cone}}(A+P). \tag{2.6}$$

The next proposition gives us a way for finding a sufficient condition to get $\operatorname{co}(A) \cap (-\operatorname{int} P) = \emptyset$, say, the convexity of $\overline{\operatorname{cone}}(A+P)$.

Proposition 2.3. *Let $A \subseteq Y$ be a nonempty set and $P \subsetneq Y$ be a convex cone such that* $\operatorname{int} P \neq \emptyset$. *The following assertions hold:*

(a) $A \cap (-\operatorname{int} P) = \emptyset \Longleftrightarrow \operatorname{cone}_+(A) \cap (-\operatorname{int} P) = \emptyset \Longleftrightarrow \overline{A} \cap (-\operatorname{int} P) = \emptyset$

(b) $A \cap (-\operatorname{int} P) = \emptyset \Longleftrightarrow A_0 \cap (-\operatorname{int} P) = \emptyset,\; \forall\, A_0,\; A+\operatorname{int} P \subseteq A_0 \subseteq \operatorname{cone}_+(A+P)$

Proof. It is straightforward. □

Remark 2.3. On combining (a) and (b), we obtain

$$A \cap (-\operatorname{int} P) = \emptyset \Longleftrightarrow B \cap (-\operatorname{int} P) = \emptyset,$$

for $B = A+\operatorname{int}\, P, A+P, \operatorname{cone}_+(A), \operatorname{cone}(A)+P, \operatorname{cone}(A+P), \operatorname{cone}_+(A+\operatorname{int}\, P)$, $\operatorname{cone}(A)$, $\operatorname{cone}(A)+\operatorname{int} P$, and certainly all of their closures.

The case when P is a halfspace deserves a special formulation.

Lemma 2.1. *Let* $P \subsetneq Y$ *be a closed and convex cone satisfying* $\operatorname{int} P \neq \emptyset$. *The following assertions are equivalent:*

(a) $P = Y \setminus -\operatorname{int} P$
(b) $P \cup (-P) = Y$
(c) $\exists\, p^* \in P^* \setminus \{\mathbf{0}\}$, $P = \Big\{ p \in Y : \langle p^*, p\rangle \geq 0 \Big\}$

Proof. (a) ⇒ (b): Obviously $P \cup (-P) \subseteq Y$. Take any $y \in Y \setminus P$, then by assumption $y \in -\operatorname{int} P \subseteq -P$, as required.

(b) ⇒ (c): Since $P \neq Y$, we take $p_0 \notin P$. Then, by an usual separation theorem for convex sets, there exist $p^* \in Y^*$, $p^* \neq \mathbf{0}$, $\alpha \in \mathbb{R}$, such that

$$\langle p^*, p_0\rangle < \alpha < \langle p^*, p\rangle \;\; \forall\, p \in P.$$

Hence $\alpha < 0$ and therefore $\langle p^*, p\rangle \geq 0$ for all $p \in P$, showing that $p^* \in P^* \setminus \{\mathbf{0}\}$ and

$$P \subseteq \Big\{ p \in Y : \; \langle p^*, p\rangle \geq 0 \Big\}. \tag{2.7}$$

Assume now that there exists $p \in Y \setminus P$ such that $\langle p^*, p\rangle \geq 0$. Since P is closed, there exists $\varepsilon > 0$ such that $p - \varepsilon p_0 \in Y \setminus P \subseteq -P$. Thus,

$$0 \leq \langle p^*, -p\rangle + \varepsilon \langle p^*, p_0\rangle \leq \varepsilon \langle p^*, p_0\rangle < 0,$$

reaching a contradiction. This proves the reverse inclusion in (2.7), which completes the proof of (c).

(c) ⇒ (a): Simply take into account that in this case $\operatorname{int} P = \{p \in Y : \langle p^*, p\rangle > 0\}$. □

If P is a halfspace we obtain an alternative theorem whatever the set A satisfies $A \cap (-\operatorname{int} P) = \emptyset$, as the following result shows.

Corollary 2.2. *Let* $A \subseteq Y$ *be any nonempty set, and* $P \subsetneq Y$ *be a closed convex cone satisfying* $P \cup (-P) = Y$. *Then,* $\operatorname{int} P \neq \emptyset$, $P = Y \setminus -\operatorname{int} P$, *and* $A + P$ *is convex and so* $A + \operatorname{int} P$ *is also convex. Consequently, the sets* $\operatorname{cone}(A+P)$, $\operatorname{cone}(A + \operatorname{int} P)$, $\operatorname{cone}(A) + P$, *are convex. Furthermore, either* $\overline{\operatorname{cone}}(A+P) = P$, *or* $\operatorname{cone}_+(A + \operatorname{int} P) = Y$ *and therefore* $\overline{\operatorname{cone}}(A+P) = Y$.
Moreover, the following assertions are equivalent.

(a) $A \cap (-\operatorname{int} P) = \emptyset$
(b) $A \subseteq P$
(c) $\operatorname{co}(A) \subseteq P$
(d) $\operatorname{co}(A) \cap (-\operatorname{int} P) = \emptyset$
(e) $\operatorname{co}(A + \operatorname{int} P) \subseteq \operatorname{int} P$

Proof. Obviously $Y \setminus P \subseteq -P$, and since P is closed, we conclude that $\operatorname{int} P \neq \emptyset$. Let $a_i \in A$, $i = 1,2$. We may assume $a_1 \in a_2 + P$. Since $a_2 + P$ is convex, we obtain that $[a_1, a_2] \subseteq a_2 + P$. Thus, $[a_1, a_2] + P \subseteq a_2 + P + P \subseteq a_2 + P$. This proves the convexity of $A + P$, and so $\operatorname{int}(A+P) = A + \operatorname{int} P$ is also convex. The convexity of $\operatorname{cone}(A) + P$ is a consequence of Corollary 2.1 (e) since $\operatorname{cone}_+(A+P)$ is convex.

Let us prove the last part. By (g) of Proposition 2.1, $P \subseteq \overline{\operatorname{cone}}(A+P)$. If $\overline{\operatorname{cone}}(A+P) \setminus P \neq \emptyset$, then there exists $x \in Y \setminus P = -\operatorname{int} P$ and nets $\{t_\alpha\}_{\alpha \in \Lambda}$, $\{a_\alpha\}_{\alpha \in \Lambda}$, $\{p_\alpha\}_{\alpha \in \Lambda}$ satisfying $t_\alpha > 0$, $a_\alpha \in A$, $p_\alpha \in P$ such that $t_\alpha(a_\alpha + p_\alpha) \to x$. Thus, we may assume $t_\alpha(a_\alpha + p_\alpha) \in -\operatorname{int} P$ for all $\alpha \in \Lambda$. This implies that $\mathbf{0} \in A + \operatorname{int} P$. It turns out that $\operatorname{cone}_+(A + \operatorname{int} P) = Y$.

The equivalences between (a), (b), (c) and (d), follow from the fact $P = Y \setminus -\operatorname{int} P$ (see the previous lemma). Clearly (b) implies (e); let us prove (a) from (e): if $x \in A \cap (-\operatorname{int} P)$ then

$$\mathbf{0} = x + (-x) \in A + \operatorname{int} P \subseteq \operatorname{co}(A + \operatorname{int} P) \subseteq \operatorname{int} P.$$

Thus, $\mathbf{0} \in \operatorname{int} P$, which implies that $P = Y$, a contradiction. □

2.3 Equivalent Formulations of Gordan-Type Alternative Theorems

The main goal of this section is to establish equivalent formulations of Gordan-type alternative theorems valid for (not necessarily pointed or closed) convex cones with possibly empty interior. This will be carried out via quasi relative and topological interior.

We recall the definition of pointedness for a cone that is not necessarily convex (see for instance [30]).

Definition 2.2. A cone $K \subseteq Y$ is called *pointed* if $x_1 + \cdots + x_k = \mathbf{0}$ is impossible for $x_1, x_2, \ldots, x_k$ in K unless $x_1 = x_2 = \cdots = x_k = \mathbf{0}$.

It is easy to see that a cone K is pointed if, and only if $\operatorname{co}(K) \cap (-\operatorname{co}(K)) = \{\mathbf{0}\}$ if, and only if $\mathbf{0}$ is a extremal point of $\operatorname{co}(K)$.

2.3.1 Via Quasi-Relative Interior

We start by noticing that

$$\operatorname{qri}(\operatorname{cone}(\operatorname{co}(A) + P)) \subseteq \operatorname{qri}(\overline{\operatorname{cone}}(\operatorname{co}(A) + P)). \tag{2.8}$$

Next theorem subsumes most alternative theorems existing in the literature.

Theorem 2.1. *Let $\emptyset \neq A \subseteq Y \neq \{\mathbf{0}\}$ and P be a convex cone such that*

$$\text{qri}\,(\text{cone}\,(\text{co}(A)+P)) \neq \emptyset \neq \text{qri}\,[\text{co}\,((A+P)\cup\{\mathbf{0}\})]\,.$$

Let us consider the following statements:

(a) $\mathbf{0} \notin \text{qri}\,(\overline{\text{cone}}\,(\text{co}(A)+P))$
(b) $\mathbf{0} \notin \text{qri}\,[\text{co}\,((A+P)\cup\{\mathbf{0}\})]$
(c) $\mathbf{0} \notin \text{qri}\,(\text{cone}\,(\text{co}(A)+P))$
(d) $\exists\ p^* \in P^* \setminus \{\mathbf{0}\}$ *such that* $\langle p^*, a\rangle \geq 0\ \ \forall\ a \in A$, *with strict inequality for some* $\tilde{a} \in \text{co}(A)+P$

In case $\text{qi}\,(\text{co}(A)+P) \neq \emptyset$, *consider also*

(e) $\mathbf{0} \notin \text{qri}\,(\text{co}(A)+P)$
(f) $\text{cone}\,(\text{qri}\,(\text{co}(A)+P))$ *is pointed*

The following hold:

$$(a) \Longleftrightarrow (b) \Longleftrightarrow (c) \Longleftrightarrow (d) \Longrightarrow (e) \Longrightarrow (f) \Longrightarrow (g).$$

Proof. The first two equivalences are a consequences of the following equalities:

$$\begin{aligned}\overline{\text{cone}}[\text{co}((A+P)\cup\{\mathbf{0}\})] &= \overline{\text{co}[\text{cone}((A+P)\cup\{\mathbf{0}\})]}\\ &= \overline{\text{co}[\text{cone}(A+P)]} = \overline{\text{cone}}[\text{co}(A+P)]\\ &= \overline{\text{cone}}[\text{cone}(\text{co}(A)+P)]\\ &= \overline{\text{cone}}[\overline{\text{cone}}(\text{co}(A)+P)].\end{aligned}$$

$(c) \Longleftrightarrow (d)$: See the proof of Proposition 2.16 in [5].
$(c) \Longrightarrow (e)$: It is obvious.
$(e) \Longrightarrow (f)$: Let $x,\ -x \in \text{cone}(\text{qi}(\text{co}(A)+P))$, $x \neq \mathbf{0}$. Thus, $x,\ -x \in \text{cone}_+$ $(\text{qri}(\text{co}(A)+P)$. Then

$$\mathbf{0} = \frac{1}{2}x + \frac{1}{2}(-x) \in \text{cone}_+(\text{qri}(\text{co}(A)+P)).$$

Hence, $\mathbf{0} \in \text{qri}(\text{co}(A)+P)$, proving the desired implication. □

Remark 2.4. Assume that $\text{qi}\,P \neq \emptyset$. Since $\text{co}(A)+\text{qi}\,P \subseteq \text{qi}(\text{co}(A)+P)$, then

$$\begin{array}{ccc}\text{cone}(\text{qi}(\text{co}(A)+P))\ \text{ is pointed} & \Longrightarrow & \text{cone}(\text{co}(A)+\text{qi}\,P)\ \text{ is pointed}\\ & & \Updownarrow\\ \text{co}(A)\cap(-\text{qi}\,P)=\emptyset & \Longleftrightarrow & \text{cone}(A+\text{qi}\,P)\ \text{ is pointed}\end{array}$$

where the last equivalence comes from [14].

By observing that

$$\mathrm{qi}[\mathrm{co}((A+P)\cup\{\mathbf{0}\})]\subseteq \mathrm{qi}(\mathrm{cone}(\mathrm{co}(A)+P)), \tag{2.9}$$

the preceding theorem implies the following result

Corollary 2.3. *Let $\emptyset \neq A \subseteq Y \neq \{\mathbf{0}\}$ and P be a convex cone such that*

$$\mathrm{qi}[\mathrm{co}((A+P)\cup\{\mathbf{0}\})]\neq\emptyset.$$

The following assertions are equivalent:

(a) $\mathbf{0}\notin \mathrm{qi}(\overline{\mathrm{cone}}(\mathrm{co}(A)+P))$
(b) $\mathbf{0}\notin \mathrm{qi}[\mathrm{co}((A+P)\cup\{\mathbf{0}\})]$
(c) $\mathbf{0}\notin \mathrm{qi}(\mathrm{cone}(\mathrm{co}(A)+P))$
(d) $\exists\, p^*\in P^*\setminus\{\mathbf{0}\}$ *such that* $\langle p^*,a\rangle\geq 0,\ \forall\, a\in A$

2.3.2 Via Topological Interior

Before establishing a similar result for topological interior, we state the following properties sharing by convex cones.

Proposition 2.4. *Let $\emptyset\neq A\subseteq Y$. Let $P\subsetneq Y$ be a convex cone. The following assertions hold.*

(a) $\mathrm{cone}_+(\mathrm{int}(A+P))\subseteq \mathrm{int}(\mathrm{cone}_+(A+P))$; *the equality holds provided* $\mathrm{int}\,P\neq\emptyset$.
(b) $\mathrm{int}(\mathrm{co}(\overline{\mathrm{cone}}(A+P)))=\mathrm{int}(\mathrm{co}(\mathrm{cone}_+(A+P)))=\mathrm{int}(\mathrm{co}(\mathrm{cone}(A+P)))=$

$$=\mathrm{int}(\overline{\mathrm{cone}}(\mathrm{co}(A)+P)).$$

(c) $\overline{\mathrm{cone}}(A+P)=\overline{\mathrm{cone}}((A\cup\{\mathbf{0}\})+P)$;
(d) *If* $\overline{\mathrm{cone}}(A+P)$ *is convex then* $\overline{\mathrm{cone}}(\mathrm{co}(A)+P)=\overline{\mathrm{cone}}(A+P)$.

Proof. (a) The inclusion is immediate. For the other, take any $x\in\mathrm{int}(\mathrm{cone}_+(A+P))$ and $v\in\mathrm{int}\,P$, we can choose $\varepsilon>0$ such that $x-\varepsilon v\in\mathrm{cone}_+(A+P)$. It follows easily that $x\in\mathrm{cone}_+(\mathrm{int}(A+P))=\mathrm{cone}_+(A+\mathrm{int}\,P)$ by Proposition 2.1(e), proving our claim.

(b) It follows from the following chain of inclusions:

$$\begin{aligned}\mathrm{int}(\mathrm{co}(\overline{\mathrm{cone}}(A+P)))&\subseteq \mathrm{int}(\overline{\mathrm{cone}}(\mathrm{co}(A+P)))=\mathrm{int}(\overline{\mathrm{cone}_+}(\mathrm{co}(A+P)))\\&=\mathrm{int}(\mathrm{cone}_+(\mathrm{co}(A+P)))\subseteq\mathrm{int}(\mathrm{cone}(\mathrm{co}(A+P)))\\&=\mathrm{int}(\mathrm{co}(\mathrm{cone}(A+P)))\subseteq\mathrm{int}(\mathrm{co}(\overline{\mathrm{cone}}(A+P))).\end{aligned}$$

(c) This follows from the fact $P\subseteq\overline{\mathrm{cone}}(A+P)$ by Proposition 2.1(d). □

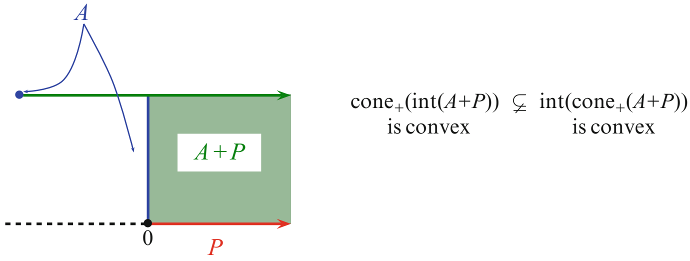

Fig. 2.1 Example 2.1(a)

Next example shows an instance where the inclusion in (a) of the previous proposition may be strict if $\operatorname{int} P = \emptyset$ but $\operatorname{int}(A+P) \neq \emptyset$; the second instance shows we cannot delete the closures in (c).

Example 2.1. Let us consider in $\mathbb{R}^2$, the cone $P = \{(t,0) \in \mathbb{R}^2 : \ t \geq 0\}$.

(a) Let $A = \operatorname{co}(\{(0,1),(0,0)\}) \cup \{(-1,1)\}$. It is easy to see that $\operatorname{int}(A+P) \neq \emptyset$, $\operatorname{cone}_+(\operatorname{int}(A+P)) \subsetneq \operatorname{int}(\operatorname{cone}_+(A+P))$. See Fig. 2.1.
(b) Take $A = \{(0,1),(0,2)\}$. Then, we obtain $\operatorname{cone}(A+P) \subsetneq \operatorname{cone}((A \cup \{\mathbf{0}\})+P)$.

Next result is the analogue to Theorem 2.1 when topological interior is employed. Likewise, it allows us to deal with cones having possibly empty interior.

Theorem 2.2. *Let $\emptyset \neq A \subseteq Y \neq \{\mathbf{0}\}$ and P be a convex cone such that*

$$\operatorname{int}[\operatorname{co}((A+P) \cup \{\mathbf{0}\})] \neq \emptyset.$$

The following statements are equivalent:

(a) $\mathbf{0} \notin \operatorname{int}(\overline{\operatorname{cone}}(\operatorname{co}(A)+P))$
(b) $\mathbf{0} \notin \operatorname{int}(\operatorname{co}(\overline{\operatorname{cone}}(A+P)))$
(c) $\mathbf{0} \notin \operatorname{int}(\operatorname{co}(\operatorname{cone}(A+P)))$
(d) $\mathbf{0} \notin \operatorname{int}[\operatorname{co}((A+P) \cup \{\mathbf{0}\})]$
(e) $\mathbf{0} \notin \operatorname{int}(\operatorname{co}(\operatorname{cone}_+(A+P)))$
(f) $\exists\, p^* \in P^* \setminus \{\mathbf{0}\}$ *such that* $\langle p^*, a\rangle \geq 0 \ \ \forall\, a \in A$

In case $\operatorname{int}(\operatorname{co}(A)+P) \neq \emptyset$, *consider also*

(g) $\mathbf{0} \notin \operatorname{int}(\operatorname{co}(A)+P)$
(h) $\operatorname{cone}(\operatorname{int}(\operatorname{co}(A)+P))$ *is pointed*

In case $\operatorname{int} P \neq \emptyset$, $(h) \Longleftrightarrow \operatorname{cone}(A+\operatorname{int} P)$ is pointed; $(g) \Longleftrightarrow \operatorname{co}(A) \cap (-\operatorname{int} P) = \emptyset$.

Proof. The equivalences among (a), (b), (c), (d), (e) and (f) follows from Corollary 2.3 and Theorem 2.1.

(h) $\Rightarrow$ (g): If $\mathbf{0} \in \operatorname{int}(\operatorname{co}(A)+P)$, then it easy to check that $Y = \operatorname{cone}_+(\operatorname{int}(\operatorname{co}(A)+P))$.
(g) $\Rightarrow$ (f): It is a consequence of a standard convex separation theorem.

We now prove the last equivalence in case $\text{int}\, P \neq \emptyset$. Indeed, from (a) and (e) of Proposition 2.1, it follows that $\text{cone}(\text{int}(\text{co}(A)+P)) = \text{co}(\text{cone}(A+\text{int}\, P))$. Taking into the account the remark made after Definition 2.2, the result follows □

The alternative theorems proved in [33, 42, 43], [10, Theorem 1.79] (where A is the image of a vector-valued function) are consequences of the following result.

Theorem 2.3. *Assume that* $\text{int}(\text{cone}_+(A+P)) \neq \emptyset$ *and*

$$\text{int}(\text{co}(\overline{\text{cone}}(A+P))) = \text{int}(\text{cone}_+(A+P)). \tag{2.10}$$

Then, exactly one of the following assertions holds:

(a) $\mathbf{0} \in \text{int}(\text{cone}_+(A+P))$
(b) $\exists\, p^* \in P^* \setminus \{\mathbf{0}\}$ *such that* $\langle p^*, a\rangle \geq 0 \;\; \forall\, a \in A$

Proof. It is a consequence of the first part of Theorem 2.2. □

Some results from [16], where $\text{int}\, P = \emptyset$, are also recovered.

When $\text{int}\, P \neq \emptyset$, the convexity of $\overline{\text{cone}}(A+P)$, or equivalently, of $\text{cone}_+(A+\text{int}\, P)$, implies that (2.10) is fulfilled, by virtue of Propositions 2.4(b) and 2.1(k). This yields the following result, which already appears in [33, 42, 43], [10, Theorem 1.79] (where A is the image of a vector-valued function).

Corollary 2.4. *Assume that* $\text{int}\, P \neq \emptyset$. *If* $\overline{\text{cone}}(A+P)$ *is convex, then, exactly one of the following assertions holds:*

(a) $A \cap (-\text{int}\, P) \neq \emptyset$
(b) $\exists\, p^* \in P^* \setminus \{\mathbf{0}\}$ *such that* $\langle p^*, a\rangle \geq 0 \;\; \forall\, a \in A$

An example showing the convexity of $\overline{\text{cone}}(A+P)$ is not necessary for the validity of the previous alternative theorem, is exhibited in [14].

Let us consider in addition to $F: C \to Y$ and a closed convex cone $P \subsetneq Y$ with $\text{int}\, P \neq \emptyset$, another mapping $G: C \to Z$, with Z being another real locally convex topological vector space and a closed convex cone $Q \subsetneq Z$.

Corollary 2.5 ([33]). *Assume that* $\overline{\text{cone}}((F\times G)(C) + (P\times Q))$ *is convex and*

$$\text{int}(\overline{\text{cone}}((F\times G)(C)+(P\times Q))) = \text{int}(\text{cone}_+((F\times G)(C)+(P\times Q))) \neq \emptyset.$$

If the following system is inconsistent:

$$x \in C,\; F(x) \in -\text{int}\, P,\; G(x) \in -Q,$$

then there exists $(p^*, q^*) \in (P^* \times Q^*) \setminus \{(\mathbf{0},\mathbf{0})\}$*such that*

$$\langle p^*, F(x)\rangle + \langle q^*, G(x)\rangle \geq 0 \;\; \forall\, x \in C.$$

The converse assertion is true if $p^* \neq \mathbf{0}$.

Proof. If the above system has no solution, then

$$(\mathbf{0},\mathbf{0}) \notin \text{int}(\text{cone}_+((F\times G)(C)+(P\times Q))).$$

Then, Theorem 2.2 applies. □

A standard constraint qualification implying $p^* \neq \mathbf{0}$ is $\overline{\text{cone}}(G(C)+Q) = Z$. When int $Q \neq \emptyset$ the latter is implied by the condition: $G(x_0) \in -\text{int } Q$ for some $x_0 \in C$.

In view of previous results, the following notion arise in a natural way. It seems to be the most general among the relaxed notions of convexity that were used in alternative theorems.

Definition 2.3. Let $P \subseteq Y$ be a closed convex cone with nonempty interior. A set $A \subseteq Y$ is called *nearly subconvexlike* if $\overline{\text{cone}}(A+P)$ is convex.

The previous notion was introduced originally in [42] when A is the image of set-valued mappings, and further developed in [32].

Proposition 2.2(a) provides a characterization of near subconvexlikeness already appeared in [32]. When int $P \neq \emptyset$, several necessary and sufficient conditions for having near subconvexlikeness appear in [10, Proposition 1.76] and [14, Proposition 3.5]. In particular, the presubconvexlikeness which is a transcription of an analogous definition for vector-valued functions given in [45], is nothing else that nearly subconvexlikeness, see Proposition 2.6 below. We also know that (Proposition 2.1(k))

$$\overline{\text{cone}}(A+P) \text{ is convex} \Longleftrightarrow \text{cone}_+(\text{int}(A+P)) \text{ is convex}.$$

However, if int $P = \emptyset$ but $\text{int}(A+P) \neq \emptyset$, one can show that there is not any relationship between the convexity of $\text{cone}_+(\text{int}(A+P))$ and the convexity of $\overline{\text{cone}}(A+P)$, see Figs. 2.2 and 2.3.

Another interesting class of mappings arising in deriving alternative theorems is the following. Given a convex set $C \subseteq X$, with X being a locally convex topological vector space, a mapping $F : C \to Y$ is called $*$-*quasiconvex* [22] if $\langle x^*, F(\cdot)\rangle$ is quasiconvex for all $x^* \in P^*$. It is called *naturally-P-quasiconvex* [38] if for all $x, y \in C$, $F([x,y]) \subseteq [F(x),F(y)] - P$. Both classes coincide as shows in [14, Proposition 3.9], [15, Theorem 2.3]. It is still valid if P has empty interior.

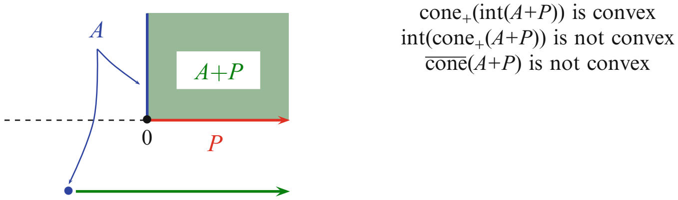

Fig. 2.2 $\text{cone}_+(\text{int}(A+P))$ convex $\not\Rightarrow \overline{\text{cone}}(A+P)$ convex

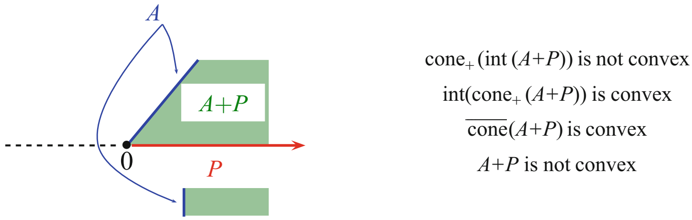

Fig. 2.3 $\overline{\text{cone}}(A+P)$ convex $\not\Rightarrow$ $\text{cone}_+(\text{int}(A+P))$ convex

In [22] it is proven that a Gordan-type alternative theorem holds for $A = F(C)$ under the $*$-quasiconvexity of F and the assumption

$$\begin{array}{l}\forall p^* \in P^*, \text{ the restriction of } \langle p^*, F(\cdot)\rangle \text{ on any} \\ \text{line segment of } C \text{ is lower semicontinuous.}\end{array} \tag{2.11}$$

We will see the naturally P-quasiconvexity of F along with (2.11) imply the convexity of $F(C)+P$; in particular, F is nearly subconvexlike, and so the alternative theorems of [22] and [38] are consequences from Theorem 2.2.

Proposition 2.5 ([14]). *Let $\emptyset \neq C \subseteq X$ be any convex set, $\emptyset \neq P \subsetneq Y$ be a closed convex cone and $F : C \to Y$ be naturally-P-quasiconvex and satisfying (2.11). Then*

$$\forall\, x, y \in C, \;\; [F(x), F(y)] \subseteq F([x,y]) + P. \tag{2.12}$$

Consequently, $F(C)+P$ is convex.

Next result supplements [10, Proposition 1.76] and [14, Proposition 3.5].

Proposition 2.6. *Let $\emptyset \neq A \subseteq Y$, $P \subseteq Y$ be a convex with* $\text{int}\, P \neq \emptyset$. *The following assertions are equivalent.*

(a) *A is nearly subconvexlike*
(b) $\text{cone}_+(\text{int}(A+P))$ *is convex*
(c) $\text{cone}(A) + \text{int}\, P$ *is convex*
(d) $\exists\, u \in \text{int}\, P$, $\forall\, x_1,\, x_2 \in A$, $\forall\, \alpha \in\ ]0,1[$, $\forall\, \varepsilon > 0$, $\exists\, \rho > 0$ *such that*

$$\varepsilon u + \alpha x_1 + (1-\alpha)x_2 \in \rho A + P \tag{2.13}$$

(e) $\exists\, u \in Y$, $\forall\, x_1,\, x_2 \in A$, $\forall\, \alpha \in\ ]0,1[$, $\forall\, \varepsilon > 0$, $\exists\, \rho > 0$ *such that (2.13) holds*
(f) $\forall\, u \in \text{int}\, P$, $\forall\, x_1,\, x_2 \in A$, $\forall\, \alpha \in\ ]0,1[$, $\exists\, \rho > 0$ *such that*

$$u + \alpha x_1 + (1-\alpha)x_2 \in \rho A + P$$

(g) $\forall\, u \in \text{int}\, P$, $\forall\, x_1,\, x_2 \in A$, $\forall\, \alpha \in\ ]0,1[$, $\exists\, \rho > 0$ *such that*

$$u + \alpha x_1 + (1-\alpha)x_2 \in \rho A + \text{int}\, P$$

(h) $\forall\, x_1,\, x_2 \in A$, $\forall\, \alpha \in\ ]0,1[$, $\exists\, u \in \text{int}\, P$, $\forall\, \varepsilon > 0$, $\exists\, \rho > 0$ *such that (2.13) holds*

Proof. From Proposition 2.1 we get the equivalences among (a), (b) and (c). The equivalences (a) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) are proved in [14, Proposition 3.5], whereas (c) $\Leftrightarrow$ (g) is proved in [10, Proposition 1.76]. The remaining implications (g) $\Rightarrow$ (f) $\Rightarrow$ (d) $\Rightarrow$ (h) $\Rightarrow$ (g) are straightforward. □

Remark 2.5. Assertion (d) refers to the notion of generalized subconvexlikeness introduced in [41], see also [43]; whereas (e) corresponds to the notion of presub-convexlikeness which is a transcription of an analogous definition for vector-valued functions given in [45].

Proposition 2.5 shows that any naturally-P-quasiconvex function satisfying (2.11) is nearly-subconvexilke. One can give some examples showing the converse is not true in general, see [14].

2.4 A Bidimensional Optimal Alternative Theorem and a Characterization of Two-Dimensionality

The bidimensional setting deserves a special treatment since, as we will see, the convexity of $\overline{\text{cone}}(A+P)$ is not only sufficient (see Theorem 2.3) but also a necessary condition to have a Gordan-type alternative theorem. In such a case, we refer it as an optimal alternative theorem, valid for convex cones with possibly empty interior under a regularity assumption. This is expressed in the next theorem.

Theorem 2.4. *Let* $P \subseteq \mathbb{R}^2$ *be a convex cone,* $A \subseteq \mathbb{R}^2$ *such that* $\text{int}(\text{cone}_+(A+P)) \neq \emptyset$*, and*

$$\text{int}(\overline{\text{cone}}(A+P)) = \text{int}(\text{cone}_+(A+P)). \tag{2.14}$$

The following assertions are equivalent:

(a) $\mathbf{0} \notin \text{int}(\text{cone}_+(A+P))$ *and* $\overline{\text{cone}}(A+P)$ *is convex.*
(b) $\mathbf{0} \notin \text{int}(\text{cone}_+(A+P))$ *and* $\text{cone}_+(A+P)$ *is convex, provided* $\mathbf{0} \in A+P$.
(c) $\exists\, p^* \in P^* \setminus \{\mathbf{0}\}$ *such that* $\langle p^*, a\rangle \geq 0\ \ \forall\, a \in A$.

Notice that when $\text{int}\, P \neq \emptyset$ *condition (2.14) is superfluous.*

Proof. (a) $\Rightarrow$ (c): It follows from Theorem 2.3; $(b) \Longrightarrow (a)$ is evident.

(c) $\Rightarrow$ (a), (c) $\Rightarrow$ (b): (We do not need (2.14)) The first part of (a) is a consequence of Theorem 2.2 (b). For the second part, we re-write the proof of [14, Theorem 4.1] with obvious changes. Certainly, $\langle p^*, a\rangle \geq 0$ for all $a \in \text{cone}(A+P)$. Choose $u \in P \setminus \{\mathbf{0}\}$. Let $y, z \in A$. Then obviously

$$\text{cone}(\{y\}) + \text{cone}(\{u\}) = \{\lambda y + \mu u : \lambda, \mu \geq 0\}$$

is a closed convex cone containing y and u and contained in $\overline{\text{cone}}(A+P)$ (if $\mathbf{0} \in A+P$, it is contained in $\text{cone}_+(A+P)$). The same is true for the cone $\text{cone}(\{z\}) + \text{cone}(\{u\})$. The two cones have the line $\text{cone}(\{u\})$ in common and their union is contained in $\overline{\text{cone}}(A+P)$, thus it is contained in the halfspace $\{x \in \mathbb{R}^2 : \langle p^*, x\rangle \geq 0\}$. Hence, the set $B \doteq (\mathrm{cone}(\{y\}) + \mathrm{cone}(\{u\})) \cup (\mathrm{cone}(\{z\}) + \mathrm{cone}(\{u\}))$ is a convex cone. Since $y, z \in B$ we deduce that $[y,z] \subseteq B \subseteq \overline{\mathrm{cone}}(A+P)$. Thus $\alpha A + (1-\alpha)A \subseteq \overline{\mathrm{cone}}(A+P)$ for all $\alpha \in\ ]0,1[$, proving the convexity of $\overline{\mathrm{cone}}(A+P)$ by Corollary 2.1(f). In case $\mathbf{0} \in A+P$, we get $\alpha A + (1-\alpha)A \subseteq \mathrm{cone}_+(A+P)$, proving the convexity of $\mathrm{cone}_+(A+P)$ by Corollary 2.1(a). □

When $\mathrm{int}\, P \neq \emptyset$ a more precise formulation of the previous theorem may be obtained.

Theorem 2.5. **([14, Theorem 4.1]** *Let* $P \subseteq \mathbb{R}^2$ *be a convex closed cone such that* $\mathrm{int}\, P \neq \emptyset$*, and* $A \subseteq \mathbb{R}^2$ *be any nonempty set satisfying* $A \cap (-\mathrm{int}\, P) = \emptyset$*. Then the following assertions are equivalent:*

(a) $\exists\, p^* \in P^* \setminus \{\mathbf{0}\}$ *such that* $\langle p^*, a\rangle \geq 0\ \ \forall\, a \in A$.
(b) $\mathrm{cone}(A+P)$ *is convex.*
(c) $\mathrm{cone}(A+\mathrm{int}\, P)$ *is convex.*
(d) $\mathrm{cone}(A)+P$ *is convex.*
(e) $\overline{\mathrm{cone}}(A+P)$ *is convex.*

Next result has its own importance from a functional analysis point of view. Indeed, such a result characterizes the two-dimensionality of any space where a Gordan-type alternative theorem holds.

Theorem 2.6. **([14, Theorem 4.2])** *Let* Y *be a locally convex topological vector space and* $P \subseteq Y$ *be a closed, convex cone such that* $\mathrm{int}\, P \neq \emptyset$ *and* $\mathrm{int}\, P^* \neq \emptyset$*. The following assertions are equivalent:*

(a) *For all sets* $A \subseteq Y$ *one has*

$$[\exists\, p^* \in P^* \setminus \{\mathbf{0}\},\ \langle p^*, a\rangle \geq 0\ \forall\, a \in A] \implies \overline{\mathrm{cone}}(A+P) \text{ is convex;}$$

(b) *For all sets* $A \subseteq Y$ *one has*

$$[\exists\, p^* \in P^* \setminus \{\mathbf{0}\},\ \langle p^*, a\rangle \geq 0\ \forall\, a \in A] \implies \mathrm{cone}(A)+P \text{ is convex;}$$

(c) *For all sets* $A \subseteq Y$ *one has*

$$[\exists\, p^* \in P^* \setminus \{\mathbf{0}\},\ \langle p^*, a\rangle \geq 0\ \forall\, a \in A] \implies \mathrm{cone}(A+\mathrm{int}\, P) \text{ is convex;}$$

(d) Y *is at most two-dimensional.*

Remark 2.6. The assumption $\mathrm{int}\, P^* \neq \emptyset$ (which corresponds to pointedness of P when Y is finite-dimensional) cannot be removed. Indeed, let $P = \{y \in Y : \langle p^*, y\rangle \geq 0\}$ where $p^* \in Y^* \setminus \{\mathbf{0}\}$. Then $P^* = \mathrm{cone}(\{p^*\})$, $\mathrm{int}\, P^* = \emptyset$. For any nonempty $A \subseteq Y$, the set $A+P$ is convex by Corollary 2.2. Thus, (a) in Theorem 2.6 holds no matter the dimension of the space Y is.

2.5 Applications to Vector Optimization

One of the important issues in optimization concerns the characterization of various notions of solutions to vector optimization problems through linear scalarization. This will be done for Benson proper efficiency and weak efficiency in case of bicriteria problems. For an theoretical treatment of these notions and others solution concepts, we refer the books [20, 26, 34]. The last subsection will be devoted to characterize the Fritz–John optimality condition.

In what follows, for a real-valued function h, by $\operatorname{argmin}_K h$ we mean the set of minima of h on K. Let X be a real vector space, $\emptyset \neq K \subseteq X$, Y be a real normed vector space. Given a vector function $F : K \to Y$ and a convex cone, possibly with empty interior, $P \subseteq Y$, we immediately obtain the following result.

Theorem 2.7. *Let $K \subseteq X$, F as above, and P a convex cone. Assume that*

$$\operatorname{int}(\operatorname{co}(F(K)) - F(\bar{x}) + P) \neq \emptyset.$$

The following assertions are equivalent:

(a)
$$\bar{x} \in \bigcup_{p^* \in P^*, p^* \neq \mathbf{0}} \operatorname{argmin}_K \langle p^*, F(\cdot) \rangle$$

(b) $\operatorname{cone}(\operatorname{int}(\operatorname{co}(F(K)) - F(\bar{x}) + P))$ *is pointed*

(b') In case $\operatorname{int} P \neq \emptyset$, *(b)* $\Leftrightarrow \operatorname{cone}(F(K) - F(\bar{x}) + \operatorname{int} P)$ *is pointed, as observed in Theorem 2.2*

Proof. It follows from Theorem 2.2 applied to $A = F(K) - F(\bar{x})$. □

2.5.1 *Characterizing Weakly Efficient Solutions Through Linear Scalarization of Bicriteria Problems*

Here, we assume that $\operatorname{int} P \neq \emptyset$. We say that $\bar{x} \in K$ is a *weakly efficient point* of F on K, shortly $\bar{x} \in E_W$, if

$$F(x) - F(\bar{x}) \notin -\operatorname{int} P, \quad \forall\, x \in K. \tag{2.15}$$

Clearly

$$\begin{aligned} \bar{x} \in E_W \;&\Leftrightarrow\; (F(K) - F(\bar{x})) \cap (-\operatorname{int} P) = \emptyset. \qquad (2.16)\\ &\Leftrightarrow\; \overline{\operatorname{cone}}(F(K) - F(\bar{x}) + P) \cap (-\operatorname{int} P) = \emptyset. \end{aligned}$$

In case $Y = \mathbb{R}^2$, we get the following theorem whose proof follows from Theorem 2.5.

Theorem 2.8. *Let $\emptyset \neq K \subseteq X$ and P be a convex cone having nonempty interior with $Y = \mathbb{R}^2$. Then, the following assertions are equivalent:*

(a)

$$\bar{x} \in \bigcup_{p^* \in P^*, p^* \neq \mathbf{0}} \operatorname{argmin}_K \langle p^*, F(\cdot) \rangle.$$

(b) $\bar{x} \in E_W$ *and* $\operatorname{cone}_+(F(K) - F(\bar{x}) + \operatorname{int} P)$ *is convex.*
(c) $\bar{x} \in E_W$ *and* $\operatorname{cone}_+(F(K) - F(\bar{x}) + P)$ *is convex.*
(d) $\bar{x} \in E_W$ *and* $\operatorname{cone}(F(K) - F(\bar{x}) + P)$ *is convex.*
(e) $\bar{x} \in E_W$ *and* $\operatorname{cone}(F(K) - F(\bar{x})) + P$ *is convex.*

2.5.1.1 The Pareto Case

We consider $P = \mathbb{R}^2_+$ and denote $\mathbb{R}^2_{++} \doteq \operatorname{int} \mathbb{R}^2_+$. Given a vector mapping $F = (f_1, f_2) : K \to \mathbb{R}^2$, we consider the problem of finding

$$\bar{x} \in K : F(x) - F(\bar{x}) \notin -\mathbb{R}^2_{++}, \quad \forall x \in K. \tag{2.17}$$

Let $\bar{x} \in E_W$ and for $i = 1, 2$, set

$$S_i^-(\bar{x}) \doteq \{x \in K : \ f_i(x) < f_i(\bar{x})\}; \ S_i^+(\bar{x}) \doteq \{x \in K : \ f_i(x) > f_i(\bar{x})\};$$

$$S_i^=(\bar{x}) \doteq \{x \in K : \ f_i(x) = f_i(\bar{x})\}.$$

Taking into account Theorem 2.8, we write $F(K) - F(\bar{x}) + \mathbb{R}^2_{++} = \Omega_1 \cup \Omega_2 \cup \Omega_3$. It follows that

$$\operatorname{cone}_+(F(K) - F(\bar{x}) + \mathbb{R}^2_{++}) = \operatorname{cone}_+(\Omega_1) \cup \operatorname{cone}_+(\Omega_2) \cup \operatorname{cone}_+(\Omega_3),$$

where

$$\Omega_1 \doteq \bigcup_{x \in S_1^-(\bar{x})} [(f_1(x) - f_1(\bar{x}), f_2(x) - f_2(\bar{x})) + \mathbb{R}^2_{++}];$$

$$\Omega_2 \doteq \bigcup_{x \in S_1^=(\bar{x})} [(f_1(x) - f_1(\bar{x}), f_2(x) - f_2(\bar{x})) + \mathbb{R}^2_{++}];$$

$$\Omega_3 \doteq \bigcup_{x \in S_1^+(\bar{x})} [(f_1(x) - f_1(\bar{x}), f_2(x) - f_2(\bar{x})) + \mathbb{R}^2_{++}].$$

Whenever $S_1^+(\bar{x}) \cap S_2^-(\bar{x}) \neq \emptyset$ and $S_1^-(\bar{x}) \cap S_2^+(\bar{x}) \neq \emptyset$, we set

$$\alpha \doteq \inf_{x \in S_1^+(\bar{x}) \cap S_2^-(\bar{x})} \frac{f_2(x) - f_2(\bar{x})}{f_1(x) - f_1(\bar{x})}, \quad \beta \doteq \sup_{x \in S_1^-(\bar{x}) \cap S_2^+(\bar{x})} \frac{f_2(x) - f_2(\bar{x})}{f_1(x) - f_1(\bar{x})}. \tag{2.18}$$

Clearly, $-\infty \leq \alpha < 0$ and $-\infty < \beta \leq 0$.
Figures 2.4–2.6 can be obtained directly

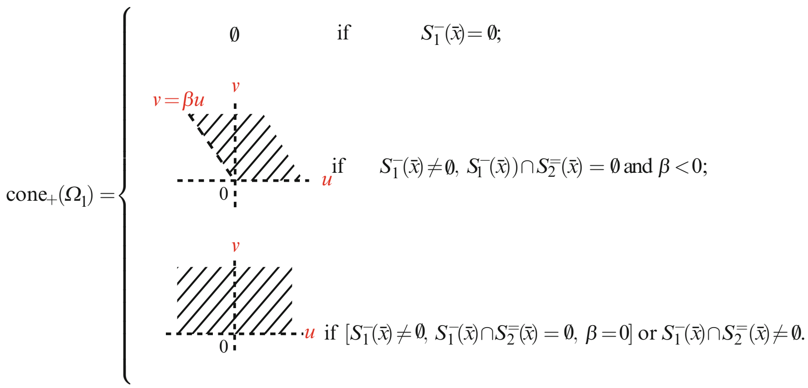

Fig. 2.4 To visualize Theorem 2.9

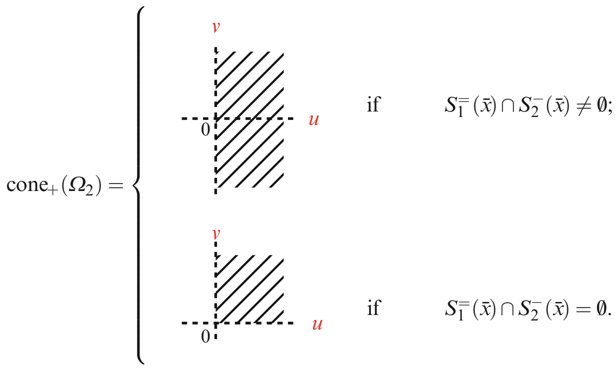

Fig. 2.5 To visualize Theorem 2.9

Notice that

$$S_1^-(\bar{x}) \cap S_2^+(\bar{x}) = \emptyset \Longleftrightarrow S_1^-(\bar{x}) \subseteq S_2^=(\bar{x}); \;\; S_1^+(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset \Longleftrightarrow S_2^-(\bar{x}) \subseteq S_1^=(\bar{x}).$$

The following theorem is immediate from the expressions of $\text{cone}_+(\Omega_i)$, $i = 1,2,3$.

Theorem 2.9. *Assume that* $\bar{x} \in E_W$. *Then,* $\text{cone}_+(F(K) - F(\bar{x}) + \mathbb{R}^2_{++})$ *is convex if, and only if any of the following assertions hold:*

(a) $S_1^-(\bar{x}) = \emptyset$
(b) $S_1^-(\bar{x}) \neq \emptyset$, $S_1^-(\bar{x}) \cap S_2^=(\bar{x}) = \emptyset$, $\beta < 0$, $S_1^=(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset$ *and, either*

(b1) $S_1^+(\bar{x}) = \emptyset$, *or*
(b2) $S_1^+(\bar{x}) \neq \emptyset$ *and* $S_1^+(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset$, *or*
(b3) $S_1^+(\bar{x}) \cap S_2^-(\bar{x}) \neq \emptyset$, $\alpha > -\infty$, $\beta \leq \alpha$

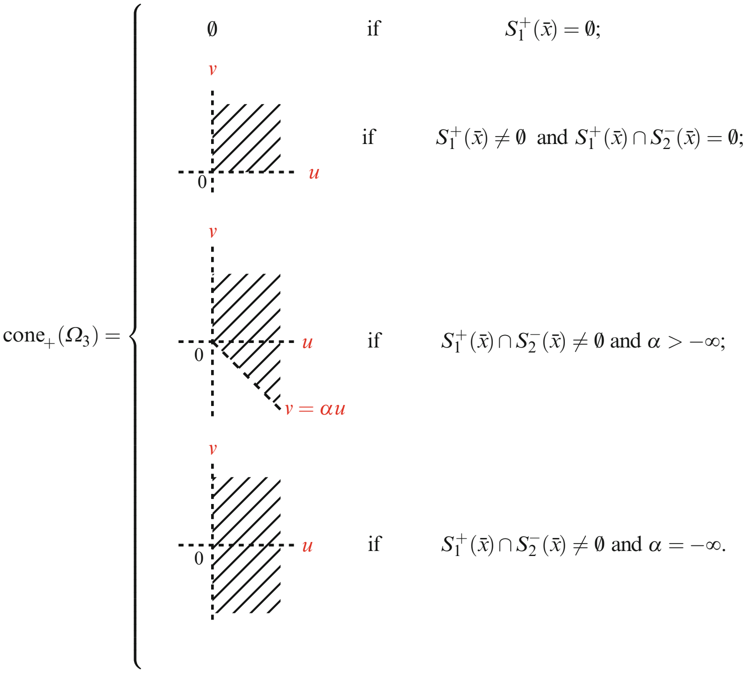

Fig. 2.6 To visualize Theorem 2.9

(c) $S_1^-(\bar{x}) \neq \emptyset$, $S_1^-(\bar{x}) \cap S_2^=(\bar{x}) = \emptyset$, $\beta = 0$, $S_1^=(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset$ *and, either*

- *(c1)* $S_1^+(\bar{x}) = \emptyset$, *or*
- *(c2)* $S_1^+(\bar{x}) \neq \emptyset$ *and* $S_1^+(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset$

(d) $S_1^-(\bar{x}) \cap S_2^=(\bar{x}) \neq \emptyset$, $S_1^=(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset$ *and, either*

- *(d1)* $S_1^+(\bar{x}) = \emptyset$, *or*
- *(d2)* $S_1^+(\bar{x}) \neq \emptyset$ *and* $S_1^+(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset$.

Proof. We omit the long but easy proof once we get Figs. 2.4–2.6. □

We also notice that

$$[S_1^+(\bar{x}) = \emptyset \ \text{ and } \ S_1^=(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset] \Longrightarrow S_2^-(\bar{x}) = \emptyset;$$

and

$$[S_1^+(\bar{x}) \neq \emptyset \ \text{ and } \ S_1^+(\bar{x}) \cap S_2^-(\bar{x}) = \emptyset] \Longrightarrow S_2^-(\bar{x}) = \emptyset.$$

Both implications assert that (b_1) (along with (b)), (b_2) (along with (b)), (c) and (d) of the previous theorem imply $S_2^-(\bar{x}) = \emptyset$. On the other hand,

$$S_i^-(\bar{x}) = \emptyset \Longleftrightarrow \bar{x} \in \operatorname{argmin}_K f_i.$$

Thus, next corollary, which follows from (b_3) (along with (b)) of Theorem 2.9, excludes situations the other situations of such a theorem.

Corollary 2.6. *Let us consider problem* (2.17) *and assume that* $\bar{x} \notin \operatorname{argmin}_K f_i$, $i = 1,2$. *Then,*

(a)

$$\bar{x} \in \bigcup_{(p_1^*,p_2^*)\in\mathbb{R}^2_+\setminus\{(0,0)\}} \operatorname{argmin}_K(p_1^* f_1 + p_2^* f_2)$$

if and only if $\bar{x} \in E_W$ *and* (b_3) $($*along with* $(b))$ *of Theorem* 2.9 *is satisfied.*

(b) If $\bar{x} \in E_W$ *and* (b_3) $($*along with* $(b))$ *holds, then any* $-\alpha \le p_1^* \le -\beta$ *satisfies*

$$\bar{x} \in \operatorname{argmin}_K(p_1^* f_1 + f_2).$$

2.5.2 Characterizing Properly Efficient Solutions Through Linear Scalarization of Bicriteria Problems

We say that $\bar{x} \in K$ is (Benson) *properly efficient point* of F on K ([2]), in short $\bar{x} \in E_{pr}$, if

$$\overline{\operatorname{cone}}(F(K) - F(\bar{x}) + P) \cap (-P) = \{\mathbf{0}\}. \tag{2.19}$$

One can easily check that if E_{pr} is nonempty, then P is pointed.

Setting

$$P^{*i} \doteq \Big\{p^* \in Y^*,\ \langle p^*, p\rangle > 0,\ \ \forall\, p \in P \setminus \{\mathbf{0}\}\Big\},$$

it can be seen that

$$\bigcup_{p^*\in P^{*i}} \operatorname{argmin}_K \langle p^*, F(\cdot)\rangle \subseteq E_{pr}. \tag{2.20}$$

Conversely, if $\bar{x} \in E_{pr}$ and $\overline{\operatorname{cone}}(F(K) - F(\bar{x}) + P)$ is convex then

$$\bar{x} \in \bigcup_{p^*\in P^{*i}} \operatorname{argmin}_K \langle p^*, F(\cdot)\rangle,$$

provided P is locally compact (use the separation result for convex cones [3, Proposition 3]).

In case $Y = \mathbb{R}^2$, we get the following theorem whose proof follows from Theorem 2.4 and the remarks above.

Theorem 2.10. *Let* $K \subseteq X$ *be a convex set and* F *as above with* $P \subseteq \mathbb{R}^2$ *being a pointed, closed, convex cone. Assume that*

$$\operatorname{int}(F(K) - F(\bar{x}) + P) \neq \emptyset.$$

Then, the following assertions are equivalent:

(a)

$$\bar{x} \in \bigcup_{p^* \in \text{int } P^*} \text{argmin}_K \langle p^*, F(\cdot) \rangle.$$

(b) $\bar{x} \in E_{pr}$ *and* $\overline{\text{cone}}(F(K) - F(\bar{x}) + P)$ *is convex.*
(c) $\bar{x} \in E_{pr}$ *and* $\text{cone}(F(K) - F(\bar{x}) + P)$ *is convex.*

2.5.2.1 The Pareto Case

We now particularize $P = \mathbb{R}^2_+$. Given a vector mapping $F = (f_1, f_2) : K \to \mathbb{R}^2$, we consider the problem of finding

$$\bar{x} \in K : \ \overline{\text{cone}}(F(K) - F(\bar{x}) + \mathbb{R}^2_+) \cap (-\mathbb{R}^2_+) = \{(0,0)\}, \tag{2.21}$$

Let $\bar{x} \in E_{pr}$ and for $i = 1, 2$, consider the sets $S_i^-(\bar{x})$, $S_i^+(\bar{x})$ and $S_i^=(\bar{x})$ as defined in the previous subsection.

By (k) of Proposition 2.1, the convexity of $\overline{\text{cone}}(F(K) - F(\bar{x}) + \mathbb{R}^2_+)$ is equivalent to the convexity of $\text{cone}_+(F(K) - F(\bar{x}) + \mathbb{R}^2_{++})$. Thus, by writing $F(K) - F(\bar{x}) + \mathbb{R}^2_{++} = \Omega_1 \cup \Omega_2 \cup \Omega_3$, we can use the same expressions for $\text{cone}_+(\Omega_i)$, $i = 1, 2, 3$ computed in the preceding section. The fact that $\bar{x} \in E_{pr}$ allows us to conclude that α, β (as defined in (2.18)) satisfy $-\infty < \alpha < 0$, $-\infty < \beta < 0$, and

$$S_1^-(\bar{x}) \subseteq S_2^+(\bar{x}); \ \ S_2^-(\bar{x}) \subseteq S_1^+(\bar{x}).$$

Thus, the preceding expressions for $\text{cone}(\Omega_i)$, $i = 1, 2, 3$, reduces to

Theorem 2.11. *Assume that* $\bar{x} \in E_{pr}$. *Then,* $\overline{\text{cone}}(F(C) - F(\bar{x}) + \mathbb{R}^2_+)$ *is convex if, and only if either* (a) *or* (b) *holds. Here,*

(a) $S_1^-(\bar{x}) \neq \emptyset$ *and, either* $S_1^+(\bar{x}) = \emptyset$ *or* $[S_2^-(\bar{x}) = \emptyset, S_1^+(\bar{x}) \neq \emptyset]$ *or* $[S_2^-(\bar{x}) \neq \emptyset, \beta \leq \alpha]$
(b) $S_1^-(\bar{x}) = \emptyset$

Proof. The proof is easy once we get Figs. 2.7–2.9. □

Corollary 2.7. *Let us consider problem* (2.21)*. Then,*

$$\bar{x} \in \bigcup_{(p_1^*, p_2^*) \in \mathbb{R}^2_{++}} \text{argmin}_K (p_1^* f_1 + p_2^* f_2)$$

if and only if either (a) *or* (b) *holds, where*

(a) $\bar{x} \in E_{pr}$, $S_1^-(\bar{x}) \neq \emptyset$ *and, either* $S_1^+(\bar{x}) = \emptyset$ *or* $[S_2^-(\bar{x}) = \emptyset, S_1^+(\bar{x}) \neq \emptyset]$ *or* $[S_2^-(\bar{x}) \neq \emptyset, \beta \leq \alpha]$
(b) $\bar{x} \in E_{pr}$ *and* $S_1^-(\bar{x}) = \emptyset$

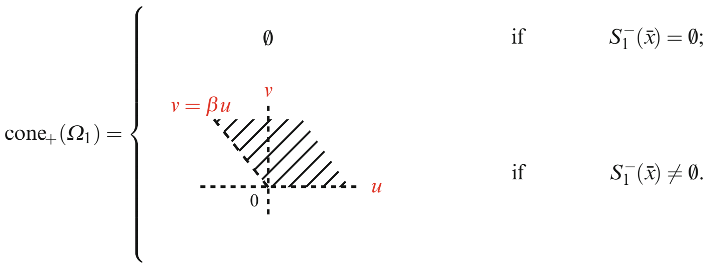

Fig. 2.7 To visualize Theorem 2.11

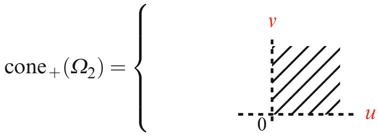

Fig. 2.8 To visualize Theorem 2.11

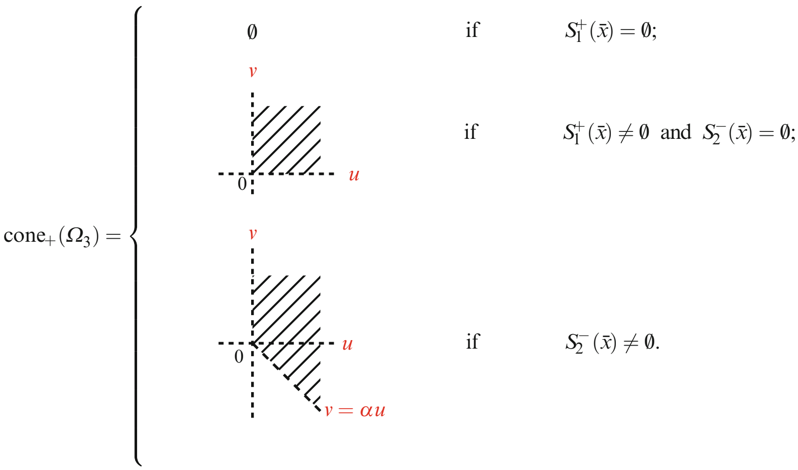

Fig. 2.9 To visualize Theorem 2.11

Corollary 2.8. *Let us consider problem* (2.21). *Then,*

(a) If $\overline{x} \in E_{pr}, S_1^-(\overline{x}) \neq \emptyset$ *and, either* $S_1^+(\overline{x}) = \emptyset$ *or* $[\ S_2^-(\overline{x}) = \emptyset, S_1^+(\overline{x}) \neq \emptyset\ ]$, *then any* p_1^* *such that* $0 < p_1^* \leq -\beta$ *satisfies*

$$\overline{x} \in \operatorname{argmin}_K (p_1^* f_1 + f_2).$$

(b) If $\overline{x} \in E_{pr}, S_1^-(\overline{x}) \neq \emptyset$ *and* $[\ S_2^-(\overline{x}) \neq \emptyset, \beta \leq \alpha\ ]$, *then any* p_1^* *such that* $-\alpha \leq p_1^* \leq -\beta$ *satisfies*

$$\bar{x} \in \mathrm{argmin}_K(p_1^* f_1 + f_2).$$

(c) If $\bar{x} \in E_{pr}$, $S_1^-(\bar{x}) = \emptyset$ *and* $S_2^-(\bar{x}) \neq \emptyset$, *then any* p_1^* *such that* $-\alpha \leq p_1^*$ *satisfies*

$$\bar{x} \in \mathrm{argmin}_K(p_1^* f_1 + f_2).$$

(d) If $\bar{x} \in E_{pr}$, $S_1^-(\bar{x}) = \emptyset$ *and* $S_2^-(\bar{x}) = \emptyset$, *then any* $(p_1^*, p_2^*) \in \mathbb{R}^2_{++}$ *satisfies*

$$\bar{x} \in \mathrm{argmin}_K(p_1^* f_1 + p_2^* f_2).$$

2.5.3 Characterizing the Fritz–John Type Optimality Conditions

For simplicity we now consider X to be a real normed vector space. It is well known that if $\bar{x}$ is a local minimum point for the real-valued differentiable function F on K, then

$$\nabla F(\bar{x}) \in (T(K;\bar{x}))^*. \tag{2.22}$$

Here, K is a (not necessarily convex) set, $T(C;\bar{x})$ denotes the *contingent cone* of C at $\bar{x} \in C$, defined as the set of vectors v such that there exist $t_k \downarrow 0$, $v_k \in X$, $v_k \to v$ such that $\bar{x} + t_k v_k \in C$ for all k; recall that C^* denotes the (positive) polar cone of C.

It is now our purpose to extend the previous optimality condition to the vector case without smoothness assumptions. More precisely, let $K \subseteq X$ be closed and consider a mapping $F : K \to \mathbb{R}^m$. A vector $\bar{x} \in K$ is a local weakly efficient solution for F on K, if there exists an open neighborhood V of $\bar{x}$ such that

$$(F(K \cap V) - F(\bar{x})) \cap (-\mathrm{int}\, P) = \emptyset. \tag{2.23}$$

Following [37], we say that a function $h : X \to \mathbb{R}$ admits a *Hadamard directional derivative* at $\bar{x} \in X$ in the direction v if

$$\lim_{(t,u)\to(0^+,v)} \frac{h(\bar{x}+tu) - h(\bar{x})}{t} \in \mathbb{R}.$$

In this case, we denote such a limit by $dh(\bar{x};v)$.

If $F = (f_1, f_2, \ldots, f_m)$, we set

$$\mathscr{F}(v) \doteq ((df_1(\bar{x};v), \ldots, df_m(\bar{x};v)), \;\; \mathscr{F}(T(K;\bar{x})) \doteq \{\mathscr{F}(v) \in \mathbb{R}^n : \; v \in T(K;\bar{x})\}.$$

It is known that if $df_i(\bar{x};\cdot)$, $i = 1, \ldots, m$, do exist in $T(K;\bar{x})$, and $\bar{x} \in K$ is a local weakly efficient solution for F on K, i.e., $\bar{x}$ satisfies (2.23), then (see for instance [37, Lemma 3.2])

$$(df_1(\bar{x};v),\ldots,df_n(\bar{x};v)) \in \mathbb{R}^n \setminus -\mathrm{int}\, P,\ \ \forall\, v \in T(K;\bar{x}), \tag{2.24}$$

or equivalently,

$$\mathscr{F}(T(K;\bar{x})) \cap (-\mathrm{int}\, P) = \emptyset.$$

The following theorems provide complete characterizations for the validity of (a) as a necessary condition for $\bar{x}$ to be a local weakly efficient solution for F on K.

Theorem 2.12. *Let $K \subseteq X$ be a closed set, $P \subseteq \mathbb{R}^n$ be a closed convex cone such that $\mathrm{int}\, P \neq \emptyset$ and $P \neq \mathbb{R}^n$. Assume that $\bar{x} \in K$ and $df_i(\bar{x};\cdot)$, $i = 1,\ldots,m$, do exist in $T(K;\bar{x})$. Then, the following assertions are equivalent:*

(a) $\exists\, (\alpha_1^*,\ldots,\alpha_m^*) \in P^* \setminus \{\mathbf{0}\},\ \ \alpha_1^* df_1(\bar{x},v) + \ldots + \alpha_m^* df_n(\bar{x},v) \geq 0\ \ \forall\, v \in T(K;\bar{x})$.
(b) $\mathrm{cone}(\mathscr{F}(T(K;\bar{x})) + \mathrm{int}\, P)$ *is pointed.*

Proof. We obtain the desired result from Corollary 2.2. □

When $Y = \mathbb{R}^2$, more precise formulations can be obtained from Theorem 2.5.

Theorem 2.13. *Let $K \subseteq X$ be a closed set, $P \subseteq \mathbb{R}^2$ be a closed convex cone such that $\mathrm{int}\, P \neq \emptyset$. Assume that $\bar{x} \in K$ and $df_i(\bar{x};\cdot)$, $i = 1,2$, do exist in $T(K;\bar{x})$. Then, the following assertions are equivalent:*

(a) $\exists\, (\alpha_1^*,\alpha_2^*) \in P^* \setminus \{(0,0)\},\ \ \alpha_1^* df_1(\bar{x},v) + \alpha_2^* df_2(\bar{x},v) \geq 0\ \ \forall\, v \in T(K;\bar{x})$.
(b) $\mathscr{F}(T(K;\bar{x})) \cap (-\mathrm{int}\, P) = \emptyset$ *and* $\mathrm{cone}(\mathscr{F}(T(K;\bar{x})) + \mathrm{int}\, P)$ *is convex.*

Proof. We apply Corollary 2.5 to obtain the desired result. □

We can go further when differentiability conditions are imposed.

Proposition 2.7. *Assume that $P = \mathbb{R}^m_+$ and $f_i : \mathbb{R}^n \to \mathbb{R}$ be differentiable for $i = 1,\ldots,m$, and $\bar{x} \in \mathbb{R}^n$. Then, for any set $A \subseteq \mathbb{R}^n$,*

$$\mathscr{F}(A) \cap (-\mathrm{int}\, \mathbb{R}^m_+) = \emptyset \ \ \Leftrightarrow \ \ \max_{1\leq i\leq m} \langle \nabla f_i(\bar{x}), v\rangle \geq 0\ \ \forall\, v \in \overline{A},$$

and the following statements are equivalent:

(a) $\mathrm{cone}(\mathscr{F}(T(K;\bar{x})) + \mathrm{int}\, \mathbb{R}^n_+)$ is pointed;
(b) $\mathscr{F}(\overline{\mathrm{co}}(T(K;\bar{x}))) \cap (-\mathrm{int}\, \mathbb{R}^m_+) = \emptyset$.
(c) $\max_{1\leq i\leq m} \langle \nabla f_i(\bar{x}), v\rangle \geq 0\ \ \forall\, v \in \overline{\mathrm{co}}(T(K;\bar{x}))$.
(d) $\mathrm{co}(\{\nabla f_i(\bar{x}) :\ i = 1,\ldots,m\}) \cap (T(K;\bar{x}))^* \neq \emptyset$.

Proof. The first part is a consequence of the linearity of $\mathscr{F}$:

$$\mathscr{F}(v) = (\langle \nabla f_1(\bar{x}), v\rangle,\ldots,\langle \nabla f_m(\bar{x}), v\rangle).$$

We already know that

$$\mathrm{cone}(\mathscr{F}(T(K;\bar{x})) + \mathrm{int}\, \mathbb{R}^m_+)) \text{ is pointed } \ \Leftrightarrow \ \mathrm{co}(\mathscr{F}(T(K;\bar{x})) \cap (-\mathrm{int}\, \mathbb{R}^m_+) = \emptyset.$$

It is not difficult to prove that $\text{co}(\mathscr{F}(T(K;\bar{x}))) = \mathscr{F}(\text{co}(T(K;\bar{x})))$ and

$$\mathscr{F}(\text{co}(T(K;\bar{x}))) \cap (-\text{int}\,\mathbb{R}^m_+) = \emptyset \Leftrightarrow \mathscr{F}(\overline{\text{co}}(T(K;\bar{x}))) \cap (-\text{int}\,\mathbb{R}^m_+) = \emptyset$$
$$\Leftrightarrow \overline{\mathscr{F}(\text{co}(T(K;\bar{x})))} \cap (-\text{int}\,\mathbb{R}^m_+) = \emptyset.$$

This and the fact that (a) of Theorem 2.12 amounts to writing

$$\text{co}(\{\nabla f_i(\bar{x}):\ i = 1,\ldots,m\}) \cap (T(K;\bar{x}))^* \neq \emptyset,$$

we get all the remaining equivalences. □

We apply the previous proposition to get the following result.

Theorem 2.14. *Let $K \subseteq X$ be a closed set, Assume that $\bar{x} \in K$ and $f_i : \mathbb{R}^n \to \mathbb{R}^2$ are differentiable functions for $i = 1, 2, \ldots, m$. Then, the following assertions are equivalent:*

(a) $\mathscr{F}(T(K;\bar{x}) \cap (-\text{int}\mathbb{R}^2_+) = \emptyset$ *and* $\text{cone}(\mathscr{F}(T(K;\bar{x})) + \mathbb{R}^2_+)$ *is convex.*
(b) $\text{co}(\{\nabla f_i(\bar{x}):\ i = 1, 2\}) \cap (T(K;\bar{x}))^* \neq \emptyset$.

Before going on some remarks are in order. Certainly, if $T(K;\bar{x})$ is convex, then (d) is a necessary optimality condition for $\bar{x}$ to be a local weakly efficient solution (this fact was point out earlier in [39], see also [9]). Thus, (d) could be considered a natural extension of (2.22). However, next example shows that (d) is not a necessary optimality condition if $T(K;\bar{x})$ is not convex. The second example shows an instance where (d) holds without the convexity of $T(K;\bar{x})$.

Example 2.2. Take the (modified) example from [1], see also [9, 40]:

$$K = \{(x_1, x_2) :\ (x_1 + 2x_2)(2x_1 + x_2) \leq 0\},\ f_i(x_1, x_2) = x_i,\ \bar{x} = (0,0) \in E_W.$$

In this case $T(K;\bar{x}) = K$, which is nonconvex, $(T(K;\bar{x}))^* = \{(0,0)\}$, and therefore (d) does not hold since $\text{co}(\{\nabla f_1(\bar{x}), \nabla f_2(\bar{x})\}) = \text{co}(\{(1,0),(0,1)\})$. Since $\mathscr{F}(v) = v$, the set

$$\text{cone}(\mathscr{F}(T(K;\bar{x})) + \mathbb{R}^2_+) = \bigcup_{t \geq 0} t(T(K;\bar{x}) + \mathbb{R}^2_+).$$

is nonconvex.

Example 2.3. Consider the same mapping F as before and

$$K = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_1 x_2 = 0\},\ \bar{x} = (0,0) \in E_W.$$

Then, (d) holds since in this case, $T(K;\bar{x}) = K$, $(T(K;\bar{x}))^* = \mathbb{R}^2_+$. Here, the set

$$\text{cone}(\mathscr{F}(T(K;\bar{x})) + \mathbb{R}^2_+) = \bigcup_{t \geq 0} t(T(K;\bar{x}) + \mathbb{R}^2_+)$$

is convex.

2.6 More About Proper Efficiency

We now present some recent developments about proper efficiency. As before, throughout this section we consider a nonempty set $A \subsetneq Y$, with Y being a locally convex topological vector space. In addition, we are given a convex cone $P \subsetneq Y$. We say that $\overline{a} \in A$ is a

- *Benson proper efficient point* if $\overline{\text{cone}}(A - \overline{a} + P) \cap (-P) = \{\mathbf{0}\}$. This is the definition given in Benson [2]. and the set is denoted by $E_{\text{pr}}(A, P)$.
- *Borwein proper efficient point* if $\overline{\text{cone}}(A - \overline{a}) \cap (-P) = \{\mathbf{0}\}$. This notion is introduced in [4] when P is pointed.

Evidently every Benson proper efficient point is also a Borwein efficient.

Proper efficiency is introduced in order to avoid efficient points satisfying some abnormal properties, in particular, efficient points for which at least one objective function exists for which the marginal trade-off between it and each of the other objective functions is infinitely large, [17], or if one prefers efficient points that allow more satisfactory characterization in terms of linear/sublinear scalarization, for instance. The starting point was the pioneer work by Kuhn and Tucker in multiojective programming problems [24].

Benson and Borwein efficiency coincide if P has a compact base, see [11]; whereas in general it is not true, as shows Example 4.3 in [11]. We say that B is a *base* for P if B is convex, $\mathbf{0} \notin \overline{B}$ and $P = \text{cone}(B)$. Obviously, the existence of a base for P implies its pointedness; likewise if $E_{\text{pr}}(A, P) \neq \emptyset$.

When the corresponding scalar function which is involved in the characterization of proper efficiency, is a continuous seminorm, we refer to [11]. This result is based in the following theorem

Theorem 2.15. ([11, Theorem 2.3]) *Let P and Q be cones in Y satisfying $P \cap Q = \{\mathbf{0}\}$, and either (a) P be a weak-closed and Q have a weak-compact base or (b) P be closed and Q have a compact base. Then, there is a pointed convex cone C such that $Q \setminus \{\mathbf{0}\} \subseteq \text{int}\, C$ and $C \cap P = \{\mathbf{0}\}$.*

Now, we present some results on interior of a polar cone, and afterwards, dual characterizations and scalarizations for Benson proper efficiency. To that purpose, we recall that Y^* is the topological dual of Y. For any convex cone $P \subseteq Y$, the quasi interior of P^*, is defined as

$$\text{qi}\, P^* = P^{*i} \doteq \{y^* \in Y^* : \langle y^*, y\rangle > 0 \,\forall\, p \in P \setminus \{\mathbf{0}\}\}.$$

A convex cone P with $\text{int}\, P \neq \emptyset$ is said to be a *solid cone*. Moreover, a convex cone P has a *base* if and only if $P^{*i} \neq \emptyset$. For a base B of P, we define B^{st} to be the set

$$B^{st} \doteq \{y^* \in Y^* : \inf_{b \in B} \langle y^*, b\rangle > 0\}.$$

For any locally convex topological vector space Y, we have various ways of introducing a locally convex topology on the dual Y^*. If $\mathscr{M}$ is any total saturated class of bounded subsets of Y [19, 23, 35], the topology of uniform convergence on the sets M of $\mathscr{M}$ is a locally convex topology on Y^*. We denote it by $\tau_{\mathscr{M}}$. Obviously $\{M^\circ : M \in \mathscr{M}\}$ is a $\mathbf{0}$-neighborhood base in $(Y^*, \tau_{\mathscr{M}})$. Particularly, we denote the topologies on Y^* of uniform convergence on bounded subsets, weakly compact (absolutely) convex subsets, and finite subsets of Y by $\beta(Y^*,Y)$, $\tau(Y^*,Y)$, and $\sigma(Y^*,Y)$, which are called the strong topology, Mackey topology, and weak topology, respectively.

Lemma 2.2. **([29, Lemma 2.1])** *Let $P \subseteq Y$ be a convex cone. If there exist a locally convex topology $\mathscr{T}$ on Y^* such that $\mathrm{int}_{\mathscr{T}} P^* \neq \emptyset$, where $\mathrm{int}_{\mathscr{T}} P^*$ denotes the interior of P^* in $(Y^*, \mathscr{T})$, then $\mathrm{int}_{\mathscr{T}}\, P^* \subseteq P^{*i}$.*

Theorem 2.16. **([29, Theorem 2.1])** *Let $P \subseteq Y$ be a convex cone. Then, $\mathrm{int}_{\tau_{\mathscr{M}}}\, P^* \neq \emptyset$ if and only if P has a base $B \in \mathscr{M}$. In this case, $\mathrm{int}_{\tau_{\mathscr{M}}}\, P^* = B^{st}$.*

Similar expressions hold for $\tau(Y^*,Y)$ and $\beta(Y^*,y)$, for details, see [21, Theorem 3.8.6], [28, Theorem 2.3], and [28, Theorem 2.2].

We now give the following general dual characterization and scalarization for Benson proper efficiency.

Theorem 2.17. **([29, Theorem 3.1.])** *Let $P \subseteq Y$ be a closed convex cone, $\overline{a} \in A \subseteq Y$. Then the following statements are equivalent:*

(a) $\overline{a} \in E_{\mathrm{pr}}(\mathrm{co}\, A, P)$.
(b) $(P^* - P^* \cap (A - \overline{a})^*)$ *is dense in* $(Y^*, \mathscr{T})$ *where* $\mathscr{T}$ *is any locally convex topology on* Y^* *which is compatible with the dual pair* (Y^*, Y) *(i.e.,* $(Y^*, \mathscr{T})^* = Y$*).*
(c) *for any weakly compact convex set* $K \subseteq P$ *and* $\mathbf{0} \notin K$*, there exists* $p^* \in P^* \cap K^{st}$ *such that* $\langle p^*, a\rangle \geq \langle p^*, \overline{a}\rangle\ \forall\, a \in A$.
(d) *for any* $p \in P \setminus \{\mathbf{0}\}$ *there exists* $p^* \in P^*$ *such that* $\langle p^*, p\rangle > 0$ *and* $\langle p^*, a\rangle \geq \langle p^*, \overline{a}\rangle\ \forall\, a \in A$.

Theorem 2.18. **([29, Theorem 3.2])** *Let $P \subseteq Y$ be a closed convex cone and $\overline{a} \in A \subseteq Y$. If there exists a locally convex topology $\mathscr{T}$ on Y^* such that $(Y^*, \mathscr{T})^* = Y$ and $\mathrm{int}_{\mathscr{T}} P^* \neq \emptyset$ then the following statements are equivalent:*

(a) $\overline{a} \in E_{\mathrm{pr}}(\mathrm{co}\, A, P)$.
(b) *There exists* $p^* \in \mathrm{int}_{\mathscr{T}} P^*$ *such that* $\langle p^*, a\rangle \geq \langle p^*, \overline{a}\rangle,\ \ \forall\, a \in A$.
(c) *There exists* $p^* \in P^{*i}$ *such that* $\langle p^*, a\rangle \geq \langle p^*, \overline{a}\rangle,\ \ \forall\, a \in A$.

Corollary 2.9. **([29, Corollary 3.1])** *Let $C \subseteq Y$ be a closed convex cone with a weakly compact base B and $\overline{a} \in A \subseteq Y$. Then the following statements are equivalent:*

(a) $\overline{a} \in E_{\mathrm{pr}}(\mathrm{co}\, A, P)$
(b) *there exists* $p^* \in B^{st}$ *such that* $\langle p^*, a\rangle \geq \langle p^*, \overline{a}\rangle\ \ \forall\, a \in A$
(c) *there exists* $p^* \in C^{*i}$ *such that* $\langle p^*, a\rangle \geq \langle p^*, \overline{a}\rangle\ \ \forall\, a \in A$

A recent notion of proper efficiency was introduced in [27]. It is equivalent to strict efficiency, strong efficiency and to super efficiency as shown in [27, Proposition 2.2], provided P is a convex cone with a (convex) bounded base.

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Chapter 3
Duality in Vector Optimization with Infimum and Supremum

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3.1 Introduction

We consider the vector optimization problem

$$\text{minimize } f : X \to \overline{Y} \text{ with respect to } \le \text{ over } S \subset X. \qquad \text{(VOP)}$$

The extended partially ordered vector space $(\overline{Y}, \le)$ is regarded to be a subset of the complete lattice $(\mathscr{I}, \preccurlyeq)$ which is defined as follows: The convex and pointed ordering cone C that induces the partial ordering $\le$ on Y is assumed to satisfy $\emptyset \neq \operatorname{int} C \neq Y$. The *infimal set* of a set $A \subset Y$ is defined by

$$\operatorname{Inf} A := \operatorname{wMin} \operatorname{cl}(A + C),$$

where some additional considerations with respect to the elements $\pm\infty$ will be added later. A set A satisfying $A = \operatorname{Inf} A$ is called *self-infimal*. The family of all self-infimal sets is denoted by $\mathscr{I}$. The space $\mathscr{I}$ is equipped with a partial ordering defined by

$$A^1 \preccurlyeq A^2 \quad :\Longleftrightarrow \quad A^1 + \operatorname{int} C \supseteq A^2 + \operatorname{int} C$$

whenever $\pm\infty$ do not occur. If a vector $y \in Y$ is identified with its infimal set $\operatorname{Inf}\{y\} \in \mathscr{I}$ and if the ordering cone C is supposed to be closed, the partially ordered set $(\mathscr{I}, \preccurlyeq)$ is an extension of the partially ordered set $(Y, \le)$. If Y is supplemented by $\pm\infty$, and $\mathscr{I}$ is defined correspondingly, $(\mathscr{I}, \preccurlyeq)$ is a complete lattice, that is, infimum and supremum of every subset exist. We assign to f an $\mathscr{I}$-valued objective function

$$\overline{f} : X \to \mathscr{I}, \qquad \overline{f}(x) := \operatorname{Inf}\{f(x)\}$$

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and consider the related problem

$$\text{minimize } \overline{f} : X \to \mathscr{I} \text{ with respect to } \preccurlyeq \text{ over } S \subseteq X. \qquad (\mathscr{V})$$

There is a close connection between the values of f and $\overline{f}$; that is, for all $x^1, x^2 \in X$ we have

$$f(x^1) \leq f(x^2) \qquad \Longleftrightarrow \qquad \overline{f}(x^1) \preccurlyeq \overline{f}(x^2).$$

Since the objective space $\mathscr{I}$ in Problem ($\mathscr{V}$) is a complete lattice, the latter correspondence allows us to develop the theory of vector optimization based on infimum and supremum.

We consider exemplary Lagrange duality and a finite dimensional variant of conjugate duality in order to demonstrate how the complete lattice $(\mathscr{I}, \preccurlyeq)$ can be used to transfer scalar duality results into a vectorial framework. Finally, we compare the results with other duality schemes from the literature and point out some advantages.

We present here a selection of concepts and results from [17] in order to make the reader familiar with some important ideas of vector optimization with infimum and supremum. For a comprehensive exposition the reader is referred to [17]. This book deals additionally with:

- Solution concepts based on the attainment of infimum
- Continuity and semicontinuity notions for $\mathscr{I}$-valued functions
- Existence of solutions
- Saddle point concepts based on infimum and supremum
- An infinite dimensional variant of conjugate duality
- Type II duality and existence of solutions to the dual problem
- Existence of saddle points
- Solution concepts and duality for linear problems
- Algorithms for linear problems

The mentioned concepts and results have been established over the last years, see e.g. [5–10, 15–18].

There are many other approaches to duality in vector optimization in the literature [3,11,19,24,25,27,28]. In the recent book by Boţ et al. [3], an overview and a general classification are given. The authors distinguish between duality via scalarization, Wolfe and Mond-Weir duality concepts and duality based on vector conjugacy. The results presented here are related in several aspects to the first and third class of duality schemes in [3]. The philosophy is, however, a completely different one. A consequent usage of the complete lattice $\mathscr{I}$ leads to concepts and results that are very similar to their scalar counterparts. Among all other approaches to duality in vector optimization the one by Tanino [27] seems to be the closest. The paper by Tanino [27] is partially based on several earlier works [12, 13, 19, 24, 26]. The conjugate in Definition 3.12 can be seen as a combination of the k-conjugate introduced in [28] and the conjugate considered in [27], see also Chap. 7 in [3]. One can sometimes observe similarities between the mentioned results from the literature and the results of this chapter. The difference is, however, that the infimality concept introduced by Nieuwenhuis [19] and used by many authors was not considered to be an infimum in a complete lattice. Set-valued optimization problems based on an ordering relation $\preccurlyeq$ on the power set have been investigated, for instance, by Kuroiwa [14], but the properties of the complete lattice were not used.

3.2 A Complete Lattice for Vector Optimization

We start this section with some basic definitions followed by three subsections: First we introduce infimal sets and provide several fundamental results concerning infimal sets. Section 3.2.2 deals with the space $\mathscr{I}$ of self-infimal sets, which is shown to be a complete lattice. The third subsection is devoted to scalarization methods for $\mathscr{I}$-valued problems, which will be used to derive duality results from their scalar counterparts.

Definition 3.1. Let $(Z,\leq)$ be a partially ordered set and let $A \subseteq Z$. An element $l \in Z$ is called *lower bound* of A if $l \leq z$ for all $z \in A$. An upper bound is defined analogously.

Next we define an *infimum* and a *supremum* for a subset A of a partially ordered set $(Z,\leq)$.

Definition 3.2. Let $(Z,\leq)$ be a partially ordered set and let $A \subseteq Z$. An element $k \in Z$ is called a *greatest lower bound* or *infimum* of $A \subseteq Z$ if k is a lower bound of A and for every other lower bound l of A we have $l \leq k$. We use the notation $k = \inf A$ for the infimum of A, if it exists.

The *least upper bound* or *supremum* is defined analogously and is denoted by $\sup A$. The lower (upper) bound of Z, if it exists, is called *least (greatest)* element.

Definition 3.3. A partially ordered set $(Z,\leq)$ is called a *complete lattice*, if the infimum and supremum exist for every subset $A \subseteq Z$.

Note that a one-sided condition is already sufficient to characterize a complete lattice. The existence of the infimum for all subsets implies the existence the supremum.

Example 3.1. The extended real numbers $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ equipped with the usual ordering $\leq$ provide a complete lattice.

The next example shows that the infimum in a partially ordered vector space, if it exists, is not related to the usual solution concepts of vector optimization. In typical problems there is no feasible point where this infimum is attained.

Example 3.2. Let $\leq$ be the componentwise ordering relation in $\mathbb{R}^q$. If the ordering relation $\leq$ is extended to $Z := \mathbb{R}^q \cup \{\pm\infty\}$ by setting $-\infty \leq z \leq +\infty$ for all $z \in Z$, $(Z,\leq)$ provides a complete lattice. The infimum of a subset $A \subseteq Z$ is

$$\inf A = \begin{cases} \left(\inf\limits_{z\in A} z_1, \dots, \inf\limits_{z\in A} z_q\right)^\top & \text{if} \qquad \exists b \in \mathbb{R}^q, \forall z \in A : b \le z \\ +\infty & \text{if} \qquad A = \emptyset \\ -\infty & \text{otherwise.} \end{cases}$$

The following example shows that the infimum in a partially ordered vector space does not need to exist.

Example 3.3. Let $Z = \mathbb{R}^3$ and let C be the polyhedral (convex) cone which is spanned by the vectors $(0,0,1)^\top$, $(0,1,1)^\top$, $(1,0,1)^\top$, $(1,1,1)^\top$. Then $(Z, \le_C)$ is not a complete lattice. For instance, there is no supremum of the finite set $\{(0,0,0)^\top, (1,0,0)^\top\}$.

Example 3.4. Let X be a nonempty set and let $\mathscr{P}(X) = 2^X$ be the power set of X. $(\mathscr{P}(X), \supseteq)$ provides a complete lattice. The infimum and supremum of a nonempty subset $\mathscr{A} \subseteq \mathscr{P}(X)$ are given as

$$\inf \mathscr{A} = \bigcup_{A \in \mathscr{A}} A \qquad \sup \mathscr{A} = \bigcap_{A \in \mathscr{A}} A.$$

Note that $X \in \mathscr{P}(X)$ is the least element and $\emptyset \in \mathscr{P}(X)$ is the greatest element in $(\mathscr{P}(X), \supseteq)$. If $\mathscr{A}$ is empty, we set $\sup \mathscr{A} = X$ and $\inf \mathscr{A} = \emptyset$.

Example 3.5. Let X be a topological space and let $\mathscr{F}(X)$ be the family of all closed subsets of X. $(\mathscr{F}(X), \supseteq)$ provides a complete lattice. The infimum and supremum of a nonempty subset $\mathscr{A} \subseteq \mathscr{F}(X)$ are given as

$$\inf \mathscr{A} = \operatorname{cl} \bigcup_{A \in \mathscr{A}} A \qquad \sup \mathscr{A} = \bigcap_{A \in \mathscr{A}} A.$$

If $\mathscr{A}$ is empty, we set again $\sup \mathscr{A} = X$ and $\inf \mathscr{A} = \emptyset$.

Let $(Y, \le)$ be a partially ordered vector space, that is, Y is partially ordered by a pointed (i.e. $C \cap (-C) = \{0\}$) convex cone C:

$$y^1 \le y^2 \quad :\Longleftrightarrow \quad y^2 - y^1 \in C.$$

An *extended partially ordered vector space* $(\overline{Y}, \le)$, where $\overline{Y} := Y \cup \{\pm\infty\}$, is defined by the additional rules

$$\begin{aligned} \forall y \in \overline{Y} : &\qquad -\infty \le y \le +\infty, \\ 0 \cdot (+\infty) = 0, &\qquad 0 \cdot (-\infty) = 0, \\ \forall \alpha > 0 : &\qquad \alpha \cdot (+\infty) = +\infty, \\ \forall \alpha > 0 : &\qquad \alpha \cdot (-\infty) = -\infty, \\ \forall y \in \overline{Y} : &\quad y + (+\infty) = +\infty + y = +\infty, \\ \forall y \in Y \cup \{-\infty\} : &\quad y + (-\infty) = -\infty + y = -\infty. \end{aligned}$$

In particular, we have $-\infty + (+\infty) = +\infty$. This type of addition is called *inf-addition*. It is the preferable choice for minimization problems, see [2, 17, 23] for further details.

If Y is additionally a topological vector space, we speak about an *extended partially ordered topological vector space* $(\overline{Y}, \leq)$. For our concerns it is sufficient the consider a topology on Y.

Note that the ordering cone C of a partially ordered vector space is always pointed and convex. Therefore, we do not mention these properties in the following.

3.2.1 Upper Closed and Infimal Sets

The infimal set of a subset A of an extended partially ordered topological vector space $\overline{Y}$ is a generalization of the infimum in $\overline{\mathbb{R}}$. But it is not an infimum in $\overline{Y}$. This already follows from the fact that an infimum in $\overline{Y}$ is an element and not a subset of $\overline{Y}$. Nevertheless, infimal sets play a crucial role in the following. They will be used to construct an appropriate complete lattice that allows us to treat vector optimization problems based on the notions infimum and supremum. Infimal and supremal sets are also used to express the infimum and supremum in this complete lattice. Upper closed sets are helpful to define infimal sets. They are closely related to infimal sets.

Definition 3.4. Let $\overline{Y}$ be an extended partially ordered topological vector space and let the ordering cone C of Y satisfy $\emptyset \neq \operatorname{int} C \neq Y$. The *upper closure* of a subset $A \subset \overline{Y}$ (with respect to C) is defined by

$$\operatorname{Cl}_+ A := \begin{cases} Y & \text{if } -\infty \in A \\ \emptyset & \text{if } A = \{+\infty\} \\ \{y \in Y |\ \{y\} + \operatorname{int} C \subseteq A \setminus \{+\infty\} + \operatorname{int} C\} & \text{otherwise.} \end{cases}$$

We continue with a characterization of upper closed sets.

Proposition 3.1. *Let $\overline{Y}$ be an extended partially ordered topological vector space with an ordering cone C such that $\emptyset \neq \operatorname{int} C \neq Y$ and let $A \subseteq \overline{Y}$. Then*

$$\operatorname{Cl}_+ A = \begin{cases} Y & \text{if } -\infty \in A \\ \emptyset & \text{if } A = \{+\infty\} \\ \operatorname{cl}\left(A \setminus \{+\infty\} + C\right) & \text{otherwise.} \end{cases}$$

Proof. Without loss of generality we can assume that $+\infty \notin A$. It remains to show that

$$B^1 := \{y \in Y |\ \{y\} + \operatorname{int} C \subseteq A + \operatorname{int} C\} = \operatorname{cl}\left(A + C\right) =: B^2.$$

(i) Let $y \in B^1$ and let $c \in \operatorname{int} C$. We have $y + \frac{1}{n} c \in A + \operatorname{int} C$ for all $n \in \mathbb{N}$. Taking the limit for $n \to \infty$, we obtain $y \in B^2$.

(ii) Let $y \in B^2$. We obtain

$$\{y\} + \operatorname{int} C \subseteq \operatorname{cl}\left(A + C\right) + \operatorname{int} C = A + C + \operatorname{int} C = A + \operatorname{int} C,$$

and hence $y \in B^1$. □

The concept of weakly minimal vectors is well-known in vector optimization. We will use it to define infimal sets and to define our complete lattice.

Definition 3.5. Let $\overline{Y}$ be an extended partially ordered topological vector space with an ordering cone C such that $\emptyset \neq \operatorname{int} C \neq Y$. The set of *weakly minimal* points of a subset $A \subseteq Y$ (with respect to C) is defined by

$$\operatorname{wMin} A := \{y \in A |\ (\{y\} - \operatorname{int} C) \cap A = \emptyset\}.$$

The next result will be used to show the existence of weakly minimal points.

Theorem 3.1 ([17]). *Let A and B be subsets of a topological vector space Y. Let B be convex and assume that* $\operatorname{cl} A \cap B \neq \emptyset$. *Then*

$$\operatorname{cl} A \cap B \subseteq \operatorname{int} A \quad \Longrightarrow \quad A \supseteq B.$$

Proof. Let $a \in \operatorname{cl} A \cap B$ and assume there is some $b \in B \setminus A$. We consider the expression

$$\overline{\lambda} := \inf\{\lambda \geq 0 |\ \lambda a + (1-\lambda) b \in A\}.$$

There exists a sequence $(\lambda_n) \to \overline{\lambda}$ such that $\lambda_n a + (1-\lambda_n) b \in A$ for all $n \in \mathbb{N}$. As B is convex and $\overline{\lambda} \in [0,1]$, we conclude

$$\overline{\lambda} a + (1-\overline{\lambda}) b \in \operatorname{cl} A \cap B \subseteq \operatorname{int} A.$$

In particular, we see that $\overline{\lambda} > 0$. On the other hand, there is a sequence $(\mu_n) \to \overline{\lambda}$ such that $\mu_n a + (1-\mu_n) b \notin A$ for all $n \in \mathbb{N}$. This yields

$$\overline{\lambda} a + (1-\overline{\lambda}) b \notin \operatorname{int} A.$$

This is a contradiction. □

The next theorem shows the existence of weakly minimal elements. It is based on Theorem 3.1.

Theorem 3.2 ([17]). *Let* $\overline{Y}$ *be an extended partially ordered topological vector space, where the ordering cone C satisfies* $\emptyset \neq \operatorname{int} C \neq Y$. *Let* $A \subseteq \overline{Y}$ *be an arbitrary set and let* $B \subseteq Y$ *be a convex set. Let* $\operatorname{Cl}_+ A \cap B \neq \emptyset$ *and* $B \setminus \operatorname{Cl}_+ A \neq \emptyset$. *Then*

$$\operatorname{wMin}\left(\operatorname{Cl}_+ A \cap B\right) \neq \emptyset.$$

Proof. Assuming that $\mathrm{wMin}\,(\mathrm{Cl}_+A \cap B)$ is empty, we get

$$\forall y \in \mathrm{Cl}_+A \cap B,\ \exists z \in \mathrm{Cl}_+A \cap B: \ \ y \in \{z\} + \mathrm{int}\, C.$$

This implies

$$\mathrm{Cl}_+A \cap B \subseteq (\mathrm{Cl}_+A \cap B) + \mathrm{int}\, C.$$

It follows

$$\begin{aligned}\mathrm{Cl}_+A \cap B &\subseteq (\mathrm{Cl}_+A \cap B) + \mathrm{int}\, C\\ &\subseteq (\mathrm{Cl}_+A + \mathrm{int}\, C) \cap (B + \mathrm{int}\, C)\\ &\subseteq \mathrm{Cl}_+A + \mathrm{int}\, C\\ &\subseteq \mathrm{int}\,\mathrm{Cl}_+A.\end{aligned}$$

Theorem 3.1 yields $\mathrm{Cl}_+A \supseteq B$, which contradicts the assumption $B \setminus \mathrm{Cl}_+A \neq \emptyset$. □

We continue with a further theorem concerning weakly minimal elements.

Theorem 3.3 ([17]). *Let $\overline{Y}$ be an extended partially ordered topological vector space, where the ordering cone C satisfies $\emptyset \neq \mathrm{int}\, C \neq Y$. Let $A \subseteq \overline{Y}$ be an arbitrary set and let $B \subseteq Y$ be an open set. Then*

$$\mathrm{wMin}\,(\mathrm{Cl}_+A \cap B) = (\mathrm{wMin}\,\mathrm{Cl}_+A) \cap B.$$

Proof. In order to prove the inclusion $\mathrm{wMin}\,(\mathrm{Cl}_+A \cap B) \subseteq (\mathrm{wMin}\,\mathrm{Cl}_+A) \cap B$, let $y \in \mathrm{wMin}\,(\mathrm{Cl}_+A \cap B)$. Of course, this implies $y \in B$ and $y \in \mathrm{Cl}_+A$. It remains to show that $(\{y\} - \mathrm{int}\, C) \cap \mathrm{Cl}_+A = \emptyset$. Assuming the contrary, we get some $z \in \mathrm{Cl}_+A$ such that $c := y - z \in \mathrm{int}\, C$. As B is open, there exists some $\varepsilon \in (0,1)$ such that $w := y - \varepsilon c \in B$. From $z \in \mathrm{Cl}_+A$ we conclude $w \in \mathrm{Cl}_+A + \mathrm{int}\, C \subseteq \mathrm{Cl}_+A$. Thus, we have $w \in (\{y\} - \mathrm{int}\, C) \cap (\mathrm{Cl}_+A \cap B)$ and hence $y \notin \mathrm{wMin}\,(\mathrm{Cl}_+A \cap B)$, a contradiction.

The opposite inclusion $\supseteq$ follows immediately from the definition. □

The following conclusion of Theorem 3.2 will play a crucial role.

Corollary 3.1. *Let $\overline{Y}$ be an extended partially ordered topological vector space, where the ordering cone C satisfies $\emptyset \neq \mathrm{int}\, C \neq Y$. For every set $A \subseteq \overline{Y}$ the following statements are equivalent:*

(a) $\emptyset \neq \mathrm{Cl}_+A \neq Y$
(b) $\mathrm{wMin}\,\mathrm{Cl}_+A \neq \emptyset$

Proof. (a) ⇒ (b): Follows from Theorem 3.2 for the choice $B = Y$.

(b) ⇒ (a): Follows from the definition of wMin. □

We now define a central concept of this exposition, an infimal set for a subset of the extended space $\overline{Y}$.

Definition 3.6. Let $\overline{Y}$ be an extended partially ordered topological vector space with an ordering cone C such that $\emptyset \neq \operatorname{int} C \neq Y$. The *infimal set* of $A \subset \overline{Y}$ (with respect to C) is defined by

$$\operatorname{Inf} A := \begin{cases} \operatorname{wMin} \operatorname{Cl}_+ A & \text{if } \emptyset \neq \operatorname{Cl}_+ A \neq Y \\ \{-\infty\} & \text{if } \operatorname{Cl}_+ A = Y \\ \{+\infty\} & \text{if } \operatorname{Cl}_+ A = \emptyset. \end{cases}$$

If A is a nonempty subset of Y and $\operatorname{cl}(A+C) \neq Y$, then $\operatorname{Inf} A = \operatorname{wMin} \operatorname{cl}(A+C)$. By Corollary 3.1, $\operatorname{Inf} A$ is always a nonempty set. Clearly, if $-\infty$ belongs to A, we have $\operatorname{Inf} A = \{-\infty\}$, in particular, $\operatorname{Inf}\{-\infty\} = \{-\infty\}$. Moreover, we have $\operatorname{Inf} \emptyset = \operatorname{Inf}\{+\infty\} = \{+\infty\}$. Furthermore, $\operatorname{Cl}_+ A = \operatorname{Cl}_+(A \cup \{+\infty\})$ holds and hence $\operatorname{Inf} A = \operatorname{Inf}(A \cup \{+\infty\})$ for all $A \subset \overline{Y}$.

We close this subsection with several useful results concerning the infimal set and the upper closure. We derive them from Theorem 3.2 (based on Theorem 3.1) and Theorem 3.3.

Theorem 3.4. *Let $\overline{Y}$ be an extended partially ordered topological vector space with an ordering cone C such that $\emptyset \neq \operatorname{int} C \neq Y$. For $A \subset \overline{Y}$ with $\emptyset \neq \operatorname{Cl}_+ A \neq Y$, it is true that*

$$\operatorname{Cl}_+ A + \operatorname{int} C \subseteq \operatorname{Inf} A + \operatorname{int} C.$$

Proof. Let $y \in \operatorname{Cl}_+ A + \operatorname{int} C$, then $\big(\{y\} - \operatorname{int} C\big) \cap \operatorname{Cl}_+ A \neq \emptyset$. We set $B := \{y\} - \operatorname{int} C$. As B is convex and open, Theorems 3.2 and Lemma 3.3 imply that

$$\emptyset \neq \operatorname{wMin}(\operatorname{Cl}_+ A \cap B) = (\operatorname{wMin} \operatorname{Cl}_+ A) \cap B.$$

Thus, there exists some $z \in \operatorname{wMin} \operatorname{Cl}_+ A = \operatorname{Inf} A$ such that $y \in \{z\} + \operatorname{int} C$, whence $y \in \operatorname{Inf} A + \operatorname{int} C$. □

Theorem 3.5. *Let $\overline{Y}$ be an extended partially ordered topological vector space with an ordering cone C such that $\emptyset \neq \operatorname{int} C \neq Y$. For $A \subseteq \overline{Y}$ with $\emptyset \neq \operatorname{Cl}_+ A \neq Y$, the following statement holds true:*

$$\operatorname{Cl}_+ A \,\cup\, (\operatorname{Inf} A - \operatorname{int} C) = Y.$$

Proof. We have $\operatorname{Cl}_+ A - \operatorname{int} C \supseteq \{z\} + \operatorname{int} C - \operatorname{int} C = Y$ for every $z \in \operatorname{Cl}_+ A$. Let $y \in Y \setminus \operatorname{Cl}_+ A$. The set $B := \{y\} + \operatorname{int} C$ is open and convex. Moreover, we have $\operatorname{Cl}_+ A \cap B \neq \emptyset$, since otherwise we get the contradiction $y \notin \operatorname{Cl}_+ A - \operatorname{int} C = Y$. We show that $B \setminus \operatorname{Cl}_+ A \neq \emptyset$. Indeed, assuming the contrary, we obtain $B \subseteq \operatorname{Cl}_+ A$ which implies the contradiction $y \in \operatorname{cl} B \subseteq \operatorname{Cl}_+ A$. Theorems 3.2 and 3.3 imply

$$\emptyset \neq \operatorname{wMin}(\operatorname{Cl}_+ A \cap B) = (\operatorname{wMin} \operatorname{Cl}_+ A) \cap B.$$

Consequently, there exists some $z \in \operatorname{wMin} \operatorname{Cl}_+ A = \operatorname{Inf} A$ such that $z \in B = \{y\} + \operatorname{int} C$. Hence $y \in \operatorname{Inf} A - \operatorname{int} C$. □

The next corollary provides a collection of useful properties with respect to infimal sets.

Corollary 3.2. *Let $\overline{Y}$ be an extended partially ordered topological vector space with an ordering cone C such that $\emptyset \neq \operatorname{int} C \neq Y$. If $A, B \subseteq \overline{Y}$ with $\emptyset \neq \mathrm{Cl}_+ A \neq Y$ and $\emptyset \neq \mathrm{Cl}_+ B \neq Y$, then:*

(a) $\mathrm{Cl}_+ A + \operatorname{int} C = \operatorname{Inf} A + \operatorname{int} C$
(b) $\operatorname{Inf} A = \{y \in Y |\ \{y\} + \operatorname{int} C \subseteq \mathrm{Cl}_+ A + \operatorname{int} C \ \wedge\ y \notin \mathrm{Cl}_+ A + \operatorname{int} C\}$
(c) $\operatorname{int} \mathrm{Cl}_+ A = \mathrm{Cl}_+ A + \operatorname{int} C$
(d) $\operatorname{Inf} A = \operatorname{bd} \mathrm{Cl}_+ A$
(e) $\operatorname{Inf} A = \mathrm{Cl}_+ A \setminus (\mathrm{Cl}_+ A + \operatorname{int} C)$
(f) $\mathrm{Cl}_+ A = \mathrm{Cl}_+ B \iff \operatorname{Inf} A = \operatorname{Inf} B$
(g) $\mathrm{Cl}_+ A = \mathrm{Cl}_+ B \iff \mathrm{Cl}_+ A + \operatorname{int} C = \mathrm{Cl}_+ B + \operatorname{int} C$
(h) $\operatorname{Inf} A = \operatorname{Inf} B \iff \operatorname{Inf} A + \operatorname{int} C = \operatorname{Inf} B + \operatorname{int} C$
(i) $\mathrm{Cl}_+ A = \operatorname{Inf} A \cup (\operatorname{Inf} A + \operatorname{int} C)$
(j) $\operatorname{Inf} A$, $(\operatorname{Inf} A - \operatorname{int} C)$ *and* $(\operatorname{Inf} A + \operatorname{int} C)$ *are pairwise disjoint*
(k) $\operatorname{Inf} A \cup (\operatorname{Inf} A - \operatorname{int} C) \cup (\operatorname{Inf} A + \operatorname{int} C) = Y$

Proof. (a) We have $\operatorname{Inf} A = \operatorname{wMin} \mathrm{Cl}_+ A \subseteq \mathrm{Cl}_+ A$ and hence $\operatorname{Inf} A + \operatorname{int} C \subseteq \mathrm{Cl}_+ A + \operatorname{int} C$. The opposite inclusion is just the statement of Theorem 3.4.
(b) Follows from the definitions of upper closure and weakly minimal points.
(c) Let $y \in \operatorname{int} \mathrm{Cl}_+ A$ and let $c \in \operatorname{int} C$. There is some $t > 0$ such that $y - tc \in \mathrm{Cl}_+ A$ and hence $y \in \mathrm{Cl}_+ A + \operatorname{int} C$. On the other hand, we have $\mathrm{Cl}_+ A + \operatorname{int} C \subseteq \mathrm{Cl}_+ A$. The set $\mathrm{Cl}_+ A + \operatorname{int} C$ is open, whence $\mathrm{Cl}_+ A + \operatorname{int} C \subseteq \operatorname{int} \mathrm{Cl}_+ A$.
(d) Follows from (b) and (c).
(e) Follows from (b).
(f) Taking the closure, $\mathrm{Cl}_+ A + \operatorname{int} C = \mathrm{Cl}_+ B + \operatorname{int} C$ implies $\mathrm{Cl}_+ A = \mathrm{Cl}_+ B$. By definition this yields $\operatorname{Inf} A = \operatorname{Inf} B$.
On the other hand, $\operatorname{Inf} A = \operatorname{Inf} B$ implies $\operatorname{Inf} A + \operatorname{int} C = \operatorname{Inf} B + \operatorname{int} C$ which is by (a) equivalent to $\mathrm{Cl}_+ A + \operatorname{int} C = \mathrm{Cl}_+ B + \operatorname{int} C$.
(g) By Proposition 3.1, we have

$$\begin{aligned} \operatorname{cl}\left(\mathrm{Cl}_+ A + \operatorname{int} C\right) &= \operatorname{cl}\left(\operatorname{cl}\,(A \setminus \{+\infty\} + C) + \operatorname{int} C\right) \\ &= \operatorname{cl}\left(A \setminus \{+\infty\} + C\right) = \mathrm{Cl}_+ A. \end{aligned}$$

The statement is now obvious.
(h) Follows from (a), (f) and (g).
(i) Let $y \in \mathrm{Cl}_+ A$. In the case where $y \in \mathrm{Cl}_+ A + \operatorname{int} C$, (i) implies $y \in \operatorname{Inf} A + \operatorname{int} C$. Otherwise, if $y \notin \mathrm{Cl}_+ A + \operatorname{int} C$, we obtain $y \in \operatorname{wMin} \mathrm{Cl}_+ A = \operatorname{Inf} A$. On the other hand, it is obvious that $\operatorname{Inf} A = \operatorname{wMin} \mathrm{Cl}_+ A \subseteq \mathrm{Cl}_+ A$ and $\operatorname{Inf} A + \operatorname{int} C \subseteq \mathrm{Cl}_+ A + \operatorname{int} C \subseteq \mathrm{Cl}_+ A$.
(j) Let $y \in \operatorname{Inf} A - \operatorname{int} C$. There exists $z \in \operatorname{Inf} A$ such that $y \in \{z\} - \operatorname{int} C$. We have $(\{z\} - \operatorname{int} C) \cap \mathrm{Cl}_+ A = \emptyset$ and hence $y \notin \mathrm{Cl}_+ A$. From (i) we get $(\operatorname{Inf} A - \operatorname{int} C) \cap \operatorname{Inf} A = \emptyset$ and $(\operatorname{Inf} A - \operatorname{int} C) \cap (\operatorname{Inf} A + \operatorname{int} C) = \emptyset$. By (a) and the definition of an infimal set, we conclude $\operatorname{Inf} A \cap (\operatorname{Inf} A + \operatorname{int} C) = \emptyset$.
(k) Follows from (i) and Theorem 3.5. $\square$

Corollary 3.3. *Let $A \subset \overline{Y}$. Then*

(a) $\operatorname{Inf}\operatorname{Inf}A = \operatorname{Inf}A$, $\operatorname{Cl}_+\operatorname{Cl}_+A = \operatorname{Cl}_+A$, $\operatorname{Inf}\operatorname{Cl}_+A = \operatorname{Inf}A$, $\operatorname{Cl}_+\operatorname{Inf}A = \operatorname{Cl}_+A$
(b) $\operatorname{Inf}(\operatorname{Inf}A + \operatorname{Inf}B) = \operatorname{Inf}(A+B)$
(c) $\alpha \operatorname{Inf}A = \operatorname{Inf}(\alpha A)$ *for* $\alpha > 0$

Proof. This follows from the definitions and results of this section. □

The set $\operatorname{wMax}A$ of *weakly maximal elements* of $A \subseteq Y$, as well as the lower closure Cl_-A and the *supremal set* $\operatorname{Sup}A$ of a subset $A \subseteq \overline{Y}$ are defined likewise. One has

$$\operatorname{Sup}A = -\operatorname{Inf}(-A) \tag{3.1}$$

and analogous results hold true, where the sup-addition has to be used.

3.2.2 The Space of Self-Infimal Sets

This subsection is devoted to properties of the space $\mathscr{I}$ of self-infimal sets. We first give a precise definition including the elements $\pm\infty \in \overline{Y}$. We introduce an addition, a multiplication by nonnegative real numbers and an ordering relation such that $(\mathscr{I}, \preccurlyeq)$ can regarded to be an extension of $(\overline{Y}, \leq)$. As the main result of this subsection, we show that $(\mathscr{I}, \preccurlyeq)$ provides a complete lattice, where the infimum and supremum of every nonempty set $\mathscr{B} \subset \mathscr{I}$ can be expressed as

$$\inf \mathscr{B} = \operatorname{Inf} \bigcup_{B \in \mathscr{B}} B, \qquad \sup \mathscr{B} = \operatorname{Sup} \bigcup_{B \in \mathscr{B}} B.$$

Expressions of this type are commonly used in vector optimization, see e.g. [27]. This means that all these statements can considered in the framework of a complete lattice. Concepts from scalar optimization that require an infimum or supremum can be reformulated in a vectorial framework. Note that in comparison to Example 3.3 no further assumptions are required for the existence of infimum and supremum.

By technical reasons we consider parallel the space of upper closed sets, which turns out to be isomorphic and isotone to $\mathscr{I}$.

Definition 3.7. Let $\overline{Y}$ be an extended partially ordered topological vector space with an ordering cone C such that $\emptyset \neq \operatorname{int} C \neq Y$. A set $A \subseteq Y$ is called an *upper closed set* if $\operatorname{Cl}_+A = A$. A subset $B \subseteq \overline{Y}$ is called *self-infimal* if $\operatorname{Inf}B = B$ holds.

Let $\mathscr{F} := \mathscr{F}_C(Y)$ be the family of all upper closed subsets of Y. In $\mathscr{F}$ we introduce an addition $\oplus : \mathscr{F} \times \mathscr{F} \to \mathscr{F}$ and a multiplication by nonnegative real numbers $\odot : \mathbb{R}_+ \times \mathscr{F} \to \mathscr{F}$ as

$$A^1 \oplus A^2 := \operatorname{cl}(A^1 + A^2),$$

$$\alpha \odot A := \operatorname{Cl}_+(\alpha \cdot A).$$

It is easy to verify that $(\mathscr{F}, \oplus, \odot)$ provides a *conlinear space [5]* which is defined as follows.

Definition 3.8. A nonempty set Z equipped with an addition $+ : Z \times Z \to Z$ and a multiplication $\cdot : \mathbb{R}_+ \times Z \to Z$ is said to be a *conlinear space* with the *neutral element* $\theta \in Z$ if for all $z, z^1, z^2 \in Z$ and all $\alpha, \beta \geq 0$ the following axioms are satisfied:

(C1) $z^1 + (z^2 + z) = (z^1 + z^2) + z$
(C2) $z + \theta = z$
(C3) $z^1 + z^2 = z^2 + z^1$
(C4) $\alpha \cdot (\beta \cdot z) = (\alpha\beta) \cdot z$
(C5) $1 \cdot z = z$
(C6) $0 \cdot z = \theta$
(C7) $\alpha \cdot (z^1 + z^2) = (\alpha \cdot z^1) + (\alpha \cdot z^2)$

Conlinear spaces are adequate to deal with convexity and cones because neither of the definitions require a multiplication by a negative real number or the existence of inverse elements. For instance, if X is a linear space, a function $f : X \to \mathscr{F}$ is said to be *convex* if for all $x^1, x^2 \in X$ and for all $\lambda \in (0,1)$ one has

$$f(\lambda x^1 + (1-\lambda)x^2) \preccurlyeq f(\lambda \odot x^1) \oplus f((1-\lambda) \odot x^2). \tag{3.2}$$

Note that the power set $\mathscr{P}(Y)$ is supposed to be a conlinear space with respect to the Minkowski-addition and the usual multiplication by nonnegative numbers. In particular, we use the rule $0 \cdot \emptyset = \{0\}$, which implies that $0 \odot \emptyset = \mathrm{Cl}_+\{0\} = \mathrm{cl}\, C$. For more details the reader is referred to [5, 17].

Let $\mathscr{I} := \mathscr{I}_C(\overline{Y})$ be the family of all self-infimal subsets of $\overline{Y}$. In $\mathscr{I}$ we introduce an addition $\oplus : \mathscr{I} \times \mathscr{I} \to \mathscr{I}$, a multiplication by nonnegative real numbers $\odot : \mathbb{R}_+ \times \mathscr{I} \to \mathscr{I}$ and an order relation $\preccurlyeq$ by

$$\begin{aligned} B^1 \oplus B^2 &:= \mathrm{Inf}\,(B^1 + B^2), \\ \alpha \odot B &:= \mathrm{Inf}\,(\alpha \cdot B), \\ B^1 \preccurlyeq B^2 \ &:\iff \mathrm{Cl}_+ B^1 \supseteq \mathrm{Cl}_+ B^2. \end{aligned}$$

Note that the definition of the addition $\oplus$ in $\mathscr{I}$ is based on the inf-addition in $\overline{Y}$. As a consequence we obtain $\{-\infty\} \oplus \{+\infty\} = \{+\infty\}$. Moreover, we get $0 \odot B = \mathrm{Inf}\,\{0\} = \mathrm{bd}\, C$ for all $B \in \mathscr{I}$. Convex functions with values in $\mathscr{I}$ are defined likewise to (3.2).

Proposition 3.2. *The space* $(\mathscr{I}, \oplus, \odot, \preccurlyeq)$ *is a partially ordered conlinear space with the neutral element* $\mathrm{Inf}\,\{0\} = \mathrm{bd}\, C$. *The spaces* $(\mathscr{F}, \oplus, \odot, \supseteq)$ *and* $(\mathscr{I}, \oplus, \odot, \preccurlyeq)$ *are isomorphic and isotone. The corresponding bijection is given by*

$$j : \mathscr{F} \to \mathscr{I}, \quad j(\cdot) = \mathrm{Inf}\,(\cdot), \quad j^{-1}(\cdot) = \mathrm{Cl}_+(\cdot).$$

Proof. By Corollary 3.3 (a), j is a bijection between $\mathscr{F}$ and $\mathscr{I}$. From Corollary 3.3 (b) we obtain that $j(A^1) \oplus j(A^2) = j(A^1 \oplus A^2)$ for all $A^1, A^2 \in \mathscr{F}$. It can easily be verified that $\alpha \odot j(A) = j(\alpha \odot A)$ for all $\alpha \leq 0$ and all $A \in \mathscr{F}$. From the definition of the ordering $\preccurlyeq$ in $\mathscr{I}$, we conclude that we have $A^1 \supset A^2$ if and only if $j\left(A^1\right) \preccurlyeq j\left(A^2\right)$. □

Proposition 3.3. *$(\mathscr{F}, \supseteq)$ and $(\mathscr{I}, \preccurlyeq)$ are complete lattices. For nonempty subsets $\mathscr{A} \subseteq \mathscr{F}$ and $\mathscr{B} \subseteq \mathscr{I}$ the infimum and supremum can be expressed by*

$$\inf \mathscr{A} = \operatorname{cl} \bigcup_{A \in \mathscr{A}} A, \qquad \sup \mathscr{A} = \bigcap_{A \in \mathscr{A}} A,$$

$$\inf \mathscr{B} = \operatorname{Inf} \bigcup_{B \in \mathscr{B}} \operatorname{Cl}_+ B, \qquad \sup \mathscr{B} = \operatorname{Inf} \bigcap_{B \in \mathscr{B}} \operatorname{Cl}_+ B.$$

Proof. For the space $(\mathscr{F}, \supseteq)$ the statements are obvious and for $(\mathscr{I}, \preccurlyeq)$ they follow from Proposition 3.2. □

As usual, if $\mathscr{A} \subseteq \mathscr{F}$ and $\mathscr{B} \subseteq \mathscr{I}$ are empty, we define the infimum (supremum) to be the greatest (least) element in the corresponding complete lattice, i.e., $\inf \mathscr{A} = \emptyset$, $\sup \mathscr{A} = Y$, $\inf \mathscr{B} = \{+\infty\}$ and $\sup \mathscr{B} = \{-\infty\}$.

For vector optimization, the following characterization of infimum and supremum is important.

Theorem 3.6 ([17, 18]). *For nonempty sets $\mathscr{B} \subseteq \mathscr{I}$, we have*

$$\inf \mathscr{B} = \operatorname{Inf} \bigcup_{B \in \mathscr{B}} B, \qquad \sup \mathscr{B} = \operatorname{Sup} \bigcup_{B \in \mathscr{B}} B.$$

Proof. (i) The expression for the infimum can be shown as follows:

$$\begin{aligned} \inf \mathscr{B} &= \operatorname{Inf} \bigcup_{B \in \mathscr{B}} \operatorname{Cl}_+ B = \operatorname{Inf} \operatorname{Cl}_+ \bigcup_{B \in \mathscr{B}} \operatorname{Cl}_+ B \\ &= \operatorname{Inf} \operatorname{Cl}_+ \bigcup_{B \in \mathscr{B}} B = \operatorname{Inf} \bigcup_{B \in \mathscr{B}} B. \end{aligned}$$

(ii) Let us prove the expression for the supremum. By Proposition 3.3, it remains to show that $\operatorname{Sup} \bigcup_{B \in \mathscr{B}} B = \operatorname{Inf} \bigcap_{B \in \mathscr{B}} \operatorname{Cl}_+ B$. We distinguish three cases:

(a) If $\{+\infty\} \in \mathscr{B}$, we have $+\infty \in \bigcup_{B \in \mathscr{B}} B$ and hence $\operatorname{Sup} \bigcup_{B \in \mathscr{B}} B = \{+\infty\}$. On the other hand, since $\operatorname{Cl}_+ \{+\infty\} = \emptyset$, we have $\operatorname{Inf} \bigcap_{B \in \mathscr{B}} \operatorname{Cl}_+ B = \operatorname{Inf} \emptyset = \{+\infty\}$.

(b) Let $\{+\infty\} \notin \mathscr{B}$ but $\{-\infty\} \in \mathscr{B}$. If $\{-\infty\}$ is the only element in $\mathscr{B}$ the assertion is obvious, otherwise we can omit this element without changing the expressions.

(c) Let $\{+\infty\} \notin \mathscr{B}$ and $\{-\infty\} \notin \mathscr{B}$. Then $B \subseteq Y$ and $\emptyset \neq \operatorname{Cl}_+ B \neq Y$ for all $B \in \mathscr{B}$, i.e., we can use the statements of Corollary 3.2. We define the sets $V := \bigcup_{B \in \mathscr{B}} (B - \operatorname{int} C) = \left(\bigcup_{B \in \mathscr{B}} B\right) - \operatorname{int} C$ and $W := \bigcap_{B \in \mathscr{B}} \operatorname{Cl}_+ B$.

We show that $V \cap W = \emptyset$ and $V \cup W = Y$. Assume there exists some $y \in V \cap W$. Then, there is some $\overline{B} \in \mathscr{B}$ such that $y \in \left(\overline{B} - \operatorname{int} C\right) \cap \operatorname{Cl}_+ \overline{B} = \emptyset$, a contradiction. Let $y \in Y \setminus W$ (we have $W \neq Y$, because otherwise $\operatorname{Cl}_+ B = Y$ holds for all $B \in \mathscr{B}$). Then there exists some $\overline{B} \in \mathscr{B}$ such that $y \notin \operatorname{Cl}_+ \overline{B}$. By Corollary 3.2 (i), (k) we obtain $y \in \overline{B} - \operatorname{int} C \subset V$.

If $\operatorname{Cl}_- V = Y$ we get $V = \operatorname{Cl}_- V - \operatorname{int} C = Y$, whence $W = \emptyset$. It follows $\operatorname{Sup} \bigcup_{B \in \mathscr{B}} B = \operatorname{Sup} V = \{+\infty\} = \operatorname{Inf} \emptyset = \operatorname{Inf} W$. Otherwise, we have $\emptyset \neq \operatorname{Cl}_- V \neq Y$ and $\emptyset \neq \operatorname{Cl}_+ W \neq Y$. By Corollary 3.2, we obtain

$$
\begin{aligned}
\operatorname{Sup} \bigcup_{B \in \mathscr{B}} B &= \{y \in Y \mid y \notin V,\ \{y\} - \operatorname{int} C \subset V\} \\
&= \{y \in Y \mid y \in W, (\{y\} - \operatorname{int} C) \cap W = \emptyset\} \\
&= \operatorname{wMin} W = \operatorname{wMin} \operatorname{Cl}_+ W = \operatorname{Inf} W,
\end{aligned}
$$

which completes the proof. □

Even though $\mathscr{I}$ is not a linear space we have the following result.

Corollary 3.4. *One has $A \in \mathscr{I}$ if and only if $-A \in \mathscr{I}$.*

Proof. Let $A \in \mathscr{I}$. Of course, we have $\sup\{A\} = A$ and Theorem 3.6 yields $\operatorname{Sup} A = A$. It follows that $-A = -\operatorname{Sup} A = \operatorname{Inf}(-A)$ and hence $-A \in \mathscr{I}$. □

Note that the last statement is not true for $A \in \mathscr{F}$. Nevertheless, it is sometimes easier to work with the complete lattice $\mathscr{F}$ in the proofs. The corresponding results for the space $\mathscr{I}$ can be obtained using Proposition 3.2.

In the following proposition we use a generalization of the Minkowski sum. For $\mathscr{A}, \mathscr{B} \subseteq \mathscr{I}$, we set

$$
\mathscr{A} \oplus \mathscr{B} := \{I \in \mathscr{I} \mid \exists A \in \mathscr{A}, \exists B \in \mathscr{B} :\ I = A \oplus B\}.
$$

Proposition 3.4. *Let $\mathscr{A}, \mathscr{B} \subseteq \mathscr{I}$, then:*

(a) $\inf \mathscr{A} \oplus \mathscr{B} = \inf \mathscr{A} \oplus \inf \mathscr{B}$

(b) $\sup \mathscr{A} \oplus \mathscr{B} \preccurlyeq \sup \mathscr{A} \oplus \sup \mathscr{B}$

Proof. (a) If $\mathscr{A} = \emptyset$, we have $\inf \mathscr{A} \oplus \mathscr{B} = \inf \mathscr{A} = \{+\infty\}$ and thus $\inf \mathscr{A} \oplus \mathscr{B} = \inf \mathscr{A} \oplus \inf \mathscr{B} = \{+\infty\}$. Otherwise, we get

$$
\begin{aligned}
\inf \mathscr{A} \oplus \mathscr{B} = \operatorname{Inf} \bigcup_{A \in \mathscr{A}, B \in \mathscr{B}} A \oplus B &= \operatorname{Inf} \bigcup_{A \in \mathscr{A}, B \in \mathscr{B}} A + B \\
&= \operatorname{Inf} \left(\bigcup_{A \in \mathscr{A}} A + \bigcup_{B \in \mathscr{B}} B \right) = \operatorname{Inf} \bigcup_{A \in \mathscr{A}} A \oplus \operatorname{Inf} \bigcup_{B \in \mathscr{B}} B \\
&= \inf \mathscr{A} \oplus \inf \mathscr{B}.
\end{aligned}
$$

(b) For all $A \in \mathscr{A}$, $B \in \mathscr{B}$ we have $A \oplus B \preccurlyeq \sup \mathscr{A} \oplus \sup \mathscr{B}$. Taking the supremum, we obtain the desired statement. □

The following example shows that Proposition 3.4 (b) does not hold with equality.

Example 3.6 ([17]). We consider the space $\mathscr{I}$ for $Y = \mathbb{R}^2$ and $C = \mathbb{R}^2_+$. Let $\mathscr{A} = \{A^1, A^2\}$ and $\mathscr{B} = \{B\}$, where we set $A^1 = \mathrm{Inf}\,\{(0,1)^\top\}$, $A^2 = \mathrm{Inf}\,\{(1,0)^\top\}$, $B = \{y \in \mathbb{R}^2 |\; y_1 + y_2 = 0\}$. Then, we have

$$A^1 \oplus B = A^2 \oplus B = \{y \in \mathbb{R}^2 |\; y_1 + y_2 = 1\},$$
$$\sup \mathscr{A} \oplus \mathscr{B} = \sup\{A^1 \oplus B, A^2 \oplus B\} = \{y \in \mathbb{R}^2 |\; y_1 + y_2 = 1\},$$
$$\sup \mathscr{A} \oplus \sup \mathscr{B} = \sup\{A^1, A^2\} \oplus B = \{y \in \mathbb{R}^2 |\; y_1 + y_2 = 2\}.$$

Whence $\sup \mathscr{A} \oplus \mathscr{B} \neq \sup \mathscr{A} \oplus \sup \mathscr{B}$.

If $(Z, +, \cdot\,)$ is a conlinear space, we denote by Z_{co} the subset of all $z \in Z$ satisfying the second distributive law

$$\text{(C8)} \qquad \alpha \cdot z + \beta \cdot z = (\alpha + \beta) \cdot z.$$

$(Z_{\mathrm{co}}, +, \cdot\,)$ is again a conlinear space. The additional axiom ensures that every singleton set of a conlinear space is convex [5, 17].

Let us consider the partially ordered conlinear spaces $(\mathscr{F}_{\mathrm{co}}, \oplus, \odot, \supseteq)$ and $(\mathscr{I}_{\mathrm{co}}, \oplus, \odot, \preccurlyeq)$, which are isomorphic and isotone [17]. The next result shows that convex functions $\mathscr{I}$-valued functions are actually $\mathscr{I}_{\mathrm{co}}$-valued.

Proposition 3.5 ([17]). *Let X be a vector space, $S \subseteq X$ a convex subset of X and $f : X \to \mathscr{I}$ a convex function. Then:*

(a) $f : X \to \mathscr{I}_{\mathrm{co}}$
(b) $\inf_{x \in S} f(x) \in \mathscr{I}_{\mathrm{co}}$

Proof. Since (a) is a special case (set $S = \{x\}$), it remains to show (b). For all $\lambda \in [0,1]$, we have

$$\begin{aligned} \inf_{x \in S} f(x) &= \inf_{y,z \in S} f(\lambda y + (1-\lambda) z) \\ &\preccurlyeq \inf_{y,z \in S} (\lambda \odot f(y) \oplus (1-\lambda) \odot f(z)) \\ &\overset{\text{Pr. 3.4 (a)}}{=} \lambda \odot \inf_{x \in S} f(x) \oplus (1-\lambda) \odot \inf_{x \in S} f(x). \end{aligned}$$

Hence $\inf_{x \in S} f(x) \in \mathscr{I}_{\mathrm{co}}$.

Of course, for every $A \in \mathscr{F}$, we have $\lambda \odot A \oplus (1-\lambda) \odot A \supseteq A$. As the spaces $(\mathscr{F}, \oplus, \odot, \supseteq)$ and $(\mathscr{I}, \oplus, \odot, \preccurlyeq)$ are isomorphic and isotone, we conclude that $\lambda \odot B \oplus (1-\lambda) \odot B \preccurlyeq B$ for every $B \in \mathscr{I}$. Consequently, the above statement holds with equality. This completes the proof. □

3.2.3 Scalarization

Let $\overline{Y}$ be an extended partially ordered locally convex space with an ordering cone C such that $\emptyset \neq \operatorname{int} C \neq Y$, and let Y^* be its topological dual. We denote by

$$C^\circ := \{y^* \in Y^* \,|\, \forall y \in C : \; y^*(y) \leq 0\}$$

the polar cone of C. Moreover, for some $c \in Y$, we set

$$B_c := \{y^* \in C^\circ \,|\, y^*(c) = -1\}.$$

Further let $\mathscr{F} = \mathscr{F}_C(Y)$ and $\mathscr{I} = \mathscr{I}_C(\overline{Y})$. Based on the support function we define a scalarizing functional depending on a parameter $y^* \in C^\circ \setminus \{0\}$. For $A \in \mathscr{I}$, we set

$$\varphi_A(y^*) := \varphi(y^*|A) := -\sigma(y^*|\operatorname{Cl}_+ A). \tag{3.3}$$

For fixed y^*, we get by (3.3) a functional from $\mathscr{I}$ to $\overline{\mathbb{R}}$. For fixed $A \in \mathscr{I}$, we consider φ_A to be a function from $C^\circ \setminus \{0\}$ into $\overline{\mathbb{R}}$, that is,

$$\varphi_A : \; C^\circ \setminus \{0\} \to \overline{\mathbb{R}}.$$

For some $\gamma \in \overline{\mathbb{R}}$ we write $\varphi_A \equiv \gamma$ whenever $\varphi_A(y^*) = \gamma$ for all $y^* \in C^\circ \setminus \{0\}$. The addition, the multiplication by positive real numbers, the ordering relation, the infimum and the supremum for the extended real-valued function φ_A are defined pointwise for all $y^* \in C^\circ \setminus \{0\}$. We use the inf-addition, i.e., $-\infty + (+\infty) = +\infty + (-\infty) = +\infty$.

Theorem 3.7 ([17]). *Let $A, B \in \mathscr{I}$ and $\alpha > 0$, then:*

(a) $[A \in \mathscr{I}_{\text{co}} \;\wedge\; \varphi_A \equiv -\infty] \iff A = \{-\infty\}$
(b) $\varphi_A \equiv +\infty \iff [\exists y^* \in C^\circ \setminus \{0\} : \varphi_A(y^*) = +\infty] \iff A = \{+\infty\}$
(c) $A \preccurlyeq B \implies \varphi_A \leq \varphi_B$
(d) $[A \in \mathscr{I}_{\text{co}} \;\wedge\; \varphi_A \leq \varphi_B] \implies A \preccurlyeq B$
(e) $\varphi_{A \oplus B} = \varphi_A + \varphi_B$
(f) $\alpha \cdot \varphi_A = \varphi_{\alpha \odot A}$

Let $\mathscr{A} \subseteq \mathscr{I}$ be nonempty, then

(g) $\varphi_{\inf \mathscr{A}} = \inf_{A \in \mathscr{A}} \varphi_A$
(h) $\varphi_{\sup \mathscr{A}} \geq \sup_{A \in \mathscr{A}} \varphi_A$

Proof. The proof is based on the properties of the well-known support function.

(a) As $\operatorname{Cl}_+ \{-\infty\} = Y$, we get $\varphi_{\{-\infty\}} = -\sigma(y^* \,|\, Y) = -\infty$ for all $y^* \in C^\circ \setminus \{0\}$. On the other hand, $\varphi_A \equiv -\infty$ implies that $\sigma(y^* \,|\, \operatorname{Cl}_+ A) = +\infty$ for all $y^* \in C^\circ \setminus \{0\}$.

Moreover, $\sigma(y^* \,|\, C) = +\infty$ for all $y^* \in Y \setminus C^\circ$. We get

$$+\infty = \sigma(y^* \,|\, \mathrm{Cl}_+ A) + \sigma(y^* \,|\, C) = \sigma(y^* \,|\, \mathrm{Cl}_+ A + C) = \sigma(y^* \,|\, \mathrm{Cl}_+ A)$$

for all $y^* \in Y \setminus \{0\}$. It follows $\mathrm{Cl}_+ A = \mathrm{cl\,co}\,\mathrm{Cl}_+ A = Y$ and hence $A = \{-\infty\}$.

(b) We have $\mathrm{Cl}_+ \{+\infty\} = \emptyset$ and so $A = \{+\infty\}$ implies $\varphi_A(y^*) = -\sigma(y^*|\emptyset) = +\infty$ for all $y^* \in C^\circ \setminus \{0\}$. If $\sigma(y^*|\mathrm{Cl}_+ A) = -\infty$ for some $y^* \in C^\circ \setminus \{0\}$, then $\mathrm{Cl}_+ A = \emptyset$. Since $A \in \mathscr{I}$, this implies $A = \{+\infty\}$.

(c) Let $A \preccurlyeq B$. We get $\mathrm{Cl}_+ A \supseteq \mathrm{Cl}_+ B$ and hence

$$\varphi_A(y^*) = -\sigma(y^*|\mathrm{Cl}_+ A) \le -\sigma(y^*|\mathrm{Cl}_+ B) = \varphi_B(y^*)$$

for all $y^* \in \mathbb{R}^q$, in particular, for all $y^* \in C^\circ \setminus \{0\}$.

(d) Let $\varphi_A \le \varphi_B$, i.e., for all $y^* \in C^\circ \setminus \{0\}$, $-\sigma(y^*|\mathrm{Cl}_+ A) \le -\sigma(y^*|\mathrm{Cl}_+ B)$ holds. By similar arguments as in the proof of (a), the latter inequality is valid for all $y^* \in Y$. As $\mathrm{Cl}_+ A$ is convex and closed we get $\mathrm{Cl}_+ A = \mathrm{cl\,co}\,\mathrm{Cl}_+ A \supseteq \mathrm{cl\,co}\,\mathrm{Cl}_+ B \supseteq \mathrm{Cl}_+ B$ and thus $A \preccurlyeq B$.

In order to prove the statements (e) to (h), let $y^* \in C^\circ \setminus \{0\}$ be arbitrarily given.

(e) If A or B equals $\{+\infty\}$, then $A \oplus B = \{+\infty\}$ and the statement follows as $\mathrm{Cl}_+ \{+\infty\} = \emptyset$. If A and B are not $\{+\infty\}$ but one of them or both equal $\{-\infty\}$ then the result follows from the fact $\mathrm{Cl}_+ \{-\infty\} = Y$. Thus we can assume $A, B \subseteq Y$. In this case we have $\mathrm{Cl}_+ A = \mathrm{cl}\,(A + C)$. It follows

$$\begin{aligned} \varphi_{A+B}(y^*) &= -\sigma(y^*|\,\mathrm{cl}\,(A+B+C)) \\ &= -\sigma(y^*|\,\mathrm{cl}\,(A+C)) - \sigma(y^*|\,\mathrm{cl}\,(B+C)) \\ &= \varphi_A(y^*) + \varphi_B(y^*). \end{aligned}$$

(f) If $A = \{+\infty\}$, then $\alpha \odot A = \{+\infty\}$ and hence

$$\alpha \cdot \varphi_A(y^*) = \varphi_{\alpha \odot A}(y^*) = +\infty.$$

If $A = \{-\infty\}$, then $\alpha \odot A = \{-\infty\}$ and thus

$$\alpha \cdot \varphi_A(y^*) = \varphi_{\alpha \odot A}(y^*) = -\infty.$$

If $A \subseteq Y$, then we have

$$\alpha \cdot \varphi_A(y^*) = -\alpha\sigma(y^*|\,\mathrm{cl}\,(A+C)) = -\sigma(y^*|\,\mathrm{cl}\,(\alpha A + C)) = \varphi_{\alpha \odot A}(y^*).$$

(g) It remains to show the statement for the case $\{+\infty\} \notin \mathscr{A}$, because omitting $\{+\infty\}$ does not change anything. If $\{-\infty\} \in \mathscr{A}$ the equality can be easily shown. Therefore let $A \subseteq Y$ for all $A \in \mathscr{A}$. We obtain

$$\varphi_{\inf \mathscr{A}}(y^*) = -\sigma\left(y^* \middle| \mathrm{Cl}_+ \bigcup_{A\in\mathscr{A}} A\right) = -\sigma\left(y^* \middle| \mathrm{cl} \bigcup_{A\in\mathscr{A}} (A+C)\right)$$
$$= \inf_{A\in\mathscr{A}} -\sigma\left(y^* | \mathrm{cl}\,(A+C)\right) = \inf_{A\in\mathscr{A}} \varphi_A(y^*).$$

(h) We have $\sup \mathscr{A} \succcurlyeq A$ and hence $\varphi_{\sup \mathscr{A}} \geq \varphi_A$ for all $A \in \mathscr{A}$. Taking the supremum we obtain the desired statement. □

Statement (h) in the last theorem does not hold with equality as the following example shows.

Example 3.7 ([17]). Let $C := \mathbb{R}^2_+$, $A := \mathrm{Inf}\left\{(0,1)^\top\right\}$ and $B := \mathrm{Inf}\left\{(1,0)^\top\right\}$. Then $\sup\{A,B\} = \mathrm{Inf}\left\{(1,1)^\top\right\}$. For $y^* = (-1,-1)^\top$, we get $\varphi_A(y^*) = \varphi_B(y^*) = 1$ but $\varphi_{\sup\{A,B\}}(y^*) = 2$.

It is well-known that a convex extended real-valued function $\xi : X \to \overline{\mathbb{R}}$ is said to be *proper* if

$$\forall x \in X:\ \xi(x) \neq -\infty \quad \wedge \quad \exists \bar{x} \in X:\ \xi(\bar{x}) \neq +\infty.$$

Likewise, a concave extended real-valued function $\eta : X \to \overline{\mathbb{R}}$ is called *proper* if the convex function $-\eta$ is proper, that is

$$\forall x \in X:\ \eta(x) \neq +\infty \quad \wedge \quad \exists \bar{x} \in X:\ \eta(\bar{x}) \neq -\infty.$$

The domain of a convex extended real-valued function $\xi : X \to \overline{\mathbb{R}}$ is the set

$$\mathrm{dom}\,\xi := \{x \in X |\ \xi(x) \neq +\infty\},$$

whereas the domain of the concave extended real-valued function $\eta : X \to \overline{\mathbb{R}}$ is the set $\mathrm{dom}\,(-\eta)$, i.e.,

$$\mathrm{dom}\,\eta := \{x \in X |\ \eta(x) \neq -\infty\}.$$

We introduce similar notions for $\mathscr{I}$-valued functions.

Definition 3.9. A convex function $f : X \to \mathscr{I}$ is said to be *proper*, if

$$\forall x \in X:\ f(x) \neq \{-\infty\} \quad \wedge \quad \exists \bar{x} \in X:\ f(\bar{x}) \neq \{+\infty\}.$$

The *domain* of the convex function f is the set

$$\mathrm{dom}\,f := \{x \in X |\ f(x) \neq \{+\infty\}\}.$$

Taking into account that $\varphi_A : C^\circ \setminus \{0\} \to \overline{\mathbb{R}}$ is a concave function, we get the following statement.

Corollary 3.5. *Let* $A \in \mathscr{I}_{\mathrm{co}}$. *Then*

$$A \in \mathscr{I}_{\mathrm{co}} \setminus \{\{-\infty\},\{+\infty\}\} \iff \varphi_A \text{ is proper.}$$

Proof. This follows from Theorem 3.7 (a), (b). □

Corollary 3.6. *Let* $f : X \to \mathscr{I}$ *be a function. The following statements are equivalent:*

(a) $f : X \to \mathscr{I}$ *is convex.*
(b) For all $y^* \in C^\circ \setminus \{0\}$, $\varphi_{f(\cdot)}(y^*) : X \to \overline{\mathbb{R}}$ *is convex.*

Moreover, for all $y^* \in C^\circ \setminus \{0\}$, *we have* $\operatorname{dom} f = \operatorname{dom} \varphi_{f(\cdot)}(y^*)$.

Proof. This follows from Theorem 3.7 (e), (f) and (a), (b), respectively. □

Note that in the preceding result, both spaces $\mathscr{I}$ and $\overline{\mathbb{R}}$ are equipped with the inf-addition.

3.3 Duality Theory

We start this section with a general duality scheme for complete-lattice-valued problems. In order to prove weak and strong duality results we need some more structure. Section 3.3.2 deals with Lagrange duality and Sect. 3.3.3 is devoted to conjugate duality. Finally, we compare our approach with classical results.

3.3.1 A General Duality Scheme

Let $p : X \to Z$, where X is an arbitrary nonempty set and $(Z, \le)$ is a complete lattice. For a nonempty subset $S \subseteq X$, we consider the optimization problem

$$\text{minimize } p : X \to Z \text{ with respect to } \le \text{ over } S, \tag{P}$$

which is called the *primal problem*. Concurrently, we consider a dual optimization problem. Let $d : V \to Z$, where V is an arbitrary nonempty set and $T \subseteq V$ is a nonempty subset, called the *dual feasible set*. We consider the *dual problem*

$$\text{maximize } d : V \to Z \text{ with respect to } \le \text{ over } T. \tag{D}$$

Definition 3.10. We say that *weak duality* holds for the pair of problems (P) and (D) if we have the implication

$$(x \in S \wedge v \in T) \implies d(v) \le p(x).$$

Definition 3.11. We say that *strong duality* holds for the pair of problems (P) and (D) if

$$\sup_{v\in T} d(v) = \inf_{x\in S} p(x).$$

It is clear that weak duality can be characterized equivalently by the inequality

$$\sup_{v\in T} d(v) \leq \inf_{x\in S} p(x).$$

Hence strong duality implies weak duality.

We now apply the general duality principle to vector optimization problems. Let X be a nonempty set and $\overline{Y}$ an extended partially ordered topological vector space. Assume that the ordering cone C of Y is closed and $\emptyset \neq \operatorname{int} C \neq Y$. Let $p : X \to \overline{Y}$ a given vector-valued objective function and $S \subseteq X$ a given feasible set. The primal problem is considered to be the vector optimization problem

$$\text{minimize } p : X \to \overline{Y} \text{ with respect to } \leq_C \text{ over } S. \tag{VOP}$$

We assign to (VOP) a corresponding $\mathscr{I}$-valued problem, i.e., a problem of type (P), where the complete lattice $(Z, \leq) = (\mathscr{I}, \preccurlyeq)$ is used. We obtain a closely related complete-lattice-valued problem, even if $\overline{Y}$ is not a complete lattice with respect to the ordering relation generated by C.

We set $p_{\mathscr{I}} : X \to \mathscr{I}$, $p_{\mathscr{I}}(x) := \operatorname{Inf}\{p(x)\}$ and assign to (VOP) the problem

$$\text{minimize } p_{\mathscr{I}} : X \to \mathscr{I} \text{ with respect to } \preccurlyeq \text{ over } S. \tag{P$_{\mathscr{I}}$}$$

Based on the properties of the complete lattice $\mathscr{I}$, we derive dual problems for special classes of problems so that weak and strong duality can be shown. We consider a set V and a feasible subset $T \subseteq V$, a dual objective function $d_{\mathscr{I}} : V \to \mathscr{I}$ and the dual problem

$$\text{maximize } d_{\mathscr{I}} : V \to \mathscr{I} \text{ with respect to } \preccurlyeq \text{ over } T. \tag{D$_{\mathscr{I}}$}$$

The optimal values of (P$_{\mathscr{I}}$) and (D$_{\mathscr{I}}$) are defined, respectively, by

$$\overline{p}_{\mathscr{I}} := \inf_{x\in S} p_{\mathscr{I}}(x) \qquad \text{and} \qquad \overline{d}_{\mathscr{I}} := \sup_{v\in T} d_{\mathscr{I}}(v).$$

3.3.2 Lagrange Duality

In this section, $\mathscr{I}$-valued optimization problems with set-valued constraints are studied. As shown above, vector optimization problems can be regarded as a subclass of $\mathscr{I}$-valued problems. Our duality result will be derived from a corresponding scalar result, which is discussed and proved at the beginning. Of course, other variants of scalar Lagrange duality could serve as a template. This section can be understood as a demonstration, how vectorial duality can be derived from corresponding scalar results.

Throughout this section, let $\overline{Y}$ be an extended partially ordered locally convex space with an ordering cone $C \subseteq Y$ such that $\emptyset \neq \operatorname{int} C \neq Y$. The topological dual space of Y is denoted by Y^*. We set $\mathscr{I} := \mathscr{I}_C\left(\overline{Y}\right)$. In order to investigate arbitrary $\mathscr{I}$-valued problems, the cone C is not required to be closed. In case we start with a vector optimization problem (VOP) and consider its lattice extension ($P_{\mathscr{I}}$), C must be closed in order to ensure that the ordering $\preccurlyeq$ in $\mathscr{I}$ is an extension of the ordering $\leq$ in $\overline{Y}$.

Let X be a linear space, U a Hausdorff locally convex space with topological dual space U^*. We denote by $\langle \cdot, \cdot \rangle$ the canonical duality map.

Let $f : X \to \mathscr{I}$, let $g : X \rightrightarrows U$ be a set-valued map and let $D \subseteq U$ be a nonempty closed convex cone. The primal problem is given as

$$\text{minimize } f : X \to \mathscr{I} \text{ w.r.t. } \preccurlyeq \text{ over } S := \{x \in X \,|\, g(x) \cap -D \neq \emptyset\}. \qquad (\mathrm{P_L})$$

The *optimal value* of (P_L) is defined by

$$\overline{p} := \inf_{x \in S} f(x).$$

The set-valued map g is said to be *D-convex* [11] if

$$\forall x_1, x_2 \in X,\ \forall \lambda \in [0,1] : \ g\big(\lambda x_1 + (1-\lambda)x_2\big) + D \supseteq \lambda g(x_1) + (1-\lambda) g(x_2).$$

The set-valued map g can be understood as a function from X into 2^U. The power set 2^U equipped with the usual Minkowski operations provides a conlinear space. The conlinear space is quasi-ordered (i.e., the ordering is reflexive and transitive) by

$$A \leq B \quad :\Longleftrightarrow \quad A + D \supseteq B + D.$$

Therefore, the notion of D-convexity can be interpreted as convexity of a function with values in this quasi-ordered conlinear space. For the origin of the mentioned quasi-ordering the reader is referred to [5].

Initially, a scalar Lagrange duality result with set-valued constraints is provided. The scalar result is used to prove strong duality for $\mathscr{I}$-valued problems. Let us consider the scalar case of Problem (P_L), i.e., let the objective function be $f : X \to \overline{\mathbb{R}}$. Note that $\overline{\mathbb{R}}$ is equipped with the inf-addition. Commonly, a scalar optimization problem is shortly denoted by

$$\hat{p} := \inf_{x \in S} f(x). \qquad (3.4)$$

The *Lagrangian* is defined [20] by

$$L : X \times U^* \to \overline{\mathbb{R}}, \qquad L(x,u^*) = f(x) + \inf_{u \in g(x)+D} \langle u^*, u \rangle . \tag{3.5}$$

The dual objective is defined as

$$\phi : U^* \to \overline{\mathbb{R}}, \quad \phi(u^*) := \inf_{x \in X} L(x,u^*)$$

and the dual problem is

$$\hat{d} := \sup_{u^* \in U^*} \phi(u^*). \tag{3.6}$$

Of course, weak duality holds, that is, $\hat{d} \le \hat{p}$. Under convexity assumptions and some constraint qualification, we obtain the following strong duality assertion, which we prove in a common way, compare [2, Proposition 4.3.5].

Theorem 3.8. *Let $f : X \to \overline{\mathbb{R}}$ be convex, let $g : X \rightrightarrows U$ be D-convex and let*

$$g(\operatorname{dom} f) \cap -\operatorname{int} D \neq \emptyset. \tag{3.7}$$

Then, strong duality between (3.4) *and* (3.6) *holds; that is, $\hat{d} = \hat{p}$. If $\hat{p}$ is finite, then there exists a solution to the dual problem* (3.6)*.*

Proof. The value function is defined by

$$v : U \to \overline{\mathbb{R}} : \qquad v(u) := \inf \{ f(x) | \; x \in X : \; g(x) \cap (\{u\} - D) \neq \emptyset \} .$$

Since f is convex and g is D-convex, v is convex. Moreover, we have $v(0) = \hat{p}$. If $\hat{p} = -\infty$, we obtain $\hat{d} = \hat{p}$ from the weak duality. Therefore, let $\hat{p} > -\infty$. For the conjugate $v^* : U^* \to \overline{\mathbb{R}}$ of v, we have

$$\begin{aligned} -v^*(-u^*) &= \inf \{ \langle u^*, u \rangle + v(u) | \; u \in U \} \\ &= \inf \{ \langle u^*, u \rangle + f(x) | \; u \in U, \; x \in X : \; g(x) \cap (\{u\} - D) \neq \emptyset \} \\ &= \inf \{ \langle u^*, u \rangle + f(x) | \; x \in X, \; u \in g(x) + D \} \\ &= \inf_{x \in X} L(x,u^*) = \phi(u^*). \end{aligned}$$

It follows $v^{**}(0) = \hat{d}$. We next show that v is lower semi-continuous at 0 (even continuous). Indeed, by (3.7) there is some $\bar{x} \in \operatorname{dom} f$ and some $\bar{z} \in -\operatorname{int} D$ such that $\bar{z} \in g(\bar{x})$. There exists some neighborhood U of 0 such that $\{\bar{z}\} - U \subset -\operatorname{int} D$. It follows that $f(\bar{x})$ is an upper bound of v on U. This implies that v is continuous at 0 [4, Lemma 2.1].

We have $v(0) = (\operatorname{lsc} v)(0) = (\operatorname{cl} v)(0)$ [22, Theorem 4]. By the biconjugation theorem, see e.g. [4, Proposition 4.1] or [29, Theorem 2.3.4], we have $\operatorname{cl} v = v^{**}$. This yields $\hat{p} = v(0) = v^{**}(0) = \hat{d}$.

If $\hat{p}$ is finite, there exists $\overline{u}^* \in \partial v(0)$ [4, Proposition 5.2]. It follows $v(0) + v^*(\overline{u}^*) = \langle \overline{u}^*, 0\rangle$. Hence $\hat{d} = \phi(-\overline{u}^*)$. Thus, $-\overline{u}^*$ solves the dual problem. □

It is typical in Lagrange duality to show that the primal problem is re-obtained from the Lagrangian. To this end we need the additional assumption that $g(x)+D$ is closed and convex for every $x \in X$. If g is D-convex, like in the strong duality theorem, $g(x)+D$ is convex for every x, but in general it is not closed. As D is assumed to be closed, the sum $g(x)+D$ is closed whenever $g(x)$ is compact. Of course, the important case of single-valued maps g is also covered.

Proposition 3.6 ([17]). *Let $f : X \to \overline{\mathbb{R}}$ be a proper function and let the set $g(x)+D$ be closed and convex for every $x \in X$. Then*

$$\sup_{u^* \in U^*} L(x,u^*) = \begin{cases} f(x) & \textit{if } x \in S \\ +\infty & \textit{otherwise.} \end{cases}$$

Proof. Note first that $x \in S$ is equivalent to $0 \in g(x)+D$. If $x \in S$ we get

$$\begin{aligned} \sup_{u^* \in U^*} L(x,u^*) &= \sup_{u^* \in U^*} \left(f(x) + \inf_{u \in g(x)+D} \langle u^*, u \rangle \right) \\ &\leq \sup_{u^* \in U^*} \left(f(x) + \langle u^*, 0 \rangle \right) = f(x). \end{aligned}$$

On the other hand

$$\sup_{u^* \in U^*} L(x,u^*) \geq L(x,0) = f(x),$$

i.e., we have equality.

Assuming $x \notin S$, we obtain $0 \notin g(x)+D$. The latter set is closed and convex. Using a separation theorem, e.g. [1, Theorem 4.54], we obtain some $\overline{u}^* \in U^*$ such that $\inf_{u \in g(x)+D} \langle \overline{u}^*, u \rangle > 0$. If we consider multiples $u_n^* := n \cdot \overline{u}^* \in U^*$, the latter expression tends to $+\infty$ for $n \to +\infty$. Since f is assumed to be proper, we have $f(x) \neq -\infty$ for every x. Hence $L(x, u_n^*) \to +\infty$, which proves the statement. □

Note that the assumptions in the last proposition are only used for the proof in the case $x \notin S$. They cannot be dropped as the following examples show.

Example 3.8 ([17]). Let $f : \mathbb{R}^2 \to \overline{\mathbb{R}}$ be a proper function such that $0 \in \operatorname{dom} f$, let $U = \mathbb{R}^2$, $D = \mathbb{R}^2_+$ and $g(x) = \{x\} + A$, where

$$A := \left\{ a \in \mathbb{R}^2 \middle| \, a_1 > 0 \wedge a_1 a_2 \leq -1 \right\}.$$

Note that the sets $g(x)$ and D are closed for all x, but the sum $g(x)+D$ is not. We have $\sup_{u^* \in \mathbb{R}^2} L(0,u^*) = f(0) \in \mathbb{R}$, but $g(0) \cap -D = \emptyset$, i.e., $0 \notin S$.

Example 3.9 ([17]). Let $f : \mathbb{R} \to \overline{\mathbb{R}}$ be a function such that $f(1) = -\infty$, let $U = \mathbb{R}$, $D = \mathbb{R}_+$ and $g(x) = \{x\}$. We have $\sup_{u^* \in \mathbb{R}} L(1,u^*) = -\infty$, but $g(1) \cap -D = \emptyset$, i.e., $1 \notin S$.

An $\mathscr{I}$-valued version of Theorem 3.8 is now considered. The Lagrangian of Problem (P_L) (with respect to $c \in Y$) is defined by

$$L_c : X \times U^* \to \mathscr{I}, \;\; L_c(x,u^*) = f(x) \oplus \inf_{u \in g(x)+D} \big(\langle u^*, u\rangle \{c\} + \mathrm{bd}\, C\big). \tag{3.8}$$

Recall that $\mathrm{Inf}\{0\} = \mathrm{bd}\, C$ plays the role of the zero element in the conlinear space $\mathscr{I}$. It is used in (3.8) to transform the vector $\langle u^*, u\rangle \{c\}$ into an element of $\mathscr{I}$. This ensures that the infimum is well-defined. Note that the vector c and the zero element $\mathrm{bd}\, C$ are the only structural differences to the Lagrangian (3.5) in the scalar case. In the special case $Y = \mathbb{R}$, $C = \mathbb{R}_+$, $c = 1$, the Lagrangian coincides with the Lagrangian (3.5) of the scalar problem (3.4).

The vector $c \in Y$ can be arbitrarily chosen for the moment. For the most assertions, however, we have to assume $c \in \mathrm{int}\, C$. For every choice of $c \in \mathrm{int}\, C$ we may have a different Lagrangian and a different corresponding dual problem, but the same duality results hold.

We continue with an $\mathscr{I}$-valued variant of Proposition 3.6.

Proposition 3.7 ([17]). *For every $x \in S$,*

$$\sup_{u^* \in U^*} L_c(x,u^*) = f(x).$$

Proof. Note that $x \in S$ is equivalent to $0 \in g(x) + D$. It follows

$$\begin{aligned} \sup_{u^* \in U^*} L_c(x,u^*) &= \sup_{u^* \in U^*} \left(f(x) \oplus \inf_{u \in g(x)+D} \big(\langle u^*, u\rangle \{c\} + \mathrm{bd}\, C\big)\right) \\ &\preccurlyeq \sup_{u^* \in U^*} \left(f(x) \oplus (\langle u^*, 0\rangle \{c\} + \mathrm{bd}\, C)\right) = f(x). \end{aligned}$$

From

$$\sup_{u^* \in U^*} L_c(x,u^*) \geq L_c(x,0) = f(x)$$

we get equality. □

We proceed with the case $x \notin S$. As in the scalar case in Proposition 3.6, some additional assumptions are required.

Proposition 3.8 ([17]). *Let $f : X \to \mathscr{I}$ be a proper function, let the set $g(x) + D$ be closed and convex for every $x \in X$ and let $c \in \mathrm{int}\, C$. Then*

$$\sup_{u^* \in U^*} L_c(x,u^*) = \begin{cases} f(x) & \text{if } x \in S \\ \{+\infty\} & \text{otherwise.} \end{cases}$$

Proof. The first case has already been shown in Proposition 3.7.

Let $x \notin S$. For all $u^* \in U^*$, we have

$$A := \sup_{u^* \in U^*} L_c(x,u^*) \succcurlyeq f(x) \oplus \inf_{u \in g(x)+D} \left(\langle u^*, u\rangle \{c\} + \operatorname{bd} C\right).$$

From Theorem 3.7 (c), (e) and (g) we get

$$\varphi_A \geq \varphi_{f(x)} + \inf_{u \in g(x)+D} \varphi_{\{\langle u^*,u\rangle\{c\}+\operatorname{bd} C\}}.$$

Let $\bar{y}^* \in \operatorname{dom} \varphi_{f(x)} \cap B_c$. Then

$$\varphi_{\{\langle u^*,u\rangle\{c\}+\operatorname{bd} C\}}(\bar{y}^*) = \langle u^*, u\rangle.$$

As shown in the proof of Proposition 3.6 (using a separation theorem), in case of $x \notin S$ there exists a sequence (u_n^*) in U^* such that $\inf_{u \in g(x)+D} \langle u_n^*, u\rangle$ tends to $+\infty$. It follows $\varphi_A(\bar{y}^*) = +\infty$. By Theorem 3.7 (b), we conclude $A = \{+\infty\}$. □

We next define the dual problem. The dual objective function (with respect to $c \in Y$) is defined by

$$\phi_c : U^* \to \mathscr{I}, \qquad \phi_c(u^*) := \inf_{x \in X} L_c(x,u^*).$$

The dual problem (with respect to $c \in Y$) associated to (P_L) is

$$\text{maximize } \phi : U^* \to \mathscr{I} \text{ with respect to } \preccurlyeq \text{ over } T \subseteq U^*, \tag{D_L}$$

where T is subset of U^*, such that $\{u^* \in U^* |\ \phi(u^*) \neq \{-\infty\}\} \subset T$. The set T is called the dual feasible set. There are important special cases, where the set T can be determined explicitly. In the linear case, for instance, a description by inequalities is possible. In the present framework we do not loose generality by setting $T = U^*$. The *dual optimal value* is denoted by

$$\bar{d}_c := \sup_{u^* \in U^*} \phi_c(u^*). \tag{3.9}$$

We start with a weak duality.

Theorem 3.9 ([17]). *Let $c \in \operatorname{int} C$. Then ($P_L$) and ($D_L$) satisfy the weak duality inequality $\bar{d}_c \preccurlyeq \bar{p}$.*

Proof. Since $\mathscr{I}$ is a complete lattice, we immediately have

$$\sup_{u^* \in U^*} \inf_{x \in X} L_c(x,u^*) \preccurlyeq \inf_{x \in X} \sup_{u^* \in U^*} L_c(x,u^*) \tag{3.10}$$

(even if L_c would be replaced by an arbitrary function from $X \times U^*$ into $\mathscr{I}$). By Proposition 3.7 we know that $\inf_{x \in X} \sup_{u^* \in U^*} L_c(x,u^*) \preccurlyeq \bar{p}$ in case of $c \in \operatorname{int} C$. □

We continue with strong duality.

Theorem 3.10 ([17]). *Suppose that $f : X \to \mathscr{I}$ is convex and $g : X \rightrightarrows U$ is D-convex. Let*

$$g(\operatorname{dom} f) \cap -\operatorname{int} D \neq \emptyset, \tag{3.11}$$

and $c \in \operatorname{int} C$. Then strong duality between ($\mathrm{P_L}$) and ($\mathrm{D_L}$) holds; that is, $\overline{p} = \overline{d}_c$.

Proof. If $\overline{p} = \{-\infty\}$, strong duality follows from weak duality. Note further that $\operatorname{dom} f$ is nonempty, hence $\overline{p} \neq \{+\infty\}$. Therefore, it remains to prove strong duality for the case $\overline{p} \in \mathscr{I} \setminus \{\{-\infty\}, \{+\infty\}\}$. We use the scalarization functional $\varphi_A : C^\circ \setminus \{0\} \to \overline{\mathbb{R}}$ $(A \in \mathscr{I})$ as introduced above. As $f : X \to \mathscr{I}$ is convex and S is a convex set (as g is D-convex), Proposition 3.5 implies $\overline{p} \in \mathscr{I}_{\mathrm{co}}$. By Corollary 3.5, $\varphi_{\overline{p}}$ is proper, in particular $\operatorname{dom} \varphi_{\overline{p}} \neq \emptyset$.

For $y^* \in B_c$ (as defined in Sect. 3.2.3) we have

$$\begin{aligned}
&\varphi\left(y^* \,\middle|\, \inf_{u \in g(x)+D} \left(\langle u^*, u\rangle \{c\} + \operatorname{bd} C\right)\right) \\
&\overset{\text{Th. 3.7 (g)}}{=} \inf_{u \in g(x)+D} \varphi\left(y^* \middle| \, \langle u^*, u\rangle \{c\} + \operatorname{bd} C\right) \\
&= \inf_{u \in g(x)+D} -\sigma\left(y^* \middle| \, \langle u^*, u\rangle \{c\} + C\right) \\
&= \inf_{u \in g(x)+D} \langle u^*, u\rangle .
\end{aligned}$$

Let $y^* \in \operatorname{dom} \varphi_{\overline{p}} \cap B_c$. By Theorem 3.8 there exists some $\overline{u}^*$ (a solution to the scalar dual problem) such that

$$\begin{aligned}
\varphi(y^* | \overline{p}) \quad &= \quad \varphi\left(y^* \,\middle|\, \inf_{g(x) \cap -D \neq \emptyset} f(x)\right) \\
&\overset{\text{Th. 3.7 (g)}}{=} \inf_{g(x) \cap -D \neq \emptyset} \varphi\left(y^* \middle| \, f(x)\right) \\
&\overset{\text{Th. 3.8}}{=} \inf_{x \in X} \left(\varphi\left(y^* \middle| \, f(x)\right) + \inf_{u \in g(x)+D} \langle \overline{u}^*, u\rangle\right) \\
&= \inf_{x \in X} \Bigg(\varphi\left(y^* \middle| \, f(x)\right) \\
&\qquad + \varphi\left(y^* \,\middle|\, \inf_{u \in g(x)+D} \left(\langle \overline{u}^*, u\rangle \{c\} + \operatorname{bd} C\right)\right)\Bigg) \\
&\overset{\text{Th. 3.7 (e), (g)}}{=} \varphi\left(y^* \middle| \, \phi_c(\overline{u}^*)\right).
\end{aligned}$$

Together we have

$$\forall y^* \in \operatorname{dom} \varphi_{\overline{p}} \cap B_c,\ \exists \overline{u}^* \in U^* : \quad \varphi(y^* | \phi_c(\overline{u}^*)) = \varphi(y^* | \overline{p}). \tag{3.12}$$

For every $A \in \mathscr{I}$ and $\alpha > 0$ it is true that $\varphi(\alpha \cdot y^*|A) = -\alpha\varphi(y^*|A)$. We conclude from (3.12) that $\varphi(y^*|\overline{d}_c) \geq \varphi(y^*|\overline{p})$ for all $y^* \in C^\circ \setminus \{0\}$. As $\overline{p} \in \mathscr{I}_{\text{co}}$, Theorem 3.7 (iv) yields $\overline{d}_c \succcurlyeq \overline{p}$. By the weak duality inequality we obtain $\overline{d}_c = \overline{p}$. □

Note that strong duality implies that (3.10) is satisfied with equality.

We know from the scalar optimization theory that strong duality results usually consist of two statements. The first one is the equality of the primal and dual optimal values and the second one is the dual attainment, that is, if the primal optimal value is finite, then a solution to the dual problem exists, see Theorem 3.8. In the current framework we cannot answer the question, whether the dual attainment holds or not.

However, a surrogate result can be shown. Usually, the dual optimal value can be expressed as

$$\overline{d}_c = \sup_{u^* \in \mathbb{R}^m} \phi_c(u^*) = \operatorname{Sup} \bigcup_{u^* \in \mathbb{R}^m} \phi_c(u^*).$$

It is shown in the following result that the supremal set can be replaced by the set of weakly maximal elements.

Theorem 3.11 ([17]). *Let all the assumptions of Theorem 3.10 be satisfied and let* $\overline{p} \neq \{-\infty\}$, *then*

$$\overline{d}_c = \operatorname{wMax} \bigcup_{u^* \in U^*} \phi_c(u^*).$$

Proof. We have $\overline{p} \notin \{\{-\infty\}, \{+\infty\}\}$ and hence $\emptyset \neq \operatorname{Cl}_+\overline{p} \neq Y$. Let

$$\overline{y} \in \operatorname{Sup} \bigcup_{u^* \in U^*} \phi_c(u^*) = \overline{d}_c = \overline{p}.$$

By Proposition 3.5 we have $\overline{p} \in \mathscr{I}_{\text{co}}$. We get $\overline{y} \notin \overline{p} + \operatorname{int} C$ and the set $\overline{p} + \operatorname{int} C$ is convex. Using a separation theorem [1, Theorem 4.46], we obtain some $\overline{y}^* \in Y^* \setminus \{0\}$ such that

$$\overline{y}^*(\overline{y}) \geq \sup_{y \in \overline{p} + \operatorname{int} C} \overline{y}^*(y).$$

Assuming that $y^* \notin C^\circ$, we get a contradiction as the supremum becomes $+\infty$. Thus we have $y^* \in C^\circ \setminus \{0\}$ and hence

$$\overline{y}^*(\overline{y}) \geq \sigma_{\overline{p} + \operatorname{int} C}(\overline{y}^*) = \sigma_{\operatorname{Cl}_+\overline{p}}(\overline{y}^*) = -\varphi_{\overline{p}}(\overline{y}^*).$$

Without loss of generality we can assume that $\overline{y}^*(c) = -1$. By (3.12) there exists some $\overline{u}^* \in U^*$ such that

$$\overline{y}^*(\overline{y}) \geq -\varphi_{\overline{p}}(\overline{y}^*) = -\varphi_{\phi_c(\overline{u}^*)}(\overline{y}^*).$$

Assuming that $\overline{y} \in \phi_c(\overline{u}^*) + \operatorname{int} C = \phi_c(\overline{u}^*) + \operatorname{int} C + \operatorname{int} C$ we obtain

$$\forall y^* \in C^\circ \setminus \{0\}: \quad y^*(\overline{y}) < \sigma_{\phi_c(\overline{u}^*) + \operatorname{int} C}(y^*) = -\varphi_{\phi_c(\overline{u}^*)}(y^*),$$

a contradiction. It follows $\overline{y} \notin \phi_c(\overline{u}^*) + \operatorname{int} C$.

On the other hand, $\{\bar{y}\} + \operatorname{int} C \subseteq \bar{p} + \operatorname{int} C \subseteq \phi_c(\bar{u}^*) + \operatorname{int} C$. We get $\bar{y} \in \operatorname{Inf} \phi_c(\bar{u}^*) = \phi_c(\bar{u}^*) \subseteq \bigcup_{u^* \in U^*} \phi_c(u^*)$. Together we have $\bar{y} \in \operatorname{wMax} \bigcup_{u^* \in U^*} \phi_c(u^*)$.
□

In [17] the problem of attainment could be solved by an alternative Lagrange dual problem, called type II. It can be shown that a solution of the dual problem (as introduced in [17]) exists under the assumptions of the strong duality theorem.

3.3.3 Conjugate Duality

As a second instance of duality, conjugate duality (also called Fenchel duality) is studied in this subsection. We restrict ourselves to the finite dimensional case because this is sufficient to promote the ideas. Infinite dimensional variants can be found in [17]. The only difference is that the constraint qualification has to be adapted, but this is also a feature of the scalar theory.

Throughout this section let $Y = \mathbb{R}^q$ and $C \subset \mathbb{R}^q$ is a convex and pointed cone such that $\emptyset \neq \operatorname{int} C \neq \mathbb{R}^q$. We set $\mathscr{I} := \mathscr{I}_C\left(\overline{Y}\right)$.

Definition 3.12. The conjugate of a function $f : \mathbb{R}^n \to \mathscr{I}$ (with respect to some fixed $c \in \mathbb{R}^q$) is defined by

$$f_c^* : \mathbb{R}^n \to \mathscr{I}, \qquad f_c^*(x^*) := \sup_{x \in \mathbb{R}^n} \big(\langle x^*, x\rangle \{c\} - f(x)\big).$$

We know that if f is an $\mathscr{I}$-valued function, so is $-f$. This follows from the fact

$$A \in \mathscr{I} \quad \Longleftrightarrow \quad -A \in \mathscr{I},$$

which was shown in Corollary 3.4. So the term $\langle x^*, x\rangle \{c\} - f(x)$ stands for a shift of $-f(x) \in \mathscr{I}$. This means that the minus sign has to be interpreted in the sense of Minkowski-addition. As a result we have $\langle x^*, x\rangle \{c\} - f(x) \in \mathscr{I}$ and the supremum in the definition of the conjugate is well-defined.

Given two functions $f : \mathbb{R}^n \to \mathscr{I}$ and $g : \mathbb{R}^m \to \mathscr{I}$ and a matrix $B \in \mathbb{R}^{m \times n}$, the primal problem ($\mathrm{P}_{\mathscr{I}}$) is considered for the objective function

$$p : \mathbb{R}^n \to \mathscr{I}, \qquad p(x) := f(x) \oplus g(Bx).$$

For the dual problem ($\mathrm{D}_{\mathscr{I}}$) we specify the objective function as

$$d_c : \mathbb{R}^m \to \mathscr{I}, \qquad d_c(u^*) := -f_c^*(B^T u^*) \oplus -g_c^*(-u^*).$$

Setting, moreover, $S = \mathbb{R}^n$ and $T = \mathbb{R}^m$, the problems ($\mathrm{P}_{\mathscr{I}}$) and ($\mathrm{D}_{\mathscr{I}}$) turn into the following problems; the conjugate (or Fenchel) primal problem

$$\text{minimize } p : \mathbb{R}^n \to \mathscr{I} \text{ with respect to } \preccurlyeq \text{ over } \mathbb{R}^n \tag{$\mathrm{P_F}$}$$

and the dual problem associated to (P_F)

$$\text{maximize } d_c : \mathbb{R}^m \to \mathscr{I} \text{ with respect to } \preccurlyeq \text{ over } \mathbb{R}^m. \tag{D_F}$$

The *optimal values* of (P_F) and (D_F) are denoted, respectively, by

$$\overline{p} := \inf_{x \in \mathbb{R}^n} p(x) \in \mathscr{I} \qquad \text{and} \qquad \overline{d}_c := \sup_{u^* \in \mathbb{R}^m} d_c(u^*) \in \mathscr{I}.$$

If $c \in \operatorname{int} C$, then B_c (as defined in Sect. 3.2.3) is a *base* of C°, that is, each nonzero element y^* of C° has a unique representation by an element b^* of the convex set B_c by $y^* = \lambda b^*$ for some $\lambda > 0$ [21, p. 25].

We continue with a conjugate duality theorem which is formulated likewise to its scalar counterpart. This high degree of analogy is one of the essential advantages of using the supremum and infimum in vector optimization.

Theorem 3.12 ([17, 18]). *The problems* (P_F) *and* (D_F) *(with arbitrary $c \in Y$) satisfy the weak duality inequality, that is, $\overline{d}_c \preccurlyeq \overline{p}$. Furthermore, let f and g be proper convex functions, $c \in \operatorname{int} C$, and let the following constraint qualification be satisfied:*

$$\operatorname{ri} \operatorname{dom} g \cap B(\operatorname{ri} \operatorname{dom} f) \neq \emptyset. \tag{3.13}$$

Then strong duality holds, that is, $\overline{d}_c = \overline{p}$.

Proof. For all $u^* \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$,

$$-f^*(B^T u^*) \oplus -g^*(-u^*) \preccurlyeq \left(f(x) - \left\langle B^T u^*, x \right\rangle\right) \oplus \left(g(u) + \langle u^*, u \rangle\right).$$

Set $u := Bx$. From $\left\langle B^T u^*, x \right\rangle = \langle u^*, Bx \rangle$, we get $d_c(u^*) \preccurlyeq p(x)$ for all $u^* \in \mathbb{R}^m$ and all $x \in \mathbb{R}^n$. Taking the supremum over $u^* \in \mathbb{R}^m$ and the infimum over $x \in \mathbb{R}^n$, we obtain the weak duality inequality $\overline{d}_c \preccurlyeq \overline{p}$.

If $\overline{p} = \{-\infty\}$, strong duality follows from weak duality. Note further that $\operatorname{dom} p$ is nonempty, hence $\overline{p} \neq \{+\infty\}$. Therefore, it remains to prove strong duality for the case $\overline{p} \in \mathscr{I} \setminus \{\{-\infty\}, \{+\infty\}\}$.

We use the scalarization functional $\varphi_A : C^\circ \setminus \{0\} \to \overline{\mathbb{R}}$ ($A \in \mathscr{I}$) as introduced in Sect. 3.2.3. As $p : \mathbb{R}^n \to \mathscr{I}$ is convex, Proposition 3.5 implies that $\overline{p} \in \mathscr{I}_{\text{co}}$.

By Corollary 3.5, (the concave function) $\varphi_{\overline{p}}$ is proper, in particular $\operatorname{dom} \varphi_{\overline{p}} \neq \emptyset$. As $c \in \operatorname{int} C$, for every $y^* \in C^\circ \setminus \{0\}$, we have $y^*(c) < 0$. We fix some $y^* \in \operatorname{dom} \varphi_{\overline{p}} \cap B_c$ and consider the extended real-valued functions $\xi : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $\eta : \mathbb{R}^m \to \overline{\mathbb{R}}$ being defined, respectively, by $\xi(x) := \varphi\big(y^* | f(x)\big)$ and $\eta(u) := \varphi\big(y^* | g(u)\big)$. It follows

$$\varphi(y^* | \overline{p}) = \inf_{x \in \mathbb{R}^n} \big(\xi(x) + \eta(Bx)\big). \tag{3.14}$$

By Corollary 3.6, ξ and η are convex, $\operatorname{dom} f = \operatorname{dom} \xi$ and $\operatorname{dom} g = \operatorname{dom} \eta$. As $y^* \in \operatorname{dom} \varphi_{\overline{p}}$ (that is $\varphi_{\overline{p}}(y^*) > -\infty$ as $\varphi_{\overline{p}} : C^\circ \setminus \{0\} \to \overline{\mathbb{R}}$ is concave), ξ and η are proper. The constraint qualification (3.13) implies a corresponding condition for the scalar problem, that is

$$\operatorname{ri} \operatorname{dom} \eta \cap B(\operatorname{ri} \operatorname{dom} \xi) \neq \emptyset.$$

A scalar duality result, e.g. [2, Theorem 3.3.5 and Exercise 20 (e) in Sect. 4.1], yields that

$$\varphi(y^*|\,\overline{p}) = \sup_{u^* \in \mathbb{R}^m} \left(-\xi^*(B^T u^*) - \eta^*(-u^*)\right).$$

Moreover, we obtain that, if $\varphi(y^*|\,\overline{p})$ is finite, the supremum is attained, that is

$$\exists \overline{u}^* \in \mathbb{R}^m : \qquad \varphi(y^*|\,\overline{p}) = -\xi^*(B^T \overline{u}^*) - \eta^*(-\overline{u}^*). \tag{3.15}$$

Furthermore, it is true that

$$\forall t \in \mathbb{R} : \;\; \varphi\big(y^*|t \cdot \{c\}\big) = -y^*(t \cdot c) = t. \tag{3.16}$$

We have

$$\begin{aligned}
\varphi\,(y^*|\,\overline{p}) &= -\xi^*(B^*\overline{u}^*) - \eta^*(-\overline{u}^*) \\
&= \inf_{x \in \mathbb{R}^n} \big(-\langle B^*\overline{u}^*, x\rangle + \xi(x)\big) + \inf_{u \in \mathbb{R}^m} \big(\langle \overline{u}^*, u\rangle + \eta(u)\big) \\
&\overset{(3.16)}{=} \inf_{x \in \mathbb{R}^n} \big(\varphi\big(y^*\big|-\langle B^*\overline{u}^*, x\rangle \cdot \{c\}\big) + \varphi\big(y^*\big|f(x)\big)\big) \\
&\quad + \inf_{u \in \mathbb{R}^m} \big(\varphi\big(y^*\big|\langle \overline{u}^*, u\rangle \cdot \{c\}\big) + \varphi\big(y^*\big|g(u)\big)\big) \\
&= \varphi\Bigg(y^*\Bigg|\inf_{x \in \mathbb{R}^n} \big(-\langle B^*\overline{u}^*, x\rangle\{c\} + f(x)\big) \\
&\qquad\qquad \oplus \inf_{u \in \mathbb{R}^m} \big(\langle \overline{u}^*, u\rangle\{c\} + g(u)\big)\Bigg) \\
&= \varphi\Bigg(y^*\Bigg|-\sup_{x \in \mathbb{R}^n} \big(\langle B^*\overline{u}^*, x\rangle\{c\} - f(x)\big) \\
&\qquad\qquad \oplus -\sup_{u \in \mathbb{R}^m} \big(\langle -\overline{u}^*, u\rangle\{c\} - g(u)\big)\Bigg) \\
&= \varphi\big(y^*\big|-f_c^*(B^*\overline{u}^*) \oplus -g_c^*(-\overline{u}^*)\big) = \varphi\,(y^*|d_c(\overline{u}^*)).
\end{aligned}$$

We deduce that

$$\forall y^* \in \operatorname{dom}\varphi_{\overline{p}} \cap B_c,\; \exists \overline{u}^* \in \mathbb{R}^m : \quad \varphi(y^*|d_c(\overline{u}^*)) = \varphi(y^*|\,\overline{p}). \tag{3.17}$$

For every $A \in \mathscr{I}$ and $\alpha > 0$, we have $\varphi(\alpha \cdot y^*|A) = -\alpha\varphi(y^*|A)$. We conclude from (3.17) that $\varphi(y^*|\,\overline{d}_c) \geq \varphi(y^*|\,\overline{p})$ for all $y^* \in C^\circ \setminus \{0\}$. As $\overline{p} \in \mathscr{I}_{\mathrm{co}}$, Theorem 3.7 (d) yields $\overline{d}_c \succcurlyeq \overline{p}$. By the weak duality inequality we obtain $\overline{d}_c = \overline{p}$. □

Again we can show that the supremal set can be replaced by the set of weakly maximal elements.

Theorem 3.13. *Let the assumptions of Theorem 3.12 be satisfied and assume* $\overline{p} \neq \{-\infty\}$*, then*

$$\overline{d}_c = \mathrm{wMax} \bigcup_{u^* \in \mathbb{R}^m} d_c(u^*).$$

Proof. The proof is the same as the proof of Theorem 3.13 but using (3.17) instead of (3.12). □

In the scalar duality theory, the constraint qualification (3.13) can be further weakened if the objective function is polyhedral. In this case it suffices to assume

$$\operatorname{dom} g \cap B(\operatorname{dom} f) \neq \emptyset \tag{3.18}$$

see e.g. [2, Corollary 5.1.9]. This is important for linear problems. Let us consider the special case $P \in \mathbb{R}^{q \times n}$,

$$f : \mathbb{R}^n \to \mathscr{I}, \quad f(x) = \mathrm{Inf}\,\{Px\}$$

and

$$g : \mathbb{R}^m \to \mathscr{I}, \quad g(u) := \begin{cases} \mathrm{Inf}\,\{0\} & \text{if } u \geq b \\ \{+\infty\} & \text{otherwise.} \end{cases}$$

The constraint qualification (3.13) in Theorem 3.12 can be weakened to (3.18), which can be expressed as

$$\exists x \in \mathbb{R}^n : \; Bx \geq b.$$

This follows by similar considerations as in the proof of Theorem 3.12, but using an adapted scalar result.

3.3.4 Connections to Classic Results

In the literature one can find duality results with a vector-valued dual objective function, see e.g. Chap. 4 in [3]. Following [17], we demonstrate in this subsection, how results of this type can be obtained from the $\mathscr{I}$-valued duality theory.

Let X be a set and let $\overline{Y}$ be an extended partially ordered topological vector space, let the ordering cone C of Y be closed and let $\emptyset \neq \operatorname{int} C \neq Y$. Let $p : X \to \overline{Y}$ be the objective function and $S \subseteq X$ be the feasible set of a vector optimization problem

$$\mathrm{wMin}_{x \in S}\, p(x), \tag{P_Y}$$

where we assume that $p[S] \subseteq Y$. One is interested in finding weakly efficient solutions to (P_Y). A vector $\overline{x} \in S$ is called a *weakly efficient solution* to (P_Y), if $p(\overline{x}) \in \mathrm{wMin}\, p[S]$.

We consider the lattice extension ($P_{\mathscr{I}}$) with the objective function $p_{\mathscr{I}} : X \to \mathscr{I}$, $p_{\mathscr{I}}(x) := \operatorname{Inf}\{p(x)\}$ as well as the dual problem ($D_{\mathscr{I}}$) as defined in Sect. 3.3.1. Let $d_{\mathscr{I}} : V \to \mathscr{I}$ be the objective function and $T \subseteq V$ the feasible set of Problem ($D_{\mathscr{I}}$). For simplicity we assume that $d_{\mathscr{I}}[T] \subseteq \mathscr{I} \setminus \{\{-\infty\}, \{+\infty\}\}$. We consider the dual vector optimization problem

$$\operatorname{wMax}_{(v,y) \in \mathscr{T}} d(v,y), \tag{D_Y}$$

where we set

$$d : V \times \overline{Y} \to \overline{Y},\ d(v,y) := y \quad \text{and} \quad \mathscr{T} := \{(v,y) \in T \times Y |\, y \in d_{\mathscr{I}}(v)\}.$$

A vector $(\bar{v},\bar{y}) \in \mathscr{T}$ is called a *weakly efficient solution* to (D_Y), if $d(\bar{v},\bar{y}) \in \operatorname{wMax} d[\mathscr{T}]$. Note that we have

$$d_{\mathscr{I}}(T) = \bigcup_{v \in T} d_{\mathscr{I}}(v) = \{y |\, (y,v) \in \mathscr{T}\} = d[\mathscr{T}]. \tag{3.19}$$

In order to formulate a weak duality assertion, we write $y^1 <_C y^2$, whenever $y^2 - y^1 \in \operatorname{int} C$.

Theorem 3.14 (Weak Duality). *The following statements are equivalent:*

(a) Weak duality between ($P_{\mathscr{I}}$) *and* ($D_{\mathscr{I}}$) *holds, that is, if* $\bar{x} \in S$ *and* $\bar{v} \in T$, *then* $d_{\mathscr{I}}(\bar{v}) \preccurlyeq p_{\mathscr{I}}(\bar{x})$.
(b) Weak duality between (P_Y) *and* (D_Y) *holds, that is, there is no* $x \in X$ *and no* $(v,y) \in \mathscr{T}$ *such that* $p(x) <_C d(v,y)$.

Proof. Let (a) be satisfied and let $\bar{x} \in S$ and $(\bar{v},\bar{y}) \in \mathscr{T}$ be given. We get $\bar{y} \in d_{\mathscr{I}}(\bar{v})$ and $d_{\mathscr{I}}(\bar{v}) \preccurlyeq p_{\mathscr{I}}(\bar{x})$. By assumption we have $d_{\mathscr{I}}(\bar{v}) \in \mathscr{I} \setminus \{\{-\infty\}, \{+\infty\}\}$ and $p[S] \subseteq Y$ implies $p_{\mathscr{I}}(\bar{x}) \in \mathscr{I} \setminus \{\{-\infty\}, \{+\infty\}\}$. We get $\emptyset \neq \operatorname{Cl}_+ p_{\mathscr{I}}(\bar{x}) \subseteq \operatorname{Cl}_+ d_{\mathscr{I}}(\bar{v}) \neq Y$ and hence $p(\bar{x}) \in \operatorname{Cl}_+ d_{\mathscr{I}}(\bar{v})$. Corollary 3.2 (i) and (j) yield $p(\bar{x}) \notin d_{\mathscr{I}}(\bar{v}) - \operatorname{int} C$. Thus we get $p(\bar{x}) \notin \{\bar{y}\} - \operatorname{int} C$. It follows that $p(\bar{x}) \not<_C d(\bar{v},\bar{y})$, i.e., (b) holds.

Let (b) be satisfied and let $\bar{x} \in S$ and $\bar{v} \in T$ be given. By assumption we have $d_{\mathscr{I}}(\bar{v}) \in \mathscr{I} \setminus \{\{-\infty\}, \{+\infty\}\}$ hence $d_{\mathscr{I}}(\bar{v})$ is a nonempty subset of Y. For all $\bar{y} \in d_{\mathscr{I}}(\bar{v})$ we have $p(\bar{x}) \not<_C d(\bar{v},\bar{y}) = \bar{y}$. We get $p(\bar{x}) \notin d_{\mathscr{I}}(\bar{v}) - \operatorname{int} C$. Corollary 3.2 (i) and (k) yield $p(\bar{x}) \in \operatorname{Cl}_+ d_{\mathscr{I}}(\bar{v})$. Hence $d_{\mathscr{I}}(\bar{v}) \preccurlyeq p_{\mathscr{I}}(\bar{x})$. □

In Theorems 3.13 and 3.11 we have shown that the dual optimal value $\bar{d}_{\mathscr{I}}$ for both dual problems (D_L) and (D_F) can be expressed as

$$\bar{d}_{\mathscr{I}} := \sup_{v \in T} d_{\mathscr{I}}(v) = \operatorname{wMax} d_{\mathscr{I}}(T). \tag{3.20}$$

As a consequence, strong duality between ($P_{\mathscr{I}}$) and ($D_{\mathscr{I}}$) entails a classical scheme of strong duality [3, 11].

Theorem 3.15 (Strong Duality). *Assume that strong duality holds between* ($\mathrm{P}_{\mathscr{I}}$) *and* ($\mathrm{D}_{\mathscr{I}}$)*, that is,* $\overline{p}_{\mathscr{I}} = \overline{d}_{\mathscr{I}}$*, and let* (3.20) *be satisfied. Then, strong duality between* (P_Y) *and* (D_Y) *holds; that is, if* $\bar{x}$ *is a weakly efficient solution to* (P_Y)*, then there exists a weakly efficient solution* $(\bar{v},\bar{y})$ *to* (D_Y) *such that* $p(\bar{x}) = d(\bar{v},\bar{y})$*.*

Proof. Let $\bar{y} = p(\bar{x}) \in \mathrm{wMin}\, p[S]$. We get

$$\begin{aligned} \mathrm{wMin}\, p[S] &\subset \mathrm{Inf}\, p[S] = \mathrm{Inf} \bigcup\nolimits_{x\in S} \mathrm{Inf}\, \{p(x)\} = \inf\nolimits_{x\in S} p_{\mathscr{I}}(x) \\ &= \overline{p}_{\mathscr{I}} = \overline{d}_{\mathscr{I}} \overset{(3.20)}{=} \mathrm{wMax}\, d_{\mathscr{I}}(T) \overset{(3.19)}{=} \mathrm{wMax}\, d[\mathscr{T}]. \end{aligned}$$

It follows $\bar{y} \in \mathrm{wMax}\, d[\mathscr{T}] \subseteq d[\mathscr{T}]$. Hence there exists $\bar{v} \in T$ such that $(\bar{v},\bar{y}) \in \mathscr{T}$ and $p(\bar{x}) = \bar{y} = d(\bar{v},\bar{y})$. □

Under the common (but restrictive) assumption that $p[S]+C$ is closed, we get also the so-called converse duality.

Theorem 3.16 (Converse Strong Duality). *Assume that strong duality holds between* ($\mathrm{P}_{\mathscr{I}}$) *and* ($\mathrm{D}_{\mathscr{I}}$) *and let* $p[S]+C$ *be closed. Then, converse strong duality between* (P_Y) *and* (D_Y) *holds; that is, if* $(\bar{v},\bar{y})$ *is a weakly efficient solution to* (D_Y)*, then* $\bar{y} \in \mathrm{wMin}\,(p[S]+C)$*.*

Proof. Let $\bar{y} = d(\bar{v},\bar{y}) \in \mathrm{wMax}\, d[\mathscr{T}]$. As $p[S]+C$ is closed, we get

$$\mathrm{Inf}\, p[S] = \mathrm{wMin}\, \mathrm{cl}\,(P[S]+C) = \mathrm{wMin}\,(p[S]+C).$$

It follows

$$\begin{aligned} \mathrm{wMax}\, d[\mathscr{T}] &\overset{(3.19)}{=} \mathrm{wMax}\, d_{\mathscr{I}}(T) \subseteq \mathrm{Sup}\, d_{\mathscr{I}}(T) = \overline{d}_{\mathscr{I}} \\ &= \ \overline{p}_{\mathscr{I}} = \mathrm{Inf}\, p[S] = \mathrm{wMin}\,(p[S]+C) \end{aligned}$$

which completes the proof. □

The opposite direction of the statements in the last two theorems can be shown when $p[S]+C$ is assumed to be closed.

Theorem 3.17. *Assume that strong duality and converse strong duality holds between* (P_Y) *and* (D_Y)*. Further let* (3.20) *be satisfied and let* $p[S] = \mathrm{cl}\,(p[S]+C)$*. Then strong duality between* ($\mathrm{P}_{\mathscr{I}}$) *and* ($\mathrm{D}_{\mathscr{I}}$) *holds.*

Proof. By the assumption $p[S] = \mathrm{cl}\,(p[S]+C)$, we get

$$\mathrm{wMin}\, p[S] = \mathrm{wMin}\, \mathrm{cl}\,(p[S]+C) = \mathrm{Inf}\, p[S] = \overline{p}_{\mathscr{I}}$$

and (3.20) yields

$$\mathrm{wMax}\, d[\mathscr{T}] = \mathrm{wMax}\, d_{\mathscr{I}}(T) = \overline{d}_{\mathscr{I}}.$$

Strong duality between (P_Y) and (D_Y) yields $\mathrm{wMin}\, p[S] \subseteq \mathrm{wMax}\, d[\mathscr{T}]$. Converse strong duality between (P_Y) and (D_Y) yields $\mathrm{wMax}\, d[\mathscr{T}] \subseteq \mathrm{wMin}\,(p[S]+C)$.

We have $p[S] \subseteq p[S]+C \subseteq \mathrm{cl}\,(p[S]+C) \subseteq p[S]$ and hence $p[S] = p[S]+C$. Together we obtain $\overline{p}_{\mathscr{I}} = \overline{d}_{\mathscr{I}}$. □

We observe that duality between (P_Y) and (D_Y) involves the existence of weakly minimal elements. In the scalar duality theory (and likewise in the $\mathscr{I}$-valued theory for suitable solution concepts and so-called type II dual problems, see [17]) we obtain the existence of a solution to the dual problem as a result. But a solution to the primal problem does not need to exist in order to get duality assertions.

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Chapter 4
Variable Ordering Structures in Vector Optimization

Gabriele Eichfelder

4.1 Introduction

In vector optimization one assumes in general that a partial ordering is given by some nontrivial convex cone K in the considered space Y. But already in 1974 in one of the first publications [37] related to the definition of optimal elements in vector optimization also the idea of variable ordering structures was given: to each element of the space a cone of dominated (or preferred) directions is defined and thus the ordering structure is given by a set-valued map. In [37] a candidate element was defined to be *nondominated* if it is not dominated by any other reference element w.r.t. the corresponding cone of this other element. Later, also another notion of optimal elements in the case of a variable ordering structure was introduced [7–9]: a candidate element is called a *minimal* (or *nondominated-like*) element if it is not dominated by any other reference element w.r.t. the cone of the candidate element.

Recently there is an increasing interest in such variable ordering structures motivated by several applications for instance in medical image registration or in portfolio optimization. For a study of such vector optimization problems with a variable ordering structure it is important to differentiate between the two mentioned optimality concepts as well as to examine the relation between the concepts. In view of applications it is also important to formulate characterizations of optimal elements by scalarizations for allowing numerical calculations.

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4.2 Variable Ordering Structure

This section introduces vector optimization problems with a variable ordering structure. In addition to optimality notions, several applications of these problems as well as their appearance in the literature are summed up. A special variable ordering structure given by a cone-valued map with images which are so-called Bishop–Phelps cones is discussed which appears to be very helpful for formulating scalarizations for characterizing optimal elements.

4.2.1 Optimality Notions

In vector optimization it is in general assumed that a *partial ordering* $\geq_K$ in a real linear space Y is given by a nontrivial convex cone $K \subset Y$. Then we write $x \leq_K y$ for $y - x \in K$. Recall that a set K is called a *cone* if $\lambda x \in K$ for all $\lambda \geq 0$ and $x \in K$. And a cone is *convex* if $K + K \subset K$. A cone satisfying $K \cap (-K) = \{0_Y\}$ is called *pointed*. According to the classical concepts, an *efficient* element $\bar{y} \in A$ of a nonempty subset A of Y w.r.t. the pointed convex cone $K \subset Y$ is defined by

$$(\{\bar{y}\} - K) \cap A = \{\bar{y}\}, \tag{4.1}$$

which is equivalent to that there is no $y \in A$ with

$$\bar{y} \in \{y\} + K \setminus \{0_Y\}. \tag{4.2}$$

In (4.2) the cone K can be interpreted as the set of dominated directions of the element y, whereas in (4.1) the cone $-K$ represents the set of preferred directions of the element $\bar{y}$. Hence, the search for efficient elements equals the search for nondominated elements, or, for the most preferred elements of the set A, respectively. (4.1) and (4.2) are equivalent, but for a variable ordering structure we have to differentiate between these two points of view: preference and domination.

In the following we assume Y to be a real topological linear space and A to be a nonempty subset of Y. Let $\mathscr{D}\colon Y \to 2^Y$ be a set-valued map with $\mathscr{D}(y)$ a pointed convex cone for all $y \in Y$. Let cl$A$, int$A$, cone$A$, conv$A$ and ∂A denote the closure, the interior, the conic hull, the convex hull and the boundary of A, respectively. Let $\mathscr{D}(A) := \bigcup_{y\in A} \mathscr{D}(y)$ denote the image of A under $\mathscr{D}$.

Based on the cone-valued map $\mathscr{D}$ one can define two different relations: for $y, \bar{y} \in Y$ we define

$$y \leq_1 \bar{y} \ \text{ if } \ \bar{y} \in \{y\} + \mathscr{D}(y) \tag{4.3}$$

and

$$y \leq_2 \bar{y} \ \text{ if } \ \bar{y} \in \{y\} + \mathscr{D}(\bar{y}). \tag{4.4}$$

We speak here of a variable ordering (structure), given by the ordering map $\mathscr{D}$, despite the binary relations given above are in general not transitive nor compatible with positive scalar multiplication, to express that the partial ordering given by a cone in most vector optimization problems in the literature is replaced by a relation defined by $\mathscr{D}$.

Relation (4.3) implies the concept of nondominated elements defined in [37, 38]. We also state the definitions of weakly and strongly nondominated elements which can be derived from the original definition of nondominated elements.

Definition 4.1.

- An element $\bar{y} \in A$ is a *nondominated element* of A w.r.t. the ordering map $\mathscr{D}$ if there is no $y \in A \setminus \{\bar{y}\}$ such that $\bar{y} \in \{y\} + \mathscr{D}(y)$, i.e., $y \not\leq_1 \bar{y}$ for all $y \in A \setminus \{\bar{y}\}$.
- An element $\bar{y} \in A$ is a *strongly nondominated* element of A w.r.t. the ordering map $\mathscr{D}$ if $\bar{y} \in \{y\} - \mathscr{D}(y)$ for all $y \in A$.
- Let $\operatorname{int} \mathscr{D}(y) \neq \emptyset$ for all $y \in A$. An element $\bar{y} \in A$ is a *weakly nondominated* element of A w.r.t. the ordering map $\mathscr{D}$ if there is no $y \in A$ such that $\bar{y} \in \{y\} + \operatorname{int} \mathscr{D}(y)$.

Example 4.1. Let $Y = \mathbb{R}^2$, the cone-valued map $\mathscr{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by

$$\mathscr{D}(y_1,y_2) := \begin{cases} \text{cone conv}\{(y_1,y_2),(1,0)\} & \text{if } (y_1,y_2) \in \mathbb{R}^2_+,\ y_2 \neq 0 \\ \mathbb{R}^2_+ & \text{otherwise} \end{cases}$$

and

$$A := \{(y_1,y_2) \in \mathbb{R}^2 \mid y_1 \geq 0,\ y_2 \geq 0,\ y_2 \geq 1 - y_1\}.$$

Then $\mathscr{D}(y_1,y_2) \subset \mathbb{R}^2_+$ for all $(y_1,y_2) \in \mathbb{R}^2$ and one can check that $\{(y_1,y_2) \in A \mid y_1 + y_2 = 1\}$ is the set of all nondominated elements of A w.r.t. $\mathscr{D}$ and that all elements of the set $\{(y_1,y_2) \in A \mid y_1 + y_2 = 1 \ \vee y_1 = 0 \ \vee \ y_2 = 0\}$ are weakly nondominated elements of A w.r.t. $\mathscr{D}$.

In Definition 4.1 the cone $\mathscr{D}(y) = \{d \in Y \mid y + d \text{ is dominated by } y\} \cup \{0_Y\}$ can be seen as the set of dominated directions for each element $y \in Y$. Note that when $\mathscr{D}(y) \equiv K$, where K is a pointed convex cone, and the space Y is partially ordered by K, the concepts of nondominated, strongly nondominated and weakly nondominated elements w.r.t. the ordering map $\mathscr{D}$ reduce to the classical concepts of efficient, strongly efficient and weakly efficient elements w.r.t. the cone K (see, for instance, [22]). Strongly nondominated is a stronger concept than nondominatedness, as it is not only demanded that $\bar{y} \in \{y\} + (Y \setminus \{\mathscr{D}(y)\})$ for all $y \in A \setminus \{\bar{y}\}$, but even $\bar{y} \in \{y\} - \mathscr{D}(y)$ for all $y \in A \setminus \{\bar{y}\}$ for $\bar{y}$ being strongly nondominated w.r.t. $\mathscr{D}$. This can be interpreted as the requirement of being far away from being dominated.

The second relation, relation (4.4), leads to the concept of minimal, also called nondominated-like, elements [7–9].

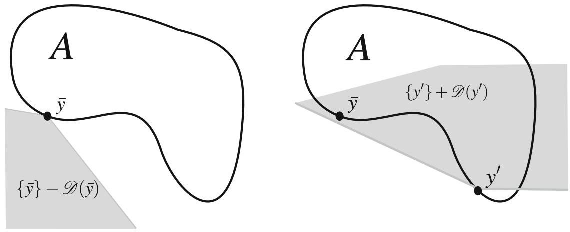


Fig. 4.1 The element $\bar{y} \in A$ is a minimal element of A w.r.t. the ordering map $\mathscr{D}$ whereas $\bar{y}$ is not a nondominated element of A w.r.t. the ordering map $\mathscr{D}$ because of $\bar{y} \in \{y'\} + \mathscr{D}(y') \setminus \{0_Y\}$

Definition 4.2.

- An element $\bar{y} \in A$ is a *minimal element* of A w.r.t. the ordering map $\mathscr{D}$ if there is no $y \in A \setminus \{\bar{y}\}$ such that $\bar{y} \in \{y\} + \mathscr{D}(\bar{y})$, i.e., $y \not\leq_2 \bar{y}$ for all $y \in A \setminus \{\bar{y}\}$.
- An element $\bar{y} \in A$ is a *strongly minimal* element of A w.r.t. the ordering map $\mathscr{D}$ if $A \subset \{\bar{y}\} + \mathscr{D}(\bar{y})$.
- Let int $\mathscr{D}(y) \neq \emptyset$ for all $y \in A$. An element $\bar{y} \in A$ is a *weakly minimal* element of A w.r.t. the ordering map $\mathscr{D}$ if there is no $y \in A$ such that $\bar{y} \in \{y\} + \operatorname{int} \mathscr{D}(\bar{y})$.

For an illustration of both optimality notions see Fig. 4.1.

The concepts of strongly minimal and strongly nondominated elements w.r.t. an ordering map $\mathscr{D}$ are illustrated in the following example.

Example 4.2. Let $Y = \mathbb{R}^2$, the cone-valued map $\mathscr{D} \colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by

$$\mathscr{D}(y_1, y_2) := \begin{cases} \mathbb{R}^2_+ & \text{if } y_2 = 0 \\ \operatorname{cone}\operatorname{conv}\{(|y_1|, |y_2|), (1,0)\} & \text{otherwise} \end{cases}$$

and

$$A := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \leq y_2 \leq 2y_1\}.$$

One can check that $(0,0) \in A$ is a strongly minimal and also a strongly nondominated element of A w.r.t. $\mathscr{D}$.

Regarding the notion of minimal elements, the cone $\mathscr{D}(y)$ for some $y \in Y$ can be viewed as the set of preferred directions: $\mathscr{D}(y) := \{d \in Y \mid y - d \text{ is preferred to } y\} \cup \{0_Y\}$. Observe that $\bar{y}$ is a minimal element of some set $A \subset Y$ w.r.t. $\mathscr{D}$ if and only if it is an efficient element of the set A with Y partially ordered by $K := \mathscr{D}(\bar{y})$.

Replacing $\mathscr{D}$ by $\tilde{\mathscr{D}}$ with $\tilde{\mathscr{D}}(y) := -\mathscr{D}(y)$ for all $y \in Y$ in the Definitions 4.1 and 4.2, we obtain corresponding concepts of (weakly, strongly) *max-nondominated* and of *maximal* elements of a set A w.r.t. the ordering map $\mathscr{D}$.

The following example illustrates that the concepts of nondominated and of minimal elements w.r.t. an ordering map $\mathscr{D}$ are not directly related.

Example 4.3. Let $Y = \mathbb{R}^2$, the cone-valued map $\mathscr{D}_1 : \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by

$$\mathscr{D}_1(y_1, y_2) := \begin{cases} \operatorname{cone}\operatorname{conv}\{(-1,1),(0,1)\} & \text{if } y_2 \geq 0, \\ \mathbb{R}^2_+ & \text{otherwise,} \end{cases}$$

and

$$A := \{(y_1,y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 \leq 1\}.$$

Then $(-1,0)$ is a nondominated but not a minimal element of A w.r.t. $\mathscr{D}_1$.

Considering instead the cone-valued map $\mathscr{D}_2 \colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ defined by

$$\mathscr{D}_2(y_1,y_2) := \begin{cases} \text{cone conv}\{(1,-1),(1,0)\} & \text{if } y_2 \geq 0, \\ \mathbb{R}^2_+ & \text{otherwise,} \end{cases}$$

then $(0,-1)$ is a minimal but not a nondominated element of A w.r.t. $\mathscr{D}_2$.

Considering instead the cone-valued map $\mathscr{D}_3 \colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ defined by

$$\mathscr{D}_3(y_1,y_2) := \begin{cases} \mathbb{R}^2_+ & \text{if } y \in \mathbb{R}^2 \setminus \{(0,-1),(-1,0)\}, \\ \{(z_1,z_2) \in \mathbb{R}^2 \mid z_1 \leq 0,\ z_2 \geq 0\} & \text{if } y = (0,-1), \\ \{(z_1,z_2) \in \mathbb{R}^2 \mid z_1 \geq 0,\ z_2 \leq 0\} & \text{if } y = (-1,0). \end{cases}$$

Then all elements of the set $\{(y_1,y_2) \in \mathbb{R}^2 \mid y_1^2 + y_2^2 = 1,\ y_1 \leq 0,\ y_2 \leq 0\}$ are minimal elements of A w.r.t. $\mathscr{D}$ but there is no nondominated element of the set A w.r.t. $\mathscr{D}$.

The two optimality concepts are only related under strong assumptions on $\mathscr{D}$:

Lemma 4.1. *(a) If $\mathscr{D}(y) \subset \mathscr{D}(\bar{y})$ for all $y \in A$ for some minimal element $\bar{y}$ of A w.r.t. $\mathscr{D}$, then $\bar{y}$ is also a nondominated element of A w.r.t. $\mathscr{D}$.*
(b) If $\mathscr{D}(\bar{y}) \subset \mathscr{D}(y)$ for all $y \in A$ for some nondominated element $\bar{y}$ of A w.r.t. $\mathscr{D}$, then $\bar{y}$ is also a minimal element of A w.r.t. $\mathscr{D}$.

Besides considering optimal elements of a set, all concepts apply also for a vector optimization problem with the image space equipped with a variable ordering structure. For that assume that X and Y are topological spaces and Y is equipped with a variable ordering structure defined by a cone-valued map $\mathscr{D} \colon Y \to 2^Y$ with $\mathscr{D}(y)$ a pointed convex cone. Let $F : X \to 2^Y$ be a given set-valued map and $S \subset X$ a nonempty set. Denote by $F(S) = \bigcup_{x \in S} F(x)$ the image of S under F. Then we consider the following vector optimization problem

$$\text{Minimize} \quad F(x) \quad \text{subject to} \quad x \in S. \qquad \text{(VOP)}$$

The various notions of nondominated and minimal elements w.r.t. the ordering map $\mathscr{D}$ for sets naturally induce corresponding notions of solutions to the optimization problem (VOP) as follows.

Definition 4.3. Let $\bar{x} \in S$ and $\bar{y} \in F(\bar{x})$. A pair $(\bar{x},\bar{y})$ is called a *"N" solution* of the problem (VOP) w.r.t. the ordering map $\mathscr{D}$, if $\bar{y}$ is a "N" element of the image set $F(S)$ respectively. Here, "N" may be (weakly, strongly, max-) nondominated, (weakly, strongly) minimal or maximal.

When F is a single-valued map $f \colon X \to Y$, we set $\bar{y} = f(\bar{x})$ in Definition 4.3.

4.2.2 Variable Ordering Structures in Applications

The examination of vector optimization problems with a variable ordering structure is motivated by some recent applications in such different areas as image registration or portfolio optimization. It turned out that the concept of efficiency in partially ordered spaces is not a sufficient tool for modeling these decision making problems. And also in game theory for N-person games variable domination structures were considered [4].

In decision-making theory the importance of variable ordering structures is also discussed and conditions are formulated which should be satisfied by such structures [16]. The cones $\mathscr{D}(y) \subset \mathbb{R}^m$ shall be closed convex ideal-symmetric cones with $\mathbb{R}^m_+ \subset \mathscr{D}(y)$ for all $y \in A \subset \mathbb{R}^m$. This is based on several requirements like monotonicity, local preferences and ideal symmetry. Here, ideal symmetric means that the cone $\mathscr{D}(y)$ is some kind of symmetric w.r.t. the direction $y - z$ pointing to the ideal point $z \in \mathbb{R}^m$ defined by $z_i := \inf\{y_i \mid y \in A\}$ for $i = 1,\ldots,m$. For the exact definitions we refer to [16]. These cones $\mathscr{D}(y)$ are in fact special Bishop–Phelps cones in the Euclidean space. Such variable orderings with images being Bishop–Phelps cones will be discussed in Sect. 4.2.4.

4.2.2.1 Variable Ordering Structures in Medical Image Registration

For modeling preferences of a totally rational decision maker in medical image registration a variable ordering structure better reflects the problem structure [34]. In medical image registration it is the aim to merge several medical images gained by different imaging methods as for instance computer tomography, magnetic resonance tomography, positron emission tomography, or ultrasound. For two data sets A and B a transformation map t, also called registration, has to be found (from a set T of allowed maps) such that some similarity measure comparing $t(A)$ and B is optimized. For some applications it is important that this transformation map is found automatically without a human decision maker. The quality of a transformation map, i.e. the similarity of the transformed data set to the target set, can be measured by a large variety of distance measures $f_i\colon (t,A,B) \to \mathbb{R}$, $i = 1,\ldots,m$, for some $m \in \mathbb{N}$. They all evaluate distinct characteristics like the sum of square differences, mutual information or cross-correlation. Different measures may lead to different best transformation maps. Some measures fail on special data sets and can lead to mathematical correct but useless results. Thus it is important to combine several measures. Possible approaches are a weighted sum of different measures. But difficulties appear as badly scaled functions or non-convex functions.

Instead, the problem can be viewed as a multiobjective optimization problem [34, 35] by arranging the several distance measures in an objective vector $f := (f_1,\ldots,f_m)^\top$. Then, for given data sets A and B, the vector optimization problem

$$\min_{t \in T} f(t,A,B)$$

has to be solved. For incorporating in the preference structure that some of the measures may fail on the given data sets, depending on the values $y \in \mathbb{R}^m$ in the image space a weighting vector $w(y) \in \mathbb{R}^m_+$ is generated. This weight can be interpreted as some kind of voting between the different measures. Also a weight component equal to zero is allowed which corresponds to the negligence of the correspondent measure, because it seems for instance to fail on the data set. This weight may also depend on gradient information, conformity and continuity aspects and reflects therefore the preferences of a totally rational decision maker who puts a higher weight on promising measures dependent on the value $y = f(t,A,B)$.

To such a weight at a point $y \in \mathbb{R}^m$ a cone of more or equally preferred directions is defined by

$$\mathscr{D}(y) := \left\{ d \in \mathbb{R}^m \;\middle|\; \sum_{i=1}^{m} \operatorname{sgn}(d_i) w_i(y) \geq 0 \right\}$$

where

$$\operatorname{sgn}(d_i) := \begin{cases} 1 & \text{if } d_i > 0, \\ 0 & \text{if } d_i = 0, \\ -1 & \text{if } d_i < 0. \end{cases}$$

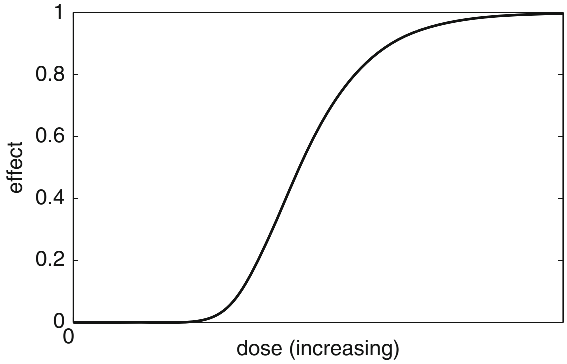
Note that for nonnegative weights $w \in \mathbb{R}^m_+$ it holds $\mathbb{R}^m_+ \subset \mathscr{D}(y)$.

4.2.2.2 Variable Ordering Structures in Intensity Modulated Radiation Therapy

In medical engineering, to be more concrete in intensity-modulated radiation therapy (IMRT), recently (Thieke, Private communication (2010)) the incorporation of variable ordering structures is considered to allow an improved modeling of the decision making problem. In IMRT one searches for an optimal treatment plan for the irradiation of a tumor with the target to spare the surrounding tissue, or at least to reduce the radiation dose delivered to the neighbored healthy organs, while destroying the tumor.

This problem is modeled as a multiobjective optimization problem with an objective for each healthy neighbored organ measuring its dose stress. For comparing different treatment plans the natural ordering in the image space, i.e. the componentwise ordering, is locally not satisfying. Instead, the set of dominated elements in the image space depends on the actual value of the objectives. As long as the response of the organs on dose variations is relatively small, which corresponds to flat parts in the dose-response curve, a change in the values for that organ in favor of an improvement of the values for other organs is more accepted than for values which correspond to steep parts in the dose-response curve, i.e. if the impact on the organ is very sensitive to variations of the dose. A dose-response curve describes thereby the caused effect to a particular volume (organ) of interest according to increasing dose, as illustrated in Fig. 4.2.

Fig. 4.2 Dose-response curve: portion of effect according to increasing dose level



4.2.2.3 Variable Ordering Structures and Equitability

Variable ordering structures play also an important role in the context of equitability in multiobjective optimization. For the concept of equitability it is assumed that the different criteria $f_1,\dots,f_m$ ($m \in \mathbb{N}$) considered in the multiobjective optimization problem are uniform in the sense of scale used, that their values are directly comparable, and that they are considered to be impartially. This makes the distribution of outcomes more important than the assignment of several outcomes to the specific criteria.

This notion is of interest for several applications as the allocation of resources, in location theory, see [25] and the references therein, or in portfolio optimization: in portfolio selection having n securities available, $x_j \in \mathbb{R}_+$ expresses the portion of the capital which is invested in the security j. Considering m equally probable scenarios, c_{ij} denotes the observed (or forecasted) rate of return of security j under scenario i. This results in an outcome matrix $C = (c_{ij})_{ij} \in \mathbb{R}^{m\times n}$. In the portfolio selection problem one has now to consider the linear optimization problem

$$\text{Maximize } Cx \text{ subject to } \sum_{j=1}^{n} x_j = 1,\ x_j \geq 0,\ j = 1,\dots,n.$$

The objective functions are uniform and it is postulated in [30] that an aggregation must be equitable to model risk averse preferences.

Equitability is a refinement of the efficiency notion w.r.t. the natural ordering cone in $\mathbb{R}^m$. One is interested in the distribution of the outcomes of the several objectives and not in their ordering, i.e. for instance a vector $(4,2,0)$ is considered to be equally good as a vector $(0,4,2)$. At the same time a principle of transfer should be satisfied stating that a transfer of any small amount from one outcome to any other relatively worse outcome is more preferred, for instance $(2,2,2)$ is considered to be better than $(4,2,0)$.

This is modeled in the following way: for the map $\Theta\colon \mathbb{R}^m \to \mathbb{R}^m$, $\Theta(y) = (\Theta_1(y),\dots,\Theta_m(y))$ with $\Theta_1(y) \geq \dots \geq \Theta_m(y)$ such that there exists a permutation

τ of $\{1,\ldots,m\}$ with $\Theta_i(y) = y_{\tau(i)}$ for $i = 1,\ldots,m$ for each $y \in \mathbb{R}^m$, the cumulative map $\overline{\Theta}\colon \mathbb{R}^m \to \mathbb{R}^m$ is defined by $\overline{\Theta}(y) = (\overline{\Theta}_1(y),\ldots,\overline{\Theta}_m(y))$ with

$$\overline{\Theta}_i(y) = \sum_{j=1}^{i} \Theta_j(y) \quad \text{for } i = 1,\ldots,m \text{ and for all } y \in \mathbb{R}^m.$$

Then the equitability relation $\preccurlyeq_e$ is defined by

$$x \preccurlyeq_e y \;\Leftrightarrow\; \overline{\Theta}_i(x) \le \overline{\Theta}_i(y) \quad \text{for all } i = 1,\ldots,m.$$

For instance $\Theta(2,4,0) = (4,2,0)$ and $\Theta(2,2,2) = (2,2,2)$ and thus $\overline{\Theta}(2,4,0) = (4,6,6)$ as well as $\overline{\Theta}(2,2,2) = (2,4,6)$. Then $(2,2,2) \preccurlyeq_e (2,4,0)$. Based on $\preccurlyeq_e$ an *equitable efficient element* $\bar{y}$ of some set $A \subset \mathbb{R}^m$ is defined as an element in A such that there exists no other element $y \in A$ with $y \preccurlyeq_e \bar{y}$. There is also a connection to the concept of efficient elements in a partially ordered space ordered by $\mathbb{R}^m_+$: y is an equitable efficient element of the set A if and only if y is efficient of the set $\overline{\Theta}(A)$ w.r.t. the cone $\mathbb{R}^m_+$.

The problem of finding equitable efficient elements is a vector optimization problem with a variable ordering structure [3]: to see this the space $Y = \mathbb{R}^m$ can be partitioned in $m!$ sectors which are non-pointed convex cones. For each sector a cone of preferred and of dominated directions is defined, i.e. $\{\mathscr{D}(y) \mid y \in A\}$ is here a family of a finite number of pointed convex cones. The ordering concept of equitability corresponds thus to a variable ordering structure with the images $\mathscr{D}(y)$ depending on the sector in which the element y is located.

4.2.3 Vector Variational Inequalities and Vector Complementarity Problems with a Variable Ordering Structure

Vector variational inequalities and their relation to vector optimization problems in partially ordered spaces have been intensively studied in the last decades since their introduction in 1980. Thus it is a natural consequence that also vector variational inequalities with variable domination cones are considered. For instance in [7] the following vector variational inequality (VVI) is studied: Let X, Y be real Banach spaces, let $S \subset X$ be a nonempty closed convex set and let $T\colon S \to \mathscr{L}(X,Y)$ be a map with $\mathscr{L}(X,Y)$ the space of continuous linear maps from X into Y. Additionally assume a set-valued map $\mathscr{C}\colon S \to 2^Y$ to be given with $\mathscr{C}(x)$ a closed pointed convex cone with nonempty interior for all $x \in S$. The task is now to find some $\bar{x} \in S$ such that

$$T(\bar{x})(x - \bar{x}) \notin -\operatorname{int}\mathscr{C}(\bar{x}) \quad \text{for all } x \in S. \tag{4.5}$$

In [28] the notion of a *generalized efficient solution* of a vector optimization problem $\min_{x\in S} F(x)$ was defined, similar to the notion given in Definition 4.3, for a vector-valued objective function F and assuming a cone-valued map $\mathscr{C}$ to be given as defined above: An element $\bar{x} \in S$ is called a generalized efficient solution if there is no $x \in S$ such that $F(x) \in F(\bar{x}) - \operatorname{int} \mathscr{C}(\bar{x})$. This notion was also used in [2] setting $S = X$ and calling it (global) vector minimum point. Clearly, if F is an injective single-valued function, we can define the cone-valued map $\mathscr{D}$ in the image space by $\mathscr{D}(y) := \mathscr{C}(x)$ with x given by the equation $y = F(x)$ for all $y \in F(S)$. Then $\bar{x}$ is a generalized efficient solution w.r.t. the ordering map $\mathscr{C}$ if and only if $(\bar{x}, F(\bar{x}))$ is a weakly minimal solution w.r.t. the ordering map $\mathscr{D}$.

Several extensions of the above VVI are considered for instance in [6, 40], see also [18] and the references therein. In this context also a nonlinear scalarization functional is used to define a so-called gap function [2]. Let $r(x) \in \operatorname{int} \mathscr{C}(x)$ be given for all $x \in S$, then the functional $\xi_r : S \times Y \to \mathbb{R}$ is defined by [9, 10]:

$$\xi_r(x,y) := \inf\{t \in \mathbb{R} \mid y \in t \cdot r(x) - \mathscr{C}(x)\} \qquad \forall x \in S,\ y \in Y.$$

Based on a similar scalarization functional also (weakly) minimal and (weakly) nondominated elements w.r.t. a variable ordering map can be characterized, see Sect. 4.4.2.2.

Variable ordering structures are also presumed in the context of vector complementarity problems [21]. For instance with T and $\mathscr{C}$ as above, setting $S := X$, and assuming that $K \subset Y$ is a convex cone, the following weak vector complementarity problem can be considered: Find some $\bar{x} \in K$ such that

$$T(\bar{x})(\bar{x}) \notin \operatorname{int} \mathscr{C}(\bar{x}) \ \wedge \ T(\bar{x})(x) \notin -\operatorname{int} \mathscr{C}(\bar{x}) \quad \forall x \in K.$$

Relations to special vector optimization problems can be examined using the notion of a generalized efficient solution as given above, as well as to special vector variational inequalities.

Finally, also in vector equilibrium problems variable domination cones are of interest [27].

4.2.4 Variable Ordering Structures Defined by Bishop–Phelps Cones

In the vector optimization problems with a variable ordering which we have studied so far, the ordering structure is defined by a cone-valued map where the images $\mathscr{D}(y)$ are arbitrary (pointed convex) cones. By imposing some weak additional assumptions on these images $\mathscr{D}(y)$, to be more concrete, to assume that they have a representation as Bishop–Phelps cones (BP cones for short), scalarization functionals can be formulated which completely characterize nondominated elements w.r.t. the variable ordering. Even sufficient optimality conditions for nondominated elements can be formulated based on these scalarizations.

It is a very natural assumption that the images $\mathscr{D}(y)$ are BP cones, as any nontrivial convex cone with a closed and bounded base in a real normed space, e.g., any closed pointed convex cone in $\mathbb{R}^n$, is representable as a BP cone [23, Remark 2.16], [26,32]. For instance, the variable ordering which was studied in [16] for describing the preferences of decision makers is based on ideal-symmetric cones which are BP cones [16, Remark 8].

The BP cones were introduced in 1962 by Bishop and Phelps [5]. They are defined as cones in a real normed space $(Y, \|\cdot\|)$ with the help of a functional from the topological dual space Y^*. The norm $\|\cdot\|_*$ in Y^* is thereby defined by

$$\|y^*\|_* := \sup_{y \neq 0_Y} \frac{|y^*(y)|}{\|y\|} \text{ for all } y^* \in Y^*.$$

Definition 4.4. For an arbitrary continuous linear functional $\phi \in Y^*$ on the real normed space $(Y, \|\cdot\|)$ the cone

$$C(\phi) := \{y \in Y \mid \|y\| \leq \phi(y)\} \tag{4.6}$$

is called *Bishop–Phelps cone*.

A cone is said to be representable as a BP cone if there exists some $\phi \in Y^*$ and a norm $\|\cdot\|$ equivalent to the norm of the space such that the cone can be written as in (4.6). Note that the definition of a BP cone introduced in [5] is slightly different from the above one which was given e.g. in [23].

Example 4.4. (a) Let $Y = \mathbb{R}^n$ and

$$C_p := \{y \in \mathbb{R}^n \mid \|(y_1, \ldots, y_{n-1})\|_p \leq y_n\}$$

for an l_p norm $\|.\|_p$ with $p \in [1, \infty]$. It has been established that C_p is a BP cone [23] with $C_p = C(\sqrt[p]{2} e_n)$ for $p \in [1, \infty)$ and $e_n := (0, \ldots, 0, 1)^\top$, and $C_\infty = C(e_n)$. Note that C_2 is the Lorentz cone (also called second-order cone or ice cream cone).

(b) Let $Y = \mathbb{R}^2$ and assume that the space is equipped with the Manhattan-norm $\|\cdot\|_1$. Then for instance for $(\phi_1, \phi_2) = (1, 1)$ we have $C(\phi_1, \phi_2) = \mathbb{R}^2_+$. Assume $\phi_1, \phi_2 \geq 1$, then $\mathbb{R}^2_+ \subset C(\phi_1, \phi_2)$, $(0, 1/\phi_2) \in C(\phi_1, \phi_2)$, $(1/\phi_1, 0) \in C(\phi_1, \phi_2)$ and

$$C(\phi_1, \phi_2) = \text{cone conv}\{y^A, y^B\}$$

with

$$y^A := \left(\frac{1 - \phi_2}{\phi_1 + \phi_2}, \frac{1 + \phi_1}{\phi_1 + \phi_2} \right)^\top \text{ and } y^B := \left(\frac{1 + \phi_2}{\phi_1 + \phi_2}, \frac{1 - \phi_1}{\phi_1 + \phi_2} \right)^\top, \tag{4.7}$$

see Fig. 4.3.

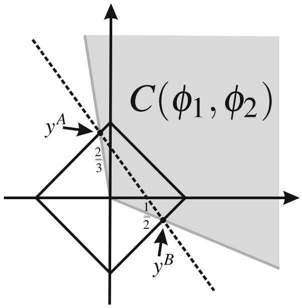


Fig. 4.3 BP cone $C(\phi_1,\phi_2)$ of Example 4.4(b) for $\phi_1 = 2$ and $\phi_2 = 3/2$, as well as the unit ball w.r.t. the Manhattan-norm and (in *dashed line*) the set $\{(y_1,y_2) \in \mathbb{R}^2 \mid (\phi_1,\phi_2)^\top (y_1,y_2) = 1\}$

BP cones have a rich mathematical structure and we recall some of their properties [23]. In the following let $(Y, \|\cdot\|)$ be a real normed space.

Proposition 4.1. *Let $\phi, \phi^1, \phi^2 \in Y^*$ be given.*

(a) $C(\phi)$ *is closed, pointed and convex.*
(b) $C(\phi) = -C(-\phi)$.
(c) *If* $\|\phi\|_* > 1$ *then* $C(\phi)$ *is nontrivial; if* $\|\phi\|_* < 1$ *then* $C(\phi) = \{0_Y\}$.
(d) $\{y \in Y \mid \|y\| < \phi(y)\} \subset int(C(\phi))$. *If* $\|\phi\|_* > 1$ *then the interior of* $C(\phi)$ *is nonempty and*

$$int(C(\phi)) = \{y \in Y \mid \|y\| < \phi(y)\}.$$

(e) $\phi \in (C(\phi))^\# := \{y^* \in Y^* \mid y^*(y) > 0\ \forall y \in C(\phi) \setminus \{0_Y\}\}$.
(f) *The set* $\{y \in C(\phi) \mid \phi(y) = 1\}$ *is a closed and bounded base for the cone* $C(\phi)$.

Assuming now that the variable ordering structure on Y is defined by a set-valued map $\mathscr{D}\colon Y \to 2^Y$ with $\mathscr{D}(y)$ representable as a BP cone for all $y \in Y$ (or for all elements y of a subset A of Y), then to any $y \in Y$ we associate a norm $\|.\|_y$ equivalent to but eventually different from the norm of the space and we define a map ℓ from Y to Y^* such that

$$\mathscr{D}(y) = C(\ell(y)) = \{u \in Y \mid \|u\|_y \le \ell(y)(u)\} \text{ for all } y \in Y.$$

Note that already in $\mathbb{R}^n$ one might need different equivalent norms to represent different nontrivial convex closed pointed cones as BP cones and this motivates us to consider the above BP cones. In $\mathbb{R}^2$ it satisfies to choose just one norm but already in $\mathbb{R}^3$ one has to use different norms to model for instance a polyhedral cone and the Lorentz cone. In an application however there might be a variable ordering structure with different cones $\mathscr{D}(y)$ but presumably they will all be of the same type, for instance all polyhedral, and can all be modeled with the same norm, compare [16, Remark 8]. In particular, when the norm $\|\cdot\|_y$ in the definition of the BP cones $\mathscr{D}(y)$ is assumed to equal the norm $\|\cdot\|$ of the space Y and is thus equal for all $y \in Y$, these cones reduce to the BP cones

$$\mathscr{D}(y) = C(\ell(y)) = \{u \in Y \mid \|u\| \le \ell(y)(u)\}. \tag{4.8}$$

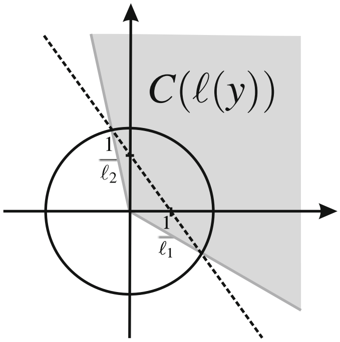


Fig. 4.4 BP cone $C(\ell(y))$ of Example 4.5 for $\ell_1 = \ell_1(y)$ and $\ell_2 = \ell_2(y)$, as well as the unit ball w.r.t. the Euclidean norm and (in *dashed line*) the line connecting the points $(1/\ell_1, 0)$ and $(0, 1/\ell_2)$

Below is an example of a variable ordering structure given by such BP cones. Even if the norm $\|\cdot\|$ does not depend on y a wide range of different cones is covered by the images $\mathscr{D}(y)$ in (4.8).

Example 4.5. Let Y be the Euclidean space $\mathbb{R}^2$, $\|.\|_y := \|.\|_2$ for all $y \in \mathbb{R}^2$ and define $\ell : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\ell(y_1, y_2) := (\ (3 + \sin y_1)/2\, , (3 + \cos y_2)/2\ ) \in [1,2] \times [1,2]. \tag{4.9}$$

Then $\mathbb{R}^2_+ \subset C(\ell(y))$ for all $y \in \mathbb{R}^2$. The cones $C(\ell(y))$ can be visualized as follows: The two extreme rays of the pointed convex cone $C(\ell(y))$ are given by two half rays starting in the origin being defined by the two intersection points of the unit circle and the line connecting the points $(1/\ell_1, 0)$ and $(0, 1/\ell_2)$, see Fig. 4.4. For instance $C(\ell(3\pi/2, \pi)) = \mathbb{R}^2_+$.

In Sect. 4.4.2.3 such special variable ordering structures are considered for the formulation of nonlinear scalarization functionals.

4.3 Basic Properties of Optimal Elements

Many properties of efficient elements in a partially ordered space are still valid for optimal elements w.r.t. a variable ordering, whereas others, see for instance Lemma 4.4, hold in general only under additional assumptions. In the following, let Y be a real topological linear space, A a nonempty subset of Y and $\mathscr{D}\colon Y \to 2^Y$ a cone-valued map with $\mathscr{D}(y)$ convex and pointed for all $y \in Y$. For both optimality concepts, for minimal and for nondominated elements w.r.t. an ordering map $\mathscr{D}$, and for the related concepts of strongly and weakly optimal elements, we can easily derive the following properties.

Lemma 4.2. *(a) Any strongly nondominated element of A w.r.t. $\mathscr{D}$ is also a nondominated element of A w.r.t. $\mathscr{D}$. Any strongly minimal element of A w.r.t. the ordering map $\mathscr{D}$ is also a minimal element of A w.r.t. $\mathscr{D}$.*

(b) *If $\mathscr{D}(A)$ is pointed, then there is at most one strongly nondominated element of A w.r.t. $\mathscr{D}$.*
(c) *Let int$(\mathscr{D}(y)) \neq \emptyset$ for all $y \in A$. Any nondominated element of A w.r.t. $\mathscr{D}$ is also a weakly nondominated element of A w.r.t. $\mathscr{D}$. Any minimal element of A w.r.t. $\mathscr{D}$ is also a weakly minimal element of A w.r.t. $\mathscr{D}$.*
(d) *If $\bar{y}$ is a strongly nondominated element of A w.r.t. $\mathscr{D}$, then the set of minimal elements of A w.r.t. $\mathscr{D}$ is empty or equals $\{\bar{y}\}$. If $\mathscr{D}(A)$ is pointed, then $\bar{y}$ is the unique minimal element of A w.r.t. $\mathscr{D}$.*
(e) *If $\bar{y} \in A$ is a strongly minimal element of A w.r.t. $\mathscr{D}$ and if $\mathscr{D}(\bar{y}) \subset \mathscr{D}(y)$ for all $y \in A$, then $\bar{y}$ is also a strongly nondominated element of A w.r.t. $\mathscr{D}$.*

Proof. (a) Let $\bar{y}$ be a strongly nondominated element of A w.r.t. $\mathscr{D}$. Then $\bar{y} \in \{y\} + \mathscr{D}(y)$ for some $y \in A$ together with $\bar{y} \in \{y\} - \mathscr{D}(y)$ implies that $\bar{y} - y \in \mathscr{D}(y) \cap (-\mathscr{D}(y))$ and due to the pointedness of $\mathscr{D}(y)$ we obtain $\bar{y} = y$. The same for a strongly minimal element of A w.r.t. $\mathscr{D}$.
(b) Let $\bar{y}$ be a strongly nondominated element of A w.r.t. $\mathscr{D}$. If $\mathscr{D}(A)$ is pointed, then $\bar{y} - y \in -\mathscr{D}(y) \subset -\mathscr{D}(A)$ implies $\bar{y} - y \notin \mathscr{D}(A)$ for all $y \in A \setminus \{\bar{y}\}$, i.e. $y \notin \{\bar{y}\} - \mathscr{D}(\bar{y})$ for all $y \in A \setminus \{\bar{y}\}$ and thus no other element of A can be strongly nondominated w.r.t. $\mathscr{D}$.
(c) Follows directly from the definitions.
(d) As $\bar{y}$ is a strongly nondominated element of A w.r.t. $\mathscr{D}$ it holds for all $y \in A \setminus \{\bar{y}\}$

$$\bar{y} \in \{y\} - \mathscr{D}(y) \tag{4.10}$$

and hence, y cannot be a minimal element of A w.r.t. $\mathscr{D}$. Next, assume there exists $y \in A$ such that $y \in \{\bar{y}\} - \mathscr{D}(\bar{y})$. Together with (4.10) we conclude

$$y - \bar{y} \in \mathscr{D}(y) \cap (-\mathscr{D}(\bar{y})) \subset \mathscr{D}(A) \cap (-\mathscr{D}(A)).$$

If $\mathscr{D}(A)$ is pointed then $y = \bar{y}$ and thus $\bar{y}$ is a minimal element of A w.r.t. $\mathscr{D}$.
(e) As $\bar{y}$ is a strongly minimal element of A w.r.t. $\mathscr{D}$ it holds under the assumptions here that $\bar{y} \in \{y\} - \mathscr{D}(\bar{y}) \subset \{y\} - \mathscr{D}(y)$ for all $y \in A$ and hence $\bar{y}$ is a strongly nondominated element of A w.r.t. $\mathscr{D}$. □

A common result is that the efficient elements of a set in a partially ordered space are a subset of the boundary of that set. The result remains true for variable ordering structures.

Lemma 4.3. *(a)* *(i)* *If int $\mathscr{D}(y) \neq \emptyset$ for all $y \in Y$ and $\bar{y} \in A$ is a weakly minimal element of the set A w.r.t. the ordering map $\mathscr{D}$, then $\bar{y} \in \partial A$.*
(ii) *If $\bar{y} \in A$ is a minimal element of the set A w.r.t. the ordering map $\mathscr{D}$ and $\mathscr{D}(\bar{y}) \neq \{0_Y\}$, then $\bar{y} \in \partial A$.*
(b) *(i)* *If $\bigcap_{y\in A}$ int $\mathscr{D}(y) \neq \emptyset$ and $\bar{y} \in A$ is a weakly nondominated element of the set A w.r.t. the ordering map $\mathscr{D}$, then $\bar{y} \in \partial A$.*
(ii) *If $\bigcap_{y\in A} \mathscr{D}(y) \neq \{0_Y\}$ and $\bar{y} \in A$ is a nondominated element of the set A w.r.t. the ordering map $\mathscr{D}$, then $\bar{y} \in \partial A$.*

Proof. (a) (i) If $\bar{y} \in \operatorname{int} A$ then for any $d \in \operatorname{int} \mathscr{D}(\bar{y})$ there exists some $\lambda > 0$ with $\bar{y} - \lambda d \in A$. Then

$$\bar{y} - \lambda d \in A \cap (\{\bar{y}\} - \operatorname{int} \mathscr{D}(\bar{y}))$$

in contradiction to $\bar{y}$ a weakly minimal element of A w.r.t. $\mathscr{D}$.
(ii) Similar to (i) but choose $d \in \mathscr{D}(\bar{y}) \setminus \{0_Y\}$.

(b) Similar to (a)(i): if $\bar{y} \in \operatorname{int} A$ then choosing $d \in \bigcap_{y \in A} \operatorname{int} \mathscr{D}(y)$ there exists $\lambda > 0$ such that

$$\bar{y} - \lambda\, d \in A \cap (\{\bar{y}\} - \operatorname{int} \mathscr{D}(\bar{y} - \lambda\, d))$$

in contradiction to $\bar{y}$ a weakly nondominated element of A w.r.t. $\mathscr{D}$.
(ii) Similar to (b)(i), but choose $d \in \left(\bigcap_{y \in A} \mathscr{D}(y)\right) \setminus \{0_Y\}$. □

The following example demonstrates that we need for instance in (b)(i) in Lemma 4.3 an assumption like

$$\bigcap_{y \in A} \operatorname{int} \mathscr{D}(y) \neq \emptyset. \tag{4.11}$$

Example 4.6. For the set $A = [1,3] \times [1,3] \subset \mathbb{R}^2$ and the ordering map $\mathscr{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$,

$$\mathscr{D}(y) := \begin{cases} \mathbb{R}^2_+ & \text{for all } y \in \mathbb{R}^2,\ y_1 \geq 2, \\ \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 \leq 0,\ z_2 \geq 0\} & \text{else,} \end{cases}$$

the point $\bar{y} = (2,2)$ is a weakly nondominated element of A w.r.t. $\mathscr{D}$ but $\bar{y} \notin \partial A$.

In view of the duality results in Sect. 4.5 we also study the relation of the optimal elements of some set A and of the set

$$M := \bigcup_{y \in A} \{y\} + \mathscr{D}(y) \tag{4.12}$$

w.r.t. the ordering map $\mathscr{D}$. Note that for $\mathscr{D}(y) = K$ for all $y \in Y$ we have $M = A + K$ and the relations are well-known, compare [22, Lemma 4.7], [37, Lemma 4.1].

Lemma 4.4. *Let M be defined as in (4.12).*

(a) (i) If $\bar{y} \in A$ is a minimal element of the set M w.r.t. $\mathscr{D}$, then it is also a minimal element of the set A w.r.t. $\mathscr{D}$.
(ii) If $\bar{y} \in A$ is a minimal element of the set A w.r.t. $\mathscr{D}$ and if $\mathscr{D}(y) \subset \mathscr{D}(\bar{y})$ for all $y \in A$, then $\bar{y}$ is also a minimal element of the set M w.r.t. $\mathscr{D}$.
(b) (i) If $\bar{y} \in M$ is a nondominated element of the set M w.r.t. $\mathscr{D}$, then $\bar{y} \in A$ and $\bar{y}$ is also a nondominated element of the set A w.r.t. $\mathscr{D}$.
(ii) If $\bar{y} \in A$ is a nondominated element of the set A w.r.t. $\mathscr{D}$, and if

$$\mathscr{D}(y + d) \subset \mathscr{D}(y) \text{ for all } y \in A \text{ and for all } d \in \mathscr{D}(y), \tag{4.13}$$

then $\bar{y}$ is a nondominated element of M w.r.t. $\mathscr{D}$.

Proof. (a) The first implication (i) follows from $A \subset M$. Next we assume for (ii) that $\bar{y}$ is a minimal element of A but not of M w.r.t. $\mathscr{D}$, i.e. there exists some $y \in A$ and $d_y \in \mathscr{D}(y) \setminus \{0_Y\}$ with $y + d_y \in \{\bar{y}\} - (\mathscr{D}(\bar{y}) \setminus \{0_Y\})$. As $\mathscr{D}(\bar{y})$ is a pointed convex cone and $\mathscr{D}(y) \subset \mathscr{D}(\bar{y})$ this implies

$$\begin{aligned} y &\in \{\bar{y}\} - (\mathscr{D}(y) \setminus \{0_Y\}) - (\mathscr{D}(\bar{y}) \setminus \{0_Y\}) \\ &\subset \{\bar{y}\} - (\mathscr{D}(\bar{y}) \setminus \{0_Y\}) - (\mathscr{D}(\bar{y}) \setminus \{0_Y\}) \\ &\subset \{\bar{y}\} - (\mathscr{D}(\bar{y}) \setminus \{0_Y\}), \end{aligned}$$

in contradiction to $\bar{y}$ a minimal element of A w.r.t. $\mathscr{D}$.

(b) (i) If $\bar{y} \in M \setminus A$ then $\bar{y} \in \{y\} + (\mathscr{D}(y) \setminus \{0_Y\})$ for some $y \in A \subset M$ in contradiction to $\bar{y}$ a nondominated element of M w.r.t. $\mathscr{D}$. Thus $\bar{y} \in A$. Due to $A \subset M$, $\bar{y}$ is then also a nondominated element of A w.r.t. $\mathscr{D}$. Next we assume for (ii) that $\bar{y}$ is a nondominated element of A w.r.t. $\mathscr{D}$ but not of M, i.e. there exists some $y \in A$ and $d_y \in \mathscr{D}(y) \setminus \{0_Y\}$ with $\bar{y} \in \{y + d_y\} + (\mathscr{D}(y + d_y) \setminus \{0_Y\})$. As $\mathscr{D}(y)$ is a pointed convex cone and $\mathscr{D}(y + d_y) \subset \mathscr{D}(y)$ this implies

$$\begin{aligned} \bar{y} &\in \{y\} + (\mathscr{D}(y) \setminus \{0_Y\}) + (\mathscr{D}(y + d_y) \setminus \{0_Y\}) \\ &\subset \{y\} + (\mathscr{D}(y) \setminus \{0_Y\}) + (\mathscr{D}(y) \setminus \{0_Y\}) \\ &\subset \{y\} + (\mathscr{D}(y) \setminus \{0_Y\}), \end{aligned}$$

in contradiction to $\bar{y}$ a nondominated element of A w.r.t. $\mathscr{D}$. □

The condition (4.13) can also be written as $\mathscr{D}(y + d) + \mathscr{D}(y) \subset \mathscr{D}(y)$ for all $y \in Y$ and all $d \in \mathscr{D}(y)$ and corresponds to the property of transitivity of a binary relation [11] as (4.13) implies: If y^1 is dominated by y^2 (in the sense of (4.3)), i.e. $y^1 \in \{y^2\} + \mathscr{D}(y^2)$, and if y^2 is dominated by y^3, i.e. $y^2 \in \{y^3\} + \mathscr{D}(y^3)$, then $y^1 \in \{y^2\} + \mathscr{D}(y^2) \subset \{y^3\} + \mathscr{D}(y^3)$, i.e. y^1 is dominated by y^3. A variable domination structure satisfying the condition (4.13) is given in the following example.

Example 4.7. Define the cone-valued map $\mathscr{D} \colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ by

$$\mathscr{D}(y_1, y_2) := \begin{cases} \{(r\cos\varphi, r\sin\varphi) \mid r \geq 0,\ \varphi \in [0, \pi/8]\} & \text{if } y_1 \geq \pi/2, \\ \{(r\cos\varphi, r\sin\varphi) \mid r \geq 0,\ \varphi \in [0, \frac{\pi}{2} + \frac{\pi}{8} - y_1]\} & \text{if } y_1 \in (\pi/8, \pi/2), \\ \mathbb{R}^2_+ & \text{if } y_1 \leq \pi/8. \end{cases}$$

Then $\mathscr{D}$ depends only on y_1 and for $y_1 \geq \bar{y}_1$ for some $y, \bar{y} \in \mathbb{R}^2$ we conclude $\mathscr{D}(y) \subset \mathscr{D}(\bar{y})$. As for any $y \in \mathbb{R}^2$ and any $d \in \mathscr{D}(y)$ we have $d_1 \geq 0$ and thus $y_1 + d_1 \geq y_1$ we conclude that (4.13) is satisfied.

In general only the cones $\mathscr{D}(y)$ for $y \in A$ are of interest for modeling a decision making problem. Thus we have the freedom of setting $\mathscr{D}(y) := \{0_Y\}$ for all $y \in Y \setminus A$. This allows us to make the assumption (4.13) dispensable for the result in Lemma 4.4(b):

Lemma 4.5. *Let $\mathscr{D}\colon Y \to 2^Y$ be given with $\mathscr{D}(y) = \{0_Y\}$ for all $y \in Y \setminus A$ and let M be defined as in (4.12). Then an element $\bar{y} \in Y$ is a nondominated element of the set A w.r.t. $\mathscr{D}$ if and only if it is a nondominated element of the set M w.r.t. $\mathscr{D}$.*

Proof. First assume $\bar{y}$ is a nondominated element of the set A w.r.t. $\mathscr{D}$. If it is not also nondominated of M w.r.t. $\mathscr{D}$ then there exists some $y \in A$ and some $d \in \mathscr{D}(y)$ such that

$$\bar{y} \in \{y+d\} + \mathscr{D}(y+d) \setminus \{0_Y\} \quad \text{with} \quad y+d \notin A. \tag{4.14}$$

Thus $y+d \in M \setminus A$ and then $\mathscr{D}(y+d) = \{0_Y\}$ in contradiction to (4.14). The other implications follows from Lemma 4.4(b)(i). □

4.4 Scalarization

Scalarization, i.e. the replacement of a vector optimization problem by an, in general parameter dependent, scalar-valued optimization problem, is an important tool for characterizing optimal elements in vector optimization. Linear functionals are the easiest way to formulate such scalarization functions, but they completely characterize weakly optimal elements only under additional assumptions as convexity. Hence, we will also consider nonlinear scalarization functionals. Again, let Y be a real topological linear space, A a nonempty subset of Y and $\mathscr{D}\colon Y \to 2^Y$ a set-valued map with $\mathscr{D}(y)$ a pointed convex cone for all $y \in Y$.

4.4.1 Linear Scalarization

A basic scalarization technique is based on continuous linear functionals l from the topological dual space Y^*. Then one examines the scalar-valued optimization problems

$$\min_{y \in A} \; l(y).$$

In finite dimensions, $Y = \mathbb{R}^m$, this scalarization is also well known as weighted sum approach, and the components $l_i \in \mathbb{R}$, $i = 1, \ldots, m$, are then denoted as weights. In the space Y partially ordered by some convex cone K we use that the elements of the dual cone are known to be monotonically increasing. Recall that the dual cone $K^* \subset Y^*$ is given by $K^* := \{l \in Y^* \mid l(y) \geq 0 \ \forall\ y \in K\}$ and the quasi-interior of the dual cone $K^\#$ is defined as $K^\# = \{l \in Y^* \mid l(y) > 0 \ \forall\ y \in K \setminus \{0_Y\}\}$. We get the following sufficient conditions for optimal elements w.r.t. a variable ordering [14, 16]:

Theorem 4.1. *Let $\bar{y} \in A$.*

(a) *(i)* *If for some $l \in (\mathscr{D}(\bar{y}))^*$*

$$l(\bar{y}) < l(y) \text{ for all } y \in A \setminus \{\bar{y}\},$$

then $\bar{y}$ is a minimal element of A w.r.t. the ordering map $\mathscr{D}$.

(ii) *If for some $l \in (\mathscr{D}(\bar{y}))^\#$*

$$l(\bar{y}) \leq l(y) \text{ for all } y \in A,$$

then $\bar{y}$ is a minimal element of A w.r.t. the ordering map $\mathscr{D}$.

(iii) *Let int $\mathscr{D}(y) \neq \emptyset$ for all $y \in A$. If for some $l \in (\mathscr{D}(\bar{y}))^* \setminus \{0_{Y^*}\}$*

$$l(\bar{y}) \leq l(y) \text{ for all } y \in A,$$

then $\bar{y}$ is a weakly minimal element of A w.r.t. the ordering map $\mathscr{D}$.

(b) *(i)* *If for some $l \in (\mathscr{D}(A))^*$*

$$l(\bar{y}) < l(y) \text{ for all } y \in A \setminus \{\bar{y}\},$$

then $\bar{y}$ is a nondominated element of A w.r.t. the ordering map $\mathscr{D}$.

(ii) *If for some $l \in (\mathscr{D}(A))^\#$*

$$l(\bar{y}) \leq l(y) \text{ for all } y \in A,$$

then $\bar{y}$ is a nondominated element of A w.r.t. the ordering map $\mathscr{D}$.

(iii) *Let int $\mathscr{D}(y) \neq \emptyset$ for all $y \in A$ and let $\mathscr{D}(A)$ be convex. If for some $l \in (\mathscr{D}(A))^* \setminus \{0_{Y^*}\}$*

$$l(\bar{y}) \leq l(y) \text{ for all } y \in A,$$

then $\bar{y}$ is a weakly nondominated element of A w.r.t. the ordering map $\mathscr{D}$.

Proof. (a) (i) If $\bar{y}$ is not a minimal element of A w.r.t. $\mathscr{D}$, then $\bar{y} - y \in \mathscr{D}(\bar{y}) \setminus \{0_Y\}$ for some $y \in A$ and as $l \in (\mathscr{D}(\bar{y}))^*$ this implies $l(\bar{y}) \geq l(y)$ in contradiction to the assumption.

(ii) If $\bar{y} - y \in \mathscr{D}(\bar{y}) \setminus \{0_Y\}$ for any $y \in A$ then we get by $l \in (\mathscr{D}(\bar{y}))^\#$ that $l(\bar{y}) > l(y)$.

(iii) If $\bar{y} - y \in \operatorname{int} \mathscr{D}(\bar{y})$ for any $y \in A$ then $l \in (\mathscr{D}(\bar{y}))^* \setminus \{0_{Y^*}\}$ implies, compare [22, Lemma 3.21], $l(\bar{y}) > l(y)$.

(b) (i) If $\bar{y}$ is not a nondominated element of A w.r.t. $\mathscr{D}$, then $\bar{y}-y \in \mathscr{D}(y)\setminus\{0_Y\}$ for some $y \in A$. As $l \in (\mathscr{D}(A))^*$ also $l \in (\mathscr{D}(y))^*$ and thus $l(\bar{y}) \geq l(y)$ in contradiction to the assumption.

(ii) If $\bar{y}-y \in \mathscr{D}(y)\setminus\{0_Y\}$ for any $y \in A$ then $l \in (\mathscr{D}(A))^{\#}$ and thus $l \in (\mathscr{D}(y))^{\#}$ implies $l(\bar{y}) > l(y)$.

(iii) If $\bar{y}-y \in \text{int } \mathscr{D}(\bar{y})$ for any $y \in A$ then $l \in (\mathscr{D}(A))^* \setminus \{0_{Y^*}\}$ and thus $l \in (\mathscr{D}(y))^* \setminus \{0_{Y^*}\}$ implies $l(\bar{y}) > l(y)$ using again [22, Lemma 3.21].

Because of $(\mathscr{D}(A))^* \subset (\mathscr{D}(\bar{y}))^*$ and $(\mathscr{D}(A))^{\#} \subset (\mathscr{D}(\bar{y}))^{\#}$ for any $\bar{y} \in A$ it suffices in (a) to consider functionals l in $(\mathscr{D}(A))^*$ and in $(\mathscr{D}(A))^{\#}$, respectively. A necessary condition for the quasi interior of a convex cone to be nonempty is the pointedness of the cone [22, Lemma 1.27]. This shows the limitation of the above results if the variable ordering structure varies too much, i.e., if $\mathscr{D}(A)$ is no longer a pointed cone. Then the quasi-interior $(\mathscr{D}(A))^{\#}$ is empty and the above characterizations can no longer be applied.

Under the additional assumption that A is a convex set also necessary conditions for weakly optimal elements and hence also for optimal elements w.r.t. a variable ordering can be formulated with the help of linear functionals.

Theorem 4.2. *Let A be convex and let int $\mathscr{D}(y) \neq \emptyset$ for all $y \in A$.*

(a) For any weakly minimal element $\bar{y} \in A$ of A w.r.t. the ordering map $\mathscr{D}$ there exists some $l \in (\mathscr{D}(\bar{y}))^* \setminus \{0_{Y^*}\}$ with

$$l(\bar{y}) \leq l(y) \text{ for all } y \in A.$$

(b) Set

$$\hat{D} := \bigcap_{y \in A} \mathscr{D}(y)$$

and let int $\hat{D}$ be nonempty. For any weakly nondominated element $\bar{y} \in A$ of A w.r.t. the ordering map $\mathscr{D}$ there exists some $l \in \hat{D}^* \setminus \{0_{Y^*}\}$ with

$$l(\bar{y}) \leq l(y) \text{ for all } y \in A.$$

Proof. (a) Since $\bar{y}$ is a weakly minimal element of A the intersection of the sets $\{\bar{y}\} - \text{int } \mathscr{D}(\bar{y})$ and A is empty. Applying a separation theorem there exists a continuous linear functional $l \in Y^* \setminus \{0_{Y^*}\}$ and a real number α with

$$l(\bar{y}-d) \leq \alpha \leq l(y) \text{ for all } d \in \mathscr{D}(\bar{y}) \text{ and for all } y \in A.$$

As $\mathscr{D}(\bar{y})$ is a cone we conclude $l(d) \geq 0$ for all $d \in \mathscr{D}(\bar{y})$ and thus $l \in (\mathscr{D}(\bar{y}))^* \setminus \{0_{Y^*}\}$, and due to $0_Y \in D(\bar{y})$ we obtain $l(\bar{y}) \leq l(y)$ for all $y \in A$.

(b) Since $\bar{y} \in A$ is a weakly nondominated element of A w.r.t. the ordering map $\mathscr{D}$ it holds $\bar{y} \notin \{y\} + \text{int } \mathscr{D}(y)$ for all $y \in A$ and thus $\bar{y} \notin \{y\} + \text{int } \hat{D}$ for all $y \in A$. Then $(\{\bar{y}\} - \text{int } \hat{D} \cap A = \emptyset$ and again with a separation theorem this results in $l(\bar{y}) \leq l(y)$ for all $y \in A$ for some $l \in \hat{D}^* \setminus \{0_{Y^*}\}$. □

The necessary condition for weakly nondominated elements w.r.t. the ordering map $\mathscr{D}$ is very weak if the cones $\mathscr{D}(y)$ for $y \in A$ vary too much, because then the cone $\hat{D}$ is very small (or even empty) and the dual cone is very large.

Example 4.8. Let $Y \in \mathbb{R}^2$ and let $\mathscr{D}$ and A be defined as in Example 4.6. The unique nondominated element w.r.t. $\mathscr{D}$ is $(2,1)$ and all the elements of the set

$$\{(2,t) \in \mathbb{R}^2 \mid t \in [1,3]\} \cup \{(t,1) \in \mathbb{R}^2 \mid t \in [1,3]\}$$

are weakly nondominated w.r.t. $\mathscr{D}$. Further, $\mathscr{D}(A) = \{(z_1,z_2) \in \mathbb{R}^2 \mid z_2 \geq 0\}$ and thus $(\mathscr{D}(A))^* = \{(z_1,z_2) \in \mathbb{R}^2 \mid z_1 = 0,\ z_2 \geq 0\}$, i.e. $(\mathscr{D}(A))^\# = \emptyset$. Let $l \in (\mathscr{D}(A))^* \setminus \{0_{Y^*}\}$ be arbitrarily chosen, i.e. $l_1 = 0$, $l_2 > 0$, and consider the scalar-valued optimization problem $\min_{y \in A} l^\top y$. Then all elements of the set $\{(t,1) \in \mathbb{R}^2 \mid t \in [1,3]\}$ are minimal solutions and hence are weakly nondominated elements of A w.r.t. $\mathscr{D}$ according to Theorem 4.1(b)(iii). All the other weakly nondominated elements w.r.t. $\mathscr{D}$ cannot be found by the sufficient condition. Because of $\operatorname{int} \hat{D} = \emptyset$, the necessary condition of Theorem 4.2(b) cannot be applied.

Based on the above scalarization results existence results for optimal elements can easily be derived.

Theorem 4.3. *Let the set A be compact and let $(\mathscr{D}(A))^\#$ be nonempty. Then there exists a minimal element and a nondominated element of the set A w.r.t. the ordering map $\mathscr{D}$.*

Proof. Let $l \in (\mathscr{D}(A))^\#$ be arbitrarily chosen. According to the Weierstraß Theorem there exists a minimal solution of the optimization problem $\min_{y \in A} l(y)$. According to Theorem 4.1 this minimal solution is then a minimal element of the set A w.r.t. $\mathscr{D}$ because of $l \in (\mathscr{D}(\bar{y}))^\#$ and also a nondominated element of A w.r.t. $\mathscr{D}$ because of $l \in (\mathscr{D}(A))^\#$.

In case the variable ordering structure is given by a cone-valued map with images Bishop–Phelps cones, i.e. $\mathscr{D}(y) = C(\ell(y))$ for all $y \in Y$, then according to Proposition 4.1(e) $\ell(y) \in (C(\ell(y)))^\# = (\mathscr{D}(y))^\#$ for all $y \in Y$. Thus, in Theorem 4.1(a) we can choose $l := \ell(\bar{y})$.

4.4.2 Nonlinear Scalarizations

As linear scalarizations are in many cases, as for instance if the set A is non-convex, not an adequate tool for characterizing optimal elements, nonlinear scalarizations are important. In the following we discuss parameter dependent nonlinear scalarization functionals based on which a characterization of optimal elements w.r.t. a variable ordering structure is possible. Especially for the case of an ordering map $\mathscr{D}$ with images Bishop–Phelps cones strong results can be achieved which are discussed in Sect. 4.4.2.3.

4.4.2.1 Hiriart-Urruty Scalarization

In this subsection we assume additionally that $(Y, \|\cdot\|)$ is a normed space. Based on the distance function $d_S\colon Y \to \mathbb{R} \cup \{+\infty\}$ for some set $S \subset Y$,

$$d_S(y) := \inf\{\|y - s\| \mid s \in S\} \text{ for all } y \in Y,$$

Hiriart-Urruty has defined in [20] the function $\Delta_S\colon Y \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\Delta_S(y) := d_S(y) - d_{Y\setminus S}(y) = \begin{cases} d_S(y) & \text{for } y \in Y \setminus S, \\ -d_{Y\setminus S}(y) & \text{for } y \in S \end{cases} \tag{4.15}$$

with $\Delta_\emptyset(y) = +\infty$ for all $y \in Y$. This function has several useful properties [20, 29, 39]:

Proposition 4.2. *Let S be a nonempty subset of Y with $S \neq Y$.*

(a) $\Delta_S(y) \in \mathbb{R}$ *for all* $y \in Y$ *and* Δ_S *is Lipschitz continuous with constant* 1.
(b) If S *is convex, then* Δ_S *is convex, and if* Δ_S *is convex, then* clS *is convex.*
(c) $\Delta_{Y\setminus S} = -\Delta_S$.
(d) $\Delta_S(y) < 0$ *if and only if* $y \in intS$; $\Delta_S(y) = 0$ *if and only if* $y \in \partial S$; $\Delta_S(y) > 0$ *if and only if* $y \in int(Y \setminus S) = Y \setminus cl(S)$;
(e) If S *is a cone, then* Δ_S *is positive homogeneous.*
(f) If S *is convex with nonempty interior, then* $\Delta_S(y) = \sup_{y^* \in B^*} \inf_{s \in S} y^*(y - s)$ *with* $B^* = \{y^* \in Y^* \mid \|y^*\|_* = 1\}$.

Note that if S is closed, then $\Delta_S(y) > 0$ if and only if $y \notin S$. For scalarization results for efficient elements in a partially ordered space the set S is replaced in (4.15) by $-K$ with K the closed pointed convex cone defining the partial ordering in the space Y. We illustrate the function defined in (4.15) in $Y = \mathbb{R}^m$ with $K = \mathbb{R}^m_+$ [39]:

Example 4.9. Let $Y = \mathbb{R}^m$ and $K = \mathbb{R}^m_+$, i.e. $S = -\mathbb{R}^m_+$.

(a) Assuming the Euclidean norm we obtain $d_{-K}(y) = \|y^+\|_2$ with y^+ defined by $y_i^+ := \max\{0, y_i\}$ for all $i = 1, \ldots, m$, and

$$d_{Y\setminus -K}(y) = \begin{cases} 0 & \text{if } y_i > 0 \quad \text{for some } i \in \{1, \ldots, m\}, \\ -\max_{i\in\{1,\ldots,m\}} y_i & \text{if } y_i \le 0 \text{ for all } i = 1 \ldots, m. \end{cases}$$

Thus

$$\Delta_{-K}(y) = \begin{cases} \|y^+\|_2 & \text{if } y \notin -K, \\ \max_{i\in\{1,\ldots,m\}} y_i & \text{if } y \in -K. \end{cases}$$

(b) Assuming the Maximum norm $\|y\|_\infty := \max_{i\in\{1,\ldots,m\}} |y_i|$ for all $y \in Y$, we obtain

$$\Delta_{-K}(y) = \max_{i\in\{1,\ldots,m\}} y_i.$$

Efficient elements in a partially ordered space Y, ordered by K as given above, can be characterized as follows [19, 39]: an element $\bar{y} \in A$ is an efficient element if and only if $\bar{y}$ is a unique minimal solution of

$$\min_{y \in A} \Delta_{-K}(y - \bar{y}),$$

i.e. if and only if

$$\Delta_{-K}(y - \bar{y}) > \Delta_{-K}(0_Y) = 0 \text{ for all } y \in A \setminus \{\bar{y}\}.$$

Also other types of optimal elements in a partially ordered space as weakly efficient elements can be characterized. By a slight modification we can also characterize minimal elements of some set A w.r.t. a variable ordering map $\mathscr{D}$, as an element $\bar{y}$ is a minimal element of A w.r.t. $\mathscr{D}$ if and only if it is an efficient element of A in the space Y partially ordered by $K := \mathscr{D}(\bar{y})$. Hence the function $\zeta_{\bar{y}} \colon Y \to \mathbb{R} \cup \{\pm\infty\}$,

$$\zeta_{\bar{y}}(y) := \Delta_{-\mathscr{D}(\bar{y})}(y - \bar{y})$$

for some element $\bar{y} \in A$ can be used to characterize minimal elements w.r.t. the ordering map $\mathscr{D}$. Note that according to Proposition 4.2 the function $\zeta_{\bar{y}}$ is convex as $\mathscr{D}(\bar{y})$ is convex.

By allowing the set S in (4.15) to vary dependently on the actual element y, also a characterization of nondominated elements w.r.t. the ordering map $\mathscr{D}$ is possible. Thus we consider additionally the function $\zeta^v_{\bar{y}} \colon Y \to \mathbb{R} \cup \{\pm\infty\}$,

$$\zeta^v_{\bar{y}}(y) := \Delta_{-\mathscr{D}(y)}(y - \bar{y}) = d_{-\mathscr{D}(y)}(y - \bar{y}) - d_{Y \setminus (-\mathscr{D}(y))}(y - \bar{y}) \quad \text{for all } y \in Y$$

for some given element $\bar{y} \in Y$. However, the function $\zeta^v_{\bar{y}}$ is in general nonconvex despite the cones $\mathscr{D}(y)$ are assumed to be convex for all $y \in Y$. The characterization results are summed up in the next theorem:

Theorem 4.4. *Let $\mathscr{D}(y)$ be a closed pointed convex cone for all $y \in Y$.*

(a) (i) $\bar{y} \in A$ is a minimal element of A w.r.t. the ordering map $\mathscr{D}$ if and only if

$$\zeta_{\bar{y}}(y) > \zeta_{\bar{y}}(\bar{y}) = 0 \textit{ for all } y \in A \setminus \{\bar{y}\}.$$

(ii) Let int $\mathscr{D}(y) \neq \emptyset$ for all $y \in A$. $\bar{y} \in A$ is a weakly minimal element of A w.r.t. the ordering map $\mathscr{D}$ if and only if

$$\zeta_{\bar{y}}(y) \geq \zeta_{\bar{y}}(\bar{y}) = 0 \textit{ for all } y \in A.$$

(b) (i) $\bar{y} \in A$ is a nondominated element of A w.r.t. the ordering map $\mathscr{D}$ if and only if

$$\zeta^v_{\bar{y}}(y) > \zeta^v_{\bar{y}}(\bar{y}) = 0 \textit{ for all } y \in A \setminus \{\bar{y}\}.$$

(ii) Let int $\mathscr{D}(y) \neq \emptyset$ *for all* $y \in A$. $\bar{y} \in A$ *is a weakly nondominated element of* A *w.r.t. the ordering map* $\mathscr{D}$ *if and only if*

$$\zeta^{v}_{\bar{y}}(y) \geq \zeta^{v}_{\bar{y}}(\bar{y}) = 0 \text{ for all } y \in A.$$

Proof. (a) (i) As $\mathscr{D}(\bar{y})$ is a pointed cone, $0_Y \in -\partial\mathscr{D}(\bar{y})$ and by Proposition 4.2 $\zeta_{\bar{y}}(\bar{y}) = \Delta_{-\mathscr{D}(\bar{y})}(0_Y) = 0$. $\bar{y}$ is a minimal element of A w.r.t. $\mathscr{D}$ if and only if $y - \bar{y} \notin -\mathscr{D}(\bar{y})$ for all $y \in A \setminus \{\bar{y}\}$ and by Proposition 4.2 this is equivalent to $\zeta_{\bar{y}}(y) = \Delta_{-\mathscr{D}(\bar{y})}(y - \bar{y}) > 0$ for all $y \in A \setminus \{\bar{y}\}$.

(ii) The proof is analogous to (i) and hence is omitted.

(b) (i) As $\mathscr{D}(\bar{y})$ is a pointed cone $\zeta^{v}_{\bar{y}}(\bar{y}) = \Delta_{-\mathscr{D}(\bar{y})}(0_Y) = 0$. $\bar{y}$ is a nondominated element of A w.r.t. $\mathscr{D}$ if and only if $y - \bar{y} \notin -\mathscr{D}(y)$ for all $y \in A \setminus \{\bar{y}\}$ and by Proposition 4.2 this is equivalent to $\zeta^{v}_{\bar{y}}(y) = \Delta_{-\mathscr{D}(y)}(y - \bar{y}) > 0$ for all $y \in A \setminus \{\bar{y}\}$.

(ii) The proof is analogous to (i) and hence is omitted. □

4.4.2.2 Pascoletti–Serafini Scalarization

Allowing two parameters $a \in Y$ and $r \in Y$ the following nonlinear scalarization functional $\psi_{a,r,K} \colon Y \to \mathbb{R} \cup \{\pm\infty\}$ is well examined in vector optimization in partially ordered spaces ordered by some nonempty convex cone $K \subset Y$ [12, 13, 17, 31]:

$$\psi_{a,r,K}(y) := \inf\{t \in \mathbb{R} \mid a + t\,r - y \in K\} \quad \text{for all} \quad y \in Y.$$

Setting $K := \mathscr{D}(\bar{y})$, $r := r(\bar{y}) \in \mathscr{D}(\bar{y}) \setminus \{0_Y\}$, and $a := 0_Y$ or $a := \bar{y}$ for some given $\bar{y} \in Y$, this functional is studied in the context of a variable ordering structure and the binary relation defined in (4.4) which corresponds to the notion of minimal elements w.r.t. a variable ordering map [9, 10, 36]. By allowing also the cone K to vary dependently on y, i.e. to consider $\mathscr{K}(y)$ with $\mathscr{K} \colon Y \to 2^Y$ a set-valued map with $\mathscr{K}(y)$ a nonempty convex cone for all $y \in Y$ we get the following scalarization function $\chi_{a,r,\mathscr{K}} \colon Y \to \mathbb{R} \cup \{\pm\infty\}$,

$$\chi_{a,r,\mathscr{K}}(y) := \inf\{t \in \mathbb{R} \mid a + t\,r - y \in \mathscr{K}(y)\} \quad \text{for all} \quad y \in Y.$$

Setting $\mathscr{K} := \mathscr{D}$, $r \in (\cap_{y \in A}\mathscr{D}(y)) \setminus \{0_Y\}$ and $a := \bar{y}$ also (weakly) nondominated elements can be characterized. We sum up necessary and sufficient conditions for minimal and for nondominated elements and sufficient conditions for weakly minimal and nondominated elements of some set A w.r.t. the ordering map $\mathscr{D}$. In the following we assume that the infima in the definitions of the above scalarizations are attained.

Theorem 4.5. *(a) (i) If* $\bar{y}$ *is a minimal element of* A *w.r.t. the ordering map* $\mathscr{D}$*, then for any pointed convex cone* $K \subset \mathscr{D}(\bar{y})$ *and* $r \in K \setminus \{0_Y\}$

$$\psi_{\bar{y},r,K}(y) > \psi_{\bar{y},r,K}(\bar{y}) = 0 \ \text{ for all } \ y \in A \setminus \{\bar{y}\}.$$

(ii) *Let int* $\mathscr{D}(y) \neq \emptyset$ *for all* $y \in A$. *If for any* $a \in Y$, $r \in Y$ *and any convex cone* $K \supset \mathscr{D}(\bar{y})$ *with int* $K \neq \emptyset$

$$\psi_{a,r,K}(y) \geq \psi_{a,r,K}(\bar{y}) \text{ for all } y \in A,$$

then $\bar{y}$ *is a weakly minimal element of* A *w.r.t. the ordering map* $\mathscr{D}$.

(iii) *If for any* $a \in Y$, $r \in Y$ *and any convex cone* $K \supset \mathscr{D}(\bar{y})$

$$\psi_{a,r,K}(y) > \psi_{a,r,K}(\bar{y}) \text{ for all } y \in A \setminus \{\bar{y}\},$$

then $\bar{y}$ *is a minimal element of* A *w.r.t. the ordering map* $\mathscr{D}$.

(b) *(i)* *If* $\bar{y}$ *is a nondominated element of* A *w.r.t. the ordering map* $\mathscr{D}$, *then for any* $r \in (\bigcap_{y \in A} \mathscr{D}(y)) \setminus \{0_Y\}$

$$\chi_{\bar{y},r,\mathscr{D}}(y) > \chi_{\bar{y},r,\mathscr{D}}(\bar{y}) = 0 \text{ for all } y \in A \setminus \{\bar{y}\}.$$

(ii) *Let int* $\mathscr{D}(y) \neq \emptyset$ *for all* $y \in A$. *If for any* $r \in (\bigcap_{y \in A} \mathscr{D}(y)) \setminus \{0_Y\}$

$$\chi_{\bar{y},r,\mathscr{D}}(y) \geq \chi_{\bar{y},r,\mathscr{D}}(\bar{y}) = 0 \text{ for all } y \in A,$$

then $\bar{y}$ *is a weakly nondominated element of* A *w.r.t. the ordering map* $\mathscr{D}$.

(iii) *If for any* $r \in (\bigcap_{y \in A} \mathscr{D}(y)) \setminus \{0_Y\}$

$$\chi_{\bar{y},r,\mathscr{D}}(y) > \chi_{\bar{y},r,\mathscr{D}}(\bar{y}) = 0 \text{ for all } y \in A \setminus \{\bar{y}\},$$

then $\bar{y}$ *is a nondominated element of* A *w.r.t. the ordering map* $\mathscr{D}$.

Proof. (a) (i) $\psi_{\bar{y},r,K}(\bar{y}) = \inf\{t \in \mathbb{R} \mid t\,r \in K\}$ and because of $r \in K \setminus \{0_Y\}$ and as K is a pointed convex cone this implies $\psi_{\bar{y},r,K}(\bar{y}) = 0$. If $\bar{y}$ is not a unique minimal solution of $\min_{y \in A} \psi_{\bar{y},r,K}(y)$, then there exists some $t \in \mathbb{R}$, $t \leq 0$, and some $y \in A \setminus \{\bar{y}\}$ such that $\bar{y} + t\,r - y \in K$. Because of $t\,r \in -K$ and as K is a convex cone, this implies $y \in \{\bar{y}\} - K \setminus \{0_Y\} \subset \{\bar{y}\} - \mathscr{D}(\bar{y}) \setminus \{0_Y\}$ in contradiction to $\bar{y}$ a minimal element of A w.r.t. $\mathscr{D}$.

(ii) For $\bar{t} := \psi_{a,r,K}(\bar{y})$ it holds $a + \bar{t}\,r - \bar{y} \in K$. If there is some $y \in A$ with $\bar{y} - y \in \text{int } \mathscr{D}(\bar{y})$, then

$$a + \bar{t}\,r - y \in K + \text{int } \mathscr{D}(\bar{y}) \subset \text{int} K.$$

Thus there exists some $\varepsilon > 0$ such that also $a + (\bar{t} - \varepsilon)\,r - y \in K$ in contradiction to $\bar{t} = \psi_{a,r,K}(\bar{y}) > \bar{t} - \varepsilon$.

(iii) The proof is similar to (b) and hence is omitted.

(b) (i) As $\mathscr{D}(\bar{y})$ is a pointed convex cone and because of $r \in \mathscr{D}(\bar{y}) \setminus \{0_Y\}$ it holds $\chi_{\bar{y},r,\mathscr{D}}(\bar{y}) = 0$. If $\bar{y}$ is not a unique minimal solution then there exists some $t \in \mathbb{R}$, $t \leq 0$, and some $y \in A \setminus \{\bar{y}\}$ such that $\bar{y} + t\,r - y \in \mathscr{D}(y)$. Because of

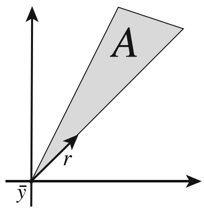


Fig. 4.5 Illustration of the Pascoletti–Serafini scalarization of Example 4.10

$t\,r \in -\mathscr{D}(y)$ and as $\mathscr{D}(y)$ is a convex cone, this implies $\bar{y} \in \{y\} + \mathscr{D}(y) \setminus \{0_Y\}$ in contradiction to $\bar{y}$ a nondominated element of A w.r.t. $\mathscr{D}$.

(ii) If there is some $y \in A$ with $\bar{y} \in \{y\} + \operatorname{int} \mathscr{D}(y)$ then this implies for any $r \in Y$ that there exists some $t < 0$ such that $(\bar{y} - y) + t\,r \in \mathscr{D}(y)$, i.e. $\bar{y} + t\,r - y \in \mathscr{D}(y)$ and hence $\chi_{\bar{y},r,\mathscr{D}}(y) < 0$ which is a contradiction.

(iii) If there is some $y \in A \setminus \{\bar{y}\}$ with $\bar{y} \in \{y\} + \mathscr{D}(y)$ then this implies $\bar{y} + 0 \cdot r - y \in \mathscr{D}(y)$, i.e. $\chi_{\bar{y},r,\mathscr{D}}(y) \le 0$ which is a contradiction. □

Example 4.10. Let $Y = \mathbb{R}^2$ and $\mathscr{D}$ and A be given as in Example 4.2. We use the sufficient criteria given in Theorem 4.5(b)(iii) to show that $\bar{y} = (0,0)$ is a nondominated element of A w.r.t. $\mathscr{D}$. We choose $r = (1,1)$, compare Fig. 4.5.Of course $\chi_{\bar{y},r,\mathscr{D}}(\bar{y}) = 0$. For $y = (y_1, y_2) \in A \setminus \{\bar{y}\}$ we have $\mathscr{D}(y) \subset \mathbb{R}^2_+$ and thus because of $y_2 \ge y_1$ for all $y \in A$

$$\begin{aligned}\chi_{\bar{y},r,\mathscr{D}}(y) &\ge \inf\{t \in \mathbb{R} \mid (t,t) \in (y_1,y_2) + \mathbb{R}^2_+\}\\ &= \inf\{t \in \mathbb{R} \mid t \ge y_2\} = y_2.\end{aligned}$$

On the other hand, for $t = y_2$ we obtain because of $(y_2 - y_1, 0) \in \mathscr{D}(y)$ for all $y \in A$ also

$$\bar{y} + t\,r - y = y_2 \cdot (1,1) - (y_1,y_2) = (y_2 - y_1, 0) \in \mathscr{D}(y),$$

i.e. $\chi_{\bar{y},r,\mathscr{D}}(y) = y_2$ for all $y \in A$, and because of $y_2 > 0$ for all $y \in A \setminus \{\bar{y}\}$ we get $\chi_{\bar{y},r,\mathscr{D}}(y) = y_2 > \chi_{\bar{y},r,\mathscr{D}}(\bar{y}) = 0$ for all $y \in A \setminus \{\bar{y}\}$.

Summarizing the previous results we get for minimal and for nondominated elements of some set A w.r.t. $\mathscr{D}$ the following complete characterization:

Corollary 4.1. *(a) $\bar{y}$ is a minimal element of A w.r.t. the ordering map $\mathscr{D}$ if and only if for any $r \in \mathscr{D}(\bar{y}) \setminus \{0_Y\}$*

$$\psi_{\bar{y},r,\mathscr{D}(\bar{y})}(y) > \psi_{\bar{y},r,\mathscr{D}(\bar{y})}(\bar{y}) = 0 \;\text{ for all }\; y \in A \setminus \{\bar{y}\}.$$

(b) $\bar{y}$ is a nondominated element of A w.r.t. the ordering map $\mathscr{D}$ if and only if for any $r \in (\bigcap_{y \in A} \mathscr{D}(y)) \setminus \{0_Y\}$

$$\chi_{\bar{y},r,\mathscr{D}}(y) > \chi_{\bar{y},r,\mathscr{D}}(\bar{y}) = 0 \;\text{ for all }\; y \in A \setminus \{\bar{y}\}.$$

4.4.2.3 Scalarization for Ordering Maps with Images Bishop–Phelps Cones

If the images of the cone-valued map $\mathscr{D}$ defining the variable ordering structure are Bishop–Phelps cones, then this special structure can be used to define nonlinear scalarization functionals which allow a complete characterization of minimal and of nondominated elements w.r.t. $\mathscr{D}$ without imposing additional assumptions. A main advantage is that we can give conditions ensuring that the scalarization is convex which allows the formulation of sufficient optimality conditions of Fermat and Lagrange type [15].

In this subsection we assume $(Y, \|\cdot\|)$ to be a real normed space and the cone-valued map $\mathscr{D}\colon Y \to 2^Y$ given by

$$\mathscr{D}(y) := C(\ell(y)) := \{u \in Y \mid \|u\|_y \leq \ell(y)(u)\} \text{ for all } y \in Y$$

with $\ell : Y \to Y^*$ a given map and with $\|\cdot\|_y$ a norm in Y for each $y \in Y$ equivalent to $\|\cdot\|$ which depends on y, compare Sect. 4.2.4. Let A be a nonempty subset of Y. For some given $\bar{y} \in Y$ we define the following scalarization functionals $\overline{\gamma}_{\bar{y}}$, $\overline{\xi}_{\bar{y}}$, $\gamma_{\bar{y}}$, $\xi_{\bar{y}}$: $Y \to \mathbb{R}$ by

$$\begin{aligned} \overline{\gamma}_{\bar{y}}(y) &:= \ell(\bar{y})(y-\bar{y}) - \|y-\bar{y}\|_{\bar{y}} \quad \text{for all } y \in Y, \\ \overline{\xi}_{\bar{y}}(y) &:= \ell(\bar{y})(y-\bar{y}) + \|y-\bar{y}\|_{\bar{y}} \quad \text{for all } y \in Y, \\ \gamma_{\bar{y}}(y) &:= \ell(y)(y-\bar{y}) - \|y-\bar{y}\|_{y} \quad \text{for all } y \in Y, \\ \xi_{\bar{y}}(y) &:= \ell(y)(y-\bar{y}) + \|y-\bar{y}\|_{y} \quad \text{for all } y \in Y. \end{aligned} \tag{4.16}$$

These four functionals allow a complete characterization of minimal and of nondominated elements w.r.t. a variable ordering structure [15]:

Theorem 4.6. *Let $\bar{y} \in A$.*

(a) *(i) $\bar{y}$ is a strongly minimal element of A w.r.t. the ordering map $\mathscr{D}$ if and only if*

$$\overline{\gamma}_{\bar{y}}(y) \geq \overline{\gamma}_{\bar{y}}(\bar{y}) = 0 \text{ for all } y \in A.$$

(ii) $\bar{y}$ is a minimal element of A w.r.t. the ordering map $\mathscr{D}$ if and only if

$$\overline{\xi}_{\bar{y}}(y) > \overline{\xi}_{\bar{y}}(\bar{y}) = 0 \text{ for all } y \in A \setminus \{\bar{y}\}.$$

(iii) Let $\|\ell(y)\|_* > 1$ (and hence, int $\mathscr{D}(y) \neq \emptyset$) for all $y \in A$. $\bar{y}$ is a weakly minimal element of A w.r.t. the ordering map $\mathscr{D}$ if and only if

$$\overline{\xi}_{\bar{y}}(y) \geq \overline{\xi}_{\bar{y}}(\bar{y}) = 0 \text{ for all } y \in A.$$

(b) *(i) $\bar{y}$ is a strongly nondominated element of A w.r.t. the ordering map $\mathscr{D}$ if and only if*

$$\gamma_{\bar{y}}(y) \geq \gamma_{\bar{y}}(\bar{y}) = 0 \text{ for all } y \in A.$$

(ii) $\bar{y}$ *is a nondominated element of A w.r.t. the ordering map* $\mathscr{D}$ *if and only if*

$$\xi_{\bar{y}}(y) > \xi_{\bar{y}}(\bar{y}) = 0 \ \text{for all } y \in A \setminus \{\bar{y}\}.$$

(iii) Let $\|\ell(y)\|_{y,*} > 1$ *(and hence, int* $\mathscr{D}(y) \neq \emptyset$*) for all* $y \in A$. $\bar{y}$ *is a weakly nondominated element of A w.r.t. the ordering map* $\mathscr{D}$ *if and only if*

$$\xi_{\bar{y}}(y) \geq \xi_{\bar{y}}(\bar{y}) = 0 \ \text{for all } y \in A.$$

Here, $\|\cdot\|_{y,*}$ *denotes the dual norm of* $\|\cdot\|_y$.

Proof. (a) The proof is similar to (b) and hence is omitted.

(b) (i) $\bar{y}$ is strongly nondominated w.r.t. the ordering map $\mathscr{D}$ if and only if $y - \bar{y} \in \mathscr{D}(y) = \{z \in Y \mid \|z\|_y \leq \ell(y)(z)\}$ for all $y \in A$ being equivalent to $\ell(y)(y-\bar{y}) - \|y-\bar{y}\|_y \geq 0$ and thus to $\gamma_{\bar{y}}(y) \geq \gamma_{\bar{y}}(\bar{y}) = 0$ for all $y \in A$.

(ii) $\bar{y}$ is a nondominated element w.r.t. the ordering map $\mathscr{D}$ if and only if $\bar{y} - y \notin \mathscr{D}(y)$ i.e. $\ell(y)(\bar{y}-y) - \|\bar{y}-y\|_y < 0$ for all $y \in A \setminus \{\bar{y}\}$ and hence if and only if $\xi_{\bar{y}}(y) = \ell(y)(y-\bar{y}) + \|y-\bar{y}\|_y > 0 = \xi_{\bar{y}}(\bar{y})$ for all $y \in A \setminus \{\bar{y}\}$.

(iii) Using Proposition 4.1(d), the proof is similar to (ii) and hence is omitted. □

Recently, the functional $\xi_{\bar{y}}$ has been used in [24, Theorem 5.8] to characterize an element $\bar{y}$ which is a properly efficient element of A in the senses of Henig or Benson in a partially ordered space. Note that there $\ell(y) := \phi \in Y^*$ for all $y \in Y$.

Example 4.11. Let Y be the Euclidean space $\mathbb{R}^2$ and the cone-valued map $\mathscr{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be defined by $\mathscr{D}(y) := C(\ell(y))$ with $\ell: \mathbb{R}^2 \to \mathbb{R}^2$ as in (4.9) and with $\|.\|_y := \|.\|_2$ for all $y \in \mathbb{R}^2$ and let

$$A := \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0,\ y_2 \geq 0,\ y_2 \geq \pi - y_1\}.$$

By Theorem 4.6(b), $\bar{y} = (0, \pi)$ is a nondominated element of A w.r.t. the ordering map $\mathscr{D}$ because it obviously holds

$$\xi_{\bar{y}}(y) = \frac{3+\sin y_1}{2}(y_1 - 0) + \frac{3+\cos y_2}{2}(y_2 - \pi) + \|(y_1,y_2)^\top - (0,\pi)^\top\|_2 > 0 = \xi_{\bar{y}}(\bar{y})$$

for all $(y_1, y_2) \in A \setminus \{(0,\pi)\}$ with $y_2 \geq \pi$. For those $y \in A \setminus \{(0,\pi)\}$ with $y_2 < \pi$ we have $0 > y_2 - \pi \geq -y_1$ and we obtain

$$\begin{aligned} \xi_{\bar{y}}(y) &\geq \frac{3+\sin y_1}{2} y_1 + \frac{3+\cos y_2}{2}(-y_1) + \sqrt{y_1^2 + (y_2-\pi)^2} \\ &> \left(\frac{\sin y_1 - \cos y_2}{2}\right) y_1 + y_1 \geq 0. \end{aligned}$$

However, it is not a strongly nondominated element of A w.r.t. $\mathscr{D}$ because

$$\gamma_{\bar{y}}(3\pi/2,0) = (1,2)(3\pi/2,-\pi)^\top - \|(3\pi/2,-\pi)^\top\|_2 < 0 = \gamma_{\bar{y}}(\bar{y}).$$

$\bar{y} = (0,\pi)$ is also a minimal element but not a strongly minimal element of A w.r.t. the ordering map $\mathscr{D}$ because it holds

$$\overline{\xi}_{\bar{y}}(y) = (3/2,1)(y_1,y_2-\pi)^\top + \|(y_1,y_2)^\top - (0,\pi)^\top\|_2 > 0 = \overline{\xi}_{\bar{y}}(\bar{y})$$

for all $y \in A \setminus \{(0,\pi)\}$, and

$$\overline{\gamma}_{\bar{y}}(3\pi/2,0) = (3/2,1)(3\pi/2,-\pi)^\top - \|(3\pi/2,-\pi)^\top\|_2 < 0 = \overline{\gamma}_{\bar{y}}(\bar{y}).$$

It is especially difficult to find scalarization functionals characterizing nondominated elements which are –at least under strong assumptions– convex. Under appropriate assumptions, the functional $\xi_{\bar{y}}$ is convex and sufficient optimality conditions for vector optimization problems as Fermat rule and Lagrange multiplier rule can be formulated even for nondominated elements [15]. Hereby, ℓ is called *monotone* if $(\ell(y_1)-\ell(y_2))(y_1-y_2) \geq 0$ for all $y_1,y_2 \in Y$. For instance for $Y = \mathbb{R}^m$ and $P \in \mathbb{R}^{m\times m}$ a positive semidefinite matrix, the map $\ell(y) := Py$ for all $y \in Y$ is linear and monotone.

Lemma 4.6. *Suppose that ℓ is linear and monotone. Then the functional $\xi_{\bar{y}}$ is convex.*

Proof. Let $y_1,y_2 \in Y$ and $\lambda \in [0,1]$, then

$$\begin{aligned}
&\ell(\lambda y_1 + (1-\lambda)y_2)(\lambda y_1 + (1-\lambda)y_2)\\
&= \lambda\ell(y_1)(\lambda y_1 + (1-\lambda)y_2) + (1-\lambda)\ell(y_2)(\lambda y_1 + (1-\lambda)y_2)\\
&= \lambda\ell(y_1)(y_1) + \lambda(1-\lambda)\ell(y_1)(y_2-y_1) + (1-\lambda)\ell(y_2)(y_2)\\
&\quad +\lambda(1-\lambda)\ell(y_2)(y_1-y_2)\\
&= \lambda\ell(y_1)(y_1) + (1-\lambda)\ell(y_2)(y_2) - \lambda(1-\lambda)(\ell(y_1)-\ell(y_2))(y_1-y_2)\\
&\leq \lambda\ell(y_1)(y_1) + (1-\lambda)\ell(y_2)(y_2).
\end{aligned}$$

As also the norm is convex and ℓ is a linear map, we conclude the convexity of the function $\xi_{\bar{y}}$. □

For a detailed examination of necessary and sufficient optimality conditions for nondominated elements of unconstrained and constrained vector-optimization problems with set-valued or single-valued objective maps we refer to [15]. The convexity of the map $\xi_{\bar{y}}$ plays also an important role in the context of duality, compare Sect. 4.5.

4.5 Duality

In vector optimization one can, under appropriate assumptions, associate a maximization problem to a minimization problem. In vector optimization with a variable ordering structure we get by that a relation between the two optimality concepts, i.e. between nondominated elements w.r.t. $\mathscr{D}$ of one set and the maximal elements w.r.t. $\mathscr{D}$, i.e. the minimal elements w.r.t. $-\mathscr{D}$, of another set. As before, let Y be a real linear space, $\mathscr{D}\colon Y \to 2^Y$ a set-valued map with $\mathscr{D}(y)$ a pointed convex cone for all $y \in Y$ and A a nonempty subset of Y. Recall that we have defined the set M in (4.12) by:

$$M := \bigcup_{y\in A} \{y\} + \mathscr{D}(y).$$

We define to the *primal set* A a so called *dual set* Q by

$$Q := Y \setminus \widetilde{M} \quad \text{with} \quad \widetilde{M} := \bigcup_{y\in A} \{y\} + \mathscr{D}(y) \setminus \{0_Y\}. \tag{4.17}$$

Lemma 4.7. *Let Q be defined as in (4.17). If $\bar{y} \in A \cap Q$, then $\bar{y}$ is a nondominated element of A w.r.t. the ordering map $\mathscr{D}$ and $\bar{y}$ is a maximal element of Q w.r.t. the ordering map $\mathscr{D}$.*

Proof. $\bar{y} \in Q$ implies $\bar{y} \notin \widetilde{M}$ and thus $\bar{y}$ is a nondominated element of A w.r.t. $\mathscr{D}$. Since $Q \cap \widetilde{M} = \emptyset$ and $\bar{y} \in A$ also

$$Q \cap (\{\bar{y}\} + (\mathscr{D}(\bar{y}) \setminus \{0_Y\})) = \emptyset$$

and thus $\bar{y} \in Q$ is a maximal element of the set Q w.r.t. the ordering map $\mathscr{D}$. □

This leads to the following duality theorem:

Theorem 4.7. *(a) If $\bar{y} \in A$ is a nondominated element of A w.r.t. the ordering map $\mathscr{D}$, then $\bar{y}$ is also a maximal element of Q w.r.t. the ordering map $\mathscr{D}$.*
(b) If $Y \setminus M$ is algebraically open then every maximal element of the set Q w.r.t. the ordering map $\mathscr{D}$ is also a nondominated element of the set A w.r.t. the ordering map $\mathscr{D}$.

Proof. (a) First assume $\bar{y} \notin Q$. Then $\bar{y} \in \widetilde{M}$ which is a contradiction to $\bar{y}$ a nondominated element of A w.r.t. $\mathscr{D}$. Thus $\bar{y} \in A \cap Q$ and the assertion follows with Lemma 4.7.
(b) Let $\bar{y}$ be an arbitrary maximal element of Q w.r.t. $\mathscr{D}$, i.e.

$$(\{\bar{y}\} + \mathscr{D}(\bar{y})) \cap Q = \{\bar{y}\}.$$

First assume $\bar{y} \notin M$. As $Y \setminus M$ is algebraically open, there exists for every $d \in \mathscr{D}(\bar{y}) \setminus \{0_Y\}$ a scalar $\lambda > 0$ with $\bar{y} + \lambda\, d \in Y \setminus M \subset Q$. As $\bar{y} + \lambda\, d \in \{\bar{y}\} + (\mathscr{D}(\bar{y}) \setminus \{0_Y\})$ this is a contradiction to the maximality of $\bar{y}$. Thus $\bar{y} \in M$.

Due to $\bar{y} \notin \widetilde{M}$ we get $\bar{y} \in A$. With Lemma 4.7 we conclude that $\bar{y}$ is a nondominated element of A w.r.t. $\mathcal{D}$. □

Example 4.12. (a) Let $Y = \mathbb{R}^2$, $A = \mathbb{R}^2_+$ and $\mathcal{D}\colon \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ be given by

$$\mathcal{D}(y) := \begin{cases} \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 \geq 0,\ z_1 + z_2 \geq 0\} & \text{if } y_2 \geq 0, \\ \mathbb{R}^2_+ & \text{else.} \end{cases}$$

Then $M = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \geq 0,\ y_1 + y_2 \geq 0\}$ and $\widetilde{M} = M \setminus \{(0,0)\}$. The set $Y \setminus M$ is algebraically open and $Q = (\mathbb{R}^2_+ \setminus M) \cup \{(0,0)\}$. Q has a single maximal element $(0,0)$ w.r.t. $\mathcal{D}$ which is also a nondominated element of A w.r.t. $\mathcal{D}$. Note, that for instance $(1,-1) \notin Q$.

(b) We consider the set $A = [1,3] \times [1,3]$ and the ordering map

$$\mathcal{D}(y) := \begin{cases} \mathbb{R}^2_+ & \text{for all } y \in \mathbb{R}^2 \text{ with } y_1 > 1, \\ \{(z_1, z_2) \in \mathbb{R}^2 \mid z_1 + z_2 \geq 0,\ z_1 - z_2 \geq 0\} & \text{else.} \end{cases}$$

The above sets are in this example

$$\widetilde{M} = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 > 1,\ y_2 \geq 2 - y_1\}$$

and $M = \widetilde{M} \cup \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = 1,\ y_2 \in [1,3]\}$. Thus

$$Q = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \leq 1 \ \vee\ (y_1 > 1 \ \wedge\ y_1 + y_2 < 2)\}.$$

The set of nondominated elements of A w.r.t. $\mathcal{D}$ is $\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = 1,\ y_2 \in [1,3]\}$ and the set of maximal elements of Q w.r.t. $\mathcal{D}$ is $\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 = 1,\ y_2 \geq 1\}$. Thus $Q \cap A$ equals the set of nondominated elements of A w.r.t. $\mathcal{D}$ which is a strict subset of the set of maximal elements of Q w.r.t. $\mathcal{D}$. Thus not all maximal elements of Q w.r.t. $\mathcal{D}$ refer to a nondominated element of A w.r.t. $\mathcal{D}$. The set $Y \setminus M$ is not algebraically open.

Additional duality results can also be gained by using characterizations gained by linear [14] or nonlinear scalarizations [15]: For the special case of a cone-valued map $\mathcal{D}$ with images BP cones the scalarization functional $\xi_{\bar{y}}$ allows the formulation of a dual optimization problem relating weakly nondominated and weakly maximal elements w.r.t. $\mathcal{D}$. For these results additional assumptions are necessary as the linearity and monotonicity of the map $\ell\colon Y \to Y^*$ defining the cone-valued map $\mathcal{D}$ by $\mathcal{D}(y) = C(\ell(y))$ for all $y \in Y$, which imply for instance the convexity of the map $\xi_{\bar{y}}$, compare Lemma 4.6 and [15, Proposition 3.17].

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Chapter 5
Strong KKT, Second Order Conditions and Non-solid Cones in Vector Optimization

Joydeep Dutta

5.1 Introduction

In this chapter we shall concentrate on studying the Karush–Kuhn–Tucker (KKT) type optimality conditions for both Pareto and weak Pareto minimum of a usual vector optimization problem, that is, a vector optimization problem with equality and inequality constraints. It is now a well known fact that the KKT conditions for scalar optimization problems play a major role in the analysis of algorithms. It is still not clear whether the KKT conditions for a vector optimization problem plays such a significant role. However that does not mean that there is really no use studying them. They do play a fundamental role in understanding the nature of the solutions of a vector optimization. Further they can always be used as an optimality certificate through which we can conclusively decide that a point is not a Pareto minimum or a weak Pareto minimum. Optimality conditions can also lead to the design of certain merit functions which can lead to robust error bounds for a convex vector optimization problems with strongly convex objective functions. Thus it is important for us to develop necessary and sufficient optimality conditions for a vector optimization problem. In our discussions in this chapter we shall be largely concentrated on the following vector optimization problem (VOP),

$$\min f(x), \quad \text{subject to} \quad x \in C,$$

where $f : X \to Y$ and $C \subseteq X$. Here X and Y are either finite dimensional spaces or infinite dimensional spaces. Further we shall consider two scenarios, namely where C is not explicitly defined through constraint functions and the case where it expressed explicitly through constraint functions. We will study the problem (VOP)

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in both the finite dimensional setting and infinite dimensional setting. However for simplicity of the exposition we will concentrate largely on the finite dimensional setting and we will study the problem (VOP) in the infinite dimensional setting in only one section but will focus on contemporary research and mainly address the question of how to develop optimality conditions in the case when the ordering cone has an empty interior. We shall provide below the list of sections that will appear in the sequel.

Section 5.2 : Tools from nonsmooth analysis
Section 5.3 : Geometric optimality conditions
Section 5.4 : Optimality conditions with explicit constraints
Section 5.5 : Second order optimality conditions
Section 5.6 : Excursions in infinite dimensions

The reader will notice that in Sect. 5.2 we describe the main tools and techniques of nonsmooth analysis that we have used in this chapter. These tools and their associated results will be given in a finite dimensional setting. The last section deals with optimality conditions in infinite dimensional spaces namely Banach spaces. The tools from nonsmooth analysis required in infinite dimensional spaces have been mentioned in that particular section.

Furthermore it is not possible to do justice to the vast topic of optimality conditions in vector optimization in just one chapter. Thus there would be many omissions and as result one might not get all the different view points in this area. Instead we try our best to collect some recent and interesting results in the area of optimality conditions for vector optimization.

It is however important on the reader's part to supplement his reading of this chapter with various other books and monographs on the subject. See for example the monographs by Jahn [26], Luc [29], Gopfert et al. [21], and the references therein.

5.2 Tools from Nonsmooth Analysis

In this section we shall try to briefly describe some important tools of nonsmooth analysis that we have used here. This will consist in the notions of various tangent and normal cones as well as the subdifferential for convex functions and the Clarke subdifferential for locally Lipschitz functions. The material in this section is largely based on the survey paper on nonsmooth optimization due to Dutta [9].

We begin by the definition of a proper convex function. This means we consider a convex function which is extended-valued i.e. $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty. +\infty\}$. A function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said to be proper if $f(x) > -\infty$ for all $x \in \mathbb{R}^n$ and $\text{dom } f = \{x : f(x) < \infty\}$ known as the domain of the function f is non-empty. The subdifferential of a proper convex function is defined as follows. Give a proper convex function $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $x \in \text{dom } f$ the subdifferential is a closed convex set $\partial f(x)$ given as

$$\partial f(x) = \{\xi \in \mathbb{R}^n : f(y) - f(x) \geq \langle \xi, y - x \rangle, \quad \forall y \in \mathbb{R}^n\}.$$

When $x \notin \text{dom } f$ we set $\partial f(x) = \emptyset$. If x is in the interior of dom f then $\partial f(x)$ is a non-empty convex and compact set. Associated with the subdifferential is the notion of the one-sided directional derivative of a convex function at $x \in \text{dom } f$ in the direction d. This is denoted as $f'(x,h)$ and is given as

$$f'(x,h) = \sup_{\xi \in \partial f(x)} \langle \xi, d \rangle.$$

Further when x is the interior of dom f we have that $f'(x,h)$ is finite. Of course $\bar{x}$ is a minimum (global) of f over $\mathbb{R}^n$ if and only if $\mathbf{0} \in \partial f(\bar{x})$.

Let us now present some important calculus rules for the subdifferential of a convex function. Of course the most simplest is that $\partial(\lambda f)(\bar{x}) = \lambda \partial f(\bar{x})$, with $\lambda \geq 0$. The next is the sum rule which says the following. Let f and g be two extended-valued proper convex functions defined on $\mathbb{R}^n$. Assume that the following qualification condition

$$\mathbf{0} \in \text{core}(\text{dom } g - \text{dom } f)$$

holds. Then

$$\partial(f+g)(x) = \partial f(x) + \partial g(x), \quad x \in \mathbb{R}^n.$$

The qualification condition under which the sum rule holds also follows from the fact that $ri(\text{dom } f) \cap ri(\text{dom } g) \neq \emptyset$. Now we will present a very important calculus rule which tells us how to calculate the subdifferential of a "max" function.

Let $f(x) = \max\{f_1(x), \ldots f_k(x)\}$ where each f_j, $j = 1, \ldots, k$ is a convex function then one has

$$\partial f(x) = \text{co} \left\{ \bigcup_{j \in J(x)} \partial f_j(x) \right\},$$

where $J(x) = \{j : f(x) = f_j(x)\}$.

We will also discuss now some present some important geometrical tools used in nonsmooth optimization. We will introduce the Bouligand tangent cone or simply the tangent cone and the normal cone to convex set. The notion of the tangent cone to an arbitrary closed set is given as follows. A vector $v \in \mathbb{R}^n$ is a tangent to a set C at $\bar{x} \in C$, if

$$\frac{x_k - \bar{x}}{t_k} \to v \quad \text{for some} \quad x_k \to \bar{x}, \quad x_k \in C \quad \text{and} \quad t_k \downarrow 0.$$

The set of all tangent vectors form a cone called the Bouligand tangent cone or simply the tangent cone and is denoted as $T_C(\bar{x})$. If C is a convex set then we have

$$T_C(\bar{x}) = \text{clcone}(C - \bar{x}).$$

A vector $v \in \mathbb{R}^n$ is said to be a normal vector to the convex set C at $\bar{x}$ if

$$\langle v, y - \bar{x} \rangle \leq 0, \quad \forall y \in C.$$

The set of all vector normal vectors v forms a cone called the *normal cone* to the convex set C at $\bar{x}$ and is denoted as $N_C(\bar{x})$. Further when C is convex the Bouligand tangent cone and the normal cone are connected through the following polarity relation,

$$(T_C(\bar{x}))^\circ = N_C(\bar{x}) \quad \text{and} \quad (N_C(\bar{x}))^\circ = T_C(\bar{x}).$$

where $A^\circ = \{w \in \mathbb{R}^n : \langle v, w \rangle \leq 0, \quad \forall v \in A\}$.

We shall now discuss the notion of Clarke subdifferential for a locally Lipschitz function. We begin with the notion of the Clarke generalized derivative which is an extension of the directional derivative of a convex function. The Clarke generalized directional derivative of f at x in the direction d is given as

$$f^\circ(x, d) = \limsup_{y \to x, \lambda \downarrow 0} \frac{f(y + \lambda d) - f(y)}{\lambda}.$$

Further note that for each given x the function $d \mapsto f^\circ(x, d)$ is sublinear in d.

The Clarke generalized gradient or the Clarke subdifferential of f at $\bar{x}$ is denoted as $\partial^\circ f(x)$ and is given as

$$\partial^\circ f(\bar{x}) = \{\xi \in \mathbb{R}^n : f^\circ(\bar{x}, d) \geq \langle \xi, d \rangle, \quad \forall v \in \mathbb{R}^n\}.$$

The set $\partial^\circ f(\bar{x})$ is nonempty, convex and compact for each $\bar{x} \in \mathbb{R}^n$. As a set-valued map $\partial^\circ f$ is locally bounded and has a closed graph and hence is upper-semicontinuous. Further if x_0 is a local minimizer of f over $\mathbb{R}^n$ then $0 \in \partial^\circ f(x)$. Note that the optimality condition is necessary and not sufficient in general. We also have

$$f^\circ(x, v) = \sup_{\xi \in \partial^\circ f(x)} \langle \xi, v \rangle.$$

Further if f is a finite-valued convex function then it is always locally Lipschitz and the Clarke generalized derivative of f and the Clarke subdifferential of f coincide with the directional derivative and the subdifferential of the convex function f. Let us also mention some important calculus rules that we would need in the sequel. The first one is the most simplest one which says that $\partial^\circ(\lambda f)(\bar{x}) = \lambda \partial^\circ f(\bar{x})$ for all $\lambda \in \mathbb{R}$. Then we have the sum rule which says that if f and g are locally Lipschitz function we have

$$\partial^\circ(f+g)(x) \subseteq \partial^\circ f(x) + \partial g(x).$$

The above inclusion can also hold in the strict sense.

It is important to note that when C is convex the Bouligand tangent cone is convex, whereas in general it is closed and need not be convex. Clarke [6] developed a notion of the tangent cone which is always a convex set irrespective of the nature of the underlying set. This notion is now known as the Clarke tangent cone which we now give below.

$$T_C^{cl}(\bar{x}) = \{v \in \mathbb{R}^n : d_C^\circ(\bar{x}, v) = 0\},$$

where d_C denotes the distance function associated with the set C and it is well known that d_C is Lipschitz.

The Clarke normal cone is defined as the polar of the Clarke tangent cone and is given as

$$N_C^{cl}(\bar{x}) = (T_C^{cl}(\bar{x}))^\circ.$$

5.3 Geometric Optimality Conditions

It is well known that the set of Pareto minimizers are contained in the set of weak Pareto minimizers. Hence any necessary optimality condition that is satisfied by weak Pareto minimizers will also hold for Pareto minimizers. Thus in this section we will concentrate on developing optimality conditions for weak Pareto minimizers for the problem (VOP) where the set C is just a closed set and need not be described explicitly through functional constraints. Let us also note that we shall consider in this chapter the Pareto and weak Pareto minimizers under the natural ordering cone $\mathbb{R}^k_+$. The reader can easily generalize the results to the case for any closed, convex and pointed cone with a non-empty interior. In this section we shall consider the problem (VOP) in the finite dimensional setting and we shall consider $X = \mathbb{R}^n$ and $Y = \mathbb{R}^r$. Thus f can be viewed as a vector function with r real-valued components. Thus one can write

$$f(x) = (f_1(x), \ldots, f_r(x)).$$

Further we will consider C to be a closed set in $\mathbb{R}^n$. It is important to note that the results in this section are fundamental and are well known. Thus instead of providing a reference for each result we shall mention some sources where these results can be found.

Theorem 5.1. *Let us consider the problem (VOP) where f is continuously differentiable (smooth) function and C is a convex set. Let $\bar{x}$ be a weak Pareto minimizer of (VOP). Then*

$$(\langle \nabla f_1(\bar{x}), v\rangle, \ldots, \langle \nabla f_k(\bar{x}), v\rangle) \notin -int\mathbb{R}^k_+ \quad \forall v \in T_C(\bar{x}).$$

Conversely if each $f_j, j = 1, \ldots, k$ is convex and C a closed convex set then the above condition is also sufficient for $\bar{x}$ to be a weak minimum.

Proof. Let us set $W = \mathbb{R}^r \setminus -\text{int}\mathbb{R}^r_+$. Let $\mathbf{0} \neq v \in T_C(\bar{x})$. Hence there exist sequences $t_k \downarrow 0$ and $v^k \to v$ such that $\bar{x} + t_k v^k \in C$. Since $\bar{x}$ is a weak Pareto minimum we have

$$(f_1(\bar{x} + t_k v^k) - f_1(\bar{x}), \ldots, f_k(\bar{x} + t_k v_k) - f_r(\bar{x})) \in W. \tag{5.1}$$

Note that W is a closed cone though not convex. Thus dividing both sides of (5.1) by t_k and passing to the limit as $t_k \to 0$ we reach our desired conclusion.

For the converse observe that since C is convex $T_C(\bar{x})$ is a convex set and using standard separation arguments or the convex version of the Gordan's theorem of the alternative (see for example Craven [8]) we have that there exists $\tau_j \geq 0$, $j = 1, \ldots r$ and $0 \neq \tau = (\tau_1, \ldots, \tau_r)$ such that

$$\sum_{j=1}^{r} \tau_j \langle \nabla f_j(\bar{x}), v\rangle \geq 0, \quad \forall v \in T_C(\bar{x}).$$

This shows that

$$\sum_{j=1}^{r} \tau_j \nabla f_j(\bar{x}) \in N_C(\bar{x}),$$

where $N_C(\bar{x})$ is the normal cone to the convex set C at $\bar{x}$. Hence from basic facts of convex optimization we see that $\bar{x}$ is a solution of the problem

$$\sum_{j=1}^{r} \tau_j f_j(x), \quad \text{subject to} \quad x \in C.$$

We will now leave it to the reader to prove from the above fact that $\bar{x}$ is a weak Pareto minimum. □

The above result can be found for example in Jahn [26]. We already know that if C is convex then $T_C(\bar{x}) = \text{clcone}(C - \bar{x})$. This allows us to phrase the optimality condition in the following form. If $\bar{x}$ is a Pareto minimum for (VOP) where C is a closed convex set then

$$(\langle \nabla f_1(\bar{x}), x - \bar{x}\rangle, \ldots, \langle \nabla f_k(\bar{x}), x - \bar{x}\rangle) \notin -\text{int}\mathbb{R}^k_+ \quad \forall x \in C. \tag{5.2}$$

This is what is known as the weak variational inequality formulation of the necessary optimality conditions. Of course we can replace the gradients of the functions by some other vector functions. That will lead us to what is known as the Stampacchia type weak vector variational inequality. See for example Charitha and

Dutta [5] for more details. In [5] they considered the more general problem of Stampacchia type weak vector variational inequality rather than a vector optimization problem itself. They developed a scalar-valued merit function and its regularization and also developed error bounds for certain classes of vector variational inequalities. We will adapt their work to the case of a convex vector optimization problem and present the error bound results for convex vector optimization problems with strongly convex objective functions. Now consider the problem (VOP) where all the functions f_i are assumed to be convex and C is a convex set. Observe that if we take $0 \neq \tau \in R^r_+$ and if $\bar{x}$ solves the scalar convex optimization problem (SP_τ)

$$\min \sum_{j=1}^{r} \tau_j f_j(x), \quad \text{subject to} \quad x \in C,$$

then $\bar{x}$ also a weak minimum of the problem (VOP). Further it is well known that

$$w - sol(VOP) = \bigcup_{\tau \in R^r_+ \setminus \{0\}} sol(SP_\tau), \tag{5.3}$$

where w-sol(VOP) denotes the set of weak minimum of the solution and $sol(SP_\tau)$ denotes the solution set of the scalar problem (SP_τ). Of course the above relation holds if the problem (VOP) is convex. Based on the approach due to Charitha and Dutta [5] we associate the following merit function or gap function with the problem (VOP) where each f_j, $j = 1, \ldots, r$ is smooth and need not be convex. The *merit function* is given as

$$\theta(x) = \min_{\tau \in S^r} \max_{y \in C} \left\langle \sum_{j=1}^{r} \tau_j \nabla f_j(x), y - x \right\rangle,$$

where S^r is the unit simples in $\mathbb{R}^r$. Again adapting [5, Theorem 2.2] in our setting we have the following result.

Theorem 5.2. *For any $x \in C$, $\theta(x) \geq 0$ and $\theta(x) = 0$, $x \in C$ if and only if $x \in sol(VOP)$.*

Proof. That fact $\theta(x) \geq 0$ for all $x \in C$ is obtained by simply setting $y = x$ in the expression for θ. Let us now consider $x_0 \in C$ such that $\theta(x_0) = 0$. Now let us set

$$\beta(x_0, \tau) = \max_{y \in C} \left\langle \sum_{j=1}^{r} \tau_j \nabla f_j(x), y - x \right\rangle.$$

Once we fix x_0 it is clear that $\beta(x_0, .)$ is a lower-semicontinuous convex function of τ. Further as $\theta(x_0) = 0$ we see that $\beta(x_0, .)$ is a proper function of τ. This shows that there exists $\tau_0 \in S^r$ such that $\beta(x_0, \tau_0) = \theta(x_0) = 0$. This shows that for all $y \in C$

$$\left\langle \sum_{j=1}^{r} (\tau_0)_j \nabla f_j(x_0), y - x_0 \right\rangle \geq 0.$$

From this we leave it to the reader to deduce that x_0 solves (VOP). We also ask the reader to try out the converse, that is, if x_0 solves (VOP) then $\theta(x_0) = 0$. This completes the proof. □

The problem with the above merit function θ is that one cannot guarantee that it will be finite-valued for all x unless there are additional conditions on C, for example C is compact. This has lead to the regularization of the gap function θ based on the *regularization* approach in the scalar case due to Fukushima [15]. The regularized version of the function θ is denoted as $\hat{\theta}$ which is given as

$$\hat{\theta}_\alpha(x) = \min_{\tau \in S^r} \max_{y \in C} \left\{ \left\langle \sum_{j=1}^{r} \tau_j \nabla f_j(x), x - y \right\rangle - \frac{\alpha}{2} ||y - x||^2 \right\}, \quad \alpha > 0.$$

Let us write

$$\phi_\alpha(x, \tau) = \max_{y \in C} \left\{ \left\langle \sum_{j=1}^{r} \tau_j \nabla f_j(x), x - y \right\rangle - \frac{\alpha}{2} ||y - x||^2 \right\}, \quad \alpha > 0.$$

This can also be equivalently written as

$$\phi_\alpha(x, \tau) = -\min_{y \in C} \left\{ \left\langle \sum_{j=1}^{r} \tau_j \nabla f_j(x), y - x \right\rangle - \frac{\alpha}{2} ||y - x||^2 \right\}, \quad \alpha > 0.$$

Now let us consider the following minimization problem

$$\min_{y \in C} \left\{ \left\langle \sum_{j=1}^{r} \tau_j \nabla f_j(x), y - x \right\rangle - \frac{\alpha}{2} ||y - x||^2 \right\}.$$

It is important to observe that the objective function in the above problem is strongly convex in y and hence coercive. Further since C is a closed and convex set there exists a unique minimum. The unique minimum of this problem is given as

$$y_\alpha(x, \tau) = proj_C \left(x - \frac{1}{\alpha} \sum_{j=1}^{r} \tau_j \nabla f_j(x) \right),$$

where $proj_C$ denotes the projection of a point on the closed convex set C. Further using that fact that the projection mapping is Lipschitz we can show that for any $\alpha >$ the function $y_\alpha(x, \tau)$ is continuous on $\mathbb{R}^n \times S^r$. This fact allows us to show that

$\hat{\theta}_\alpha$ is a finite-valued function. We will now turn our attention to develop the error bound for the problem (VOP).

We will now consider the problem (VOP) where each function f_j, $j = 1,\dots,r$ is a strongly convex function with $\mu_j > 0, j = 1,\dots,r$ as the modulus of strong convexity. Note that for each $j = 1,\dots,r$ the gradient vector ∇f_j is also strongly monotone with the modulus of strong monotonicity $\mu_j > 0$. Thus for any $\tau \in S^r$ we have $\sum_{j=1}^r \tau_j \nabla f_j$ to be strongly monotone with modulus of monotonicity $\mu = \min\{\mu_1,\dots,\mu_r\}$. Note that for each τ there is a unique solution to (SP). However as we change the τ the solution will in general change. This is general the solution set of (VOP) even under the assumption of strong convexity need not be singleton. With these basic facts we shall state the following result on error bound.

Theorem 5.3. *Let us consider the problem (VOP) where each f_j, $j = 1,\dots,r$ is strongly convex with the modulus of strong convexity $\mu_j > 0$. Let $\mu = \min\{\mu_1,\dots,\mu_r\}$ and further $\alpha > 0$ be so chosen that $\alpha < 2\mu$. As before let w-sol (VOP) denote the set of weak Pareto minimum of the problem (VOP). Then for any $x \in C$ we have*

$$d(x, w-sol(VOP)) \leq \frac{1}{\sqrt{\mu - \frac{\alpha}{2}}} \sqrt{\hat{\theta}_\alpha(x)}.$$

Proof. Let us observe that in our notations we have

$$\hat{\theta}_\alpha = \min_{\tau \in S^r} \phi_\alpha(x,\tau).$$

Now consider any $x \in C$. Thus there exists $\tau^* \in S^r$ depending of course on x, such that

$$\hat{\theta}_\alpha = \phi_\alpha(x,\tau^*).$$

Now as all f_j, $j = 1,\dots,r$ are strongly convex with $\mu_j > 0$ as the modulus of strong convexity it is clear that $\sum_{j=1}^r \tau_j^* \nabla f_j$ is strongly monotone with $\mu = \min\{\mu_1,\dots,\mu_r\}$. Also observe that the problem (SP_{τ^*}) has a unique solution. Let us denote this solution as x^*. Note that x^* is also a solution of (VOP). Now we have for any $y \in C$

$$\hat{\theta}_\alpha(x) = \phi_\alpha(x,\tau^*) \geq \left\langle \sum_{j=1}^r \tau_j^* \nabla f_j(x), x - y \right\rangle - \frac{\alpha}{2}||y - x||^2.$$

Thus in particular for $y = x^*$ we have

$$\hat{\theta}_\alpha(x) \geq \left\langle \sum_{j=1}^r \tau_j^* \nabla f_j(x), x - x^* \right\rangle - \frac{\alpha}{2}||x - x^*||^2.$$

Now strong monotonicity of $\sum_{j=1}^r \tau_j \nabla f_j$ shows that

$$\hat{\theta}_\alpha(x) \geq \left\langle \sum_{j=1}^{r} \tau_j^* \nabla f_j(x^*), x - x^* \right\rangle + \left(\mu - \frac{\alpha}{2}\right) ||x - x^*||^2.$$

Since x^* solves (SP_{τ^*}) and (SP_{τ^*}) is a convex optimization problem it is well known that

$$\left\langle \sum_{j=1}^{r} \tau_j^* \nabla f_j(x^*), x - x^* \right\rangle \geq 0.$$

Noting that $2\mu > \alpha$ we have

$$||x - x^*|| \leq \frac{1}{\sqrt{\mu - \frac{\alpha}{2}}} \sqrt{\hat{\theta}_\alpha(x)}.$$

This shows that

$$d(x, w - sol(VOP)) \leq \frac{1}{\sqrt{\mu - \frac{\alpha}{2}}} \sqrt{\hat{\theta}_\alpha(x)}.$$

Hence the result. □

Let us now focus on the case when the objective function of (VOP) is convex but not differentiable. Here is the basic result.

Theorem 5.4. *Let us consider the problem (VOP) where each component function of f is convex but need not be differentiable and C be a closed convex set. If $\bar{x}$ is weak Pareto minimum then, the following two conditions hold.*

(a) $(f_1'(\bar{x}, v), \ldots f_k'(\bar{x}, v)) \notin -int\mathbb{R}^k_+$ *for all* $v \in T_C(\bar{x})$
(b) There exists scalars $\tau_j \geq 0$, $j = 1, \ldots, k$ *with* $\tau = (\tau_1, \ldots \tau_k)^T \neq \mathbf{0}$ *such that* $\mathbf{0} \in \sum_{j=1}^{k} \tau_j \partial f_j(\bar{x}) + N_C(\bar{x})$

Conversely if any of the above two conditions hold then $\bar{x}$ is a weak Pareto minimum.

Proof. Let $0 \neq w \in T_C(\bar{x})$. Hence there exists a sequence $\{w^k\}$ such that $w^k \to w$ and a real sequence $t_k \downarrow 0$ such that $\bar{x} + t_k w^k \in C$. Since $\bar{x}$ is a weak Pareto minimum we have

$$(f_1(\bar{x} + t_k v^k) - f_1(\bar{x}), \ldots, f_r(\bar{x} + t_k v^k) - f_r(\bar{x})) \in W.$$

The conclusion is now reached by passing to the limit as $k \to \infty$. Now once the conclusion in i) is reached it is simple to observe that using the Gordan's Theorem of the Alternative one can conclude that there exists $\tau_j \geq 0, j = 1, \ldots, r$ and all not zero such that

$$\sum_{j=1}^{r} \tau_j f_j'(\bar{x}, v) \geq 0 \quad \forall v \in T_C(\bar{x}).$$

This can be equivalently given as

$$\sum_{j=1}^{r} \tau_j f_j'(\bar{x}, v) + \delta_{T_C(\bar{x})}(v) \geq 0 \quad \forall v \in \mathbb{R}^n.$$

Now it is well known in convex analysis (see for example Rockafellar [34]) that

$$\sigma_{(T_C(\bar{x}))^\circ}(v) = \delta_{T_C(\bar{x})}.$$

where σ denotes the support function of a convex set. This shows that

$$\sum_{j=1}^{r} \tau_j f_j'(\bar{x}, v) + \sigma_{(T_C(\bar{x}))^\circ}(v) \geq 0 \quad \forall v \in \mathbb{R}^n.$$

Now using the standard rules of support function calculus (see for example [34]) and the sum rule for convex subdifferentials we deduce that

$$\mathbf{0} \in \sum_{j=1}^{r} \tau_j \partial f_j(\bar{x}) + (T_C(\bar{x}))^\circ$$

Knowing that $(T_C(\bar{x}))^\circ = N_C(\bar{x})$ we can now conclude the result. □

So now we can consider the more general problem where the component functions of f are locally Lipschitz functions which need not be differentiable everywhere and C is just a closed set and need not be convex. Let us consider $\bar{x} \in C$ to be a weak minimum of (VOP). Then using the same approach as in the convex case but using the limit supremum instead of just the limit we have

$$(f_1^+(\bar{x}, v), \ldots f_r^+(\bar{x}, v)) \in W \quad \forall v \in T_C(\bar{x}),$$

where $f^+(\bar{x}, v)$ denotes the *upper Dini directional derivative* (see [7]) at $\bar{x}$ in the direction v. Since $f_j^\circ(\bar{x}, v) \geq f_j^+(\bar{x}, v)$ for all $j = 1, \ldots, r$, we have

$$(f_1^\circ(\bar{x}, v), \ldots, f_r^\circ(\bar{x}, v)) \in (f_1^+(\bar{x}, v), \ldots, f_r^+(\bar{x}, v)) + \mathbb{R}_+^r.$$

Since $W + \mathbb{R}_+^r \subset W$ we have

$$(f_1^\circ(\bar{x}, v), \ldots, f_r^\circ(\bar{x}, v)) \in W \quad \forall v \in T_C(\bar{x}).$$

Further since $T_C^{cl}(\bar{x}) \subseteq T_C(\bar{x})$ we have

$$(f_1^\circ(\bar{x}, v), \ldots, f_r^\circ(\bar{x}, v)) \in W \quad \forall v \in T_C^{cl}(\bar{x}),$$

where $T_C^{cl}(\bar{x})$ denotes the Clarke tangent cone to C at $\bar{x}$ (see Clarke [6]) for details. We would now request the reader to show from the above discussions and using the approach taken for the convex case that if $\bar{x}$ is a weak Pareto minimum of (VOP) we have that there exists $0 \neq \tau \in \mathbb{R}^r_+$ such that

$$\mathbf{0} \in \sum_{j=1}^{r} \tau_j \partial^\circ f_j(\bar{x}) + N_C^{cl}(\bar{x}),$$

where $\partial^\circ f_j(\bar{x})$ denote the Clarke subdifferential of f at $\bar{x}$ and $N_C^{cl}(\bar{x})$ denotes the Clarke normal cone to C at $\bar{x}$. For more detail on these objects see for example Clarke [6].

5.4 Optimality Conditions with Explicit Constraints

The term “explicit constraints” refer to the case where the feasible set C is described by functional constraints which is usually given in terms of equalities are inequalities. Of course there can be more general constraints given in terms of cones called conic constraints. However for simplicity we shall focus only on inequality constraints. It is important however to note that there has been a huge number of studies on optimality conditions for vector optimization problem with inequality constraints. See for example Jahn [26], Ehrgott [13], and Eichfielder [14] and the references there in. So we will not list down the details here but rather move directly to what we believe is an interesting and important issue as per the optimality conditions of a vector optimization condition is concerned. Let us set

$$C = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, \ldots, m\}$$

Now consider a point $\bar{x}$ to be a Pareto minimum of (VOP). Then then by the Chankong–Haimes scalarization we know that $\bar{x}$ solves each of the problem (P_j), $j = 1, \ldots, r$ where (P_j) is given by

$$\min f_j(x)$$

$$\text{subject to}$$

$$f_k(x) \leq f_k(\bar{x}), \quad k \neq r$$

$$g_i(x) \leq 0, \quad i = 1, \ldots, m$$

Assume now that the problem data is locally Lipschitz. Now if for some $q \in \{1, \ldots, r\}$ a suitable constraint qualification holds for the problem (P_q) then from Clarke [6] we have that there exists scalars $\tau_j \geq 0$, $j \neq q$ and $\lambda_i \geq 0$ such that:

(1) $\mathbf{0} \in \partial^\circ f_q(\bar{x}) + \sum_{j=1, j\neq q}^{r} \tau_j \partial^\circ f_j(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial^\circ g_i(\bar{x})$
(2) $\lambda_i g_i(\bar{x}) = 0$ for all $i = 1, \ldots, m$

In the above inclusion if we set $\tau_q = 1$ then we can conclude that if $\bar{x}$ is a Pareto minimum for (VOP) then there exist $\tau \in \mathbb{R}^r_+$ and $\lambda \in \mathbb{R}^m_+$ such that:

(1) $\mathbf{0} \in \sum_{j=1}^r \tau_j \partial^\circ f_j(\bar{x}) + \sum_{i=1}^m \lambda_i \partial^\circ g_i(\bar{x})$
(2) $\lambda_i g_i(\bar{x}) = 0$ for all $i = 1, \ldots, m$
(3) $\mathbf{0} \neq \tau$

The above conditions are usually called the Karush–Kunh–Tucker or KKT conditions for the problem (VOP) or the Lagrange multiplier rule associated with the problem (VOP). Of course if the problem data is convex then the above necessary condition holds and we will just have to replace the Clarke subdifferential by the convex subdifferential. Now consider again the problem (VOP) with convex data. Now if we have a triplet $(\bar{x}, \tau, \lambda) \in \mathbb{R}^n \times \mathbb{R}^r_+ \times \mathbb{R}^m_+$ with $\bar{x}$ feasible and which satisfies the KKT conditions then $\bar{x}$ is a weak Pareto minimum of (VOP) which need not necessarily be a Pareto minimum. For $\bar{x}$ to be Pareto minimum we have must have that $(\bar{x}, \tau, \lambda)$ must satisfy the KKT conditions with $\tau_j > 0$ for all $j = 1, \ldots, r$. So it will be important to see under what condition at least in the convex case the necessary *KKT conditions* for the Parto minimum will holds with $\tau_j > 0$ for all $j = 1, \ldots, m$. This would then allow us to differentiate a Pareto minimum from a weak Pareto minimum through KKT conditions. When the KKT condition holds with $\tau_j > 0$ for all $j = 1, \ldots, r$ then we say that the *strong KKT condition* holds. For the convex case the necessary KKT conditions hold for a special class of Pareto minimum points called *Geoffrion proper efficient points*. Let us begin by defining what one means by a Geoffrion proper efficient point .

We have the following result

Theorem 5.5. *Let us consider the problem (VOP) with convex data, i.e. each f_j, $j = 1, \ldots, r$ is a convex function and each g_i, $i = 1, \ldots, m$ is a convex function. Assume that the Slater constraint qualifications hold. Then $\bar{x}$ is a Geoffrion proper efficient point if and only if the strong KKT conditions hold.*

Proof. It has been shown in Geoffrion [18] that $\bar{x}$ is a Geoffrion proper efficient solution of (VOP) with convex data if and only if there exists scalars $\tau_j > 0$, $j = 1, \ldots, r$ such that $\bar{x}$ solves the problem

$$\min \sum_{j=1}^r \tau_j f_j(x) \quad \text{subject to} \quad g_i(x) \leq 0, \quad i = 1, \ldots, m.$$

Now as the Slater constraint qualification condition holds then from it is a well known fact in scalar convex optimization that there exists scalars $\lambda_i \geq 0$ such that

$$\mathbf{0} \in \partial \left(\sum_{j=1}^r \tau_j f_j \right)(\bar{x}) + \sum_{i=1}^m \lambda_i \partial g_i(\bar{x}).$$

and

$$\lambda_i g_i(\bar{x}) = 0.$$

Now using the subdifferential sum rule the first of the above two conditions become

$$\mathbf{0} \in \sum_{j=1}^{r} \tau_j \partial f_j(\bar{x}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{x}).$$

This proves that when $\bar{x}$ is a Geoffrion proper efficient point then the strong KKT conditions hold. We leave the easy proof of the converse to the reader. □

Now the interesting question is whether strong KKT holds true even when the problem is non-convex. It seems yes but even in the differentiable case it holds under quiet strong assumptions. However it is interesting to know what are these conditions since the strong KKT conditions allows us to separate a Pareto minimum point from a weak Pareto minimum which is quiet important from the point of view of applications. It is interesting that the first idea of strong KKT conditions in fact come from the seminal 1951 paper of Kuhn and Tucker [28] where the first optimality conditions about vector optimization are given. In fact they introduce a class of proper efficient points called points for a smooth vector optimization problem. In our current setting if we assume that the problem data in (VOP) is smooth then the notion of a Kuhn–Tucker proper efficient point is as follows.

An efficient point or a Pareto minimum $\bar{x}$ of (VOP) with inequality constraints is said to be Kuhn–Tucker proper efficient if the following system

$$\langle \nabla f_j(\bar{x}), d \rangle \leq 0, \quad j = 1, \ldots, r,$$
$$\langle \nabla f_j(\bar{x}), d \rangle < 0, \quad \text{at least one} \quad j$$
$$\langle \nabla g_i(\bar{x}), d \rangle \leq 0, \quad i \in I(\bar{x})$$

has no solution d in $\mathbb{R}^n$. This formulation allows us to use the Tucker's Theorem of Alternative to guarantee that the holds. Maeda [30] provided the qualification conditions under which one can guarantee that a given efficient point is also a Kuhn–Tucker proper efficient point. We shall briefly discuss the results of Maeda [30] here. Given the problem (VOP), Maeda constructs the following sets. We shall begin by defining the sets Q^j and Q as follows. For each a given x_0 and $j = 1, \ldots, r$ we have

$$Q^j(x_0) = \{x \in \mathbb{R}^n : g(x) \leq \mathbf{0}, f_k(x) \leq f_k(x_0), \quad j = 1, \ldots, r, j \neq k\}$$

and

$$Q(x_0) = \{x \in \mathbb{R}^n : g(x) \leq \mathbf{0}, \quad f(x) \leq f(x_0)\},$$

where $g(x) = (g_1(x), \ldots, g_m(x))^T$ and $f(x) = (f_1(x), \ldots, f_r(x))^T$ the vector inequality is taken component-wise. Maeda [30] then constructs the linearizing cone at x_0 as follows

$$L(Q, x_0) = \{d \in \mathbb{R}^n : \langle \nabla f_j(x_0), d \rangle \leq 0, j = 1, \ldots, r \quad \langle \nabla g_i(x_0), h \rangle \leq 0, i \in I(x_0)\}$$

It is simple to see that $L(Q,x_0)$ is a closed convex cone. Maeda [30] proved that for any feasible solution of (VOP) one has

$$\bigcap_{j=1}^{r} \text{clco}T_{Q^j}(x_0) \subseteq L(Q,x_0).$$

The idea of the proof is based on proving that for all $j = 1,\dots,r$ we have

$$T_{Q^j}(x_0) \subseteq L(Q^j,x_0).$$

where

$$L(Q^j,x_0) = \{d \in \mathbb{R}^n : \langle \nabla f_k(x_0), d\rangle \leq 0, k = 1,\dots,r, \quad k \neq j \langle \nabla g_i(x_0), h\rangle \leq 0, i \in I(x_0)\}$$

For the detailed proof we refer the reader to Maeda [30]. The reverse inclusion need not hold in general and thus it is reasonable to assume that reverse inclusion can be considered as a qualification condition, that is,

$$L(Q,x_0) \subseteq \bigcap_{j=1}^{r} \text{clco}T_{Q^j}(x_0).$$

The above inclusion is referred to by Maeda [30] as the Generalized Guignard Constraint Qualification, (GGCQ) for short. The main result in Maeda [30] can be presented as follows.

Theorem 5.6. *Let x_0 be a Pareto minimum point for (VOP) with inequality constraints and smooth data. Let the GGCQ hold at x_0. Then x_0 is a Kuhn–Tucker proper efficient and hence satisfies the strong KKT condition.*

Proof. Let us assume that x_0 is not a Kuhn–Tucker proper efficient. Hence there exists a $d \in \mathbb{R}^n$ such that following system

$$\begin{aligned} &\langle \nabla f_j(\bar{x}), d\rangle \leq 0, \quad j = 1,\dots,r, \\ &\langle \nabla f_j(\bar{x}), d\rangle < 0, \quad \text{at least one} \quad j \\ &\langle \nabla g_i(\bar{x}), d\rangle \leq 0, \quad i \in I(\bar{x}) \end{aligned}$$

is consistent. Without loss of generality let us assume that

$$\begin{aligned} &\langle \nabla f_1(x_0), d\rangle < 0, \\ &\langle \nabla f_j(x_0), d\rangle \leq 0, \quad j = 2,\dots,r. \end{aligned}$$

It is clear that $d \in L(Qx_0)$. Now from GGCQ we have that $d \in \overline{coT_{Q^j}(x_0)}$ for all $j = 1, \rightarrow r$ and hence in particular $j = 1$. Thus there exists a sequence $\{d_m\}$ in $coT_{Q^1}(x_0)$ such that $d = \lim_{m\to\infty} d_m$. Now for each d_m there exists a number $K_m \in \mathbb{N}$ and scalars $\lambda_{mk} \geq 0$ with vector $d_{mk} \in T_{Q^1}(x_0)$, with $k = 1, \ldots, K_m$ such that

$$\sum_{k=1}^{K_m} \lambda_{mk} = 1$$

and

$$\sum_{k=1}^{K_m} \lambda_{mk} d_{mk} = d$$

The next steps of the proof is quiet simple. Noting that $d_{mk} \in T_{Q^1}(x_0)$ for each k, we have that there exists $x_{mk}^N \rightarrow x_0$ with $x_{mk}^n \in Q^1$ for all n and $t_{mk}^n > 0$ such that

$$t_{mk}^n (x_{mk}^n - x_0) \rightarrow d_{mk}.$$

Since x_{mk}^n is in Q^1 we observe that

$$f_j(x_{mk}^n) \leq f_j(x_0), \quad j = 2, \ldots r$$
$$g_i(x_{mk}^n) \leq 0, \quad \forall i \in I(x_0).$$

Since x_0 is a Pareto minimum point we must have $f_1(x_{mk}^n) \geq f_1(x_0)$. Now using this facts we leave it to the reader to reach a contradiction to the fact d is the solution of the system of inequalities given at the beginning of the proof. Thus it shows that x_0 is a Kuhn–Tucker efficient point. We have already mentioned before why the Kuhn–Tucker proper efficient point also satisfies the strong KKT conditions. □

For an extension of the approach of Maeda to the nonsmooth case see the recent paper of Giorgi et al. [20] and the references there in.

5.5 Second Order Optimality Conditions

In this section we will discuss second order conditions for a vector optimization problems. To the baser of our knowledge it seems that the second order conditions for vector optimization problems have yet to appear in a book on vector optimization and thus we take this opportunity to introduce some basic results on second order optimality conditions. We will concentrate only when (VOP) has smooth data and we will provide references for the interested reader to look for further details as well as the nonsmooth case. We skip studying the second order conditions since we believe that this is an evolving area and considerable more research is needed to make it a more coherent. For simplicity of the presentation we shall consider the problem (VOP) with inequality constraints only as we have done in the previous section. Of course the date of the problem (VOP) is smooth. In the study of second order conditions we need to understand the notion of lexicographic ordering and the notion of a second order tangent set. Second order tangent sets defined in different ways by different authors but they are closely related.

We will only need to consider lexicographic ordering in $\mathbb{R}^2$. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be two points in $\mathbb{R}^2$. Then $x \leq_{lex} y$ if $x_1 < y_1$ or $x_1 = y_1$ and $x_2 \leq y_2$. Further $x <_{lex} y$ if $x_1 < y_1$ or $x_1 = y_1$ and $x_2 < y_2$. To the best of our knowledge one of the earliest work in this area is due to Wang [36] and Aghezzaf and Hachimi [1] and for the more recent contributions in this area see for example Maeda [31], Bigi [3], Aghezzaf and Hacimi [2], Guerraggio and Luc [17], Ginchev et al. [19] and the references there in. Aghezzaf and Hachimi [1] introduced the following second order tangent set to a closed set C at $\bar{x} \in C$ as the set

$$T_C^2(\bar{x}) = \left\{ (y,z) \in \mathbb{R}^n \times \mathbb{R}^n : \exists t_n \downarrow 0, \quad \text{such that} \quad \bar{x} + t_n y + \frac{1}{2} t_n^2 z + o(t_n^2) \in C \right\}.$$

Also observe that $\bar{x} + t_n y + \frac{1}{2} t_n^2 z + o(t_n^2) \in C$ implies that

$$\bar{x} + t_n y + \frac{1}{2} t_n^2 \left(z + \frac{o(t_n^2)}{t_n^2} \right) \in C.$$

Thus if we set $z_n = z + \frac{o(t_n^2)}{t_n^2}$ then $z_n \to z$ and hence we can alternatively and equivalently write $T_C^2(\bar{x})$ as

$$T_C^2(\bar{x}) = \left\{ (y,z) \in \mathbb{R}^n \times \mathbb{R}^n : \exists z_n \to z, \quad t_n \downarrow 0, \quad \text{such that} \quad \bar{x} + t_n y + \frac{1}{2} t_n^2 z_n \in C \right\}.$$

A curious fact to observe here is that the set $T_C^2(\bar{x})$ is higher dimensional then that of C while $T_C(\bar{x})$ is embedded in the same dimensional space as is C. Maeda [31] defines the notion of a second order tangent set in the following way. The second order tangent set to the set C at $\bar{x}$ in the direction y is given as

$$T_C^2(\bar{x}, y) = \left\{ z \in \mathbb{R}^n : \exists z_n \to z, \quad t_n \downarrow 0, \quad \text{such that} \quad \bar{x} + t_n y + \frac{1}{2} t_n^2 z_n \in C \right\}.$$

We would request the reader to figure out the relation between the two tangent sets defined above. We would like to mention that the both the sets $T_C^2(\bar{x})$ and $T_C^2(\bar{x}, y)$ are in fact sets and not cones in general and are convex if the underlying set C is convex.

We shall now define the work of Aghezzaf and Hachimi [1] in a fair amount of details. At this stage is quiet natural to define the notion of a second order linearizing set which is given as follows,

$$L_2(\bar{x}) = \{ (y,z) \in \mathbb{R}^n \times \mathbb{R}^n : G_i(\bar{x}, y, z) \leq_{lex} 0, \quad \forall i \in I(\bar{x}) \},$$

where

$$G_i(\bar{x}, y, z) = (\langle \nabla g_i(\bar{x}), y \rangle, \langle \nabla g_i(\bar{x}), z \rangle + \langle y, \nabla^2 g_i(\bar{x}) y \rangle).$$

The notion of a critical direction plays an important role in the analysis of second order optimality condition in vector optimization. Let $\bar{x}$ be a vector which is feasible to (VOP) consisting only of inequality constraints. A vector d is said to be a critical direction of (VOP) if

$$\begin{aligned} \langle \nabla f_j(\bar{x}), d \rangle &\leq 0, \quad j = 1, \ldots, r \\ \langle \nabla f_j(\bar{x}), d \rangle &= 0, \quad \text{for some} \quad j \\ \langle \nabla g_i(\bar{x}), d \rangle &\leq 0, \quad i = 1, \ldots, m. \end{aligned}$$

In a manner similar to that of $G_i(\bar{x}, y, z)$ we can also define $F_j(\bar{x}, y, z)$. Our first result in this section is a geometric one which we present below is a second-order necessary optimality condition. Also it is important note that result that is presented below holds for any arbitrary closed set C.

Theorem 5.7 ([1]). *Consider the problem (VOP) where the objective function is a continuous vector function and C a closed set. Let $\bar{x}$ be a Pareto minimum or a weak Pareto minimum then,*

$$T^2_{f(C)}(f(\bar{x})) \cap \Omega = \emptyset,$$

where

$$\Omega = \{(y, z) \in \mathbb{R}^r \times \mathbb{R}^r : (y_j, z_j) <_{lex} (0, 0)\}.$$

Proof. Since C is a closed set and since f is continuous we have that $f(C)$ is also a closed set. If $\bar{z} = f(\bar{x})$ is an isolated point of $f(C)$ we see that $T^2_{f(C)}(\bar{z}) = \{(0, 0)\}$ and thus the result is trivially satisfied. If it is not an isolated point then we reason as follows. Let us assume that

$$(y, z) \in T^2_{f(C)}(\bar{z}) \cap \Omega$$

Hence by definition of the second order tangent set we see that there exist sequences $z_n \in f(C)$ and $t_n \to 0$ such that

$$z_n = \bar{z} + t_n y + \frac{1}{2} t_n^2 z + o(t_n^2).$$

For n sufficiently large we can assume we can assume without loss of generality that $z_n \neq \bar{z}$. Now for each $i = 1, \ldots, r$ we have

$$(z_n)_i = \bar{z}_i + t_n y_i + \frac{1}{2} t_n^2 z_i + o(t_n^2)_i.$$

Since $(y, z) \in \Omega$ it is clear that we have to deal with two cases; namely $y_i < 0$ and $y_i = 0$ (in which case $z_i < 0$).

Let us begin by assuming that $y_i < 0$. Let us now observe that

$$t_n \left(\frac{1}{2} z_i + \frac{o(t_n^2)_i}{t_n^2} \right) \to 0,$$

as $n \to \infty$.

Now we can write

$$(z_n)_i = \bar{z}_i + t_n \left(y_i + t_n \left(\frac{1}{2} z_i + \frac{o(t_n^2)_i}{t_n^2} \right) \right).$$

Since $y_i < 0$ it is clear that for n sufficiently large we have

$$y_i + t_n \left(\frac{1}{2} z_i + \frac{o(t_n^2)_i}{t_n^2} \right) < 0.$$

Hence $(z_n)_i < \bar{z}_i$ for all n sufficiently large.

Now if $y_i = 0$, then we also have $z_i = 0$ and

$$(z_n)_i = \bar{z}_i + \frac{1}{2} t_n^2 \left(z_i + \frac{o(t_n^2)_i}{t_n^2} \right)$$

Since $z_i < 0$ we have that for n sufficiently large n

$$z_i + \frac{o(t_n^2)_i}{t_n^2} < 0.$$

This shows that $(z_n)_i < \bar{z}_i$ for n sufficiently large. This shows that $z_n - \bar{z} \in -\text{int}\mathbb{R}^r_+$ for n sufficiently large. Since $z_n \in f(C)$, there exists $x_n \in C$ such that $z_n = f(x_n)$ and hence

$$f(x_n) - f(\bar{x}) \in -\text{int}\mathbb{R}^r_+.$$

This contradicts the fact that $\bar{x}$ is a Pareto (weak Pareto) minimum for (VOP). □

Using the above geometric conditions we are now going to provide an algebraic necessary second order optimality condition for the problem (VOP). We will now consider the case where the set C is described by inequality constraints.

Theorem 5.8. *Let $\bar{x}$ be a Pareto or weak Pareto minimum for (VOP) with twice continuously differentiable data. Assume that the second order Abadie constraint qualification holds at $\bar{x}$, i.e.*

$$L_2(\bar{x}) = T_C^2(\bar{x}).$$

Then the following system

$$F_j(\bar{x}, y, z) <_{lex} 0, \quad j = 1, \dots r,$$
$$G_i(\bar{x}, y, z) \leq_{lex} 0, \quad i = 1, \dots m$$

has no solution $(y, z) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proof. We will take the help of the previous theorem to prove this result. Let us consider $(y, z) \in T_C^2(\bar{x})$. Then there exists sequences $t_n \downarrow 0$ such that

$$x_n = x + t_n y + \frac{1}{2} t_n^2 z + o(t_n^2) \in C.$$

Now by Taylor's expansion for each $j = 1, \dots, r$, we have

$$f_j(x_n) = f_j(\bar{x}) + t_n \langle \nabla f_j(\bar{x}), y \rangle + \frac{1}{2}(\langle \nabla f_j(\bar{x}), z \rangle + \langle y, \nabla^2 f_j(\bar{x}) y \rangle) + o(t_n^2).$$

The reader should carefully verify the Taylor's expansion since a lot of terms gets pulled into the term $o(t_n^2)$. The above analysis shows that the vector

$$(F_1(\bar{x}, y, z), \dots, F_r(\bar{x}, y, z)) \in T^2_{f(C)}(f(\bar{x}).$$

Hence from Theorem 5.7 it is clear that

$$F_j(\bar{x}, y, z) \not<_{lex} (0, 0), \quad j = 1, \dots, r.$$

Further as $L_2(\bar{x}) = T_C^2(\bar{x})$ it is clear that (y, z) also satisfies

$$G_i(\bar{x}, y, z) \leq_{lex} (0, 0), \quad \forall i \in I(\bar{x}).$$

This allows us to conclude the result. □

The above result leads to the more verifiable second order necessary conditions which we will present below. We do not present the proof here as the proof can be found in [1]

Theorem 5.9 ([1]). *Consider the problem (VOP) with inequality constraints and twice continuously differentiable data . Let* $\bar{x}$ *be a Pareto minimum or weak Pareto minimum for (VOP). Assume that the second order ACQ holds. Then there exists* $0 \neq \tau \in \mathbb{R}^r_+$ *and* $\lambda \in \mathbb{R}^m_+$ *such that for each critical direction y we have*

(i) $\sum_{j=1}^r \tau_j \nabla f_j(\bar{x}) + \sum_{i=1}^m \lambda_i \nabla g_i(\bar{x}) = 0$
(ii) $\langle y, (\sum_{j=1}^r \tau_j \nabla^2 f_j(\bar{x}) + \sum_{i=1}^m \lambda_i \nabla^2 g_i(\bar{x}))y \rangle \geq 0$
(iii) $\tau_j = 0$ *for all* $j \neq B(\bar{x}, y) = \{j : \langle \nabla f_j(\bar{x}), y \rangle = 0\}$
(iv) $\lambda_i = 0,$ *for all* $i \neq E(\bar{x}, y) = \{i \in I(\bar{x}) : \langle \nabla g_i(\bar{x}), y \rangle\}$

In the case of scalar optimization it is well known that the second order sufficient conditions plays a crucial role since it allows us to determine whether a KKT point is a strict local minimum or not. Thus from the point of view of multiobjective optimization it might be interesting to have a second order sufficient condition to see whether a KKT point is also a Pareto efficient point. In [1] the authors develop a second order sufficient condition by assuming some generalized convex assumption on the data. One might observe that why does one need to have a second order sufficient condition when the problem data satisfies certain generalized convexity condition. Note that even if the problem data is convex the first order KKT conditions will only return us a weak Pareto minimum. But if we consider the second order conditions even with convex data then we will get back a Pareto minimum. Thus in some sense the second order sufficient condition when (VOP) is a twice differentiable convex problem is an alternative approach to the strong KKT conditions which as we have seen in the previous section allows us to differentiate between a Pareto minimum and a weak Pareto minimum. However we would like to stress that Pareto minimum is more important from the point of view of practice. We shall now present a modified version of the result in [1] taking care of some technical issues. However for the simplicity of the presentation we consider the problem data to be convex.

Theorem 5.10. *Assume that the data of the problem (VOP) is twice continuously differentiable and convex. Let $\bar{x}$ be a feasible point of (VOP). Assume that for each critical direction $y \neq 0$ there exists a vector $0 \neq \tau \in \mathbb{R}^r_+$ and scalars $\lambda_i \geq 0$, $i \in I(\bar{x})$ such that the following holds*

(i) $0 = \sum_{j=1}^{r} \tau_j \nabla f_j(\bar{x}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x})$, *where* $\tau_j = 0$ *for all* $j \neq B(\bar{x},y) = \{j : \langle \nabla f_j(\bar{x}), y\rangle = 0\}$ *and* $\lambda_i = 0$, *for all* $i \neq E(\bar{x},y) = \{i \in I(\bar{x}) : \langle \nabla g_i(\bar{x}), y\rangle\}$
(ii) $\langle y, (\sum_{j=1}^{r} \tau_j \nabla^2 f_j(\bar{x}) + \sum_{i\in I(\bar{x})} \lambda_i \nabla^2 g_i(\bar{x}))y\rangle > 0$

Then $\bar{x}$ is a Pareto minimum for (VOP)

Proof. Let us assume that $\bar{x}$ is not a Pareto minimum. Hence the following system

$$f_j(x) \leq f_j(\bar{x}); \quad j = 1,\ldots,r$$
$$f_j(x) < f_j(\bar{x}), \quad \text{for some} \quad j$$
$$g_i(x) \leq 0, \quad i = 1,\ldots,m$$

has a solution $x \in \mathbb{R}^n$. Let us assume that $x \in \mathbb{R}^n$ is a solution of the above system. Then by convexity of the problem data we can conclude that

$$\langle \nabla f_j(\bar{x}), x - \bar{x}\rangle \leq 0, \quad j = 1,\ldots,r$$
$$\langle \nabla g_i(\bar{x}), x - \bar{x}\rangle \leq 0.$$

Now set $d = x - \bar{x}$. We now consider two different case. First let us assume that $\langle \nabla f_j(\bar{x}), d\rangle < 0$ for all $j = 1,\ldots,r$. Hence d solves the system

$$\langle \nabla f_j(\bar{x}), d\rangle < 0, \quad j = 1,\ldots r$$

$$\langle \nabla g_i(\bar{x}), d \rangle \le 0, \quad i = 1, \ldots, m.$$

Now by application of the Motzkin alternative theorem it is simple to show that the condition (i) in the hypothesis of the theorem is contradicted.

On the other hand assume that there exists $q \in \{1, \ldots, r\}$ such that $\langle \nabla f_q(\bar{x}), x - \bar{x} \rangle = 0$. Hence $d = x - \bar{x} \neq 0$ is a critical direction of the problem (VOP). From the convexity of f_j we have that for all $\lambda \in (0,1)$ and for all j,

$$f_j(\bar{x} + \lambda d) - f(\bar{x}) \le \lambda (f_j(x) - f_j(\bar{x})).$$

Hence we have $f_j(\bar{x} + \lambda d) - f_j(\bar{x}) \le 0$. Now by using Taylor's expansion we have for all $j = 1, \ldots, r$

$$\langle \nabla f_j(\bar{x}), d \rangle + \frac{\lambda}{2} (\langle d, \nabla^2 f_j(\bar{x}) d \rangle + \frac{o(\lambda^2)}{\lambda^2}. \tag{5.4}$$

We also have for all $i \in I(\bar{x})$,

$$\langle g_i(\bar{x}), d \rangle + \frac{\lambda}{2} (\langle d, \nabla^2 g_i(\bar{x}) d \rangle + \frac{o(\lambda^2)}{\lambda^2}. \tag{5.5}$$

Then by adding (5.4) and (5.5) we have that

$$\left\langle \sum_{j=1}^{r} \tau_j \nabla f_j(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla g_i(\bar{x}), d \right\rangle$$
$$+ \frac{\lambda}{2} \left(\left\langle d, \left(\sum_{j=1}^{r} \tau_j \nabla^2 f_j(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla^2 g_i(\bar{x}) \right) d \right\rangle + \frac{o(\lambda^2)}{\lambda^2} \right) \le 0.$$

Using (i) and noting that $\lambda > 0$ we have

$$\left\langle d, \left(\sum_{j=1}^{r} \tau_j \nabla^2 f_j(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla^2 g_i(\bar{x}) \right) d \right\rangle + \frac{o(\lambda^2)}{\lambda^2} \le 0.$$

As $\lambda \downarrow 0$ we have

$$\left\langle d, \left(\sum_{j=1}^{r} \tau_j \nabla^2 f_j(\bar{x}) + \sum_{i \in I(\bar{x})} \lambda_i \nabla^2 g_i(\bar{x}) \right) d \right\rangle \le 0.$$

This is clearly a contradiction to (ii) and hence the result. □

The next natural question is can we remove the convexity hypothesis from the above result. Such a result can be found in [36] (see [36, Theorem 3.5]) where there is no convexity assumption on the data. However the directions y satisfy some slightly modified condition. Further the approach due to Wang [36] uses a Fritz

John type first order condition along with a strict second order condition similar to the one given in the above theorem to reach the conclusion. Further it is important to observe that Wang [36] also characterizes a Pareto minimum.

5.6 Excursions in Infinite Dimensions

We would like to now take a brief tour of vector optimization in infinite dimensions. We will however be brief and we will not discuss each and every detail here. Our discussion in this section is largely based on the work of Durea et al. [11] and Durea and Dutta [10]. It is important to note the cone that induces the partial order can have an empty interior or in other words is non-solid. In such scenario there is no way we can consider weak Pareto minimum which is much for easy to handle from the mathematical point of view. For details of important space whose natural ordering cone or the positive cone has an empty interior see for example Jahn [26]. However it is not that all important spaces have the positive cone with empty interior. Note for example the space l^∞, the space BV of all functions of bounded variation on $\mathbb{R}$ or the space $C(\Omega)$ of all continuous real-valued functions on the compact Hausdorff space Ω have their positive cone or natural ordering cone with non-empty interior. On other hand many other important spaces like l^p, L^p, $1 \leq p < \infty$ have non-solid positive cones.

The question now is how can we handle Pareto minima with non-solid ordering cone. We will focus first on the approach due to Durea et al. [11].

Let us first put in the basic framework that we will be using in the infinite dimensional setting. Let X and Y Banach spaces over the real field $\mathbb{R}$. The symbols U_X and S_X denote the closed unit ball and the unit sphere in X, where X is a given Banach space. For any Banach space X the topological dual of X is denoted by X^*. For a positive ε and for an element $x \in X$, we denote the open ball of radius ε centered in x by $B(x,\varepsilon)$. As usual, for a set $C \subset X$, we denote by δ_C the indicator function of C ($\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ if $x \notin C$) and by d_C the distance function with respect to C, $d_C(x) = d(x,C) := \inf_{c\in C} \|x - c\|$ for every $x \in X$ (by convention, $d(x,\emptyset) = +\infty$). We will denote by K as the closed convex and pointed cone that induces the partial order on the space Y and in what follows we will assume that K has a non-empty interior. We shall also recall the notion of Pareto minimum with respect to K. For a non-empty set $A \subset Y$, a point $\overline{a} \in A$ is called Pareto minimum of A with respect to K if $(A - \overline{a}) \cap -K = \{\mathbf{0}\}$. We denote the set of Pareto minimum points of A w.r.t. K by $\mathrm{Min}(A \mid K)$. If $f : X \to Y$ is a vector-valued function and $S \subset X$ is a non-empty set, a point $\overline{x} \in S$ is said to be Pareto minimizer of f over S with respect to K if $f(\overline{x})$ is a Pareto minimum of $f(S)$ with respect to K.

We shall first see how Durea et al. [11] handles the convex case in the scenario of non-solid cones. We present here necessary optimality conditions for the minimization of a function $f : X \to Y$ over a closed set $S \subset X$. We derive a necessary condition for Pareto minimizers when the function f is K-convex and S is a closed convex subset of X. Let us recall the definition of a K-convex function:

For any $x, y \in X$ and $\lambda \in [0,1]$ it holds

$$\lambda f(y) + (1-\lambda) f(x) - f(\lambda y + (1-\lambda)x) \in K.$$

It is important to note that even if f is K-convex and the set S is a closed convex set, the image set $f(S)$ needs not to be convex. However, the set $f(S)+K$ is a convex set under the convexity hypothesis on f and S. On the other hand

$$\min(f(S)|K) = \min(f(S)+K|K). \tag{5.6}$$

This fact will indeed be a key argument in the following result due to Durea et al. [11].

Theorem 5.11. *Let us consider a K-convex function $f : X \to Y$ and let S be a closed convex subset of X. Assume that f is a continuously Frechet differentiable function. Further, assume that the set $f(S)$ has a non-empty interior. Let $\bar{x}$ be a Pareto minimizer of f over S with respect to the ordering cone K which has an empty interior. Then there exists $v \in K^* \setminus \{\mathbf{0}\}$ such that*

$$\mathbf{0} \in f'(\bar{x})^* v + N(S, \bar{x}), \tag{5.7}$$

where $f'(\bar{x})$ is the Frechet derivative of f at $\bar{x}$ and $f'(\bar{x})^*$ is the adjoint of the Frechet derivative of f at $\bar{x}$ and $N(S,\bar{x})$ denotes the normal cone to the closed convex set S at the point $\bar{x}$.

Proof. Since $\bar{x}$ is a Pareto minimizer of f over S we have using (5.6)

$$(f(S)+K) \cap (f(\bar{x}) - K) = \{f(\bar{x})\}.$$

Since the interior of $f(S)$ is non-empty the interior of $f(S)+K$ is also non-empty. Noting the fact that the Pareto minimum point $f(\bar{x})$ lies on the boundary of $f(S)+K$ we get

$$\operatorname{int}(f(S)+K) \cap (f(\bar{x}) - K) = \emptyset$$

taking into account the above expression.

Thus by applying a standard separation technique from convex analysis we conclude that there exists $v \in Y^*$ with $v \neq \mathbf{0}$ such that

$$v(z) \geq v(w), \quad \forall z \in f(S)+K, \quad \text{and} \quad \forall w \in (f(\bar{x}) - K). \tag{5.8}$$

Now for any given arbitrary $x \in S$ and $k \in K$ from (5.8) we have

$$v(f(x)+k) \geq v(f(\bar{x}) - k). \tag{5.9}$$

By setting $k = 0$ we see that $v(f(x)) \geq v(f(\bar{x}))$ for all $x \in S$. We will now show that $v \in K^*$. On the contrary assume that there exists $k \in K$ such that $v(k) < \mathbf{0}$. From

(5.9) we have

$$v(f(x)) \geq v(f(\bar{x})) - v(k), \quad \forall x \in S.$$

However, since K is a cone the right hand side of the above expression can be made arbitrarily large so as to exceed $v(f(x))$ for any given $x \in S$. This leads to a contradiction and thus $v \in K^*$. Thus we have proved that $\bar{x}$ is a minimum of the convex function $v(f(x))$ over S. Thus from the well known optimality condition in convex optimization we get

$$\mathbf{0} \in (v \circ f)'(\bar{x}) + N(S, \bar{x}).$$

The result now follows by applying the standard chain rule of differentiation. □

Remark 5.1. It is important to note that even in the convex case the above expression is only a necessary condition for the existence of a Pareto minimum and not a sufficient condition. Further, observe that the loss of interiority condition of the ordering cone had to be compensated by the interiority assumption on the image set $f(S)$.

We shall now focus on the nonsmooth case. We shall now introduce some tools from nonsmooth analysis in infinite dimensions which will be used in the sequel. We will concentrate on some concrete subdifferentials: The subdifferential of Mordukhovich (∂_M), which satisfies exact calculus rules on Asplund spaces and furthermore, the proximal subdifferential (∂_P), which satisfies fuzzy calculus rules on Hilbert spaces. Since this is an excursion into vector optimization in infinite dimension rather than a detailed study we have not explicitly provided the definitions of the subdifferential here. The details of Mordukhovich subdifferential is also provided in this volume in the chapter written by Boris Mordukhovich and [32] and [33]. For the proximal subdifferential in the setting of a Hilbert space see Clarke et al. [7]. The best calculus rules for the Mordukhovich subdifferential are in the setting of Asplund spaces which are special classes of Banach spaces over which convex functions are generically Frechet differentiable. In fact every reflexive Banach space is also an Asplund space. Now under the assumptions that X is an Asplund space we shall present here some calculus rules associated with Mordukhovich subdifferential as given [11]. If $f_1, f_2 : X \to \overline{\mathbb{R}}$, $\bar{x} \in \mathrm{Dom}\, f_1 \cap \mathrm{Dom}\, f_2$ and f_1 is Lipschitz around $\bar{x}$ and f_2 is l.s.c. around $\bar{x}$, then

$$\partial_M(f_1 + f_2)(\bar{x}) \subset \partial_M f_1(\bar{x}) + \partial_M f_2(\bar{x}).$$

A function $f : X \to Y$ is strictly Lipschitzian at $\bar{x}$ if it is locally Lipschitzian around this point and there exists a neighborhood V of the origin in X s.t. the sequence $(t_k^{-1}(f(x_k + t_k v) - f(x_k)))_{k \in \mathbb{N}}$ contains a norm convergent subsequence whenever $v \in V, x_k \to \bar{x}$, $t_k \downarrow \mathbf{0}$. It is clear that this notion reduces to local Lipschitz continuity if Y is finite dimensional. For more details see [32]. Let us now present the chain rule associated with the Mordukhovich subdifferential. If X and Y are Asplund spaces, $f : X \to Y$ is strictly Lipschitzian at $\bar{x}$ and $\varphi : Y \to \mathbb{R}$ is Lipschitz around $f(\bar{x})$, then

$$\partial_M(\varphi \circ f)(\overline{x}) \subset \bigcup_{y^* \in \partial_M \varphi(f(\overline{x}))} \partial_M(y^* \circ f)(\overline{x}).$$

For the fuzzy sum rules the following notations were used in [11]:

- $u \xrightarrow{f} x$ means that $u \to x$ and $f(u) \to f(x)$; note that if f is continuous, then $u \xrightarrow{f} x$ is equivalent with $u \to x$.
- $x^* \in \|\cdot\|^* - \limsup\limits_{u \to x} \partial f(u)$ means that for every $\varepsilon > \mathbf{0}$ there exist x_ε and x^*_ε such that $x^*_\varepsilon \in \partial f(x_\varepsilon)$ and $\|x_\varepsilon - x\| < \varepsilon$, $\|x^*_\varepsilon - x^*\| < \varepsilon$; the notation $x^* \in \|\cdot\|^* - \limsup\limits_{u \xrightarrow{f} x} \partial f(u)$ has a similar interpretation and it is equivalent with $x^* \in \|\cdot\|^* - \limsup\limits_{u \to x} \partial f(u)$ provided that f is continuous.

As we will see that in the setting of a non-solid ordering cone it is indeed quiet difficult to handle a Pareto minimum since the multipliers in the Lagrange multiplier rules or KKT conditions turn out to be trivially zero. So the question how can we modify the notion of Pareto minimum in order to develop necessary optimality conditions with non-trivial multipliers. This lead Durea et al. [11] to introduce the notion of (ε, e)-Pareto minimum. Let us consider a fixed element $e \in K$ with $\|e\| = 1$. For a positive ε, we say that $\overline{a} \in A$ is an (ε, e)-Pareto minimum of A with respect to K if $(A - \overline{a}) \cap (-K - \varepsilon e) = \emptyset$. The set of all these minima is denoted by $(\varepsilon, e) - \mathrm{Min}(A \mid K)$. As above, for a vector-valued function $f : X \to Y$ and a non-empty set $S \subset X$, a point $\overline{x} \in S$ is said to be (ε, e)-Pareto minimizer of f over S with respect to K if $f(\overline{x})$ is an (ε, e)-Pareto minimum of $f(S)$ with respect to K.

It is important to observe that the notion of (ε, e)-Pareto optima that we have defined here is a slightly different version that the standard one found in the literature, i.e., $(A - \overline{a}) \cap ((-K \setminus \{\mathbf{0}\}) - \varepsilon e) = \emptyset$. For any $\overline{a} \in A$ with $(A - \overline{a}) \cap (-K - \varepsilon e) = \emptyset$ it follows that $\overline{a}$ is an (ε, e)-Pareto minimum in the standard sense. The reverse is not true. Notice that if a point $\overline{a}$ is an (ε, e)-Pareto minimum for A w.r.t. K in the standard sense, then it is an $(\varepsilon + \delta, e)$-Pareto minimum for A w.r.t. K in our sense for every positive δ (taking into account that K is pointed).

However, we will use our concept of (ε, e)-Pareto minimizers of f over S with respect to K defined above in order to get nontrivial multipliers $y^* \neq \mathbf{0}$ using certain properties of the subdifferential of the distance function (see (5.10)).

Observe that an interesting part of this definition is the following. Viewing it in a slightly informal manner it is interesting to observe that one can in fact want to refer as (ε, e)-Pareto minimum to those points which under very small perturbation will leave the feasible objective set. Points lying very near to the efficient frontier will exhibit such behavior under small perturbations and thus from the practical point of view we are indeed talking about solutions that are very close to the Pareto frontier. Thus this notion is more practically oriented notion of approximate minimum for a vector optimization problem. The next proposition from [11] justifies the concept of (ε, e)-solution.

Proposition 5.1. *The following relation holds:*

$$Min(A \mid K) = \bigcap_{e \in K \cap S_Y} \bigcap_{\varepsilon > 0} (\varepsilon, e) - Min(A \mid K).$$

Proof. It is clear that for every positive ε and for every $e \in K \cap S_Y$ the pointedness of the cone K implies $-K - \varepsilon e \subset -K \setminus \{\mathbf{0}\}$. We deduce:

$$\mathrm{Min}(A \mid K) \subset \bigcap_{e \in K \cap S_Y} \bigcap_{\varepsilon > 0} (\varepsilon, e) - \mathrm{Min}(A \mid K).$$

For the converse inclusion, let us take $\bar{y} \in \bigcap_{e \in K \cap S_Y} \bigcap_{\varepsilon > 0} (\varepsilon, e) - \mathrm{Min}(A \mid K)$ and suppose that there exists $y \in A$ s.t. $y - \bar{y} \in -K \setminus \{\mathbf{0}\}$. Then for an $\varepsilon > 0$ small enough,

$$(y - \bar{y}) - \varepsilon \left\| y - \bar{y} \right\|^{-1} (y - \bar{y}) \in -K \setminus \{\mathbf{0}\}.$$

Consequently,

$$y - \bar{y} \in -K - \varepsilon \left\| y - \bar{y} \right\|^{-1} (\bar{y} - y),$$

whence $\bar{y} \notin (\varepsilon, \left\| y - \bar{y} \right\|^{-1} (\bar{y} - y)) - \mathrm{Min}(A \mid K)$. Since we arrived at contradiction, the proof is complete. □

Now it is important to know that in this set up can we scalarize the vector problem, that is, can we find a function which when composed with the vector function results in a scalar optimization problem. In [11] the oriented distance function introduced by Hiriart-Urruty [23] is used. In general, for a non-empty set $A \subset Y$, $A \neq Y$, the *oriented distance function* $\Delta_A : Y \to \mathbb{R}$ is given as $\Delta_A(y) = d_A(y) - d_{Y \setminus A}(y)$. We list below some important properties of the oriented distance.

Proposition 5.2 ([37]).

(a) Δ_A *is Lipschitzian of rank* 1.
(b) If A *is convex, then* Δ_A *is convex and if* A *is a cone, then* Δ_A *is positively homogeneous.*
(c) If A *is a closed convex cone and* $y_1, y_2 \in Y$ *with* $y_1 - y_2 \in A$, *then* $\Delta_A(y_1) \leq \Delta_A(y_2)$.

From the above proposition it is clear that the functional Δ_{-K} is a convex, positively homogeneous and a Lipschitzian function with Lipschitz modulus 1. Note that, the emptiness of the interior of K implies that the closure of $Y \setminus (-K)$ is Y itself, so the second distance function in the expression of Δ_{-K} reduces to 0 Hence, in fact $\Delta_{-K} = d_{-K}$.

Further it is well known that for a convex closed subset A of Y the normal cone at a point $\bar{a} \in A$ is given as

$$N_A(\bar{a}) = \{y^* \in Y^* \mid y^*(a - \bar{a}) \leq 0, \ \forall a \in A\}.$$

For the convex continuous function d_A, the classical Fenchel subdifferential is given by the following formula (see, for example, [4]):

$$\partial d_A(y) = \begin{cases} S_{X^*} \cap N_{A_y}(y), \text{ if } y \notin A \\ U_{X^*} \cap N_A(y), \text{ if } y \in A, \end{cases} \tag{5.10}$$

where $A_y := A + d_A(y)U_Y$.

Furthermore, for the special case of the convex functional Δ_{-K} it holds for every $y \in Y$, $\partial \Delta_{-K}(y) \subset K^*$. Indeed, for $y^* \in \partial \Delta_{-K}(y)$ it holds

$$y^*(z-y) \leq \Delta_{-K}(z) - \Delta_{-K}(y), \quad \forall z \in Y. \tag{5.11}$$

From Proposition 5.2, (iii), it follows $\Delta_{-K}(u+y) \leq \Delta_{-K}(y)$ for every $u \in -K$ and whence $y^*(u) \leq 0$ with (5.11). This implies that for every $y \in Y$

$$\partial \Delta_{-K}(y) \subset K^*$$

holds.

From the point of view of deriving the Lagrange multiplier rules it will be important to see if $\partial \Delta_{-K-\varepsilon e}(y) \subset K^*$. This fact was proved in [11] and we present their proof through the following remark.

Remark 5.2. In this section we will always consider $\operatorname{int} K = \emptyset$. Further,the interior of $-K - \varepsilon e$ is empty too (being a subset of $-K$), whence $\Delta_{-K-\varepsilon e}(y) = d_{-K-\varepsilon e}(y)$. In order to show $\partial \Delta_{-K-\varepsilon e}(y) \subset K^*$ for every $y \in Y$ we take $y^* \in \partial \Delta_{-K-\varepsilon e}(y) = \partial d_{-K-\varepsilon e}(y)$ for a fixed $y \in Y$. Then for every $k \in K$ one has

$$\begin{aligned} y^*(-k-\varepsilon e - y) &\leq d_{-K-\varepsilon e}(-k-\varepsilon e) - d_{-K-\varepsilon e}(y) \\ &= -d_{-K-\varepsilon e}(y) \leq 0. \end{aligned}$$

This yields $y^*(k) \geq -\varepsilon y^*(e) - y^*(y)$. Because $y \in Y$ (the reference point) is the same for every $k \in K$ and $y^* \in \partial d_{-K-\varepsilon e}(y)$ in this relation we obtain $y^* \in K^*$: Indeed, if there would exist $k \in K$ s.t. $y^*(k) < 0$, then $y^*(nk) \to -\infty$ as $n \to \infty$ and we get a contradiction with above inequality since, obviously, $nk \in K$ for every natural n and $-\varepsilon y^*(e) - y^*(y)$ is a constant once we have chosen y^* from $\partial d_{-K-\varepsilon e}(y)$. So we get for every $y \in Y$

$$\partial \Delta_{-K-\varepsilon e}(y) \subset K^*.$$

The basic result linking the concept of (ε, e)-Pareto minima with the scalarizing functional is the following.

Theorem 5.12 ([11]). *Assume $\varepsilon > 0$, $e \in K$, $||e|| = 1$. If a point $\bar{y} \in A \subset Y$ is an (ε, e)-Pareto minimum of A with respect to K, then $\bar{y}$ is an ε-solution of the problem*

$$\min_{y \in A} \Delta_{-K-\varepsilon e}(y - \bar{y}).$$

Proof. The proof is based on the obvious inequality $d_{-K-\varepsilon e}(\mathbf{0}) \leq \varepsilon$ since for every $y \in A$ we have

$$d_{-K-\varepsilon e}(\mathbf{0}) \leq \varepsilon < d_{-K-\varepsilon e}(y-\bar{y}) + \varepsilon,$$

whence $\bar{y}$ is an ε-solution over A for the scalar problem $\min_{y\in A} \Delta_{-K-\varepsilon e}(y-\bar{y})$. □

We shall now state our main optimality contains for (ε,e)-Pareto minimum. However in order to do so we need to use the well known Ekeland variational principle which we will state below. We present it in the form given in Guler [22]

Theorem 5.13 (Ekeland's Variational Principle). *Let (X,d) be a given metric space with metric d. Let $f : X \to \overline{\mathbb{R}}$ be a lower-semicontinuous function which is bounded below on X. For $\varepsilon > 0$, let x be such that*

$$f(x) \leq \inf f(x) + \varepsilon.$$

Then for any $\lambda > 0$ there exists x_ε such that

(a) $f(x_\varepsilon) \leq f(x)$
(b) $d(x,x_\varepsilon) \leq \lambda$
(c) $f(x_\varepsilon) < f(z) + \frac{\varepsilon}{\lambda} d(z,x_\varepsilon), \quad \forall z \in M, z \neq x_\varepsilon$

Theorem 5.14 ([11]). *Let X,Y be Asplund spaces, K be a closed convex pointed cone in Y with empty interior, S be a closed subset of X and $f : X \to Y$ be a strictly Lipschitzian function on S. Assume $\varepsilon > 0$ and $e \in K$, $||e|| = 1$. If $\bar{x}$ is an (ε,e)-Pareto minimizer of f over S with respect to K, then there exist $x \in B(\bar{x},\sqrt{\varepsilon}) \cap S$ and $y^* \in S_{Y^*} \cap K^*$ s.t.*

$$\mathbf{0} \in \partial_M(y^* \circ f)(x) + \sqrt{\varepsilon} U_{X^*} + N_{\partial_M}(S,x).$$

Proof. We consider the function $\varphi : X \to Y$ given by $\varphi(x) = f(x) - f(\bar{x})$. Following Theorem 5.12, $\bar{x}$ is an ε-minimum point over S for the functional $z : X \to \mathbb{R}$ defined by $z(x) = (\Delta_{-K-\varepsilon e} \circ \varphi)(x)$. Whence, from Ekeland's variational principle applied for z on S (as a complete metric space), we get an element $x \in B(\bar{x},\sqrt{\varepsilon}) \cap S$ which is a minimum point on S for the perturbed function $z(\cdot) + \sqrt{\varepsilon}\,\|\cdot - x\|$. Applying the exact calculus rules of Mordukhovich subdifferential, we have

$$\begin{aligned}
\mathbf{0} &\in \partial_M \left(z(\cdot) + \sqrt{\varepsilon}\,\|\cdot - x\| + I_S(\cdot)\right)(x) \\
&\subset \partial_M(\Delta_{-K-\varepsilon e} \circ \varphi)(x) + \sqrt{\varepsilon} U_{X^*} + \partial_M I_S(x) \\
&\subset \bigcup\nolimits_{y^* \in \partial_M \Delta_{-K-\varepsilon e}(f(x)-f(\bar{x}))} \partial_M(y^* \circ f)(x) + \sqrt{\varepsilon} U_{X^*} + N_{\partial_M}(S,x).
\end{aligned}$$

Therefore, we get that there exists $y^* \in \partial_M \Delta_{-K-\varepsilon e}(f(x) - f(\bar{x}))$ s.t. $\mathbf{0} \in \partial_M(y^* \circ f)(x) + \sqrt{\varepsilon} U_{X^*} + N_{\partial_M}(S,x_0)$. Since $\Delta_{-K-\varepsilon e} = d_{-K-\varepsilon e}$ is a convex function (cf. Proposition 5.2), its subdifferential in the sense of Mordukhovich coincides with the subdifferential in the sense of convex analysis. Taking into account the formula for the subdifferential of the distance function as given above, and the fact that $f(x) - f(\bar{x}) \notin -K - \varepsilon e$ (from the definition of (ε, e)-minimum) we get together with Remark 5.2 the assertion $y^* \in S_{Y^*} \cap K^*$. □

The next result will be in the setting of a Hilbert space for the unconstrained case

Theorem 5.15 ([11]). *Let X, Y be Hilbert spaces, K be a closed convex pointed cone in Y with empty interior and $f : X \to Y$ be a locally Lipschitzian function. Assume $\varepsilon > 0$ and $e \in K$, $||e|| = 1$. If $\bar{x} \in X$ is a (ε, e)-Pareto minimizer of f over X with respect to K, then there exist $x \in B(\bar{x}, \frac{2}{3}\sqrt{\varepsilon})$ and $y^* \in S_{Y^*} \cap K^*$ s.t.*

$$\mathbf{0} \in \partial_P (y^* \circ f)(x) + \frac{5}{3}\sqrt{\varepsilon} U_{X^*}.$$

Proof. As above, $\bar{x}$ is an unconstrained ε-minimum point for the functional $z : X \to \mathbb{R}$ defined by $z(x) = (\Delta_{-K-\varepsilon e} \circ f)(x)$. Once again, from Ekeland's variational principle we get an element $x^1 \in B(\bar{x}, \sqrt{\varepsilon})$ which is a minimum point on X for the perturbed function $z(\cdot) + \sqrt{\varepsilon}\left\|\cdot - x^1\right\|$. Whence, applying the fuzzy calculus rules of the proximal subdifferential, we can find $x^2 \in B(x^1, 3^{-1}\sqrt{\varepsilon})$, $x^3 \in B(x^1, 3^{-1}\sqrt{\varepsilon})$, s.t.

$$\mathbf{0} \in \partial_P z(x^2) + \partial_P \left(\sqrt{\varepsilon}\left\|\cdot - x^1\right\|\right)(x^3) + 3^{-1}\sqrt{\varepsilon} U_{X^*}.$$

Since z is a composite function we can apply the fuzzy calculus for its subdifferential to get $x^4 \in B(x^2, 3^{-1}\sqrt{\varepsilon})$, and $y^* \in \partial_P \Delta_{-K-\varepsilon e}(f(x^4) - f(\bar{x}))$ with

$$\mathbf{0} \in \partial_P (y^* \circ f)(x^4) + \partial_P \left(\sqrt{\varepsilon}\left\|\cdot - x^1\right\|\right)(x^3) + \frac{2}{3}\sqrt{\varepsilon} U_{X^*}.$$

Since $\sqrt{\varepsilon}||\cdot - x^1||$ is a convex function the proximal subdifferential of this function coincides with the usual subdifferential of a convex function. Further, we also know that

$$\partial \left(\sqrt{\varepsilon}\left\|\cdot - x^1\right\|\right)(x^3) \subset \sqrt{\varepsilon} U_{X^*},$$

where ∂ denotes the subdifferential of a convex function. Hence we conclude that

$$\mathbf{0} \in \partial_P (y^* \circ f)(x^4) + \frac{5}{3}\sqrt{\varepsilon} U_{X^*}.$$

Of course, taking into account the above estimations and the fact that $f(x^4) - f(\bar{x}) \notin -K - \varepsilon e$ one has with (5.10) and Remark 5.2:

$$\left\|x^4 - \bar{x}\right\| \leq \frac{2}{3}\sqrt{\varepsilon} \text{ and } y^* \in S_{Y^*} \cap K^*.$$

The proof is complete taking $x = x^4$. □

As we have seen above it has been difficult to handle Pareto minimum points in the non-convex in the scalarization framework we used in the above discussion in this section. One might wonder why we had to consider the scalarization framework involving the oriented distance function. Unfortunately in the infinite dimensional scenario for the non-convex case the more robust scalarization due to Gerth and Widner [16] which is based on a non-convex separation result but needs the interior of the ordering cone to be non-empty. However our cone was having an interior and thus scalarization technique of Gerth and Widner [16] could not be used. Further the scalarization technique of Gerth and Widner [16] is best suited in dealing with a weak Pareto minimum. In Durea and Dutta [10] developed an approach through which Pareto minimum could be treated in the setting of a ordering cone with empty interior. They develop a notion of the dialating cone which has a non-empty interior and the pareto minimum point is then viewed as a weak Pareto minimum with respect to the dialating cone. Then one can apply the scalarization technique of Gerth and Widner [16] in order to develop optimality conditions.

Let us begin by defining the notion of a weak Pareto minimum in the infinite dimensional scenario. If $\operatorname{int} K \neq \emptyset$, then a point $\bar{y} \in A$ is called weak Pareto minimum point of A with respect to K if $(A - \bar{y}) \cap -\operatorname{int} K = \emptyset$. The set of all weak Pareto minimum points of A is denoted as $\mathrm{WMin}(A|K)$. The approach in [10] is based on the notion of a proper Pareto minimum in the sense of Henig. The notion of a Henig proper Pareto minimum is given as follows. A vector $\bar{y} \in A$ is a proper Pareto minimum point (in the Henig's sense) or a Henig proper Pareto minimum of A with respect to K if there exits a closed convex and pointed cone $Q \subset Y$ with non-empty interior such that $K \setminus \{0\} \subset \operatorname{int} Q$ and $\bar{y} \in \mathrm{Min}(A \mid Q)$. The set of Henig proper efficient points will be denoted by $\mathrm{PMin}(A|K)$. In the sequel we will need the notion of a quasi-interior of the cone K which is denoted as K^* and is

$$K^{\sharp} := \{y^* \in Y^* \mid y^*(y) > 0, \forall y \in K \setminus \{0\}\}'$$

where Y^* is the topological dual of Y. It is clear that if K and Q are two cones with $K \setminus \{0\} \subset \operatorname{int} Q$ then $Q^* \subset K^*$ and $Q^* \setminus \{0\} \subset K^{\sharp}$.

The notion of a base of cone plays an important role in the sequel. The optimality conditions that we will develop will depend on this notion. A convex set B is said to be a base for the cone K if $0 \notin \operatorname{cl} B$ and $K = coneB$, where cl denotes the topological closure and $coneB := [0, \infty)B$ is the cone generated by B. A cone which admits a base is called based. The results that follow will be under the assumption that the ordering cone admits a base.

We shall now study the notion of dilating cones which will play a central role in our study. We will show that these cones will have a non-empty interior. Thus these cones will allow us to view the Pareto minimum as a weak Pareto minimum when the original cone has an empty interior. We will first present a result which will lay the theoretical foundation of constructing these cones.

Lemma 5.1 ([21]). *Let $K \subset Y$ be a closed convex cone with a base B and take $\delta = d(0,B) > 0$. For $\varepsilon \in (0,\delta)$, consider $B_\varepsilon = \{y \in Y \mid d(y,B) \leq \varepsilon\}$ and $K_\varepsilon = [0,\infty)B_\varepsilon$, the cone generated by B_ε. Then*

(a) K_ε is a closed convex cone for every $\varepsilon \in (0,\delta)$.
(b) If $0 < \gamma < \varepsilon < \delta$, $K \setminus \{0\} \subset K_\gamma \setminus \{0\} \subset \operatorname{int} K_\varepsilon$.
(c) $K = \cap_{\varepsilon \in (0,\delta)} K_\varepsilon = \cap_{n \in \mathbb{N}} K_{\varepsilon_n}$ where $(\varepsilon_n) \subset (0,\delta)$ converges to 0.

The cone K_ε constructed in the above result is termed as a *dilating cone* or a *Henig dilating cone*. It is important to note that the cone K_ε indeed has a non-empty interior. This is due to the fact that B_ε has a non-empty interior. In fact the set $\{y \in Y : d(y,B) < \varepsilon\}$ is nonempty since it contains all sets $B_{\varepsilon'}$ with $0 < \varepsilon' < \varepsilon$. Further the since $y \mapsto d(y,B)$ is Lipschitz and hence is continuous, it is simple to show that the set $\{y \in Y : d(y,B) < \varepsilon\}$ is an open set. Moreover since B is convex we have $y \mapsto d(y,B)$ is a convex function and hence

$$\operatorname{int} B_\varepsilon = \{y \in Y : d(y,B) < \varepsilon\}.$$

As we shall see, the advantage of considering this construction is that we can provide the exact form of the elements in the dual cone of such a dilating cone w.r.t. the elements in the dual cone of the original ordering cone.

We will now introduce the notion of *asymptotically compact* subset of Y in the general framework when Y is a locally convex space endowed with a topology τ compatible with the duality system (Y, Y^*). A subset A of Y is called τ-asymptotically compact (τ-a.c. for short) if there exists a neighborhood U of $\mathbf{0}$ in (Y,τ) s.t. $U \cap [0,1]A$ is relatively compact. In the case when τ is the strong topology, then we shall simply use the term "asymptotically compact." If the topology τ in question is the weak topology we call it "asymptotically weakly compact." This notion was studied in detail by Zălinescu in [35] who proved a powerful non-convex closedness criterion for the difference of two sets.

Let us now introduce the various notions from nonsmooth analysis that will play a pivotal role in our analysis. The main tools are approximate subdifferential and approximate normal cone due to Ioffe [24, 25]. However we will also discuss a stronger notion of Lipschitz continuity which valid in infinite dimensional spaces. This is called strongly compactly Lipschitz property which is defined for a vector-valued map. Though the definition of the approximate subdifferential can be given in a general setting but we will only define them for the case of locally Lipschitz scalar-valued functions since this is what our main concern here. We however mention how the approximate subdifferential can be defined for a lower-semicontinuous function. Here our presentation of the material follows along the lines of Durea and Dutta [10].

Let X be a Banach space and $f : X \to \mathbb{R}$ be a locally Lipschitz function; then its lower Dini directional derivative at $\bar{x} \in X$ in the direction $h \in X$ is

$$f^-(\bar{x},h) := \liminf_{t \downarrow 0} \frac{f(\bar{x}+th) - f(\bar{x})}{t}.$$

Then the Dini lower subdifferential of f at $\bar{x}$ is given as

$$\partial^- f(\bar{x}) := \{x^* \in X^* \mid f^-(\bar{x},h) \geq x^*(h), \forall h \in X\}.$$

Let $\mathscr{L}$ be a closed subspace of X. One sets:

$$\partial^-_{\mathscr{L}} f(\bar{x}) := \{x^* \in X^* \mid f^-(\bar{x},h) \geq x^*(h), \forall h \in \mathscr{L}\}$$

If $\mathscr{F}$ denotes the collection of finite dimensional subspaces of X, then the approximate subdifferential $\partial_a f(\bar{x})$ of f at $\bar{x}$ is

$$\partial_a f(\bar{x}) := \bigcap_{\mathscr{L} \in \mathscr{F}} \limsup_{u \to \bar{x}} \partial^-_{\mathscr{L}} f(u).$$

The normal cone to a closed set $S \subset X$ at a point $\bar{x} \in S$ is given as

$$N_a(S,\bar{x}) := \bigcup_{t \geq 0} t \partial_a d_S(\bar{x}).$$

If $f : X \to \overline{\mathbb{R}}$ is a lower semicontinuous function, then the *approximate subdifferential* $\partial_a f(\bar{x})$ of f at $\bar{x} \in \operatorname{Dom} f$ is given as

$$\partial_a f(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N_a(\operatorname{epi} f, (\bar{x}, f(\bar{x})))\}.$$

One can show that for a closed set $S \subset X$ and for $\bar{x} \in S$, $N_a(S,\bar{x}) = \partial_a \delta_S(\bar{x})$, where δ_S represents the indicator function of S.

Since we would be dealing with set-valued objective and constraint functions it is important to state the differentiability notion that we would like to use for set-valued maps. The differentiability notion for set-valued maps that we use in this paper is the *approximate coderivative* of Ioffe. This is given as follows. The *approximate coderivative* of F at a point $(\bar{x},\bar{y}) \in \operatorname{Gr} F$ is given as the set-valued map $D^*_a F(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$

$$D^*_a F(\bar{x},\bar{y})(y^*) := \{x^* \in X^* \mid (x^*, -y^*) \in N_a(\operatorname{Gr} F, (\bar{x},\bar{y}))\}.$$

When F is single-valued then we denote the coderivative simply as $D^*_a F(\bar{x})(y^*)$.

In order to prove the necessary optimality conditions we need the following nonsmooth calculus rules.

- Let X be a Banach space. If $\bar{x}$ is a local minimum for a lower semicontinuous function $f : X \to \overline{\mathbb{R}}$, then $0 \in \partial_a f(\bar{x})$.
- Let X be a Banach space and let $f : X \to \mathbb{R}$ be a locally Lipschitz function, $g : X \to \overline{\mathbb{R}}$ be a proper lower-semicontinuous function. Then for every $x \in \operatorname{Dom} g$ one has $\partial_a (f+g)(x) \subset \partial_a f(x) + \partial_a g(x)$ (see [20, Lemma 2.5]).

Further for the development of necessary optimality conditions for the problem $(\mathscr{V}\mathscr{P})$ one also needs calculus rules for the composition of two functions. Consider a locally Lipschitz function $F : X \to Y$ and let $g : Y \to \mathbb{R}$ be locally Lipschitz. Then the function $f = g \circ F$ is a locally Lipschitz and using Theorem 7.5 in Ioffe [25], it was shown in Dutta and Tammer [20, Lemma 2.6] that

$$\partial_a f(x) \subset \bigcup_{y^* \in \partial g_a(F(x))} D_a^* F(x)(y^*).$$

Of course it is important to know whether one can estimate the approximate subdifferential of the composite function f in terms of the approximate subdifferential. This is possible when the function F is assumed to be *strongly compactly Lipschitzian*. A function $F : X \to Y$ is said to be *strongly compactly Lipschitzian* at $x_0 \in X$, if there exist a multifunction $R : X \rightrightarrows \text{Comp}(Y)$, where $\text{Comp}(Y)$ denotes the set of all norm compact subsets of Y and a function $r : X \times X \to \mathbb{R}_+$ satisfying

(1) $\lim_{x \to x_0, v \to 0} r(x,v) = 0$
(2) There exists $\alpha > 0$ such that

$$\frac{F(x+tv) - F(x)}{t} \in R(v) + ||v|| r(x,v) \mathbb{B}_Y,$$

for all $x \in x_0 + \alpha \mathbb{B}_X$, $v \in \alpha \mathbb{B}_X$, and $t \in (0, \alpha)$, where $\mathbb{B}_X$ and $\mathbb{B}_Y$ denote the unit ball in the spaces X and Y respectively

(3) $R(0) = \{0\}$ and R is upper-semicontinuous as a set-valued map

Under the assumption that F is strongly compactly Lipschitzian at $x \in X$ the approximate subdifferential of the composite function f is given as

$$\partial_a f(x) \subset \bigcup_{y^* \in \partial g_a(F(x))} \partial_a \langle y^*, F \rangle (x).$$

For a proof of this fact see [27]. The above result is due to that fact that when F is strongly compactly Lipschitzian at $\bar{x}$ then it was shown in [27] that

$$D_a^* F(\bar{x})(y^*) = \partial_a \langle y^*, F \rangle (\bar{x}), \quad \forall y^* \in Y^*. \tag{5.12}$$

We shall now present few results from [10] which are pivotal in deriving the necessary optimality conditions for problem (VOP) in the sequel. The first result shows under what condition a vector which is a Pareto minimum with respect to the ordering cone K, with empty interior is also a Pareto minimum with respect to the associated dialating cone. We shall not present the proof which can be found in [10].

Theorem 5.16. *Let $A \subset Y$ and $\bar{y} \in \text{Min}(A \mid K)$. Suppose that* $\text{cone}(A - \bar{y})$ *is (weakly) closed and K admits a (weakly) closed base B. If* $\text{cone}(A - \bar{y})$ *or B is (weakly) a.c. then there exists $\varepsilon \in (0, d(0,B))$ s.t. $\bar{y} \in \text{Min}(A \mid K_\varepsilon)$.*

A set which is τ-compact is automatically τ-a.c. Hence the above result, Theorem 5.16, implies the following corollary.

Corollary 5.1. *Let $A \subset Y$ and $\overline{y} \in \operatorname{Min}(A \mid K)$. Suppose that $\operatorname{cone}(A - \overline{y})$ is (weakly) closed and K admits a (weakly) compact base B. Then there exists $\varepsilon \in (0, d(0,B))$ s.t. $\overline{y} \in \operatorname{Min}(A \mid K_\varepsilon)$.*

In Durea and Dutta [10] an explicit computation is made of he dual cone of the dialating cone associated with ordering cone K. This is pivotal for characterizing the Lagrangian multipliers in the necessary optimality conditions which we present below.

Lemma 5.2. *Let $K \subset Y$ be a closed convex cone with a base B. For every $\varepsilon \in (0, d(0,B))$,*

$$K_\varepsilon^* = \{y^* \in Y^* \mid \inf_{b \in B} y^*(b) \geq \varepsilon \|y^*\|\}.$$

In particular, $K_\varepsilon^* \subset K^*$ and $K_\varepsilon^* \setminus \{0\} \subset K^\sharp$.

Proof. Let $\varepsilon \in (0, d(0,B))$, $b \in B$ and $y^* \in K_\varepsilon^*$. Since $D(b,\varepsilon) \subset B_\varepsilon$, for every $u \in D(0,1)$, $y^*(b + \varepsilon u) \geq 0$. Therefore,

$$y^*(b) + \varepsilon \inf_{u \in D(0,1)} y^*(u) \geq 0$$

whence $y^*(b) \geq \varepsilon \|y^*\|$. Since b was arbitrarily chosen in B, we obtain the first inclusion. For the converse inclusion, let us consider $y^* \in Y^*$ s.t. $\inf_{b \in B} y^*(b) \geq \varepsilon \|y^*\|$ and $u \in B_\varepsilon$. It is enough to prove that $y^*(u) \geq 0$. For every $n \in \mathbb{N} \setminus \{0\}$, there exists $u_n \in B$ s.t. $\|u_n - u\| \leq \varepsilon + n^{-1}$. One has

$$\begin{aligned} y^*(u) &= y^*(u - u_n) + y^*(u_n) \\ &\geq -\|y^*\| \, \|u - u_n\| + \varepsilon \|y^*\| \\ &\geq -\|y^*\| \, (\varepsilon + n^{-1}) + \varepsilon \|y^*\| \\ &= -n^{-1} \|y^*\| \end{aligned}$$

for every $n \in \mathbb{N} \setminus \{0\}$. We conclude that $y^*(u) \geq 0$. □

As we have mentioned earlier that the assumption of the interiority of the ordering cone allows us to have a powerful scalarization result. We present here the result as given in [12].

Lemma 5.3. *Let $Q \subset Y$ be a closed convex cone with nonempty interior. Then for every $e \in \operatorname{int} Q$ the functional $s_e : Y \to \mathbb{R}$ given by*

$$s_e(y) = \inf\{\lambda \in \mathbb{R} \mid \lambda e \in y + Q\} \tag{5.13}$$

is continuous, sublinear, strictly-$\operatorname{int} Q$-monotone and the following relations hold:

(a) $\partial s_e(0) = \{v^* \in Q^* \mid v^*(e) = 1\}$.
(b) For every $u \in Y$, $\partial s_e(u) \neq \emptyset$ and $\partial s_e(u) = \{v^* \in Q^* \mid v^*(e) = 1, v^*(u) = s_e(u)\}$.

In particular, for every $u \in Y$ and $v^* \in \partial s_e(u)$, $\|e\|^{-1} \leq \|v^*\| \leq d(e, \mathrm{bd}\, Q)^{-1}$, where bd$Q$ *denotes the topological boundary of* Q*. Moreover,* s_e *is* $d(e, \mathrm{bd}(Q))^{-1}$*–Lipschitz.*

Using the scalarization result mentioned above we shall now state the necessary optimality conditions derived in Durea and Dutta [10].

In the sequel Y is a general Banach space and K is a closed convex and pointed cone unless otherwise stated. First, we present necessary optimality conditions for a point to be Pareto minimum of a closed set.

Theorem 5.17 ([10]). *Let $A \subset Y$ be a closed set and $\bar{y} \in \mathrm{Min}(A \mid K)$. Suppose that* cone$(A - \bar{y})$ *is (weakly) closed and K admits a (weakly) closed base B. If* cone$(A - \bar{y})$ *or B is (weakly) a.c., then there exists $\varepsilon > 0$ such that for every $e \in K \setminus \{0\}$ there exists $y^* \in Y^*$ with $y^*(e) = 1$, $\inf_{b \in B} y^*(b) \geq \varepsilon \|y^*\|$ and $-y^* \in N_a(A, \bar{y})$.*

Proof. In our conditions, there exists a positive ε s.t. $\bar{y} \in \mathrm{Min}(A \mid K_\varepsilon) \subset \mathrm{WMin}(A \mid K_\varepsilon)$ (cf. Theorem 5.16). We can apply Lemma 5.3 for the cone $Q := K_\varepsilon$ and the element $e \in K \setminus \{0\} \subset \mathrm{int}\, K_\varepsilon$. Then $\bar{y}$ is a minimum point over A for the functional $s_e(\cdot - \bar{y})$ and then, by the infinite penalization method, $\bar{y}$ is a minimum point without constraints for $s_e(\cdot - \bar{y}) + \delta_A$. Therefore,

$$\mathbf{0} \in \partial_a (s_e(\cdot - \bar{y}) + \delta_A)(\bar{y})$$

and since the first function is locally Lipschitz and the second one is lower-semicontinuous, we have

$$\mathbf{0} \in \partial_a (s_e(\cdot - \bar{y}))(\bar{y}) + \partial_a \delta_A(\bar{y}).$$

Moreover, the functional $s_e(\cdot - \bar{y})$ is sublinear and hence by using (i) in Lemmas 5.3 and 5.2 we obtain

$$\begin{aligned} \partial_a (s_e(\cdot - \bar{y}))(\bar{y}) &= \partial s_e(0) = \{y^* \in K_\varepsilon^* \mid y^*(e) = 1\} \\ &= \{y^* \in Y^* \mid \inf_{b \in B} y^*(b) \geq \varepsilon \|y^*\|, y^*(e) = 1\}. \end{aligned}$$

On the other hand, $\partial_a \delta_A(\bar{y}) = N_a(A, \bar{y})$ whence the conclusion. □

For weak Pareto minima, following result is obtained in [10] by applying the same technique of proof and Lemma 5.3 directly for the cone K (supposed to have non-empty interior).

Proposition 5.3. *Let $A \subset Y$ be a closed set, K be with nonempty interior and $\bar{y} \in$* WMin$(A \mid K)$*. Then for every $e \in \mathrm{int}\, K$ there exists $y^* \in Y^*$ s.t. $y^*(e) = 1$ s.t. $-y^* \in N_a(A, \bar{y})$.*

It is also important to observe that in the above theorem we use the condition that the cone cone$(A - \bar{y})$, where $\bar{y} \in \mathrm{Min}(A \mid K)$ is closed or weakly closed. This condition may appear to be a very strong one. So we shall now like to provide an example where it holds and an example where it does not hold. For simplicity we restrict ourselves to finite dimensions while giving these examples. Consider the two dimensional plane $\mathbb{R}^2$. Consider the cone K to be either $\mathbb{R}^2_+$ or the cone given as

$$K = \{(x_1, x_2) \in \mathbb{R}^2_+ \mid x_1 = x_2\}.$$

Consider the set A which is given as the square obtained by joining the points

$$(-1,0), (0,1), (1,0), (0,-1)$$

This means that A is given as

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid |x_1| + |x_2| \leq 1\}.$$

The efficient frontier of the set $\mathrm{Min}(A \mid K)$, with respect to any of the two cones mentioned above is given as the line segment joining the points $(-1,0)$ and $(0,-1)$. It is not difficult to see that in this case the set $\mathrm{cone}(A - \bar{y})$ is closed for all $\bar{y} \in \mathrm{Min}(A \mid K)$.

On the other hand consider the set A to be the closed unit ball, that is,

$$A = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1\}.$$

If we consider any of the cones mentioned above as the ordering cone then it is easy to see that the set $\mathrm{Min}(A \mid K)$ is given as the circular arc joining the points $(-1,0)$ and $(0,-1)$. In this case it is easy to see that for any $\bar{y} \in \mathrm{Min}(A \mid K)$ the set $\mathrm{cone}(A - \bar{y})$ is not closed.

If one observes the above results then it will be clear that above analysis was done on the image space or the objective space. However it will be important to have an optimality condition based on the decision space or the space of decision variables. Thus we will now present a necessary optimality condition for the problem (VOP).

Theorem 5.18. *Consider the problem (VOP) where the function f is locally Lipschitz and the set C is a closed set in X. Let $\bar{x}$ be a Pareto minimum for (VOP) and assume that f is strongly compactly Lipschitzian at $\bar{x}$. Suppose that* $\mathrm{cone}(f(C) - f(\bar{x}))$ *is (weakly) closed and K admits a (weakly) closed base B. Further assume that* $\mathrm{cone}(f(C) - f(\bar{x}))$ *or B is (weakly) asymptotically compact. Then there exists $\varepsilon > 0$ such that for every $e \in K \setminus \{0\}$ there exists $y^* \in Y^*$ with $y^*(e) = 1$ and $\inf_{b \in B} y^*(b) \geq \varepsilon \|y^*\|$ which satisfies*

$$0 \in \partial_a \langle y^*, f \rangle(\bar{x}) + N_a(C, \bar{x}).$$

Proof. Since $\bar{x}$ is a Pareto minimum of (VOP) and since either B or $\mathrm{cone}(f(C) - f(\bar{x}))$ is (weakly) asymptotically compact we can apply Theorem 5.16 to conclude that there exists $\varepsilon > 0$ such that $f(\bar{x}) \in \mathrm{Min}(f(C)|K_\varepsilon)$. Hence $f(\bar{x}) \in \mathrm{WMin}(f(C)|K_\varepsilon)$. Hence by applying Lemma 5.3 with $Q = K_\varepsilon$ we conclude that for any given $e \in K \setminus \{0\}$ such that there exists a continuous sublinear functional $s_e : Y \to \mathbb{R}$ such that $s_e(f(x) - f(\bar{x}) \geq 0$ for all $x \in C$. Hence by setting $\phi(x) = f(x) - f(\bar{x})$ we see that $\bar{x}$ solves the following scalar problem,

$$\min s_e \circ \phi(x) \quad \text{subject to} \quad x \in C.$$

Hence $\bar{x}$ is also an unconstrained minimizer of the function $s_e \circ \phi + \delta_C$. Hence

$$\mathbf{0} \in \partial_a(s_e \circ \phi + \delta_C)(\bar{x}).$$

Observe that $s_e \circ \phi$ is locally Lipschitz and δ_C is lower-semicontinuous. Hence by applying the sum rule we have

$$\mathbf{0} \in \partial_a(s_e \circ \phi)(\bar{x}) + N_a(C,\bar{x}).$$

Now applying the chain rule we have

$$\mathbf{0} \in \bigcup_{y^* \in \partial s_e(\phi(\bar{x}))} D^*\phi(\bar{x})(y^*) + N_a(C,\bar{x})$$

Observe that $\partial s_e(\phi(\bar{x})) = \partial s_e(0)$. Hence we can conclude using (i) in Lemma 5.3 that there exists $y^* \in Y^*$ such that $y^*(e) = 1$ and

$$\mathbf{0} \in D^*\phi(\bar{x})(y^*) + N_a(C,\bar{x}).$$

Since it is easy to see that $\partial s_e(0) \subset K^*_\varepsilon \setminus \{0\}$ we see using Lemma 5.2 that y^* satisfies

$$\inf_{b \in B} y^*(b) \geq \varepsilon \, \|y^*\| \, .$$

Further since f is strongly compactly Lipschitzian then so is ϕ and we can obtain our result by first applying the scalarization rule

$$D^*_a\phi(\bar{x})(y^*) = \partial_a\langle y^*, \phi\rangle(\bar{x})$$

and then noting that

$$\partial_a\langle y^*, \phi\rangle(\bar{x}) = \partial_a\langle y^*, f\rangle(\bar{x}).$$

This completes the proof. □

It is now important to see whether the above necessary optimality conditions can be applied to some specific problems in vector optimization. We apply the above results to what is called the vector control-approximation problem (see [21]). For completeness of our exposition we mention briefly the statement of the vector control-approximation problem. The vector control-approximation problem consists of vector minimizing the function $f: X \to Y$ where X and Y are Banach spaces and the function f is given as

$$f(x) = f_1(x) + \sum_{i=1}^{n} \alpha_i |||A_i(x) - a^i|||,$$

where $f_1 : X \to Y$ is locally Lipschitz and each $A_i : X \to Z$ is a linear map and Z is a Banach space. We also have $\alpha_i \geq 0$ for all $i = 1, \dots, n$. The symbol $|||\cdot|||$ denotes a continuous map from Z to the ordering cone K called the vector norm which satisfies the following properties:

(1) $|||z||| = 0$ if and only if $z = 0$
(2) $|||\lambda z||| = |\lambda| |||z|||, \quad \forall \lambda \in \mathbb{R}$
(3) $|||z_1 + z_2||| = |||z_1||| + |||z_2||| - k$, where $k \in K$

In Jahn [26] it has been shown that if K is a nontrivial convex cone and $x^* \in K^\sharp$ then the set

$$B = \{x \in K \mid x^*(x) = 1\}$$

is a base for the cone K. This construction is definitely possible if $K^\sharp$ is non-empty. In [10] an effort was made to find a space whose ordering cone has a base that can be written in terms of the elements of its quasi-interior. The *vector-control approximation* problem can then be viewed in that light.

Now let us consider $Y = L_2(\Omega)$ where Ω is a non-empty subset of $\mathbb{R}^n$ and $L_2(\Omega)$ is the well known space of square Lebesgue-integrable functions $f : \Omega \to \mathbb{R}$. The space $L_2(\Omega)$ is a Hilbert space and hence $(L_2(\Omega))^* = L_2(\Omega)$ and the natural ordering cone in $L_2(\Omega)$ is given as

$$L_2^+(\Omega) = \{f \in L_2(\Omega) \mid f(x) \geq 0 \quad \text{almost everywhere on} \quad \Omega\}.$$

We denote the dual cone of $L_2^+(\Omega)$ and $(L_2^+(\Omega))^*$. However it is interesting to note that $(L_2^+(\Omega))^* = L_2^+(\Omega)$, i.e. it is self dual. Further the quasi-interior of its dual cone which is denotes as $L_2^+(\Omega)^\sharp$ is non-empty (see for example [26]). Thus by considering an element $x^* \in L_2^+(\Omega)^\sharp$ then

$$B = \{x \in L_2^+(\Omega) \mid x^*(x) = 1\}$$

is a base for $L_2^+(\Omega)$ and it is weakly compact and hence weakly asymptotically compact and thus we have all the assumptions required to apply Theorem 5.18.

Let us now consider the control-approximation problem with the assumption that $Y = L_2(\Omega)$ and $K = L_2^+(\Omega)$. Now from the above discussion it is clear that one can easily construct a base which is weakly closed and weakly compact and thus is weakly asymptotically compact. Now consider any $x^* \in (L_+^2(\Omega))^\sharp$ of norm 1 and consider the base B of K given as

$$B = \{x \in L_2^+(\Omega) \mid x^*(x) = 1\}.$$

Then $x^* \in B$ and $d(0, B) = 1$.

Proposition 5.4. *Consider the vector control-approximation problem with* $Y = L_2(\Omega)$ *where* Ω *is a subset of* $\mathbb{R}^n$. *Let* $\bar{x}$ *be a Pareto minimum for the control-approximation problem. Suppose that* $\operatorname{cone}(f(X) - f(\bar{x}))$ *is weakly closed. Then for each* $x^* \in (L_2^+(\Omega))^\sharp$ *with* $||x^*|| = 1$ *there exists* $\varepsilon \in (0,1)$ *such that for any* $b \in B = \{x \in L_2^+(\Omega) : x^*(x) = 1\}$ *there exists* $y^* \in L_2(\Omega)$ *such that*

(a) $\mathbf{0} \in \partial_a \langle y^*, f_1 \rangle(\bar{x}) + \partial \langle y^*, \sum_{i=1}^n \alpha_i |||A_i(\cdot) - a^i||| \rangle(\bar{x})$.
(b) $y^*(b) = 1$.
(c) $\inf_{b' \in B} y^*(b') \geq \varepsilon \|y^*\|$.

where ∂f *denotes the subdifferential map of a convex function.*

Proof. let $\bar{x}$ be a Pareto minimum for the control-approximation problem with $Y = L_2(\Omega)$. Suppose that $\operatorname{cone}(f(X) - f(\bar{x}))$ is weakly closed. Then using Theorem 5.16 we conclude that there exists $0 < \varepsilon < 1$ such that $f(\bar{x}) \in \operatorname{Min}(f(X) \mid K_\varepsilon)$. Thus by applying Theorem 5.18 we see that for each $b \in B$ there exists $y^* \in L_2(\Omega)$ such that $y^*(b) = 1$ and $\inf_{b' \in B} y^*(b') \geq \varepsilon ||y^*||$ such that

$$\mathbf{0} \in \partial_a \langle y^*, f \rangle(\bar{x})$$

Observe that f is locally Lipschitz and it is clear that the function $||| \cdot |||$ is a $L_2^+(\Omega)$-convex function which is continuous and hence $y^*(\sum_{i=1}^n \alpha_i |||A_i(x) - a^i|||)$ is a continuous convex function and thus using the sum rule we have

$$\mathbf{0} \in \partial_a \langle y^*, f_1 \rangle(\bar{x}) + \partial \left\langle y^*, \sum_{i=1}^n \alpha_i |||A_i(\cdot) - a^i||| \right\rangle(\bar{x}).$$

□

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Chapter 6
Optimality Conditions and Image Space Analysis for Vector Optimization Problems

Giandomenico Mastroeni

6.1 Introduction

At the beginning of last century, W. Pareto introduced, in the field of Economics the idea of considering the simultaneous extremization of more that one objective function, namely a Vector Optimization Problem (VOP). This new concept has much influenced the Theory of Economics and the mathematical theory of extrema, but, in the first half of the century, only a few applications has been developed. After the second world war, in some fields of Engineering [53] and in the context of Industrial Systems, Logistics and Management Science, there has been an increasing request of mathematical models for optimizing situations with concurrent objectives, and nowadays, besides the above mentioned applications, VOP also arises in the field of statistics, approximation theory and cooperative game theory [18, 33].

The aim of this chapter is to present the most important results and the main tools in the analysis of optimality conditions for VOP.

A preliminary section is concerned with scalarization methods which consist in replacing the original VOP with a family of suitable optimization problems having a real valued objective function, which is in general obtained by replacing the vector objective function f with a linear or nonlinear transformation of f: the scalarization is said to be linear or nonlinear, accordingly. To deepen the analysis of linear scalarization, a brief overview of the main generalized convexity concepts existing in the literature is presented. In the nonlinear case, particular attention will be given to the Gerstewitz-Luc scalarization function [43]. Section 6.3 is devoted to Gâteaux differentiable VOP with abstract constraints of the form $x \in K$, where K is a convex set. Optimality conditions are expressed by means of vector variational

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inequalities, where the operator is the Gâteaux derivative of the objective function. Section 6.4 presents the image space analysis for VOP with cone constraints of the form $g(x) \in D$, where D is a closed and convex cone. In particular, following a vector separation scheme in the image space, generalized vector Lagrangian functions associated with VOP are introduced, and scalar and vector saddle points conditions are derived. Section 6.5 is concerned with Kuhn–Tucker type first order optimality conditions for Gâteaux differentiable VOP with cone constraints and for semidifferentiable [24] VOP with inequality constraints.

We preliminarily recall some basic definitions and notations that will be used throughout this chapter.

Let X and Y be locally convex Hausdorff topological vector spaces. A set $A \subseteq Y$ is said to be *convex* iff, for any $x_1, x_2 \in A$, $\alpha \in [0,1]$, we have $\alpha x_1 + (1-\alpha)x_2 \in A$. The interior, the closure and the frontier of A will be denoted by $\operatorname{int} A$, $\operatorname{cl} A$ and $\operatorname{frt} A$, respectively. $\mathbf{0}$ will denote the null vector, regardless the space we work in.

A is called a *cone* iff, for any $\alpha \geq 0$, $\alpha A \subset A$. $\operatorname{cone}(A) := \{y \in Y : y = \alpha a, \alpha \geq 0, a \in A\}$ is the *cone generated* by the set A. A convex cone A is called *pointed* iff $A \cap (-A) = \{\mathbf{0}\}$. $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \geq \mathbf{0}\}$.

If $B \subseteq Y$, $A \pm B := \{y \in Y : y = a \pm b,\ a \in A,\ b \in B\}$.

Let $C \subset Y$ be a nonempty, pointed, closed and convex cone with nonempty interior. For the sake of simplicity, $C_{\mathbf{0}}$ and $\overset{\mathbf{0}}{C}$, will denote $C \setminus \{\mathbf{0}\}$ and the interior of C, respectively. Then, (Y, C) is a Hausdorff topological vector space with a partial ordering defined by

$$y_1 \leq_C y_2 \iff y_2 - y_1 \in C, \quad y_1, y_2 \in Y.$$

Similarly, we will use the notations:

$$y_1 \not\leq_{C_{\mathbf{0}}} y_2 \iff y_2 - y_1 \notin C_{\mathbf{0}}; \quad y_1 \not\leq_{\overset{\mathbf{0}}{C}} y_2 \iff y_2 - y_1 \notin \overset{\mathbf{0}}{C}.$$

A Vector Optimization Problem (denoted by VOP) is defined by:

$$\min_C\ f(x), \quad s.t. \quad x \in K \tag{6.1}$$

where $f : X \to Y$ is a vector-valued function and $K \subseteq X$ is a nonempty subset.

Definition 6.1.

- The point $\bar{x} \in K$ is said to be a *vector minimum point* (for short, v.m.p.) of VOP, iff $f(\bar{x}) \not\geq_{C_{\mathbf{0}}} f(K)$.
- The point $\bar{x} \in K$ is said to be a *weak v.m.p.* of VOP, iff $f(\bar{x}) \not\geq_{\overset{\mathbf{0}}{C}} f(K)$.

When $C = \mathbb{R}_+$ and $f : X \to \mathbb{R}$, then Definition 6.1 collapses to the classic definition of minimum point of the function f on the set K.

The sets of v.m.p. and weak v.m.p. of VOP will be denoted by $sol(VOP)$ and $sol(VOP)^w$, respectively.

Definition 6.2. Let $A \subset Y$ be a nonempty subset.

- A point $\overline{y} \in A$ is called a *minimum (maximum) point* of A, iff

$$\overline{y} \not\geq_{C_{\mathbf{0}}} y, \quad (y \not\geq_{C_{\mathbf{0}}} \overline{y}), \quad \forall y \in A.$$

- A point $\overline{y} \in A$ is called a *weak minimum (maximum) point* of A, iff

$$\overline{y} \not\geq_{\overset{\mathbf{0}}{C}} y, \quad (y \not\geq_{\overset{\mathbf{0}}{C}} \overline{y}), \quad \forall y \in A.$$

  The sets of all minimum (maximum) points of A are denoted by $\mathrm{Min}_{C_{\mathbf{0}}} A$ ($\mathrm{Max}_{C_{\mathbf{0}}} A$) and similarly for weak minimum (maximum) points.

Observe that $\overline{x}$ is a (weak) v.m.p. of VOP iff $\overline{y} := f(\overline{x})$ is a (weak) minimum point of the set $A := f(K)$.

By $L(X,Y)$, we denote the set of all linear continuous functions from X into Y. For $l \in L(X,Y)$, the value of l at x is denoted by $\langle l,x \rangle$. Let X^*, Y^* be the topological dual spaces of X and Y. We recall that, for $\Lambda \in L(X,Y)$, the *adjoint operator* $\Lambda^* : Y^* \to X^*$ is defined by the equality

$$\langle \Lambda^* y^*, x \rangle = \langle y^*, \Lambda x \rangle, \quad \text{for all } y^* \in Y^* \text{and } x \in X.$$

We define

$$C^* := \{f \in Y^* : \langle f,x \rangle \geq 0, \ \forall x \in C\},$$
$$C^+ := \{f \in Y^* : \langle f,x \rangle > 0, \ \forall x \in C_{\mathbf{0}}\}.$$

C^* is called the dual cone (or positive polar cone) of C. Next lemma states some fundamental properties of C^* (see e.g. [33, 34]).

Lemma 6.1. *Let C be a pointed, closed and convex cone of a locally convex Hausdorff topological vector space Y and let $\overset{\mathbf{0}}{C} \neq \emptyset$. Then:*

(a) $x \in C \Leftrightarrow \langle f,x \rangle \geq 0, \quad \forall f \in C^*$

(b) $x \in \overset{\mathbf{0}}{C} \Leftrightarrow \langle f,x \rangle > 0, \quad \forall f \in C^* \setminus \{\mathbf{0}\}$

Definition 6.3. Let $f : X \to Y$ be a vector valued function.

- f is said to be *directionally differentiable* at a point $\overline{x} \subseteq X$, iff, for any $v \in X$,

$$f'(\overline{x}; v) := \lim_{t \to 0^+} \frac{f(\overline{x} + tv) - f(\overline{x})}{t}$$

  exists and is finite. If f is directionally differentiable at every $x \in K$, then f is said to be *directionally differentiable* on K.
- f is said to be *Gâteaux differentiable* at $\overline{x} \in K$, iff there exists a linear continuous operator $Df(\overline{x}) : X \to Y$, such that, for any $v \in X$:

$$\langle Df(\overline{x}), v \rangle = f'(\overline{x}; v).$$

$Df(\bar{x})$ is called the *Gâteaux derivative* of f at $\bar{x}$. If f is Gâteaux differentiable at every $x \in K$, then f is said to be *Gâteaux differentiable* on K.

- Let X, Y be Banach spaces. f is said to be *Fréchet differentiable* at $\bar{x} \in K$, iff there exists a linear continuous operator $\Phi(\bar{x}) \in L(X,Y)$, such that:

$$\lim_{x \to \bar{x}} \frac{\|f(x) - f(\bar{x}) - \langle \Phi(\bar{x}), x - \bar{x} \rangle\|}{\|x - \bar{x}\|} = 0.$$

$\Phi(\bar{x})$ is called the Fréchet derivative of f at $\bar{x}$. If f is Fréchet differentiable at every $x \in K$, then f is said to be *Fréchet differentiable* on K.

The subdifferential of the function $f : X \to \mathbb{R}$ at $\bar{x} \in X$ is defined by

$$\partial f(\bar{x}) := \{x^* \in X^* : f(x) - f(\bar{x}) \geq \langle x^*, x - \bar{x} \rangle, \ \forall x \in X\}.$$

6.2 Generalized Convex Functions and Scalarization Methods

A scalarization method consists in replacing VOP with a family of optimization problems having a real valued objective function and in such a way that the set of optimal solutions of this family coincides with $sol(VOP)^w$, or, at least, with a suitable subset of it.

We observe that the definitions of v.m.p. and weak v.m.p. of VOP can be equivalently be expressed in terms of disjunction of suitable subsets of the image space Y where the objective function f runs. Namely, $\bar{x} \in K$ is a v.m.p. of VOP if and only if

$$(f(\bar{x}) - f(K)) \cap C_{\mathbf{0}} = \emptyset, \tag{6.2}$$

and $\bar{x} \in K$ is a weak v.m.p. of VOP if and only if

$$(f(\bar{x}) - f(K)) \cap \overset{\mathbf{0}}{C} = \emptyset. \tag{6.3}$$

One of the most important tools for proving (6.2) and (6.3) are linear and nonlinear separation theorems.

We say that the hyperplane of equation $\langle \gamma, x \rangle = \alpha$, with $\gamma \in Y^* \setminus \{\mathbf{0}\}$, $\alpha \in \mathbb{R}$, separates the sets $A, B \subseteq Y$, iff

$$\langle \gamma, x \rangle \geq \alpha, \quad \forall x \in A, \quad \text{and} \quad \langle \gamma, x \rangle \leq \alpha, \quad \forall x \in B.$$

We will show that the existence of a separating hyperplane for the sets $f(\bar{x}) - f(K)$ and C_o turns out to be equivalent to a scalarization of the VOP.

We recall the following separation theorem (see, e.g., [31, 33]) that will be a key tool in the analysis developed in this chapter.

Theorem 6.1. *Let A and B be convex subsets of a Hausdorff locally convex topological vector space, and let the interior of one of them, e.g., A, be non empty. Then there exists a hyperplane that separates A and B if and only if* $(\text{int}\,A) \cap B = \emptyset$.

Remark 6.1. We observe that if A and B are two nonempty subsets of Y such that $\mathbf{0} \in \text{frt}\,A$ and $\mathbf{0} \in \text{frt}\,B$, then the hyperplane of equation $\langle \gamma, x \rangle = 0$, $\gamma \in Y^* \setminus \{\mathbf{0}\}$, separates A and B iff it separates the set $A - B$ and $\mathbf{0}$.

It is simple to see that (6.2) and (6.3) can be reduced to the disjunction of the null vector $\mathbf{0} \in Y$ and a suitable conical extension of the set $f(K) - f(\bar{x})$.

Lemma 6.2. *(a)* $\bar{x} \in K$ *is a v.m.p. of VOP if and only if*

$$\mathbf{0} \notin f(K) - f(\bar{x}) + C_{\mathbf{0}}, \tag{6.4}$$

(b) $\bar{x} \in K$ *is a weak v.m.p. of VOP if and only if*

$$\mathbf{0} \notin f(K) - f(\bar{x}) + \overset{\mathbf{0}}{C}. \tag{6.5}$$

Proof. It is enough to observe that, for any $A, B \subset Y$, $B \cap A = \emptyset$ iff $\mathbf{0} \notin A - B$. Setting $B := f(\bar{x}) - f(K)$ and $A := C_{\mathbf{0}}$ or $A := \overset{\mathbf{0}}{C}$, then, (a) and (b) follow, respectively. □

From the previous considerations it follows that the analysis of the convexity properties of the image set $f(K)$ and of its conical extensions is a basic step in order to apply linear separation theorems. Such properties are equivalently expressed by means of generalized convexity assumptions on the function f.

6.2.1 Generalized Convex Functions

Definition 6.4. Let X and Y be Hausdorff topological vector spaces and K be a convex subset of X. $f : X \to Y$ is *C-convex* on K, iff, for any $x_1, x_2 \in K, \alpha \in [0,1]$,

$$f(\alpha x_1 + (1-\alpha)x_2) \leq_C \alpha f(x_1) + (1-\alpha) f(x_2).$$

When $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, the previous definition collapses to the classic convexity.

C-convexity is the natural generalization of convexity to a vector valued function. Next result shows a characterization of a C-convex function in terms of the properties of its C-epigraph, defined by:

$$\text{epi}_C f := \{(x,y) \in X \times Y : y \in f(x) + C,\ x \in K\}.$$

Proposition 6.1 ([43]). *$f : X \to Y$ is C-convex on the convex set $K \subseteq X$, if and only if* $\text{epi}_C f$ *is a convex set. Moreover, f is C-convex on K, if and only if $\phi \circ f$ is a convex function on K, for every $\phi \in C^*$.*

Proposition 6.2 ([10]). *Let K be a nonempty convex subset of X. Assume that f is Gâteaux differentiable on K. Then f is C-convex on K if and only if, for every $x, y \in K$,*

$$f(y) - f(x) \geq_C \langle Df(x), y - x \rangle,$$

where $Df(x)$ is the Gâteaux derivative of f at $x \in K$.

The concept of convexity has been generalized in several ways. Some of the most important ones are summarized in the next definition.

Definition 6.5. Let X and Y be real topological vector spaces, $K \subseteq X$, $\mathscr{A} \subseteq Y$ a convex cone, and $f : X \to Y$.

- f is said to be *$\mathscr{A}$-convexlike* on K, iff, for any $x_1, x_2 \in K$ and any $\alpha \in]0,1[$, there exists $x_3 \in K$, such that:

$$\alpha f(x_1) + (1 - \alpha) f(x_2) - f(x_3) \in \mathscr{A}.$$

- f is said to be *$\mathscr{A}$-subconvexlike* on K, iff there exists an $a_0 \in \mathscr{A}$, such that for any $x_1, x_2 \in K$, for any $\alpha \in]0,1[$ and $\varepsilon > 0$, there exists $x_3 \in K$ such that:

$$\varepsilon a_0 + \alpha f(x_1) + (1 - \alpha) f(x_2) - f(x_3) \in \mathscr{A}.$$

- f is said to be *closely $\mathscr{A}$-convexlike* on K, iff the set $\text{cl}(f(K) + \mathscr{A})$ is convex.
- f is said to be *$\mathscr{A}$-preconvexlike* on K, iff for any $x_1, x_2 \in K$ and $\alpha \in]0,1[$, there exists $x_3 \in K$ and $\rho > 0$, such that:

$$\alpha f(x_1) + (1 - \alpha) f(x_2) - \rho f(x_3) \in \mathscr{A}.$$

- Suppose that $\text{int}\, \mathscr{A} \neq \emptyset$. f is said to be *generalized $\mathscr{A}$-subconvexlike* on K, iff there exists an $a_0 \in \text{int}\, \mathscr{A}$, such that for any $x_1, x_2 \in K$, $\alpha \in]0,1[$ and $\varepsilon > 0$, there exists $x_3 \in K$ and $\rho > 0$, such that:

$$\varepsilon a_0 + \alpha f(x_1) + (1 - \alpha) f(x_2) - \rho f(x_3) \in \mathscr{A}.$$

Convexlike mappings were considered by Fan [19]. $\mathscr{A}$-subconvexlike mappings were introduced by Jeyakumar [35]. The class of closely $\mathscr{A}$-convexlike mappings was considered in [9]. The definitions of $\mathscr{A}$-preconvexlike function and generalized $\mathscr{A}$-subconvexlike function were introduced in [63] and [60], respectively.

Next results state a characterization of $\mathscr{A}$-convexlike and $\mathscr{A}$-subconvexlike mappings, in terms of the properties of suitable conical extensions of their images.

Proposition 6.3 ([57]). *$f : X \to Y$ is $\mathscr{A}$-convexlike on $K \subseteq X$, if and only if the set $f(K) + \mathscr{A}$ is convex.*

Proof. It is enough to observe that an $\mathscr{A}$-convexlike function on K, can be equivalently defined as a function f, such that:

$$(1-\alpha)f(K) + \alpha f(K) \subseteq f(K) + \mathscr{A}, \qquad \forall \alpha \in]0,1[.$$

□

Proposition 6.4 ([10]). *Suppose that* int $\mathscr{A} \neq \emptyset$. *The following statements hold:*

(a) *$f : X \to Y$ is $\mathscr{A}$-subconvexlike on $K \subseteq X$ if and only if the set $f(K) + \text{int}\,\mathscr{A}$ is convex.*
(b) *$f : X \to Y$ is generalized $\mathscr{A}$-subconvexlike on $K \subseteq X$ if and only if the set* cone $f(K) + \text{int}\,\mathscr{A}$ *is convex.*

Remark 6.2. The following relations hold:

$$f\ \mathscr{A}\text{-convex} \Rightarrow f\ \mathscr{A}\text{-convexlike} \Rightarrow \text{f}\ \mathscr{A}\text{-subconvexlike}.$$

Proposition 6.4 shows that, if f is $\mathscr{A}$-subconvexlike then it is also generalized $\mathscr{A}$-subconvexlike, provided that int $\mathscr{A} \neq \emptyset$. The reverse implication does not hold as shown by the following example.

Example 6.1. Let $K := \left\{(x_1, x_2) \in \mathbb{R}^2_+ : x_1^2 + x_2^2 > 1\right\}$, $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x_1, x_2) = (x_1, x_2)$ and $\mathscr{A} := \mathbb{R}^2_+$. It is simple to see that:

$$f(K) + \text{int}\,\mathscr{A} = \left\{(x_1, x_2) \in \text{int}\,\mathbb{R}^2_+ : x_1^2 + x_2^2 > 1\right\}$$
$$cone\ f(K) + \text{int}\,\mathscr{A} = \text{int}\,\mathbb{R}^2_+.$$

Therefore, f is not $\mathscr{A}$-subconvexlike on K, but it is generalized $\mathscr{A}$-subconvexlike on K.

Next results are concerned with $\mathscr{A}$-preconvexlike functions. The definition of preconvexlike function does not require that the set $\mathscr{A}$ has nonempty interior, which allows possible extensions of the applications to a VOP where the ordering cone C has empty interior (see e.g. [5]). It is easy to see that, if int $\mathscr{A} \neq \emptyset$, then:

$$f\ \mathscr{A} - preconvexlike \quad \Rightarrow \quad f\ generalized\ \mathscr{A} - subconvexlike.$$

Proposition 6.5. *([63]) $f : X \to Y$ is $\mathscr{A}$-preconvexlike on $K \subseteq X$ if and only if the set*

$$\bigcup_{t>0} tf(K) + \mathscr{A}$$

is convex.

Proof. Let $a \in \mathscr{A}$, $x_1, x_2 \in K$. Since $\mathbf{0} \in \mathscr{A}$, then:

$$f(x_i) \in \bigcup_{t>0} tf(K) + \mathscr{A}, \quad i = 1,2.$$

If $\bigcup_{t>0} tf(K) + \mathscr{A}$ is convex, then:

$$\bar{y} := \alpha f(x_1) + (1-\alpha) f(x_2) \in \bigcup_{t>0} tf(K) + \mathscr{A}, \quad \forall \alpha \in [0,1].$$

Therefore, $\exists x_3 \in K$, and $\bar{t} > 0$ such that:

$$\bar{y} \in \bar{t} f(x_3) + \mathscr{A},$$

which shows, by setting $\rho = \bar{t}$, that f is $\mathscr{A}$-preconvexlike on K.

Vice versa, assume that f is $\mathscr{A}$-preconvexlike on K.

Let

$$y_1, y_2 \in \bigcup_{t>0} tf(K) + \mathscr{A}, \quad \alpha \in]0,1[.$$

Then, there exist $x_1, x_2 \in K$, $\gamma_i > 0, x_i \in K$, and $s_i \in \mathscr{A}$, $i = 1,2$, such that:

$$y_i = \gamma_i f(x_i) + s_i, \quad i = 1,2.$$

Let $\alpha \in]0,1[$ and set

$$\bar{y} := \alpha y_1 + (1-\alpha) y_2 = \alpha \gamma_1 f(x_1) + (1-\alpha)\gamma_2 f(x_2) + s_0,$$

where $s_0 = \alpha s_1 + (1-\alpha) s_2 \in \mathscr{A}$.

Set

$$\gamma := \alpha\gamma_1 + (1-\alpha)\gamma_2, \quad \delta := (\alpha\gamma_1)/\gamma.$$

Then $\gamma > 0$, $\delta \in]0,1[$, and $\bar{y} = \gamma(\delta f(x_1) + (1-\delta) f(x_2)) + s_0$.

By definition of $\mathscr{A}$-preconvexlike function, for the above $x_1, x_2 \in K$, we can find $x_3 \in K$, $a \in \mathscr{A}$ and $\rho > 0$, satisfying:

$$\delta f(x_1) + (1-\delta) f(x_2) = \rho f(x_3) + a.$$

Therefore,

$$\bar{y} := \gamma(\rho f(x_3) + a) + s_0 \in \bigcup_{t>0} tf(K) + \mathscr{A},$$

and the proof is complete. □

Similarly, it is possible to prove the next result.

Proposition 6.6. *If* $f : X \to Y$ *is* $\mathscr{A}$*-preconvexlike on* $K \subseteq X$ *then the set* cone $f(K) + \mathscr{A}$ *is convex.*

Example 6.2. Let K and f be defined as in Example 6.1 and set $\mathscr{A} := \{(x_1,x_2) \in \mathbb{R}^2_+ : x_2 = 0\}$. Then

$$\bigcup_{t>0} t f(K) + \mathscr{A} = \mathbb{R}^2_+ \setminus \{(0,0)\},$$

so that f is $\mathscr{A}$-preconvexlike on K.

Further characterizations of generalized convex functions cab be found in Chap. 2.

6.2.2 Scalarization

In the field of vector optimization, scalarization consists in replacing VOP with a family $P(c), c \in \Xi$, of optimization problems, with a real valued objective function:

$$\min\ s_c(x), \quad s.t. \quad x \in K, \tag{6.6}$$

where $s_c : X \to \mathbb{R},\ c \in \Xi$, a set of parameters. We require that the set of weak v.m.p. of VOP, or at least a suitable subset, coincides with the set of optimal solutions of $P(c), c \in \Xi$.

When $s_c(x) := \langle c, f(x)\rangle$ is a linear tranformation of the objective function f of VOP, we obtain a, so called, linear scalarization of VOP and (6.6) becomes:

$$\min_{x\in K} \langle c^*, f(x)\rangle, \tag{6.7}$$

where $c^* \in C^* \setminus \{\mathbf{0}\}$. We denote (6.7) by VOP(c^*). It is easy to show that VOP(c^*) admits an optimal solution $\bar{x}$ iff the sets $(f(\bar{x}) - f(K))$ and $\overset{\mathbf{0}}{C}$ are linearly separable.

Lemma 6.3. *(a) If $\bar{x} \in K$ is a global minimum point of $VOP(c^*)$, with $c^* \in C^* \setminus \{\mathbf{0}\}$ then the hyperplane of equation $\langle c^*, y\rangle = 0$ separates the sets $(f(\bar{x}) - f(K))$ and $\overset{\mathbf{0}}{C}$, and $\bar{x}$ is a weak v.m.p. of VOP.*

(b) If the hyperplane of equation $\langle c^*, y\rangle = 0$, $c^* \neq \mathbf{0}$, separates the sets $(f(\bar{x}) - f(K))$ and $\overset{\mathbf{0}}{C}$, with $\bar{x} \in K$, then $c^* \in C^* \setminus \{\mathbf{0}\}$, and $\bar{x}$ is a global minimum point of $VOP(c^*)$.

Proof. (a) Suppose that $\bar{x}$ is an optimal solution of $VOP(c^*)$, $c^* \in C^* \setminus \{\mathbf{0}\}$, i.e.

$$\langle c^*, f(\bar{x}) - f(x)\rangle \le 0, \quad \forall x \in K. \tag{6.8}$$

Since $c^* \in C^* \setminus \{\mathbf{0}\}$, we have that $\langle c^*, y\rangle > 0$, $\forall y \in \overset{\mathbf{0}}{C}$, and (6.3) is fulfilled, which implies that $\bar{x}$ is a weak v.m.p. of VOP.

(b) Suppose that the hyperplane of equation $\langle c^*, y\rangle = 0$, $c^* \neq \mathbf{0}$, separates the sets $(f(\bar{x}) - f(K))$ and $\overset{\circ}{C}$, i.e. (6.8) holds or, equivalently, $\bar{x}$ is an optimal solution of $VOP(c^*)$, and $\langle c^*, y\rangle \geq 0$, $\forall y \in \overset{\circ}{C}$.

By continuity, we have that $\langle c^*, y\rangle \geq 0$, $\forall y \in C$, i.e., $c^* \in C^* \setminus \{\mathbf{0}\}$, which completes the proof. □

From Lemma 6.3 it follows that whenever $VOP(c^*)$, $c^* \in C^* \setminus \{\mathbf{0}\}$, has an optimal solution $\bar{x}$, then $\bar{x} \in sol(VOP)^w$. The reverse inclusion holds under suitable generalized convexity hypotheses that guarantee the linear separation of the sets $(f(\bar{x}) - f(K))$ and $\overset{\circ}{C}$, so that $sol(VOP)^w$ can be characterized in terms of $sol(VOP(c^*))$, $c^* \in C^* \setminus \{\mathbf{0}\}$.

Theorem 6.2 ([60]). *Assume that $f - f(\bar{x})$ is a generalized C-subconvexlike mapping on K. Then $\bar{x} \in K$ is a weak v.m.p. of VOP, if and only if there exists $c^* \in C^* \setminus \{\mathbf{0}\}$, such that $\bar{x}$ is a global minimum point of $VOP(c^*)$.*

Proof. Suppose that $\bar{x} \in K$ is a weak v.m.p. of VOP. Then,

$$(f(\bar{x}) - f(K)) \cap \overset{\circ}{C} = \emptyset,$$

which is equivalent to the condition:

$$\text{cone}(f(\bar{x}) - f(K)) \cap \overset{\circ}{C} = \emptyset,$$

and in turn,

$$0 \notin \text{cone}(f(K) - f(\bar{x})) + \overset{\circ}{C}.$$

From Proposition 6.4 (b), the set $\text{cone}(f(K) - f(\bar{x})) + \overset{\circ}{C}$ is convex, and since it has nonempty interior, applying the separation theorem for convex set (see Theorem 6.1) and taking into account Remark 6.1, it follows that the sets $f(\bar{x}) - f(K)$ and $\overset{\circ}{C}$ are linearly separable. By Lemma 6.3 (b), $\bar{x}$ is a global minimum point of $VOP(c^*)$.

The converse implication follows from Lemma 6.3 (a). □

Remark 6.3. We observe that most of the generalized convexity assumptions on f ensure that $f - f(\bar{x})$ is a generalized C-subconvexlike mapping. From Definition 6.5, it is immediate that, if f is $\mathscr{A}$-subconvexlike, then $f - f(\bar{x})$ is also $\mathscr{A}$-subconvexlike, so that $f - f(\bar{x})$ is generalized $\mathscr{A}$-subconvexlike, provided that int $\mathscr{A} \neq \emptyset$ (see Proposition 6.4). Therefore, in the previous theorem, the assumption that $f - f(\bar{x})$ is generalized C-subconvexlike on K can be replaced by the one that f is C-convexlike or C-subconvexlike on K.

We refer to Chaps. 2, 4, 12, [10, 20, 32] and the references therein, for further interesting developments on linear scalarization.

Let us consider nonlinear scalarization and define the Gerstewitz-Luc scalarization function, which is one of the most studied in the field of Vector Optimization [43].

Definition 6.6. Given a fixed $e \in \overset{\mathbf{0}}{C}$ and $a \in Y$, the *Gerstewitz-Luc scalarization function* is defined by:

$$\xi_{ea}(y) = \min\{t \in \mathbb{R} : y \in a + te - C\}, \quad y \in Y. \tag{6.9}$$

We recall the main properties of the function ξ_{ea}.

Proposition 6.7 ([10]). *The nonlinear scalarization function ξ_{ea} is well-defined, that is, the minimum in (6.9) is attained.*

Definition 6.7.

- A function $\psi : Y \longrightarrow \mathbb{R}$ is *monotone* iff, for any $y_1, y_2 \in Y$,

$$y_1 \geq_C y_2 \;\Rightarrow\; \psi(y_1) \geq \psi(y_2).$$

- ψ is *strictly monotone* iff, for any $y_1, y_2 \in Y$,

$$y_1 \geq_{\overset{\circ}{C}} y_2 \;\Rightarrow\; \psi(y_1) > \psi(y_2).$$

- ψ is *strongly monotone* iff, for any $y_1, y_2 \in Y$,

$$y_1 \geq_{C_\mathbf{0}} y_2 \;\Rightarrow\; \psi(y_1) > \psi(y_2).$$

It can be proved that, ξ_{ea} is a continuous, convex and strictly monotone function on Y. However, ξ_{ea} is not strongly monotone [10, 43].

The following result [10] states an equivalent characterization of the function ξ_{ea} in the particular case where $Y = \mathbb{R}^\ell$, $C = \mathbb{R}^\ell_+$.

Proposition 6.8. *Let $Y = \mathbb{R}^\ell$, $C = \mathbb{R}^\ell_+$ and $e = (e_1, ..., e_\ell)^T \in \overset{\mathbf{0}}{C}$. Then, for any $a \in \mathbb{R}^\ell$ and $y \in \mathbb{R}^\ell$, we have:*

$$\xi_{ea}(y) = \max_{1 \leq i \leq \ell} \left\{ \frac{y_i - a_i}{e_i} \right\}.$$

Next theorem characterizes weak minimum points of a set A (see Definition 6.2) in terms of global minimum points of the function ξ_{ea}.

Theorem 6.3. *Let $e \in \overset{\circ}{C}$ and $A \subset Y$. Then $\bar{y}$ is a weak minimum point of A iff, there exists $a \in Y$ such that:*

$$\xi_{ea}(\bar{y}) = \min \xi_{ea}(y), \quad y \in A. \tag{6.10}$$

Proof. Let $e \in \overset{\circ}{C}$ and assume that for some $a \in Y$, (6.10) is fulfilled. Ab absurdo, if $\bar{y}$ is not a weak minimum point of A, then there exists $y \in A$ such that $y \leq_{\overset{\circ}{C}} \bar{y}$. Since ξ_{ea} is strictly monotone on Y, then $\xi_{ea}(y) < \xi_{ea}(\bar{y})$, which contradicts (6.10).

Conversely, assume that $\bar{y} \in A$ is a weak minimum point of A, i.e.,

$$y \notin \bar{y} - \overset{\mathbf{0}}{C}, \qquad \forall y \in A. \tag{6.11}$$

Set $a := \bar{y}$, by Definition 6.6, we have $\xi_{e\bar{y}}(y) = \min\{t \in \mathbb{R} : y \in \bar{y} + te - C\}$. Since, for every $t > 0$ and $e \in C$, we have $-C \subseteq te - C$, then :

$$\bar{y} \in \bar{y} - C \subseteq \bar{y} + te - C, \quad \forall t > 0$$

which implies

$$\min\{t \in \mathbb{R} : \bar{y} \in \bar{y} + te - C\} = \xi_{e\bar{y}}(\bar{y}) \leq 0.$$

We will prove that (6.11) implies

$$\xi_{e\bar{y}}(y) \geq 0, \quad \forall y \in A. \tag{6.12}$$

Ab absurdo, if $\xi_{e\bar{y}}(y) < 0$, for some $y \in A$, then, set $\tilde{t} := \xi_{e\bar{y}}(y) < 0$, we would have:

$$y \in \bar{y} + \tilde{t}e - C \subset \bar{y} - \overset{\mathbf{0}}{C},$$

which contradicts (6.11). Since $\xi_{e\bar{y}}(\bar{y}) \leq 0$, from (6.12) it follows that $\xi_{e\bar{y}}(y) = 0$, so that (6.10) holds. □

Let us apply Theorem 6.3 to VOP.

Corollary 6.1. *Let $e \in \overset{\mathbf{0}}{C}$. Then, $\bar{x} \in sol(VOP)^w$ if and only if there exists $a \in f(K)$ such that:*

$$\xi_{ea}(\bar{y}) = \min \xi_{ea}(y), \quad y \in f(K),$$

where $\bar{y} := f(\bar{x})$.

Proof. It is enough to recall that $\bar{x}$ is a weak v.m.p. for VOP if and only if $\bar{y} := f(\bar{x})$ is weak minimum point of the set $f(K)$, taking into account that, by the proof of Theorem 6.3 it follows that, if $\bar{x} \in sol(VOP)^w$, then $\bar{y} := f(\bar{x})$ is a global minimum point of problem:

$$\min \xi_{e\bar{y}}(y), \quad y \in f(K).$$

□

Under the hypotheses of Proposition 6.8, we obtain the following well known result.

Corollary 6.2. *Let $Y = \mathbb{R}^\ell$, $C = \mathbb{R}^\ell_+$. Then, $sol(VOP)^w = \bigcup_{a \in f(K)} sol(P(a))$, where $P(a)$ is the scalar constrained extremum problem defined by:*

$$\min_{x \in K} \max_{1 \leq i \leq \ell} (f_i(x) - a_i), \quad a \in f(K).$$

Proof. It is enough to apply Corollary 6.1, taking into account Proposition 6.8 where we have set $e := (1, \ldots, 1) \in \operatorname{int} \mathbb{R}^\ell_+$. □

Further developments of the analysis on nonlinear scalarization can be found in Chaps. 4, 11, 12, [21, 32, 43] and, more recently in [37], where it is shown that $\bar{y} \in A \subset Y$ is a proper v.m.p. of A, in the sense of Benson [6], if and only if the scalar problem

$$\min_{y \in A} [\langle y^*, y - \bar{y} \rangle + \alpha \| y - \bar{y} \|]$$

attains its minimum at $\bar{y}$, where Y is a normed space and

$$(y^*, \alpha) \in C^a := \{(y^*, \alpha) \in Y^* \times \mathbb{R}_+ : \langle y^*, y \rangle - \alpha \|y\| > 0, \forall y \in C_{\mathbf{0}}\}.$$

C^a can be regarded as an augmented dual cone associated with C.

6.3 Connections with Vector Variational Inequalities

Variational Inequalities (for short, VI) were introduced half a century ago by Stampacchia [38], inspired by Signorini Problem and the related work of Fichera in early sixties in the field of Calculus of Variations. In parallel with the development of VOP, in [22] the Theory of Vector Variational Inequalities (for short, VVI) was proposed. Since then, VVI have extensively been studied because they have shown to be a powerful tool in many fields of optimization: from the classic optimality conditions for constrained extremum problems to the equilibrium conditions for network flow and economic equilibrium problems [10, 26, 30, 38].

In the present section, we will introduce VVI and consider the main connections with VOP. When the objective function f of the vector problem is Gâteaux differentiable and the feasible set is convex, necessary optimality conditions for a VOP can be formulated in terms of a VVI, where the operator is defined by the Gâteaux derivative of f. Suitable generalized convexity assumptions ensure that a weak VVI is a sufficient optimality condition for a weak vector minimum point. In such a context, the Minty VVI is of particular importance since it provides a necessary and sufficient optimality condition for a v.m.p. of a finite dimensional convex VOP.

Definition 6.8. Let $T : K \to L(X, Y)$ and $K \subseteq X$ be a nonempty subset.

- A *VVI* consists in finding $\bar{x} \in K$ such that:

$$\langle T(\bar{x}), x - \bar{x} \rangle \not\leq_{C_0} \mathbf{0}, \quad \forall x \in K. \tag{6.13}$$

- A *weak VVI* consists in finding $\bar{x} \in K$ such that:

$$\langle T(\bar{x}), x - \bar{x} \rangle \not\leq_{\substack{\mathbf{0}\\C}} \mathbf{0}, \quad \forall x \in K. \tag{6.14}$$

When $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, then both (6.13) and (6.14) shrink to the scalar VI [4, 38], which consists in finding $\bar{x} \in K$, such that:

$$\langle T(\bar{x}), x - \bar{x} \rangle \geq 0, \quad \forall x \in K. \tag{6.15}$$

A further fundamental VVI that will be analysed in this section, is the, so-called, Minty VVI.

Definition 6.9.

- The *Minty VVI* consists in finding $\bar{x} \in K$ such that:

$$\langle T(x), \bar{x} - x \rangle \not\geq_{C_0} \mathbf{0}, \ \forall x \in K. \tag{6.16}$$

- The *weak Minty VVI* consists in finding $\bar{x} \in K$ such that

$$\langle T(x), \bar{x} - x \rangle \not\geq_{\substack{\mathbf{0}\\C}} \mathbf{0}, \ \forall x \in K. \tag{6.17}$$

When $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, then both (6.16) and (6.17) collapse to the scalar *Minty VI* [**?**], which consists in finding $\bar{x} \in K$, such that:

$$\langle T(x), \bar{x} - x \rangle \leq 0, \quad \forall x \in K. \tag{6.18}$$

We note that, unlike (6.15), (6.18) has not been considered as the mathematical model of a class of equilibrium problems, but merely as a tool for carrying out proofs mainly of existence theorems for VI.

We recall that a mapping $T : X \to L(X,Y)$ is called *C-monotone* on $K \subseteq X$ iff, for every $x, y \in K$,

$$\langle T(x) - T(y), x - y \rangle \geq_C 0.$$

It is well known that, under the hypothesis of C-monotonicity of the mapping T, weak VVI and weak Minty VVI are equivalent [11].

Another important generalization of VVI is obtained by considering a set-valued mapping $T : K \rightrightarrows L(X,Y)$, which leads to the following definition of set-valued (weak) VVI.

Definition 6.10. Let $T : K \rightrightarrows L(X,Y)$. The *set-valued weak VVI* consists in finding $\bar{x} \in K$ such that:

$$\exists \bar{t} \in T(\bar{x}) : \langle \bar{t}, x - \bar{x} \rangle \not\leq_{\substack{\mathbf{0}\\C}} \mathbf{0}, \qquad \forall x \in K. \tag{6.19}$$

Defining the mapping T as a suitable subdifferential of the objective function f, optimality conditions for non differentiable VOP can be expressed in terms of a set-valued VVI. We refer to Chap. 7 for the extension of analysis to vector variational-like inequalities.

6.3.1 Optimality Conditions for Differentiable VOP

Under the assumption of Gâteaux differentiability of f on the set K, optimality conditions for VOP can be expressed by the VVI and weak VVI obtained by defining $T := Df$, the Gâteaux derivative of f, and the feasible set K as the one of VOP.

Definition 6.11. Let f be Gâteaux differentiable on the convex set K, with Gâteaux derivative Df. Consider the following VVI:

- Find $\overline{x} \in K$ such that:

$$\langle Df(\overline{x}), x - \overline{x} \rangle \not\leq_{C_0} \mathbf{0}, \quad \forall x \in K. \tag{6.20}$$

- Find $\overline{x} \in K$ such that:

$$\langle Df(\overline{x}), x - \overline{x} \rangle \not\leq_{\overset{\circ}{C}} \mathbf{0}, \quad \forall x \in K. \tag{6.21}$$

Next theorem [10, 11] states the relationships between VVI, weak VVI and VOP.

Theorem 6.4. *The following statements hold.*

(a) If $\overline{x}$ is a weak v.m.p. of VOP, then $\overline{x}$ solves (6.21).
(b) If f is C-convex on K and $\overline{x}$ solves (6.21), then $\overline{x}$ is a weak v.m.p. of VOP.
(c) If f is C-convex on K and $\overline{x}$ solves (6.20), then $\overline{x}$ is a v.m.p. of VOP.

Proof. (a) Assume that $\overline{x} \in K$ is a weak v.m.p. of VOP. Then, for any $x \in K$, we have that $\overline{x} + t(x - \overline{x}) \in K$, $\forall t \in]0,1[$. Then,

$$f(\overline{x} + t(x - \overline{x})) - f(\overline{x}) \not\leq_{\overset{\circ}{C}} \mathbf{0}, \quad \forall t \in]0,1[,$$

which implies:

$$\frac{f(\overline{x} + t(x - \overline{x})) - f(\overline{x})}{t} \not\leq_{\overset{\circ}{C}} \mathbf{0}, \quad \forall t \in]0,1[.$$

Taking the limit for $t \downarrow 0$, we obtain (6.21).

(b) Assume that $\overline{x}$ solves (6.21). Since f is C-convex on K, then, by Proposition 6.2, it follows that

$$f(\overline{x}) - f(x) \in -\langle Df(\overline{x}), x - \overline{x} \rangle - C, \quad \forall x \in K,$$

which implies that $f(\bar{x}) - f(x) \in (Y \setminus \overset{\circ}{C}) - C$. Thus

$$f(\bar{x}) - f(x) \not\geq_{\overset{\circ}{C}} \mathbf{0}, \quad \forall x \in K,$$

i.e., $\bar{x}$ is a weak v.m.p. of VOP.

(c) Assume that $\bar{x}$ is not a v.m.p. of VOP. Then, there exists $x \in K$, such that $f(\bar{x}) - f(x) \geq_{C_0} \mathbf{0}$. Since f is C-convex on K, then, for any $x \in K$:

$$f(x) - f(\bar{x}) - \langle Df(\bar{x}), x - \bar{x} \rangle \geq_C \mathbf{0}.$$

Therefore, we have:

$$-\langle Df(\bar{x}), x - \bar{x} \rangle \geq_C f(\bar{x}) - f(x) \geq_{C_0} \mathbf{0},$$

which contradicts (6.20). □

Next result is a direct consequence of the application of scalarization techniques to VVI (6.21) (see e.g. [40]).

Theorem 6.5. *Let $\bar{x} \in K$. The following statements hold.*

(a) If $\bar{x}$ is a weak v.m.p. of VOP, then $\exists c^* \in C^* \setminus \{\mathbf{0}\}$ such that

$$\langle Df(\bar{x})^* c^*, x - \bar{x} \rangle \geq 0, \quad \forall x \in K, \tag{6.22}$$

where $Df(\bar{x})^*$ denotes the adjoint operator of $Df(\bar{x})$.

(b) If f is C-convex on K and $\exists c^* \in C^* \setminus \{\mathbf{0}\}$ such that (6.22) holds, then $\bar{x}$ is a weak v.m.p. of VOP.

(c) If f is C-convex on K and $\exists c^* \in C^+$ such that (6.22) holds, then $\bar{x}$ is a v.m.p. of VOP.

Proof. (a) If $\bar{x} \in K$ is a weak v.m.p. of VOP, then by Theorem 6.4 (a), (6.21) is fulfilled. Defining the set

$$\mathscr{K} := \{z \in Y : z = \langle Df(\bar{x}), x - \bar{x} \rangle,\ x \in K\},$$

this is equivalent to say that

$$\mathscr{K} \cap (-\overset{\circ}{C}) = \emptyset.$$

It is easy to see that $\mathscr{K}$ is convex and $\mathbf{0} \in \mathscr{K}$. Applying Theorem 6.1, there exists $c^* \in Y^* \setminus \{\mathbf{0}\}$ such that:

$$\inf_{z \in \mathscr{K}} \langle c^*, z \rangle \geq \sup_{v \in -C} \langle c^*, v \rangle. \tag{6.23}$$

Since $\mathbf{0} \in \mathscr{K} \cap (-C)$, from (6.23) it follows that $c^* \in C \setminus \{\mathbf{0}\}$ and that

$$\langle c^*, z \rangle \geq 0, \quad \forall z \in \mathscr{K},$$

or equivalently,

$$\langle c^*, Df(\bar{x})(x - \bar{x}) \rangle \geq 0, \quad \forall x \in K,$$

(where, for simplicity of notation $Df(\bar{x})(x-\bar{x}) = \langle Df(\bar{x}), x-\bar{x} \rangle$). Recalling the definition of adjoint operator, (6.22) follows.

(b) Suppose that $\bar{x} \in K$ and $\exists c^* \in C^* \setminus \{\mathbf{0}\}$ such that (6.22) holds. Ab absurdo, assume that $\bar{x}$ is not a weak v.m.p. of VOP. Therefore, there exists $\hat{x} \in K$ such that $f(\bar{x}) - f(\hat{x}) \geq_{\overset{\mathbf{0}}{C}} \mathbf{0}$. Then, since f is C-convex on K, we have:

$$f(\hat{x}) - f(\bar{x}) \geq_C \langle Df(\bar{x}), \hat{x} - \bar{x} \rangle,$$

which implies that

$$\langle Df(\bar{x}), \hat{x} - \bar{x} \rangle \leq_{\overset{\mathbf{0}}{C}} \mathbf{0}.$$

Since $c^* \in C^* \setminus \{\mathbf{0}\}$, by Lemma 6.1 it follows that

$$\langle c^*, Df(\bar{x})(\hat{x} - \bar{x}) \rangle < 0,$$

or, equivalently,

$$\langle Df(\bar{x})^* c^*, \hat{x} - \bar{x} \rangle < 0,$$

which contradicts (6.22).

(c) The proof of this statement is analogous to the one of (b), where $\overset{\mathbf{0}}{C}$ is replaced with $C_{\mathbf{0}}$ and taking into account that, if $c^* \in C^+$, then $\langle c^*, z \rangle > 0$, $\forall z \in C_{\mathbf{0}}$. □

Remark 6.4. Note that (6.22) is the classic first order optimality condition for VOP(c^*). Moreover, we recall that from Theorem 6.2 it follows that, if f is C-convex on K, then the set of the optimal solutions of the family of VOP(c^*), with $c^* \in C^* \setminus \{\mathbf{0}\}$, coincides with the set of weak v.m.p. of VOP.

When $C = \mathbb{R}^\ell_+$, a vector minimum point of VOP can be completely characterized by the Minty VVI, defined by (6.16).

Theorem 6.6 ([25]). *Assume that $X = \mathbb{R}^n$, $Y = \mathbb{R}^\ell$ and $C = \mathbb{R}^\ell_+$. Let f be a Fréchet differentiable C-convex function on K and let $T(x) = \nabla f(x) := (\nabla f_1(x), \ldots, \nabla f_\ell(x))$. Then, $\bar{x}$ is a v.m.p. of VOP if and only if it is a solution to Minty VVI.*

Proof. Suppose that $\bar{x}$ is a v.m.p. of VOP. Ab absurdo, if $\bar{x}$ is not a solution to Minty VVI, then there exists $\hat{x} \in K$, such that:

$$\langle T(\hat{x}), \bar{x} - \hat{x} \rangle \geq_{C_{\mathbf{0}}} \mathbf{0}.$$

Since f is C-convex, we have:

$$f(\bar{x}) - f(\hat{x}) \geq_C \langle T(\hat{x}), \bar{x} - \hat{x} \rangle \geq_{C_0} \mathbf{0}.$$

Then

$$f(\hat{x}) - f(\bar{x}) \leq_{C_0} \mathbf{0}, \tag{6.24}$$

which contradicts that $\bar{x}$ is a v.m.p. of VOP.

Conversely, let $\bar{x}$ be a solution to Minty VVI. Ab absurdo, assume that $\bar{x}$ is not a v.m.p. of VOP, i.e., there exists $\hat{x} \in K$ such that (6.24) is fulfilled. Since K is convex, then $x(\alpha) := \alpha\bar{x} + (1-\alpha)\hat{x} \in K$, $\forall \alpha \in [0,1]$. Since f is C-convex and by (6.24), it follows that

$$f(x(\alpha)) - f(\bar{x}) \leq_C (\alpha - 1)f(\bar{x}) + (1-\alpha)f(\hat{x}) = (1-\alpha)(f(\hat{x}) - f(\bar{x})) \leq_{C_0} \mathbf{0},$$
$$\forall \alpha \in]0,1[.$$

Because of the Lagrange Mean Value Theorem, there exist $\overline{\alpha}_i \in]0,1[$, such that:

$$f_i(x(\alpha)) - f_i(\bar{x}) = \langle \nabla f_i(x(\overline{\alpha}_i)), x(\alpha) - \bar{x} \rangle, \quad i = 1,..,\ell,$$

so that

$$(\alpha - 1)\langle \nabla f_i(x(\overline{\alpha}_i)), \bar{x} - \hat{x} \rangle \leq 0, \quad i = 1,..,\ell,$$

or,

$$\langle \nabla f_i(x(\overline{\alpha}_i)), \bar{x} - \hat{x} \rangle \geq 0, \quad i = 1,..,\ell, \tag{6.25}$$

where, for at least one i, the inequality is strictly fulfilled. Recalling that f_i is convex on K if and only if ∇f_i is monotone on K [50], the following inequality holds, for any $\overline{\alpha}_r$, $\overline{\alpha}_s$, $r,s = 1,..,\ell$:

$$\langle \nabla f_i(x(\overline{\alpha}_r)) - \nabla f_i(x(\overline{\alpha}_s)), x(\overline{\alpha}_r) - x(\overline{\alpha}_s) \rangle \geq 0, \quad i = 1,..,\ell. \tag{6.26}$$

Since $x(\overline{\alpha}_r) - x(\overline{\alpha}_s) = (\overline{\alpha}_r - \overline{\alpha}_s)(\bar{x} - \hat{x})$, then, multiplying (6.26) by $(\overline{\alpha}_r - \overline{\alpha}_s)$, for $\overline{\alpha}_r > \overline{\alpha}_s$, we obtain:

$$\langle \nabla f_i(x(\overline{\alpha}_r)), \bar{x} - \hat{x} \rangle \geq \langle \nabla f_i(x(\overline{\alpha}_s)), \bar{x} - \hat{x} \rangle \geq 0, \quad i = 1,..,\ell,$$

which implies, taking into account (6.25), that, for $\overline{\alpha} := \max\{\overline{\alpha}_1,..,\overline{\alpha}_\ell\}$

$$\langle T(x(\overline{\alpha})), \bar{x} - \hat{x} \rangle \geq_{C_0} \mathbf{0}.$$

Multiplying both sides of the previous inequality by $1 - \overline{\alpha}$, we obtain:

$$\langle T(x(\overline{\alpha})), \bar{x} - x(\overline{\alpha}) \rangle \geq_{C_0} \mathbf{0},$$

which contradicts that $\bar{x}$ is a solution to Minty VVI. □

As we will see in the next subsection, the conclusions of Theorem 6.6 still hold, if f_i is assumed to be pseudoconvex on K, for $i = 1,..,\ell$, [61]. This theorem is further generalized to a weak Minty Vector Variational-like Inequality with a variable ordering cone in [3].

Relations between Minty VVI and the proper efficiency of a solution [6] are investigated in [14].

6.3.2 Optimality Conditions for Nondifferentiable VOP

The analysis, developed so far, has been extended to the nondifferentiable case by several authors (see e.g., [15, 41, 59]). In this section, we will outline only some of the results existing in the literature, and, for the sake of simplicity, we will consider a finite dimensional VOP, where $X = \mathbb{R}^n$, $Y = \mathbb{R}^\ell$, $C = \mathbb{R}^\ell_+$ and $K \subseteq X$ is a convex subset (see Chap. 7).

Let us define the *upper Dini directional derivative* of $\phi : \mathbb{R}^n \to \mathbb{R}$ at $\bar{x}$:

$$\phi^D(\bar{x};y) = \limsup_{t\downarrow 0} \frac{\phi(\bar{x}+ty) - \phi(x)}{t}, \tag{6.27}$$

where $y \in \mathbb{R}^n$. In place of (6.21), consider the generalized VVI that consist in finding $\bar{x} \in K$, such that:

$$f^D(\bar{x};x-\bar{x}) := (f_1^D(\bar{x};x-\bar{x}),\ldots,f_\ell^D(\bar{x};x-\bar{x})) \not\leq_{\overset{\mathbf{0}}{C}} \mathbf{0}, \quad \forall x \in K. \tag{6.28}$$

By means of the upper Dini directional derivative it is possible to introduce the definition of pseudoconvexity for a nondifferentiable function.

Definition 6.12 ([59]). $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be D^+*-pseudoconvex* on $K \subseteq \mathbb{R}^n$, iff, $\forall x,y \in K$,

$$\phi(x) < \phi(y) \quad \textit{implies} \quad \phi^D(y;x-y) < 0.$$

Next result is a straightforward extension of Theorem 6.4 (a) and (b) and it is analogous to [59, Theorem 2.2].

Theorem 6.7 ([59]). *Assume that there exist finite the upper Dini derivatives $f_i^D(\bar{x};\cdot)$ of f_i at $\bar{x}$, $i = 1,\ldots,\ell$. The following statements hold.*

(a) If $\bar{x}$ is a weak v.m.p. of VOP, then $\bar{x}$ solves (6.28).
(b) If f_i is D^+pseudoconvex on K, $i = 1,\ldots,\ell$, and $\bar{x}$ solves (6.28), then $\bar{x}$ is a weak v.m.p. of VOP.

Proof. (a) Let $\overset{\mathbf{0}}{C} := \operatorname{int} \mathbb{R}^\ell_+$ and assume that $\bar{x} \in K$ is a weak v.m.p. of VOP. Since K is convex, for any $x \in K$, we have that $\bar{x}+t(x-\bar{x}) \in K$, $\forall t \in]0,1[$. Then,

$$f(\bar{x}+t(x-\bar{x})) - f(\bar{x}) \not\leq_{\overset{\mathbf{0}}{C}} \mathbf{0}, \quad \forall t \in]0,1[,$$

which implies:

$$\frac{f(\bar{x}+t(x-\bar{x}))-f(\bar{x})}{t} \not\leq_{\substack{\mathbf{0}\\C}} \mathbf{0}, \quad \forall t \in]0,1[.$$

Taking the limit sup for $t \downarrow 0$, we obtain

$$f^D(\bar{x};x-\bar{x}) = \limsup_{t\downarrow 0} \frac{f(\bar{x}+t(x-\bar{x}))-f(\bar{x})}{t} \not\leq_{\substack{\mathbf{0}\\C}} \mathbf{0}, \quad \forall x \in K,$$

i.e., $\bar{x}$ is a solution to (6.28).

(b) Let $\bar{x}$ be a solution to (6.28). Ab absurdo, suppose that $\bar{x}$ is not a weak v.m.p. of VOP. Then $\exists \tilde{x} \in K$ such that:

$$f_i(\tilde{x}) < f_i(\bar{x}), \quad i = 1,..,\ell.$$

Since f_i is D^+-pseudoconvex on K, $i = 1,..,\ell$, then, we have:

$$f^D(\bar{x};\tilde{x}-\bar{x}) \leq_{\substack{\mathbf{0}\\C}} \mathbf{0},$$

which contradicts (6.28). □

In parallel with the classic VVI, for a nondifferentiable VOP, we can consider the following generalization of the Minty VVI, that consists in finding $\bar{x} \in K$ such that:

$$f^D(x;\bar{x}-x) := (f_1^D(x;\bar{x}-x),\dots,f_\ell^D(x;\bar{x}-x)) \not\geq_{C_{\mathbf{0}}} \mathbf{0}, \quad \forall x \in K. \tag{6.29}$$

Theorem 6.8 ([2]). *Let $K \subseteq \mathbb{R}^n$ be a convex set and suppose that the following assumptions hold.*

(i) For each $i \in \{1,..,\ell\}$, and for every $x \in K$ there exists finite the upper Dini derivative $f_i^D(x;\bar{x}-x)$ of $f_i : \mathbb{R}^n \to \mathbb{R}$.
(ii) For each $i \in \{1,..,\ell\}$, f_i is upper semicontinuous and D^+pseudoconvex on K.
(iii) For every $x \in K$ and every $y \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $f^D(x;y) \geq -f^D(x;-y)$.

Then, $\bar{x}$ is a v.m.p. of VOP if and only if it is a solution to the generalized Minty VVI defined by (6.29).

Theorem 6.8 extends to the nondifferentiable case Theorem 6.6 of the previous subsection. We refer to Chap. 7 for a deeper analysis of the properties of the upper Dini directional derivative and for an extension of the previous results by means of nonsmooth vector variational-like inequalities.

In the particular case where $f : \mathbb{R}^n \to \mathbb{R}^\ell$ is a $\mathbb{R}^\ell_+$-convex function on K, or equivalently, each of its components f_i is convex on K, $i = 1,..,\ell$, the following set-valued VVI can be associated with VOP:

$$\text{Find } \bar{x} \in K \quad \text{s.t.} \quad \exists \xi_i \in \partial f_i(\bar{x}),\ i = 1,..,\ell : \langle \xi, x - \bar{x} \rangle \not\leq_{\overset{\mathbf{0}}{C}} \mathbf{0}, \quad \forall x \in K, \tag{6.30}$$

where $\overset{\mathbf{0}}{C} := \text{int}\, \mathbb{R}^{\ell}_{+}$, $\xi = (\xi_1,..,\xi_\ell)$ and $\partial f_i(\bar{x})$ denotes the subdifferential at $\bar{x}$ of the convex function f_i, $i = 1,..,\ell$.

From statements (i) and (ii) of Theorem 6.4, it follows that, if f is a differentiable C-convex function, then the set of weak solutions of VOP coincides with the set of solutions of the VVI (6.21). Next theorem provides an analogous result in the nondifferentiable finite dimensional case.

Theorem 6.9 ([41]). *Let $X = \mathbb{R}^n$, $Y = \mathbb{R}^\ell$, $C = \mathbb{R}^\ell_+$ and assume that $f : X \to Y$ is C-convex on the convex set $K \subseteq \mathbb{R}^n$. Then $\bar{x}$ is a weak v.m.p. of VOP, if and only if $\bar{x}$ solves (6.30).*

Further developments can be found in [41], where, in particular, the scalarization of (6.30) is analysed. There exists a wide literature on scalarization of set-valued VVI: in particular, we mention the work of Konnov, Li, Luc and Yang (see [10, 26, 29, 39, 43] and the references therein).

6.4 Image Space Analysis and Saddle Point Optimality Conditions

The Image Space Analysis [27] is a unifying scheme for studying constrained extremum problems, variational inequalities, and, more generally, can be applied to any kind of problem that can be expressed under the form of the impossibility of a parametric system.

In this approach, the impossibility of such a system is reduced to the disjunction of two suitable subsets, $\mathscr{K}$ and $\mathscr{H}$, of the Image Space (in short, IS) associated with the given problem. Such a disjunction can be proved by showing that the two sets lie in two disjoint level sets of a suitable, possibly vector, separation functional.

In the particular case of a VOP, by means of the Image Space Analysis, several topics can be developed, as Lagrangian-type necessary optimality conditions, saddle point sufficient conditions, regularity and duality [28]. In the present section, we will present the Image Space Analysis for VOP and we will derive sufficient optimality conditions for VOP arising from the existence of a vector separation in the IS associated with VOP. In particular, we will show that the existence of a vector separation in the IS is equivalent to a vector saddle point condition for a suitable vector Lagrangian function associated with VOP. Finally, we analyse the particular case of a scalar Lagrangian function and we outline some connections with duality theory for VOP.

6.4.1 Image of VOP

Let $X, Y, \mathscr{Z}$, be locally convex Hausdorff topological vector spaces, $f : X \to Y$ and $g : X \to \mathscr{Z}$. Consider the VOP defined by:

$$\min_C f(x), \quad s.t. \quad x \in K := \{x \in S : g(x) \geq_D \mathbf{0}\}, \tag{6.31}$$

where $S \subseteq X$ is a convex subset, $C \subset Y$ and $D \subset \mathscr{Z}$ are nonempty closed, pointed and convex cones, and C has nonempty interior.

We observe that $\bar{x} \in K$ is a v.m.p. of VOP if and only if the following system

$$f(\bar{x}) - f(x) \geq_{C_0} \mathbf{0}\ ,\ g(x) \geq_D \mathbf{0}\ ,\ x \in S\ , \tag{6.32}$$

is impossible.

Let us introduce the map $A_{\bar{x}} : X \longrightarrow Y \times \mathscr{Z}$, defined by

$$A_{\bar{x}}(x) := (f(\bar{x}) - f(x), g(x)), \tag{6.33}$$

and the sets

$$\mathscr{K}_{\bar{x}} := \{(u,v) \in Y \times \mathscr{Z} : u = f(\bar{x}) - f(x),\ v = g(x),\ x \in S\} = A_{\bar{x}}(S),$$

$$\mathscr{H}_{C_0} := \{(u,v) \in Y \times \mathscr{Z} : u \geq_{C_0} \mathbf{0},\ v \geq_D \mathbf{0}\}.$$

$\mathscr{K}_{\bar{x}}$ is called the *image* associated with VOP.

Now, observe that system (6.32) is impossible if and only if

$$\mathscr{K}_{\bar{x}} \cap \mathscr{H}_{C_0} = \emptyset\ . \tag{6.34}$$

Hence, $\bar{x} \in K$ is a v.m.p. of VOP if and only if (6.34) holds.

In general, the image set is not convex even when the involved functions are convex. To overcome this difficulty, we introduce a regularization of the image $\mathscr{K}_{\bar{x}}$ with respect to the cone $\operatorname{cl} \mathscr{H}_{C_0}$: the set

$$\mathscr{E}_{\bar{x}} := \mathscr{K}_{\bar{x}} - \operatorname{cl} \mathscr{H}_{C_0} = \{(u,v) \in Y \times \mathscr{Z} : u \leq_C f(\bar{x}) - f(x),\ v \leq_D g(x),\ x \in S\}$$

is called the *extended image* associated with VOP.

The following statement motivates the introduction of $\mathscr{E}_{\bar{x}}$:

Proposition 6.9. *Condition (6.34) holds if and only if*

$$\mathscr{E}_{\bar{x}} \cap \mathscr{H}_{C_0} = \emptyset. \tag{6.35}$$

Proof. We observe that $\mathscr{H}_{C_0} + \operatorname{cl} \mathscr{H}_{C_0} = \mathscr{H}_{C_0}$. Since

$$\mathscr{E}_{\bar{x}} - \mathscr{H}_{C_0} = \mathscr{K}_{\bar{x}} - (\operatorname{cl} \mathscr{H}_{C_o} + \mathscr{H}_{C_0}) = \mathscr{K}_{\bar{x}} - \mathscr{H}_{C_0},$$

we have that

$$\mathbf{0} \in \mathscr{E}_{\bar{x}} - \mathscr{H}_{C_\mathbf{0}} \quad \text{iff} \quad \mathbf{0} \in \mathscr{K}_{\bar{x}} - \mathscr{H}_{C_\mathbf{0}},$$

so that (6.35) is equivalent to (6.34). □

The optimality condition for $\bar{x}$ can be expressed in terms of the disjunction between the sets $\mathscr{E}_{\bar{x}}$ and $\mathscr{H}_{C_\mathbf{0}}$.

Proposition 6.10. *$\bar{x} \in K$ is a v.m.p. of VOP, if and only if (6.34), or equivalently (6.35), holds.*

Similarly, by defining

$$\mathscr{H}_{\underset{C}{\mathbf{0}}} := \{(u,v) \in Y \times \mathscr{Z} : u \geq_{\underset{C}{\mathbf{0}}} \mathbf{0},\ v \geq_D \mathbf{0}\},$$

we obtain the following result:

Proposition 6.11. *$\bar{x} \in K$ is a weak v.m.p. of VOP iff*

$$\mathscr{K}_{\bar{x}} \cap \mathscr{H}_{\underset{C}{\mathbf{0}}} = \emptyset\ , \tag{6.36}$$

or, equivalently,

$$\mathscr{E}_{\bar{x}} \cap \mathscr{H}_{\underset{C}{\mathbf{0}}} = \emptyset. \tag{6.37}$$

Observe that, since *cl* $\mathscr{H}_{\underset{C}{\mathbf{0}}} = \text{cl}\ \mathscr{H}_{C_\mathbf{0}}$, then $\mathscr{E}_{\bar{x}} = \mathscr{K}_{\bar{x}} - \text{cl}\ \mathscr{H}_{\underset{C}{\mathbf{0}}}$, so that the extended image in (6.37) coincides with $\mathscr{E}_{\bar{x}}$.

Suitable generalized convexity assumptions on the functions involved, ensure the convexity of the extended image $\mathscr{E}_{\bar{x}}$.

Proposition 6.12 ([57]). *The set $\mathscr{E}_{\bar{x}}$ is convex, if and only if $-A_{\bar{x}}$ is $(C \times D)$-convexlike on S, where $A_{\bar{x}}$ is defined by (6.33).*

Proof. From Proposition 6.3, $-A_{\bar{x}}$ is $(C \times D)$-convexlike on S, if and only if

$$-A_{\bar{x}}(S) + (C \times D)$$

is a convex set. This is equivalent to the fact that

$$\mathscr{E}_{\bar{x}} := \mathscr{K}_{\bar{x}} - \text{cl}\ \mathscr{H}_{C_\mathbf{0}} = A_{\bar{x}}(S) - (C \times D)$$

is a convex set, which completes the proof. □

In particular, we obtain the following result.

Corollary 6.3. *If $-A_{\bar{x}}$ is $(C \times D)$-convex on S, then $\mathscr{E}_{\bar{x}}$ is a convex set.*

Proof. It is enough to note that if $-A_{\bar{x}}$ is $(C \times D)$-convex on S, then $-A_{\bar{x}}$ is $(C \times D)$-convexlike on S, so that the thesis follows from Proposition 6.12. □

6.4.2 Vector Separation in the Image Space

Conditions (6.35), or (6.37), can be proved by showing that $\mathscr{E}_{\bar{x}}$ and $\mathscr{H}_{C_\mathbf{0}}$, or $\mathscr{H}_{\underset{C}{\mathbf{0}}}$, lie in two disjoint level sets of a suitable vector functional.

Definition 6.13. Let Ω be a set of parameters and Q be a pointed, convex cone in $\mathbb{R}^k$. A function $w : Y \times \mathscr{Z} \longrightarrow \mathbb{R}^k$, depending on the parameter $(\Theta, \Lambda) \in \Omega$, is a *separation function* iff, $\forall(\Theta, \Lambda) \in \Omega$,

$$w(u,v;\Theta,\Lambda) \geq_Q \mathbf{0}, \quad \forall(u,v) \in \mathscr{H}_{C_\mathbf{0}}.$$

If $k > 1$, then w is a *vector separation function*; if $k = 1$, then Definition 6.13 recovers Definition 1.1 given by Giannessi in [23], with $Z = Q^c$.

In order to introduce a suitable class of vector separation functions, we need to extend the classic concept of polar of a cone.

Definition 6.14. Let $D \subset \mathscr{Z}$ be a convex cone and Q be a closed, pointed, convex cone in $\mathbb{R}^k$. The *vector polar* of D with respect to Q is given by

$$D^*_Q := \{\phi \in (\mathscr{Z}^*)^k : \phi d \geq_Q \mathbf{0} \ , \ \forall d \in D\},$$

where

$$\phi d = \begin{pmatrix} \langle \phi_1, d \rangle \\ \vdots \\ \langle \phi_k, d \rangle \end{pmatrix}, \quad \phi_i \in \mathscr{Z}^*, \ i = 1,..,k.$$

At $k = 1$, D^*_Q becomes either the *positive* or the *negative polar cone* of D, according to $Q = \mathbb{R}_+$ or $Q = \mathbb{R}_-$, respectively.

Now, we can introduce the above mentioned class of vector separation functions which will allow us to prove (6.34) or (6.36). Let $\Omega := C^*_Q \times D^*_Q$ and $w : Y \times \mathscr{Z} \times \Omega \to \mathbb{R}^k$, given by

$$w = w(u,v;\Theta,\Lambda) := \Theta u + \Lambda v \ , \quad \Theta \in C^*_Q \ , \quad \Lambda \in D^*_Q \ . \tag{6.38}$$

We observe that w is a linear separation function, according to Definition 6.13.

For any $\Theta \in C^*_Q$ and $\Lambda \in D^*_Q$, set

$$lev_Q \, w(u,v;\Theta,\Lambda) := \{(u,v) \in Y \times \mathscr{Z} : w(u,v;\Theta,\Lambda) \geq_Q \mathbf{0}\}.$$

The following result is proved in [28].

Proposition 6.13. *Let $Q_\mathbf{0} := Q \setminus \{\mathbf{0}\}$ and let w be given by (6.38).*

$$(a) \qquad \mathscr{H}_{C_{\mathbf{0}}} \subset \bigcap_{\Theta \in (C_{\mathbf{0}})^*_{Q_{\mathbf{0}}}, \Lambda \in D^*_Q} lev_{Q_{\mathbf{0}}} w(u,v;\Theta,\Lambda). \tag{6.39}$$

$$(b) \qquad \mathscr{H}_{\overset{\mathbf{0}}{C}} \subset \bigcap_{\Theta \in (\overset{\mathbf{0}}{C})^*_{\overset{\mathbf{0}}{Q}}, \Lambda \in D^*_Q} lev_{\overset{\mathbf{0}}{Q}} w(u,v;\Theta,\Lambda). \tag{6.40}$$

Proof. (a) $(u,v) \in \mathscr{H}_{C_{\mathbf{0}}} \iff u \in C_{\mathbf{0}}, v \in D$. Therefore, $\forall \Theta \in (C_{\mathbf{0}})^*_{Q_{\mathbf{0}}}$ and $\forall \Lambda \in D^*_Q$, we have $\Theta u + \Lambda v \in Q + Q_{\mathbf{0}} = Q_{\mathbf{0}}$, since Q is pointed; hence $w(u,v;\Theta,\Lambda) \geq_{Q_{\mathbf{0}}} \mathbf{0}$ and (6.39) holds.

(b) $(u,v) \in \mathscr{H}_{\overset{\mathbf{0}}{C}} \iff u \in \overset{\mathbf{0}}{C}, v \in D$. Therefore, $\forall \Theta \in (\overset{\mathbf{0}}{C})^*_{\overset{\mathbf{0}}{Q}}$ and $\forall \Lambda \in D^*_Q$, we have $\Theta u + \Lambda v \in \overset{\mathbf{0}}{Q} + Q = \overset{\mathbf{0}}{Q}$; hence $w(u,v;\Theta,\Lambda) \geq_{\overset{\mathbf{0}}{Q}} \mathbf{0}$ and (6.40) holds. □

Corollary 6.4. *Let $\overline{x} \in K$.*

(a) If there exist $\overline{\Theta} \in (C_{\mathbf{0}})^*_{Q_{\mathbf{0}}}$ and $\overline{\Lambda} \in D^*_Q$ such that

$$w(u,v;\overline{\Theta},\overline{\Lambda}) \not\geq_{Q_{\mathbf{0}}} \mathbf{0}, \quad \forall (u,v) \in \mathscr{K}_{\overline{x}}, \tag{6.41}$$

then $\overline{x}$ is a v.m.p. of VOP.

(b) If there exist $\Theta \in (\overset{\mathbf{0}}{C})^*_{\overset{\mathbf{0}}{Q}}$ and $\Lambda \in D^*_Q$, such that:

$$w(u,v;\overline{\Theta},\overline{\Lambda}) \not\geq_{\overset{\mathbf{0}}{Q}} \mathbf{0}, \quad \forall (u,v) \in \mathscr{K}_{\overline{x}}, \tag{6.42}$$

then $\overline{x}$ is a weak v.m.p. of VOP.

Proof. (a) From Proposition 6.13 (a), it follows that (6.41) implies that (6.34) is fulfilled.

(b) From Proposition 6.13 (b), it follows that (6.42) implies that (6.36) is fulfilled. □

The existence of a vector separation between $\mathscr{K}_{\overline{x}}$ and $\mathscr{H}_{C_{\mathbf{0}}}$ can be characterized by means of a vector saddle point condition on a suitable *vector Lagrangian function* associated with VOP, $\mathscr{L} : C^*_Q \times D^*_Q \times X \to \mathbb{R}^k$, defined by

$$\mathscr{L}(\Theta;\Lambda,x) := \Theta f(x) - \Lambda g(x)\ . \tag{6.43}$$

Definition 6.15. Let $\overline{\Theta} \in C^*_Q$.

- $(\overline{\Lambda},\overline{x}) \in D^*_Q \times S$ is a *vector saddle point* for $\mathscr{L}(\overline{\Theta};\Lambda,x)$ on $D^*_Q \times S$ iff $(\overline{\Theta},\overline{\Lambda}) \neq (\mathbf{0},\mathbf{0})$, and

$$\mathscr{L}(\overline{\Theta};\overline{\Lambda},x) \not\leq_{Q_{\mathbf{0}}} \mathscr{L}(\overline{\Theta};\overline{\Lambda},\overline{x}) \not\leq_{Q_{\mathbf{0}}} \mathscr{L}(\overline{\Theta};\Lambda,\overline{x}), \quad \forall x \in S,\ \forall \Lambda \in D^*_Q. \tag{6.44}$$

- $(\overline{\Lambda},\overline{x}) \in D_Q^* \times S$ is a *weak vector saddle point* for $\mathscr{L}(\overline{\Theta};\Lambda,x)$ on $D_Q^* \times S$ iff $(\overline{\Theta},\overline{\Lambda}) \neq (\mathbf{0},\mathbf{0})$, and

$$\mathscr{L}(\overline{\Theta};\overline{\Lambda},x) \not\leq_{\substack{\mathbf{0}\\Q}} \mathscr{L}(\overline{\Theta};\overline{\Lambda},\overline{x}) \not\leq_{\substack{\mathbf{0}\\Q}} \mathscr{L}(\overline{\Theta};\Lambda,\overline{x}), \quad \forall x \in S,\ \forall \Lambda \in D_Q^*. \tag{6.45}$$

Concerning the class of separation functions (6.38), we can state the following result.

Theorem 6.10. *The following statements are equivalent.*

(a) $\overline{x} \in K$ *and there exists* $(\overline{\Theta},\overline{\Lambda}) \in C_Q^* \times D_Q^*$ *such that:*

$$w(u,v;\overline{\Theta},\overline{\Lambda}) \not\geq_{Q_\mathbf{0}} \mathbf{0}, \quad \forall (u,v) \in \mathscr{K}_{\overline{x}}; \tag{6.46}$$

(b) $(\overline{\Lambda},\overline{x}) \in D_Q^* \times S$ *is a vector saddle point for* $\mathscr{L}(\overline{\Theta};\Lambda,x)$ *on* $D_Q^* \times S$*, with* $\overline{\Theta} \in C_Q^*$*.*

Proof. (a) $\Rightarrow$ (b). (a) is equivalent to the condition

$$\overline{\Theta}(f(\overline{x}) - f(x)) + \overline{\Lambda} g(x) \not\geq_{Q_\mathbf{0}} \mathbf{0}, \quad \forall x \in S. \tag{6.47}$$

By setting $x = \overline{x}$ in (6.47), we obtain $\overline{\Lambda} g(\overline{x}) \not\geq_{Q_\mathbf{0}} \mathbf{0}$. Since $\overline{\Lambda} \in D_Q^*$ and $\overline{x} \in K$, we have that $\overline{\Lambda} g(\overline{x}) \geq_Q \mathbf{0}$ and therefore

$$\overline{\Lambda} g(\overline{x}) = \mathbf{0}. \tag{6.48}$$

Taking into account (6.48), condition (6.47) is equivalent to

$$\mathscr{L}(\overline{\Theta};\overline{\Lambda},\overline{x}) \not\geq_{Q_\mathbf{0}} \mathscr{L}(\overline{\Theta};\overline{\Lambda},x), \quad \forall x \in S. \quad (\overline{\Theta},\overline{\Lambda}) \neq (\mathbf{0},\mathbf{0})$$

In order to show the second inequality in (6.44), observe that for every $\Lambda \in D_Q^*$, we have $\Lambda g(\overline{x}) \geq_Q \mathbf{0}$. Therefore,

$$-\mathscr{L}(\overline{\Theta};\overline{\Lambda},\overline{x}) + \mathscr{L}(\overline{\Theta};\Lambda,\overline{x}) = -\Lambda g(\overline{x}) \not\geq_{Q_\mathbf{0}} \mathbf{0}\ , \quad \forall \Lambda \in D_Q^*,$$

taking into account that Q is a pointed cone and (6.48).
(b) $\Rightarrow$ (a). We preliminarily prove that $\overline{x} \in K$. From the second inequality in (6.44), we draw:

$$(\overline{\Lambda} - \Lambda) g(\overline{x}) \not\geq_{Q_\mathbf{0}} \mathbf{0}, \quad \forall \Lambda \in D_Q^*.$$

Set $A := \{y \in \mathbb{R}^k : y = (\overline{\Lambda} - \Lambda) g(\overline{x}),\ \Lambda \in D_Q^*\}$. Then A is a convex set such that

$$A \cap Q_\mathbf{0} = \emptyset.$$

Since $Q_\mathbf{0}$ is a convex cone, $\exists c^* \in Q^* \setminus \{\mathbf{0}\}$ such that

$$\langle c^*, (\overline{\Lambda} - \Lambda) g(\overline{x}) \rangle \leq 0, \quad \forall \Lambda \in D_Q^*. \tag{6.49}$$

Ab absurdo, assume that $\bar{x} \notin K$. Then $\exists \lambda^* \in D^*$ such that

$$\langle \lambda^*, g(\bar{x}) \rangle < 0. \tag{6.50}$$

Actually, if (6.50) is not fulfilled for any $\lambda^* \in D^*$, then, by Lemma 6.1 (a), we have that $g(\bar{x}) \geq_D \mathbf{0}$. Consider $\Lambda_\alpha \in D_Q^*$ defined, for every $y \in \mathscr{Z}$, by the relation

$$\Lambda_\alpha y := \alpha \langle \lambda^*, y \rangle e, \quad \alpha > 0,$$

where $e \in \overset{\mathbf{0}}{Q}$. Substituting Λ_α in (6.49) we obtain:

$$\alpha \langle c^*, e \rangle \cdot \langle \lambda^*, g(\bar{x}) \rangle \geq \langle c^*, \overline{\Lambda} g(\bar{x}) \rangle, \quad \forall \alpha > 0. \tag{6.51}$$

Since the limit for $\alpha \to +\infty$, in the left-hand side of (6.51), is $-\infty$, we achieve a contradiction, which proves that $\bar{x} \in K$. From the condition

$$-\mathscr{L}(\overline{\Theta}; \overline{\Lambda}, \bar{x}) + \mathscr{L}(\overline{\Theta}; \Lambda, \bar{x}) \not\geq_{Q_\mathbf{0}} \mathbf{0}\ , \quad \forall \Lambda \in D_Q^*,$$

computed for $\Lambda \equiv \mathbf{0}$, we obtain $\overline{\Lambda} g(\bar{x}) \not\geq_{Q_\mathbf{0}} \mathbf{0}$ and, since $\overline{\Lambda} \in D_Q^*$ and $\bar{x} \in K$, we have (6.48). Similarly to the proof of the reverse implication, exploiting the complementarity relation (6.48), we have that the condition

$$\mathscr{L}(\overline{\Theta}; \overline{\Lambda}, \bar{x}) \not\geq_{Q_\mathbf{0}} \mathscr{L}(\overline{\Theta}; \overline{\Lambda}, x)\ , \quad \forall x \in S,$$

is equivalent to (6.47), and (a) is proved. □

Next theorem shows the relationships between the existence of a weak vector saddle point and vector separation. In the present case the complementarity relation (6.48) may not be fulfilled (see [7, Example 4.1]); hence, it must be taken as an additional assumption.

Theorem 6.11. *The following statements are equivalent.*

(a) $\bar{x} \in K$ *and there exists* $(\overline{\Theta}, \overline{\Lambda}) \in C_Q^* \times D_Q^* (\overline{\Theta}, \overline{\Lambda}) \neq (\mathbf{0}, \mathbf{0})$ *such that:*

$$w(u, v; \overline{\Theta}, \overline{\Lambda}) \not\geq_{\overset{\mathbf{0}}{Q}} \mathbf{0}, \quad \forall (u, v) \in \mathscr{K}_{\bar{x}}, \quad \textit{and} \quad \overline{\Lambda} g(\bar{x}) = \mathbf{0}; \tag{6.52}$$

(b) $(\overline{\Lambda}, \bar{x}) \in D_Q^* \times S$ *is a weak vector saddle point of* $\mathscr{L}(\overline{\Theta}, \Lambda, x)$ *on* $D_Q^* \times S$, *with* $\overline{\Theta} \in C_Q^*$ *and* $\overline{\Lambda} g(\bar{x}) = \mathbf{0}$.

Proof. It is analogous to that of Theorem 6.10. □

Exploiting the separation results in the image space, we are now in position to obtain vector saddle point optimality conditions for VOP defined by (6.31).

Proposition 6.14. *Let $Q \subset \mathbb{R}^k$ be a convex, pointed cone with nonempty interior and let $\overline{x} \in K$.*

(a) If there exists $(\overline{\Theta}, \overline{\Lambda}) \in (C_{\mathbf{0}})^*_{Q_{\mathbf{0}}} \times D^*_Q$ such that

$$\overline{\Theta}(f(\overline{x}) - f(x)) + \overline{\Lambda} g(x) \not\geq_{Q_{\mathbf{0}}} \mathbf{0}, \quad \forall x \in S, \tag{6.53}$$

then $\overline{x}$ is a v.m.p. of VOP.

(b) If there exists $(\overline{\Theta}, \overline{\Lambda}) \in (\overset{\mathbf{0}}{C})^*_{\underset{Q}{\mathbf{0}}} \times D^*_Q$ such that

$$\overline{\Theta}(f(\overline{x}) - f(x)) + \overline{\Lambda} g(x) \not\geq_{\underset{Q}{\mathbf{0}}} \mathbf{0} \;, \quad \forall x \in S\,, \tag{6.54}$$

then $\overline{x}$ is a weak v.m.p. of VOP.

Proof. It follows from Corollary 6.4, taking into account that (6.41) and (6.42) are equivalent to (6.53) and (6.54), respectively. □

Coupling Proposition 6.14 and Theorem 6.10 or Theorem 6.11, vector saddle point sufficient optimality conditions for VOP can be derived. Consider the vector Lagrangian function associated with VOP and defined by (6.43).

Theorem 6.12. *Let $\overline{x} \in S$.*

(a) If there exists $(\overline{\Theta}, \overline{\Lambda}) \in (C_{\mathbf{0}})^*_{Q_{\mathbf{0}}} \times D^*_Q$ such that $(\overline{\Lambda}, \overline{x})$ is a vector saddle point for $\mathscr{L}(\overline{\Theta}; \Lambda, x)$ on $D^*_Q \times S$, then $\overline{x}$ is a v.m.p. of VOP.

(b) If there exists $(\overline{\Theta}, \overline{\Lambda}) \in (\overset{\mathbf{0}}{C})^*_{\underset{Q}{\mathbf{0}}} \times D^*_Q$ such that $(\overline{\Lambda}, \overline{x})$ is a weak vector saddle point for $\mathscr{L}(\overline{\Theta}; \Lambda, x)$ on $D^*_Q \times S$ with $\overline{\Lambda} g(\overline{x}) = \mathbf{0}$, then $\overline{x}$ is a weak v.m.p. of VOP.

Proof. (a) We preliminarily observe that $\overline{\Theta} \in (C_{\mathbf{0}})^*_{Q_{\mathbf{0}}}$ implies that $\overline{\Theta} \in C^*_Q$. From Theorem 6.10, it follows that (6.41) holds, and by Proposition 6.14 (a), we obtain that $\overline{x}$ is a v.m.p. of VOP.

(b) It is similar to the one of (a), replacing Theorem 6.10 with Theorem 6.11 and Proposition 6.14 (a) with Proposition 6.14 (b). □

For general results concerning the existence of saddle points of vector valued functions, see e.g., [54, 55] and references therein.

6.4.3 Scalar Separation in the Image Space

Definition 6.13 introduces a general concept of separation in the IS. The separation functional can be a nonlinear vector function. In the previous subsection, we have analysed the particular case of a linear vector separation. In the present one we address our attention to linear and nonlinear scalar separation.

Let $\Omega := (\mathscr{Q} \times \Xi)$ be a set of parameters, we introduce the class of functions $w : Y \times \mathscr{Z} \times \Omega \longrightarrow \mathbb{R}$, given by:

$$w(u,v;\theta,\lambda) := \alpha(u;\theta) + \gamma(v;\lambda), \quad \theta \in \mathscr{Q}, \lambda \in \Xi, \tag{6.55}$$

where $\alpha : Y \times \mathscr{Q} \to \mathbb{R}$, $\gamma : \mathscr{Z} \times \Xi \longrightarrow \mathbb{R}$, fulfil the following conditions:

$$\bigcap_{\theta \in \mathscr{Q}} \text{lev}_{\geq 0}\alpha(\cdot;\theta) \supseteq C_{\mathbf{0}}, \qquad \bigcap_{\lambda \in \Xi} \text{lev}_{\geq 0}\gamma(\cdot;\lambda) \supseteq D, \tag{6.56}$$

where, for a fixed $\lambda \in \Xi$,

$$\text{lev}_{\geq a}\gamma(\cdot;\lambda) := \{v \in \mathscr{Z} : \gamma(v;\lambda) \geq a\}, \quad a \in \mathbb{R},$$

denotes the level set of the function $\gamma(\cdot;\lambda)$ and similarly for $\text{lev}_{\geq a}\alpha(\cdot;\theta)$.

Conditions (6.56) guarantee that (6.55) is a separation function according to Definition 6.13, where we have set $Q := \mathbb{R}_+$. The function α can be taken as a scalarization function for VOP.

Let us analyse, at first, a scalar linear separation function, i.e., in (6.55), we set:

$$\alpha(u;\theta) = \langle \theta, u \rangle, \quad \theta \in Y^*, \qquad \gamma(v;\lambda) = \langle \lambda, v \rangle, \quad \lambda \in \mathscr{Z}^*.$$

In such a case, (6.46) becomes:

$$\langle \theta^*, u \rangle + \langle \lambda^*, v \rangle \leq 0, \qquad \forall (u,v) \in \mathscr{K}_{\bar{x}}, \tag{6.57}$$

for a suitable $(\theta^*, \lambda^*) \in C^* \times D^*$, $(\theta^*, \lambda^*) \neq \mathbf{0}$.

If (6.57) holds, then we say that the sets $\mathscr{K}_{\bar{x}}$ and $\mathscr{H}_{C_{\mathbf{0}}}$ admit a linear separation.

Remark 6.5. In the given space X, (6.57) is equivalent to the condition:

$$\langle \theta^*, f(\bar{x}) - f(x) \rangle + \langle \lambda^*, g(x) \rangle \leq 0, \quad \forall x \in S. \tag{6.58}$$

Next result shows that a linear functional separates $\mathscr{K}_{\bar{x}}$ and $\mathscr{H}_{C_{\mathbf{0}}}$ iff it separates $\mathscr{E}_{\bar{x}}$ and $\mathscr{H}_{C_{\mathbf{0}}}$.

Proposition 6.15. *Let* $(\theta^*, \lambda^*) \in C^* \times D^*$, $(\theta^*, \lambda^*) \neq \mathbf{0}$*, Then (6.57) is equivalent to*

$$\langle \theta^*, u \rangle + \langle \lambda^*, v \rangle \leq 0, \qquad \forall (u,v) \in \mathscr{E}_{\bar{x}}. \tag{6.59}$$

Proof. Suppose that (6.57) holds. Let $(h_1, h_2) \in \text{cl } \mathscr{H}_{C_{\mathbf{0}}} = (C \times D)$. Since $\langle \theta^*, -h_1 \rangle + \langle \lambda^*, -h_2 \rangle \leq 0$, then

$$\langle \theta^*, u - h_1 \rangle + \langle \lambda^*, v - h_2 \rangle \leq 0, \quad \forall (u,v) \in \mathscr{K}_{\bar{x}}, \quad \forall (h_1, h_2) \in \text{cl } \mathscr{H}_{C_{\mathbf{0}}},$$

and (6.59) holds. It is obvious that (6.59) implies (6.57), since $\mathscr{K}_{\bar{x}} \subseteq \mathscr{E}_{\bar{x}}$. □

We observe that, since $\mathrm{cl}\ \mathscr{H}_{\underset{C}{\mathbf{0}}} = \mathrm{cl}\ \mathscr{H}_{C_{\mathbf{0}}}$, then a continuous linear functional separates $\mathscr{E}_{\bar{x}}$ and $\mathscr{H}_{C_{\mathbf{0}}}$ iff it separates $\mathscr{E}_{\bar{x}}$ and $\mathscr{H}_{\underset{C}{\mathbf{0}}}$.

The existence of a linear separation is ensured by suitable generalized convexity assumptions on the function $-A_{\bar{x}}$, defined by (6.33).

Proposition 6.16. *Let $\bar{x}$ be a weak v.m.p. of VOP and assume that $\overset{\mathbf{0}}{D} \neq \emptyset$. If $-A_{\bar{x}}$ is generalized $(C \times D)$-subconvexlike on S, then $\mathscr{K}_{\bar{x}}$ and $\mathscr{H}_{C_{\mathbf{0}}}$ admit a linear separation.*

Proof. We have to prove that there exist $\theta^* \in C^*$ and $\lambda^* \in D^*$, $(\theta, \lambda) \neq \mathbf{0}$ such that (6.57) holds. Recall that $\bar{x}$ is a weak v.m.p. iff $\mathscr{K}_{\bar{x}}$ and $\mathscr{H}_{\underset{C}{\mathbf{0}}}$ are disjoint convex sets (see Proposition 6.11). This implies that

$$\mathscr{K}_{\bar{x}} \cap int \mathscr{H}_{\underset{C}{\mathbf{0}}} = \emptyset \tag{6.60}$$

We observe that $\mathrm{int}\ \mathscr{H}_{\underset{C}{\mathbf{0}}} = \overset{\mathbf{0}}{C} \times \overset{\mathbf{0}}{D}$ is nonempty. Moreover, (6.60) is equivalent to:

$$\mathrm{cone}\ (\mathscr{K}_{\bar{x}}) \cap \mathrm{int}\ \mathscr{H}_{\underset{C}{\mathbf{0}}} = \emptyset,$$

and, recalling that $\mathscr{K}_{\bar{x}} = A_{\bar{x}}(S)$, to the condition:

$$\mathbf{0} \notin \mathrm{cone}\ (-A_{\bar{x}}(S)) + (\overset{\mathbf{0}}{C} \times \overset{\mathbf{0}}{D}). \tag{6.61}$$

Since $-A_{\bar{x}}$ is generalized $(C \times D)$-subconvexlike on S, then, by Proposition 6.4 (b), the set $\mathrm{cone}\ (-A_{\bar{x}}(S)) + (\overset{\mathbf{0}}{C} \times \overset{\mathbf{0}}{D})$ is convex and, moreover it has nonempty interior.

Applying the separation theorem for convex sets (Theorem 6.1), we have that there exist $(\theta^*, \lambda^*) \in Y^* \times \mathscr{Z}^*$, $(\theta^*, \lambda^*) \neq \mathbf{0}$, such that

$$\langle \theta^*, u \rangle + \langle \lambda^*, v \rangle \leq 0, \qquad \forall (u,v) \in \mathrm{cone}\ (A_{\bar{x}}(S)) - (\overset{\mathbf{0}}{C} \times \overset{\mathbf{0}}{D}). \tag{6.62}$$

Observe that, by continuity, (6.62) holds for any $(u,v) \in \mathrm{cl}\ [\mathrm{cone}\ (A_{\bar{x}}(S)) - (\overset{\mathbf{0}}{C} \times \overset{\mathbf{0}}{D})]$. Recalling that for any sets M_1 and M_2 in a Hausdorff topological vector space, $\mathrm{cl}\ (M_1 + M_2) \supseteq \mathrm{cl}\ M_1 + \mathrm{cl}\ M_2$, we have:

$$\mathrm{cl}\ [\mathrm{cone}\ (A_{\bar{x}}(S)) - (\overset{\mathbf{0}}{C} \times \overset{\mathbf{0}}{D})] \supseteq \mathrm{cl}\ \mathrm{cone}\ (A_{\bar{x}}(S)) - (C \times D) \supseteq A_{\bar{x}}(S) - (C \times D) = \mathscr{E}_{\bar{x}}.$$

From (6.62) we have:

$$\langle \theta^*, u \rangle + \langle \lambda^*, v \rangle \leq 0, \qquad \forall (u,v) \in \mathscr{E}_{\bar{x}}. \tag{6.63}$$

To complete the proof, we only need to show that $\theta^* \in C^*$ and $\lambda^* \in D^*$. To this aim, observe that, since $\bar{x} \in K$, then:

$$(u,v) := (\mathbf{0}, g(\bar{x})) - (\mathbf{0}, g(\bar{x}) + d) \in \mathscr{E}_{\bar{x}}, \quad \forall d \in D.$$

From (6.63), it follows that

$$\langle \lambda^*, -d \rangle \leq 0, \quad \forall d \in D,$$

i.e., $\lambda^* \in D^*$. Similarly, the point

$$(u,v) := (0, g(\bar{x})) - (c, g(\bar{x})) \in \mathscr{E}_{\bar{x}}, \quad \forall c \in C,$$

and from (6.63),

$$\langle \theta^*, -c \rangle \leq 0, \quad \forall c \in C,$$

i.e., $\theta^* \in C^*$. Taking into account Proposition 6.15, we have that (6.57) holds. □

We observe that the existence of a linear separation, i.e., (6.57) is fulfilled, does not imply that $\mathscr{K}_{\bar{x}} \cap \mathscr{H}_{\overset{\mathbf{0}}{C}} = \emptyset$. In order to ensure the disjunction between the two sets (or between $\mathscr{K}_{\bar{x}}$ and $\mathscr{H}_{C_{\mathbf{0}}}$), some restrictions on the choice of the multiplier θ^* must be imposed.

Proposition 6.17. *Assume that condition (6.57) holds or, equivalently, the sets $\mathscr{K}_{\bar{x}}$ and $\mathscr{H}_{C_{\mathbf{0}}}$ admit a linear separation.*

(a) If $\theta^* \in C^+$ then (6.34) is fulfilled.
(b) If $\theta^* \in C^* \setminus \{\mathbf{0}\}$ then (6.36) is fulfilled.

Proof. (a) Ab absurdo, suppose that $\mathscr{K}_{\bar{x}} \cap \mathscr{H}_{C_{\mathbf{0}}} \neq \emptyset$. Therefore, $\exists z \in K$ such that $f(\bar{x}) - f(z) \in C_{\mathbf{0}}$. Then, taking into account that $\theta^* \in C^+$, we have

$$0 < \langle \theta^*, f(\bar{x}) - f(z) \rangle \leq \langle \theta^*, f(\bar{x}) - f(z) \rangle + \langle \lambda^*, g(z) \rangle \leq 0, \tag{6.64}$$

which is impossible.

(b) Ab absurdo, suppose that $\mathscr{K}_{\bar{x}} \cap \mathscr{H}_{\overset{\mathbf{0}}{C}} \neq \emptyset$. Following the proof of part (b), $\exists z \in K$ such that $f(\bar{x}) - f(z) \in \overset{\mathbf{0}}{C}$. Then, taking into account that $\theta^* \in C^* \setminus \{\mathbf{0}\}$ and Lemma 6.1 (b) , we have (6.64), which is impossible. □

Next result states a sufficient condition that guarantees that the hypothesis of the Proposition 6.17 (b) is fulfilled.

Theorem 6.13. *Assume that $\overset{\mathbf{0}}{D} \neq \emptyset$ and that condition (6.57) holds. If there exists $y \in S$ such that $g(y) \in \overset{\mathbf{0}}{D}$, then we can suppose that $\theta^* \neq \mathbf{0}$ in (6.57).*

Proof. Ab absurdo, suppose that $\theta^* = \mathbf{0}$ in (6.57). Then, $\lambda^* \neq \mathbf{0}$ and, since $g(y) \in \overset{\mathbf{0}}{D}$, by Lemma 6.1 (b), we have:

$$0 < \langle \lambda^*, g(y) \rangle \leq 0,$$

a contradiction. □

Setting $Q = \mathbb{R}^+$ in (6.43), we obtain that the (scalar) Lagrangian function associated with VOP, given by

$$L(\theta; \lambda, x) = \langle \theta, f(x) \rangle - \langle \lambda, g(x) \rangle, \quad (\theta, \lambda) \in C^* \times D^*. \tag{6.65}$$

Next result is a direct consequence of Theorem 6.10.

Theorem 6.14. *$\mathcal{K}_{\bar{x}}$ and $\mathcal{H}_{C_\mathbf{0}}$ admit a linear separation with $\bar{x} \in K$, iff there exist $\theta^* \in C^*$ and $\lambda^* \in D^*$ with $(\theta^*, \lambda^*) \neq \mathbf{0}$, such that $(\lambda^*, \bar{x})$ is a saddle point for $L(\theta^*; \lambda, x)$ on $D^* \times S$.*

In particular, we obtain the following sufficient optimality conditions.

Corollary 6.5. *Let $\bar{x} \in S$.*

(a) If there exist $\theta^* \in C^+$ and $\lambda^* \in D^*$ such that $(\lambda^*, \bar{x})$ is a saddle point for $L(\theta^*; \lambda, x)$ on $D^* \times S$, then $\bar{x}$ is a v.m.p. of VOP.
(b) If there exist $\theta^* \in C^* \setminus \{\mathbf{0}\}$ and $\lambda^* \in D^*$ such that $(\lambda^*, \bar{x})$ is a saddle point for $L(\theta^*; \lambda, x)$ on $D^* \times S$, then $\bar{x}$ is a weak v.m.p. of VOP.

Proof. (a) It follows from Theorem 6.12 (a), taking into account that, if $Q = \mathbb{R}^+$, then $C^+ = (C_\mathbf{0})^*_{Q_\mathbf{0}}$.
(b) We observe that, since $\theta^* \in C^* \setminus \{\mathbf{0}\}$, then by Lemma 6.1 (b), we have that $\theta^* \in (\overset{\mathbf{0}}{C})^*_{\overset{\mathbf{0}}{Q}}$. The thesis follows from Theorem 6.12 (b), taking into account that, if $Q = \mathbb{R}^+$, then (6.44) and (6.45) in Definition 6.15 are equivalent, so that the complementarity condition $\overline{\Lambda} g(\bar{x}) = \langle \lambda^*, g(\bar{x}) \rangle = 0$ is fulfilled. □

Under generalized convexity and suitable regularity assumptions, the existence of a saddle point for $L(\theta^*; \lambda, x)$ is a necessary and sufficient condition for the existence of a weak optimal solution to VOP.

Theorem 6.15. *Let $-A_{\bar{x}}$ be a generalized $(C \times D)$-subconvexlike function on S and suppose that $\overset{\mathbf{0}}{D} \neq \emptyset$ and there exists $\hat{x} \in S$ such that $g(\hat{x}) \in \overset{\mathbf{0}}{D}$. Then, $\bar{x}$ is a weak v.m.p. of VOP, iff there exists $\theta^* \in C^* \setminus \{\mathbf{0}\}$ and $\lambda^* \in D^*$ such that $(\lambda^*, \bar{x})$ is a saddle point for $L(\theta^*; \lambda, x)$ on $D^* \times S$.*

Proof. Taking into account Corollary 6.5 (b), we only need to prove that, if $\bar{x}$ is a weak v.m.p. of VOP, then there exists $\theta^* \in C^* \setminus \{\mathbf{0}\}$ and $\lambda^* \in D^*$ such that $(\lambda^*, \bar{x})$ is a saddle point for $L(\theta^*; \lambda, x)$ on $D^* \times S$. By Proposition 6.16, we have that $\mathcal{K}_{\bar{x}}$ and $\mathcal{H}_{C_\mathbf{0}}$ admit a linear separation, i.e., there exist $(\theta^*, \lambda^*) \in (C^* \times D^*)$ such that

$$\langle \theta^*, u \rangle + \langle \lambda^*, v \rangle \leq 0, \quad \forall (u,v) \in \mathscr{K}_{\bar{x}}.$$

From Theorem 6.13 it follows that $\theta^* \neq \mathbf{0}$ and by Theorem 6.14, $(\lambda^*, \bar{x})$ is a saddle point for $L(\theta^*; \lambda, x)$ on $D^* \times S$. □

Let us consider nonlinear scalar separation functions. To this aim, the scalarization approach (see Sect. 6.2.2) is a useful tool: by making use of (6.9), it is possible to define a nonlinear scalar separation function w in the image space. Let us set, in (6.55),

$$\alpha(u;\theta) := \xi_{\theta\mathbf{0}}(u) = \min\{t \in \mathbb{R} : u \in t\theta - C\}, \qquad \theta \in \overset{\mathbf{0}}{C}.$$

In such a case, (6.55) becomes:

$$w(u,v;\theta,\lambda) := \xi_{\theta\mathbf{0}}(u) + \gamma(v;\lambda), \quad (\theta,\lambda) \in \Omega := \overset{\mathbf{0}}{C} \times \Xi, \tag{6.66}$$

where $\gamma : \mathscr{Z} \times \Xi \to \mathbb{R}$, is such that

$$\bigcap_{\lambda \in \Xi} \operatorname{lev}_{\geq 0} \gamma(\cdot;\lambda) \supseteq D. \tag{6.67}$$

We prove that the function $\xi_{\theta\mathbf{0}}$ fulfils the first of (6.56), where $\mathscr{Q} := \overset{\mathbf{0}}{C}$.

Lemma 6.4. *For any* $\theta \in \overset{\mathbf{0}}{C}$*:*

$$\xi_{\theta\mathbf{0}}(u) > 0, \quad \forall u \in C_{\mathbf{0}}.$$

Proof. Ab absurdo, assume that there exist $u \in C_{\mathbf{0}}$ and $\theta \in \overset{\mathbf{0}}{C}$ such that $\xi_{\theta\mathbf{0}}(u) \leq 0$, i.e., there exist $\bar{t} \leq 0$ and $\bar{c} \in C$ such that

$$u = \bar{t}\theta - \bar{c}. \tag{6.68}$$

Since $\bar{t} \leq 0$ and C is a convex cone, then (6.68) implies that $u \in -C$, which is impossible, because $u \in C_{\mathbf{0}}$ and C is pointed. □

Some possible choices of functions $\gamma(v,\lambda)$ that fulfil (6.67) are the following:

(i) $\gamma(v,\lambda) = \langle \lambda, v \rangle, \quad \lambda \in D^*$
(ii) $\gamma(v,\lambda) = -\lambda \min\{\|v - d\|,\, d \in D\}, \quad \lambda \in \mathbb{R}_+ \setminus \{0\}$
(3i) $\gamma(v,\lambda) = \sup_{z \leq v}[\langle \lambda, z \rangle - c\|z\|^2], \quad \lambda \in \mathbb{R}^m,\ c \in \mathbb{R}_+ \setminus \{0\},\ D = \mathbb{R}^m_+.$

It is immediate to see that when γ is defined as in (i) or (ii), then (6.67) is fulfilled. As regards case (3i), this is proved in [44]. We refer to [27] for further examples.

As a consequence of Lemma 6.4, we obtain the following sufficient optimality condition for VOP.

Proposition 6.18. *Assume that $\bar{x} \in K$ and that $\gamma : \mathscr{Z} \times \Xi \to \mathbb{R}$ fulfils (6.67). If there exist $\bar{\theta} \in \overset{\mathbf{0}}{C}$ and $\bar{\lambda} \in \Xi$ such that*

$$\xi_{\bar{\theta}\mathbf{0}}(f(\bar{x}) - f(x)) + \gamma(g(x); \bar{\lambda}) \le 0, \quad \forall x \in S, \tag{6.69}$$

then $\bar{x}$ is a v.m.p. of VOP.

Proof. Equation (6.69) is equivalent to:

$$w(u, v; \bar{\theta}, \bar{\lambda}) \le 0, \quad \forall (u, v) \in \mathscr{K}_{\bar{x}}.$$

Condition (6.67) implies that for any $\lambda \in \Xi$:

$$\gamma(\lambda; v) \ge 0, \quad \forall v \in D,$$

and, recalling that $\mathscr{H}_{C_\mathbf{0}} = C_\mathbf{0} \times D$, from Lemma 6.4, we obtain:

$$w(u, v; \bar{\theta}, \bar{\lambda}) = \xi_{\bar{\theta}\mathbf{0}}(u) + \gamma(v; \bar{\lambda}) > 0, \quad \forall (u, v) \in \mathscr{H}_{C_\mathbf{0}}.$$

This shows that (6.34) is fulfilled, so that $\bar{x}$ is a v.m.p. of VOP. □

6.4.4 Connections with Duality

Saddle point sufficient optimality conditions for VOP are closely related to duality theory for VOP. In the present subsection we will outline the main connections between the separation scheme in the image space and duality theory for VOP.

If we say to be *regular* the subclass of separation functions which fulfil (6.39), duality arises from the problem of finding a regular separation function such that (6.41) holds.

In order to develop the analysis, we have to extend the concept of vector minimality (or maximality) to a set-valued function.

Definition 6.16 ([52]). Let $\Phi : X \rightrightarrows Y$. $\bar{x} \in K$ is called a *minimum (maximum)* point of Φ on $K \subseteq X$ iff there exists $\bar{y} \in \Phi(\bar{x})$ such that $\bar{y}$ is a minimum (maximum) point of the set $\cup_{x \in K} \Phi(x)$. $\bar{y}$ is called an optimal value of Φ.

Consider the vector Lagrangian function given by (6.43) and define the following set-valued optimization problem $VD(\Theta)$:

$$\operatorname*{Max}_{\Lambda \in D_Q^*}{}_{Q_\mathbf{0}} \min_{x \in S}{}_{Q_\mathbf{0}} \mathscr{L}(\Theta; \Lambda, x). \tag{6.70}$$

For $\Theta \in C_Q^*$, (6.70) defines a family of *vector dual problems* associated with VOP. Next theorem clarifies the properties of $VD(\Theta)$.

Theorem 6.16. *Let $\bar{x} \in K$ and $\overline{\Theta} \in C_Q^*$. Then, there exists $\overline{\Lambda} \in D_Q^*$ such that*

$$w(u,v;\overline{\Theta},\overline{\Lambda}) \not\geq_{Q_0} \mathbf{0}, \quad \forall (u,v) \in \mathscr{K}_{\bar{x}}, \tag{6.71}$$

iff $\overline{\Lambda}$ is an optimal solution of $VD(\overline{\Theta})$ and $\overline{\Theta} f(\bar{x})$ is an optimal value.

Proof. Let $\boldsymbol{\Phi}(\overline{\Theta},\Lambda)$ be the set of optimal values of the problem

$$\min_{x\in S}{}_{Q_0} \mathscr{L}(\overline{\Theta};\Lambda,x) = \min_{x\in S}{}_{Q_0} [\overline{\Theta} f(x) - \Lambda g(x)]. \tag{6.72}$$

Assume that (6.71) holds. Observe that (6.71) is equivalent to the condition

$$\overline{\Theta}(f(\bar{x}) - f(x)) + \overline{\Lambda} g(x) \not\geq_{Q_0} \mathbf{0}, \quad \forall x \in S. \tag{6.73}$$

Computing (6.73) for $x = \bar{x}$, we have $\overline{\Lambda} g(\bar{x}) \not\geq_{Q_0} \mathbf{0}$. Since $x \in K$, then $\overline{\Lambda} g(\bar{x}) \geq_Q \mathbf{0}$, so that $\overline{\Lambda} g(\bar{x}) = \mathbf{0}$. Therefore, (6.73) can be written as

$$\overline{\Theta} f(\bar{x}) - \overline{\Lambda} g(\bar{x}) - [\overline{\Theta} f(x) - \overline{\Lambda} g(x)] \not\geq_{Q_0} \mathbf{0}, \quad \forall x \in S,$$

and $\overline{\Theta} f(\bar{x}) \in \boldsymbol{\Phi}(\overline{\Theta},\overline{\Lambda})$. We have to prove that

$$z - \overline{\Theta} f(\bar{x}) \not\geq_{Q_0} \mathbf{0}, \quad \forall z \in \boldsymbol{\Phi}(\overline{\Theta},\Lambda),\ \forall \Lambda \in D_Q^*. \tag{6.74}$$

Fixed $\Lambda \in D_Q^*$, let $z \in \boldsymbol{\Phi}(\overline{\Theta},\Lambda)$. Since z is an optimal value of (6.72), we have

$$z - [\overline{\Theta} f(\bar{x}) - \Lambda g(\bar{x})] \not\geq_{Q_0} \mathbf{0}.$$

As $\Lambda g(\bar{x}) \geq_Q \mathbf{0}$, then $z - \overline{\Theta} f(\bar{x}) \not\geq_{Q_0} \mathbf{0}$, which proves (6.74).

Vice versa, assume that $\overline{\Theta} f(\bar{x})$ is an optimal value of $VD(\overline{\Theta})$, i.e.,

$$\overline{\Theta} f(\bar{x}) \in \operatorname*{Max}_{\Lambda\in D_Q^*}{}_{Q_0} \boldsymbol{\Phi}(\overline{\Theta},\Lambda).$$

Therefore, there exists $\overline{\Lambda} \in D_Q^*$ such that $\overline{\Theta} f(\bar{x}) \in \boldsymbol{\Phi}(\overline{\Theta},\overline{\Lambda})$, i.e.,

$$\overline{\Theta} f(\bar{x}) - [\overline{\Theta} f(x) - \overline{\Lambda} g(x)] \not\geq_{Q_0} \mathbf{0}, \quad \forall x \in S,$$

which is equivalent to (6.71). □

As an application of the previous theorem, let us consider the particular case where $Y = \mathbb{R}^\ell$, $Q = C \subseteq \mathbb{R}^\ell$ and $\overline{\Theta} = I_\ell$, the identity map on Y. Then, (6.70) becomes:

$$\operatorname*{Max}_{\Lambda\in D_Q^*}{}_{Q_0} \min_{x\in S}{}_{Q_0} [f(x) - \Lambda g(x)] \tag*{$VD(I_\ell)$}$$

$VD(I_\ell)$ and its weak form, where $Q_{\mathbf{0}}$ is replaced by $\overset{\mathbf{0}}{Q}$, have extensively been studied in the literature, in particular, weak and strong duality for $VD(I_\ell)$ and saddle point conditions for the vector Lagrangian function $\mathscr{L}(I_\ell;\Lambda,x)$ (see e.g., [7, 8, 42, 43, 52, 56, 58]).

Let $\Delta_1 := \{f(x) : x \text{ is a v.m.p. of VOP}\}$ be the set of the optimal values of VOP and Δ_2 the set of optimal values of $VD(I_\ell)$ and define $\Delta := \Delta_1 - \Delta_2$. Δ is called *duality gap*.

Theorem 6.17. *Let $Y = \mathbb{R}^\ell$, $Q = C \subseteq \mathbb{R}^\ell$. There exists a vector saddle point $(\overline{\Lambda},\bar{x})$ for $\mathscr{L}(I_\ell;\Lambda,x)$ on $D_C^* \times S$ if and only if $\mathbf{0} \in \Delta$.*

Proof. Let $(\overline{\Lambda},\bar{x})$ be a vector saddle point for $\mathscr{L}(I_\ell;\Lambda,x)$. By Theorem 6.12 (a), $\bar{x}$ is a v.m.p. of VOP so that $f(\bar{x}) \in \Delta_1$. By Theorem 6.10, (6.71) is fulfilled with $\overline{\Theta} = I_\ell$ and $Q_{\mathbf{0}} = C_{\mathbf{0}}$. Theorem 6.16 implies that $f(\bar{x}) \in \Delta_2$ so that $\mathbf{0} \in \Delta$.

Vice versa, assume that $\mathbf{0} \in \Delta$, then there exists $\bar{x} \in K$ such that $f(\bar{x}) \in \Delta_2$. By Theorem 6.16, we have that there exists $\overline{\Lambda} \in D_C^*$ such that (6.71) is fulfilled with $\overline{\Theta} = I_\ell$ and $Q_{\mathbf{0}} = C_{\mathbf{0}}$ and by Theorem 6.10, where we have set $Q = C$ and $\overline{\Theta} = I_\ell$, we obtain that $(\overline{\Lambda},\bar{x})$ is a vector saddle point for $\mathscr{L}(I_\ell;\Lambda,x)$ on $D_C^* \times S$. □

For an extensive analysis on duality for VOP, we refer to Chap. 3, [8] and the references therein.

6.5 Necessary Optimality Conditions

In this section we consider first order optimality conditions for VOP in the case where the feasible set K is defined by means of cone constraints. We extend the analysis of Gâteaux differentiable problems, that we have already considered, in Sect. 6.3, for VOP with feasible region given by a general convex set. Next, we analyse finite dimensional G-semidifferentiable VOP, in the sense of Giannessi [24, 28], in the presence of inequality constraints. As shown in [62], the concept of G-semiderivative embeds, as special cases, several kinds of generalized derivatives as, Dini, Dini-Hadamard and Clarke directional derivatives, which allows us to derive necessary optimality conditions in terms of the concepts of Clarke generalized differentiability, and quasidifferentiability in the sense of Pshenichnyi, Demyanov and Rubinov [12, 16, 17, 47].

6.5.1 Necessary Optimality Conditions for Differentiable VOP

Consider the VOP defined by (6.31) where we assume that f and g are Gâteaux differentiable at the point $\bar{x} \in S$, and denote by Df and Dg their Gâteaux derivatives. Moreover, we suppose that D is a closed convex cone, with nonempty interior. Let

$$Lf(\overline{x},x) := f(\overline{x}) + \langle Df(\overline{x}), x-\overline{x}\rangle,$$
$$Lg(\overline{x},x) := g(\overline{x}) + \langle Dg(\overline{x}), x-\overline{x}\rangle,$$

be the linearizations of f and g at the point $\overline{x}$, and consider the linearized image associated with VOP at the point $\overline{x} \in S$ and defined by

$$\mathscr{K}_{\overline{x}}^{\mathscr{L}} := \{(u,v) \in Y \times \mathscr{Z} : u = -\langle Df(\overline{x}), x-\overline{x}\rangle,\ v = g(\overline{x}) + \langle Dg(\overline{x}), x-\overline{x}\rangle,\ x \in S\}$$

We observe that $\mathscr{K}_{\overline{x}}^{\mathscr{L}} = LA_{\overline{x}}(S)$, where $LA_{\overline{x}}$ is the linearization at $\overline{x}$ of the function $A_{\overline{x}}$ defined by (6.33).

The conical extension of the linearized image is given by

$$\mathscr{E}_{\overline{x}}^{\mathscr{L}} := \mathscr{K}_{\overline{x}}^{\mathscr{L}} - cl\,\mathscr{H}_{C_{\mathbf{0}}} = LA_{\overline{x}}(S) - (C \times D).$$

Necessary optimality conditions for VOP, at the point $\overline{x} \in K$, can be expressed in terms of suitable properties of the set $\mathscr{E}_{\overline{x}}^{\mathscr{L}}$.

Next definition is analogous to the one given by Robinson for a Fréchet differentiable VOP [48].

Definition 6.17. We say that the *Fritz–John optimality conditions* for VOP are satisfied at the point $\overline{x} \in K$, iff there exists $\theta^* \in C^*$ and $\lambda^* \in D^*$, not both zero, such that:

$$\langle Df(\overline{x})^*\theta^* - Dg(\overline{x})^*\lambda^*, x-\overline{x}\rangle \geq 0, \quad \forall x \in S, \tag{6.75}$$
$$\langle \lambda^*, g(\overline{x})\rangle = 0, \tag{6.76}$$

where the asterisk on a linear operator denotes the adjoint.

We say that the *Kuhn–Tucker optimality conditions* hold at $\overline{x} \in K$ if the Fritz–John conditions are satisfied at $\overline{x}$, with $\theta^* \neq \mathbf{0}$.

Observe that, if $\overline{x} \in \overset{\mathbf{0}}{S}$, then (6.75) collapses into:

$$Df(\overline{x})^*\theta^* - Dg(\overline{x})^*\lambda^* = \mathbf{0}. \tag{6.77}$$

We need the following preliminary result.

Lemma 6.5 ([9]). *Let $A \subseteq Y$ be a nonempty set and $P \subseteq Y$ a convex cone with* int $P \neq \emptyset$. *Then*

$$A + \text{int}\, P = \text{int}\,(A+P).$$

Proof. Let $x \in \text{int}\,(A+P)$. Therefore, there exists a neighborhood U of $\mathbf{0}_Y$ such that

$$x - U \subseteq A + P. \tag{6.78}$$

Let $p_0 \in \operatorname{int} P$. For a sufficiently small $\alpha > 0$, we have that $\alpha p_0 \in U$ and from (6.78) it follows that the vector $z = x - \alpha p_0 \in A + P$. We observe that $x - \operatorname{int} P$ is a neighborhood of z, since it is an open set containing z. Consequently, we have

$$(A+P) \cap (x - \operatorname{int} P) \neq \emptyset.$$

Then, there exist $a \in A$, $p \in P$ such that $a + p \in x - \operatorname{int} P$. This implies that:

$$x \in a + p + \operatorname{int} P \subseteq A + P + \operatorname{int} P \subseteq A + \operatorname{int} P.$$

This proves that $\operatorname{int}(A+P) \subseteq A + \operatorname{int} P$. To prove the opposite inclusion, it is enough to note that $A + \operatorname{int} P$ is an open subset of $A + P$, which implies that

$$A + \operatorname{int} P \subseteq \operatorname{int}(A+P),$$

and the proof is complete. □

Proposition 6.19. *If $\bar{x} \in K$ is a weak v.m.p. of VOP then $\mathbf{0} \notin \operatorname{int} \mathscr{E}_{\bar{x}}^{\mathscr{L}}$.*

Proof. We observe that, since we assume that $\overset{\mathbf{0}}{D} \neq \emptyset$, then, from Lemma 6.5 it follows that

$$\operatorname{int} \mathscr{E}_{\bar{x}}^{\mathscr{L}} := \operatorname{int}(LA_{\bar{x}}(S) - (C \times D)) = LA_{\bar{x}}(S) - (\overset{\mathbf{0}}{C} \times \overset{\mathbf{0}}{D}). \tag{6.79}$$

Ab absurdo, suppose that $\mathbf{0} \in \operatorname{int} \mathscr{E}_{\bar{x}}^{\mathscr{L}}$, i.e., set $\mathbf{0} = \mathbf{0}_{Y \times \mathscr{Z}}$, there exist $\tilde{x} \in S, c \in \overset{\mathbf{0}}{C}$ and $d \in \overset{\mathbf{0}}{D}$ such that

$$\mathbf{0}_Y = -\langle Df(\bar{x}), \tilde{x} - \bar{x} \rangle - c, \tag{6.80}$$

$$\mathbf{0}_{\mathscr{Z}} = g(\bar{x}) + \langle Dg(\bar{x}), \tilde{x} - \bar{x} \rangle - d. \tag{6.81}$$

Setting $x(\alpha) := \bar{x} + \alpha(\tilde{x} - \bar{x}), \quad \alpha \in (0,1)$, we have $\alpha(\tilde{x} - \bar{x}) = x(\alpha) - \bar{x}$, so that, multiplying (6.80) and (6.81) by $\alpha \in (0,1)$, we obtain:

$$\mathbf{0}_Y = -\langle Df(\bar{x}), \alpha(\tilde{x} - \bar{x}) \rangle - \alpha c, \qquad \mathbf{0}_{\mathscr{Z}} = \alpha g(\bar{x}) + \langle Dg(\bar{x}), \alpha(\tilde{x} - \bar{x}) \rangle - \alpha d,$$

and, $\forall \alpha \in (0,1)$, we have:

$$-\langle Df(\bar{x}), x(\alpha) - \bar{x} \rangle = -\langle Df(\bar{x}), \alpha(\tilde{x} - \bar{x}) \rangle = \alpha c \in \overset{\mathbf{0}}{C}, \tag{6.82}$$

$$\langle Dg(\bar{x}), x(\alpha) - \bar{x} \rangle = \langle Dg(\bar{x}), \alpha(\tilde{x} - \bar{x}) \rangle = \alpha(d - g(\bar{x})) \in \overset{\mathbf{0}}{D} - \alpha g(\bar{x}). \tag{6.83}$$

Since f and g are Gâteaux differentiable at $\bar{x}$, we have:

$$\begin{pmatrix} f(x(\alpha)) \\ g(x(\alpha)) \end{pmatrix} = \begin{pmatrix} Lf(\bar{x}, x(\alpha)) + r_f(\alpha, \tilde{x} - \bar{x}) \\ Lg(\bar{x}, x(\alpha)) + r_g(\alpha, \tilde{x} - \bar{x}) \end{pmatrix},$$

where

$$\lim_{\alpha\downarrow 0}\frac{r_f(\alpha,\tilde{x}-\overline{x})}{\alpha}=\mathbf{0}_Y,\quad \lim_{\alpha\downarrow 0}\frac{r_g(\alpha,\tilde{x}-\overline{x})}{\alpha}=\mathbf{0}_{\mathscr{Z}}.$$

Therefore, for $0<\alpha<\alpha_1$ sufficiently small, from (6.82), we have:

$$f(\overline{x})-Lf(\overline{x},x(\alpha))-r_f(\alpha,\tilde{x}-\overline{x})=-\langle Df(\overline{x}),x(\alpha)-\overline{x}\rangle-r_f(\alpha,\tilde{x}-\overline{x})\in\overset{\circ}{C},$$

and, in turn,

$$f(\overline{x})-f(x(\alpha))\in\overset{\circ}{C},\quad 0<\alpha<\alpha_1. \tag{6.84}$$

Similarly, for $0<\alpha<\alpha_2$ sufficiently small, from (6.83), we obtain:

$$g(x(\alpha))=g(\overline{x})+\langle Dg(\overline{x}),x(\alpha)-\overline{x}\rangle+r_g(\alpha,\tilde{x}-\overline{x})\in g(\overline{x})+\overset{\circ}{D}-\alpha g(\overline{x}),$$

which implies

$$g(x(\alpha))\in(1-\alpha)g(\overline{x})+\overset{\circ}{D}\subseteq D+\overset{\circ}{D}\subseteq D,\quad 0<\alpha<\alpha_2. \tag{6.85}$$

For $\alpha<\min\{\alpha_1,\alpha_2\}$, (6.84) and (6.85) contradict that $\overline{x}$ is a weak v.m.p. of VOP. □

Theorem 6.18. *(a) If $\overline{x}\in K$ is a weak v.m.p. of VOP (6.31), then, the Fritz–John conditions are satisfied at $\overline{x}$.*

(b) Assume that there exists $\tilde{x}\in S$ such that

$$g(\overline{x})+\langle Dg(\overline{x}),\tilde{x}-\overline{x}\rangle\in\overset{\circ}{D}. \tag{6.86}$$

If $\overline{x}\in K$ is a weak v.m.p. of VOP, then, the Kuhn–Tucker conditions are satisfied at $\overline{x}$.

Proof. (a) If $\overline{x}\in K$ is a weak v.m.p. of VOP then, by Proposition 6.19, $\mathbf{0}\notin \operatorname{int}\mathscr{E}_{\overline{x}}^{\mathscr{L}}$. Note that, $\mathbf{0}:=\mathbf{0}_{Y\times\mathscr{Z}}$ belongs to the set $\mathscr{E}_{\overline{x}}^{\mathscr{L}}$. Actually, $LA_{\overline{x}}(\overline{x})=(\mathbf{0}_Y,g(\overline{x}))$, and

$$(\mathbf{0}_Y-c,g(\overline{x})-d)\in\mathscr{E}_{\overline{x}}^{\mathscr{L}},\quad \text{for } c=\mathbf{0}_Y,\ d=g(\overline{x})\in D.$$

Since $\overset{\circ}{C}$ and $\overset{\circ}{D}$ are nonempty, then by (6.79), it follows that $\operatorname{int}\mathscr{E}_{\overline{x}}^{\mathscr{L}}\neq\emptyset$, so that, by Theorem 6.1, there exists a closed supporting hyperplane for $\mathscr{E}_{\overline{x}}^{\mathscr{L}}$ at $\mathbf{0}$, i.e., there exist $(\theta^*,\lambda^*)\in Y^*\times Z^*$, not both zero, such that:

$$\langle\theta^*,f(\overline{x})-Lf(\overline{x},x)-c\rangle+\langle\lambda^*,Lg(\overline{x},x)-d\rangle\le 0,\quad \forall x\in S,\ \forall c\in C,\ \forall d\in D. \tag{6.87}$$

Setting $x=\overline{x}$, $d=g(\overline{x})$, in (6.87), we obtain:

$$\langle \theta^*, c \rangle \geq 0, \quad \forall c \in C, \quad \text{i.e.,} \quad \theta^* \in C^*.$$

Similarly, setting $x = \bar{x}$, $c = \mathbf{0}$, in (6.87), we have:

$$\langle \lambda^*, g(\bar{x}) \rangle \leq \langle \lambda^*, d \rangle, \quad \forall d \in D,$$

which implies, taking into account that D is a cone, that

$$\langle \lambda^*, d \rangle \geq 0, \quad \forall d \in D \quad \text{i.e.,} \quad \lambda^* \in D^*.$$

Computing (6.87) for $x = \bar{x}$, $c = \mathbf{0}$, $d = \mathbf{0}$, we get $\langle \lambda^*, g(\bar{x}) \rangle \leq 0$, and, since $\lambda^* \in D^*$ and $\bar{x} \in K$, we have:

$$\langle \lambda^*, g(\bar{x}) \rangle = 0.$$

Thus, (6.76) holds. Finally, setting $c = \mathbf{0}$, $d = \mathbf{0}$ in (6.87), we obtain:

$$\langle \theta^*, Df(\bar{x})(x - \bar{x}) \rangle - \langle \lambda^*, Dg(\bar{x})(x - \bar{x}) \rangle \geq 0, \quad \forall x \in S, \tag{6.88}$$

(where, for simplicity of notation, $Df(\bar{x})(x - \bar{x}) = \langle Df(\bar{x}), x - \bar{x} \rangle$ and similarly for $Dg(\bar{x})(x - \bar{x})$). By recalling the definition of adjoint operator, we obtain that (6.88) is equivalent to (6.75), which completes the proof of part (a).

(b) We have to show that $\theta^* \neq \mathbf{0}$ in (6.87). Ab absurdo, assume that $\theta^* = \mathbf{0}$. Then, setting $x = \tilde{x}$ and $d = \mathbf{0}$, in (6.87), taking into account (6.86) and Lemma 6.1 (b), we obtain:

$$0 < \langle \lambda^*, g(\bar{x}) + Dg(\bar{x})(\tilde{x} - \bar{x}) \rangle \leq 0,$$

a contradiction. □

In the particular case where $Y = \mathbb{R}^\ell$, $\mathscr{Z} = \mathbb{R}^m$, Theorem 6.18 leads to the classic formulation of Fritz–John and Kuhn–Tucker optimality conditions.

Corollary 6.6. *Let $Y = \mathbb{R}^\ell$, $\mathscr{Z} = \mathbb{R}^m$, $D = \mathbb{R}^m_+$, S be a subset of X with nonempty interior, $\bar{x} \in \overset{\mathbf{0}}{S}$, and assume that $f_i : S \to \mathbb{R}$, $i = 1, .., \ell$ and $g_j : S \to \mathbb{R}$, $j = 1, .., m$, are Gâteaux differentiable at $\bar{x}$.*

(a) If $\bar{x} \in K$ is a weak v.m.p of VOP (6.31), then there exists $(\theta^*, \lambda^*) \in \mathbb{R}^\ell \times \mathbb{R}^m$, with $(\theta^*, \lambda^*) \neq \mathbf{0}$ such that $(\bar{x}, \theta^*, \lambda^*)$ is a solution of the following system:

$$\begin{cases} \sum_{i=1}^{\ell} \theta_i Df_i(x) - \sum_{j=1}^{m} \lambda_j Dg_j(x) = \mathbf{0}, \\ \langle \lambda, g(x) \rangle = 0, \\ \theta \in C^*, \ \lambda \geq 0, \ g(x) \geq \mathbf{0}, \ x \in S. \end{cases} \tag{6.89}$$

(b) Moreover, if there exists $z \in X$ such that

$$\langle Dg_j(\bar{x}), z\rangle > 0, \quad j \in J(\bar{x}) := \{j : g_j(\bar{x}) = 0\}, \tag{6.90}$$

then the solution $(\bar{x}, \theta^*, \lambda^*)$ *of* (6.89) *is such that* $\theta^* \neq \mathbf{0}$.

Proof. (a) It follows from Theorem 6.18 (a), taking into account that, since $\bar{x} \in \overset{\mathbf{0}}{S}$, then (6.75) collapses to (6.77).

(b) We show that (6.90) implies (6.86), so that the thesis follows from Theorem 6.18 (b). To this aim set $x(\alpha) = \bar{x} + \alpha z \in S$ for $\alpha \leq \alpha_0 \in (0,1)$ sufficiently small. Then, for any $j = 1,..,m$, there exists $\alpha_j \in (0,1)$ such that

$$g_j(\bar{x}) + \langle Dg_j(\bar{x}), x(\alpha) - \bar{x}\rangle = g_j(\bar{x}) + \langle Dg_j(\bar{x}), \alpha z\rangle > 0, \quad 0 < \alpha \leq \alpha_j.$$

Defining $\tilde{x} := x(\overline{\alpha})$, $\overline{\alpha} : = \min\{\alpha_j : j=0,..,m\}$, we obtain that (6.86) holds.

□

The previous results can be extended to a VOP where the feasible region K contains both equality and inequality constraints. For a deeper analysis of these generalizations we refer to [33, 48].

6.5.2 Semidifferentiable Problems

In this subsection, we will consider necessary optimality conditions for nondifferentiable VOP. We will make use of the concept of semidifferentiability of a function, introduced by Giannessi in [24], for developing in an axiomatic way the theory of generalized directional derivatives. In the particular case where the (upper) G-semiderivative coincides with the (upper) Dini directional derivative, necessary optimality conditions in terms of the concepts of quasidifferentiability introduced by Pshenichnyi, Demyanov and Rubinov [16, 17, 47] are obtained. Connections with Clarke generalized differentiability [12] are also provided.

We will consider the VOP defined by (6.31) in the finite dimensional case, i.e., we set $X := \mathbb{R}^n$, $Y := \mathbb{R}^\ell$, $S := X$ and $D := \mathbb{R}^m_+$. $f : X \to \mathbb{R}^\ell$ and $g : X \to \mathbb{R}^m$.

We preliminarily recall the concept of G-semidifferentiability [24]. Denote by G a given subset of the set, say $\mathscr{G}$, of positively homogeneous functions of degree one on X, by $\mathscr{C} \subseteq \mathscr{G}$ the set of convex positively homogeneous functions, and by $\mathscr{L}$ the set of linear functions on X.

Definition 6.18 ([24]).

- A function $\phi : X \to \mathbb{R}$ is said *lower G-semidifferentiable at* $\bar{x} \in X$ iff there exist functions $\underline{\mathscr{D}}_G\phi : X \times X \to \mathbb{R}$ and $\varepsilon_\phi : X \times X \to \mathbb{R}$ such that:
  1. $\underline{\mathscr{D}}_G\phi(\bar{x}; \cdot) \in G$;
  2. $\phi(x) - \phi(\bar{x}) = \underline{\mathscr{D}}_G\phi(\bar{x}; x - \bar{x}) + \varepsilon_\phi(\bar{x}; x - \bar{x})$, $\forall x \in X$;
  3. $\liminf\limits_{x \to \bar{x}} \dfrac{\varepsilon_\phi(\bar{x}; x - \bar{x})}{\|x - \bar{x}\|} \geq 0$;

4. For every pair (h,ε) of functions (of the same kind of $\underline{\mathscr{D}}_G\phi$ and ε_ϕ, respectively), which satisfy 1–3, we have epi$h \supseteq$ epi$\underline{\mathscr{D}}_G\phi$.

$\underline{\mathscr{D}}_G\phi(\bar{x};x-\bar{x})$ is called the *lower G-semiderivative of ϕ at $\bar{x}$.*

- A function $\phi : X \to \mathbb{R}$ is said *upper G-semidifferentiable at $\bar{x} \in X$* iff there exist functions $\overline{\mathscr{D}}_G\phi : X \times X \to \mathbb{R}$ and $\varepsilon_\phi : X \times X \to \mathbb{R}$ such that:

1'. $\overline{\mathscr{D}}_G\phi(\bar{x};\cdot) \in G$;
2'. $\phi(x)-\phi(\bar{x}) = \overline{\mathscr{D}}_G\phi(\bar{x};x-\bar{x}) + \varepsilon_\phi(\bar{x};x-\bar{x})\,, \quad \forall x \in X$;
3'. $\limsup\limits_{x\to\bar{x}} \dfrac{\varepsilon_\phi(\bar{x};x-\bar{x})}{\|x-\bar{x}\|} \le 0$;
4'. for every pair (h,ε) of functions (of the same kind of $\overline{\mathscr{D}}_G\phi$ and ε_ϕ, respectively), which satisfy 1'–3', we have epi$h \subseteq$ epi$\overline{\mathscr{D}}_G\phi$.

$\overline{\mathscr{D}}_G\phi(\bar{x};x-\bar{x})$ is called the *upper G-semiderivative* of ϕ at $\bar{x}$.

- When lower and upper G-semiderivatives coincide at $\bar{x}$, then ϕ is said to be *G-differentiable* at $\bar{x}$.

Remark 6.6. From Definition 6.18, it is immediate that

$$\underline{\mathscr{D}}_G\phi = -\overline{\mathscr{D}}_{(-G)}(-\phi), \tag{6.91}$$

so that ϕ is lower G-semidifferentiable at $\bar{x}$ if and only if $-\phi$ is upper $(-G)$-semidifferentiable at $\bar{x}$. Moreover, we observe that the set of $\mathscr{L}$-differentiable functions coincides with the set of Fréchet differentiable functions.

Definition 6.19. Let $G \subseteq \mathscr{C}$. The *generalized subdifferential* of a lower (or upper) G-semidifferentiable function ϕ at $\bar{x}$, denoted by $\partial_G\phi(\bar{x})$, is defined as the subdifferential at $\bar{x}$ of the convex function $\underline{\mathscr{D}}_G\phi(\bar{x};x-\bar{x})$ (or $\overline{\mathscr{D}}_G\phi(\bar{x};x-\bar{x})$), that is,

$$\partial_G\phi(\bar{x}) = \partial\underline{\mathscr{D}}_G\phi(\bar{x},0) \quad (or\ \partial_G\phi(\bar{x}) = \partial\overline{\mathscr{D}}_G\phi(\bar{x},0)).$$

Remark 6.7. If $G \subseteq (-\mathscr{C})$ then the generalized superdifferential of a lower (or upper) G-semidifferentiable function ϕ is defined as the superdifferential of its concave approximation $\underline{\mathscr{D}}_G\phi$ (or $\overline{\mathscr{D}}_G\phi$).

For $G = \mathscr{G}$, the class of G-differentiable functions at $\bar{x}$ coincides with the class of B-differentiable functions at $\bar{x}$, in the sense of Robinson [49].

Let us recall the main properties of G-semidifferentiable functions, that will be used in what follows.

Proposition 6.20 ([46]). *Suppose that G satisfies the following conditions:*

(i) $\psi_1, \psi_2 \in G \quad implies \quad \psi_1+\psi_2 \in G$
(ii) $\psi \in G \quad implies \quad \alpha\psi \in G, \quad \forall \alpha > 0$

(a) If ϕ_1, ϕ_2 and $\phi_1 + \phi_2$ are upper G-semidifferentiable at $\bar{x}$ then

$$\overline{\mathscr{D}}_G \phi_1(\bar{x}; x-\bar{x}) + \overline{\mathscr{D}}_G \phi_2(\bar{x}; x-\bar{x}) \leq \overline{\mathscr{D}}_G(\phi_1+\phi_2)(\bar{x}; x-\bar{x}), \quad \forall x \in X.$$

(b) If ϕ is upper (lower) G-semidifferentiable at $\bar{x}$ then $\forall \alpha > 0$, $\alpha\phi$ is upper (lower) G-semidifferentiable at $\bar{x}$ with

$$\alpha \overline{\mathscr{D}}_G \phi(\bar{x}; x-\bar{x}) \quad (\alpha \underline{\mathscr{D}}_G \phi(\bar{x}; x-\bar{x}))$$

as upper (lower) G-semiderivative.

The separation approach in the image space is a source for deriving necessary optimality conditions for VOP: following the line developed in [24] for the scalar case, necessary optimality conditions can be achieved by separating $\mathscr{H}_{\mathbf{0}}$ from a suitable approximation of the image $\mathscr{K}_{\bar{x}}$. Here, such an approximation is obtained by assuming, for every $i = 1, \ldots, \ell$, and $j = 1, \ldots, m$, G-semidifferentiability of f_i and g_j and by replacing them with their G-semiderivatives. Let us introduce the index sets $I := \{1, \ldots, \ell\}$, $J := \{1, \ldots, m\}$ and $J^0(\bar{x}) := \{j \in J : g_j(\bar{x}) = 0,\ \varepsilon_{g_j}(\bar{x}; x-\bar{x}) \not\equiv 0\}$. Next lemma is analogous to the classic Linearization Lemma of Abadie [1].

Lemma 6.6. *Let the functions f_i, $i \in I$, be upper Φ-semidifferentiable and g_j, $j \in J$, be lower Γ-semidifferentiable at $\bar{x}$, where $\Phi, \Gamma \subseteq \mathscr{G}$. If $\bar{x} \in K$ is a weak v.m.p. of VOP (6.31), then the system (in the unknown x):*

$$\begin{cases} \overline{\mathscr{D}}_\Phi f_i(\bar{x}; x-\bar{x}) < 0\ , \ i \in I \\ \underline{\mathscr{D}}_\Gamma g_j(\bar{x}; x-\bar{x}) > 0\ , \ j \in J^0(\bar{x}) \\ g_j(\bar{x}) + \underline{\mathscr{D}}_\Gamma g_j(\bar{x}; x-\bar{x}) \geq 0\ , \ j \in J \setminus J^0(\bar{x}) \\ x \in X \end{cases} \tag{6.92}$$

is impossible.

Proof. $\bar{x}$ is a weak v.m.p. of VOP iff the following system is impossible

$$\begin{cases} f(\bar{x}) - f(x) > 0 \\ g(x) \geq 0,\ x \in X. \end{cases} \tag{6.93}$$

Taking into account the semidifferentiability of f and g at $\bar{x}$, (6.93) can be rewritten as

$$\begin{cases} -(\overline{\mathscr{D}}_\Phi f_i(\bar{x}; x-\bar{x}) + \varepsilon_{f_i}(\bar{x}; x-\bar{x})) > 0, \quad i \in I \\ g_j(\bar{x}) + \underline{\mathscr{D}}_\Gamma g_j(\bar{x}; x-\bar{x}) + \varepsilon_{g_j}(\bar{x}; x-\bar{x}) \geq 0, \quad j \in J \\ x \in X. \end{cases} \tag{6.94}$$

Ab absurdo, suppose that (6.92) be possible, and let $\hat{x}$ be a solution of (6.92). Now, we prove that $x(\alpha) := (1-\alpha)\bar{x} + \alpha\hat{x}$ is a solution of (6.92), $\forall \alpha \in]0,1]$. Actually, $x(\alpha) - \bar{x} = \alpha(\hat{x} - \bar{x})$ and substituting it in (6.92), taking into account the positive homogeneity of the semiderivatives, (6.92) becomes

$$\begin{cases} \alpha \underline{\mathscr{D}}_{\Phi} f_i(\bar{x}; \hat{x} - \bar{x}) < 0, \; i \in I \\ \alpha \underline{\mathscr{D}}_{\Gamma} g_j(\bar{x}; \hat{x} - \bar{x}) > 0, \; j \in J^0(\bar{x}) \\ g_j(\bar{x}) + \alpha \underline{\mathscr{D}}_{\Gamma} g_j(\bar{x}; \hat{x} - \bar{x}) \geq 0, \; j \in J \setminus J^0(\bar{x}). \end{cases} \tag{6.95}$$

In order to prove (6.95) it is enough to show that the inequalities

$$g_j(\bar{x}) + \alpha \underline{\mathscr{D}}_{\Gamma} g_j(\bar{x}; \hat{x} - \bar{x}) \geq 0 \tag{6.96}$$

are fulfilled for every $j \in J \setminus J^0(\bar{x})$ such that $g_j(\bar{x}) > 0$: in fact, since (6.96) holds for $\alpha = 1$, then it is fulfilled for every $\alpha \in]0,1]$.
Observe that the remainders satisfy the inequalities:

$$\limsup_{x \to \bar{x}} \frac{\varepsilon_{f_i}(\bar{x}; x - \bar{x})}{\|x - \bar{x}\|} \leq 0, \; i \in I, \quad \liminf_{x \to \bar{x}} \frac{\varepsilon_{g_j}(\bar{x}; x - \bar{x})}{\|x - \bar{x}\|} \geq 0, \; j \in J.$$

From the definitions of lim sup and lim inf, for every fixed $\delta > 0$, $\exists \alpha_\delta > 0$ such that $\forall \alpha \in]0, \alpha_\delta]$, we have:

$$\frac{\varepsilon_{f_i}(\bar{x}; \alpha(\hat{x} - \bar{x}))}{\|\alpha(\hat{x} - \bar{x})\|} \leq \delta, \; i \in I, \tag{6.97}$$

$$\frac{\varepsilon_{g_j}(\bar{x}; \alpha(\hat{x} - \bar{x}))}{\|\alpha(\hat{x} - \bar{x})\|} \geq -\delta, \; j \in J. \tag{6.98}$$

Suppose that $j \in J^0(\bar{x})$. Because of the positive homogeneity of $\overline{\mathscr{D}}_{\Phi} f_i$ and $\underline{\mathscr{D}}_{\Gamma} g_j$, we get:

$$\begin{aligned} &\overline{\mathscr{D}}_{\Phi} f_i(\bar{x}; \alpha(\hat{x} - \bar{x})) + \varepsilon_{f_i}(\bar{x}; \alpha(\hat{x} - \bar{x})) \\ &\quad = \left[\frac{\overline{\mathscr{D}}_{\Phi} f_i(\bar{x}; \hat{x} - \bar{x})}{\|\hat{x} - \bar{x}\|} + \frac{\varepsilon_{f_i}(\bar{x}; \alpha(\hat{x} - \bar{x}))}{\alpha\|\hat{x} - \bar{x}\|} \right] \cdot \alpha\|\hat{x} - \bar{x}\|, \quad i \in I, \\ &\underline{\mathscr{D}}_{\Gamma} g_j(\bar{x}; \alpha(\hat{x} - \bar{x})) + \varepsilon_{g_j}(\bar{x}; \alpha(\hat{x} - \bar{x})) \\ &\quad = \left[\frac{\underline{\mathscr{D}}_{\Gamma} g_j(\bar{x}; \hat{x} - \bar{x})}{\|\hat{x} - \bar{x}\|} + \frac{\varepsilon_{g_j}(\bar{x}; \alpha(\hat{x} - \bar{x}))}{\alpha\|\hat{x} - \bar{x}\|} \right] \cdot \alpha\|\hat{x} - \bar{x}\|, \quad j \in J^0(\bar{x}). \end{aligned}$$

Taking into account (6.97) and (6.98), from the above equalities we obtain that there exist $\delta > 0$ and $\alpha_\delta > 0$, such that $\forall \alpha \in]0, \alpha_\delta]$ the following inequalities hold:

$$\overline{\mathscr{D}}_{\Phi} f_i(\bar{x}; \alpha(\hat{x} - \bar{x})) + \varepsilon_{f_i}(\bar{x}; \alpha(\hat{x} - \bar{x})) < 0, \; i \in I,$$

$$\underline{\mathscr{D}}_\Gamma g_j(\bar{x};\alpha(\hat{x}-\bar{x}))+\varepsilon_{g_j}(\bar{x};\alpha(\hat{x}-\bar{x}))>0,\ j\in J^0(\bar{x}).$$

When $j \in J\setminus J^0(\bar{x})$, we consider two cases: $g_j(\bar{x})=0$ and $\varepsilon_{g_j}\equiv 0$; or $g_j(\bar{x})>0$. In the latter, choosing α small enough, we have

$$g_j(\bar{x})+\underline{\mathscr{D}}_\Gamma g_j(\bar{x};\alpha(\hat{x}-\bar{x}))+\varepsilon_{g_j}(\bar{x};\alpha(\hat{x}-\bar{x}))\geq 0; \tag{6.99}$$

while, in the former, the inequality (6.99) is fulfilled $\forall\alpha\geq 0$. Therefore $\exists\overline{\alpha}\in]0,\alpha_\delta]$ such that $x(\overline{\alpha})$ is a solution of (6.94), which contradicts the hypothesis. □

Following the analysis developed in the previous section, the impossibility of system (6.92) leads, by standard separation arguments, to obtain Lagrangian type optimality conditions for VOP. Let us define

$$\overline{\mathscr{D}}_\Phi f(\bar{x};x-\bar{x}):=(\overline{\mathscr{D}}_\Phi f_1(\bar{x};x-\bar{x}),\dots,\overline{\mathscr{D}}_\Phi f_\ell(\bar{x};x-\bar{x})),$$

$$\underline{\mathscr{D}}_\Gamma g(\bar{x};x-\bar{x})=(\underline{\mathscr{D}}_\Gamma g_1(\bar{x};x-\bar{x}),\dots,\underline{\mathscr{D}}_\Gamma g_m(\bar{x};x-\bar{x})),$$

and introduce the sets:

$$\mathscr{K}_{\bar{x}}^G:=\{(u,v)\in\mathbb{R}^\ell\times\mathbb{R}^m: u=-\overline{\mathscr{D}}_\Phi f(\bar{x};x-\bar{x}),\ v=g(\bar{x})+\underline{\mathscr{D}}_\Gamma g(\bar{x};x-\bar{x}),\ \ x\in X\},$$

and its conical extension

$$\mathscr{E}_{\bar{x}}^G:=\mathscr{K}_{\bar{x}}^G-cl\mathscr{H}_{\underset{C}{\mathbf{0}}}=\mathscr{K}_{\bar{x}}^G-(\mathbb{R}_+^\ell\times\mathbb{R}_+^m).$$

The following result is a direct consequence of Lemma 6.6.

Proposition 6.21. *If $\bar{x}\in K$ is a weak v.m.p. of VOP, then* $\mathbf{0}\notin\operatorname{int}\mathscr{E}_{\bar{x}}^G$.

Proof. From Lemma 6.6 it follows that system (6.92) is impossible. This implies the impossibility of the system

$$-\overline{\mathscr{D}}_\Phi f_i(\bar{x};x-\bar{x})>0,\ i\in I,\quad g_j(\bar{x})+\underline{\mathscr{D}}_\Gamma g_j(\bar{x};x-\bar{x})>0,\ j\in J,\quad x\in X,$$

or, equivalently,

$$\mathscr{K}_{\bar{x}}^G\cap\operatorname{int}(\operatorname{cl}\mathscr{H}_{\underset{C}{\mathbf{0}}})=\emptyset, \tag{6.100}$$

Observe that (6.100) holds iff

$$\mathbf{0}\notin\mathscr{K}_{\bar{x}}^G-\operatorname{int}(\operatorname{cl}\mathscr{H}_{\underset{C}{\mathbf{0}}})=\operatorname{int}(\mathscr{K}_{\bar{x}}^G-\operatorname{cl}\mathscr{H}_{\underset{C}{\mathbf{0}}})=\operatorname{int}\mathscr{E}_{\bar{x}}^G,$$

where the first equality is due to Lemma 6.5, which completes the proof. □

The following results are obtained in the case where the approximations of $f_i, i\in I$ and $g_j, j\in J$, are given by convex and concave positively homogeneous functions, respectively.

Theorem 6.19. *Suppose that the following assumptions are fulfilled.*

(i) f_i*,* $i \in I$*, is upper* Φ*-semidifferentiable at* $\bar{x}$*, and* $\Phi \subseteq \mathscr{C}$*.*
(ii) g_j*,* $j \in J$*, is lower* Γ*-semidifferentiable at* $\bar{x}$*, and* $\Gamma \subseteq (-\mathscr{C})$*.*

(a) If $\bar{x} \in K$ *is a weak v.m.p. of VOP* (6.31)*, then there exists* $(\theta^*, \lambda^*) \in \mathbb{R}^\ell_+ \times \mathbb{R}^m_+$*, with* $(\theta^*, \lambda^*) \neq \mathbf{0}$*, such that:*

$$\langle \theta^*, \overline{\mathscr{D}}_\Phi f(\bar{x}; x - \bar{x}) \rangle - \langle \lambda^*, \underline{\mathscr{D}}_\Gamma g(\bar{x}; x - \bar{x}) \geq 0, \quad \forall x \in X;$$
$$\langle \lambda^*, g(\bar{x}) \rangle = 0\,. \tag{6.101}$$

(b) Moreover, if there exists $z \in X$ *such that*

$$\underline{\mathscr{D}}_\Gamma g_j(\bar{x}; z - \bar{x}) > 0, \quad j \in J(\bar{x}) := \{ j \in J : g_j(\bar{x}) = 0 \}, \tag{6.102}$$

then we can suppose $\theta^* \neq \mathbf{0}$ *in (6.101).*

Proof. (a) Since the functions $\overline{\mathscr{D}}_\Phi f_i(\bar{x}; \cdot)$, $i \in I$ and $-\underline{\mathscr{D}}_\Gamma g_j(\bar{x}, \cdot)$, $j \in J$, are convex, then the set $\mathscr{E}^G_{\bar{x}}$ is convex (see Corollary 6.3). Since $\bar{x}$ is a weak v.m.p. of VOP then, by Proposition 6.21, $\mathbf{0} \notin \operatorname{int} \mathscr{E}^G_{\bar{x}}$. Note that, $\mathbf{0}$ belongs to the set $\mathscr{E}^G_{\bar{x}}$. Actually, $(\mathbf{0}, g(\bar{x})) \in \mathscr{K}^G_{\bar{x}}$, since it is the image point related to $x := \bar{x}$. Therefore,

$$(\mathbf{0} - c, g(\bar{x}) - d) \in \mathscr{E}^G_{\bar{x}}, \quad \text{for } c = \mathbf{0},\ d = g(\bar{x}) \geq \mathbf{0}.$$

By Theorem 6.1, there exists a supporting hyperplane for $\mathscr{E}^G_{\bar{x}}$ at $\mathbf{0}$, i.e., there exists a vector $(\theta^*, \lambda^*) \in \mathbb{R}^\ell \times \mathbb{R}^m$, $(\theta^*, \lambda^*) \neq \mathbf{0}$ such that:

$$\langle \theta^*, -\overline{\mathscr{D}}_\Phi f(\bar{x}; x - \bar{x}) - c \rangle + \langle \lambda^*, g(\bar{x}) + \underline{\mathscr{D}}_\Gamma g(\bar{x}; x - \bar{x}) - d \rangle \leq 0,$$
$$\forall x \in X, \quad \forall (c, d) \in (\mathbb{R}^\ell_+ \times \mathbb{R}^m_+). \tag{6.103}$$

Setting $x = \bar{x}$, $d = g(\bar{x})$, in (6.103), we obtain:

$$\langle \theta^*, c \rangle \geq 0, \quad \forall c \in \mathbb{R}^\ell_+,$$

so that $\theta^* \in \mathbb{R}^\ell_+$. Similarly, setting $x = \bar{x}$, $c = \mathbf{0}$, in (6.103), we have:

$$\langle \lambda^*, g(\bar{x}) \rangle \leq \langle \lambda^*, d \rangle, \quad \forall d \in \mathbb{R}^m_+,$$

which implies that $\lambda^* \in \mathbb{R}^m_+$. Computing (6.103) for $x = \bar{x}$, $c = \mathbf{0}$, $d = \mathbf{0}$ we obtain:

$$\langle \lambda^*, g(\bar{x}) \rangle \leq 0,$$

which implies that $\langle \lambda^*, g(\bar{x}) \rangle = 0$, since $\bar{x}$ is a feasible point. Finally, setting $c = \mathbf{0}$, $d = \mathbf{0}$ in (6.103) we obtain (6.101).

(b) Assume that (6.102) holds and, ab absurdo, that (6.101) is fulfilled with $\theta^* = \mathbf{0}$. Then, $\lambda^* \neq \mathbf{0}$ and, from (6.101) and (6.102) we obtain:

$$0 < \langle \lambda^*, \underline{\mathscr{D}}_\Gamma g(\bar{x}; z-\bar{x})\rangle \le 0,$$

a contradiction. This completes the proof. □

Consider the Lagrangian function $L(\theta;\lambda,x) := \langle \theta, f(x)\rangle - \langle \lambda, g(x)\rangle$ associated with VOP.

Theorem 6.20. *Let $\Phi \subseteq \mathscr{C}$. Suppose that $f_i, i \in I$, and $-g_j,\ j \in J$, are upper Φ-semidifferentiable functions at $\bar{x} \in K$, as well as any of their combination $\sum_{i\in I}\theta_i f_i - \sum_{j\in J}\lambda_j g_j$ with $(\theta,\lambda) \in \mathbb{R}^\ell_+ \times \mathbb{R}^m_+$.*

(a) If $\bar{x}$ is a weak v.m.p. of VOP (6.31), then there exists $(\theta^*,\lambda^*) \in \mathbb{R}^\ell \times \mathbb{R}^m$, with $(\theta^*,\lambda^*) \neq \mathbf{0}$, such that $(\bar{x},\theta^*,\lambda^*)$ is a solution of the following system:

$$\begin{cases} \mathbf{0} \in \partial_\Phi L(\theta,\lambda;x) \\ \langle \lambda, g(x)\rangle = 0 \\ \theta \ge \mathbf{0},\ \lambda \ge \mathbf{0},\ g(x) \ge \mathbf{0},\ x \in X. \end{cases} \tag{6.104}$$

(b) Moreover, if there exists $z \in X$ such that

$$\underline{\mathscr{D}}_\Phi(-g_j)(\bar{x}; z-\bar{x}) < 0, \quad j \in J(\bar{x}) := \{j \in J : g_j(\bar{x}) = 0\}, \tag{6.105}$$

then the solution $(\bar{x},\theta^*,\lambda^*)$ of (6.104) is such that $\theta^* \neq \mathbf{0}$.

Proof. (a) Set $\Gamma = -\Phi$, from Theorem 6.19 it follows that there exists a non zero vector $(\theta^*,\lambda^*) \in (\mathbb{R}^\ell_+ \times \mathbb{R}^m_+)$ such that:

$$\sum_{i=1}^{\ell} \theta_i^* \overline{\mathscr{D}}_\Phi f_i(\bar{x}; x-\bar{x}) - \sum_{j=1}^{m} \lambda_j^* \underline{\mathscr{D}}_{(-\Phi)} g_j(\bar{x}; x-\bar{x}) \ge 0, \quad \forall x \in X,$$

and $\langle \lambda^*, g(\bar{x})\rangle = 0$.

From the hypotheses, the function $L(\theta^*;\lambda^*,x)$ is upper Φ-semidifferentiable at $\bar{x}$ and the following relations hold:

$$\begin{aligned} \overline{\mathscr{D}}_\Phi L(\theta^*;\lambda^*,\bar{x}; x-\bar{x}) &\ge \sum_{i=1}^{\ell} \theta_i^* \overline{\mathscr{D}}_\Phi f_i(\bar{x}; x-\bar{x}) + \sum_{j\in J} \lambda_j^* \overline{\mathscr{D}}_\Phi(-g_j)(\bar{x}; x-\bar{x}) \\ &= \sum_{i=1}^{\ell} \theta_i^* \overline{\mathscr{D}}_\Phi f_i(\bar{x}; x-\bar{x}) - \sum_{j\in J} \lambda_j^* \underline{\mathscr{D}}_{(-\Phi)} g_j(\bar{x}; x-\bar{x}) \ge 0, \\ &\quad \forall x \in X. \end{aligned}$$

The first inequality follows from Proposition 6.20, while for the equality, it is enough to recall that $-\underline{\mathscr{D}}_\Phi g_j = \overline{\mathscr{D}}_{(-\Phi)}(-g_j),\ j \in J$. The previous relations imply that

$$\mathbf{0} \in \partial_\Phi L(\theta^*;\lambda^*,\bar{x}).$$

(b) The statement again follows from Theorem 6.19, taking into account that for $\Gamma = -\Phi$, we have $\underline{\mathscr{D}}_\Gamma g_j = -\overline{\mathscr{D}}_\Phi(-g_j)$, for every $j \in J(\bar{x})$, so that (6.105) collapses to (6.102), which completes the proof. □

In order to relate the optimality conditions for G-semidifferentiable VOP, with existing results, we report some useful characterizations of the upper G-semiderivative, due to N.D. Yen, under suitable assumptions on the set G [62].

Let $\mathscr{G}_o \subset \mathscr{G}$ be the set of continuous and positively homogeneous functions on X.

Theorem 6.21. *Assume that $G \subset \mathscr{G}_o$ fulfils the property*

$$G + \mathscr{C} \subseteq G. \tag{6.106}$$

If $\phi : X \to \mathbb{R}$ is upper G-semidifferentiable at $\bar{x}$, then

$$\overline{\mathscr{D}}_G\phi(\bar{x};\bar{v}) = \phi^{DH}(\bar{x};\bar{v}) := \limsup_{t\downarrow 0, v\to\bar{v}} \frac{\phi(\bar{x}+tv)-\phi(\bar{x})}{t}.$$

ϕ^{DH} is called the *upper Dini-Hadamard* directional derivative.

Remark 6.8. Observe that, if $G = \mathscr{C}$, $G = \mathscr{C} - \mathscr{C}$, the set of differences of two convex positively homogeneous functions, or $G = \mathscr{G}_o$, then (6.106) is fulfilled.

Moreover, in [62] it is shown that, if G fulfils (6.106), then a necessary and sufficient condition for ϕ to be upper G-semidifferentiable at $\bar{x}$ is that $\phi^{DH}(\bar{x},\cdot) \in G$.

Let us recall the concepts of quasidifferentiability introduced by Pshenichnyi, Demyanov and Rubinov [16, 17, 47].

Definition 6.20.

- The function $\phi : X \to \mathbb{R}$ is said to be *quasidifferentiable* in the sense of Pshenichnyi at $\bar{x}$, if ϕ is directionally differentiable at $\bar{x}$ (see Definition 6.3) and $\phi'(\bar{x};\cdot) \in \mathscr{C}$;
- ϕ is said to be *quasidifferentiable* at $\bar{x}$, if ϕ is directionally differentiable at $\bar{x}$ and $\phi'(\bar{x};\cdot) \in \mathscr{DC} := \mathscr{C} - \mathscr{C}$
- ϕ is said to be *upper Dini quasidifferentiable* at $\bar{x}$, if $\phi^D(\bar{x};\cdot) \in \mathscr{DC}$, where ϕ^D is defined by (6.27)

From Theorem 7.43 and Remark 6.8 next result follows [62].

Proposition 6.22. *(a) If*

$$\phi^{DH}(\bar{x};v) = \phi^D(\bar{x};v), \quad \forall v \in X, \tag{6.107}$$

then ϕ is upper $\mathscr{DC}$-semidifferentiable at $\bar{x}$ if and only if it is upper Dini quasidifferentiable at $\bar{x}$. In that case, the function $\phi^D(\bar{x};\cdot)$ is the upper $\mathscr{DC}$-semiderivative.

(b) If ϕ is directionally differentiable at $\bar{x}$ and (6.107) holds, then ϕ is upper $\mathscr{D}\mathscr{C}$-semidifferentiable at $\bar{x}$ if and only if it is quasidifferentiable at $\bar{x}$. In that case, $\phi'(\bar{x};\cdot)$ is the upper $\mathscr{D}\mathscr{C}$-semiderivative.

(c) If ϕ is directionally differentiable at $\bar{x}$ and (6.107) holds, then ϕ is upper $\mathscr{C}$-semidifferentiable at $\bar{x}$ if and only if it is quasidifferentiable in the sense of Pshenichnyi at $\bar{x}$. In that case, $\phi'(\bar{x};\cdot)$ is the upper $\mathscr{C}$-semiderivative.

We recall that $\phi : X \to \mathbb{R}$ is said to be *locally Lipschitz at* $\bar{x} \in X$, if there exists a scalar $L > 0$ and a neighbourhood V of $\bar{x}$ such that:

$$\|\phi(x^2) - \phi(x^1)\| \leq L\|x^2 - x^1\|, \quad \forall x^1, x^2 \in V.$$

Assume that $\phi : X \to \mathbb{R}$ is locally Lipschitz at $\bar{x}$ and let us consider the relations between the upper G-semiderivative and *Clarke generalized directional derivative* at $\bar{x}$, defined [12] by

$$\phi^C(\bar{x};v) := \limsup_{t\downarrow 0, x\to\bar{x}} \frac{\phi(x+tv) - \phi(\bar{x})}{t}, \quad v \in X.$$

Consider the following subset of $\mathscr{G}$:

$$\mathscr{C}^o := \{\gamma \in \mathscr{G} : \text{epi } \gamma \subseteq \text{cl HC } ((\bar{x}, \phi(\bar{x})), epi\ \phi)\},$$

where HC denotes the *hypertangent cone* (see e.g. [12,51]). In [24] it is shown that a locally Lipschitz function at $\bar{x}$ is upper $\mathscr{C}^o$-semidifferentiable at $\bar{x}$ and the upper $\mathscr{C}^o$-semiderivative collapses to the Clarke generalized directional derivative of ϕ, i.e.,

$$\overline{\mathscr{D}}_{\mathscr{C}^o}\phi(\bar{x};v) = \phi^C(\bar{x};v), \quad \forall v \in X.$$

Furthermore, in [62] it is proved that the upper $\mathscr{C}$-semiderivative of a locally Lipschitz function ϕ at $\bar{x}$ coincides with $\phi^o(\bar{x};\cdot)$ if and only if $\phi^C(\bar{x};\cdot) = \phi^D(\bar{x};\cdot)$.

The literature on optimality conditions for non differentiable VOP is very wide. Besides the above mentioned results, further important developments of the analysis on this topic can be found in Chaps. 5, 7, 12, 13 and in [13,36,43,45].

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Chapter 7
Nonsmooth Invexities, Invariant Monotonicities and Nonsmooth Vector Variational-Like Inequalities with Applications to Vector Optimization

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7.1 Introduction

It is well known that the convexity of functions plays a vital role in mathematical economics, engineering, management, optimization theory, etc. This concept in linear spaces relies on the possibility of connecting any two points of the space by the line segment between them. Since convexity is often not enjoyed the real problems, several classes of functions have been defined and studied for the purpose of weakening the limitations of convexity. In 1981, Hanson [33] realized that the convexity requirement, utilized to prove sufficient optimality conditions for a differentiable mathematical programming problem, can be further weakened by substituting the linear term $y-x$ appearing in the definition of differentiable convex, pseudoconvex and quasiconvex functions with an arbitrary vector-valued function. In view of this idea, Hanson [33] (see also Craven [12]) introduced the concept of invexity by replacing the linear term $y-x$ in the definition of convex function by a vector-valued function $\eta(y,x)$. After the invention of invex functions, a large number of papers on this topic has appeared in the literature on different directions with different applications. Kaul and Kaur [42] introduced and studied the concepts of strictly pseudoinvexity, pseudoinvexity and quasiinvexity with their applications in nonlinear programming problem. Ben-Israel and Mond [9] introduced a new

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generalization of convex sets and convex functions, and Craven [13] called them invex sets and pre-invex functions, respectively. It is showed in [9] that the class of invex functions is equivalent to the class of functions whose stationary points are global minima. Such class of pre-invex functions is further studied by Weir and Mond [71] and Weir and Jeyakumar [70]. The initial results on invexity and pre-invexity inspired a great deal of subsequent work which has greatly expanded the role and applications of these classes of functions in nonlinear optimization and other branches of pure and applied sciences. In 1991, Pini [60] presented the notion of pre-pseudoinvex and pre-quasiinvex functions and established the relationships between invexity and generalized invexity. Mohan and Neogy [58] proved that under certain assumptions, an invex function is pre-invex and a quasiinvex function is pre-quasiinvex. Recently, certain characterizations and applications of pre-invex functions, strictly pre-invex functions, pre-quasiinvex functions and semistrictly pre-quasiinvex functions have been studied in [51, 52, 72, 73, 75]; See also the references therein.

The concepts of monotone and pseudomonotone operators, closely related to convexity and pseudoconvexity, were introduced by Karamardian [39, 40] with further applications in nonlinear complementarity problems and variational inequalities. It is well known that the convexity of a real-valued function is equivalent to the monotonicity of the corresponding gradient function. Karamardian and Schaible [41] studied the relationships between seven kinds of monotone operators and convexities. The monotonicity has played a very important role in the study of the existence and solution methods of variational inequalities and complementarity problems. For further details on monotone operators, generalized monotone operators, convexity of functions and generalized convexities of functions, we refer to [7, 17, 26, 32, 46, 68] and the references therein. On the other hand, several people studied invariant monotonicity with its applications to the existence of solutions of variational-like inequalities. In the recent past, several people have studied the relationships among different kinds of invariant monotonicities and different kinds of invexities; See, for example, [36, 59, 65, 66, 76, 78] and the references therein.

The theory of vector variational inequalities (VVIs) was initiated by Giannessi [21] in 1980. Since then it has been gown up in different directions. One of such direction is the application of VVIs to vector optimization. In the last two decades, VVIs and their generalizations have been used as tools to solve vector optimization problems (VOPs); See, for example, [1, 3, 5, 14–16, 21, 24, 25, 44, 47–50, 66, 74, 77] and the references therein. The (vector) optimization problem may have a nonsmooth objective function. Therefore, Crespi et al. [14, 15] introduced the Minty variational inequality for scalar-valued functions defined by means of Dini lower directional derivative. More recently, the same authors extended their formulation to the vector case in [16]. They also have established the relations between a Minty vector variational inequality (MVVI) and the solutions of vector minimization problem (both ideal and weakly efficient but not efficient) solutions. Crespi et al. [16] used the scalarization method to obtain their results. In [3], Ansari and Lee introduced both the Minty and the Stampacchia type vector variational inequalities (MVVIs and SVVIs, respectively) defined by means of Dini upper directional derivative. By using the MVVI, they provided a necessary and sufficient condition for an efficient solution of VOP for pseudoconvex functions involving Dini upper directional derivative. They established the relationship between the MVVI and the SVVI under upper sign continuity. Some relationships among efficient solutions, weakly efficient solutions, solutions of the SVVI and solutions of the MVVI are discussed. They also presented an existence result for the solutions of weak MVVI and the weak SVVI. Their approach seems to be more direct than the one adopted in [16]. They extended the results of [24, 77] for pseudoconvex functions involving Dini upper directional derivative. On the other hand, several authors studied VOP by using Stampacchia vector variational-like inequalities (SVVLIs). Recently, Ruiz-Garzón et al. [66] established some relationships between Stampacchia vector variational-like inequality (SVVLI) and VOP. They studied the existence of a weakly efficient solution of VOP for differentiable but pseudoinvex functions by using SVVLI. Mishra and Wang [57] established relationships between Stampacchia vector variational-like inequality problems and nonsmooth vector optimization problems under nonsmooth invexity. Recently, Yang and Yang [74] gave some relationships between MVVLI and VOP for differentiable but pseudoinvex vector-valued functions. In particular, they extended the results of Giannessi [24] and Yang et al. [77] for differentiable but pseudoinvex vector-valued functions. They provided the necessary and sufficient conditions for a point to be a solution of VOP for differentiable but pseudoinvex functions, is that, the point be a solution of a Minty vector variational-like inequality problem. They also considered SVVLI and proved its equivalence with MVVLI under continuity assumption. Very recently, in [1], we considered generalized Minty vector variational-like inequality problems, generalized Stampacchia vector variational-like inequality problem and nonsmooth vector optimization problem under nonsmooth invexity. We studied the relationship among these problems under nonsmooth invexity. We also considered the weak formulations of generalized Minty vector variational-like inequality problem and generalized Stampacchia vector variational-like inequality problem and gave some relationships between the solutions of these problems and a weakly efficient solution of vector optimization problem.

Rest of the chapter is organized as follows. In the next section, we review some elementary concepts from nonlinear analysis, convex analysis and invex analysis. In Sect. 7.3, we study directional derivatives, Gâteaux derivative, Dini (lower and upper) directional derivative, Dini–Hadamard (lower and upper) directional derivative, Clarke directional derivative and their properties. By treating these directional derivatives as a bifunction, in Sect. 7.4, we introduce different kinds of invexities for such a bifunction. We present several properties of these invexities. Section 7.5 is devoted to the different kinds of invariant monotonicities. Several relations between different kinds of invexities and different kinds of invariant monotonicities are also given. In Sect. 7.6, we introduce vector variational-like inequalities for bifunctions in such a way that if we treat the Dini upper directional derivative of a function as a bifunction, then we get, so called, the nonsmooth vector variational-like inequalities involving Dini upper directional derivative. Some existence results for these kinds of vector variational-like inequalities are presented. At the end, we study vector optimization problem by using our vector variational-like inequalities. We give several relationships among the weakly efficient solutions and efficient solutions of the VOP, and the solutions of our vector variational-like inequalities.

7.2 Preliminaries

Throughout the chapter, **0** will be considered as a zero vector in the corresponding vector space.

Definition 7.1. Let X be a vector space. A mapping $f : X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be

- *Positively homogeneous* if for all $x \in X$ and all $r > 0$, $f(rx) = rf(x)$
- *Sublinear* if it is positively homogeneous and

$$f(x+y) \leq f(x) + f(y), \quad \text{for all } x, y \in X$$

- *Subodd* if for all $x \in X \setminus \{\mathbf{0}\}$, $f(x) \geq -f(-x)$

  The extension of the addition to $\mathbb{R} \cup \{\pm\infty\}$ is given by

$$(+\infty) + r = +\infty \quad \text{for all } \; r \in \mathbb{R} \cup \{\pm\infty\},$$
$$(-\infty) + r = -\infty \quad \text{for all } \; r \in \mathbb{R} \cup \{-\infty\}.$$

This extension of the addition provides the following equivalence:

$$r + s \geq 0 \quad \text{if and only if} \quad r \geq -s \quad \text{for all } r, s \in \mathbb{R} \cup \{\pm\infty\}. \tag{7.1}$$

Remark 7.1. (a) By using (7.1), f is subodd if and only if $f(x) + f(-x) \geq 0$ for all $x \in X$.
(b) If f is sublinear and is not constant with value $-\infty$ such that $f(0) \geq 0$, then f is subodd.

Definition 7.2 (Hemicontinuous Function). Let K be a nonempty convex subset of a topological vector space X. A function $f : K \to \mathbb{R}$ is said to be

- *Lower hemicontinuous* if the function $t \mapsto f(x + t(y - x))$ is lower semicontinuous on $[0, 1]$
- *Upper hemicontinuous* if the function $t \mapsto f(x + t(y - x))$ is upper semicontinuous on $[0, 1]$
- *Hemicontinuous* if the function $t \mapsto f(x + t(y - x))$ is continuous on $[0, 1]$

Definition 7.3. Let K be a nonempty convex subset of a topological vector space X. A set-valued map $P : K \to 2^K$ is said to be a *KKM map* if for every finite subset $\{x_1, x_2, \ldots, x_n\}$ of K,

$$\mathrm{co}\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} P(x_i),$$

where $\mathrm{co}\{x_1, x_2, \ldots, x_n\}$ denotes the convex hull of $\{x_1, x_2, \ldots, x_n\}$.

The following Fan-KKM theorem and the Browder type fixed point theorem for set-valued maps will be the key tools to establish existence results for solutions of nonsmooth vector variational-like inequalities.

Theorem 7.1 (Fan-KKM Theorem [19]). *Let K be a nonempty convex subset of a Hausdorff topological vector space X and $P : K \to 2^K$ be a KKM map such that $P(x)$ is closed for all $x \in K$, and $P(x)$ is compact for at least one $x \in K$. Then $\bigcap_{x \in K} P(x) \neq \emptyset$.*

Theorem 7.2 ([4]). *Let K be a nonempty convex subset of a Hausdorff topological vector space X, and let $P, Q : K \to 2^K$ be two set-valued maps. Assume that the following conditions hold.*

(i) For each $x \in K$, $\mathrm{co}P(x) \subseteq Q(x)$ and $P(x)$ is nonempty.
(ii) For each $y \in K$, $P^{-1}(y) = \{x \in K : y \in P(x)\}$ is open in K.
(iii) If K is not compact, assume that there exist a nonempty compact convex subset B of K and a nonempty compact subset D of K such that for each $x \in K \setminus D$ there exists $\tilde{y} \in B$ such that $\tilde{y} \in P(x)$.

Then, there exists $\bar{x} \in K$ such that $\bar{x} \in Q(\bar{x})$.

7.2.1 Convexity

Definition 7.4. Let K be a nonempty convex subset of a vector space X. A function $f : K \to \mathbb{R}$ is said to be

- *Convex* if for all $x, y \in K$ and all $t \in [0,1]$,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

- *Strictly convex* if for all $x, y \in K$, $x \neq y$ and all $t \in (0,1)$,

$$f(tx + (1-t)y) < tf(x) + (1-t)f(y)$$

- *Quasiconvex* if for all $x, y \in K$ and all $t \in [0,1]$,

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

- *Strictly quasiconvex* if for all $x, y \in K$, $x \neq y$ and all $t \in (0,1)$,

$$f(tx + (1-t)y) < \max\{f(x), f(y)\}$$

- *Semistrictly quasiconvex* if for all $x,y \in K$, $f(x) \neq f(y)$ and all $t \in (0,1)$,

$$f(tx+(1-t)y) < \max\{f(x), f(y)\}$$

Proposition 7.1 ([6, 54]). *Let K be a nonempty open convex subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a differentiable function. Then,*

(a) f is convex if and only if for all $x,y \in K$,

$$f(y) - f(x) \geq \langle \nabla f(x), y-x \rangle.$$

(b) f is strictly convex if and only if for all $x,y \in K$, $x \neq y$,

$$f(y) - f(x) > \langle \nabla f(x), y-x \rangle.$$

(c) f is quasiconvex if and only if for all $x,y \in K$,

$$f(y) \leq f(x) \quad \textit{implies} \quad \langle \nabla f(x), y-x \rangle \leq 0$$

Definition 7.5 ([53]). Let K be a nonempty open subset of $\mathbb{R}^n$. A differentiable function $f : K \to \mathbb{R}$ is said to be

- *Pseudoconvex* if for all $x,y \in K$,

$$\langle \nabla f(x), y-x \rangle \geq 0 \quad \text{implies} \quad f(y) \geq f(x),$$

  equivalently,

$$f(y) < f(x) \quad \text{implies} \quad \langle \nabla f(x), y-x \rangle < 0$$

- *Strictly pseudoconvex* if for all $x,y \in K$, $x \neq y$,

$$\langle \nabla f(x), y-x \rangle \geq 0 \quad \text{implies} \quad f(y) > f(x),$$

  equivalently,

$$f(y) \leq f(x) \quad \text{implies} \quad \langle \nabla f(x), y-x \rangle < 0.$$

Theorem 7.3 ([20, 54]). *Let K be a nonempty open convex subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a differentiable function. Then,*

(a) f is convex if and only if for all $x,y \in K$,

$$\langle \nabla f(x) - \nabla f(y), x-y \rangle \geq 0. \tag{7.2}$$

(b) f is strictly convex if and only if for all $x,y \in K$, $x \neq y$, the strict inequality holds in (7.2).

(c) f is pseudoconvex if and only if for all $x,y \in K$, $x \neq y$,

$$\langle \nabla f(x), y-x \rangle \geq 0 \quad \textit{implies} \quad \langle \nabla f(y), y-x \rangle \geq 0. \tag{7.3}$$

(d) f *is quasiconvex if and only if for all* $x,y \in K$, $x \neq y$,

$$\langle \nabla f(x), y-x\rangle > 0 \quad \textit{implies} \quad \langle \nabla f(y), y-x\rangle \geq 0. \tag{7.4}$$

(e) f *is semistrictly quasiconvex and quasiconvex if it is pseudoconvex.*

For the proof of this result, we refer to [8, Theorems 3.3.4 and 3.5.11] and [26, Theorems 2.12.1, 2.12.2], respectively. We remark that the part (c) and (d) of above theorem is due to Karamardian [40] and Karamardian and Schaible [41].

If inequality (7.2) (respectively, implications (7.3) and (7.4)) holds, then ∇f is called monotone (respectively, pseudomonotone and quasimonotone).

Several books and monographs on convex analysis are available in the literature. But here, for further details on convex functions, generalized convex functions and their properties, we refer to [6–8, 26, 54, 61] and the references therein.

7.2.2 Invexity

Definition 7.6. Let K be a nonempty subset of a vector space X and $\eta : K \times K \to X$ be a map. The set K is said to be *invex* w.r.t. η if for all $x,y \in K$ and all $t \in [0,1]$, we have $x + t\eta(y,x) \in K$.

Remark 7.2. It can be easily seen that any subset of X is invex w.r.t. $\eta(y,x) = \mathbf{0}$ for all $x,y \in X$, where $\mathbf{0}$ is the zero vector of the vector space X. Mohan and Neogy [58] have pointed out that the definition of an invex set essentially says that there is a path starting from x which is contained in K. It is not required that y should be one of the end points of the path. However, if we demand that x should be an end point of the path for every pair x,y, then $\eta(y,x) = y - x$, reducing to convexity.

We say that the map η is *skew* if for all $x,y \in K$,

$$\eta(y,x) + \eta(x,y) = \mathbf{0}.$$

Condition A. [76] Let X be a vector space, $K \subseteq X$ be an invex set w..r.t. $\eta : K \times K \to X$ and $g : K \to \mathbb{R}$ be a function. Then,

$$g(x + \eta(y,x)) \leq g(y), \quad \text{for all } x,y \in K.$$

Condition C. [58] Let X be a vector space and $K \subseteq X$ be an invex set w.r.t. $\eta : K \times K \to X$. Then, for all $x,y \in K$, $t \in [0,1]$,

(a) $\eta(x, x + t\eta(y,x)) = -t\eta(y,x)$.
(b) $\eta(y, x + t\eta(y,x)) = (1-t)\eta(y,x)$.

Obviously, the map $\eta(y,x) = y - x$ satisfies Condition C. The examples of the map η that satisfies Condition C are given in [75, 76].

Remark 7.3. It is shown in [78] that the Condition C implies that

$$\eta(x+s\eta(y,x),x) = s\eta(y,x), \quad \text{for all } s \in [0,1].$$

The following example shows that a map η that satisfies the Condition C may not be affine in the first argument and may not be skew and vice-versa.

Example 7.1 ([78]). Let $K \subseteq \mathbb{R}$ be a nonempty set. Consider the map $\eta : K \times K \to \mathbb{R}$ defined by

$$\eta(y,x) = \begin{cases} y-x, & \text{if } x \geq 0,\ y \geq 0 \\ y-x, & \text{if } x \leq 0,\ y \leq 0 \\ -2-x, & \text{if } x \leq 0,\ y > 0 \\ 2-x, & \text{if } x > 0,\ y \leq 0. \end{cases}$$

Then, it is easy to see that η satisfies Condition C, but it is not affine in the first argument and not skew.

Consider the map $\eta : K \times K \to \mathbb{R}$ defined by $\eta(y,x) = 3(y-x)$ for all $x,y \in K \subseteq \mathbb{R}$. Then, η is affine in the first argument and skew, but does not satisfies Condition C.

The following lemma can be easily proved.

Lemma 7.1 ([59]). *Let K be a nonempty convex subset of a vector space X and $\eta : K \times K \to X$ be a map. If η is affine in the first argument and skew, then it is also affine in the second argument.*

The following definitions of invex functions, pseudoinvex functions, strictly pseudoinvex functions and quasiinvex functions were introduced and studied in [9, 12, 33, 42].

Definition 7.7. Let K be a nonempty open subset of $\mathbb{R}^n$ and $\eta : K \times K \to X$ be a map. A differentiable function $f : K \to \mathbb{R}$ is said to be

- *Invex* w.r.t. η if for all $x,y \in K$,

$$f(y) - f(x) \geq \langle \nabla f(x), \eta(y,x) \rangle$$

- *Pseudoinvex* w.r.t. η if for all $x,y \in K$,

$$\langle \nabla f(x), \eta(y,x) \rangle \geq 0 \quad \text{implies} \quad f(y) \geq f(x),$$

  equivalently,

$$f(y) < f(x) \quad \text{implies} \quad \langle \nabla f(x), \eta(y,x) \rangle < 0$$

- *Strictly pseudoinvex* w.r.t. η if for all $x,y \in K$, $x \neq y$,

$$\langle \nabla f(x), \eta(y,x) \rangle \geq 0 \quad \text{implies} \quad f(y) > f(x),$$

  equivalently,

$$f(y) \leq f(x) \quad \text{implies} \quad \langle \nabla f(x), \eta(y,x) \rangle < 0$$

- *Quasiinvex* w.r.t. η if for all $x, y \in K$,

$$f(y) \leq f(x) \quad \text{implies} \quad \langle \nabla f(x), \eta(y,x) \rangle \leq 0$$

- *Semistrictly quasiinvex* w.r.t. η if for all $x, y \in K$ with $f(x) \neq f(y)$,

$$f(y) \leq f(x) \quad \text{implies} \quad \langle \nabla f(x), \eta(y,x) \rangle < 0$$

The name "invex" was given by Craven [13] and stands for "invariant convex."

Definition 7.8. Let X be a vector space and K be an invex set w.r.t. $\eta : K \times K \to X$. A function $f : K \to \mathbb{R}$ is said to be

- *Pre-invex* w.r.t. η if for all $x, y \in K$ and all $t \in [0,1]$,

$$f(x + t\eta(y,x)) \leq tf(y) + (1-t)f(x)$$

- *Strictly pre-invex* w.r.t. η if for all $x, y \in K$, $x \neq y$ and all $t \in (0,1)$,

$$f(x + t\eta(y,x)) < tf(y) + (1-t)f(x)$$

- *Pre-quasiinvex* w.r.t. η if for all $x, y \in K$ and all $t \in [0,1]$,

$$f(x + t\eta(y,x)) \leq \max\{f(x), f(y)\}$$

- *Strictly pre-quasiinvex* w.r.t. η if for all $x, y \in K$, $x \neq y$ and all $t \in (0,1)$,

$$f(x + t\eta(y,x)) < \max\{f(x), f(y)\}$$

- *Semistrictly pre-quasiinvex* w.r.t. η if for all $x, y \in K$, $f(x) \neq f(y)$ and all $t \in (0,1)$,

$$f(x + t\eta(y,x)) < \max\{f(x), f(y)\}$$

Proposition 7.2. ([58, Theorems 2.1 and 2.2]) *Let $K \subseteq \mathbb{R}^n$ be a nonempty open invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a differentiable function. Suppose that η satisfies the Condition C.*

(a) If f is invex w.r.t. η, then it is pre-invex w.r.t. the same η.
(b) If f is quasiinvex w.r.t. η, then it is pre-quasiinvex w.r.t. the same η.

For further details on pre-invex, pre-quasiinvex and semistrictly pre-quasiinvex functions, we refer to [51, 52, 58, 65, 72, 73, 75] and the references therein.

Definition 7.9 (η-Hemicontinuous Function). Let X be a topological vector space, $K \subseteq X$ be a nonempty invex set w.r.t. $\eta : K \times K \to X$. A function $f : K \to \mathbb{R}$ is said to be

- η-*lower hemicontinuous* if the function $t \mapsto f(x + t\eta(y,x))$ is lower semicontinuous on $[0,1]$.
- η-*upper hemicontinuous* if the function $t \mapsto f(x + t\eta(y,x))$ is upper semicontinuous on $[0,1]$.
- η-*hemicontinuous* if the function $t \mapsto f(x + t\eta(y,x))$ is continuous on $[0,1]$.

It is well known that the theory of monotonicities (in the sense of Karamardian) plays a vital role in optimization, mathematical economics, nonlinear analysis, variational inequalities, etc. The concept of monotonicity (in the sense of Karamardian) has been extended by replacing the linear term $y - x$ involved in the definition of a monotone map by a vector-valued function $\eta(y,x)$. Such kind of monotonicity is called invex or invariant monotonicity. Several authors have done a lot in this direction. For further detail on this topic, we refer to [59, 65, 76, 78] and the references therein.

7.3 Directional Derivatives

In this section, we discuss directional derivatives, Gâteaux derivative, Dini directional derivatives, Dini–Hadamard directional derivatives and Clarke directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$. Some basic properties of these derivatives are also presented.

Definition 7.10. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a function and $x \in \mathbb{R}^n$ be a point where f is finite.

- The *right-sided directional derivative* of f at x in the direction $d \in \mathbb{R}^n$ is defined by
$$f'_+(x;d) = \lim_{t \to 0^+} \frac{f(x+td) - f(x)}{t} \tag{7.5}$$
if the said limit exist, finite or not.
- The *left-sided directional derivative* of f at x in the direction $d \in \mathbb{R}^n$ is defined by
$$f'_-(x;d) = \lim_{t \to 0^-} \frac{f(x+td) - f(x)}{t} \tag{7.6}$$
if the said limit exist, finite or not.

For $d = \mathbf{0}$ the zero vector in $\mathbb{R}^n$, both $f'_+(x;\mathbf{0})$ and $f'_-(x;\mathbf{0})$ are defined to be zero. It can be easily seen that
$$-f'_+(x;-d) = f'_-(x;d).$$

If $f'_+(x;d)$ exists and $f'_+(x;d) = f'_-(x;d)$, then it is called *directional derivative* of f at x in the direction d. Thus, the *directional derivative* of f at x in the direction $d \in \mathbb{R}^n$ is defined by

$$f'(x;d) = \lim_{t \to 0} \frac{f(x+td) - f(x)}{t} \tag{7.7}$$

provided that the said limit exist, finite or not.

Remark 7.4. (a) If $f'(x;d)$ exists, then $f'(x;-d) = -f'(x;d)$.
(b) If $d = (0,0,\ldots,0,1,0,\ldots,0,0) = e_i$, where 1 is on the ith place, then $f'(x;e_i) = \dfrac{\partial f(x)}{\partial x_i}$ the partial derivative of f with respect to x_i.

The following result ensures the existence of $f'_+(x;d)$ when f is a convex function.

Theorem 7.4. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be an extended convex function and let x be a point in $\mathbb{R}^n$ where f is finite. Then, for each direction $d \in \mathbb{R}^n$, the ratio $\frac{f(x+td)-f(x)}{t}$ is a nondecreasing function of $t > 0$, so that $f'_+(x;d)$ exists for every direction d and*

$$f'_+(x;d) = \inf_{t>0} \frac{f(x+td) - f(x)}{t}. \tag{7.8}$$

Moreover, $f'_+(x;d)$ is a convex and positively homogeneous function of d with $f'_-(x;d) \leq f'_+(x;d)$.

For the proof of Theorem 7.4, we refer to [26, Theorem 2.6.1, pp. 95].

Definition 7.11 (Gâteaux Derivative). A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *Gâteaux differentiable* at a point $x \in \mathbb{R}^n$ if the directional derivative $f'(x;d)$ exists for all $d \in \mathbb{R}^n$, that is, if

$$f^G(x;d) = \lim_{t \to 0} \frac{f(x+td) - f(x)}{t} \tag{7.9}$$

exists for all $d \in \mathbb{R}^n$. $f^G(x;d)$ is called *Gâteaux derivative* of f at x in the direction d.

Theorem 7.5. *Let K be a nonempty open convex subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a convex function. If f is Gâteaux differentiable at $x \in K$ (that is, if $f'(x;d)$ exists for all d), then $f^G(x;d)$ is linear in d. Conversely, if $f'_+(x;d)$ is linear in d, then f is Gâteaux differentiable.*

For the proof of Theorem 7.5, we refer to [30, Theorem 2, pp. 118] (See also [61, pp. 113]).

Remark 7.5. (a) A nonconvex function $f : X \to \mathbb{R}$ may be Gâteaux differentiable at a point but the Gâteaux derivative may not be linear at that point. For example, consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1,x_2) = \begin{cases} \dfrac{x_1^2 x_2}{x_1^2 + x_2^2}, & \text{if } x = (x_1,x_2) \neq (0,0) \\ 0, & \text{if } x = (x_1,x_2) = (0,0). \end{cases}$$

For $d = (d_1,d_2) \neq (0,0)$ and $t \neq 0$, we have

$$\frac{f((0,0)+t(d_1,d_2)) - f((0,0))}{t} = \frac{d_1^2 d_2}{d_1^2 + d_2^2}.$$

Then,

$$f^G((0,0);d) = \lim_{t \to 0} \frac{f((0,0)+t(d_1,d_2)) - f((0,0))}{t} = \frac{d_1^2 d_2}{d_1^2 + d_2^2}.$$

Therefore, f is Gâteaux differentiable at $(0,0)$, but $f^G((0,0);d)$ is not linear in d.

(b) For a real-valued function f on $\mathbb{R}^n$, the partial derivative may exist at a point but f may not be Gâteaux differentiable at that point. For example, consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1,x_2) = \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2}, & \text{if } x = (x_1,x_2) \neq (0,0) \\ 0, & \text{if } x = (x_1,x_2) = (0,0). \end{cases}$$

For $d = (d_1,d_2) \neq (0,0)$ and $t \neq 0$, we have

$$\frac{f((0,0)+t(d_1,d_2)) - f((0,0))}{t} = \frac{d_1 d_2}{t(d_1^2 + d_2^2)}.$$

Then,

$$\lim_{t \to 0} \frac{f((0,0)+t(d_1,d_2)) - f((0,0))}{t} = \lim_{t \to 0} \frac{d_1 d_2}{t(d_1^2 + d_2^2)}$$

exists only if $d = (d_1,0)$ or $d = (0,d_2)$. That is, $f^G(\mathbf{0};\mathbf{0})$ does not exist but $\dfrac{\partial f(0,0)}{\partial x_1} = 0 = \dfrac{\partial f(0,0)}{\partial x_2}$, where $\mathbf{0} = (0,0)$ is the zero vector in $\mathbb{R}^2$.

(c) The existence, linearity and continuity of $f^G(x;d)$ in d do not imply the continuity of the function f. For example, consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x_1,x_2) = \begin{cases} \dfrac{x_1^3}{x_2}, & \text{if } x=(x_1,x_2) \neq (0,0) \\ 0, & \text{if } x=(x_1,x_2)=(0,0). \end{cases}$$

Then,

$$f^G((0,0);d) = \lim_{t\to 0} \frac{t^3 d_1^3}{t^2 d_2} = 0 \quad \text{for all } d=(d_1,d_2)\in\mathbb{R}^2 \text{ with } (d_1,d_2)\neq(0,0).$$

Thus, $f^G(\mathbf{0};d)$ exists and it is continuous and linear in d but f is discontinuous at $(0,0)$. The function f is Gâteaux differentiable but not continuous. Hence a Gâteaux differentiable function is not necessarily continuous.

(d) The Gâteaux derivative $f^G(x;d)$ of a function f is positively homogeneous in the second argument, that is, $f^G(x;rd) = rf^G(x;d)$ for all $r>0$. But, as we have seen in part (a), $f^G(x;d)$ is not linear in d.

The following theorem whose proof can be found in [30, pp. 120], shows that the partial derivatives and Gâteaux derivative are same if the function f is convex.

Theorem 7.6. *Let K be a nonempty convex subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a convex function. If the partial derivatives of f at $x \in K$ exist, then f is Gâteaux differentiable at x.*

For nonconvex functions, we can not expect that in general the limit in the definitions of directional derivative and Gâteaux derivative exists. Therefore, other kinds of generalized directional derivatives were developed which could be useful in the applications for nonconvex case. The simplest way to define such kinds of generalized directional derivatives is to replace the limit operation by the limit superior and limit inferior.

Definition 7.12 (Dini Directional Derivative). Let $f : \mathbb{R}^n \to \mathbb{R}\cup\{\pm\infty\}$ be a function and $x\in\mathbb{R}^n$ be a point where f is finite.

- The *Dini upper directional derivative* at the point $x\in\mathbb{R}^n$ in the direction $d\in\mathbb{R}^n$ is defined by

$$\begin{aligned} f^D(x;d) &= \limsup_{t\to 0^+} \frac{f(x+td)-f(x)}{t} \\ &= \inf_{s>0} \sup_{0<t<s} \frac{f(x+td)-f(x)}{t} \end{aligned}$$

- The *Dini lower directional derivative* at the point $x\in\mathbb{R}^n$ in the direction $d\in\mathbb{R}^n$ is defined by

$$\begin{aligned} f_D(x;d) &= \liminf_{t\to 0^+} \frac{f(x+td)-f(x)}{t} \\ &= \sup_{s>0} \inf_{0<t<s} \frac{f(x+td)-f(x)}{t} \end{aligned}$$

- Similarly, we can define

$$f_{-}^{D}(x;d) = \limsup_{t\to 0^-} \frac{f(x+td)-f(x)}{t}$$

and

$$f_{D}^{-}(x;d) = \liminf_{t\to 0^-} \frac{f(x+td)-f(x)}{t}.$$

Since for each $d \in \mathbb{R}^n$ one has

$$f_{-}^{D}(x;d) = -f_D(x;-d) \quad \text{and} \quad f_{D}^{-}(x;d) = -f^D(x;-d),$$

therefore, it is quite obvious to deal only with the Dini upper directional derivative and the Dini lower directional derivative.

By using the definition, it can be easily seen that for any $x,d \in \mathbb{R}^n$ one has

$$f_D(x;d) \le f^D(x;d)$$

and, of course, when we get the equality in the above inequality, we obtain the right-sided direction derivative $f'(x;d)$. Naturally, in general, this equality does not ensure the convexity of the directional derivative which would be very important for the application. We further note that if the Dini upper and Dini lower directional derivatives in a direction d are finite at a given point, then the function is continuous at that point along the direction d. But the converse need not be true in general. For example, consider the real-valued function $f(x) = \sqrt{|x|}$ for all $x \in \mathbb{R}$. Then f is continuous, but its Dini upper and Dini lower directional derivatives at $x = 0$ in the direction $d = 1$ and $d = -1$ are infinite. One of the most important features of the Dini upper and Dini lower directional derivatives is that they always exist even when the function is discontinuous. Although they are not necessarily finite.

We observe that the function $f(x) = |x|$ does not have a derivative at the point $x = 0$ but does have one-sided derivative $f'_{+}(0) = 1$ and $f'_{-}(0) = -1$. The following example shows that a continuous function may not have even one-sided directional derivative at a point but it may have Dini derivative at that point. For further detail, we refer to [69].

Example 7.2. Consider the function

$$f(x) = \begin{cases} |x|\left|\cos\left(\frac{1}{x}\right)\right|, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Since $\left|\cos\left(\frac{1}{x}\right)\right| \le 1$ for all $x \neq 0$,

$$\lim_{x\to 0} f(x) = 0 = f(0),$$

so f is continuous at $x=0$. It is clear that f is also continuous at all other points of $\mathbb{R}$, so f is a continuous function.

The oscillatory behavior of f is such that the sets

$$\left\{x : \left|\cos\left(\frac{1}{x}\right)\right| = 1\right\} \quad \text{and} \quad \left\{x : \left|\cos\left(\frac{1}{x}\right)\right| = 0\right\}$$

both have zero as two-sided limit point. Thus, each of the sets

$$\{x : f(x) = |x|\} \quad \text{and} \quad \{x : f(x) = 0\}$$

has zero as two-sided limit point. Inspection of the difference quotient reveals that

$$\limsup_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = 1, \quad \text{while} \quad \liminf_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = 0,$$

so $f'_+(x)$ does not exist at $x=0$. Similarly, $f'_+(0)$ does not exist.

However,

$$f^D(0) = 1, \quad f_D(0) = 0, \quad f^D_-(0) = 0, \quad f_D^-(0) = -1,$$

and , elsewhere $f'(x)$ exists and all four Dini derivatives have that value.

In the next example, the given function is not continuous.

Example 7.3. Consider the discontinuous function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

Then, at every rational x,

$$f^D(x) = 0, \quad f_D(x) = -\infty, \quad f^D_-(x) = \infty, \quad f_D^-(x) = 0.$$

For x irrational, there are similar values for the Dini derivatives.

The following result presents some elementary properties and calculus rules for Dini upper (lower) directional derivative. For further study, we refer to [18, 26–29, 32, 37, 55].

Theorem 7.7. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be functions. The following assertions hold.*

(a) Homogeneity: $f^D(x;d)$ is positively homogeneous in d, that is, for all $r>0$ we have $f^D(x;rd) = rf^D(x;d)$.

(b) Scalar multiple: For $r>0$, $(rf)^D(x;d) = rf^D(x;d)$, and for $r<0$, $(rf)^D(x;d) = rf_D(x;d)$.

(c) Sum rule: $(f+g)^D(x;d) \leq f^D(x;d) + g^D(x;d)$ *provided that the sum in the right hand side exists.*

(d) Product rule: $(fg)^D(x;d) \leq [g(x)f]^D(x;d) + [f(x)g]^D(x;d)$ *provided that the sum in the right hand side exists, the functions* f *and* g *are continuous at* x*, and that either of the following conditions is satisfied:* $f(x) \neq 0$*;* $g(x) \neq 0$*;* $f^D(x;d)$ *is finite; and* $g^D(x;d)$ *is finite.*

(e) Quotient rule: $\left(\frac{f}{g}\right)^D(x;d) \leq \frac{[g(x)f]^D(x;d)+[-f(x)g]^D(x;d)}{[g(x)]^2}$ *provided that the expression in the right hand side exists and the function* g *is continuous at* x*.*

If, in addition, the functions f *and* g *are directionally differentiable at* x*, then the inequalities in the last three assertions become equalities.*

The proof follows immediately from the definition. Properties and calculus rules for Dini lower directional derivative can be obtained in a similar manner. The next result shows that Dini upper and Dini lower directional derivatives are convenient tools for characterizing an extremum of a function.

Theorem 7.8. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a function. Then the following assertions hold.*

(a) If $f(x) \leq f(x+td)$ *(respectively,* $f(x) \geq f(x+td)$*) for all* $t > 0$ *sufficiently small, then* $f_D(x;d) \geq 0$ *(respectively,* $f^D(x;d) \leq 0$*). In particular, if* f *is directionally differentiable at* x*, and* $f(x) \leq f(y)$ *(respectively,* $f(x) \geq f(y)$*) for every* y *in a small neighborhood of* x*, then its directional derivative at this point is positive (respectively, negative). Consequently, if* $f'(x;d)$ *is linear in* d*, it vanishes in all directions.*

(b) If $f^D(x+td) \geq 0$ *for all* $t \in (0,1)$ *and if the function* $t \mapsto f(x+td)$ *is continuous on* $[0,1]$*, then* $f(x) \leq f(x+d)$*.*

For the proof of this theorem, we refer to [37, Theorem 1.1.4].

We present the following mean-value theorem due to Diewert [18].

Theorem 7.9 (Diewert Mean Value Theorem). ([18, Theorem 1]) *Let* $f : [a,b] \to \mathbb{R}$ *be an upper hemicontinuous function of one variable defined over the closed interval* $[a,b]$*. Then, there exists* $c \in [a,b)$ *such that*

$$f_D(c) \leq f^D(c) \leq \frac{f(b)-f(a)}{b-a}.$$

We write $f^D(c)$ *in place of* $f^D(c;d)$ *if the directional vector* d *is the scalar* 1*, that is,*

$$f^D(c) = \limsup_{t \to c^+} \frac{f(t)-f(c)}{t-c}$$

and

$$f_D(c) = \liminf_{t \to c^+} \frac{f(t)-f(c)}{t-c}.$$

The following form of mean value theorem for a function defined on a nonempty convex subset of $\mathbb{R}^n$ can be easily derived.

Theorem 7.10. *Let K be a nonempty convex subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ be upper hemicontinuous on K. Then, for any pair x, y of distinct points of K, there exists $w \in [x, y)$ such that*

$$f(y) - f(x) \geq f^D(w; y - x),$$

where $[x, y)$ denotes the line segment joining x and y including the endpoint x. In other words, there exists $t \in [0, 1)$ such that

$$f(y) - f(x) \geq f^D(w; y - x), \quad \text{where } w = x + t(y - x).$$

For the mean value theorem for Dini lower directional derivative, we refer to [18, Theorem 1] and [37, Theorem 1.1.5].

Theorem 7.11. ([18, Corollary 4]) *Let g be an upper semicontinuous function of one variable defined over the interval $[a, b]$. If for all $c \in [a, b)$, $g^D(c) \geq 0$ (respectively, $g^D(c) \leq 0$), then g is a nondecreasing (respectively, nonincreasing) function over $[a, b]$.*

Definition 7.13. Let K be a nonempty subset of $\mathbb{R}^n$ and $x \in K$ be a given point. A function $f : K \to \mathbb{R}$ is said to be *locally Lipschitz* around $x \in K$ if for some $k > 0$

$$|f(y) - f(z)| \leq k\|y - z\|, \quad \text{for all } y, z \in N(x), \tag{7.10}$$

where $N(x)$ is the neighborhood of x.

The function f is said to be *Lipschitz* on K if the inequality (7.10) holds for all $y, z \in K$.

The class of Lipschitz functions is quite large. It is invariant under usual operations of sum, product, and quotient. Lipschitz functions are continuous, but not always directionally differentiable. For example, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 0] \cup [1, \infty) \\ -2x + \frac{2}{3^n}, & \text{if } x \in \left[\frac{2}{3^{n+1}}, \frac{1}{3^n}\right) \\ 2x - \frac{2}{3^{n+1}}, & \text{if } x \in \left[\frac{1}{3^{n+1}}, \frac{2}{3^{n+1}}\right), \end{cases}$$

for $n = 0, 1, 2, 3, \ldots$. Then f is Lipschitz on $\mathbb{R}$ with Lipschitz constant $k = 2$. However, for $x = 0$ and $d = 1$ we have $f^D(x; d) = 1$ and $f_D(x; d) = 0$, which shows that f is not directional differentiable at x. The following result shows that the Lipschitz functions can be characterized by their Dini upper and Dini lower directional derivatives.

Theorem 7.12. *Let K be a nonempty open subset of $\mathbb{R}^n$ and $f : K \to \mathbb{R}$ be a function. Then f is Lipschitz on K with Lipschitz constant $k > 0$ if and only if for all $x \in K$ and all $d \in \mathbb{R}^n$,*

$$\max\left\{f_D(x;d), f^D(x;d)\right\} \leq k\|d\|.$$

For the proof of this theorem, we refer [37, Proposition 1.1.6].

The extension of the Dini (upper and lower) directional derivative is the Dini–Hadamard (upper and lower) directional derivative.

Definition 7.14 (Dini–Hadamard Directional Derivative). Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ be a function and $x \in \mathbb{R}^n$ be a point where f is finite.

- The *Dini–Hadamard upper directional derivative* of f at x in the direction $d \in \mathbb{R}^n$ is defined by
$$f^{DH}(x;d) = \limsup_{\substack{v\to d\\ t\to 0^+}} \frac{f(x+tv)-f(x)}{t}. \tag{7.11}$$
- The *Dini–Hadamard lower directional derivative* of f at x in the direction $d \in \mathbb{R}^n$ is defined by
$$f_{DH}(x;d) = \liminf_{\substack{v\to d\\ t\to 0^+}} \frac{f(x+tv)-f(x)}{t}. \tag{7.12}$$

The following example shows that the Dini (upper and lower) directional derivative and Dini–Hadamard (upper and lower) directional derivative are not same.

Example 7.4. Let $X = \mathbb{R}^2$ and $f : X \to \mathbb{R}$ be a function defined by

$$f(x_1,x_2) = \begin{cases} 0, & \text{if } x_2 = 0\\ x_1 + x_2, & \text{if } x_2 \neq 0. \end{cases}$$

Let $x = (0,0)$ and $d = e_1 = (1,0)$. Then we can easily calculate that

$$f^{DH}(x;d) = 1 \quad \text{and} \quad f^D(x;d) = 0.$$

The above example shows that the Dini upper (lower) directional derivative could not coincide with the Dini–Hadamard upper (lower) directional derivative. The following result shows that they are same if f is locally Lipschitz around x.

Theorem 7.13. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz around x. Then for every $d \in \mathbb{R}^n$,*

$$f^{DH}(x;d) = f^D(x;d) \quad \text{and} \quad f_{DH}(x;d) = f_D(x;d).$$

For the proof of this theorem, we refer to [27, Theorem 2.3].

From the definitions of Dini upper (lower) directional derivative and Dini–Hadamard upper (lower) directional derivative, one can easily obtain the following relations.

$$[-f]^{DH}(x;d) = -f_{DH}(x;d), \quad [-f]_{DH}(x;d) = -f^{DH}(x;d)$$
$$f_{DH}(x;d) \leq f_D(x;d) \leq f^D(x;d) \leq f^{DH}(x;d).$$

Theorem 7.14. **([27, Theorem 2.1])** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Then $f_{DH}(x;d)$ (respectively, $f^{DH}(x;d)$) is a lower (respectively, upper) semicontinuous function in d.*

The following result provides an optimality condition for an unconstrained optimization problem.

Theorem 7.15. **([26, Theorem 4.4.1, pp. 383])** *If $x \in \mathbb{R}^n$ is a local minimum point of the function $f : \mathbb{R}^n \to \mathbb{R}$, then $f_{DH}(x;d) \geq 0$ for all $d \in \mathbb{R}^n$.*

We mention another kind of directional derivative, known as Clarke directional derivative.

Definition 7.15 (Clarke Directional Derivative). Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz around a given point $x \in \mathbb{R}^n$ and d be any other vector in $\mathbb{R}^n$. The *Clarke directional derivative* of f at x in the direction d is defined by

$$\begin{aligned} f^C(x;d) &= \limsup_{\substack{y \to x \\ t \to 0^+}} \frac{f(y+td)-f(y)}{t} \\ &= \inf_{\varepsilon,\delta>0} \sup_{0<t<\delta,\ 0<\|y-x\|<\varepsilon} \frac{f(y+td)-f(y)}{t}, \end{aligned}$$

where, of course, y is a vector in $\mathbb{R}^n$ and t is a positive scalar.

Proposition 7.3. **([11, Proposition 2.1.1])** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz around a given point $x \in \mathbb{R}^n$. Then,*

(a) The function $d \mapsto f^C(x;d)$ is finite, positively homogeneous and subadditive on $\mathbb{R}^n$ and satisfies

$$\left|f^C(x;d)\right| \leq k\|d\|.$$

(b) $f^C(x;d)$ is upper semicontinuous as a function of $(x;d)$ and, as a function of d alone, satisfies Lipschtiz condition on $\mathbb{R}^n$.
(c) $f^C(x;-d) = (-f)^C(x;d)$.

Remark 7.6. Since positive homogeneity and subadditivity imply convexity, $f^C(x;d)$ is a convex function of d.

Theorem 7.16. **([11, Proposition 2.2.2])** *If f is locally Lipschitz around x and has a Gâteaux derivative $f^G(x;d)$, then $f^G(x;d) \leq f^C(x;d)$ for all d.*

Theorem 7.17 ([10]). *If f has a Gâteaux derivative $f^G(x;d)$ which is convex in d and upper semicontinuous in x, then $f^G(x;d) = f^C(x;d)$.*

Remark 7.7. In general, the existence of a Gâteaux derivative does not imply the existence of a finite Clarke directional derivative. Conversely, the existence of a Clarke directional derivative does not imply the existence of a Gâteaux derivative.

The following simple example further clarifies the relationship among Dini, Clarke and Gâteaux directional derivatives.

Example 7.5. Consider the function

$$f(x) = \begin{cases} ax, & \text{if } x \geq 0 \\ bx, & \text{if } x < 0. \end{cases}$$

Then, we can easily calculate that

$$f^G(0;d) = f_D(0;d) = f^D(0;d) = \begin{cases} ay, & \text{if } y \geq 0 \\ by, & \text{if } y < 0 \end{cases}$$

and

$$f^C(0;d) = \begin{cases} \max\{a,b\} \cdot y, & \text{if } y \geq 0 \\ \min\{a,b\} \cdot y, & \text{if } y < 0. \end{cases}$$

Several other kinds of derivatives are defined in the literature; See, for example, [22, 23, 26, 34, 35, 38, 56, 62–64] and the references therein.

7.4 Nonsmooth Invexities

As we have seen above, most of the generalized directional derivatives are positively homogeneous and subodd in their second argument. Some of these derivatives are also subadditive in their second argument. Therefore, instead of considering each generalized directional derivative separately, we consider an abstract two variable function $h : K \times \mathbb{R}^n \to \mathbb{R}$ such that $h(x;0) = 0$ for all $x \in K$.

Definition 7.16. Let $K \subseteq \mathbb{R}^n$ be a nonempty set, $\eta : K \times K \to \mathbb{R}^n$ be a map and $h : K \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction. A function $f : K \to \mathbb{R}$ is said to be

- *h-invex* w.r.t. η if for all $x, y \in K$,

$$f(y) - f(x) \geq h(x; \eta(y,x))$$

- *Strictly h-invex* w.r.t. η if for all $x, y \in K$ with $x \neq y$,

$$f(y) - f(x) > h(x; \eta(y,x))$$

- *h-pseudoinvex* w.r.t. η if for all $x, y \in K$ with $x \neq y$,

$$f(y) < f(x) \quad \text{implies} \quad h(x; \eta(y,x)) < 0,$$

  equivalently,

$$h(x; \eta(y,x)) \geq 0 \quad \text{implies} \quad f(y) \geq f(x)$$

- *Strictly h-pseudoinvex* w.r.t. η if for all $x, y \in K$ with $x \neq y$,

$$f(y) \leq f(x) \quad \text{implies} \quad h(x; \eta(y,x)) < 0,$$

  equivalently,

$$h(x; \eta(y,x)) \geq 0 \quad \text{implies} \quad f(y) > f(x)$$

- *h-quasiinvex* w.r.t. η if for all $x, y \in K$,

$$f(y) \leq f(x) \quad \text{implies} \quad h(x; \eta(y,x)) \leq 0,$$

  equivalently,

$$h(x; \eta(y,x)) > 0 \quad \text{implies} \quad f(y) > f(x)$$

- *Semistrictly h-quasiinvex* w.r.t. η if for all $x, y \in K$ with $f(x) \neq f(y)$,

$$f(y) \leq f(x) \quad \text{implies} \quad h(x; \eta(y,x)) < 0,$$

  equivalently,

$$h(x; \eta(y,x)) \geq 0 \quad \text{implies} \quad f(y) > f(x)$$

Remark 7.8. (a) The above definitions can be applied for the extended real-valued bifunction $h : K \times \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$.
(b) From the above definition, the following relations can be easily verified.

$$\begin{array}{ccc} \text{strict } h\text{-invexity} & \Rightarrow & h\text{-invexity} \\ \Downarrow & & \Downarrow \\ \text{strict } h\text{-pseudoinvexity} & \Rightarrow & h\text{-pseudoinvexity} \\ \Downarrow & & \\ h\text{-quasiinvexity} & & \end{array}$$

With the help of the examples one can easily show that neither the reverse implications nor the following additional ones are true in general:

$$h\text{-quasiinvexity} \Rightarrow h\text{-pseudoinvexity}, \quad h\text{-pseudoinvexity} \Rightarrow h\text{-quasiinvexity}.$$

(c) If the bifunction $h(x; \eta(y,x)) = f'_+(x; \eta(y,x))$

$$\big(\text{respectively,} \quad h(x; \eta(y,x)) = f'_-(x; \eta(y,x)),$$
$$h(x; \eta(y,x)) = f^G(x; \eta(y,x)), \quad h(x; \eta(y,x)) = f^D(x; \eta(y,x)),$$
$$h(x; \eta(y,x)) = f_D(x; \eta(y,x)), \quad h(x; \eta(y,x)) = f^{DH}(x; \eta(y,x)),$$
$$h(x; \eta(y,x) = f_{DH}(x; \eta(y,x)), \quad h(x; \eta(y,x)) = f^C(x; \eta(y,x))\big),$$

then in the above definition, we replace h by UD (respectively, LD, G, D^+, D_-, DH^+, DH^-, C). That is, if $h(x;\eta(y,x)) = f'_+(x;\eta(y,x))$ (respectively, $h(x;\eta(y,x)) = f'_-(x;\eta(y,x))$,

$$h(x;\eta(y,x)) = f^G(x;\eta(y,x)), \quad h(x;\eta(y,x)) = f^D(x;\eta(y,x)),$$

$$h(x;\eta(y,x)) = f_D(x;\eta(y,x)), \quad h(x;\eta(y,x)) = f^{DH}(x;\eta(y,x)),$$

$$h(x;\eta(y,x)) = f_{DH}(x;\eta(y,x)), \quad h(x;\eta(y,x)) = f^C(x;\eta(y,x))),$$

then h-invexity is called UD-invexity (respectively, LD-invexity, G-invexity, D^+-invexity, D_--invexity, DH^+-invexity, DH^--invexity, C-invexity). Other kinds of invexities can be defined in a similar manner.

The following result is a simple consequence of the above definitions.

Theorem 7.18. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$. Let $f : K \to \mathbb{R}$ be a function and $h, g : K \times \mathbb{R}^n \to \mathbb{R}$ be bifunctions such that $g(x;d) \le h(x;d)$ for all $x \in K$ and all $d \in \mathbb{R}^n$. If f is h-invex (respectively, h-quasiinvex, h-pseudoinvex and strictly h-pseudoinvex) w.r.t. η, then it is g-invex (respectively, g-quasiinvex, g-pseudoinvex and strictly g-pseudoinvex) w.r.t. the same η.*

The following theorem is the extension of [58, Theorem 2.1] to h-invexity.

Theorem 7.19. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that the Condition C holds. Let $h : K \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction that is sublinear in the second argument and for all $x \in K$, $h(x;\mathbf{0}) = 0$. If $f : K \to \mathbb{R}$ is (strictly) h-invex w.r.t. η, then it is (strictly) pre-invex w.r.t. the same η;*

Proof. Suppose that $x, y \in K$ and $t \in (0,1)$. Since K is invex, we have $\hat{x} = y + t\eta(x,y) \in K$. By h-invexity of f, we have

$$f(x) - f(\hat{x}) \ge h(\hat{x};\eta(x,\hat{x})). \tag{7.13}$$

Similarly, the condition of h-invexity applied to the pair $y, \hat{x}$ yields

$$f(y) - f(\hat{x}) \ge h(\hat{x};\eta(y,\hat{x})). \tag{7.14}$$

Multiplying inequality (7.13) by t and inequality (7.14) by $(1-t)$ and then adding the resultants, we obtain

$$tf(x) + (1-t)f(y) - f(\hat{x}) \ge th(\hat{x};\eta(x,\hat{x})) + (1-t)h(\hat{x};\eta(y,\hat{x})). \tag{7.15}$$

By Condition C, we have

$$t\eta(x,\hat{x}) + (1-t)\eta(y,\hat{x}) = t(1-t)\eta(x,y) - t(1-t)\eta(x,y) = \mathbf{0}.$$

Since h is sublinear in the second argument, we obtain

$$th(\hat{x};\eta(x,\hat{x}))+(1-t)h(\hat{x};\eta(y,\hat{x}))\geq h(\hat{x};t\eta(x,\hat{x})+(1-t)\eta(y,\hat{x}))=0.$$

Inequality (7.15) yields the conclusion. □

The following theorem provides the converse of Theorem 7.19.

Theorem 7.20. *Let $K\subseteq\mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta:K\times K\to\mathbb{R}^n$. Let $h:K\times\mathbb{R}^n\to\mathbb{R}$ be a bifunction such that for all $x\in K$, $h(x;\cdot)\leq f^D(x;\cdot)$. If f is (strictly) pre-invex w.r.t. η, then it is (strictly) h-invex w.r.t. the same η.*

Proof. Let f be pre-invex w.r.t. η. Then, we have

$$f(x+t\eta(y,x))\leq f(x)+t(f(y)-f(x)),\quad \text{for all } t\in[0,1].$$

Therefore,

$$\frac{f(x+t\eta(y,x))-f(x)}{t}\leq f(y)-f(x),\quad \text{for all } t\in[0,1].$$

Taking $\limsup$ as $t\to 0^+$, we obtain

$$f^D(x;\eta(y,x))=\limsup_{t\to 0^+}\frac{f(x+t\eta(y,x))-f(x)}{t}\leq f(y)-f(x).$$

Since $h(x;\cdot)\leq f^D(x;\cdot)$, we have $h(x;\eta(y,x))\leq f(y)-f(x)$. Hence, f is h-invex w.r.t. η. □

Theorem 7.21. *Let $K\subseteq\mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta:K\times K\to\mathbb{R}^n$. Let $h:K\times\mathbb{R}^n\to\mathbb{R}$ be a bifunction such that for all $x\in K$, $h(x;\cdot)\leq f^D(x;\cdot)$. If f is pre-quasiinvex (respectively, semistrictly pre-quasiinvex) w.r.t. η, then it is h-quasiinvex (respectively, semistrictly h-quasiinvex) w.r.t. the same η.*

Proof. Let $f(y)\leq f(x)$ for all $x,y\in K$. Then, by the pre-quasiinvexity of f w.r.t. η, we obtain $f(x+t\eta(y,x))\leq f(x)$ for all $t\in[0,1]$. Therefore,

$$\frac{f(x+t\eta(y,x))-f(x)}{t}\leq 0\quad \text{for all } t\in[0,1].$$

Taking $\limsup$ as $t\to 0^+$, we obtain

$$f^D(x;\eta(y,x))=\limsup_{t\to 0^+}\frac{f(x+t\eta(y,x))-f(x)}{t}.$$

Since $h(x;\cdot)\leq f^D(x;\cdot)$, we have $h(x;\eta(y,x))\leq 0$. Hence, f is h-quasiinvex w.r.t. η. □

Following the technique of [58, Theorem 2.2], we derive the following result which is the converse of the above theorem.

Theorem 7.22. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that the Condition C holds. Let $h : K \times \mathbb{R}^n \to \mathbb{R}$ be positively homogeneous and subodd in the second argument such that for all $x \in K$, $h(x;\cdot) \leq f^D(x;\cdot)$. If f is continuous and h-quasiinvex w.r.t. η, then it is pre-quasiinvex w.r.t. the same η.*

Proof. Let $x, y \in K$ and $f(y) \leq f(x)$. Consider the set

$$\Omega = \{z \in K : z = x + t\eta(y,x),\ f(z) > f(x),\ 0 \leq t \leq 1\}.$$

In order to show that f is pre-quasiinvex w.r.t. η, we have to show that $\Omega = \emptyset$. If $\Omega \neq \emptyset$, then by continuity of f, the set

$$\tilde{\Omega} = \{z \in K : z = x + t\eta(y,x),\ f(z) > f(x),\ 0 < t < 1\}$$

is also nonempty. Therefore, it is sufficient to show that $\tilde{\Omega} = \emptyset$.

Suppose that $z \in \tilde{\Omega}$. Then $z = x + \tilde{t}\eta(y,x)$ for some $\tilde{t} \in (0,1)$ and $f(z) > f(x) \geq f(y)$. Applying the definition of h-quasiinvexity to the pair z and y, we obtain

$$h(z;\eta(y,z)) \leq 0. \tag{7.16}$$

Similarly, by applying h-quasiinvexity to the pair z and x, we get

$$h(z;\eta(x,z)) \leq 0. \tag{7.17}$$

By Condition C and, the suboddness and positive homogeneity of h in the second argument, inequalities (7.16) and (7.17) can be written as

$$0 \geq h(z;\eta(y,z)) = (1-\tilde{t})h(z;\eta(y,x))$$

and

$$0 \geq h(z;\eta(x,z)) = h(z;-\tilde{t}\eta(y,x)) \geq -\tilde{t}h(z;\eta(y,x)).$$

Since $\tilde{t} \in (0,1)$, we have

$$h(z;\eta(y,x)) = 0. \tag{7.18}$$

Note that (7.18) holds for all $z \in \tilde{\Omega}$. Now suppose that $\tilde{\Omega} \neq \emptyset$ and let $z \in \tilde{\Omega}$ and $z = x + \tilde{t}\eta(y,x)$. By the continuity of f, we can find $0 \leq t_1 \leq \tilde{t} < t_2 < 1$ such that for all $t \in (t_1,t_2)$, we have

$$f(x + t\eta(y,x)) > f(x) \quad \text{and} \quad f(x + t_1\eta(y,x)) = f(x).$$

Let $g(t) = f(x + t\eta(y,x))$. Then we have $g(t_1) = f(x)$. By applying Diewert Mean Value Theorem 7.9 to the function $g : [t_1,t_2] \to \mathbb{R}$, there exists $\hat{t} \in (t_1,t_2)$ such that

$$g^D(\hat{t}) \leq \frac{g(t_2) - g(t_1)}{t_2 - t_1},$$

that is,

$$f^D(x + \hat{t}\eta(y,x); \eta(y,x)) \leq \frac{f(x + t_2\eta(y,x)) - f(x)}{t_2 - t_1}.$$

The right-hand side is positive by our hypothesis, but the left-hand side is zero by (7.18) as $x + \hat{t}\eta(y,x) \in \tilde{\Omega}$, by construction. Therefore, we have a contradiction and the proof follows. The proof is similar when $f(x) \leq f(y)$. □

Theorem 7.23. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that the Condition C holds. Let $h : K \times \mathbb{R}^n \to \mathbb{R}$ be positively homogeneous and subodd in the second argument. If $f : K \to \mathbb{R}$ is h-pseudoinvex w.r.t. η, then it is pre-quasiinvex w.r.t. the same η.*

Proof. It is sufficient to show that if there exist $x, y, z \in K$ such that $z = x + t\eta(y,x)$ for $t \in [0,1]$ and $f(z) > f(x)$, $f(z) > f(y)$ one is led to a contradiction. The h-pseudoinvexity of f w.r.t. η ensures that

$$h(z; \eta(x,z)) < 0 \quad \text{and} \quad h(z; \eta(y,z)) < 0.$$

From Condition C, $\eta(x,z) = -t\eta(y,x)$ and $\eta(y,z) = (1-t)\eta(y,x)$. By positive homogeneity and suboddness of h in the second argument, we have

$$h(z; \eta(y,x)) > 0 \quad \text{and} \quad h(z; \eta(y,x)) < 0,$$

a desired contradiction. □

Proposition 7.4. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that the Condition C holds. Let $h : K \times \mathbb{R}^n \to \mathbb{R}$ be positively homogeneous and subodd in the second argument, and let $f : K \to \mathbb{R}$ be h-pseudoinvex w.r.t. η and upper semicontinuous. If*

$$f^D(x; \eta(y,x)) > 0 \quad \textit{implies} \quad g(t) = f(x + t\eta(y,x)) \tag{7.19}$$

is a decreasing function on $[0,1]$, then f is semistrictly pre-quasiinvex w.r.t. η.

Proof. Without loss of generality, we may assume that $f(y) > f(x)$. Suppose that there exists $t^* \in (0,1)$ such that

$$f(x + t^*\eta(y,x)) \geq f(y). \tag{7.20}$$

By Theorem 7.23, f is pre-quasiinvex w.r.t. η, and hence, the strict inequality in (7.20) cannot hold. So, we must have

$$f(x + t^*\eta(y,x)) = f(y) > f(x) \quad \text{for } t^* \in (0,1). \tag{7.21}$$

Taking $\hat{t} \in [0,1]$, and define $g(t) = f(x + t\eta(y,x))$ for $t \in [0,\hat{t}]$. Then, inequality (7.21) becomes

$$g(t^*) > g(0). \tag{7.22}$$

Suppose $g^D(t) \leq 0$ for $t \in [0,t^*)$. Then, by Theorem 7.11, g is a nonincreasing function over $[0,t^*]$, that is,

$$g(0) \geq g(t^*),$$

which contradicts (7.22). Thus, our supposition that $g^D(t) \leq 0$ for $t \in [0,t^*)$ is false. Hence, we must have $t_0 \in [0,t^*)$ such that

$$g^D(t_0) > 0. \tag{7.23}$$

But (7.23) and (7.19) imply that $g(t)$ is a decreasing function for $t \in [t_0,t^*]$ which contradicts (7.22). Hence, our supposition that there exists $t^* \in (0,1)$ such that $f(x + t^*\eta(y,x)) \geq f(y)$ is false. Therefore, f is semistrictly quasiinvex w.r.t. η. □

Theorem 7.24. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that the Condition C holds. Let $h : K \times \mathbb{R}^n \to \mathbb{R}$ be positively homogeneous and subodd in the second argument. If $f : K \to \mathbb{R}$ is h-pseudoinvex w.r.t. η and upper semicontinuous, then it is semistrictly pre-quasiinvex w.r.t. the same η.*

Proof. Let $x,y \in K$. Then, $x + t\eta(y,x) \in K$ since K is invex. Let $f^D(x;\eta(y,x)) > 0$. Then by Proposition 7.4, we need only to show that the function $g(t) = f(x + t\eta(y,x))$ is decreasing over $[0,1]$. Moreover, since $f^D(x;\eta(y,x)) = g^D(0) > 0$, we must have

$$g(t) > g(0) \quad \text{for all } t \in [0,1]. \tag{7.24}$$

Suppose that g is not decreasing over $[0,1]$. Then since it is nonincreasing, g must be a constant function over a subinterval of $[0,1]$, say over $[t_1,t_2]$, where $0 < t_1 < t_2 \leq 1$. But then $g^D(t_2) = 0$, and since g is pseudoinvex w.r.t. η, $g(0) \geq g(t_2)$ contradicting (7.24). Hence our supposition is false, and g is a decreasing function over $[0,1]$. □

Theorem 7.25. ([74, Theorem 2.2]) *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that the Condition C holds. If $f : K \to \mathbb{R}$ is semistrictly pre-quasiinvex w.r.t. η, then it is pre-quasiinvex w.r.t. the same η.*

Proposition 7.5. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$. Let $h : K \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction and $f : K \to \mathbb{R}$ be h-invex w.r.t. η. If*

$$h(x;y-x) \leq h(x;\eta(y,x)) \tag{7.25}$$

for all $x,y \in K$ such that $f(y) < f(x)$, then f is h-pseudoconvex, that is, for all $x,y \in K$, $f(y) < f(x)$ implies that $h(x;y-x) < 0$. Moreover, if the strict inequality holds in (7.25), then f is strictly h-pseudoconvex, that is, for all $x,y \in K$, $x \neq y$, $f(y) \leq f(x)$ implies that $h(x;y-x) < 0$.

Proof. Let $x, y \in K$ and $f(y) < f(x)$. Then by the condition of h-invexity and inequality (7.25), we have

$$\begin{aligned} h(x; y-x) &= h(x; y-x) - h(x; \eta(y,x)) + h(x; \eta(y,x)) \\ &\leq h(x; y-x) - h(x; \eta(y,x)) + f(y) - f(x) \\ &< h(x; y-x) - h(x; \eta(y,x)) \leq 0. \end{aligned}$$ □

Proposition 7.6. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$. Let $h : K \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction and $f : K \to \mathbb{R}$ be h-invex w.r.t. η such that*

$$h(x; y-x) > 0 \quad \textit{implies} \quad h(x; \eta(y,x)) \geq h(x; y-x) \tag{7.26}$$

for all $x, y \in K$. Then f is h-quasiinvex w.r.t. the same η.

Proof. Let $h(x; y-x) > 0$. Then by the condition of h-invexity of f, we have

$$\begin{aligned} f(y) - f(x) &\geq h(x; \eta(y,x)) \\ &= h(x; \eta(y,x)) - h(x; y-x) + h(x; y-x) \\ &> h(x; \eta(y,x)) - h(x; y-x) \geq 0. \end{aligned}$$ □

Remark 7.9. Propositions 7.6 and 7.5 extend and generalize [60, Proposition 2.5] and [60, Proposition 2.7], respectively, for nondifferentiable and h-invex functions.

7.5 Invariant Monotonicities

Definition 7.17. Let $K \subseteq \mathbb{R}^n$ be a nonempty set and $\eta : K \times K \to \mathbb{R}^n$ be a map. A bifunction $h : K \times \mathbb{R}^n \to \mathbb{R}$ is said to be

- *Invariant monotone* w.r.t. η if for every pair of points $x, y \in K$, we have
$$h(x; \eta(y,x)) + h(y; \eta(x,y)) \leq 0$$
- *Strictly invariant monotone* w.r.t. η if for every pair of distinct points $x, y \in K$, we have
$$h(x; \eta(y,x)) + h(y; \eta(x,y)) < 0$$
- *Invariant pseudomonotone* w.r.t. η if for every pair of distinct points $x, y \in K$, we have
$$h(x; \eta(y,x)) \geq 0 \quad \text{implies} \quad h(y; \eta(x,y)) \leq 0, \tag{7.27}$$

equivalently,

$$h(y;\eta(x,y)) > 0 \quad \text{implies} \quad h(x;\eta(y,x)) < 0$$

- *Strictly invariant pseudomonotone* w.r.t. η if for every pair of distinct points $x,y \in K$, we have

$$h(x;\eta(y,x)) \geq 0 \quad \text{implies} \quad h(y;\eta(x,y)) < 0,$$

  equivalently

$$h(y;\eta(x,y)) \geq 0 \quad \text{implies} \quad h(x;\eta(y,x)) < 0$$

- *Invariant quasimonotone* w.r.t. η if for every pair of distinct points $x,y \in K$, we have

$$h(x;\eta(y,x)) > 0 \quad \text{implies} \quad h(y;\eta(x,y)) \leq 0,$$

  equivalently,

$$h(y;\eta(x,y)) > 0 \quad \text{implies} \quad h(x;\eta(y,x)) \leq 0$$

Remark 7.10. If the bifunction h is replaced by f'_+ (respectively, f'_-, f^G, f^D, f_D, f^{DH}, f_{DH}, f^C), then invariant monotonicity is called UD-invariant monotonicity (respectively, LD-invariant monotonicity, G-invariant monotonicity, D^+-invariant monotonicity, D_--invariant monotonicity, DH^+-invariant monotonicity, DH^--invariant monotonicity, C-invariant monotonicity). Similarly, we can define other kinds of invariant monotonicities.

We have the following relations among these kinds of monotonicities.

strictly invariant monotonicity	$\Rightarrow$	invariant monotonicity
$\Downarrow$		$\Downarrow$
strictly invariant pseudomonotonicity	$\Rightarrow$	invariant pseudomonotonicity
		$\Downarrow$
		invarinat quasimonotonicity

Lemma 7.2. *Let $K \subseteq \mathbb{R}^n$ be a nonempty set. A bifunction $h : K \times \mathbb{R}^n \to \mathbb{R}$ is invariant pseudomonotone if and only if for every pair of distinct points $x,y \in K$, we have*

$$h(x;\eta(y,x)) > 0 \quad \textit{implies} \quad h(y;\eta(x,y)) < 0. \tag{7.28}$$

Proof. The implication (7.28) is equivalent to the following implication:

$$h(y;\eta(x,y)) \geq 0 \quad \text{implies} \quad h(x;\eta(y,x)) \leq 0.$$

Interchanging x and y, we get (7.27). □

The following result, which generalizes [43, Proposition 2.1], is a direct consequence of the definition of invariant quasimonotonicity.

Lemma 7.3. *Let $K \subseteq \mathbb{R}^n$ be a nonempty set and $\eta : K \times K \to \mathbb{R}^n$ be a map. The bifunction $h : K \times \mathbb{R}^n \to \mathbb{R}$ is invariant quasimonotone w.r.t. η if and only if for every pair of distinct points $x, y \in K$, we have*

$$\min\{h(x;\eta(y,x)),\ h(y;\eta(x,y))\} \leq 0.$$

The following lemma directly follows from the definition.

Lemma 7.4. *Let $K \subseteq \mathbb{R}^n$ be a nonempty set and $\eta : K \times K \to X$ be a map. Let $h, g : K \times \mathbb{R}^n \to \mathbb{R}$ be bifunctions such that $g(x;d) \leq h(x;d)$ for all $x \in K$ and all $d \in \mathbb{R}^n$. If $h(x;d)$ is invariant monotone (respectively, strictly invariant monotone, invariant pseudomonotone, invariant strictly pseudomonotone and invariant quasimonotone) w.r.t. η, then $g(x;d)$ is invariant monotone (respectively, strictly invariant monotone, invariant pseudomonotone, strictly invariant pseudomonotone and invariant quasimonotone) w.r.t. the same η.*

Proof. We only consider the case of invariant quasimonotone w.r.t. η. Suppose that for all $x, y \in K$, $x \neq y$,

$$g(x;\eta(y,x)) > 0.$$

Since $g(x;d) \leq h(x;d)$, we have $h(x;\eta(y,x)) > 0$. By invariant quasimonotonicity of h, we obtain $h(y;\eta(x,y)) \leq 0$. Again, applying $g(x;d) \leq h(x;d)$, we get $g(y;\eta(x,y)) \leq 0$, and hence, g is invariant quasimonotone. □

Theorem 7.26. *Let $K \subseteq \mathbb{R}^n$ be a nonempty set, $\eta : K \times K \to \mathbb{R}^n$ be a map and $h : K \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction.*

(a) If $f : K \to \mathbb{R}$ is h-invex (respectively, strictly h-invex) w.r.t. η, then h is invariant monotone (respectively, strictly invariant monotone) w.r.t. the same η.
(b) If $f : K \to \mathbb{R}$ is strictly h-pseudoinvex w.r.t. η, then h is strictly invariant pseudomonotone w.r.t. the same η.
(c) If $f : K \to \mathbb{R}$ is h-quasiinvex w.r.t. η, then h is invariant quasimonotone w.r.t. the same η.

Proof. (a) Let f be h-invex w.r.t. η. Then for all $x, y \in K$,

$$f(y) - f(x) \geq h(x;\eta(y,x)). \tag{7.29}$$

By interchanging x and y, we obtain

$$f(x) - f(y) \geq h(y;\eta(x,y)). \tag{7.30}$$

Adding inequalities (7.29) and (7.30), we get

$$h(x;\eta(y,x)) + h(y;\eta(x,y)) \leq 0,$$

and hence, h is invariant monotone w.r.t. η.

(b) Let f be strictly h-pseudoinvex w.r.t. η. Then for all $x, y \in K$, $x \neq y$,

$$h(x;\eta(y,x)) \geq 0 \quad \text{implies} \quad f(y) > f(x).$$

Suppose that for all $x, y \in K$, $x \neq y$,

$$h(x;\eta(y,x)) \geq 0 \quad \text{but} \quad h(y;\eta(x,y)) \geq 0.$$

By using the second inequality and strict h-pseudoinvexity of f, we have $f(y) < f(x)$, which is a contradiction.

(c) Suppose that f is h-quasiinvex w.r.t. η. Then for all $x, y \in K$ with $x \neq y$,

$$f(y) \leq f(x) \quad \text{implies} \quad h(x;\eta(y,x)) \leq 0.$$

Let $x, y \in K$ be such that $h(x;\eta(y,x)) > 0$. Then $f(y) > f(x)$. By h-quasiinvexity of f, we have $h(y;\eta(x,y)) \leq 0$. Hence h is invariant quasimonotone w.r.t. η. □

Corollary 7.1. *Let $K \subseteq \mathbb{R}^n$ be a nonempty set, $\eta : K \times K \to \mathbb{R}^n$ be a map and $h : K \times \mathbb{R}^n \to \mathbb{R}$ be a bifunction. If $f : K \to \mathbb{R}$ is h-invex w.r.t. η, then h is invariant pseudomonotone w.r.t. the same η.*

Proof. From Theorem 7.26 (a), h-invexity of f implies invariant monotonicity of h. Since invariant monotonicity implies invariant pseudomonotonicity, we obtain the desired result. □

Theorem 7.27. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that the Condition C holds. Let the bifunction $h : K \times \mathbb{R}^n \to \mathbb{R}$ be subodd and positively homogeneous in the second argument such that for all $x \in K$, $h(x;\cdot) \leq f^D(x;\cdot)$. If $f : K \to \mathbb{R}$ is h-pseudoinvex w.r.t. η, then h is invariant pseudomonotone w.r.t. the same η. In addition, if η is skew, f is η-upper hemicontinuous on K and for all $x, y \in K$, $x \neq y$, $f(y) < f(x)$ implies $f(x + \eta(y,x)) < f(x)$, and h is invariant pseudomonotone w.r.t. η, then f is h-pseudoinvex w.r.t. the same η.*

Proof. Assume that f is h-pseudoinvex w.r.t. η. Let $x, y \in K$ be such that $h(x;\eta(y,x)) > 0$. Since $h(x;d) \leq f^D(x;d)$, we have $f^D(x;\eta(y,x)) > 0$. Then

$$\limsup_{t \to 0^+} \frac{f(x + t\eta(y,x)) - f(x)}{t} > 0.$$

Therefore, for some $t \in (0,1)$, we have $f(x + t\eta(y,x)) > f(x)$. Since by Theorem 7.23, f is pre-quasiinvex w.r.t. η, so we get

$$f(y) \geq f(x + t\eta(y,x)) > f(x).$$

Applying the definition of h-pseudoinvexity to the pair x and y, we obtain $h(y;\eta(x,y)) < 0$. The results follows from Lemma 7.2.

Assume that h is invariant pseudomontone w.r.t. η. Let $x,y \in K$, $x \neq y$, such that $f(y) < f(x)$. Then by hypothesis $f(x+\eta(y,x)) < f(x)$. We claim that

$$h(x;\eta(y,x)) < 0. \tag{7.31}$$

Let $g : [0,1] \to \mathbb{R}$ be a function defined as

$$g(t) = f(x+t\eta(y,x)).$$

Then by Diewert Mean Value Theorem 7.9, there exists $t \in [0,1)$ such that $g^D(t) \leq g(1)-g(0) = f(x+\eta(y,x)) - f(x)$, that is,

$$h(z;\eta(y,x)) \leq f^D(z;\eta(y,x)) \leq f(x+\eta(y,x)) - f(x) < 0,$$

where $z = x+t\eta(y,x)$ for $t \in [0,1)$. When $t = 0$, then the inequality (7.31) holds. When $t > 0$, then by Condition C, Remark 7.3 and positive homogeneity of h is the second argument, we successively obtain

$$h(z;\eta(z,x)) = h(z;\eta(x+t\eta(y,x),x)) = h(z;t\eta(y,x)) = th(z;\eta(y,x)) < 0.$$

By skewness of η and suboddness of h in the second argument, we have $h(z;\eta(x,z)) > 0$. The invariant pseudomonotonicity of h w.r.t. η implies that $h(x;\eta(z,x)) < 0$. The Condition C, Remark 7.3 and positive homogeneity of h in the second argument yield (7.31). ☐

Remark 7.11. Theorem 7.27 extends and generalizes [67, Theorem 5.2] for h-pseudoinvex functions w.r.t. η.

Theorem 7.28. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that the Condition C holds. Let $h : K \times \mathbb{R}^n \to \mathbb{R}$ be invariant pseudomonotone w.r.t. η and, positively homogeneous and subodd in the second argument. Let $f : K \to \mathbb{R}$ be a function such that for all $x,y \in K$, $x \neq y$*

$$f(y) < f(x) \quad \textit{implies} \quad h(x+t\eta(y,x);\eta(y,x)) < 0 \quad \textit{for some } t \in (0,1). \tag{7.32}$$

Then, f is h-pseudoinvex w.r.t. the same η.

Proof. Let $x,y \in K$ be such that $x \neq y$ and

$$h(x;\eta(y,x)) \geq 0. \tag{7.33}$$

We claim that $f(x) \leq f(y)$. Assume to the contrary that $f(x) > f(y)$. Then by implication (7.32), we have

$$h(x+t\eta(y,x);\eta(y,x)) < 0 \quad \text{for some } t \in (0,1). \tag{7.34}$$

From Condition C, we have $\eta(x, x+s\eta(y,x)) = -s\eta(y,x)$ for all $s \in [0,1]$. By using positive homogeneity and suboddness of h in the second argument, the above inequality (7.34) can be written as

$$h(x+t\eta(y,x);\eta(x,x+t\eta(y,x))) > 0 \quad \text{for some } t \in (0,1). \tag{7.35}$$

Since h is invariant pseudomonotone w.r.t. η, inequality (7.35) implies that

$$h(x;\eta(x+t\eta(y,x),x)) < 0 \quad \text{for some } t \in (0,1). \tag{7.36}$$

By using Remark 7.3 and the fact that $t \in (0,1)$, inequality (7.36) becomes $h(x;\eta(y,x)) < 0$ which is a contradiction of inequality (7.33). Thus, f is h-pseudoinvex w.r.t. η. □

The following theorem is the converse of Theorem 7.26 (b).

Theorem 7.29. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that the Condition C holds. Let $h : K \times \mathbb{R}^n \to \mathbb{R}$ be strictly invariant pseudomonotone w.r.t. η and $f : K \to \mathbb{R}$ be a function such that for all $x, y \in K$, $x \neq y$*

$$f(y) \le f(x) \quad \textit{implies} \quad h(x+t\eta(y,x);\eta(y,x)) \le 0 \quad \textit{for some } t \in (0,1). \tag{7.37}$$

Then, f is strictly h-pseudoinvex w.r.t. η.

Proof. From Condition C, we have

$$\eta(x+s\eta(y,x),x) = s\eta(y,x) \quad \text{for all } s \in [0,1],$$

and following the same argument as in the proof of Theorem 7.28, we get the conclusion. □

Remark 7.12. Theorems 7.28 and 7.29 extend and generalize [78, Theorem 2.1] for nondifferentiable functions.

The following theorem is the converse of Theorem 7.26 (c).

Theorem 7.30. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that the Condition C holds. Let $h : K \times \mathbb{R}^n \to \mathbb{R}$ be invariant quasimonotone w.r.t. η and, subodd and positively homogeneous in the second argument. Let $f : K \to \mathbb{R}$ be a function such that for all $x, y \in K$, $x \neq y$*

$$f(x) \ge f(y) \quad \textit{implies} \quad h(x+t\eta(y,x);\eta(y,x)) < 0 \quad \textit{for some } t \in (0,1). \tag{7.38}$$

Then, f is h-quasiinvex w.r.t. the same η.

Proof. Assume to the contrary that f is not h-quasiinvex w.r.t. η. Then there exist $x, y \in K$ such that

$$f(y) \le f(x), \tag{7.39}$$

but

$$h(x;\eta(y,x)) > 0. \tag{7.40}$$

Then by the inequalities (7.38) and (7.39), we have

$$h(x+t\eta(y,x);\eta(y,x)) < 0 \quad \text{for some } t \in (0,1).$$

By Condition C (a), we have $\eta(x,x+t\eta(y,x)) = -t\eta(y,x)$. Therefore, by positive homogeneity and suboddness of h in the second argument, we obtain

$$h(x+t\eta(y,x);\eta(x,x+t\eta(y,x))) > 0 \quad \text{for some } t \in (0,1). \tag{7.41}$$

Since h is invariant quasimonotone w.r.t. η, the inequality (7.41) implies that

$$h(x;\eta(x+t\eta(y,x),x)) \leq 0 \quad \text{for some } t \in (0,1). \tag{7.42}$$

From Condition C, Remark 7.3 and the fact that $t \in (0,1)$, the inequality (7.42) becomes $h(x;\eta(y,x)) \leq 0$ which is a contradiction of the inequality (7.40). Thus, f is h-quasiinvex w.r.t. η. □

Remark 7.13. Theorem 7.30 extends and generalizes [78, Theorem 3.1].

Now we present some necessary conditions of (strict) invariant pseudomonotonicity and invariant quasimonotonicity of h with the condition that the under lying set K is convex and η is affine in the first argument and skew instead of invexity of K and Condition C, respectively.

Theorem 7.31. *Let K be a nonempty convex subset of $\mathbb{R}^n$ and $\eta : K \times K \to \mathbb{R}^n$ be skew and affine in the first argument. Let $h : K \times \mathbb{R}^n \to \mathbb{R}$ be invariant pseudomonotone w.r.t. η and positively homogeneous in the second argument. If $f : K \to \mathbb{R}$ is a function such that for all $x,y \in K$, $x \neq y$*

$$f(x) > f(y) \quad \textit{implies} \quad h(z;\eta(x,z)) > 0 \tag{7.43}$$

for some z which lies on the line segment joining x and y, then f is h-pseudoinvex w.r.t. η.

Proof. Let $x,y \in K$ be such that

$$h(x;\eta(y,x)) \geq 0. \tag{7.44}$$

We claim that $f(y) \geq f(x)$. Assume to the contrary that $f(y) < f(x)$. Then by the inequality (7.43), we have

$$h(z;\eta(x,z)) > 0, \tag{7.45}$$

where $z = tx + (1-t)y$ for some $t \in (0,1)$. Since h is invariant pseudomonotone w.r.t. η, by Lemma 7.2, the inequality (7.45) becomes $h(x;\eta(z,x)) < 0$. By the skewness of η, we have $\eta(x,x) = \mathbf{0}$. Since η is affine in the first argument, by positive homogeneity of h is the second argument, we have

$$\begin{aligned} 0 > h(x;\eta(z,x)) &= h(x;t\eta(x,x) + (1-t)\eta(y,x)) \\ &= h(x;(1-t)\eta(y,x)) \\ &= (1-t)h(x;\eta(y,x)). \end{aligned}$$

Therefore, $h(x;\eta(y,x)) < 0$ which is a contradiction of the inequality (7.44). Hence, f is h-pseudoinvex w.r.t. η. □

Theorem 7.32. *Let K be a nonempty convex subset of $\mathbb{R}^n$ and $\eta : K \times K \to \mathbb{R}^n$ be skew and affine in the first argument. Let $h : K \times \mathbb{R}^n \to \mathbb{R}$ be strictly invariant pseudomonotone w.r.t. η and positively homogeneous in the second argument. If $f : K \to \mathbb{R}$ is a function such that for all $x,y \in K$, $x \neq y$*

$$f(x) \geq f(y) \quad \textit{implies} \quad h(z;\eta(x,z)) \geq 0 \tag{7.46}$$

for some z which lies on the line segment joining x and y, then f is strictly h-pseudoinvex w.r.t. η.

Proof. Let $x,y \in K$ be such that $x \neq y$ and

$$h(x;\eta(y,x)) \geq 0. \tag{7.47}$$

We claim that $f(y) > f(x)$. Assume to the contrary that $f(y) \leq f(x)$. Then by the implication (7.46), we have

$$h(z;\eta(x,z)) \geq 0, \tag{7.48}$$

where $z = tx + (1-t)y$ for some $t \in (0,1)$. Since h is strictly invariant pseudomonotone w.r.t. η, the inequality (7.48) implies that

$$h(x;\eta(x,z)) > 0. \tag{7.49}$$

Rest of the proof follows on the lines of the proof of Theorem 7.31. □

Theorem 7.33. *Let K be a nonempty convex subset of $\mathbb{R}^n$ and $\eta : K \times K \to \mathbb{R}^n$ be skew and affine in the first argument. Let $h : K \times \mathbb{R}^n \to \mathbb{R}$ be invariant quasimonotone w.r.t. η and positively homogeneous in the second argument. If $f : K \to \mathbb{R}$ is a function such that for all $x,y \in K$, $x \neq y$*

$$f(y) \leq f(x) \quad \textit{implies} \quad h(z;\eta(x,z)) > 0 \tag{7.50}$$

for some z which lies on the line segment joining x and y, then f is h-quasiinvex w.r.t. η.

Proof. Assume to the contrary that f is not h-quasiinvex w.r.t. η. Then there exist $x,y \in K$ such that

$$f(y) \leq f(x), \tag{7.51}$$

but

$$h(x;\eta(y,x)) > 0 \tag{7.52}$$

Then by the implication (7.50) and inequality (7.51), we have

$$h(z;\eta(x,z)) > 0, \tag{7.53}$$

where $z = tx + (1-t)y$ for some $t \in (0,1)$. Since h is invariant quasimonotone w.r.t. η, the inequality (7.53) implies that

$$h(y;\eta(z,y)) \leq 0. \tag{7.54}$$

Rest of the proof follows on the lines of the proof of Theorem 7.31. □

Remark 7.14. Theorems 7.31, 7.32 and 7.33 extend and generalize [59, Theorems 2.2, 2.1 and 2.3].

7.6 Nonsmooth Vector Variational-Like Inequalities

In the sequel, we adopt the following ordering relations. We consider the cones $C = \mathbb{R}^\ell_+$, $C_\mathbf{0} := \mathbb{R}^\ell_+ \setminus \{\mathbf{0}\}$ and $\overset{\circ}{C} := \text{int}\, \mathbb{R}^\ell_+$, where $\mathbb{R}^\ell_+$ is the nonnegative orthant of $\mathbb{R}^\ell$ and $\mathbf{0}$ is the origin of $\mathbb{R}^\ell$; let D be a subset of $\mathbb{R}^\ell$. Then for all $x, y \in D$,

$$x \geq_C y \;\Leftrightarrow\; x - y \in C; \quad x \not\geq_C y \;\Leftrightarrow\; x - y \notin C;$$
$$x \leq_C y \;\Leftrightarrow\; y - x \in C; \quad x \not\leq_C y \;\Leftrightarrow\; y - x \notin C;$$
$$x \geq_{C_\mathbf{0}} y \Leftrightarrow x - y \in C_\mathbf{0}; \quad x \not\geq_{C_\mathbf{0}} y \Leftrightarrow x - y \notin C_\mathbf{0};$$
$$x \leq_{C_\mathbf{0}} y \Leftrightarrow y - x \in C_\mathbf{0}; \quad x \not\leq_{C_\mathbf{0}} y \Leftrightarrow y - x \notin C_\mathbf{0};$$
$$x \geq_{\overset{\circ}{C}} y \;\Leftrightarrow\; x - y \in \overset{\circ}{C}; \quad x \not\geq_{\overset{\circ}{C}} y \;\Leftrightarrow\; x - y \notin \overset{\circ}{C};$$
$$x \leq_{\overset{\circ}{C}} y \;\Leftrightarrow\; y - x \in \overset{\circ}{C}; \quad x \not\leq_{\overset{\circ}{C}} y \;\Leftrightarrow\; y - x \notin \overset{\circ}{C}.$$

Let K be a nonempty subset of $\mathbb{R}^n$ and $\eta : K \times K \to \mathbb{R}^n$ be a given map. Let $h = (h_1, \ldots, h_\ell) : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be a vector-valued function. We consider the following nonsmooth vector variational-like inequality problems, namely, Stampacchia type vector variational-like inequality problems and Minty type vector variational-like inequality problems.

Stampacchia vector variational-like inequality problem (SVVLIP): Find $\bar{x} \in K$ such that

$$h(\bar{x};\eta(y,\bar{x})) = \big(h_1(\bar{x};\eta(y,\bar{x})), \ldots, h_\ell(\bar{x};\eta(y,\bar{x}))\big) \not\leq_{C_\mathbf{0}} \mathbf{0}, \quad \text{for all } y \in K. \tag{7.55}$$

Minty vector variational-like inequality problem (MVVLIP): Find $\bar{x} \in K$ such that

$$h(y;\eta(\bar{x},y)) = \big(h_1(y;\eta(\bar{x},y)),\dots,h_\ell(y;\eta(\bar{x},y))\big) \not\geq_{C_0} \mathbf{0}, \quad \text{for all } y \in K. \tag{7.56}$$

If we replace the order relation $\not\leq_{C_0}$ by $\not\leq_{\overset{\circ}{C}}$ in (7.55) and $\not\geq_{C_0}$ by $\not\geq_{\overset{\circ}{C}}$ in (7.56), then we get the following weak formulations of (SVVLIP) and (MVVLIP):
Weak Stampacchia vector variational-like inequality problem (WSVVLIP): Find $\bar{x} \in K$ such that

$$h(\bar{x};\eta(y,\bar{x})) = \big(h_1(\bar{x};\eta(y,\bar{x})),\dots,h_\ell(\bar{x};\eta(y,\bar{x}))\big) \not\leq_{\overset{\circ}{C}} \mathbf{0}, \quad \text{for all } y \in K. \tag{7.57}$$

Weak Minty vector variational-like inequality problem (WMVVLIP): Find $\bar{x} \in K$ such that

$$h(y;\eta(\bar{x},y)) = \big(h_1(y;\eta(\bar{x},y)),\dots,h_\ell(y;\eta(\bar{x},y))\big) \not\geq_{\overset{\circ}{C}} \mathbf{0}, \quad \text{for all } y \in K. \tag{7.58}$$

Let $f = (f_1,\dots,f_\ell) : \mathbb{R}^n \to \mathbb{R}^\ell$ be a vector-valued function and $f^D(x;d) = \big(f_1^D(x;d),\dots,f_\ell^D(x;d)\big)$. When $h(x;\cdot) = f^D(x;\cdot)$, the (SVVLIP) and (MVVLIP) become the following *nonsmooth vector variational-like inequality problems*:
(NSVVLIP): Find $\bar{x} \in K$ such that

$$f^D(\bar{x};\eta(y,\bar{x})) = \big(f_1^D(\bar{x};\eta(y,\bar{x})),\dots,f_\ell^D(\bar{x};\eta(y,\bar{x}))\big) \not\leq_{C_0} \mathbf{0}, \quad \text{for all } y \in K. \tag{7.59}$$

(NMVVLIP): Find $\bar{x} \in K$ such that

$$f^D(y;\eta(\bar{x},y)) = \big(f_1^D(y;\eta(\bar{x},y)),\dots,f_\ell^D(y;\eta(\bar{x},y))\big) \not\geq_{C_0} \mathbf{0}, \quad \text{for all } y \in K. \tag{7.60}$$

As above, if we replace the order relation $\not\leq_{C_0}$ by $\not\leq_{\overset{\circ}{C}}$ in (7.59) and $\not\geq_{C_0}$ by $\not\geq_{\overset{\circ}{C}}$ in (7.60), then we get the following weak formulations of (NSVVLIP) and (NMVVLIP):
(NWSVVLIP): Find $\bar{x} \in K$ such that

$$f^D(\bar{x};\eta(y,\bar{x})) = \big(f_1^D(\bar{x};\eta(y,\bar{x})),\dots,f_\ell^D(\bar{x};\eta(y,\bar{x}))\big) \not\leq_{\overset{\circ}{C}} \mathbf{0}, \quad \text{for all } y \in K. \tag{7.61}$$

(NWMVVLIP): Find $\bar{x} \in K$ such that

$$f^D(y;\eta(\bar{x},y)) = \big(f_1^D(y;\eta(\bar{x},y)),\dots,f_\ell^D(y;\eta(\bar{x},y))\big) \not\geq_{\overset{\circ}{C}} \mathbf{0}, \quad \text{for all } y \in K. \tag{7.62}$$

If we consider the Dini derivative (upper or lower) as a bifunction $h(x;d)$, with x referring to a point in $\mathbb{R}^n$ and d referring to a direction from $\mathbb{R}^n$, then (7.55), (7.56), (7.57) and (7.58) are equivalent to (7.59), (7.60), (7.61) and (7.62), respectively.

Definition 7.18. Let $K \subseteq \mathbb{R}^n$ be a nonempty set and $\eta : K \times K \to \mathbb{R}^n$ be a map. A vector-valued bifunction $h = (h_1, \ldots, h_\ell) : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ is said to be:

- *C-pseudomonotone* w.r.t. η if for all $x, y \in K$,

$$h(x; \eta(y,x)) \not\leq_{C_0} \mathbf{0} \quad \text{implies} \quad h(y; \eta(x,y)) \not\geq_{C_0} \mathbf{0};$$

- *Weakly C-pseudomonotone* w.r.t. η if for all $x, y \in K$,

$$h(x; \eta(y,x)) \not\leq_{\overset{\circ}{C}} \mathbf{0} \quad \text{implies} \quad h(y; \eta(x,y)) \not\geq_{\overset{\circ}{C}} \mathbf{0};$$

- *C-properly subodd* if

$$h(x; d_1) + h(x; d_2) + \cdots + h(x; d_m) \geq_C \mathbf{0},$$

for every $d_i \in \mathbb{R}^n$ with $\sum_{i=1}^m d_i = \mathbf{0}$ and $x \in K$.

The definition of proper suboddness is considered in [47]. Of course, if $m = 2$, the definition of proper suboddness reduces to the definition of suboddness.

The definition of positive homogeneity of a vector-valued function $g : \mathbb{R}^n \to \mathbb{R}^\ell$ can be derived in a natural way, that is, g is *positively homogeneous* if for all $x \in \mathbb{R}^n$ and all $r > 0$, $g(rx) = rg(x)$.

We introduce the notion of η-upper sign continuity for the bifunction h, which extends the concept of upper sign continuity introduced in [31] and further studied in [2].

Definition 7.19. Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$. A vector-valued bifunction $h = (h_1, \ldots, h_\ell) : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ is said to be η*-upper sign continuous* (respectively, *weakly* η*-upper sign continuous*) if for all $x, y \in K$ and $t \in (0, 1)$,

$$h(x + t\eta(y,x); \eta(y,x)) \not\leq_{C_0} \mathbf{0} \quad \text{implies} \quad h(x; \eta(y,x)) \not\leq_{C_0} \mathbf{0}$$

$$\Big(\text{respectively,} \quad h(x + t\eta(y,x); \eta(y,x)) \not\leq_{\overset{\circ}{C}} \mathbf{0} \quad \text{implies} \quad h(x; \eta(y,x)) \not\leq_{\overset{\circ}{C}} \mathbf{0}\Big).$$

Remark 7.15. If η is skew and h is η-upper hemicontinuous in the first argument, then it is η-upper sign continuous. But the converse is not true in general.

The following result provides the relations between (SVVLIP) and (MVVLIP) in the setting of invex sets.

Proposition 7.7. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that the Condition C (a) holds. Let the vector-valued bifunction $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be C-properly subodd, C-pseudomonotone w.r.t. η and η-upper sign continuous such that for each fixed $x \in K$, $h(x; \cdot)$ is positively homogeneous. Then, $\bar{x} \in K$ is a solution of the (SVVLIP) if and only if it is a solution of the (MVVLIP).*

Proof. The C-pseudomonotonicity of h w.r.t. η implies that every solution of the (SVVLIP) is a solution of the (MVVLIP).

Conversely, let $\bar{x} \in K$ be a solution of the (MVVLIP). Then,

$$h(y;\eta(\bar{x},y)) \not\geq_{C_0} \mathbf{0}, \quad \text{for all } y \in K. \tag{7.63}$$

Since K is invex, we have $y_t = \bar{x} + t\eta(y,\bar{x}) \in K$ for all $t \in (0,1)$, and therefore, (7.63) becomes

$$h(y_t;\eta(\bar{x},y_t)) \not\geq_{C_0} \mathbf{0}.$$

By Condition C (a), $\eta(\bar{x},y_t) = -t\eta(y,\bar{x})$, and thus,

$$h(y_t;-t\eta(y,\bar{x})) \not\geq_{C_0} \mathbf{0}.$$

By positive homogeneity and C-proper suboddness of h, we have

$$h(y_t;\eta(y,\bar{x})) \not\leq_{C_0} \mathbf{0}.$$

Thus, the η-upper sign continuity of h yields $\bar{x} \in K$ is a solution of (SVVLIP). □

The following result gives the equivalence between (SVVLIP) and (MVVLIP) in the setting of convex sets.

Proposition 7.8. *Let $K \subseteq \mathbb{R}^n$ be a nonempty convex set and $\eta : K \times K \to \mathbb{R}^n$ be skew and affine in the first argument. Let the vector-valued bifunction $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be C-properly subodd, C-pseudomonotone w.r.t. η and η-upper sign continuous such that for each fixed $x \in K$, $h(x;\cdot)$ is positively homogeneous. Then, $\bar{x} \in K$ is a solution of the (SVVLIP) if and only if it is a solution of the (MVVLIP).*

Proof. The C-pseudomonotonicity of h w.r.t. η implies that every solution of the (SVVLIP) is a solution of the (MVVLIP).

Conversely, let $\bar{x} \in K$ be a solution of the (MVVLIP). Then,

$$h(y;\eta(\bar{x},y)) \not\geq_{C_0} \mathbf{0}, \quad \text{for all } y \in K. \tag{7.64}$$

Since K is convex, we have $y_t = \bar{x} + t(y - \bar{x}) \in K$ for all $t \in (0,1)$, and therefore, (7.64) becomes

$$h(y_t;\eta(\bar{x},y_t)) \not\geq_{C_0} \mathbf{0}.$$

Since η is affine in the first argument and skew, by Lemma 7.1, η is also affine in the second argument. Since $\eta(x,x) = \mathbf{0}$ by skewness of η, we obtain

$$h(y_t;\eta(\bar{x},y_t)) = h(y_t;t\eta(\bar{x},y) + (1-t)\eta(\bar{x},\bar{x})) = h(y_t;t\eta(\bar{x},y)) \not\geq_{C_0} \mathbf{0}.$$

By positive homogeneity of h in the second argument, we have

$$h(y_t;\eta(\bar{x},y)) \not\geq_{C_0} \mathbf{0}.$$

Since $\eta(y,\bar{x}) + \eta(\bar{x},y) = \mathbf{0}$ by skewness of η, the C-proper suboddness of h implies that

$$h(y_t;\eta(y,\bar{x})) \not\leq_{C_0} \mathbf{0}.$$

The η-upper sign continuity of h yields $\bar{x} \in K$ is a solution of (SVVLIP). □

Similarly, we can prove the following results which provide the equivalence between (WSVVLIP) and (WMVVLIP) in the setting of invex sets or convex sets.

Proposition 7.9. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that the Condition C (a) holds. Let the vector-valued bifunction $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be C-properly subodd, weakly C-pseudomonotone w.r.t. η and weakly η-upper sign continuous such that for each fixed $x \in K$, $h(x;\cdot)$ is positively homogeneous. Then, $\bar{x} \in K$ is a solution of the (WSVVLIP) if and only if it is a solution of the (WMVVLIP).*

Proposition 7.10. *Let $K \subseteq \mathbb{R}^n$ be a nonempty convex set and $\eta : K \times K \to \mathbb{R}^n$ be skew and affine in the first argument. Let the vector-valued bifunction $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be C-properly subodd, weakly C-pseudomonotone w.r.t. η and weakly η-upper sign continuous such that for each fixed $x \in K$, $h(x;\cdot)$ is positively homogeneous. Then, $\bar{x} \in K$ is a solution of the (WSVVLIP) if and only if it is a solution of the (WMVVLIP).*

Corollary 7.2. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that the Condition C (a) holds. Let the vector-valued bifunction $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be C-properly subodd, weakly C-pseudomonotone w.r.t. η and continuous in the first argument such that for each fixed $x \in K$, $h(x;\cdot)$ is positively homogeneous. Then, $\bar{x} \in K$ is a solution of the (WSVVLIP) if and only if it is a solution of the (WMVVLIP).*

Corollary 7.3. *Let $K \subseteq \mathbb{R}^n$ be a nonempty convex set and $\eta : K \times K \to \mathbb{R}^n$ be skew and affine in the first argument. Let the vector-valued bifunction $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be C-properly subodd and weakly C-pseudomonotone w.r.t. η and continuous in the first argument such that for each fixed $x \in K$, $h(x;\cdot)$ is positively homogeneous. Then, $\bar{x} \in K$ is a solution of the (WSVVLIP) if and only if it is a solution of the (WMVVLIP).*

We consider the *ε-perturbed Stampacchia vector variational-like inequality problem* (ε-PSVVLIP) of finding $\bar{x} \in K$ for which there exists $\bar{\varepsilon} \in (0,1)$ such that

$$h(\bar{x}+\varepsilon\eta(y,\bar{x});\eta(y,\bar{x})) \not\leq_{C_0} \mathbf{0}, \quad \text{for all } y \in K \text{ and all } \varepsilon \in (0,\bar{\varepsilon}). \tag{7.65}$$

Proposition 7.11. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that the Condition C (a) holds. Let the vector-valued bifunction $h = (h_1, \ldots, h_\ell) : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be C-properly subodd such that it is positively homogeneous in the second argument. Then, $\bar{x} \in K$ is a solution of the (ε-PSVVLIP) if it is a solution of the (MVVLIP). Furthermore, if h is C-pseudomonotone w.r.t. η and, η is skew and satisfies the Condition C (b), then every solution of (ε-PSVVLIP) is a solution of the (MVVLIP).*

Proof. Let $\overline{x}$ be a solution of the (MVVLIP). Then

$$h(y;\eta(\overline{x},y)) \not\geq_{C_0} \mathbf{0}, \quad \text{for all } y \in K. \tag{7.66}$$

Since K is invex, we have $x_\varepsilon := \overline{x} + \varepsilon\eta(z,\overline{x}) \in K$ for all $z \in K$ and all $\varepsilon \in [0,1]$. Taking $y = x_\varepsilon$ with $\overline{\varepsilon} = 1$ and $\varepsilon \in (0,\overline{\varepsilon})$ in (7.66), we have

$$h(x_\varepsilon;\eta(\overline{x},x_\varepsilon)) \not\geq_{C_0} \mathbf{0}.$$

As in the proof of Proposition 7.7, by Condition C (a), and positive homogeneity and C-proper suboddness of h, we obtain

$$h(x_\varepsilon;\eta(z,\overline{x})) \not\leq_{C_0} \mathbf{0}, \quad \text{for all } z \in K \text{ and all } \varepsilon \in (0,\overline{\varepsilon}).$$

Thus, $\overline{x} \in K$ is a solution of the (ε-PSVVLIP).

Conversely, suppose that $\overline{x} \in K$ is a solution of the (ε-PSVVLIP), but not a solution of the (MVVLIP). Then, there exists $z \in K$ such that

$$h(z;\eta(\overline{x},z)) \geq_{C_0} \mathbf{0}.$$

Since K is invex, we have $x_\varepsilon := \overline{x} + \varepsilon\eta(z,\overline{x}) \in K$ for all $\varepsilon \in [0,1]$. By skewness of η and Condition C (b), we have $\eta(x_\varepsilon,z) = (1-\varepsilon)\eta(\overline{x},z)$. The positive homogeneity of $h(x;\cdot)$ in the second argument implies that

$$h(z;\eta(\overline{x},z)) = \frac{1}{1-\varepsilon}h(z;\eta(x_\varepsilon,z)) \geq_{C_0} \mathbf{0}, \quad \text{for all } \varepsilon \in \ (0,1);$$

thus,

$$h(z;\eta(x_\varepsilon,z)) \geq_{C_0} \mathbf{0}, \quad \text{for all } \varepsilon \in (0,1).$$

By the C-pseudomonotonicity of h w.r.t. η, we obtain

$$h(x_\varepsilon;\eta(z,x_\varepsilon)) \leq_{C_0} \mathbf{0}, \quad \text{for all } \varepsilon \in (0,1).$$

Since $\eta(z,x_\varepsilon) = (1-\varepsilon)\eta(z,\overline{x})$ by Condition C (b), and since $h(x;\cdot)$ is positively homogeneous, we have

$$h(x_\varepsilon;\eta(z,\overline{x})) \leq_{C_0} \mathbf{0}, \quad \text{for all } \varepsilon \in (0,1),$$

which contradicts our supposition that $\overline{x}$ is a solution of the (ε-PSVVLIP). □

Remark 7.16. Proposition 7.9 extends and generalizes [3, Propostion 3.1], and therefore, [24, Proposition 2] and [77, Theorem 3.2] for nondifferentaible and noconvex functions.

We present some existence results for the solutions of (SVVLIP), (MVVLIP), (WSVVLIP) and (WMVVLIP) with our without boundedness assumption on the underlying set K.

Theorem 7.34. *Let $K \subseteq \mathbb{R}^n$ be a nonempty, convex and bounded set and $\eta : K \times K \to \mathbb{R}^n$ be skew and, affine and lower semicontinuous in the first argument. Let $h = (h_1, \ldots, h_\ell) : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be C-properly subodd, positively homogeneous in the second argument and C-pseudomonotone w.r.t. η such that for each $i \in \mathscr{I} = \{1, 2, \ldots, \ell\}$ and for every fixed $x \in K$, $h_i(x; \cdot)$ is lower semicontinuous. Then, there exists a solution $\bar{x} \in K$ of the (MVVLIP).*

Furthermore, if h is η-upper sign continuous, then $\bar{x} \in K$ is a solution of the (SVVLIP).

Proof. For all $y \in K$, we define two set-valued maps $P, Q : K \to 2^K$ by

$$P(y) = \left\{x \in K : h(x; \eta(y, x)) \not\leq_{C_0} \mathbf{0}\right\}$$

and

$$Q(y) = \left\{x \in K : h(y; \eta(x, y)) \not\geq_{C_0} \mathbf{0}\right\}.$$

Then, P is a KKM map. Indeed, let $\{y_1, y_2, \ldots, y_m\}$ be a finite subset of K and let $\hat{x} \in \text{co}\{y_1, y_2, \ldots, y_m\}$. Then $\hat{x} = \sum_{i=1}^m t_i y_i$ with $t_i \geq 0$ and $\sum_{i=1}^m t_i = 1$. If $\hat{x} \notin \bigcup_{i=1}^m P(y_i)$, then $h(\hat{x}; \eta(y_i, \hat{x})) \leq_{C_0} \mathbf{0}$ for all $i = 1, 2, \ldots, m$. Since C_0 is a convex cone and $t_i \geq 0$ with $\sum_{i=1}^m t_i = 1$, we have

$$\sum_{i=1}^m t_i h(\hat{x}; \eta(y_i, \hat{x})) \leq_{C_0} \mathbf{0}. \tag{7.67}$$

Since η is skew, we have $\eta(x, x) = \mathbf{0}$. By affineness of η in the first argument, we have

$$\sum_{i=1}^m t_i \eta(y_i, \hat{x}) = \eta\left(\sum_{i=1}^m t_i y_i, \hat{x}\right) = \eta(\hat{x}, \hat{x}) = \mathbf{0}.$$

By C-proper suboddness of h, we have

$$\sum_{i=1}^m h(\hat{x}; t_i \eta(y_i, \hat{x})) \geq_C \mathbf{0}.$$

The positive homogeneity of h yields

$$\sum_{i=1}^m t_i h(\hat{x}; \eta(y_i, \hat{x})) \geq_C \mathbf{0},$$

a contradiction of (7.67). Hence P is a KKM map. The C-pseudomonotonicity of h w.r.t. η implies that $P(y) \subseteq Q(y)$ for all $y \in K$; hence, Q is a KKM map.

We claim that $Q(y)$, for all $y \in K$, is a closed set in K. Indeed, let $\{x_n\}$ be a sequence in $Q(y)$ which converges to $x \in K$. Then,

$$h(y; \eta(x_n, y)) \not\geq_{C_0} \mathbf{0}.$$

Since η is lower semicontinuous in the first argument and each h_i is lower semicontinuous in the second argument, we have

$$h_i(y;\eta(x,y)) \leq \liminf_{n\to\infty} h_i(y;\eta(x_n,y)) \leq 0, \quad \text{for all } i \in \mathscr{I},$$

with strict inequality holds for some i. Therefore,

$$h(y;\eta(x,y)) \not\geq_{C_0} \mathbf{0},$$

and hence, $x \in Q(y)$. Thus, $Q(y)$ is closed in K.

Since K is bounded, it follows that $Q(y)$ is compact for all $y \in K$. By Fan-KKM Theorem 7.1,

$$\bigcap_{y\in K} Q(y) \neq \emptyset,$$

that is, there exists $\bar{x} \in K$ such that

$$h(y;\eta(\bar{x},y)) \not\geq_{C_0} \mathbf{0}, \quad \text{for all } y \in K.$$

Thus, $\bar{x} \in K$ is a solution of the (MVVLIP).

By Proposition 7.8, $\bar{x} \in K$ is a solution of the (SVVLIP). □

By using the similar argument as in the proof of above theorem and applying Proposition 7.10, we can easily derive the following existence result for solutions of (WSVVLIP) and (WMVVLIP).

Theorem 7.35. *Let $K \subseteq \mathbb{R}^n$ be a nonempty, convex and bounded set and $\eta : K \times K \to \mathbb{R}^n$ be skew and, affine and lower semicontinuous in the first argument. Let $h = (h_1,\ldots,h_\ell) : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be C-properly subodd, positively homogeneous in the second argument and weakly C-pseudomonotone w.r.t. η such that for all $i \in \mathscr{I}$ and for each fixed $x \in K$, $h_i(x;\cdot)$ is lower semicontinuous. Then, there exists a solution $\bar{x} \in K$ of the (WMVVLIP).*

Furthermore, if h is weakly η-upper sign continuous, then $\bar{x} \in K$ is a solution of the (WSVVLIP).

Remark 7.17. (a) Theorem 7.34 (respectively, Theorem 7.35) also holds if we replace the boundedness assumption on the set K by the following coercivity condition:
There exist a nonempty compact convex subset D of K and $\tilde{y} \in D$ such that for all $x \in K \setminus D$, $h(\tilde{y},\eta(x,\tilde{y})) \geq_{C_0} \mathbf{0}$ (respectively, $h(\tilde{y},\eta(x,\tilde{y})) \geq_{\overset{\circ}{C}} \mathbf{0}$).
Indeed, by above coercivity condition, $Q(y)$ is a closed subset of the compact set D, and hence, compact.
(b) Theorems 7.34 and 7.35 extend and generalize [3, Theorem 5.1] and [47, Theorem 2.2].

Definition 7.20. Let $K \subseteq \mathbb{R}^n$ be a nonempty convex set. A vector-valued function $g : K \to \mathbb{R}^\ell$ is said to be *C-convex* if for all $x,y \in K$ and all $t \in [0,1]$,

$$g(tx+(1-t)y) \leq_C tg(x)+(1-t)g(y).$$

When K is not necessarily bounded, then we also have the following existence results for solutions of (SVVLIP), (MVVLIP), (WSVVLIP) and (WMVVLIP).

Theorem 7.36. *Let $K \subseteq \mathbb{R}^n$ be a nonempty convex set and $\eta : K \times K \to \mathbb{R}$ be skew and lower semicontinuous in the first argument. Let $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be C-pseudomonotone w.r.t. η such that for each $i \in \mathscr{I}$ and for all $x \in K$, $h_i(x;\cdot)$ is lower semicontinuous, and the set $D = \{y \in K : h(x,\eta(y,x)) \leq_{C_0} \mathbf{0}\}$ is convex for all $x \in K$. Assume that there exist a nonempty compact convex subset B of K and a nonempty compact subset D of K such that for all $x \in K \setminus D$, there exists $\tilde{y} \in B$ such that $h(\tilde{y},\eta(x,\tilde{y})) \geq_{C_0} \mathbf{0}$. Then, there exists a solution $\overline{x} \in K$ of (MVVLIP).*

Furthermore, if h is η-upper sign continuous and for each fixed $x \in K$, $h(x;\cdot)$ is positively homogeneous, then $\overline{x} \in K$ is a solution of the (SVVLIP).

Proof. For each $x \in K$, define set-valued maps $P,Q : K \to 2^K$ by

$$P(x) = \{y \in K : h(y,\eta(x,y)) \geq_{C_0} \mathbf{0}\}$$

and

$$Q(x) = \{y \in K : h(x,\eta(y,x)) \leq_{C_0} \mathbf{0}\}.$$

By hypothesis, for each $x \in K$, $Q(x)$ is convex. By C-pseudomonotonicity of h, $P(x) \subseteq Q(x)$ for all $x \in K$. Thus, $\mathrm{co}P(x) \subseteq Q(x)$ for all $x \in K$.

For each $y \in K$, the complement of $P^{-1}(y)$ in K is

$$[P^{-1}(y)]^c = \{x \in K : h(y,\eta(x,y)) \not\geq_{C_0} \mathbf{0}\}.$$

As we have seen in Theorem 7.34 that $[P^{-1}(y)]^c$ is closed in K, and hence, $P^{-1}(y)$ is open in K.

Assume that for all $x \in K$, $P(x)$ is nonempty. Then all the conditions of Theorem 7.2 are satisfied, and therefore, there exists $\hat{x} \in K$ such that $\hat{x} \in Q(\hat{x})$. It follows that

$$\mathbf{0} = h(\hat{x},\eta(\hat{x},\hat{x})) \leq_{C_0} \mathbf{0},$$

a contradiction because $\eta(\hat{x},\hat{x}) = \mathbf{0}$. Hence, our assumption $P(x)$ is nonempty for all $x \in K$ is false. Therefore, there exists $\overline{x} \in K$ such that $P(\overline{x}) = \emptyset$. This implies that for all $y \in K$,

$$h(y,\eta(\overline{x},y)) \not\geq_{C_0} \mathbf{0}.$$

Thus, $\overline{x} \in K$ is a solution of the (MVVLIP).

By Proposition 7.8, $\overline{x} \in K$ is a solution of the (SVVLIP). □

Theorem 7.37. *Let $K \subseteq \mathbb{R}^n$ be a nonempty convex set and $\eta : K \times K \to \mathbb{R}$ be skew and lower semicontinuous in the first argument. Let $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be weakly*

C-pseudomonotone w.r.t. η *such that for each* $i \in \mathscr{I}$ *and for all* $x \in K$, $h_i(x;\cdot)$ *is lower semicontinuous, and the set* $\tilde{D} = \{y \in K : h(x,\eta(y,x)) \leq_{\overset{\circ}{C}} \mathbf{0}\}$ *is convex for all* $x \in K$*. Assume that there exist a nonempty compact convex subset* B *of* K *and a nonempty compact subset* D *of* K *such that for all* $x \in K \setminus D$*, there exists* $\tilde{y} \in B$ *such that* $h(\tilde{y},\eta(x,\tilde{y})) \geq_{\overset{\circ}{C}} \mathbf{0}$*. Then, there exists a solution* $\overline{x} \in K$ *of (MVVLIP).*

Furthermore, if h *is* η*-upper sign continuous and for each fixed* $x \in K$, $h(x;\cdot)$ *is positively homogeneous, then* $\overline{x} \in K$ *is a solution of the (SVVLIP).*

Remark 7.18. For all $x \in K$, the set $D = \{y \in K : h(x,\eta(y,x)) \leq_{C_\mathbf{0}} \mathbf{0}\}$ is convex, if η is affine in the first argument and h is C-convex in the second argument.

Indeed, let $y_1, y_2 \in D$. Since $C_\mathbf{0}$ is a convex cone, for all $t \in (0,1)$, we have

$$th(x;\eta(y_1,x)) \leq_{C_\mathbf{0}} \mathbf{0} \quad \text{and} \quad (1-t)h(x;\eta(y_2,x)) \leq_{C_\mathbf{0}} \mathbf{0}.$$

By adding these relations, we get

$$th(x;\eta(y_1,x)) + (1-t)h(x;\eta(y_2,x)) \leq_{C_\mathbf{0}} \mathbf{0}. \tag{7.68}$$

Since h is C-convex in the second argument, we have

$$h(x;t\eta(y_1,x)+(1-t)\eta(y_2,x)) \leq_C th(x;\eta(y_1,x)) + (1-t)h(x;\eta(y_2,x)). \tag{7.69}$$

Combining relations (7.68) and (7.69), we obtain

$$h(x;t\eta(y_1,x)+(1-t)\eta(y_2,x)) \leq_{C_\mathbf{0}} \mathbf{0}.$$

Since η is affine in the first argument, we get

$$h(x;\eta(ty_1+(1-t)y_2,x)) \leq_{C_\mathbf{0}} \mathbf{0},$$

and hence, $ty_1 + (1-t)y_2 \in D$. Thus, for all $x \in K$, D is a convex set.

Similarly, we can prove that $\tilde{D}$ is convex for all $x \in K$.

7.7 Nonsmooth Vector Optimization

Let $f = (f_1,\ldots,f_\ell) : \mathbb{R}^n \to \mathbb{R}^\ell$ be a vector-valued function. The *vector optimization problem* (VOP) is defined as follows:

$$\min f(x) \quad \text{subject to } x \in K, \tag{7.70}$$

where $f(x) = (f_1(x),\ldots,f_\ell(x))$.

A point $\overline{x} \in K$ is said to be an *efficient solution* (respectively, *weakly efficient solution*) of (VOP) if and only if

$$f(\overline{x}) \not\geq_{C_\mathbf{0}} f(y), \quad \text{for all } y \in K, \tag{7.71}$$

$$\left(\text{respectively, } f(\overline{x}) \not\geq_{\overset{\circ}{C}} f(y), \quad \text{for all } y \in K\right).$$

It is clear that every efficient solution is a weakly efficient solution.

Definition 7.21. Let $h = (h_1, \dots, h_\ell) : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be a vector-valued bifunction and $\eta : K \times K \to \mathbb{R}^n$ be a map. A vector-valued function $f = (f_1, \dots, f_\ell) : K \to \mathbb{R}^\ell$ is said to be *h-invex* (respectively, *strictly h-invex, h-pseudoinvex* and *strictly h-pseudoinvex*) w.r.t. η if for each $i \in \mathscr{I}$, f_i is h_i-invex (respectively, strictly h_i-invex, h_i-pseudoinvex and strictly h_i-pseudoinvex) w.r.t. η.

Remark 7.19. If $h(x; \eta(y,x)) = D^+ f(x; \eta(y,x))$, then h-invexity (respectively, strictly h-invexity, h-pseudoinvexity and strictly h-pseudoinvexity) w.r.t. η is called D^+-invexity (respectively, strictly D^+-invexity, D^+-pseudoinvexity and strictly D^+-pseudoinvexity) w.r.t. η.

The following result provides the relationship among the efficient solution and weakly efficient solution of the (VOP) and the solutions of (NSVVLIP), (NWSVVLIP) and (WSVVLIP).

Theorem 7.38. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ and $f : K \to \mathbb{R}^\ell$ be a vector-valued function. Then, every (respectively, weakly) efficient solution of the (VOP) is a solution of (NSVVLIP) (respectively, (NWSVVLIP)).*

Proof. Let $\bar{x}$ be an efficient solution of the (VOP). Then,

$$f(\bar{x}) \not\geq_{C_0} f(y), \quad \text{for all } y \in K.$$

Since K is invex, we have $\bar{x} + t\eta(y, \bar{x}) \in K$ for all $t \in [0,1]$; thus,

$$\frac{f(\bar{x} + t\eta(y,\bar{x})) - f(\bar{x})}{t} \not\leq_{C_0} \mathbf{0}, \quad \text{for all } t \in (0,1).$$

Taking the lim sup as $t \to 0^+$, we obtain

$$f^D(\bar{x}; \eta(y,\bar{x})) = \limsup_{t \to 0^+} \frac{f(\bar{x} + t\eta(y,\bar{x})) - f(\bar{x})}{t} \not\leq_{C_0} \mathbf{0}, \quad \text{for all } y \in K.$$

Hence, $\bar{x}$ is a solution of the (NSVVLIP).

Similarly, we can prove that every weakly efficient solution of the (VOP) is a solution of (NWSVVLIP). □

Theorem 7.39. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ and $f : K \to \mathbb{R}^\ell$ be a vector-valued function. If f is h-pseudoinvex w.r.t. η, then every solution of (WSVVLIP) is a weakly efficient solution of the (VOP).*

Proof. Assume that $\bar{x} \in K$ is a solution of (WSVVLIP), but not a weakly efficient solution of the (VOP). Then, there exists $y \in K$ such that

$$f(\bar{x}) \geq_{\overset{\circ}{C}} f(y). \tag{7.72}$$

That is, $f_i(\bar{x}) > f_i(y)$ for all $i = 1, 2, \ldots, \ell$. By h_i pseudoinvexity of f_i, we obtain

$$h_i(\bar{x}; \eta(y, \bar{x})) < 0, \quad \text{for all } i = 1, 2, \ldots, \ell,$$

that is,

$$h(\bar{x}; \eta(y, \bar{x})) \leq_{\overset{\circ}{C}} \mathbf{0}.$$

Thus, $\bar{x}$ is not a solution of (WSVVLIP), a contradiction of our assumption. Hence, $\bar{x} \in K$ is a weakly efficient solution of the (VOP). □

Corollary 7.4. *Let $K \subseteq \mathbb{R}^n$ be a nonempty set and $f : K \to \mathbb{R}^\ell$ be a h-invex vector-valued function. Then, every solution of the (WSVVLIP) is a weakly efficient solution of the (VOP).*

Proof. Since every h-invex function is h-pseudoinvex, we obtain the desired result from Theorem 7.39. □

Theorem 7.40. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that the Condition C holds. Let $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be positively homogeneous and subodd in the second argument such that for each fixed $x \in K$, $h(x; \cdot) \leq f^D(x; \cdot)$. If $f : K \to \mathbb{R}^\ell$ is strictly h-pseudoinvex w.r.t. η and upper semicontinuous then, every solution of the (WSVVLIP) is an efficient solution of the (VOP).*

Proof. Assume that $\bar{x} \in K$ is a solution of (WSVVLIP), but not an efficient solution of the (VOP). Then, there exists $y \in K$ such that

$$f(\bar{x}) \geq_{C_0} f(y). \tag{7.73}$$

Clearly, strictly h-pseudoinvex w.r.t. η implies h-pseudoinvexity w.r.t. the same η. From Theorems 7.23 and 7.24, each f_i is semistrictly pre-quasiinvex and pre-quasiinvex. Then by Theorem 7.21, we have

$$h(\bar{x}; \eta(y, \bar{x})) \leq_{C_0} \mathbf{0}. \tag{7.74}$$

If there exists some $i \in \mathscr{I}$ such that $h_i(\bar{x}; \eta(y, \bar{x})) = 0$, then by the strict h-pseudoinvexity of f, we have $f_i(y) > f_i(\bar{x})$, which contradicts (7.73). Thus, (7.74) implies that $h(\bar{x}; \eta(y, \bar{x})) \leq_{\overset{\circ}{C}} \mathbf{0}$ contradicting our assumption that $\bar{x}$ is a solution of (SWVVLIP). □

Theorem 7.41. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$, $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be a vector-valued function and $f : K \to \mathbb{R}^\ell$ be strictly h-pseudoinvex w.r.t. η. Then, every weakly efficient solution of the (VOP) is a solution of the (MVVLIP).*

Proof. Assume that $\bar{x} \in K$ is a weakly efficient solution of the (VOP), but not a solution on the (MVVLIP). Then, there exists $y \in K$ such that

$$h(y; \eta(\bar{x}, y)) = \big(h_1(y; \eta(\bar{x}, y)), \ldots, h_\ell(y; \eta(\bar{x}, y))\big) \geq_{C_0} \mathbf{0}.$$

The strict h-pseudoinvexity of f w.r.t. η implies

$$f(\bar{x}) - f(y) \geq_{\overset{\circ}{C}} \mathbf{0},$$

a contradiction of our assumption that $\bar{x}$ is a weakly efficient solution of (VOP). □

Theorem 7.42. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that the Condition C holds. Let $h : K \times \mathbb{R}^n \to \mathbb{R}^\ell$ be positively homogeneous and subodd in the second argument such that for each $i \in \mathscr{I}$ and all $x \in K$, $h_i(x;\cdot) \leq f_i^D(x;\cdot)$, and let $f : K \to \mathbb{R}^\ell$ be h-pseudoinvex w.r.t. η. Then, every solution of the (SVVLIP) is an efficient solution of the (VOP).*

Proof. Suppose that $\bar{x} \in K$ is a solution of the (SVVLIP), but not an efficient solution of the (VOP). Then, there exists $y \in K$ such that $f(\bar{x}) \geq_{C_0} f(y)$. Since each f_i is h_i-pseudoinvex w.r.t. η, by Theorem 7.23, each f_i is pre-quasiinvex w.r.t. η. By Theorem 7.21, each f_i is h_i-quasiinvex w.r.t. η. Thus, by using the h-pseudoinvexity and h-quasiinvexity of f w.r.t. η, we get $h(\bar{x};\eta(y,\bar{x})) \leq_{C_0} \mathbf{0}$, which contradicts the fact that $\bar{x}$ is a solution of (SVVLIP). Hence $\bar{x}$ is an efficient solution of the (VOP). □

Theorem 7.43. *Let $K \subseteq \mathbb{R}^n$ be a nonempty set and $f : K \to \mathbb{R}^\ell$ be a vector-valued function such that $-f$ is h-invex w.r.t. η, that is, $f(y) - f(x) \leq_{C_0} h(x;\eta(y,x))$ for all $x, y \in K$. Then, every efficient solution of the (VOP) is a solution of the (SVVLIP).*

Proof. Assume that $\bar{x}$ is an efficient of the (VOP), but not a solution of the (SVVLIP). Then, there exists $y \in K$ such that

$$h(\bar{x};\eta(y,\bar{x})) \leq_{C_0} \mathbf{0}. \tag{7.75}$$

Since $-f$ is h-invex w.r.t. η, we have

$$f(y) - f(\bar{x}) \leq_{C_0} h(\bar{x};\eta(y,\bar{x})). \tag{7.76}$$

Combining (7.75) and (7.76), we obtain

$$f(\bar{x}) \geq_{C_0} f(y),$$

a contradiction of our assumption that $\bar{x}$ is an efficient solution of the (VOP). Hence, $\bar{x}$ is a solution of the (SVVLIP). □

Similarly, we can prove the following result.

Theorem 7.44. *Let $K \subseteq \mathbb{R}^n$ be a nonempty set and $f : K \to \mathbb{R}^\ell$ be a vector-valued function such that $-f$ is strictly h-invex w.r.t. η, that is, $f(y) - f(x) \leq_{\overset{\circ}{C}} h(x;\eta(y,x))$ for all $x, y \in K$. Then, every weakly efficient solution of the (VOP) is a solution of the (SVVLIP).*

Theorem 7.45. *Let $K \subseteq \mathbb{R}^n$ be a nonempty invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$. If $f : K \to \mathbb{R}^\ell$ is a strictly D^+-invex function w.r.t. η, then every weakly efficient solution of the (VOP) is an efficient of the (VOP).*

Proof. Assume that $\bar{x}$ is a weakly efficient solution of the (VOP), but not an efficient solution of the (VOP). Then, there exists $y \in K$ such that

$$f(\bar{x}) \geq_{C_0} f(y). \tag{7.77}$$

Since f is strictly D^+-invex w.r.t. η, we have

$$f(y) - f(\bar{x}) \geq_{\overset{\circ}{C}} f^D(\bar{x};\eta(y,\bar{x})). \tag{7.78}$$

Combining relations (7.77) and (7.78), we obtain

$$f^D(\bar{x};\eta(y,\bar{x})) \leq_{\overset{\circ}{C}} \mathbf{0}.$$

Thus, $\bar{x}$ is not a solution of the (NWSVVLIP). By using Theorem 7.38 (a), we see that $\bar{x}$ is not a weakly efficient solution of the (VOP), contradicting our assumption. □

Theorem 7.46. *Let $K \subseteq \mathbb{R}^n$ be an invex set w.r.t. $\eta : K \times K \to \mathbb{R}^n$ such that the Condition C holds and η is skew. For each $i \in \mathscr{I} = \{1,\ldots,\ell\}$, let $f_i : K \to \mathbb{R}$ be upper semicontinuous and, for all $x \in K$, let $h_i(x;\cdot)$ be positively homogeneous and subodd such that $h_i(x;\cdot) \leq f_i^D(x;\cdot)$. For each $i \in \mathscr{I}$, let f_i be D^+-pseudoinvex w.r.t. η. Then, $\bar{x} \in K$ is a solution of the (MVVLIP) if and only if it is an efficient solution of the (VOP).*

Proof. Let $\bar{x} \in K$ be a solution of the (MVVLIP), but not an efficient solution of the (VOP). Then, there exists $z \in K$ such that

$$f(\bar{x}) \geq_{C_0} f(z). \tag{7.79}$$

Set $z(t) := \bar{x} + t\eta(z,\bar{x})$ for all $t \in [0,1]$. Since K is invex, $z(t) \in K$ for all $t \in [0,1]$. Also since each f_i is D^+-pseudoinvex w.r.t. η, it follows from Theorems 7.23 and 7.24 that f_i is pre-quasiinvex and semistrictly pre-quasiinvex w.r.t. the same η. By using pre-quasiinvexity, semistrictly pre-quasiinvexity and the relation (7.79), we get

$$f_i(\bar{x}) \geq_{C_0} f_i(z(t)), \quad \text{for all } t \in (0,1).$$

That is,

$$f_i(\bar{x}) \geq f_i(z(t)), \quad \text{for all } t \in (0,1) \text{ and all } i = 1,\ldots,\ell, \tag{7.80}$$

with strict inequality holds in inequality (7.80) for some k such that $1 \leq k \leq \ell$. For each $i \in \mathscr{I}$, let $g_i : [0,1] \to \mathbb{R}$ be a function defined by

$$g_i(t) = f_i(\bar{x} + t\eta(z,\bar{x})).$$

Then, by Diewert Mean-Value Theorem 7.9, there exists $t_i \in [0,t)$, for all $t \in (0,1)$, such that

$$\frac{1}{t}\left(f_i(z(t)) - f_i(\bar{x})\right) \geq f_i^D\left(z(t_i);\eta(z,\bar{x})\right), \quad \text{for all } i = 1,\ldots,\ell. \tag{7.81}$$

Combining inequalities (7.80) and (7.81), we obtain

$$f_i^D\left(z(t_i);\eta(z,\bar{x})\right) \leq 0, \quad \text{for all } i = 1,\ldots,\ell,$$

with strict inequality holds for some k such that $1 \leq k \leq \ell$. Since, for each fixed $x \in K$, $h_i(x;\cdot) \leq f_i^D(x;\cdot)$, we have

$$h_i\left(z(t_i);\eta(z,\bar{x})\right) \leq 0, \quad \text{for all } i = 1,\ldots,\ell, \tag{7.82}$$

with strict inequality holds for some k such that $1 \leq k \leq \ell$. From Condition C and Remark 7.3, we have $\eta(z(t_i),\bar{x}) = \eta(\bar{x}+t_i\eta(z,\bar{x}),\bar{x}) = t_i\eta(z,\bar{x})$ for all $i = 1,\ldots,\ell$. By positive homogeneity of each h_i in the second argument, we have

$$h_i(z(t_i);\eta(z(t_i),\bar{x})) \leq 0, \quad \text{for all } i = 1,\ldots,\ell, \tag{7.83}$$

with strict inequality holds for some k such that $1 \leq k \leq \ell$.

Suppose that $t_1,t_2,\ldots,t_\ell$ are all equal. Then, by inequality (7.83) and suboddness of each h_i, we obtain

$$\big(h_1\left(z(t_1);\eta(\bar{x},z(t_1))\right),\ldots,h_\ell\left(z(t_\ell);\eta(\bar{x},z(t_\ell))\right)\big) \geq_{C_0} \mathbf{0},$$

which contradicts our assumption that $\bar{x}$ is a solution of the (MVVLIP).

Consider the case when $t_1,t_2,\ldots,t_\ell$ are not equal. By Condition C, we have

$$\eta\left(z(t_1),z(t_2)\right) = \frac{t_1-t_2}{t_1}\eta\left(z(t_1),\bar{x}\right) = \frac{t_1-t_2}{t_2}\eta\left(z(t_2),\bar{x}\right) \tag{7.84}$$

and

$$\eta\left(z(t_2),z(t_1)\right) = \frac{t_2-t_1}{t_1}\eta\left(z(t_1),\bar{x}\right) = \frac{t_2-t_1}{t_2}\eta\left(z(t_2),\bar{x}\right). \tag{7.85}$$

Case 7.1. If $t_1 > t_2$, then from inequality (7.83), relation (7.84) and positive homogeneity of each h_i, we have

$$0 \geq h_1\left(z(t_1);\frac{t_1-t_2}{t_1}\eta(z(t_1),\bar{x})\right) = h_1\big(z(t_1);\eta(z(t_1),z(t_2))\big).$$

By suboddness of each h_i in the second argument, we obtain

$$h_1\big(z(t_1);\eta(z(t_2),z(t_1))\big) \geq 0,$$

with strict inequality holds for $k = 1$.

Since each f_i is D^+-pseudoinvex w.r.t. η and $h_i(x;\cdot) \leq f_i^D(x;\cdot)$, by Lemma 7.4, f_i is h_i-pseudoinvex w.r.t. η; further, by Theorem 7.27, h_i is invariant pseudomonotone w.r.t. η. Therefore, we have

$$h_1\big(z(t_2);\eta(z(t_1),z(t_2))\big) \leq 0,$$

with strict inequality holds for $k = 1$. The relation (7.84) and positive homogeneity of $h_i(x;\cdot)$ imply that

$$h_1\big(z(t_2);\eta(z(t_2),\bar{x})\big) \leq 0,$$

with strict inequality holds for $k = 1$.

Case 7.2. If $t_1 < t_2$, then from inequality (7.83), relation (7.85) and positive homogeneity of each h_i, we have

$$0 \geq h_2\left(z(t_2);\frac{t_2-t_1}{t_2}\eta(z(t_2),\bar{x})\right) = h_2\big(z(t_2);\eta(z(t_2),z(t_1))\big).$$

By suboddness of each h_i in the second argument, we obtain

$$h_2\big(z(t_2);\eta(z(t_1),z(t_2))\big) \geq 0,$$

with strict inequality holds for $k = 2$. As above, each h_i is pseudomonotone; therefore,

$$h_2\big(z(t_1);\eta(z(t_2),z(t_1))\big) \leq 0,$$

with strict inequality holds for $k = 2$. Again as above, by using the relation (7.85) and positive homogeneity of $h_i(x;\cdot)$, we get

$$h_2\big(z(t_1);\eta(z(t_1),\bar{x})\big) \leq 0,$$

with strict inequality holds for $k = 2$.

For the case $t_1 \neq t_2$, let $\hat{t} = \min\{t_1,t_2\}$. Then, we have

$$h_i\big(z(\hat{t});\eta(z(\hat{t}),\bar{x})\big) \leq 0, \quad \text{for all } i = 1,2.$$

By continuing this process, we can find $t^* \in (0,1)$ such that

$$h_i\big(z(t^*);\eta(z(t^*),\bar{x})\big) \leq 0, \quad \text{for all } i = 1,2,\ldots,\ell,$$

with strict inequality holds for some k such that $1 \leq k \leq \ell$. This contradicts our supposition that $\bar{x}$ is a solution of (MVVLIP).

Conversely, suppose that $\bar{x} \in K$ is an efficient of the (VOP), but not a solution of the (MVVLIP). Then, there exists $z \in K$ such that

$$h(z;\eta(\bar{x},z)) \geq_{C_0} \mathbf{0},$$

that is,

$$h_i(z;\eta(\bar{x},z)) \geq 0, \quad \text{for all } i \in \mathscr{I},$$

with strict inequality holds for some i. Since $h_i(z;\cdot) \leq f_i^D(z;\cdot)$ for all $i \in \mathscr{I}$,

$$f_i^D(z;\eta(\bar{x},z)) \geq 0, \quad \text{for all } i \in \mathscr{I},$$

with strict inequality holds for some i. Since each f_i is D^+-pseudoinvex w.r.t. η,

$$f_i(\bar{x}) \geq f_i(z), \quad \text{for all } i \in \mathscr{I}.$$

Let $j \in \mathscr{I}$ be such that $f_j^D(z;\eta(\bar{x},z)) > 0$. It follows from Theorems 7.21 and 7.23 that f_j is h-quasiinvex w.r.t. η, and hence, $f_j(\bar{x}) > f_j(z)$. Thus, $f(z) \leq_{C_0} f(\bar{x})$; hence, $\bar{x}$ is not an efficient solution of the (VOP). This is a contradiction.

Remark 7.20. Theorem 7.40 extends and generalizes [3, Theorem 3.1], and therefore, also generalizes the necessary part of [24, Proposition 1] and [77, Theorem 3.1] for nondifferentiable and nonconvex functions.

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Chapter 8
Optimality Conditions for Approximate Solutions of Convex Semi-Infinite Vector Optimization Problems

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8.1 Introduction

Efficient solutions for vector optimization problems are well recognized to be important and have attracted many mathematicians since the solutions have crucial meanings when decision makers determine one solution with conflicting objective functions. On the other hands, from computational point of view, algorithms which have been used in the literature to solve nonlinear optimization problems often give rise to approximate solutions (ε-approximate solution) for the problems. This explains why approximate solutions to optimization problems in general and especially to vector optimizations have attracted much attention from many authors (see, e.g., [13, 14, 17–20, 23, 26, 27, 29, 30] and the references therein). These works mainly devoted to the study of the existence of ε-approximate solutions, ε-optimality conditions, and ε-duality results for several kinds of optimization problems. Recently, efforts were devoted to the study of properties of ε-approximate solutions (for scalar problems), ε-Pareto optimality conditions and ε-duality theorems for vector optimization problems with finite constraints (see [11, 12, 21, 22, 24, 27, 28, 31]) while approximate solutions for scalar convex and nonconvex problems with infinitely many constraints were examined recently in [26] (such class of problems was studied also in [2, 3]).

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In this chapter, we consider a convex semi-infinite vector optimization problem which consists of a finite number of convex objective functions with infinitely many convex constraints and an abstract constraint set in locally convex Hausdorff spaces setting. Such a problem is called convex semi-infinite vector problem (CSIVP).

For $\varepsilon = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p) \in \mathbb{R}^p_+$, we introduce the notions of ε-efficient solutions and weakly ε-efficient solutions for (CSIVP) which collapse to efficient solution and weakly efficient solutions when $\varepsilon = \mathbf{0} \in \mathbb{R}^p$.

We concern the efficient, weakly efficient solutions, and also ε-efficient solutions and weakly ε-efficient solutions for (CSIVP). One of the methods to deal with vector optimization problems is that of scalarzing them (see, e.g., [9, 25, 31]). However, by this procedure, the resulting scalarized problems may not satisfy the Slater or any interior-type constraint qualification conditions. The main aim of the chapter is to introduce an alternative constraint qualification conditions (called closedness conditions) for scalarized problem associated to (CSIVP) under which, optimality conditions for ε-efficient and weakly ε-efficient solutions are established. Optimality conditions for efficient and weakly efficient solutions of (CSIVP) are then obtained as special cases. The results were obtained by using a version of Farkas lemma for systems of infinitely many convex constraints and under a regularity condition (so-called closedness condition) expressed in terms of epigraphs of conjugate functions.

The rest of the chapter is organized as follows: In Sect. 8.2, we give a preliminary results which will be useful in the sequel and obtain, as a consequence of the early works, a generalized Farkas lemma associated to the constraint systems of the problem (CSIVP) in consideration. In Sect. 8.3, we introduce the definitions of ε-efficient solutions and weakly ε-solutions for (CSIVP) and then the closedness conditions which will be used to establish optimality conditions in the next sections. The optimality conditions for ε-efficient solutions and efficient solutions for (CSIVP) were established in Sect. 8.4. In the last section, we establish optimality conditions for weakly ε-efficient solutions to (CSIVP). Numerical examples are given to illustrate the meaning of the results.

8.2 Preliminaries

Throughout this chapter, X denotes a locally convex Hausdorff topological vector space with its topological dual, X^*, endowed with weak*-topology, and $C \subset X$ is a closed convex subset.

Let T be an arbitrary (possibly infinite) index set and $\mathbb{R}^T$ the product space with product topology. Let $\mathbb{R}^{(T)}$ denote the generalized finite sequence space which consists of all sequences $\lambda = (\lambda_t)_{t\in T}$ such that $\lambda_t \in \mathbb{R}$ for each $t \in T$ and $\operatorname{supp}\lambda := \{t \in T \mid \lambda_t \neq 0\}$ is a finite subset of T. For $(\lambda_t)_{t\in T} \in \mathbb{R}^{(T)}$ and $(x_t)_{t\in T} \in \mathbb{R}^T$, we understand that

$$\sum_{t\in T} \lambda_t x_t = \sum_{t\in \operatorname{supp}\lambda} \lambda_t x_t, \quad \forall x \in \mathbb{R}^T, \quad \forall \lambda \in \mathbb{R}^{(T)}.$$

Denote

$$\mathbb{R}_+^{(T)} := \Big\{\lambda = (\lambda_t) \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0, t \in T, \lambda_t = 0 \text{ for all but a finite number of } t \in T\Big\}.$$

It is clear that $\mathbb{R}_+^{(T)}$ is a convex cone in $\mathbb{R}^{(T)}$, see, for example, [10, p. 48].

We now recall some notations and basic results which will be used in the sequel. Since we always deal with weak*-topology in dual spaces, for a subset $D \subset X^*$, the closure of D in the weak*-topology, will be denoted by cl$D$; the convex cone generated by $D \cup \{0\}$ by cone D. Let I be an arbitrary index, $\{X_i, i \in I\}$ be a family of subsets of X, let $\mathfrak{I}$ be the collection of all nonempty finite subsets of I, then

$$\begin{aligned}\text{cone}(\textstyle\bigcup_{i\in I} X_i) &= \textstyle\bigcup_{J\in\mathfrak{I}} \text{cone}(\bigcup_{j\in J} X_j)\\ &= \textstyle\bigcup_{J\in\mathfrak{I}} (\sum_{j\in J} \text{cone} X_j).\end{aligned} \tag{8.1}$$

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. The *conjugate function* of f, f^*, is defined as

$$f^* : X^* \to \mathbb{R} \cup \{+\infty\}$$

$$f^*(v) := \sup\{v(x) - f(x) \mid x \in \text{dom} f\},$$

where dom$f := \{x \in X \mid f(x) < +\infty\}$ is the *effective domain* of f. The *epigraph* of f is defined by

$$\text{epi} f := \{(x, r) \in X \times \mathbb{R} \mid x \in \text{dom} f, f(x) \leq r\}.$$

The set (possibly empty)

$$\partial f(a) := \{v \in X^* \mid f(x) - f(a) \geq v(x-a), \forall x \in \text{dom} f\}$$

is the *subdifferential* of the convex function f at $a \in \text{dom} f$.

For a nonempty closed convex set $C \in X$ and a point $a \in C$, the *normal cone* to C at a is defined as

$$N_C(\bar{x}) := \partial \delta_C(a) = \{x^* \in X^* \mid x^*(x - \bar{x}) \leq 0, \forall x \in C\}.$$

For $\varepsilon \geq 0$, the ε-*subdifferential* of f at $a \in \text{dom} f$ is defined as the set (possibly empty)

$$\partial_\varepsilon f(a) = \{v \in X^* \mid f(x) - f(a) \geq v(x-a) - \varepsilon, \forall x \in \text{dom} f\}.$$

If $\varepsilon > 0$, then $\partial_\varepsilon f(a)$ is nonempty and is a weak*-closed subset of X^*. When $\varepsilon = 0$, $\partial_0 f(a)$ collapses to $\partial f(a)$ [32]. If $a \in \text{dom} f$, then epif^* has a representation as follows (see [15]):

$$\text{epi} f^* = \bigcup_{\varepsilon \geq 0} \{(v, v(a) + \varepsilon - f(a)) \mid v \in \partial_\varepsilon f(a)\}. \tag{8.2}$$

Note that, for $\varepsilon_1, \varepsilon_2 \geq 0$ and $z \in \text{dom} f \cap \text{dom} g$,

$$\partial_{\varepsilon_1} f(z) + \partial_{\varepsilon_2} g(z) \subset \partial_{\varepsilon_1+\varepsilon_2}(f+g)(z),$$

and for $\mu > 0, \varepsilon \geq 0, z \in \text{dom} f$, we have (see [45, p. 83])

$$\mu \partial_\varepsilon f(z) = \partial_{\mu\varepsilon}(\mu f)(z).$$

If g is sublinear (i.e., convex and positively homogeneous of degree one), then $\partial_\varepsilon g(0) = \partial g(0)$ for all $\varepsilon \geq 0$. If $\widetilde{g}(x) = g(x) - k,\ x \in X,\ k \in R$, then $\text{epi}\tilde{g}^* = \text{epi} g^* + (0,k)$. It is worth noting that if g is sublinear, then $\text{epi} g^* = \partial g(0) \times R_+$. Moreover, if g is sublinear and if $\widetilde{g}(x) = g(x) - k,\ x \in X,\ k \in \mathbb{R}$, then

$$\text{epi}\widetilde{g}^* = \partial g(0) \times [k, \infty).$$

Now, let $g, h : X \to \mathbb{R} \cup \{\infty\}$ be proper l.s.c. convex functions. Then

$$\text{epi }(g+h)^* = \text{cl}(\text{epi} g^* + \text{epi} h^*).$$

If at least one of them is continuous at some point of $\text{dom} g \cap \text{dom} h$ then the closure in the right hand side of the previous equality can be dropped. More concretely, one has (see, e.g., [1, 3]):

Lemma 8.1. *Let $g, h : X \to \mathbb{R} \cup \{\infty\}$ be proper lower semicontinuous convex functions. The following statements are equivalent:*

(a) $\text{epi} g^* + \text{epi} h^*$ *is weak***- closed.*
(b) $\text{epi}(g+h)^* = \text{epi} g^* + \text{epi} h^*$.
(c) $(g+h)^*(x^*) = \min_{u \in X^*}\{g^*(u) + h^*(x^* - u)\},\ \ \forall x^* \in X^*$.

Now we give a version of generalized Farkas lemma for convex infinite system which can comes easily from the results in recently works [2, 3].

Lemma 8.2. *Let $h : X \to \mathbb{R}$ be a continuous convex function and $g_t : X \to \mathbb{R} \cup \{+\infty\}$, $t \in T$, proper lower semi-continuous convex functions. Let further, $\alpha \in \mathbb{R}$ and C be a closed convex subset of X. Assume that $F := \{x \in C \mid g_t(x) \leq 0$ for all $t \in T\} \neq \emptyset$. Then the following statements are equivalent:*

(a) $x \in F \quad \Longrightarrow \quad h(x) \geq \alpha$.
(b) $(0, -\alpha) \in \text{epi} h^* + \text{cl}\left[\text{cone}\left(\bigcup_{t \in T} \text{epi} g_t\right)^* + \text{epi}\delta_C^*\right]$.

Proof. Let $\tilde{h}(x) := h(x) - \alpha$. It is easy to see that (i) is equivalent to $(\tilde{h} + \delta_F)^*(0) \leq 0$ or,

$$0 \in \text{epi }(\tilde{h} + \delta_F)^*. \tag{8.3}$$

Note that

$$\text{epi }(\tilde{h} + \delta_F)^* = \text{cl}\left(\text{epi}\tilde{h}^* + \text{epi}\delta_F^*\right).$$

Since $\tilde{h}$ is continuous (and hence, $\text{epi}\tilde{h}^* + \text{epi}\delta_F^*$ is closed), it follows from Lemma 8.1 that (8.3) is equivalent to

$$0 \in \text{epi}\tilde{h}^* + \text{epi}\delta_F^*.$$

On the other hand, by [2, Theorem 4.1],

$$\text{epi}\delta_F^* = \text{cl cone}\left(\bigcup_{t\in T} \text{epi}g_t^* \cup \text{epi}\delta_C^*\right).$$

Thus (a) holds if and only if

$$0 \in \text{epi}\tilde{h}^* + \text{cl cone}\left(\bigcup_{t\in T} \text{epi}g_t^* \cup \text{epi}\delta_C^*\right),$$

which is equivalent to (b) since $\text{epi}\tilde{h}^* = \text{epi}h^* + (0,\alpha)$ and

$$\text{cone}\left(\bigcup_{t\in T} \text{epi}g_t^* \cup \text{epi}\delta_C^*\right) = \text{cone}\left(\bigcup_{t\in T} \text{epi}g_t\right)^* + \text{epi}\delta_C^*.$$

The proof is complete. □

8.3 Approximate Solutions and Constraint Qualification Conditions

We consider the following convex semi-infinite vector optimization problem (CSIVP):

$$\begin{aligned}\text{(CSIVP) Minimize } & (f_1(x),\cdots,f_p(x))\\ \text{subject to } & g_t(x) \le 0,\ t\in T,\\ & x\in C,\end{aligned}$$

where $f_i : X \to \mathbb{R}$, $i = 1,2,\cdots,p$ are continuous convex functions and $g_t : X \to \mathbb{R}\cup\{+\infty\}$, $t \in T$, are proper lower semicontinuous (l.s.c.) convex functions, and T is an arbitrary (possibly infinite) set. Let $\varepsilon = (\varepsilon_1,\cdots,\varepsilon_p) \in R_+^p$ and let F be the feasible set of (CSIVP), i.e., $F := \{x \in C \mid g_t(x) \le 0, \forall t \in T\}$. Assume that $F \neq \emptyset$.

The definitions of ε-efficient solutions and weakly ε-efficient solutions for vector optimization problems were given in [24]. The modification of these notions for (CSIVP) are as follows.

Definition 8.1 (ε-efficient Solution). A point $\bar{x} \in F$ is said to be an *ε-efficient solution* of (CSIVP) if there does not exist $x \in F$ such that

$$f_i(x) \le f_i(\bar{x}) - \varepsilon_i \text{ for all } i = 1, \cdots p,$$
$$f_j(x) < f_j(\bar{x}) - \varepsilon_j \text{ for some } j.$$

Definition 8.2 (Weakly ε-Efficient Solution). A point $\bar{x} \in F$ is said to be a *weakly ε-efficient solution* of (CSIVP) if there does not exist $x \in F$ such that

$$f_i(x) < f_i(\bar{x}) - \varepsilon_i \text{ for all } i = 1, \cdots p.$$

When $\varepsilon = 0$, then Definitions 8.1 and 8.2 collapse to the definition of efficient solution and weakly efficient solution of (CSIVP), respectively [25]. When $p = 1$, then (CSIVP) becomes an scalar (ordinary) convex infinite optimization and Definition 8.1 becomes the definition of ε-solution given in [27].

We introduce Geoffrion's definition of solution for (CSIVP) with bounded tradeoffs, called properly efficient solution [9].

Definition 8.3 (Properly Efficient Solution [9]). A point $\bar{x} \in F$ is said to be a *properly efficient solution* of (CSIVP) if $\bar{x} \in F$ is an efficient solution of (CSIVP) and there exists $M > 0$ such that for each $i \in \{1, \cdots, p\}$, we have

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \le M$$

for some j such that $f_j(x) > f_j(\bar{x})$ whenever $x \in F$ and $f_i(x) < f_i(\bar{x})$.

Here, we denote the set of all properly efficient solutions of (CSIVP), the set of all efficient solutions of (CSIVP), the set of all weakly efficient solutions of (CSIVP), the set of all ε-efficient solutions of (CSIVP), and the set of all weakly ε-efficient solutions of (CSIVP) by PrEff(CSIVP), Eff(CSIVP), WEff(CSIVP), ε-Eff(CSIVP) and ε-WEff(CSIVP), respectively. When ε is a positive vector, it is clear from defnitions that

$$\begin{aligned}&\text{PrEff(CSIVP)} \subset \text{Eff(CSIVP)} \subset \text{WEff(CSIVP)}\\ &\subset \varepsilon - \text{Eff(CSIVP)} \subset \varepsilon - \text{WEff(CSIVP)}.\end{aligned}$$

Let $\bar{x} \in F$ and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_p) \in R^p_+$. Let $\tilde{f}_i(x) := f_i(x) - f_i(\bar{x}) + \varepsilon_i$, $\hat{f}_i(x) := f_i(x) - f_i(\bar{x})$, $i = 1, \cdots, p$.

We introduce three closedness conditions. They serve as regularity conditions for the Problem (CSIVP) in consideration.

- The pair $((g_t)_{t \in T}, C)$ is said to satisfy the *closedness condition* **(CC)** if

$$\textbf{(CC)} \quad \text{cone}\left(\bigcup_{t \in T} \text{epi} g_t\right)^* + \text{epi}\delta_C^* \text{ is weak}^*\text{-closed.}$$

- The triple $((f_i),(g_t)_{t\in T},C)$ is said to satisfy the *closedness condition* $(\mathbf{CC}_{\overline{\mathbf{x}}})$ at $\overline{x}$ if

$$(\mathbf{CC}_{\overline{\mathbf{x}}})\quad \text{cone}\left(\bigcup_{t\in T}\text{epi}g_t^* \cup \bigcup_{i=1}^{p}\text{epi}\hat{f}_i^*\right)+\text{epi}\delta_C^* \text{ is weak}^*\text{-closed.}$$

- The triple $((f_i),(g_t)_{t\in T},C)$ is said to satisfy the *closedness condition* $(\mathbf{CC1}_{\overline{\mathbf{x}}})$ at $\overline{x}$ if

$$(\mathbf{CC1}_{\overline{\mathbf{x}}})\quad \text{cone}\left(\bigcup_{t\in T}\text{epi}g_t^* \cup \bigcup_{i=1}^{p}\text{epi}\tilde{f}_i^*\right)+\text{epi}\delta_C^* \text{ is weak}^*\text{-closed.}$$

Constraints qualification conditions of this type have been successfully used in [2, 3, 10, 16] to establish optimality conditions, duality, stability for convex/DC infinite (single objective) problems, problems with parameters [5], and for equilibrium problems with DC cost functions [8]. Several new versions of generalized Farkas lemmas for systems involving DC functions are also established under these conditions as well [3, 6, 7]. It is shown that this type of conditions are weaker than many other ones known in the literature (see e.g., [3, 4, 16]). In particular, for (single objective) convex problems with infinitely many constraints, it is shown in [4] that it is weaker than the Slater one.

8.4 Optimality Conditions for ε-Efficient/Efficient Solutions of (CSIVP)

In this section we will give optimality conditions for ε-efficient solutions of (CSIVP), using conjugate theory in convex analysis and generalized version of Farkas lemma for systems of infinitely many inequalities (see Lemma 8.2. Optimality for efficient solutions of (CSIVP) are also obtained as a special case by letting $\varepsilon = 0$.

For $z\in F$, denote

$$F(z)=\{x\in X \mid f_i(x)\le f_i(z)-\varepsilon_i,\quad \forall\, i=1,\cdots p\}.$$

Notice that even though $\overline{x}\in F$ is an ε-efficient solution of (CSIVP), $F\cap F(\overline{x})$ may be empty. However, if $\varepsilon=0$, that is, $\overline{x}\in F$ is an efficient solution of (CSIVP), then it holds $F\cap F(\overline{x})\neq\emptyset$.

The following simple example shows that $F\cap F(z)$ may be empty for (CSIVP) even for the case where the index set T is finite.

Example 8.1. Consider the following convex vector optimization problem:

$$\begin{array}{lll}(\text{CVP1}) & \text{Minimize} & (x_1,x_2)\\ & \text{subject to} & (x_1,x_2)\in F:=\left\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1^2+x_2^2\le 1\right\}.\end{array}$$

Let $\varepsilon = (\varepsilon_1, \varepsilon_2) = (1,2)$ and $f(x_1,x_2) = (f_1(x_1,x_2), f_2(x_1,x_2)) = (x_1,x_2)$. Then $(0,0)$ is an ε-efficient solution of (CVP1), $f_1(0,0) - \varepsilon_1 = -1$, and $f_2(0,0) - \varepsilon_2 = -2$. However, we have

$$\begin{aligned}
&F \cap F(0,0) \\
&= F \cap \left\{(x_1,x_2) \in \mathbb{R}^2 \mid f_1(x_1,x_2) \leq f_1(0,0) - \varepsilon_1,\ f_2(x_1,x_2) \leq f_2(0,0) - \varepsilon_2\right\} \\
&= F \cap \left\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \leq -1,\ x_2 \leq -2\right\} \\
&= \emptyset.
\end{aligned}$$

The next result gives a simple characterization of ε-efficient solution of (CSIVP) which is similar to the one appeared in [24] and plays an important role in the study of this kind of solutions.

Proposition 8.1. *Let $\bar{x} \in F$. Then $\bar{x}$ is an ε-efficient solution of (CSIVP) if and only if either $F \cap F(\bar{x}) = \emptyset$ or,*

$$\sum_{i=1}^{p} f_i(x) = \sum_{i=1}^{p} f_i(\bar{x}) - \sum_{i=1}^{p} \varepsilon_i, \textit{ for any } x \in F \cap F(\bar{x}).$$

Proof. $(\Rightarrow)$ Let $\bar{x}$ be an ε-efficient solution of (CSIVP). Then $F \cap F(\bar{x}) = \emptyset$ or $F \cap F(\bar{x}) \neq \emptyset$. Suppose that $F \cap F(\bar{x}) \neq \emptyset$. Then for any $x \in F \cap F(\bar{x})$ and all $i = 1, \cdots p$, $f_i(x) \leq f_i(\bar{x}) - \varepsilon_i$. Hence the ε-efficiency of $\bar{x}$ yields

$$f_i(x) = f_i(\bar{x}) - \varepsilon_i,$$

for any $x \in F \cap F(\bar{x})$ and all $i = 1, \cdots p$. Thus we have, for all $x \in F \cap F(\bar{x})$,

$$\sum_{i=1}^{p} f_i(x) = \sum_{i=1}^{p} f_i(\bar{x}) - \sum_{i=1}^{p} \varepsilon_i.$$

$(\Leftarrow)$ Suppose that $F \cap F(\bar{x}) = \emptyset$. Then there does not exist $x \in F$ such that $x \in F(\bar{x})$, that is, there does not exist $x \in F$ such that $f_i(x) \leq f_i(\bar{x}) - \varepsilon_i$ for all $i = 1, \cdots, p$. Hence there does not exist $x \in F$ such that

$$\begin{aligned}
&f_i(x) \leq f_i(\bar{x}) - \varepsilon_i, \text{ for all } i = 1, \cdots p, \\
&f_j(x) < f_j(\bar{x}) - \varepsilon_j, \text{ for some } j.
\end{aligned}$$

Therefore $\bar{x}$ is an ε-efficient solution of (CSIVP).

Assume that $F \cap F(\bar{x}) \neq \emptyset$ and

$$\sum_{i=1}^{p} f_i(x) = \sum_{i=1}^{p} f_i(\bar{x}) - \sum_{i=1}^{p} \varepsilon_i \text{ for all } x \in F \cap F(\bar{x}). \tag{8.4}$$

Suppose to the contrary that $\overline{x}$ is not an ε-efficient solution of (CSIVP). Then there exist $\hat{x} \in F$ and an index j such that

$$f_i(\hat{x}) \leq f_i(\overline{x}) - \varepsilon_i \;\; i = 1, \cdots p,$$

$$f_j(\hat{x}) < f_j(\overline{x}) - \varepsilon_j \;\text{ for some } j.$$

Therefore, $\hat{x} \in F \cap F(\overline{x})$ and $\sum_{i=1}^{p} f_i(\hat{x}) < \sum_{i=1}^{p} f_i(\overline{x}) - \sum_{i=1}^{p} \varepsilon_i$, which contradicts the inequality (8.4). The proof is complete. □

The following result is a direct consequence of Proposition 8.1.

Corollary 8.1. *Let $\overline{x} \in F$. Assume that $F \cap F(\overline{x}) \neq \emptyset$. Then $\overline{x}$ is an ε-efficient solution of (CSIVP) if and only if $\overline{x}$ is a $\sum_{i=1}^{p} \varepsilon_i$-solution of the scalar convex infinite problem :*

$$\begin{aligned} &\text{Minimize } \textstyle\sum_{i=1}^{p} f_i(x) \\ &\text{subject to } g_t(x) \leq 0, \; t \in T, \\ &\qquad\qquad f_i(x) - f_i(\overline{x}) + \varepsilon_i \leq 0, \; i = 1, \cdots, p, \\ &\qquad\qquad x \in C. \end{aligned}$$

As a consequence, when $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_p) = 0$, we get:

Corollary 8.2. *Let $\overline{x} \in F$. Then $\overline{x}$ is an efficient solution of (CSIVP) if and only if $\overline{x}$ is a solution of the scalar convex infinite problem:*

$$\begin{aligned} &\text{Minimize } \textstyle\sum_{i=1}^{p} f_i(x) \\ &\text{subject to } g_t(x) \leq 0, \; t \in T, \\ &\qquad\qquad f_i(x) - f_i(\overline{x}) \leq 0, \; i = 1, \cdots, p, \\ &\qquad\qquad x \in C. \end{aligned} \tag{8.5}$$

It is worth observing that for the scalar convex infinite problem in Corollary 8.1 (and also, the scalar convex infinite problem in Corollary 8.2), constraint qualification conditions of interior-type, such as Slater one, fail to hold. However, optimality condition for (CSIVP) can be established under our constraint qualification conditions $(\mathbf{CC}_{\overline{\mathbf{x}}})$ and $(\mathbf{CC1}_{\overline{\mathbf{x}}})$.

We are now in a position to establish a necessary and sufficient optimality condition for ε-efficient solutions of (CSIVP) using the characterization given in Proposition 8.1. Recall that $\tilde{f}_i(x) = f_i(x) - f_i(\overline{x}) + \varepsilon_i$, $\hat{f}_i(x) = f_i(x) - f_i(\overline{x})$, $i = 1, \cdots, p$. It is worth noticing that $\text{epi}\tilde{f}_i^* = \text{epi}f_i^* + (0, f_i(\overline{x}) - \varepsilon_i)$, $\text{epi}\hat{f}_i^* = \text{epi}f_i^* + (0, f_i(\overline{x}))$.

Theorem 8.1 (Optimality Conditions for ε-Efficient Solutions for (CSIVP)). *Let $\overline{x} \in F$. Assume that $F \cap F(\overline{x}) \neq \emptyset$ and that $(CC_{\overline{x}})$ holds. Then the following are equivalent.*

(a) $\overline{x}$ is an ε-efficient solution of (CSIVP)

(b) It holds

$$\left(0, \sum_{i=1}^{p}[\varepsilon_i - f_i(\bar{x})]\right) \in \sum_{i=1}^{p} \text{epi} f_i^* + \text{cone}\left[\bigcup_{t\in T} \text{epi} g_t^* \cup \bigcup_{i=1}^{p} \text{epi} \tilde{f}_i^*\right] + \text{epi}\delta_C^*;$$

(c) There exist $\alpha_i \geq 0,\ i = 1,\cdots,p,\ (\lambda_t) \in R_+^{(T)},\ \beta_t \geq 0,\ t \in T,\ \gamma_j \geq 0,\ \mu_j \geq 0,$ $j = 1,\cdots,p,\ \delta \geq 0$ *such that*

$$0 \in \sum_{i=1}^{p} \partial_{\alpha_i} f_i(\bar{x}) + \sum_{j=1}^{p} \mu_k \partial_{\gamma_j} f_j(\bar{x}) + \sum_{t\in T} \lambda_t \partial_{\beta_t} g_t(\bar{x}) + \partial_\delta \delta_C(\bar{x}),$$

$$0 = \sum_{i=1}^{p}\Big[\alpha_i + \mu_i\gamma_i - (1+\mu_i)\varepsilon_i\Big] + \delta + \sum_{t\in T} \lambda_t \Big[\beta_t - g_t(\bar{x})\Big].$$

Proof. (a) ⇔ (b): Since $F \cap F(\bar{x}) \neq \emptyset$, by Proposition 8.1, (a) is equivalent to

$$\sum_{i=1}^{p} f_i(x) \geq \sum_{i=1}^{p} f_i(\bar{x}) - \sum_{i=1}^{p} \varepsilon_i,\ \forall x \in F \cap S(\bar{x}),$$

which is equivalent to

$$\left(0, \sum_{i=1}^{p}[\varepsilon_i - f_i(\bar{x})]\right) \in \text{epi}\left(\textstyle\sum_{i=1}^{p} f_i\right)^* + \text{cl cone}\left[\bigcup_{t\in T} \text{epi} g_t^* \cup \bigcup_{i=1}^{p} \text{epi} \tilde{f}_i^*\right] + \text{epi}\delta_C^*,$$

thanks to Farkas lemma (Lemma 8.2) and the continuity of f_i, $i = 1,\cdots,p$.

The equivalence of (a) and (b) follows from the closedness condition $(CC_{\bar{x}})$ and the fact that

$$\text{epi}\left(\sum_{i=1}^{p} f_i\right)^* = \sum_{i=1}^{p} \text{epi} f_i^*,$$

which is a direct consequence of the continuity of f_i and Lemma 8.1.
(b) ⇔ (b): Applying the representation (8.2) in Sect. 8.2 to epif_i^*, epig_t^* and epiδ_C^* (taking also (8.1) in Sect. 8.2 into account), (b) is equivalent to the fact that there exist $\alpha_i \geq 0,\ u_i \in \partial_{\alpha_i} f_i(\bar{x}),\ i = 1,\cdots,p,\ (\lambda_t) \in \mathbb{R}_+^{(T)},\ \beta_t \geq 0, t \in T,\ v_t \in \partial_{\beta_t} g_t(\bar{x}),$ $\gamma_j \geq 0,\ \mu_j \geq 0,\ \omega_j \in \partial_{\gamma_j} f_j(\bar{x}),\ j = 1,\cdots,p,\ \delta \geq 0,\ \nu \geq 0,\ w \in \partial_\delta \delta_C(\bar{x})$ such that

$$0 = \sum_{i=1}^{p}(u_i + \mu_i\omega_i) + \sum_{t\in T} \lambda_t v_t + w$$

and

$$0=\sum_{i=1}^{p}\Big[\alpha_i+\mu_i\gamma_i-(1+\mu_i)\varepsilon_i\Big]+\delta+\sum_{t\in T}\lambda_t\Big[\beta_t-g_t(\bar{x})\Big].$$

Thus, (b) is equivalent to (c). □

It is worth mentioning that the conclusion of Theorem 8.1 holds regardless of the fact that $\bar{x}$ belongs to $F\cap F(\bar{x})$ or not.

We give an example to illustrate the use of Proposition 8.1 and Theorem 8.1.

Example 8.2. Consider the following convex semi-infinite vector optimization problem:

$$\text{(CSIVP1)}\qquad \text{Minimize}\quad (x_1,x_2)$$
$$\text{subject to}\quad \max\{-x_1,0\}-tx_2\le 0\quad \forall t\in(1,2].$$

Let $f_1(x_1,x_2)=x_1$, $f_2(x_1,x_2)=x_2$, $g_t(x_1,x_2)=\max\{-x_1,0\}-tx_2$, $\forall t\in(1,2]$, and $C=\mathbb{R}^2$. Then (CSIVP1) becomes:

$$\begin{aligned}\text{Minimize}\quad & (f_1(x_1,x_2),f_2(x_1,x_2))\\ \text{subject to}\quad & g_t(x_1,x_2)\le 0\quad \forall t\in(1,2],\\ & (x_1,x_2)\in C.\end{aligned}$$

Let F be the set of all feasible solutions of the above problem (CSIVP1) and let $\varepsilon=(\varepsilon_1,\varepsilon_2)=\left(\frac{1}{8},\frac{1}{8}\right)$. In fact, we can easily check that the set of all ε-efficient solutions of (CSIVP1) is $\{(x_1,x_2)\in\mathbb{R}^2\mid x_1+x_2\ge 0,\ x_1+x_2\le\frac{1}{4},\ x_1-x_2\le 0\}\cup\{(x_1,x_2)\in\mathbb{R}^2\mid x_1\ge 0,\ x_2<\frac{1}{8},\ x_1-x_2>0\}$.

Let, for any $(z_1,z_2)\in F$, $F(z_1,z_2)=\big\{(x_1,x_2)\in\mathbb{R}^2\mid f_i(x_1,x_2)\le f_i(z_1,z_2)-\varepsilon_i,\ i=1,2\}$. Then $\{(x_1,x_2)\in\mathbb{R}^2\mid F\cap F(x_1,x_2)=\emptyset\}=\{(x_1,x_2)\in\mathbb{R}^2\mid \max\{-x_1,0\}\le x_2<\max\{-x_1+\frac{1}{4},\frac{1}{8}\}\}$. Hence, from Proposition 8.1, this set is a subset of the set of all ε-efficient solution of (CSIVP1). Let $(\bar{x}_1,\bar{x}_2)=\left(\frac{1}{8},\frac{1}{8}\right)$ and $\widetilde{f}_i(x_1,x_2)=f_i(x_1,x_2)-f_i(\bar{x}_1,\bar{x}_2)+\varepsilon_i$, $i=1,2$. Then we can easily to see that

$$\begin{aligned}&\operatorname{epi}g_t^*=[-1,0]\times\{-t\}\times\mathbb{R}_+,\\ &\operatorname{epi}f_1^*=\operatorname{epi}\widetilde{f}_1^*=\{(1,0)\}\times\mathbb{R}_+,\\ &\operatorname{epi}f_2^*=\operatorname{epi}\widetilde{f}_2^*=\{(0,1)\}\times\mathbb{R}_+,\\ &\operatorname{epi}\delta_C^*=\{(0,0)\}\times\mathbb{R}_+,\\ &(0,0,\textstyle\sum_{i=1}^{2}(\varepsilon_i-\widetilde{f}_i(\bar{x}_1,\bar{x}_2))=(0,0,0).\end{aligned}$$

Hence,

$$\begin{aligned}&\sum_{i=1}^{2}\mathrm{epi}f_i^* + \mathrm{cone}\left(\bigcup_{t\in T}\mathrm{epi}g_t^* \cup \mathrm{epi}\widetilde{f_1^*}\cup \mathrm{epi}\widetilde{f_2^*}\right) + \mathrm{epi}\delta_C^*\\ &= \mathrm{cone}\left(\bigcup_{t\in T}\mathrm{epi}g_t^* \cup \mathrm{epi}\widetilde{f_1^*}\cup \mathrm{epi}\widetilde{f_2^*}\right) + \mathrm{epi}\delta_C^*\\ &= \mathbb{R}\times\mathbb{R}\times\mathbb{R}_+,\end{aligned}$$

and so, the closedness condition ($CC_{\overline{x}}$) holds. By Theorem 8.1, $(\overline{x}_1,\overline{x}_2)$ is an ε-efficient solution of (CSIVP1).

Letting $\varepsilon_i = 0,\ i = 1,\cdots,p$ we get necessary and sufficient conditions for efficient solutions to (CSIVP).

Theorem 8.2 (Optimality Conditions for Efficient Solutions for (CSIVP)). *Let $\overline{x}\in F$. Assume that $(CC_{\overline{x}})$ holds. Then the following statements are equivalent:*

(a) $\overline{x}$ is an efficient solution of (CSIVP).

(b) $\sum_{i=1}^{p}(0,-f_i(\overline{x})) \in \sum_{i=1}^{p}\mathrm{epi}f_i^* + \mathrm{cone}\left[\bigcup_{t\in T}\mathrm{epi}g_t^* \ \cup\ \bigcup_{i=1}^{p}\mathrm{epi}\hat{f}_i^*\right] + \mathrm{epi}\delta_C^*.$

(c) there exist $\mu_i > 0,\ i = 1,\cdots,p,\ \sum_{i=1}^{p}\mu_i = 1$ and $(\lambda_t)\in R_+^{(T)}$ such that

$$0\in\sum_{i=1}^{p}\mu_i\partial f_i(\overline{x}) + \sum_{t\in T}\lambda_t\partial g_t(\overline{x}) + N_C(\overline{x})$$

and

$$\lambda_t g_t(\overline{x}) = 0,\ t\in T.$$

(d) $\overline{x}$ is a properly efficient solution of (CSIVP).

Proof. Take $\varepsilon_i = 0$ for all $i = 1,\cdots,p$. It follows from Theorem 8.1 that (a), (b), (c) are equivalent. Also, it is clear from the definitions of efficient and properly solutions that (d) implies (a). So, it is sufficient to prove that (c) implies (d).

Suppose that (c) holds. Then there exist $u_i\in\partial f_i(\overline{x}),\ i = 1,\cdots,p,\ v_t\in\partial g_t(\overline{x})$, $t\in T$ and $\omega\in N_C(\overline{x})$ such that

$$0 = \sum_{i=1}^{p}\mu_i u_i + \sum_{t\in T}\lambda_t v_t + \omega.$$

Let $\overline{x}\in F$ be any fixed. Then since $\omega\in N_C(\overline{x})$, $\omega(x-\overline{x})\le 0$. Moreover, since $\lambda_t g_t(x)\le 0,\ \lambda_t g_t(\overline{x}) = 0,\ t\in T$ and $v_t\in\partial g_t(\overline{x}),\ t\in T$, we have

$$0\ge\lambda_t g_t(x) - \lambda_t g_t(\overline{x}) \ge \lambda_t v_t(x-\overline{x}),\ \ t\in T.$$

So, we have,

$$\sum_{t \in T} \lambda_t v_t(x - \bar{x}) + \omega(x - \bar{x}) \leq 0$$

and hence,

$$\sum_{i=1}^{p} \mu_i u_i(x - \bar{x}) \geq 0.$$

Thus we have,

$$\sum_{i=1}^{p} \mu_i f_i(x) \geq \sum_{i=1}^{p} \mu_i f_i(\bar{x}).$$

It now follows from [9, Theorem 1] that $\bar{x}$ is a properly efficient solution of (CSIVP). □

Theorem 8.2 gives a sufficient criteria for the properness of efficient solutions of (CSIVP). In fact, if $\bar{x}$ is an efficient solution of (CSIVP) and if the closedness condition $(CC_{\bar{x}})$ holds then $\bar{x}$ is a properly efficient solution of (CSIVP). The following example illustrates this special feature.

Example 8.3. Consider the following convex infinite vector optimization problem:

$$\begin{array}{lll} \text{(CSIVP2)} & \text{Minimize} & f(x) := (x, x^2) \\ & \text{subject to} & x \in F := \{x \in R \mid g_t(x) := x - t \leq 0,\ t \in (0,1)\}. \end{array}$$

Then we can easily check that the set of all efficient solutions of (CSIVP2) is $\{x \in R : x \leq 0\}$. Let $f_1(x) = x,\ f_2(x) = x^2,\ g_t(x) = x - t,\ t \in (0,1)$ and $C = R$. Let $A_{\bar{x}} = \text{cone}\Big[\bigcup_{t \in T} \text{epi}g_t^* \ \cup \ \bigcup_{i=1}^{p} (\text{epi}f_i^* + (0, f_i(\bar{x}))\Big] + \text{epi}\delta_C^*$. When $\bar{x} = 0$ then $A_{\bar{x}} = \{(x,y) : x \geq 0,\ y \geq 0\} \cup \{(x,y) \in \mathbb{R}^2 : x < 0,\ y > 0\}$, and it is not a closed set. However, if $\bar{x} < 0$, $A_{\bar{x}} = \{(x,y) \in \mathbb{R}^2 : y \geq \bar{x}x\}$ then $A_{\bar{x}}$ is closed, and so by Theorem 8.2, $\bar{x}$ is a properly efficient solution of (CSIVP2). In fact, $\bar{x}$ with $\bar{x} < 0$ is a properly efficient solution of (CSIVP2), but $\bar{x} = 0$ is an efficient solution of (CSIVP2) which is not proper.

We now pay attention to the case where the index set T is finite, say, $T = \{1, 2, \cdots, m\}$. The problem (CSIVP) collapses to the usual convex vector optimization problem (CVP):

$$\begin{array}{rl} \text{(CVP) Minimize} & (f_1(x), \cdots, f_p(x)) \\ \text{subject to} & g_j(x) \leq 0,\ j = 1, 2, \cdots, m, \\ & x \in C. \end{array}$$

Proposition 8.1 and Corollaries 8.1, 8.2 hold for (CVP). In the following results, for the simplicity of representation, by the conditions $(CC_{\bar{x}})$, $(CC1_{\bar{x}})$ for (CVP) we mean the same formulas as in previous sections, just replace T by $\{1, 2, \cdots, m\}$. As consequences of Theorems 8.1, 8.2, we get the following corollary.

Corollary 8.3 (Optimality Conditions for ε-Efficient Solutions of (CVP)). *For the Problem (CVP), let $\bar{x} \in F$. Assume that $T = \{1, 2, \cdots, m\}$, $F \cap F(\bar{x}) \neq \emptyset$ and that $(CC1_{\bar{x}})$ holds. Then the following conditions are equivalent.*

(a) $\bar{x}$ is an ε-efficient solution of (CVP).
(b) It holds

$$\left(0, \sum_{i=1}^{p} [\varepsilon_i - f_i(\bar{x})]\right) \in \sum_{i=1}^{p} \mathrm{epi} f_i^* + \mathrm{cone}\left[\bigcup_{j=1}^{m} \mathrm{epi} g_j^* \cup \bigcup_{i=1}^{p} \mathrm{epi} \tilde{f}_i^*\right] + \mathrm{epi} \delta_C^*;$$

(c) There exist $\alpha_i \geq 0$, $\gamma_i \geq 0$, $\mu_i \geq 0$, $i = 1, \cdots, p$, $(\lambda_j) \in \mathbb{R}^m_+$, $\beta_j \geq 0$, $j = 1, \cdots, m$, $\delta \geq 0$ such that

$$0 \in \sum_{i=1}^{p} \partial_{\alpha_i} f_i(\bar{x}) + \sum_{i=1}^{p} \mu_i \partial_{\gamma_j} f_i(\bar{x}) + \sum_{j=1}^{m} \lambda_j \partial_{\beta_j} g_j(\bar{x}) + \partial_\delta \delta_C(\bar{x}),$$

$$0 = \sum_{i=1}^{p} \left[\alpha_i + \mu_i \gamma_i - (1 + \mu_i)\varepsilon_i\right] + \delta + \sum_{j=1}^{m} \lambda_j \left[\beta_j - g_j(\bar{x})\right].$$

Corollary 8.4 (Optimality Conditions for Efficient Solutions of (CVP)). *For the Problem (CVP), let $\bar{x} \in F$. Assume that $(CC_{\bar{x}})$ holds. Then the following statements are equivalent:*

(a) $\bar{x}$ is an efficient solution of (CVP).
(b) $\left(0, -\sum_{i=1}^{p} f_i(\bar{x})\right) \in \sum_{i=1}^{p} \mathrm{epi} f_i^* + cone\left[\bigcup_{j=1}^{m} \mathrm{epi} g_j^* \cup \bigcup_{i=1}^{p} \left(\mathrm{epi} \hat{f}_i^*\right)\right] + \mathrm{epi} \delta_C^*.$
(c) There exist $\mu_i > 0$, $i = 1, \cdots, p$ and $(\lambda_j) \in R^m_+$ such that

$$0 \in \sum_{i=1}^{p} \mu_i \partial f_i(\bar{x}) + \sum_{j=1}^{m} \lambda_j \partial g_j(\bar{x}) + N_C(\bar{x})$$

and

$$\lambda_j g_j(\bar{x}) = 0, \; j = 1, \cdots, m.$$

(d) $\bar{x}$ is a properly efficient solution of (CVP).

8.5 Optimality Conditions for Weakly ε-Efficient Solutions of (CSIVP)

In this section, we will establish optimality conditions for weakly ε-efficient solutions of (CSIVP). We first need the following theorem which can be easily obtained using separation theorem.

Proposition 8.2. *Let $\bar{x} \in F$. Then $\bar{x}$ is a weakly ε-efficient solution of (CSIVP) if and only if there exist $\mu_i \geq 0$ $i = 1, \cdots, p$, $\sum_{i=1}^{p} \mu_i = 1$ such that*

$$\sum_{i=1}^{p} \mu_i f_i(x) \geq \sum_{i=1}^{p} \mu_i f_i(\bar{x}) - \sum_{i=1}^{p} \mu_i \varepsilon_i \ \ \textit{for any } x \in F.$$

We now in a position to establish a necessary and sufficient optimality condition for weakly ε-efficient solutions of (CSIVP).

Theorem 8.3 (Optimality Conditions for Weakly ε-Efficient Solutions for (CSIVP)). *Let $\bar{x} \in F$ and assume that* cone $\left(\bigcup_{t \in T} \mathrm{epi} g_t^* \right) + \mathrm{epi}\delta_C^*$ *is weak*-closed. Then the following statements are equivalent.*

(a) $\bar{x}$ is a weakly ε-efficient solution of (CSIVP).

(b) There exist $\mu_i \geq 0$, $i = 1, \cdots p$, $\sum_{i=1}^{p} \mu_i = 1$, such that

$$\sum_{i=1}^{p} \mu_i (0, \varepsilon_i - f_i(\bar{x})) \in \sum_{i=1}^{p} \mathrm{epi}(\mu_i f_i)^* + \mathrm{cone} \bigcup_{t \in T} \mathrm{epi} g_t^* + \mathrm{epi}\delta_C^*.$$

(c) There exist $\mu_i \geq 0$, $i = 1, \cdots p$, $\sum_{i=1}^{p} \mu_i = 1$, $\alpha_i \geq 0$, $i = 1, \cdots, p$, $(\lambda_t) \in \mathbb{R}_+^{(T)}$, $\beta_t \geq 0, t \in T$, $\gamma \geq 0$, $\nu \geq 0$ such that

$$0 \in \sum_{i=1}^{p} \mu_i \partial_{\alpha_i} f_i(\bar{x}) + \sum_{t \in T} \lambda_t \partial_{\beta_t} g_t(\bar{x}) + \nu \partial_\gamma \delta_C(\bar{x})$$

and

$$0 = \sum_{i=1}^{p} \mu_i (\alpha_i - \varepsilon_i) + \nu \delta + \sum_{t \in T} \lambda_t \left[\beta_t - g_t(\bar{x}) \right].$$

Proof. It follows from Lemma 8.2 and Proposition 8.2 that (a) is equivalent to (b). Using (8.1) in Sect. 8.2 and applying representation (8.2) in Sect. 8.2 to epif_i^*, epig_t^* and epiδ_C^*, it is seen that (b) is equivalent to the fact that there exist $\mu_i \geq 0$, $i = 1, \cdots p$, $\sum_{i=1}^{p} \mu_i = 1$, $\alpha_i \geq 0$, $u_i \in \partial_{\alpha_i} f_i(\bar{x})$, $i = 1, \cdots, p$, $(\lambda_t) \in \mathbb{R}_+^{(T)}$, $\beta_t \geq 0$, $t \in T$, $v_t \in \partial_{\beta_t} g_t(\bar{x})$, $\gamma \geq 0$, $\nu \geq 0$, $\omega \in \partial_\gamma \delta_C(\bar{x})$ such that

$$0 = \sum_{i=1}^{p} \mu_i u_i + \sum_{t \in T} \lambda_t v_t + \nu \omega,$$

and

$$0 = \sum_{i=1}^{p} \mu_i(\alpha_i - \varepsilon_i) + \nu\delta + \sum_{t\in T} \lambda_t \left[\beta_t - g_t(\bar{x})\right].$$

Thus (b) is equivalent to (c). □

The following examples give an illustration of Theorem 8.3.

Example 8.4. Let $B_1 = \left\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \le 1\right\}$ and $T_1 = \left\{(x_1,x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\right\}$. Let $f_1(x_1,x_2) = x_1$, $f_2(x_1,x_2) = x_2$, $g_{(t_1,t_2)}(x_1,x_2) = t_1x_1 + t_2x_2 - 1$ for any $(t_1,t_2) \in T_1$ and $C = \mathbb{R}^2$. Consider the following linear semi-infinite vector optimization problem:

$$\begin{array}{lll} \text{(LSIVP)} & \text{Minimize} & (x_1,x_2) \\ & \text{subject to} & g_{(t_1,t_2)}(x_1,x_2) \le 0 \quad \forall (t_1,t_2) \in T_1, \\ & & (x_1,x_2) \in C. \end{array}$$

Let $\bar{x} = (\bar{x}_1,\bar{x}_2) = \left(-\frac{1}{\sqrt{2}} + \frac{1}{8}, -\frac{1}{\sqrt{2}} + \frac{1}{8}\right)$, $\varepsilon = (\varepsilon_1,\varepsilon_2) = \left(\frac{1}{8},\frac{1}{8}\right)$ and $\mu = (\mu_1,\mu_2) = \left(\frac{1}{2},\frac{1}{2}\right)$. We can check that

$$\begin{aligned} \text{cone}\left(\bigcup_{(t_1,t_2)\in T_1} \text{epi}g^*_{(t_1,t_2)}\right) + \text{epi}\delta^*_C &= \left\{(x_1,x_2,\alpha) \in \mathbb{R}^3 \mid \sqrt{x_1^2 + x_2^2} \le \alpha\right\}, \\ \text{epi}(\mu_1 f_1)^* &= \{(\tfrac{1}{2},0)\} \times \mathbb{R}_+, \\ \text{epi}(\mu_2 f_2)^* &= \{(0,\tfrac{1}{2})\} \times \mathbb{R}_+, \\ \textstyle\sum_{i=1}^{2} \mu_i(0,0,\varepsilon_i - f_i(\bar{x})) &= \left(0,0,\tfrac{1}{\sqrt{2}}\right), \\ \text{epi}(\mu_1 f_1)^* + \text{epi}(\mu_2 f_2)^* + \text{cone}&\left(\bigcup_{(t_1,t_2)\in T_1} \text{epi}g^*_{(t_1,t_2)}\right) + \text{epi}\delta^*_C \\ &= \left\{(x_1,x_2,\alpha) \in R^3 \mid \sqrt{\left(x_1 - \tfrac{1}{2}\right)^2 + \left(x_2 - \tfrac{1}{2}\right)^2} \le \alpha\right\}. \end{aligned}$$

The set $\text{cone}\left(\bigcup_{(t_1,t_2)\in F_1} \text{epi}g^*_{(t_1,t_2)}\right) + \text{epi}\delta^*_C$ is closed, and

$$\sum_{i=1}^{2} \mu_i(0,0,\varepsilon_i - f_i(\bar{x})) \in \text{epi}(\mu_i f_i)^* + \text{epi}(\mu_2 f_2)^* + \text{cone}\left(\bigcup_{(t_1,t_2)\in F_1} \text{epi}g^*_{(t_1,t_2)}\right) + \text{epi}\delta^*_C.$$

By Theorem 8.3, $(\bar{x}_1,\bar{x}_2)$ is a weakly ε-efficient solution of (LSIVP).

In fact, the set of all ε-efficient solutions of (LSIVP) coincides with the set of weakly ε-efficient solution of (LSIVP), which is the following one:

$$\left\{(x_1,x_2)\in\mathbb{R}^2 \mid x_2\le x_1+1,\ x_2\ge x_1-1,\ x_1^1 \right.$$
$$\left. +x_2^2\le 1,\ \left(x_1-\frac{1}{8}\right)^2+\left(x_2-\frac{1}{8}\right)^2\ge 1\right\}.$$

Example 8.5. Let $B_1=\left\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1^2+x_2^2\le 1\right\}$ and $T_1=\left\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1^2+x_2^2=1\right\}$. Let $f_1(x_1,x_2)=x_1,\ f_2(x_1,x_2)=x_2,\ g_{(t_1,t_2)}(x_1,x_2)=t_1x_1+t_2x_2-1$ for any $(t_1,t_2)\in T_1$ and $C=\mathbb{R}^2$. Slightly modifying Example 8.4, we consider the following convex semi-infinite vector optimization problem:

$$\begin{array}{lll}
\text{(CSIVP3)} & \text{Minimize} & (x_1,x_2^2) \\
 & \text{subject to} & g_{(t_1,t_2)}(x_1,x_2)\le 0 \quad \forall(t_1,t_2)\in T_1, \\
 & & (x_1,x_2)\in C.
\end{array}$$

Let $\bar{x}=(\bar{x}_1,\bar{x}_2)=\left(0,\frac{1}{\sqrt{8}}\right)$, $\varepsilon=(\varepsilon_1,\varepsilon_2)=\left(\frac{1}{8},\frac{1}{8}\right)$ and $\mu=(\mu_1,\mu_2)=(0,1)$. It is easy to verify that

$$\text{cone}\left(\bigcup_{(t_1,t_2)\in T_1}\text{epi}g^*_{(t_1,t_2)}\right)+\text{epi}\delta^*_C=\left\{(x_1,x_2,\alpha)\in R^3 \mid \sqrt{x_1^2+x_2^2}\le\alpha\right\},$$
$$\text{epi}(\mu_1 f_1)^*=\{(0,0)\}\times\mathbb{R}_+,$$
$$\text{epi}(\mu_2 f_2)^*=\{(0,x_2,\tfrac{1}{4}x_2^2+\alpha)\mid x_2\in\mathbb{R},\ \alpha\ge 0\},$$
$$\textstyle\sum_{i=1}^2\mu_i(0,0,\varepsilon_i-f_i(\bar{x}))=(0,0,0).$$

Therefore, the set $\text{cone}\left(\bigcup_{(t_1,t_2)\in T_1}\text{epi}g^*_{(t_1,t_2)}\right)+\text{epi}\delta^*_C$ is closed and

$$\sum_{i=1}^{2}\mu_i(0,0,\varepsilon_i-f_i(\bar{x}))\in\text{epi}(\mu_i f_i)^*+\text{epi}(\mu_2 f_2)^*+\text{cone}\left(\bigcup_{(t_1,t_2)\in T_1}\text{epi}g^*_{(t_1,t_2)}\right)+\text{epi}\delta^*_C.$$

Due to Theorem 8.3, $(\bar{x}_1,\bar{x}_2)$ is a weakly ε-efficient solution of (LSIVP).

In fact, for this problem, the set of all weakly ε-efficient solutions is $\{(x_1,x_2)\in B_1\ :\ |x_2|\le\frac{1}{\sqrt{8}}\}$ and the set of all ε-efficient solutions is

$$\left\{(x_1,x_2)\in B_1\ :\ |x_2|\le\frac{1}{\sqrt{8}}\right\}\setminus\left[\left\{(x_1,\frac{1}{\sqrt{8}})\in B_1\ :\ x_1>-\frac{7}{8}\right\}\right.$$

$$\cup \left\{ (x_1, -\frac{1}{\sqrt{8}}) \in B_1 \ : \ x_1 > -\frac{7}{8} \right\} \Big].$$

Remark 8.1. In Example 8.2 of the previous section, the set

$$\left\{ (x_1, x_2) \in R^2 \ : \ \max\{-x_1, 0\} \le x_2 \le \max\{-x_1 + \frac{1}{4}, \frac{1}{8}\} \right\}$$

is the set of all weakly ε-efficient solution of (CSIVP1). However, the set of all its ε-efficient solution is $\left\{(x_1,x_2) \in R^2 \ : \ x_1 + x_2 \ge 0,\ x_1 + x_2 \le \frac{1}{4},\ x_1 - x_2 \le 0\right\} \cup \left\{(x_1,x_2) \in R^2 \ : \ x_1 \ge 0,\ x_2 < \frac{1}{8},\ x_1 - x_2 > 0\right\}$. It is also worth observing that $(\bar{x}_1, \bar{x}_2) = \left(1, \frac{1}{8}\right)$ is a weakly ε-efficient solution of (CSIVP1), but cone $\left(\bigcup_{t \in T} \text{epi} g_t^*\right) + \text{epi}\delta_C^*$ is not closed. However, when we reformulate the problem (CSIVP1) to an equivalent one with a finitely many constraints, Theorem 8.3 can be applied to the resulting problem as shown in the next example.

Example 8.6. Consider the following vector optimization problem which is equivalent to (CSIVP2) in Example 8.2.

$$\begin{array}{lll} \text{(VP)} & \text{Minimize} & (x_1, x_2) \\ & \text{subject to} & \max\{-x_1, 0\} - x_2 \le 0. \end{array}$$

Let $f_1(x_1,x_2) = x_1$, $f_2(x_1,x_2) = x_2$, $g(x_1,x_2) = \max\{-x_1, 0\} - x_2$, and $C = \mathbb{R}^2$. Then (VP) becomes

$$\begin{array}{lll} \text{(VP)} & \text{Minimize} & (f_1(x_1,x_2), f_2(x_1,x_2)) \\ & \text{subject to} & g(x_1,x_2) \le 0, \\ & & (x_1,x_2) \in C. \end{array}$$

Then $\text{cone}(\text{epi} g^*) + \text{epi}\delta_C^* = \text{cone}([-1,0] \times \{-1\} \times \mathbb{R}_+) + \{(0,0)\} \times \mathbb{R}_+ = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \le 0, x_2 \le 0, x_1 \ge x_2\} \times \mathbb{R}_+$ and hence $\text{cone}(\text{epi} g^*) + \text{epi}\delta_C^*$ is closed. Let $\varepsilon = (\varepsilon_1, \varepsilon_2) = \left(\frac{1}{8}, \frac{1}{8}\right)$, $(\bar{x}_1, \bar{x}_2) = \left(1, \frac{1}{8}\right)$ and $(\mu_1, \mu_2) = (0, 1)$. Then we have,

$$\begin{aligned} \sum_{i=1}^{2} \mu_i (0, \varepsilon_i - f_i(\bar{x})) \in \sum_{i=1}^{2} & \text{epi}\,(\mu_i f_i)^* + \text{cone}\,(\text{epi} g^*) + \text{epi}\delta_C^* \\ &= \left\{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 \le x_1 + 1,\ x_1 \le 0\right\}. \end{aligned}$$

Hence, by Theorem 8.3, $(\bar{x}_1, \bar{x}_2)$ is a weakly ε-efficient solution of (VP).

We give an example of a simple problem where the Slater regularity condition does not holds, but our regularity condition does.

Example 8.7. Consider the following convex vector optimization problem:

$$\text{(CVP2)} \qquad \text{Minimize} \quad (x, x^2)$$
$$\qquad\qquad\qquad \text{subject to} \quad x \in F := \{x \in \mathbb{R} \mid g(x) \leq 0\},$$

where

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}$$

Let $f_1(x) = x$, $f_2(x) = x^2$, $\varepsilon = (\varepsilon_1, \varepsilon_2) = \left(\frac{1}{8}, \frac{1}{8}\right)$, $\mu = (0,1)$ and $\bar{x} = 0$. Then it is obvious the Slater condition does not hold. However,

$$\begin{aligned}
&\textstyle\sum_{i=1}^2 \mu_i(0, \varepsilon_i - f_i(\bar{x})) = \left(0, \frac{1}{8}\right), \\
&\textstyle\sum_{i=1}^2 \mathrm{epi}(\mu_i f_i)^* = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq \frac{1}{4}x_1^2\right\}, \\
&\mathrm{epi} g^* = \partial g(0) \times \mathbb{R}_+ = [0,1] \times \mathbb{R}_+, \\
&\mathrm{epi}\delta_C^* = \{(0,0)\} \times \mathbb{R}_+,
\end{aligned}$$

and $\mathrm{cone}\,(\mathrm{epi} g^*) + \mathrm{epi}\delta_C^* = \mathbb{R}_+^2$ is closed, and hence, the regularity condition in Theorem 8.3 holds. On the other hands, since

$$\sum_{i=1}^2 \mathrm{epi}\,(\mu_i f_i)^* + \mathrm{cone}\,(\mathrm{epi} g^*) + \mathrm{epi}\delta_C^* = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq \frac{1}{4}x_1^2,\ x_1 \leq 0\right\} \cup \mathbb{R}_+^2,$$

it holds

$$(0, \sum_{i=1}^2 \mu_i(\varepsilon_i - f_i(\bar{x})) \in \sum_{i=1}^2 \mathrm{epi}\,(\mu_i f_i)^* + \mathrm{cone}\,(\mathrm{epi} g^*) + \mathrm{epi}\delta_C^*.$$

Consequently, by Theorem 8.3, $\bar{x}$ is a weakly ε-efficient solution of (CVP2).

Taking $\varepsilon_i = 0,\ i = 1, \cdots, p$ in Theorem 8.3 we get the following result for weakly efficient solutions of (CSIVP).

Theorem 8.4 Optimality Conditions for Weakly Efficient Solutions for (CSIVP). *Let $\bar{x} \in F$ and assume that* $\mathrm{cone}\left(\bigcup_{t \in T} \mathrm{epi} g_t^*\right) + \mathrm{epi}\delta_C^*$ *is weak*-closed. Then the following statements are equivalent.*

(a) $\bar{x}$ is a weakly efficient solution of (CSIVP).

(b) There exist $\mu_i \geq 0,\ i = 1, \cdots p,\ \sum_{i=1}^p \mu_i = 1$, such that

$$\sum_{i=1}^p (0, -f_i(\bar{x})) \in \sum_{i=1}^p \mathrm{epi}(\mu_i f_i)^* + cone \bigcup_{t \in T} \mathrm{epi} g_t^* + \mathrm{epi}\delta_C^*.$$

(c) There exist $\mu_i \geq 0,\ i = 1, \cdots p,\ \sum_{i=1}^{p} \mu_i = 1$, *and* $(\lambda_t) \in \mathbb{R}_+^{(T)}$ *such that*

$$0 \in \sum_{i=1}^{p} \mu_i \partial f_i(\overline{x}) + \sum_{t \in T} \lambda_t \partial g_t(\overline{x}) + N_C(\overline{x})$$

and

$$\lambda_t g_t(\overline{x}) = 0,\ t \in T.$$

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Chapter 9
Linear Fractional and Convex Quadratic Vector Optimization Problems

Nguyen Dong Yen

9.1 Introduction

Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex set, $\varphi = (\varphi_1, \ldots, \varphi_m) : \Omega \to \mathbb{R}^m$ a continuously differentiable function defined on an open set $\Omega \subseteq \mathbb{R}^n$ which contains K as a subset. The *standard vector optimization problem* given by the *constraint set* K and the vector *objective function* φ is written formally as follows:

$$\text{(VP)} \qquad \text{Minimize } \varphi(x) \quad \text{subject to} \quad x \in K.$$

As usual, we denote by $\mathbb{R}^m_+$ the nonnegative orthant in $\mathbb{R}^m$ and by $\mathrm{int}\,\mathbb{R}^m_+$ the interior of that orthant. A point $\bar{x} \in K$ is said to be a *Pareto solution* (or an *efficient solution*) of (VP) if $(\varphi(K) - \varphi(\bar{x})) \cap \left(-\mathbb{R}^m_+ \setminus \{\mathbf{0}\}\right) = \emptyset$. If $\bar{x} \in K$ satisfies the condition $(\varphi(K) - \varphi(\bar{x})) \cap \left(-\mathrm{int}\,\mathbb{R}^m_+\right) = \emptyset$, then one says that $\bar{x}$ is a *weak Pareto solution* (or a *weakly efficient solution*) of (VP).

The following first-order necessary and sufficient optimality conditions are well known. A proof can be found, for instance, in [16, Theorem 3.1].

Theorem 9.1. *Let $\bar{x} \in K$. The following assertions hold:*

(a) If $\bar{x}$ is a weak Pareto solution of (VP)*, then there exists $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ such that*

$$\left\langle \sum_{i=1}^{m} \xi_i \nabla \varphi_i(\bar{x}), x - \bar{x} \right\rangle \geq 0 \quad \text{for every } x \in K. \tag{9.1}$$

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(b) *If the restriction of each component of φ on K is a convex function and if there exists $\xi \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ such that* (9.1) *is valid, then $\bar{x}$ is a weak Pareto solution of* (VP).

(c) *If the restriction of each component of φ on K is a convex function and if there exists $\xi \in \operatorname{int}\mathbb{R}^m_+ \setminus \{\mathbf{0}\}$ such that* (9.1) *is valid, then $\bar{x}$ is a Pareto solution of* (VP).

Let us denote the Pareto solution set and the weak Pareto solution set of (VP) respectively by $\mathrm{Sol(VP)}$ and $\mathrm{Sol}^w\mathrm{(VP)}$. Of course, $\mathrm{Sol(VP)} \subseteq \mathrm{Sol}^w\mathrm{(VP)}$. Concerning the vector optimization problem (VP), the next questions are of importance:

1. When the set $\mathrm{Sol(VP)}$ $\big($resp., $\mathrm{Sol}^w\mathrm{(VP)}\big)$ is nonempty? [Solution existence.]
2. Under which circumstances, the set $\mathrm{Sol(VP)}$ $\big($resp., $\mathrm{Sol}^w\mathrm{(VP)}\big)$ is bounded/connected/path-connected/contractible? [Properties of the solution sets.]
3. Is the set $\mathrm{Sol(VP)}$ $\big($resp., $\mathrm{Sol}^w\mathrm{(VP)}\big)$ stable, in a sense to be made precise, when the problem data K and φ undergo small perturbations? [Solution stability.]
4. How to find at least one element of $\mathrm{Sol(VP)}$ $\big($resp., $\mathrm{Sol}^w\mathrm{(VP)}\big)$? How to find the whole set $\mathrm{Sol(VP)}$ (resp., $\mathrm{Sol}^w\mathrm{(VP)}$)? [Solution methods.]

Answers to some questions from those listed above can be given in broad settings. For example, from the solution existence theorems in [21] it follows that: *If there exists an element $u \in K$ such that the set $K_\varphi(u) := \{x \in K : \varphi(x) \le \varphi(u)\}$ is bounded, then* $\mathrm{Sol(VP)}$ *is nonempty.* (Since $K_\varphi(u)$ is a level set of the restriction of φ on K and $K_\varphi(u)$ is closed by our assumptions, we may term the last fact "Solution existence under the level sets compactness condition".) But, it is worthy to stress that *specific properties of the solution sets, solution stability, and effective solution methods are available just for special classes of problems*. Being unable to address adequately the vast related literature, we confine ourselves to a few remarks as follows: (a) Basic results on linear vector optimization problems can be found in [21]; (b) Interesting results on linear fractional vector optimization problems were obtained in [4, 5, 28]; (c) Solution properties of strictly quasiconvex vector minimization problems have been studied intensively in the last three decades [1, 2, 6, 10, 14, 27, 29]; (d) Detailed investigations on strongly convex vector optimization problems were done in [16, 31]; (e) Several results on piecewise linear vector optimization problems have been established recently in [30, 34].

This chapter surveys some existing results on solution stability and connectedness of the solution sets of linear fractional vector optimization problems and of convex quadratic vector optimization problems. Our main concern is the situation where the constraint set is unbounded. Note that linear fractional vector optimization problems and convex quadratic vector optimization problems are two important classes of vector optimization problems. Although both the classes contain linear vector optimization problems as an important subclass, they have different features. For instance, the first class contains many nonconvex vector optimization problems, while the second one is composed entirely by convex vector optimization problems. Despite to this, one can treat linear fractional vector optimization problems and convex quadratic vector optimization problems by a single tool: *monotone affine* *vector variational inequality*. In other words, the concept of *vector variational inequality*, which is rooted in Giannessi's well-known paper [7] (see also the book [8]), provides us with an unified approach to studying (VP) when K is a polyhedral convex set, i.e., K is the intersection of finitely many closed half-spaces of $\mathbb{R}^n$, and φ has one of the following forms:

(i) [**Linear fractional vector functions**] $\varphi = (\varphi_1, \ldots, \varphi_m)$ where

$$\varphi_i(x) = \frac{a_i^\top x + \alpha_i}{b_i^\top x + \beta_i} \qquad (i = 1, \ldots, m),$$

for some $a_i \in \mathbb{R}^n, b_i \in \mathbb{R}^n, \alpha_i \in \mathbb{R}$, and $\beta_i \in \mathbb{R}$, provided that $b_i^\top x + \beta_i > 0$ for all $i \in \{1, \cdots, m\}$ and $x \in K$. (Here and in the sequel, the superscript $^\top$ denotes the matrix transposition.)

(ii) [**Convex quadratic vector functions**] $\varphi = (\varphi_1, \ldots, \varphi_m)$ where

$$\varphi_i(x) = \frac{1}{2} x^\top M_i x + q_i^\top x \qquad (i = 1, \ldots, m),$$

with $M_1, \ldots, M_m$ being symmetric positive semidefinite $n \times n$ matrices, $q_1, \ldots, q_m \in \mathbb{R}^n$.

If φ is of type (i), then putting

$$\Omega = \{x \in \mathbb{R}^n \, : \, b_i^\top x + \beta_i > 0, \ \forall i = 1, \ldots, m\}$$

we see that φ is a continuously differentiable function defined on Ω. The corresponding problem (VP) is denoted by (VP_1). In the special case where $b_i = 0$ and $\beta_i = 1$ for all $i = 1, \ldots, m$, (VP_1) is a linear vector optimization problem.

If φ is of type (ii), then the corresponding problem (VP) is denoted by (VP_2). In the special case where $M_i = [0]$ for all $i = 1, \ldots, m$, (VP_2) is a linear vector optimization problem.

A topological space X is said to be *connected* if one cannot represent $X = U \cup V$ where U, V are nonempty open sets of X with $U \cap V = \emptyset$. If for any pair $x, y \in X$ there is a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, then X is said to be *path-connected*. One says that X is *contractible* if there exist $x_* \in X$ and a continuous map $H : [0,1] \times X \to X$ such that $H(0,x) = x$ and $H(1,x) = x_*$ for all $x \in X$. Clearly, contractibility implies path-connectedness, which yields connectedness.

A nonempty subset $A \subseteq X$ of a topological space X is said to be a *connected component* (or a *component*) of X if A (equipped with the induced topology) is connected and it is not a proper subset of any connected subset of X. Every connected component of X is a closed subset. The (cardinal) *number of connected components* of X is denoted by $\chi(X)$.

A multifunction $G : X \rightrightarrows Y$ between two topological spaces is said to be *upper semicontinuous* (usc for brevity) at $u \in X$ if for any open set $V \subseteq Y$ with $G(u) \subseteq V$ there exists a neighborhood U of u such that $G(u') \subseteq V$ for all $u' \in U$. If $G(u) \neq \emptyset$

and for any open set $V \subseteq Y$ with $G(u) \cap V \neq \emptyset$ there exists a neighborhood U of u such that $G(u') \cap V \neq \emptyset$ for all $u' \in U$, then G is said to be *lower semicontinuous* (lsc for brevity) at $u \in X$.

The rest of this chapter has four sections. The next one presents several results on solution stability and connectedness of the solution sets of monotone affine variational inequalities. The subsequent two sections show how these results can be used for studying solution stability and connectedness of the solution sets of (VP_1) and of (VP_2). The final section states some open problems which are worthy further investigations.

9.2 Monotone Affine Variational Inequalities

Introduced by Giannessi [7] in 1980, the concept of vector variational inequality (VVI for brevity) has generated a strong stream of related research works; see e.g. [8]. VVI is one of the most important types of vector equilibrium problems and it can serve as an adequate tool for studying vector optimization problems (see for instance [16, 31, 32]).

We denote the inner product and the norm of an Euclidean space respectively by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. It is customary to represent vectors in an Euclidean space as rows of real numbers but interpret them as column vectors when the vectors participate in a matrix computation. The norm of a matrix $M \in \mathbb{R}^{n \times r}$ is given by the formula $\|M\| = \max\{\|Mx\| : x \in \mathbb{R}^n, \|x\| \leq 1\}$. Given a nonempty closed convex set $K \subseteq \mathbb{R}^n$ and vector-valued functions $F_i : K \to \mathbb{R}^n$ $(i = 1, \dots, m)$, we put $F = (F_1, \dots, F_m)$ and

$$F(x)(u) = (\langle F_1(x), u \rangle, \dots, \langle F_m(x), u \rangle), \quad \forall x \in K, \ \forall u \in \mathbb{R}^n.$$

Let $\Sigma = \{\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m_+ : \sum_{i=1}^m \xi_i = 1\}$. The relative interior of Σ is described by the formula

$$\mathrm{ri}\Sigma = \Sigma \cap (\mathrm{int}\mathbb{R}^n_+) = \{\xi \in \Sigma : \xi_i > 0 \text{ for all } i = 1, \dots, m\}.$$

The vector variational inequality [7, p. 167] defined by F, K and the cone $\mathbb{R}^m_+$ is the problem:

$$\text{(VVI)} \qquad \text{Find } x \in K \text{ such that } F(x)(y - x) \not\leq_{\mathbb{R}^m_+ \setminus \{\mathbf{0}\}} \mathbf{0}, \quad \forall y \in K,$$

where the inequality means that $F(x)(x - y) \notin \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$. As in [3], to this problem we associate the following one:

$$\text{(VVI)}^w \qquad \text{Find } x \in K \text{ such that } F(x)(y - x) \not\leq_{\mathrm{int}\mathbb{R}^m_+} \mathbf{0}, \quad \forall y \in K.$$

with the inequality indicating that $F(x)(x-y) \notin \text{int}\,\mathbb{R}^m_+$. The solution sets of (VVI) and (VVI)^w are denoted respectively by $\text{Sol}(\text{VVI})$ and $\text{Sol}^w(\text{VVI})$. The elements of the first set (resp., second set) are said to be the *Pareto solutions* (resp., the *weak Pareto solutions*) of (VVI).

For $m = 1$, it holds $F = F_1 : K \to \mathbb{R}^n$, hence (VVI) and (VVI)^w coincide with the classical *variational inequality* problem [15, p. 13]:

$$\text{(VI)} \qquad \text{Find } x \in K \text{ such that } \langle F(x), y - x\rangle \geq 0, \quad \forall y \in K.$$

Denote the solution set of the latter by $\text{Sol}(\text{VI})$. If $\langle F(y) - F(x), y - x\rangle \geq 0$ for any pair $(x, y) \in K \times K$, then (VI) is said to be a *monotone variational inequality*. For each $\xi \in \Sigma$, consider the variational inequality

$$\text{(VI)}_\xi \qquad \text{Find } x \in K \text{ such that } \left\langle \sum_{i=1}^{m} \xi_i F_i(x), y - x \right\rangle \geq 0, \quad \forall y \in K,$$

and denote its solution set by $\text{Sol}(\text{VI})_\xi$. Taking the union of $\text{Sol}(\text{VI})_\xi$ on $\xi \in \text{ri}\Sigma$ (resp., on $\xi \in \Sigma$) we can find a part of $\text{Sol}(\text{VVI})$ (resp., the whole set $\text{Sol}^w(\text{VVI})$).

Theorem 9.2 ([16, 19]). *It holds*

$$\bigcup_{\xi \in \text{ri}\Sigma} \text{Sol}(\text{VI})_\xi \subseteq \text{Sol}(\text{VVI}) \subseteq \text{Sol}^w(\text{VVI}) = \bigcup_{\xi \in \Sigma} \text{Sol}(\text{VI})_\xi. \tag{9.2}$$

If K is a polyhedral convex set, then the first inclusion in (9.2) *holds as equality.*

Definition 9.1.

- Problem (VI) is said to be an *affine variational inequality* (or AVI) if K is a polyhedral convex set and $F(x) = Mx + q$ for all $x \in K$, where $M \in \mathbb{R}^{n\times n}$ is a square matrix and $q \in \mathbb{R}^n$. The problem and its solution set are denoted by $\text{AVI}(M, q, K)$ and $\text{Sol}(\text{AVI}(M, q, K))$, respectively.
- For problem (VI), if $\langle F(y) - F(x), y - x\rangle \geq 0$ for all $x, y \in K$, then one says that F is *monotone* on K and (VI) is a *monotone variational inequality*.

The reader is referred to [17] for a detailed information about AVIs, to [18, 20, 24, 25] and the references therein for stability results on AVIs with perturbed constraint sets and other related models. Note that if $M \in \mathbb{R}^{n\times n}$ is a positive semidefinite matrix, i.e. $\langle Mv, v\rangle \geq 0$ for all $v \in \mathbb{R}^n$, then the affine operator $F(x) = Mx + q$ (with $q \in \mathbb{R}^n$ being fixed) is monotone on K. The converse is true if $\text{int}K \neq \emptyset$. Note that the solution set of a monotone AVI is closed and convex.

Definition 9.2.

- Problem (VVI) is said to be an *affine vector variational inequality* (or AVVI) if K is a polyhedral convex set and there exist matrices $M_i \in \mathbb{R}^{n\times n}$ and vectors $q_i \in \mathbb{R}^n$ $(i = 1, \ldots, m)$ such that $F_i(x) = M_i x + q_i$ for all $i = 1, \ldots, m$

and $x \in K$. The problem and its solution set are denoted respectively by $\mathrm{AVVI}(\omega, K)$, Sol(AVVI(ω, K)) and Sol^w(AVVI(ω, K)), where the data set $\omega := (M_1, \ldots, M_m, q_1, \ldots, q_m) \in \mathbb{R}^{(n\times n)\times m} \times \mathbb{R}^{n\times m}$ is interpreted as a parameter.

- One says that (VVI) is a *monotone vector variational inequality* if the problems $\mathrm{VI}(F_i, K)$ $(i = 1, \ldots, m)$ are monotone.

The next solution stability theorem for monotone AVIs is a fundamental result.

Theorem 9.3. ([23, Theorem 2]) *Let $K \subseteq \mathbb{R}^n$ be a nonempty polyhedral convex set, $M \in \mathbb{R}^{n\times n}$ a positive semidefinite matrix, and $q \in \mathbb{R}^n$. The following two properties are equivalent:*

(a) The solution set $\mathrm{Sol}(\mathrm{AVI}(M, q, K))$ *is nonempty and bounded.*
(b) There exists $\varepsilon > 0$ such that for each $\widetilde{M} \in \mathbb{R}^{n\times n}$ and each $\widetilde{q} \in \mathbb{R}^n$ with

$$\max\{\|\widetilde{M} - M\|,\ \|\widetilde{q} - q\|\} < \varepsilon, \tag{9.3}$$

the set $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, K))$ *is nonempty.*

When (a) is valid, there exist constants $\varepsilon > 0$, $\rho > 0$ and $\ell > 0$ such that if $(\widetilde{M}, \widetilde{q}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n$, $\widetilde{M}$ is positive semidefinite, and (9.3) is valid, then the set $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, K))$ *is nonempty,*

$$\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, K)) \subseteq \overline{B}(0, \rho),$$

and

$$\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, K)) \subseteq \mathrm{Sol}(\mathrm{AVI}(M, q, K)) + \ell(\|\widetilde{M} - M\| + \|\widetilde{q} - q\|)\overline{B}(0, 1).$$

Here $\overline{B}(u, \delta)$ denotes the closed ball centered at u with radius δ.

A new proof of Theorem 9.3 can be found in [17, Chap. 7]. The following result, which is an analogue (in a relaxed form) of Theorem 9.3 for the weak Pareto solution sets of monotone AVVIs, has been obtained recently [33]. For the convenience of the reader, the proof of [33] is recalled herein.

Theorem 9.4. ([33, Theorem 3.1]) *Suppose that $K \subseteq \mathbb{R}^n$ is a nonempty polyhedral convex set, $M_1, \ldots, M_m \in \mathbb{R}^{n\times n}$ are positive semidefinite matrices, and $q_1, \ldots, q_m \in \mathbb{R}^n$. Let*

$$\omega := (M_1, \ldots, M_m, q_1, \ldots, q_m) \in \mathbb{R}^{(n\times n)\times m} \times \mathbb{R}^{n\times m}.$$

Consider the properties:

(a) The solution set $\mathrm{Sol}^w(\mathrm{AVVI}(\omega, K))$ *is nonempty and bounded.*
(b) There exists $\varepsilon > 0$ such that for all $\widetilde{M}_1, \ldots, \widetilde{M}_m \in \mathbb{R}^{n\times n}$ and $\widetilde{q}_1, \ldots, \widetilde{q}_m \in \mathbb{R}^n$ with

$$\max_{i\in\{1,\ldots,m\}} \max\{\|\widetilde{M}_i - M_i\|,\ \|\widetilde{q}_i - q_i\|\} < \varepsilon, \tag{9.4}$$

the set $\mathrm{Sol}^w(\mathrm{AVVI}(\widetilde{\omega},K))$*, where* $\widetilde{\omega} := (\widetilde{M}_1,\ldots,\widetilde{M}_m,\widetilde{q}_1,\ldots,\widetilde{q}_m)$*, is nonempty.*

One has (a) $\Rightarrow$ *(b). The reverse implication is true if either* K *is compact or* $m = 1$ *but, in general, it does not hold if* K *is noncompact and* $m \geq 2$*.*

When (a) is valid, given any $\alpha > 0$ *there exist constants* $\varepsilon > 0,\ \rho > 0$ *such that if* $(\widetilde{M}_i,\widetilde{q}_i) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n$*,* $\widetilde{M}_i$ *is positive semidefinite for all* $i = 1,\ldots,m$*, and* (9.4) *is valid, then the set* $\mathrm{Sol}^w(\mathrm{AVVI}(\widetilde{\omega},K))$ *is nonempty,*

$$\mathrm{Sol}^w(\mathrm{AVVI}(\widetilde{\omega},K)) \subseteq \overline{B}(0,\rho), \tag{9.5}$$

and

$$\mathrm{Sol}^w(\mathrm{AVVI}(\widetilde{\omega},K)) \subseteq \mathrm{Sol}^w(\mathrm{AVVI}(\omega,K)) + \alpha B(0,1), \tag{9.6}$$

with $B(0,1)$ *denoting the open unit ball in* $\mathbb{R}^n$*. In particular, the solution map* $\mathrm{Sol}^w(\mathrm{AVVI}(\cdot,K))$ *is upper semicontinuous at* ω*.*

The proof of Theorem 9.4 is based on Theorem 9.3 and the forthcoming lemma.

Lemma 9.1. ([33, Lemma 3.2]) *Under the assumptions of Theorem 9.4, if (a) holds then*

$$\left\{\xi \in \Sigma \,:\, \mathrm{Sol(VI)}_\xi \neq \emptyset\right\} = \Sigma, \tag{9.7}$$

where in the formulation of the problem $\mathrm{(VI)}_\xi$ *one puts*

$$\sum_{i=1}^{m} \xi_i F_i(x) = \sum_{i=1}^{m} \xi_i (M_i x + q_i).$$

Proof. By (9.2),

$$\mathrm{Sol}^w(\mathrm{AVVI}(\omega,K)) = \bigcup_{\xi\in\Sigma} \mathrm{Sol(VI)}_\xi. \tag{9.8}$$

For every $\xi \in \Sigma$, we set

$$M(\xi) = \sum_{i=1}^{m} \xi_i M_i \quad \text{and} \quad q(\xi) = \sum_{i=1}^{m} \xi_i q_i, \tag{9.9}$$

and observe that $M(\xi) \in \mathbb{R}^{n\times n}$ is positive semidefinite by our assumptions. Since

$$\mathrm{Sol}(\mathrm{AVI}(M(\xi),q(\xi),K)) = \mathrm{Sol(VI)}_\xi,$$

the set on the left-hand side is bounded by (9.8) and the assumptions of the lemma. When the set is nonempty, by applying the first assertion of Theorem 9.3 to the triple $\{M(\xi),q(\xi),K\}$ we find a constant $\varepsilon(\xi) > 0$ such that if

$$\max\{\|\widetilde{M} - M(\xi)\|,\ \|\widetilde{q} - q(\xi)\|\} < \varepsilon(\xi) \tag{9.10}$$

then $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M},\widetilde{q},K)) \neq \emptyset$.

Denote the left-hand side of (9.7) by Ω. Property (a), which guarantees that $\mathrm{Sol}^w(\mathrm{AVVI}(\omega,K)) \neq \emptyset$, and (9.8) imply that $\Omega \neq \emptyset$. If we can show that Ω is both open and closed in the induced topology of Σ, then (9.7) follows from the connectedness of Σ.

To obtain the openness, fix any $\xi \in \Omega$ and choose $\delta > 0$ as small as

$$\delta\sqrt{m}\left(\max_{i\in\{1,\dots,m\}} \|M_i\|\right) < \varepsilon(\xi) \quad \text{and} \quad \delta\sqrt{m}\left(\max_{i\in\{1,\dots,m\}} \|q_i\|\right) < \varepsilon(\xi). \tag{9.11}$$

Then, for every $\xi' \in B(\xi,\delta)\cap\Sigma$, where $B(\xi,\delta)$ denotes the open ball centered at ξ with radius δ, we have

$$\begin{aligned}
\|M(\xi') - M(\xi)\| &= \|\sum_{i=1}^{m}(\xi'_i - \xi_i)M_i)\| \\
&\le \sum_{i=1}^{m}|\xi'_i - \xi_i|\|M_i\| \\
&\le \sqrt{m}\|\xi' - \xi\|\left(\max_{i\in\{1,\dots,m\}} \|M_i\|\right) \\
&\le \delta\sqrt{m}\left(\max_{i\in\{1,\dots,m\}} \|M_i\|\right) < \varepsilon(\xi).
\end{aligned}$$

Similarly, $\|q(\xi') - q(\xi)\| < \varepsilon(\xi)$. Thus the pair $(\widetilde{M},\widetilde{q}) := (M(\xi'),q(\xi'))$ satisfies (9.10). Hence

$$\mathrm{Sol}(\mathrm{AVI}(M(\xi'),q(\xi'),K)) \neq \emptyset.$$

We have shown that $B(\xi,\delta)\cap\Sigma \subseteq \Omega$.

Now, in order to get the desired closedness of Ω, we take any sequence $\{\xi^{(j)}\} \subseteq \Omega$ with $\lim_{j\to\infty} \xi^{(j)} = \overline{\xi} \in \Sigma$. Our aim is to prove that $\overline{\xi} \in \Omega$. For each $j \in \mathbb{N}$, select a solution $x^{(j)} \in \mathrm{Sol}(\mathrm{VI})_{\xi^{(j)}}$. As $\mathrm{Sol}(\mathrm{VI})_{\xi^{(j)}} \subseteq \mathrm{Sol}^w(\mathrm{AVVI}(\omega,K))$ for all j and the set on the right-hand side is bounded, we may assume that $\lim_{j\to\infty} x^{(j)} = \bar{x} \in K$. By our choice of $x^{(j)}$,

$$\left\langle M(\xi^{(j)})x^{(j)} + q(\xi^{(j)}), y - x^{(j)} \right\rangle \ge 0, \quad \forall y \in K,\ \forall j \in \mathbb{N}.$$

Passing the last inequality to the limit as $j \to \infty$, we get

$$\langle M(\overline{\xi})\bar{x} + q(\overline{\xi}), y - \bar{x}\rangle \ge 0, \quad \forall y \in K.$$

It follows that $\bar{x} \in \mathrm{Sol}(\mathrm{VI})_{\overline{\xi}}$; hence $\overline{\xi} \in \Omega$. □

Proof (Proof of Theorem 9.4). Suppose that (a) holds. For every $\xi \in \Sigma$, we define $M(\xi)$ and $q(\xi)$ by (9.9). Since $M(\xi)$ is positive semidefinite, the set

$\mathrm{Sol}(\mathrm{AVI}(M(\xi),q(\xi),K)) = \mathrm{Sol}(\mathrm{VI})_\xi$ is nonempty by (9.7) and bounded by (9.8). The first claim of Theorem 9.3 yields a constant $\varepsilon(\xi) > 0$ such that if (9.10) is fulfilled then $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M},\widetilde{q},K)) \neq \emptyset$. Take any $\varepsilon \in (0,\varepsilon(\xi))$. For every $\widetilde{M}_1,\dots,\widetilde{M}_m \in \mathbb{R}^{n\times n}$ and $\widetilde{q}_1,\dots,\widetilde{q}_m \in \mathbb{R}^n$ satisfying (9.4), we put

$$\widetilde{M}(\xi) = \sum_{i=1}^{m} \xi_i \widetilde{M}_i \quad \text{and} \quad \widetilde{q}(\xi) = \sum_{i=1}^{m} \xi_i \widetilde{q}_i \quad (\forall \xi \in \Sigma). \tag{9.12}$$

It is clear that

$$\begin{aligned} \|\widetilde{M}(\xi) - M(\xi)\| &= \|\sum_{i=1}^{m} \xi_i (\widetilde{M}_i - M_i)\| \\ &\le \sum_{i=1}^{m} \xi_i \|\widetilde{M}_i - M_i\| \\ &\le \varepsilon \sum_{i=1}^{m} \xi_i = \varepsilon < \varepsilon(\xi). \end{aligned}$$

Similarly, $\|\widetilde{q}(\xi) - q(\xi)\| < \varepsilon(\xi)$. Therefore, $(\widetilde{M},\widetilde{q}) := \left(\widetilde{M}(\xi),\widetilde{q}(\xi)\right)$ satisfies (9.10). It follows that

$$\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}(\xi),\widetilde{q}(\xi),K)) \neq \emptyset.$$

Hence, according to Theorem 9.2, the solution set $\mathrm{Sol}^w(\mathrm{AVVI}(\widetilde{\omega},K))$ is nonempty.

If K is compact and (b) is valid, by choosing $\widetilde{M}_i = M_i$ and $\widetilde{q}_i = q_i$ for all $i = 1,\dots,m$ we can assert that $\mathrm{Sol}^w(\mathrm{AVVI}(\omega,K))$ is nonempty and bounded, i.e., (a) holds. If $m = 1$, the implication (b) $\Rightarrow$ (a) follows from Theorem 9.3. Now, suppose that K is noncompact and $m \ge 2$. Let $M_1 \in \mathbb{R}^{n\times n}$ be any positive definite matrix (that is, $\langle M_1 v, v\rangle > 0$ for any $v \neq 0$), $q_1 \in \mathbb{R}^n$ be arbitrarily chosen. Let $M_i = [0]$ and $q_i = 0$ for $i = 2,\dots,m$. Setting $\xi^{(1)} = (1,0,\dots,0) \in \Sigma$ we observe that $(\mathrm{VI})_{\xi^{(1)}}$ is a strongly monotone variational inequality, hence the problem has a unique solution (see e.g. [15]). It follows from this fact and Theorems 9.2, 9.3, that (b) is valid. To see that the boundedness stated in (a) is not available here, it suffices to take $\xi^{(2)} = (0,1,\dots,0) \in \Sigma$ and observe that $\mathrm{Sol}(\mathrm{VI})_{\xi^{(2)}} = K$.

To prove the last assertion of the theorem, assume that (a) is satisfied. For each $\xi \in \Sigma$, let $M(\xi)$ and $q(\xi)$ be defined as in (9.9). Since $\mathrm{Sol}(\mathrm{AVI}(M(\xi),q(\xi),K))$ is nonempty and bounded by our assumption and Lemma 9.1, applying Theorem 9.3 we find $\varepsilon(\xi) > 0$, $\rho(\xi) > 0$ and $\ell(\xi) > 0$ such that if $(\widetilde{M},\widetilde{q}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n$, $\widetilde{M}$ is positive semidefinite, (9.10) is fulfilled, then $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M},\widetilde{q},K)) \neq \emptyset$,

$$\mathrm{Sol}(\mathrm{AVI}(\widetilde{M},\widetilde{q},K)) \subseteq \overline{B}(0,\rho(\xi)), \tag{9.13}$$

and

$$\mathrm{Sol}(\mathrm{AVI}(\widetilde{M},\widetilde{q},K)) \subseteq \mathrm{Sol}(\mathrm{AVI}(M(\xi),q(\xi),K))$$

$$+\ell(\xi)(\|\widetilde{M}-M(\xi)\|+\|\widetilde{q}-q(\xi)\|)\overline{B}(0,1). \qquad (9.14)$$

By reducing $\varepsilon(\xi)>0$, if necessary, we can assume that $2\varepsilon(\xi)\ell(\xi)<\alpha$. Choose $\gamma(\xi)>0$ as small as

$$2\gamma(\xi)\sqrt{m}\left(\max_{i\in\{1,\dots,m\}}\|M_i\|\right)<\varepsilon(\xi) \quad \text{and} \quad 2\gamma(\xi)\sqrt{m}\left(\max_{i\in\{1,\dots,m\}}\|q_i\|\right)<\varepsilon(\xi). \qquad (9.15)$$

As Σ is compact, there exist $\xi^1,\dots,\xi^k\in\Sigma$ such that

$$\Sigma\subseteq B\left(\xi^1,\gamma(\xi^1)\right)\cup\cdots\cup B\left(\xi^k,\gamma(\xi^k)\right).$$

Set

$$\varepsilon=2^{-1}\min_{j\in\{1,\dots,k\}}\varepsilon(\xi^j), \quad \rho=\max_{j\in\{1,\dots,k\}}\rho(\xi^j).$$

Given any positive semidefinite matrices $\widetilde{M}_1,\dots,\widetilde{M}_m\in\mathbb{R}^{n\times n}$ and vectors $\widetilde{q}_1,\dots,\widetilde{q}_m\in\mathbb{R}^n$ satisfying (9.4), we want to check (9.5) and (9.6). According to Theorem 9.2,

$$\mathrm{Sol}^w(\mathrm{AVVI}(\widetilde{\omega},K))=\bigcup_{\xi\in\Sigma}\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}(\xi),\widetilde{q}(\xi),K)), \qquad (9.16)$$

where $\widetilde{M}(\xi)$ and $\widetilde{q}(\xi)$ are defined by (9.12). For every $\xi\in\Sigma$, there exists an index $j\in\{1,\dots,k\}$ such that $\xi\in B\left(\xi^j,\gamma(\xi^j)\right)$. Taking account of (9.15), we have

$$\begin{aligned}\|\widetilde{M}(\xi)-M(\xi^j)\| &= \|\sum_{i=1}^m\xi_i(\widetilde{M}_i-M_i)+\sum_{i=1}^m(\xi_i-\xi_i^j)M_i\| \\ &\le \sum_{i=1}^m\xi_i\|\widetilde{M}_i-M_i\|+\sum_{i=1}^m|\xi_i-\xi_i^j|\|M_i\| \\ &\le \varepsilon\sum_{i=1}^m\xi_i+\|\xi-\xi^j\|\sqrt{m}\left(\max_{i\in\{1,\dots,m\}}\|M_i\|\right) \\ &\le \varepsilon+\gamma(\xi^j)\sqrt{m}\left(\max_{i\in\{1,\dots,m\}}\|M_i\|\right) \\ &< \varepsilon+2^{-1}\varepsilon(\xi^j)\le\varepsilon(\xi^j)\end{aligned}$$

and, similarly, $\|\widetilde{q}(\xi)-q(\xi^j)\|<\varepsilon(\xi^j)$. Hence $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}(\xi),\widetilde{q}(\xi),K))\neq\emptyset$,

$$\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}(\xi),\widetilde{q}(\xi),K))\subseteq\overline{B}(0,\rho(\xi^j))\subseteq\overline{B}(0,\rho),$$

and

$$\begin{aligned}\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}(\xi),\widetilde{q}(\xi),K)) &\subseteq \mathrm{Sol}(\mathrm{AVI}(M(\xi^j),q(\xi^j),K))\\ &\quad +\ell(\xi^j)(\|\widetilde{M}(\xi)-M(\xi^j)\|+\|\widetilde{q}(\xi)-q(\xi^j)\|)\overline{B}(0,1)\\ &\subseteq \mathrm{Sol}^w(\mathrm{AVVI}(\omega,K))+2\ell(\xi^j)\varepsilon(\xi^j)\overline{B}(0,1).\\ &\subseteq \mathrm{Sol}^w(\mathrm{AVVI}(\omega,K))+\alpha B(0,1).\end{aligned}$$

The last estimates are valid for each $\xi \in \Sigma$. Combining this with (9.16), we obtain (9.5) and (9.6). By the compactness of $\mathrm{Sol}^w(\mathrm{AVVI}(\omega,K))$, for any open set $V \subseteq \mathbb{R}^n$ containing $\mathrm{Sol}^w(\mathrm{AVVI}(\omega,K))$, there is $\alpha > 0$ such that

$$\mathrm{Sol}^w(\mathrm{AVVI}(\omega,K))+\alpha B(0,1) \subseteq V.$$

Hence, applying the result obtained just now, we find a neighborhood U of ω in $\mathbb{R}^{n\times(n\times m)} \times \mathbb{R}^{n\times m}$ such that for any $\widetilde{\omega} = (\widetilde{M}_1,\ldots,\widetilde{M}_m,\widetilde{q}_1,\ldots,\widetilde{q}_m) \in U$ with $\widetilde{M}_i$ being positive semidefinite for $i = 1,\ldots,m$, the inclusion $\mathrm{Sol}^w(\mathrm{AVVI}(\widetilde{\omega},K)) \subseteq V$ holds. This establishes the desired usc property and completes the proof. □

The solution map $\mathrm{Sol}(\mathrm{AVVI}(\cdot,K))$ is "less stable" than $\mathrm{Sol}^w(\mathrm{AVVI}(\cdot,K))$. Namely, we have the following result.

Theorem 9.5. **([33, Theorem 3.3])** *Let K, M_i, q_i and ω be as in Theorem 9.4. Consider the properties:*

(a') The solution set $\mathrm{Sol}(\mathrm{AVVI}(\omega,K))$ is nonempty and bounded.
(b') There exists $\varepsilon > 0$ such that for all $\widetilde{M}_1,\ldots,\widetilde{M}_m \in \mathbb{R}^{n\times n}$ and $\widetilde{q}_1,\ldots,\widetilde{q}_m \in \mathbb{R}^n$ satisfying (9.4) the set $\mathrm{Sol}(\mathrm{AVVI}(\widetilde{\omega},K))$, where $\widetilde{\omega} := (\widetilde{M}_1,\ldots,\widetilde{M}_m,\widetilde{q}_1,\ldots,\widetilde{q}_m)$, is nonempty.

One has (a') ⇒ (b'). The reverse implication is true if either K is compact or $m = 1$ but, in general, it does not hold if K is noncompact and $m \geq 2$.

If the weak Pareto solution set $\mathrm{Sol}^w(\mathrm{AVVI}(\omega,K))$ is nonempty and bounded, then for any $\alpha > 0$ there exist constants $\varepsilon > 0$, $\rho > 0$ such that if $(\widetilde{M}_i,\widetilde{q}_i) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n$, $\widetilde{M}_i$ is positive semidefinite for all $i = 1,\ldots,m$, and (9.4) is valid, the set $\mathrm{Sol}(\mathrm{AVVI}(\widetilde{\omega},K))$ is nonempty,

$$\mathrm{Sol}(\mathrm{AVVI}(\widetilde{\omega},K)) \subseteq \overline{B}(0,\rho),$$

and

$$\mathrm{Sol}(\mathrm{AVVI}(\widetilde{\omega},K)) \subseteq \mathrm{Sol}^w(\mathrm{AVVI}(\omega,K))+\alpha B(0,1).$$

The next analogue of Lemma 9.1 is useful for the proof of Theorem 9.5. The reader is referred to [33] for more details.

Lemma 9.2. **([33, Lemma 3.4])** *Under the assumptions of Theorem 9.5, if (a') holds then*

$$\left\{\xi \in \mathrm{ri}\Sigma \,:\, \mathrm{Sol}(\mathrm{VI})_\xi \neq \emptyset\right\} = \mathrm{ri}\Sigma,$$

where in the formulation of the problem $(\mathrm{VI})_\xi$ one puts

$$\sum_{i=1}^{m} \xi_i F_i(x) = \sum_{i=1}^{m} \xi_i (M_i x + q_i).$$

Besides, the equality (9.7) *is also valid.*

Connectedness of the weak Pareto solution set of a monotone AVVI is discussed in the forthcoming theorem. For the convenience of the reader, the proof of [33] is reproduced below.

Theorem 9.6. ([33, Theorem 4.1]) *Suppose that $K \subseteq \mathbb{R}^n$ is a nonempty polyhedral convex set, $M_1, \ldots, M_m \in \mathbb{R}^{n\times n}$ are positive semidefinite matrices, and $q_1, \ldots, q_m \in \mathbb{R}^n$. Let*

$$\omega = (M_1, \ldots, M_m, q_1, \ldots, q_m).$$

The following assertions are valid:

(a) If $\mathrm{Sol}^w(\mathrm{AVVI}(\omega, K))$ is bounded, then it is connected.
(b) If $\mathrm{Sol}^w(\mathrm{AVVI}(\omega, K))$ is disconnected, then each connected component of the solution set is unbounded.

Proof. Clearly, the second assertion implies the first one. To prove (b), suppose on the contrary that $\mathrm{Sol}^w(\mathrm{AVVI}(\omega, K))$ is disconnected, but it has a bounded connected component A. Since $A \neq \mathrm{Sol}^w(\mathrm{AVVI}(\omega, K))$, by (9.2) one has

$$\{\xi \in \Sigma : \mathrm{Sol(VI)}_\xi \neq \emptyset,\ \mathrm{Sol(VI)}_\xi \subseteq A\} \neq \Sigma.$$

Denote the left-hand side of the last inequality by Σ_0. As $\mathrm{Sol(VI)}_\xi$ is a convex set and A is a connected component of $\mathrm{Sol}^w(\mathrm{AVVI}(\omega, K))$, $\mathrm{Sol(VI)}_\xi \subseteq A$ whenever $\mathrm{Sol(VI)}_\xi \cap A \neq \emptyset$. Thus

$$\Sigma_0 = \{\xi \in \Sigma : \mathrm{Sol(VI)}_\xi \cap A \neq \emptyset\}.$$

It follows that $\Sigma_0 \neq \emptyset$. If we could show that Σ_0 is both open and closed in the induced topology of Σ then, due to the connectedness of the latter set, $\Sigma_0 = \Sigma$. This would contradict the inequality $\Sigma_0 \neq \Sigma$ and complete the proof.

To show that Σ_0 is open in the induced topology of Σ, we fix a point $\xi \in \Sigma_0$. Let $M(\xi)$ and $q(\xi)$ be as in (9.9). Using Theorem 9.3 for the triple $\{M(\xi), q(\xi), K\}$ we find constants $\varepsilon(\xi) > 0$, $\rho(\xi) > 0$ and $\ell(\xi) > 0$ such that if $(\widetilde{M}, \widetilde{q}) \in \mathbb{R}^{n\times n} \times \mathbb{R}^n$, $\widetilde{M}$ is positive semidefinite, (9.10) is fulfilled, then $\mathrm{Sol}(\mathrm{AVI}(\widetilde{M}, \widetilde{q}, K)) \neq \emptyset$ and the inclusions (9.13), (9.14) hold. Choose $\delta > 0$ satisfying (9.11). As in the proof of Lemma 9.2, for every $\xi' \in B(\xi, \delta) \cap \Sigma$, $(\widetilde{M}, \widetilde{q}) := (M(\xi'), q(\xi'))$ satisfies the condition (9.10). Hence

$$\mathrm{Sol}(\mathrm{AVI}(M(\xi'), q(\xi'), K)) \neq \emptyset.$$

Moreover, by (9.13) and (9.14) we have

$$\mathrm{Sol}(\mathrm{AVI}(M(\xi'), q(\xi'), K)) \subseteq \overline{B}(0, \rho(\xi)),$$

and

$$\begin{aligned}\mathrm{Sol}(\mathrm{AVI}(M(\xi'),q(\xi'),K)) \subseteq\ & \mathrm{Sol}(\mathrm{AVI}(M(\xi),q(\xi),K)) \\ & +\ell(\xi)(\|M(\xi')-M(\xi)\|+\|q(\xi')-q(\xi)\|)\overline{B}(0,1).\end{aligned}$$

This means that the multifunction

$$\mathrm{Sol}(\mathrm{AVI}(M(\cdot),q(\cdot),K)) : B(\xi,\delta)\cap\Sigma \rightrightarrows \mathbb{R}^n$$

is uniformly bounded on $B(\xi,\delta)\cap\Sigma$ and upper-Lipschitz at ξ. In particular, the multifunction is usc at ξ. It is not difficult to show that $\mathrm{Sol}(\mathrm{AVI}(M(\cdot),q(\cdot),K))$ is usc at every point of $B(\xi,\delta)\cap\Sigma$. Since $\mathrm{Sol}(\mathrm{AVI}(M(\xi'),q(\xi'),K))$ is a nonempty convex set for each $\xi'\in B(\xi,\delta)\cap\Sigma$, the usc property of $\mathrm{Sol}(\mathrm{AVI}(M(\cdot),q(\cdot),K))$ implies (see [29]) that the image set

$$W := \bigcup\left\{\mathrm{Sol}(\mathrm{AVI}(M(\xi'),q(\xi'),K))\ :\ \xi'\in B(\xi,\delta)\cap\Sigma\right\}$$

is connected. As $\mathrm{Sol}(\mathrm{VI})_{\xi}=\mathrm{Sol}(\mathrm{AVI}(M(\xi),q(\xi),K))$ has a nonempty intersection with A by the choice of ξ, and A is a connected component of $\mathrm{Sol}^w(\mathrm{AVVI}(\omega,K))$, we can assert that $W\subseteq A$. Hence $B(\xi,\delta)\cap\Sigma\subseteq\Sigma_0$.

It remains to prove that Σ_0 is closed in the induced topology of Σ. Take any sequence $\{\xi^{(j)}\}\subseteq\Sigma_0$ with $\lim\limits_{j\to\infty}\xi^{(j)}=\overline{\xi}\in\Sigma$. We are going to show that $\overline{\xi}\in\Sigma_0$. For each $j\in\mathbb{N}$, select a solution $x^{(j)}\in\mathrm{Sol}(\mathrm{VI})_{\xi^{(j)}}\subseteq A$. There is no loss of generality in assuming that $\lim\limits_{j\to\infty}x^{(j)}=\bar{x}\in K$. Since

$$\left\langle M(\xi^{(j)})x^{(j)}+q(\xi^{(j)}),y-x^{(j)}\right\rangle\geq 0,\quad \forall y\in K,\ \forall j\in\mathbb{N},$$

we get

$$\langle M(\overline{\xi})\bar{x}+q(\overline{\xi}),y-\bar{x}\rangle\geq 0,\quad \forall y\in K.$$

It follows that $\bar{x}\in\mathrm{Sol}(\mathrm{VI})_{\overline{\xi}}\subseteq\mathrm{Sol}^w(\mathrm{AVVI}(\omega,K))$. Because $x^{(j)}\in A$ for all $j\in\mathbb{N}$ and A is a closed subset of $\mathrm{Sol}^w(\mathrm{AVVI}(\omega,K))$, we have $\bar{x}\in A$; hence $\overline{\xi}\in\Sigma_0$. □

The Pareto solution sets of monotone AVVIs enjoy some connectedness properties similar to those just obtained for the weak Pareto solution sets.

Theorem 9.7. ([33, Theorem 4.2]) *Under the assumptions of Theorem 9.6, the following assertions hold:*

(a') If $\mathrm{Sol}(\mathrm{AVVI}(\omega,K))$ *is bounded, then it is connected.*
(b') If $\mathrm{Sol}(\mathrm{AVVI}(\omega,K))$ *is disconnected, then each connected component of the solution set is unbounded.*

Proof. One can proceed similarly as in proving Theorem 9.6. The only change is that instead of $\xi \in \Sigma$ one considers $\xi \in \mathrm{ri}\Sigma$. □

9.3 Linear Fractional Vector Optimization Problems

We now present some basic information about the *linear fractional vector optimization problem* (or LFVOP). More details can be found in [28, 32] and [17, Chap. 8].

Let $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \ldots, m)$ be m *linear fractional functions*, that is

$$\varphi_i(x) = \frac{a_i^\top x + \alpha_i}{b_i^\top x + \beta_i}$$

for some $a_i \in \mathbb{R}^n, b_i \in \mathbb{R}^n, \alpha_i \in \mathbb{R}$, and $\beta_i \in \mathbb{R}$. Suppose that $b_i^\top x + \beta_i > 0$ for all $i \in \{1, \ldots, m\}$ and $x \in K$. Put $\varphi(x) = (\varphi_1(x), \ldots, \varphi_m(x))$ and consider the vector optimization problem

$$(\mathrm{VP}_1) \qquad \text{Minimize} \quad \varphi(\mathrm{x}) \text{ subject to } \quad \mathrm{x} \in \mathrm{K}.$$

As observed in the first section, for $\Omega := \{x \in \mathbb{R}^n : b_i^\top x + \beta_i > 0, \ \forall i = 1, \ldots, m\}$, φ is continuously differentiable on Ω. Thus (VP_1) is a special case of (VP). Since following necessary and sufficient optimality conditions for (VP_1) cannot be obtained as corollaries of Theorem 9.1 stating optimality conditions for the general problem (VP), a detailed proof is provided here for the clarity of our presentation.

Theorem 9.8 ([4, 22]). *Let $x \in K$. The following assertions hold:*

(a) $x \in \mathrm{Sol}(\mathrm{VP}_1)$ *if and only if there exists* $\xi = (\xi_1, \ldots, \xi_m) \in \mathrm{ri}\Sigma$ *such that*

$$\left\langle \sum_{i=1}^{m} \xi_i \left[\left(b_i^\top x + \beta_i\right) a_i - \left(a_i^\top x + \alpha_i\right) b_i \right], y - x \right\rangle \geq 0, \quad \forall y \in K. \tag{9.17}$$

(b) $x \in \mathrm{Sol}^w(\mathrm{VP}_1)$ *if and only if there exists* $\xi = (\xi_1, \ldots, \xi_m) \in \Sigma$ *such that* (9.17) *holds.*

(c) If $K = \{x \in \mathbb{R}^n : Cx \leq d\}$ *with C being an $r \times n$ matrix, d an $r-$dimensional column vector, then condition* (9.17) *is satisfied if and only if there exists* $\mu = (\mu_1, \ldots, \mu_r)$, $\mu_j \geq 0$ *for all* $j = 1, \ldots, r$, *such that*

$$\sum_{i=1}^{m} \xi_i \left[\left(b_i^\top x + \beta_i\right) a_i - \left(a_i^\top x + \alpha_i\right) b_i \right] + \sum_{j \in J(x)} \mu_j C_j^\top = 0, \tag{9.18}$$

where C_j denotes the j-th row of the matrix C and $J(x) = \{j : C_j x = d_j\}$ denotes the active index set of x.

Lemma 9.3. *[22] Let $\psi(x) = (a^\top x + \alpha)/(b^\top x + \beta)$ be a linear fractional function. Suppose that $b^\top x + \beta \neq 0$ for every $x \in K$. Then for any $x, y \in K$, it holds*

$$\psi(y) - \psi(x) = \frac{b^\top x + \beta}{b^\top y + \beta} \langle \nabla \psi(x), y - x \rangle, \tag{9.19}$$

where $\nabla \psi(x)$ denotes the gradient of ψ at x.

Proof. By the definition of gradient,

$$\begin{aligned} &\langle \nabla \psi(x), y - x \rangle \\ &= \lim_{t \downarrow 0} \frac{1}{t} \left[\psi(x + t(y-x)) - \psi(x) \right] \\ &= \lim_{t \downarrow 0} \frac{1}{t} \left[\frac{a^\top (x + t(y-x)) + \alpha}{b^\top (x + t(y-x)) + \beta} - \frac{a^\top x + \alpha}{b^\top x + \beta} \right] \\ &= \frac{a^\top (y-x)(b^\top x + \beta) - b^\top (y-x)(a^\top x + \alpha)}{(b^\top x + \beta)^2}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} &\frac{b^\top x + \beta}{b^\top y + \beta} \langle \nabla \psi(x), y - x \rangle \\ &= \frac{a^\top (y-x)(b^\top x + \beta) - b^\top (y-x)(a^\top x + \alpha)}{(b^\top y + \beta)(b^\top x + \beta)} \\ &= \frac{(b^\top x + \beta)(a^\top y + \alpha) - (a^\top x + \alpha)(b^\top y + \beta)}{(b^\top y + \beta)(b^\top x + \beta)} \\ &= \psi(y) - \psi(x), \end{aligned}$$

which completes the proof. □

Proof (Proof of Theorem 9.8). (a) We claim that $x \in \mathrm{Sol}(\mathrm{VP}_1)$ if and only if

$$Q_x(K - x) \cap \left(-\mathbb{R}^m_+\right) = \{\mathbf{0}\}, \tag{9.20}$$

where

$$Q_x = \begin{pmatrix} (b_1^\top x + \beta_1) a_1^\top - (a_1^\top x + \alpha_1) b_1^\top \\ \vdots \\ (b_m^\top x + \beta_m) a_m^\top - (a_m^\top x + \alpha_m) b_m^\top \end{pmatrix}$$

is an $m \times n$ matrix and $Q_x(K - x) = \{Q_x(y - x) \,:\, y \in K\}$. Indeed, $x \notin \mathrm{Sol}(\mathrm{VP}_1)$ if and only if there exist $y \in K$ and i_0 such that

$$\varphi_i(y) \leq \varphi_i(x) \;\; \forall i \in \{1, \ldots, m\}, \quad \varphi_{i_0}(y) < \varphi_{i_0}(x).$$

By Lemma 9.3, the last system of inequalities is equivalent to the following one:

$$\langle \nabla \varphi_i(x), y-x\rangle \le 0 \ \ \forall i \in \{1,\ldots,m\}, \quad \big\langle \nabla \varphi_{i_0}(x), y-x\big\rangle < 0. \tag{9.21}$$

Since

$$\langle \nabla \varphi_i(x), y-x\rangle = \frac{a_i^\top(y-x)(b_i^\top x+\beta_i) - b_i^\top(y-x)(a_i^\top x+\alpha_i)}{(b_i^\top x+\beta_i)^2},$$

we can rewrite (9.21) as follows

$$\begin{cases} \left[(b_i^\top x+\beta_i)a_i^\top - (a_i^\top x+\alpha_i)b_i^\top\right](y-x) \le 0 \ \ \forall i \in \{1,\ldots,m\}, \\ \left[(b_{i_0}^\top x+\beta_{i_0})a_{i_0}^\top - (a_{i_0}^\top x+\alpha_{i_0})b_{i_0}^\top\right](y-x) < 0. \end{cases}$$

Therefore, $x \notin \mathrm{Sol}(\mathrm{VP}_1)$ if and only if there exists $y \in K$ such that

$$Q_x(y-x) \in -\mathbb{R}^m_+ \quad \text{and} \quad Q_x(y-x) \ne \mathbf{0}.$$

Our claim has been proved.

It is clear that $D := Q_x(K-x)$ is a polyhedral convex set. Hence, by [27, Corollary 19.7.1], $P := \mathrm{cone} D$ is a polyhedral convex cone. In particular, P is a closed convex cone. It is easily seen that (9.20) is equivalent to the property $P \cap \left(-\mathbb{R}^m_+\right) = \{\mathbf{0}\}$. Setting

$$P^+ = \{z \in \mathbb{R}^m : \langle z, v\rangle \ge 0 \quad \forall v \in P\},$$

we have $P^+ \cap \mathrm{int}\,\mathbb{R}^k_+ \ne \emptyset$. Indeed, on the contrary, suppose that $P^+ \cap \mathrm{int}\,\mathbb{R}^k_+ = \emptyset$. Then, by the separation theorem, there exists $\xi \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ such that

$$\langle \xi, u\rangle \le 0 \le \langle \xi, z\rangle \quad \forall u \in \mathrm{int}\,\mathbb{R}^m_+,\ \forall z \in P^+.$$

This implies that $\xi \in -\mathbb{R}^m_+$ and $\xi \in (P^+)^+ = P$. So we get $\xi \in P \cap (-\mathbb{R}^m_+) = \{\mathbf{0}\}$, a contradiction.

Fix any $\widetilde{\xi} \in P^+ \cap \mathrm{int}\,\mathbb{R}^m_+$. For $\xi := \widetilde{\xi}/(\widetilde{\xi}_1 + \cdots + \widetilde{\xi}_m)$, we have $\xi \in P^+ \cap \mathrm{ri}\,\Sigma$. Since $\langle \xi, v\rangle \ge 0$ for every $v \in P$, we deduce that

$$\langle Q_x^\top \xi, y-x\rangle = \langle \xi, Q_x(y-x)\rangle \ge 0$$

for every $x \in K$. Hence (9.17) is valid.

(b) It is easily seen that $x \in \mathrm{Sol}^w(\mathrm{VP})$ if and only if

$$Q_x(K-x) \cap \left(-\mathrm{int}\,\mathbb{R}^m_+\right) = \emptyset.$$

Using the separation theorem we find a multiplier $\xi = (\xi_1, \ldots, \xi_m) \in \Sigma$ satisfying (9.17).

(c) It suffices to apply the well-known Farkas Lemma [27, p. 200]. □

Condition (9.17) can be rewritten in the form of a parametric affine variational inequality

$$(\mathrm{VI})'_\xi \qquad \langle M(\xi)x + q(\xi), y - x\rangle \geq 0, \quad \forall y \in K,$$

with

$$M(\xi) := \sum_{i=1}^{n} \xi_i M_i, \quad q(\xi) := \sum_{i=1}^{n} \xi_i q_i,$$

where

$$M_i = a_i b_i^\top - b_i a_i^T, \quad q_i = \beta_i a_i - \alpha_i b_i \quad (i = 1, \ldots, m). \tag{9.22}$$

Observe that, for each $i \in \{1, \ldots, m\}$, the pair (M_i, q_i) is defined solely by the coefficients of the linear fractional function φ_i. Since $M_i^\top = -M_i$, we have $\langle M_i v, v\rangle = 0$ for every $v \in \mathbb{R}^n$. Thus M_i is a positive semidefinite matrix for $i = 1, \ldots, m$, and $(\mathrm{VI})'_\xi$ is a monotone AVI for every $\xi \in \Sigma$. Denote by $\Phi(\xi)$ the solution set of the problem $(\mathrm{VI})'_\xi$ and consider the multifunction $\Phi : \Sigma \rightrightarrows \mathbb{R}^n$, $\xi \mapsto \Phi(\xi)$. By virtue of the above-mentioned optimality conditions for (VP_1),

$$\mathrm{Sol}(\mathrm{VP}_1) = \bigcup_{\xi \in \mathrm{ri}\Sigma} \Phi(\xi) = \Phi(\mathrm{ri}\Sigma), \tag{9.23}$$

and

$$\mathrm{Sol}^w(\mathrm{VP}_1) = \bigcup_{\xi \in \Sigma} \Phi(\xi) = \Phi(\Sigma). \tag{9.24}$$

By formulae (9.23), (9.24), and Theorem 9.2, the efficient solution set (resp., the weakly efficient solution set) of the vector optimization problem (VP_1) coincides with the Pareto solution set (resp., the weak Pareto solution set) of the monotone AVVI defined by K and the affine functions

$$F_i(x) = M_i x + q_i \quad (i = 1, \ldots, m).$$

In other words, *the first-order necessary and sufficient optimality condition* (9.17) *of a LFVOP can be treated as a special monotone affine VVI*. This valuable observation of [32], which has been used also in [12], will allow us to derive several results on (VP_1) from the corresponding results on monotone AVVIs.

Consider the linear fractional vector optimization problem (VP_1). Let M_i and q_i be given by (9.22), $\omega = (M_1, \ldots, M_m, q_1, \ldots, q_m)$. It is convenient for us to denote the solution sets of (VP_1) corresponding to the data set ω by $\mathrm{Sol}^w(\omega, \mathrm{VP}_1)$ and $\mathrm{Sol}(\omega, \mathrm{VP}_1)$. It follows from (9.23), (9.24) and Theorem 9.2 that

$$\mathrm{Sol}^w(\omega, \mathrm{VP}_1) = \mathrm{Sol}^w(\mathrm{AVVI}(\omega, K)), \quad \mathrm{Sol}(\omega, \mathrm{VP}_1) = \mathrm{Sol}(\mathrm{AVVI}(\omega, K)).$$

Suppose that (VP_1) undergoes a small perturbation: The original data set ω is replaced by a new one

$$\widetilde{\omega} = (\widetilde{M}_1, \ldots, \widetilde{M}_m, \widetilde{q}_1, \ldots, \widetilde{q}_m)$$

where, similarly as in (9.22),

$$\widetilde{M}_i = \widetilde{a}_i \widetilde{b}_i^\top - \widetilde{b}_i \widetilde{a}_i^\top, \quad \widetilde{q}_i = \widetilde{\beta}_i \widetilde{a}_i - \widetilde{\alpha}_i \widetilde{b}_i \quad (i = 1, \ldots, m).$$

The new solution sets are denoted by $\mathrm{Sol}^w(\widetilde{\omega}, \mathrm{VP}_1)$ and $\mathrm{Sol}(\widetilde{\omega}, \mathrm{VP}_1)$.

As in [33], specializing the results discussed in the previous section to the case of the monotone AVVI associated with (VP_1), we get the next statements.

Theorem 9.9 ([32]). *If* $\mathrm{Sol}^w(\omega, \mathrm{VP}_1)$ *is nonempty and bounded, then for any* $\alpha > 0$ *there exist constants* $\varepsilon > 0$, $\rho > 0$ *such that if* (9.4) *is fulfilled, then the set* $\mathrm{Sol}^w(\widetilde{\omega}, \mathrm{VP}_1)$ *is nonempty,*

$$\mathrm{Sol}^w(\widetilde{\omega}, \mathrm{VP}_1) \subseteq \overline{B}(0, \rho),$$

and

$$\mathrm{Sol}^w(\widetilde{\omega}, \mathrm{VP}_1) \subseteq \mathrm{Sol}^w(\omega, \mathrm{VP}_1) + \alpha B(0,1).$$

In particular, the solution map $\mathrm{Sol}^w(\cdot, \mathrm{VP}_1)$ *is upper semicontinuous at* ω.

Theorem 9.10. *If* $\mathrm{Sol}^w(\omega, \mathrm{VP}_1)$ *is nonempty and bounded, then for any* $\alpha > 0$ *there exist constants* $\varepsilon > 0$, $\rho > 0$ *such that if* (9.4) *is fulfilled, then the set* $\mathrm{Sol}(\widetilde{\omega}, \mathrm{VP}_1)$ *is nonempty,*

$$\mathrm{Sol}(\widetilde{\omega}, \mathrm{VP}_1) \subseteq \overline{B}(0, \rho),$$

and

$$\mathrm{Sol}(\widetilde{\omega}, \mathrm{VP}_1) \subseteq \mathrm{Sol}^w(\omega, \mathrm{VP}_1) + \alpha B(0,1).$$

Theorem 9.11 ([10]). *The following assertions are valid:*

(a) If $\mathrm{Sol}^w(\omega, \mathrm{VP}_1)$ *is bounded, then it is connected.*
(b) If $\mathrm{Sol}^w(\omega, \mathrm{VP}_1)$ *is disconnected, then each connected component of the solution set is unbounded.*

Theorem 9.12 ([10]). *The following assertions hold:*

(a') (Phu 1998, private communication) If $\mathrm{Sol}(\omega, \mathrm{VP}_1)$ *is bounded, then it is connected.*
(b') If $\mathrm{Sol}(\omega, \mathrm{VP}_1)$ *is disconnected, then each connected component of the solution set is unbounded.*

Remark 9.1. In the previous section, the method of [32] for proving the result in Theorem 9.9 has been developed furthermore to obtain Theorem 9.4.

Remark 9.2. It is likely that the assumption of Theorem 9.10 is not enough for getting the usc property of the multifunction $\mathrm{Sol}(\cdot, \mathrm{VP}_1)$ at ω.

Remark 9.3. The result in Theorem 9.11, which is due to [10], has been proved by another method.

Remark 9.4. The second assertion of Theorem 9.12 improves a claim in [10, Theorem 3.2], which requires additionally that $\mathrm{Sol}(\omega, \mathrm{VP}_1)$ is closed and the number of connected components of $\mathrm{Sol}(\omega, \mathrm{VP}_1)$ is finite. Note also that the method of proving [10, Theorem 3.2] is quite different from that of the proof of Theorem 9.7.

The next question arises in a natural way.

Question 9.1. At most, how many connected components the solution set $\mathrm{Sol}(\mathrm{VP}_1)$ (resp., $\mathrm{Sol}^w(\mathrm{VP}_1)$) may have?

A long time ago, Choo and Atkins [5] have constructed an example of a two-dimensional bicriteria LFVOP where the Pareto solution set coincides with the weak Pareto solution set and has two connected components.

No examples of LFVOPs with three or more connected components in the Pareto solution set (or in the weak Pareto solution set) had been known until the year 2005 when Hoa et al. [11] obtained the next result.

Theorem 9.13. *For any integer $m \geq 2$, there exists a linear fractional vector optimization problem whose Pareto solution set coincides with the weak Pareto solution set and has exactly m connected components.*

To prove Theorem 9.13, we will use the construction and the arguments of [11]. Consider problem (VP_1) where $n = m$, $m \geq 2$,

$$K = \left\{ x \in \mathbb{R}^m : x_1 \geq 0, x_2 \geq 0, \ldots, x_m \geq 0, \sum_{k=1}^{m} x_k \geq 1 \right\},$$

and

$$\varphi_i(x) = \frac{-x_i + \frac{1}{2}}{\sum_{k=1}^{m} x_k - \frac{3}{4}} \qquad (i = 1, \ldots, m).$$

For simplicity, let us denote this LFVOP by (P_1). Here we have

$$a_1 = (-1, 0, 0, \ldots, 0)^\top,\ a_2 = (0, -1, 0, \ldots, 0)^\top, \ldots,$$
$$a_m = (0, 0, 0, \ldots, -1)^\top,\ \alpha_1 = \alpha_2 = \cdots = \alpha_m = \tfrac{1}{2},$$

and

$$b_1 = b_2 = \cdots = b_m = (1, 1, 1, \ldots, 1)^\top, \quad \beta_1 = \beta_2 = \cdots = \beta_m = -\frac{3}{4}.$$

Setting

$$C = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix},$$

we see that $K = \{x \in \mathbb{R}^n : Cx \le d\}$. We are going to compute the sets $\mathrm{Sol}(\mathrm{P}_1)$ and $\mathrm{Sol}^w(\mathrm{P}_1)$.

According to Theorem 9.8, for any $x \in K$, $x \in \mathrm{Sol}(\mathrm{P}_1)$ if and only if there exist $\xi_1 > 0, \dots, \xi_m > 0$ and $\mu_1 \ge 0, \dots, \mu_{m+1} \ge 0$ such that

$$\sum_{i=1}^{m} \xi_i \left[\left(b_i^\top x + \beta_i \right) a_i - \left(a_i^\top x + \alpha_i \right) b_i \right] + \sum_{j \in J(x)} \mu_j C_j^\top = 0, \tag{9.25}$$

where $J(x) := \{j \in \{1, \dots, m+1\} : C_j x = d_j\}$, C_j is the $j-$th row of C, and d_j is the $j-$th component of d. Also, for any $x \in K$, $x \in \mathrm{Sol}^w(\mathrm{P}_1)$ if and only if there exist $\xi_1 \ge 0, \dots, \xi_m \ge 0$, not all zeroes, and $\mu_1 \ge 0, \dots, \mu_r \ge 0$ such that (9.25) holds.

Lemma 9.4. *For $m \ge 2$, it holds*

$$\begin{aligned} \mathrm{Sol}(\mathrm{P}_1) = \mathrm{Sol}^w(\mathrm{P}_1) = {} & \{(x_1, 0, \dots, 0)^\top : x_1 \ge 1\} \\ & \cup \{(0, x_2, \dots, 0)^\top : x_2 \ge 1\} \\ & \dots \dots \dots \\ & \cup \{(0, \dots, 0, x_m)^\top : x_m \ge 1\}. \end{aligned}$$

Let $m \ge 2$. Consider problem (P_1) and fix any $x \in K$. It is easy to see that $|J(x)| \le m$, where $|J(x)|$ denotes the number of elements in $J(x)$. In order to obtain Lemma 9.4, we need to prove a several claims.

Claim. If $|J(x)| = 0$, then $x \notin \mathrm{Sol}^w(\mathrm{P}_1)$ (so $x \notin \mathrm{Sol}(\mathrm{P}_1)$).

Proof. Since $x \in K$ and since $J(x) = \emptyset$, we have $x_1 > 0, \dots, x_m > 0$ and $\sum_{i=1}^m x_i > 1$. Suppose the claim were false. Then we could find $\xi_1 \ge 0, \dots, \xi_m \ge 0$, not all zeroes, such that

$$\sum_{i=1}^{m} \xi_i \left[\left(b_i^\top x + \beta_i \right) a_i - \left(a_i^\top x + \alpha_i \right) b_i \right] = 0. \tag{9.26}$$

To shorten notation, we set

$$Q_i(x, \xi) = -\xi_i \left(\sum_{k=1}^{m} x_k - \frac{3}{4} \right) + \xi_1 \left(x_1 - \frac{1}{2} \right) + \dots + \xi_m \left(x_m - \frac{1}{2} \right) \tag{9.27}$$

for $i = 1, \dots, m$. It is easy to verify that (9.26) is equivalent to the following system:

$$Q_i(x, \xi) = 0 \qquad (i = 1, \dots, m). \tag{9.28}$$

Comparing the first and the i-th equalities of (9.28) we deduce that $\xi_i = \xi_1$. Therefore $\xi_m = \cdots = \xi_1$. Substituting this into the first equality of (9.28) yields

$$\xi_1 \left(\frac{3}{4} - \frac{m}{2} \right) = 0;$$

hence $\xi_1 = 0$. This implies $\xi_m = \cdots = \xi_1 = 0$, a contradiction. □

Claim. If $1 \le |J(x)| \le m-2$, then $x \notin \mathrm{Sol}^w(\mathrm{P}_1)$ (so $x \notin \mathrm{Sol}(\mathrm{P}_1)$).

Proof. For $m = 2$, the claim is obvious. So we will assume that $m \ge 3$. Due to the symmetry of the expressions defining K and φ_i $(i = 1, \dots, m)$ with respect to the variables $x_1, \dots, x_m$, it suffices to consider the following two cases: (i) Case 1: $J(x) = \{1, \dots, j\},\ 1 \le j \le m-2$; (ii) Case 2: $J(x) = \{1, \dots, j, m+1\},\ 1 \le j \le m-3$.

We first consider Case 1. Suppose the claim were false. Then we could find $\xi_1 \ge 0, \dots, \xi_m \ge 0$, not all zero, and $\mu_1 \ge 0, \dots, \mu_{m+1} \ge 0$ such that (9.25) holds. Since $J(x) = \{1, \dots, j\},\ 1 \le j \le m-2$, (9.25) is equivalent to the system

$$Q_i(x, \xi) = \mu_i \ \ (i = 1, \dots, j), \quad Q_i(x, \xi) = 0 \ \ (i = j+1, \dots, m), \tag{9.29}$$

where $Q_i(x, \xi)$ $(i = 1, \dots, m)$ are defined by (9.27). From the last $m - j$ equalities of (9.29) we obtain $\xi_{j+1} = \cdots = \xi_m$. Substituting $x_1 = \cdots = x_j = 0$ and $\xi_{j+1} = \cdots = \xi_m$ into the $(j+1)$-th equality of (9.29) we get

$$\xi_{j+1} \left(\frac{3}{4} - \frac{m-j}{2} \right) - \frac{1}{2} \sum_{k=1}^{j} \xi_i = 0$$

or, equivalently,

$$\xi_{j+1} \left(\frac{3}{2} - m + j \right) = \sum_{k=1}^{j} \xi_i. \tag{9.30}$$

The condition $j \le m-2$ implies $\frac{3}{2} - m + j < 0$. Hence from (9.30) it follows that $\xi_{j+1} = \xi_1 = \cdots = \xi_j = 0$. Then we have $\xi_1 = \cdots = \xi_m = 0$, which is impossible.

We now consider Case 2. Suppose, contrary to our claim, that $x \in \mathrm{Sol}^w(\mathrm{P}_1)$. Then we could find $\xi_1 \ge 0, \dots, \xi_m \ge 0$, not all zero, and $\mu_1 \ge 0, \dots, \mu_{m+1} \ge 0$ such that (9.25) holds. Since $J(x) = \{1, \dots, j, m+1\},\ 1 \le j \le m-3$, (9.25) is equivalent to the following system:

$$Q_i(x, \xi) = \mu_i + \mu_{m+1} \ \ (i = 1, \dots, j), \quad Q_i(x, \xi) = \mu_{m+1} \ \ (i = j+1, \dots, m). \tag{9.31}$$

It is easily seen that the last $m-j$ equalities in (9.31) imply $\xi_{j+1}=\cdots=\xi_m$. Substituting $x_1=\cdots=x_j=0$ and $\xi_m=\cdots=\xi_{j+1}$ into the $(j+1)$-th equality of (9.29) we get

$$\xi_{j+1}\left(\frac{3}{4}-\frac{m-j}{2}\right)-\frac{1}{2}\sum_{k=1}^{j}\xi_i=\mu_{m+1}$$

or, equivalently,

$$\xi_{j+1}\left(\frac{3}{2}-m+j\right)=\sum_{k=1}^{j}\xi_i+2\mu_{m+1}. \tag{9.32}$$

Since $j\le m-3=0$, $\frac{3}{2}-m+j<0$. Hence (9.32) yields $\xi_{j+1}=0$ and $\xi_1=\cdots=\xi_j=0$. Then we have $\xi_1=\cdots=\xi_m=0$, which is impossible. □

Claim. If $|J(x)|=m-1$, then $x\in \mathrm{Sol}(\mathrm{P}_1)$ if and only if $m+1\notin J(x)$. Similarly, $x\in \mathrm{Sol}^w(\mathrm{P}_1)$ if and only if $m+1\notin J(x)$.

Proof. Suppose that $x\in K$, $|J(x)|=m-1$ and $m+1\in J(x)$. By symmetry, we can assume that $J(x)=\{1,\ldots,j,m+1\}$, where $j=m-2$. Arguing similarly as in Case 2 in the proof of Claim 9.3, we can conclude that $x\notin \mathrm{Sol}^w(\mathrm{P}_1)$ (hence $x\notin \mathrm{Sol}(\mathrm{P}_1)$.

We now consider the case $x\in K$, $|J(x)|=m-1$ and $m+1\notin J(x)$. Due to the symmetry of the expressions defining K and $\varphi_1,\ldots,\varphi_m$ w.r.t. $x_1,x_2,\ldots,x_m$, there is no loss of generality in assuming that $J(x)=\{1,\ldots,m-1\}$. In order to prove that $x\in \mathrm{Sol}(\mathrm{P}_1)$, it suffices to show that there exist $\xi_1>0,\ldots,\xi_m>0$ and $\mu_1\ge 0,\ldots,\mu_{m+1}\ge 0$ satisfying (9.25). In the case under consideration, it is clear that (9.25) is equivalent to the following system:

$$Q_i(x,\xi)=\mu_i\ \ (i=1,\ldots,m-1),\quad Q_m(x,\xi)=0. \tag{9.33}$$

Substituting $x_1=\cdots=x_{m-1}=0$ into the last equality of (9.33) yields

$$\xi_m=2\left(\xi_1+\cdots+\xi_{m-1}\right).$$

Choosing $\xi_1=\cdots=\xi_{m-1}=1$ and $\xi_m=2(m-1)$, we see that the last equality of (9.33) is valid. Subtracting this equality from the i-th equality $(i=1,\ldots,m-1)$ of that system, we get

$$(\xi_m-\xi_i)\left(x_m-\frac{3}{4}\right)=\mu_i.$$

Therefore, choosing $\mu_i=(2m-3)\left(x_m-\frac{3}{4}\right)>0$ (recall that $m\ge 2$ and $x_m>1$) for $i=1,\ldots,m-1$, we see that the first $m-1$ equalities of (9.33) are valid. We have proved that $x=(0,\ldots,0,x_m)^\top\in \mathrm{Sol}(\mathrm{P}_1)$ provided $x_m>1$. By symmetry, $x=(0,\ldots,0,x_j,0,\ldots,0^\top\in \mathrm{Sol}(\mathrm{P}_1)$ for any $j\in\{1,\ldots,m-1\}$, provided that $x_j>1$. □

Claim. If $|J(x)|=m$, then $x\in \mathrm{Sol}(\mathrm{P}_1)$ if and only if $m+1\in J(x)$. Similarly, $x\in \mathrm{Sol}^w(\mathrm{P}_1)$ if and only if $m+1\in J(x)$.

Proof. Suppose that $x \in K$ and $|J(x)| = m$.

If $m+1 \notin J(x)$, then we have $x_1 = \cdots = x_m = 0$. This is impossible, because the constraint $x_1 + \cdots + x_m \geq 1$ is not satisfied.

We now consider the case $x \in K$, $|J(x)| = m$, and $m+1 \in J(x)$. If $J(x) = \{1,\dots,m-1,m+1\}$ then $x = (0,\dots,0,1^\top$. Clearly, (9.25) is equivalent to the following system:

$$Q_i(x,\xi) = \mu_i + \mu_{m+1} \ \ (i = 1,\dots,m-1), \quad Q_m(x,\xi) = \mu_{m+1}. \tag{9.34}$$

Choosing $\mu_{m+1} = 0$, $\xi_1 = \cdots = \xi_{m-1} = 1$, and $\xi_m = 2(m-1)$,

$$\mu_i = (2m-3)\left(x_m - \frac{3}{4}\right) > 0$$

for $i = 1,\dots,m-1$, we see that system (9.34) is satisfied. Therefore $x = (0,\dots,0,1)^\top \in \mathrm{Sol}(\mathrm{P}_1)$ provided $x_m > 1$. Similarly, the vectors $x = (1,0,\dots,0^\top,\dots,$ $x = (0,\dots,0,1,0)^\top$ also belong to $\mathrm{Sol}(\mathrm{P}_1)$. □

Summarizing the results stated in Claims 9.3–9.3 we obtain the assertion of Lemma 9.4 which implies the result stated in Theorem 9.13.

Since (VP_1) is a very simple nonlinear vector optimization problem, one may think that the connected components of their solution sets must have very simple topologies. To our surprise, it is not so! Namely, following Huy and Yen [13] we now have deal with a LFVOP whose Pareto solution set is *connected by line segments* (hence it is path-connected), but not contractible. This means that the unique connected component in the Pareto solution set is enough complicated in respect to its topological structure.

Consider problem (VP_1), where

$$\begin{aligned} K &= \{x = (x_1,x_2,x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 \geq 2,\ x_1 + x_2 - 2x_3 \leq 2, \\ &\qquad x_1 - 2x_2 + x_3 \leq 2,\ -2x_1 + x_2 + x_3 \leq 2\}, \\ \varphi_i(x) &= \frac{-x_i}{x_1 + x_2 + x_3 - 1} \qquad (i = 1,2,3). \end{aligned}$$

Denote this particular problem by (P_2). It is easily seen that if $x = (x_1,x_2,x_3) \in K$ then $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$.

We shall show that $\mathrm{Sol}(\mathrm{P}_2)$ and $\mathrm{Sol}^w(\mathrm{P}_2)$ are path-connected, but not contractible.

Let $x \in K$. Then according to Theorem 9.8, $x \in \mathrm{Sol}(\mathrm{P}_2)$ if and only if there exist $\xi_1 > 0$, $\xi_2 > 0$, $\xi_3 > 0$ and $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\mu_3 \geq 0$, $\mu_4 \geq 0$ such that

$$\begin{aligned}
&\xi_1\left[(x_1+x_2+x_3-1)\begin{pmatrix}1\\0\\0\end{pmatrix}-x_1\begin{pmatrix}1\\1\\1\end{pmatrix}\right]\\
&+\xi_2\left[(x_1+x_2+x_3-1)\begin{pmatrix}0\\1\\0\end{pmatrix}-x_2\begin{pmatrix}1\\1\\1\end{pmatrix}\right]\\
&+\xi_3\left[(x_1+x_2+x_3-1)\begin{pmatrix}0\\0\\1\end{pmatrix}-x_3\begin{pmatrix}1\\1\\1\end{pmatrix}\right]+\sum_{j\in J(x)}\mu_j C_j^\top=0,
\end{aligned}\tag{9.35}$$

where $J(x)=\{j\in\{1,2,3,4\}:C_jx=d_j\}$, C_j and d_j are the j-th row and the j-th component of

$$C=\begin{pmatrix}-1&-1&-1\\1&1&-2\\1&-2&1\\-2&1&1\end{pmatrix}\quad\text{and}\quad d=\begin{pmatrix}-2\\2\\2\\2\end{pmatrix},$$

respectively. Similarly, $x\in\mathrm{Sol}^w(P_2)$ if and only if there exist $\xi_1\ge 0,\ \xi_2\ge 0,\ \xi_3\ge 0$, not all zeroes, and $\mu_1\ge 0,\ \mu_2\ge 0,\ \mu_3\ge 0,\ \mu_4\ge 0$ satisfying (9.35).

Let $x\in K$ and let $|J(x)|$ stand for the number of elements of $J(x)$. We will use the following abbreviations:

$$\begin{aligned}
Q_1(x,\xi)&=\xi_1(x_1+x_2+x_3-1)-\xi_1x_1-\xi_2x_2-\xi_3x_3,\\
Q_2(x,\xi)&=\xi_2(x_1+x_2+x_3-1)-\xi_1x_1-\xi_2x_2-\xi_3x_3,\\
Q_3(x,\xi)&=\xi_3(x_1+x_2+x_3-1)-\xi_1x_1-\xi_2x_2-\xi_3x_3.
\end{aligned}$$

We consider the following four cases.

Case 1: $|J(x)|=0$. If $x\in\mathrm{Sol}(\mathrm{P}_2)$ then there exist $\xi_1>0,\ \xi_2>0,\ \xi_3>0$ satisfying (9.35). Since $J(x)=\emptyset$, (9.35) equivalent to the system

$$Q_1(x,\xi)=0,\quad Q_2(x,\xi)=0,\quad Q_3(x,\xi)=0.\tag{9.36}$$

Subtracting the second equality of (9.36) from the first one, we obtain $(\xi_1-\xi_2)(x_1+x_2+x_3-1)=0$. Since $x_1+x_2+x_3\ge 2$, it follows that $\xi_1=\xi_2$. Similarly, from the first and the third equalities of (9.36) we can deduce that $\xi_1=\xi_3$. Substituting $\xi_3=\xi_3=\xi_1$ into the first equality of (9.36) gives $-\xi_1=0$. Therefore $\xi_1=\xi_2=\xi_3=0$. We have thus shown that $x\notin\mathrm{Sol}(\mathrm{P}_2)$. The same arguments show that $x\notin\mathrm{Sol}^w(\mathrm{P}_2)$.

Case 2: $|J(x)|=1$. We have three subcases:

Subcase 2.1: $J(x)=\{1\}$. Suppose that $x\in\mathrm{Sol}(\mathrm{P}_2)$. Then, by the optimality condition (9.35), there exist $\xi_1>0,\ \xi_2>0,\ \xi_3>0$ and $\mu_1\ge 0$ such that

$$Q_1(x,\xi)-\mu_1=0,\quad Q_2(x,\xi)-\mu_1=0,\quad Q_3(x,\xi)-\mu_1=0.\tag{9.37}$$

Arguing similarly as in Case 1 from (9.37) we get $\xi_1 = \xi_2 = \xi_3$. Substituting $\xi_3 = \xi_2 = \xi_1$ into the first equality of (9.37) gives $-\xi_1 = \mu_1$. This implies $\xi_1 = \xi_2 = \xi_3 = 0$, a contradiction. Therefore $x \notin \mathrm{Sol}(\mathrm{P}_2)$. Similarly, $x \notin \mathrm{Sol}^w(\mathrm{P}_2)$.

Subcase 2.2: $J(x) = \{2\}$. By (9.35), $x \in \mathrm{Sol}(\mathrm{P}_2)$ if and only if there exist $\xi_1 > 0, \xi_2 > 0,\ \xi_3 > 0$ and $\mu_2 \geq 0$ such that

$$Q_1(x,\xi) + \mu_2 = 0, \quad Q_2(x,\xi) + \mu_2 = 0, \quad Q_3(x,\xi) - 2\mu_2 = 0. \tag{9.38}$$

From the first two equalities of (9.38) we obtain $\xi_1 = \xi_2$. The first and the third equalities of (9.38) imply

$$(\xi_1 - \xi_3)(x_1 + x_2 + x_3 - 1) + 3\mu_2 = 0.$$

Hence

$$\mu_2 = \frac{1}{3}(\xi_3 - \xi_1)(x_1 + x_2 + x_3 - 1). \tag{9.39}$$

If we choose $\xi_1 = \xi_2 = 1$, then the first equalities of (9.38) becomes

$$x_3 - 1 - \xi_3 x_3 + \frac{1}{3}(\xi_3 - 1)(x_1 + x_2 + x_3 - 1) = 0,$$

or $(\xi_3 - 1)(x_1 + x_2 - 2x_3 - 1) = 3$. Since $x_1 + x_2 - 2x_3 = 2$, it follows that $\xi_3 = 4$. By (9.39), $\mu_2 = 3x_3 + 1 > 0$. Therefore, if we choose $\xi_1 = \xi_2 = 1,\ \xi_3 = 4$ and $\mu_2 = 3x_3 + 1$, then (9.38) is fulfilled. We have shown that $x \in \mathrm{Sol}(\mathrm{P}_2)$. This implies that $x \in \mathrm{Sol}^w(\mathrm{P}_2)$.

Subcase 2.3: $J(x) = \{3\}$ or $J(x) = \{4\}$. Analysis similar to that in Subcase 2.2 shows that $x \in \mathrm{Sol}(\mathrm{P}_2) \subseteq \mathrm{Sol}^w(\mathrm{P}_2)$.

Case 3: $|J(x)| = 2$. This case also has three subcases:

Subcase 3.1: $J(x) = \{1, 2\}$. Since $x_1 + x_2 + x_3 = 2$ and $x_1 + x_2 - 2x_3 = 2$, we have $x_3 = 0$. By (9.35), $x \in \mathrm{Sol}(\mathrm{P}_2)$ if and only if there exist $\xi_1 > 0,\ \xi_2 > 0,\ \xi_3 > 0,\ \mu_1 \geq 0$ and $\mu_2 \geq 0$ such that

$$\begin{aligned} &Q_1(x,\xi) - \mu_1 + \mu_2 = 0, \quad Q_2(x,\xi) - \mu_1 + \mu_2 = 0,\\ &Q_3(x,\xi) - \mu_1 - 2\mu_2 = 0. \end{aligned} \tag{9.40}$$

For $\xi_1 = \xi_2 = 1,\ \xi_3 = 4$, and $\mu_1 = 0,\ \mu_2 = 1$, we see that (9.40) is satisfied. Hence $x \in \mathrm{Sol}(\mathrm{P}_2) \subseteq \mathrm{Sol}^w(\mathrm{P}_2)$.

Subcase 3.2: $J(x) = \{1, 3\}$ or $J(x) = \{1, 4\}$. Analysis similar to that in the Subcase 3.1 shows that $x \in \mathrm{Sol}(\mathrm{P}_2) \subseteq \mathrm{Sol}^w(\mathrm{P}_2)$.

Subcase 3.3: $J(x) = \{2, 3\}$. Since $x_1 + x_2 - 2x_3 = 2$ and $x_1 - 2x_2 + x_3 = 2$, we have $x_2 = x_3$. By (9.35), $x \in \mathrm{Sol}(\mathrm{P}_2)$ if and only if there exist $\xi_1 > 0, \xi_2 > 0, \xi_3 > 0,\ \mu_2 \geq 0$ and $\mu_3 \geq 0$ such that

$$Q_1(x,\xi) + \mu_2 + \mu_3 = 0, \quad Q_2(x,\xi) + \mu_2 - 2\mu_3 = 0, \quad Q_3(x,\xi) - 2\mu_2 + \mu_3 = 0. \tag{9.41}$$

It is easy to verify that for $\xi_1 = \xi_2 = 1,\ \xi_3 = 4,\ \mu_2 = 2x_3 + 1$, and $\mu_3 = 0$, (9.41) is fulfilled. Thus $x \in \mathrm{Sol}(\mathrm{P}_2) \subseteq \mathrm{Sol}^w(\mathrm{P}_2)$.

Subcase 3.4: $J(x) = \{2,\ 4\}$ or $J(x) = \{3,\ 4\}$. Similarly as in Subcase 3.3, we have $x \in \mathrm{Sol}(\mathrm{P}_2) \subseteq \mathrm{Sol}^w(\mathrm{P}_2)$.

Case 4: $|J(x)| = 3$.

Subcase 4.1: $J(x) = \{1,\ 2,\ 3\}$. In this subcase, $x \in \mathrm{Sol}(\mathrm{P}_2)$ if and only if there exist $\xi_1 > 0,\ \xi_2 > 0,\ \xi_3 > 0,\ \mu_1 \geq 0,\ \mu_2 \geq 0$ and $\mu_3 \geq 0$ such that

$$\begin{aligned} &Q_1(x,\xi) - \mu_1 + \mu_2 + \mu_3 = 0, \quad Q_2(x,\xi) - \mu_1 + \mu_2 - 2\mu_3 = 0, \\ &Q_3(x,\xi) - \mu_1 - 2\mu_2 + \mu_3 = 0. \end{aligned} \tag{9.42}$$

Since (9.42) is satisfied with $\xi_1 = \xi_2 = 1,\ \xi_3 = 4,\ \mu_1 = \mu_3 = 0$ and $\mu_2 = 1$, we have $x \in \mathrm{Sol}(\mathrm{P}_2) \subseteq \mathrm{Sol}^w(\mathrm{P}_2)$.

Subcase 4.2: $J(x) = \{1,\ 2,\ 4\}$, or $J(x) = \{1,\ 3,\ 4\}$. Arguing as in Subcase 4.1 we have $x \in \mathrm{Sol}(\mathrm{P}_2) \subseteq \mathrm{Sol}^w(\mathrm{P}_2)$.

From what has already been said, it follows that both the sets $\mathrm{Sol}^w(\mathrm{P}_2)$ and $\mathrm{Sol}(\mathrm{P}_2)$ coincide with the surrounding surface of the parallelepiped K; that is

$$\mathrm{Sol}^w(\mathrm{P}_2) = \mathrm{Sol}(\mathrm{P}_2) = F_1 \cup F_2 \cup F_3,$$

where

$$\begin{aligned} F_1 &= \{x : x_1 = 2 - x_2 + 2x_3,\ x_3 \geq 0,\ x_3 \leq x_2 \leq 2 + x_3\} \\ F_2 &= \{x : x_2 = 2 - x_3 + 2x_1,\ x_1 \geq 0,\ x_1 \leq x_3 \leq 2 + x_1\} \\ F_3 &= \{x : x_3 = 2 - x_1 + 2x_2,\ x_2 \geq 0,\ x_2 \leq x_1 \leq 2 + x_2\}. \end{aligned}$$

So $\mathrm{Sol}^w(\mathrm{P}_2)$ and $\mathrm{Sol}(\mathrm{P}_2)$ are not contractible. Since any two points belonging to the surrounding surface of the parallelepiped K can be joined by a curve composed of (not more than two) line segments which are contained in the surrounding surface, the just mentioned solution sets are *connected by line segments* (hence they are path-connected).

9.4 Convex Quadratic Vector Optimization Problems

Necessary and sufficient optimality conditions for a *convex quadratic vector optimization problem* (or convex QVOP) can be rewritten as a monotone AVVI. To see this, we consider the problem

$$(\mathrm{VP}_2) \qquad \text{Minimize } f(x) \quad \text{subject to } x \in K,$$

where $K \subseteq \mathbb{R}^n$ is a polyhedral convex set, $f(x) = (f_1(x), \ldots, f_m(x))$,

$$f_i(x) = \frac{1}{2}x^\top M_i x + q_i^\top x \quad (i = 1, \ldots, m),$$

with $M_1,\ldots,M_m$ being symmetric positive semidefinite $n \times n$ matrices, $q_1,\ldots,q_m \in \mathbb{R}^n$. The efficient solution set and the weakly efficient solution set of (VP_2) are abbreviated respectively to $\mathrm{Sol}(\mathrm{VP}_2)$ and $\mathrm{Sol}^w(\mathrm{VP}_2)$. Since $\nabla f_i(x) = M_i x + q_i$, putting $F_i(x) = M_i x + q_i$ we get the next statement, which expresses well-known necessary and sufficient optimality conditions for convex QVOPs, as a direct corollary of Theorem 9.1.

Theorem 9.14. *Let $x \in K$. The following assertions are valid:*

(a) $x \in \mathrm{Sol}^w(\mathrm{VP}_2)$ if and only if there exists $\xi \in \Sigma$ such that $x \in \mathrm{Sol}(\mathrm{VI})_\xi$.
(b) If there exists $\xi \in \mathrm{ri}\,\Sigma$ such that $x \in \mathrm{Sol}(\mathrm{VI})_\xi$, then $x \in \mathrm{Sol}(\mathrm{VP}_2)$.

It is important to stress that *the sufficient optimality stated in assertion (b) of Theorem 9.14 is not a necessary one*. The forthcoming very useful counterexample was proposed by one of the two referees of [33].

Example 9.1. Let $n = 1$, $m = 2$, and $K = \mathbb{R}$, $f(x) = (x^2, (x-1)^2)$ for every $x \in \mathbb{R}$. It is easy to show that $\mathrm{Sol}(\mathrm{VP}_2) = [0,1]$. Given any $\xi = (\xi_1, \xi_2) \in \mathrm{ri}\,\Sigma$, we see that

$$\xi_1 F_1(x) + \xi_2 F_2(x) = 2\xi_1 x + 2\xi_2(x-1) = 2(x - \xi_2),$$

hence $(\mathrm{VI})_\xi$ has the unique solution $x = \xi_2$. Consequently,

$$\mathrm{Sol}(\mathrm{VP}_2) = [0,1] \neq \mathrm{Sol}(\mathrm{AVVI}(\omega, K)) = \bigcup_{\xi \in \mathrm{ri}\,\Sigma} \mathrm{Sol}(\mathrm{VI})_\xi = (0,1).$$

Now, it is convenient for us to denote the solution sets of (VP_2) corresponding to the data set ω by $\mathrm{Sol}^w(\omega, \mathrm{VP}_2)$ and $\mathrm{Sol}(\omega, \mathrm{VP}_2)$. Combining Theorem 9.14 with Theorem 9.2 yields

$$\mathrm{Sol}(\omega, \mathrm{VP}_2) \supseteq \mathrm{Sol}(\mathrm{AVVI}(\omega, K)) \text{ and } \mathrm{Sol}^w(\omega, \mathrm{VP}_2) = \mathrm{Sol}^w(\mathrm{AVVI}(\omega, K)), \quad (9.43)$$

where $\omega = (M_1,\ldots,M_m,q_1,\ldots,q_m)$.

Suppose that (VP_2) undergoes a small perturbation: The original data set ω is replaced by a new one

$$\widetilde{\omega} = (\widetilde{M}_1,\ldots,\widetilde{M}_m,\widetilde{q}_1,\ldots,\widetilde{q}_m)$$

where the matrices $\widetilde{M}_i$ $(i = 1,\ldots,m)$ are symmetric. The new solution sets are denoted by $\mathrm{Sol}^w(\widetilde{\omega}, \mathrm{VP}_2)$ and $\mathrm{Sol}(\widetilde{\omega}, \mathrm{VP}_2)$.

Applying Theorems 9.4 and 9.6 to the monotone AVVI associated with (VP_2) we get the following two statements.

Theorem 9.15. *If $\mathrm{Sol}^w(\omega, \mathrm{VP}_2)$ is nonempty and bounded, then for any $\alpha > 0$ there exist constants $\varepsilon > 0,\ \rho > 0$ such that if (9.4) is fulfilled, then the set $\mathrm{Sol}^w(\widetilde{\omega}, \mathrm{VP}_2)$ is nonempty,*

$$\mathrm{Sol}^w(\widetilde{\omega}, \mathrm{VP}_2) \subseteq \overline{B}(0, \rho),$$

and

$$\mathrm{Sol}^w(\widetilde{\omega},\mathrm{VP}_2) \subseteq \mathrm{Sol}^w(\omega,\mathrm{VP}_2)+\alpha B(0,1).$$

In particular, the solution map $\mathrm{Sol}^w(\cdot,\mathrm{VP}_2)$ *is upper semicontinuous at* ω.

Theorem 9.16. *The following assertions are valid:*

(a) If $\mathrm{Sol}^w(\omega,\mathrm{VP}_2)$ *is bounded, then it is connected.*
(b) If $\mathrm{Sol}^w(\omega,\mathrm{VP}_2)$ *is disconnected, then each connected component of the solution set is unbounded.*

As shown by Example 9.1, the inclusion in (9.43) may be strict. Hence Theorem 9.5 do not implies the forthcoming statement. To make the presentation as complete as possible, we will give a detailed proof.

Theorem 9.17. *If* $\mathrm{Sol}^w(\omega,\mathrm{VP}_2)$ *is nonempty and bounded, then for any* $\alpha>0$ *there exist constants* $\varepsilon>0,\ \rho>0$ *such that if* (9.4) *is fulfilled, then the set* $\mathrm{Sol}(\widetilde{\omega},\mathrm{VP}_1)$ *is nonempty,*

$$\mathrm{Sol}(\widetilde{\omega},\mathrm{VP}_2) \subseteq \overline{B}(0,\rho), \tag{9.44}$$

and

$$\mathrm{Sol}(\widetilde{\omega},\mathrm{VP}_2) \subseteq \mathrm{Sol}^w(\omega,\mathrm{VP}_2)+\alpha B(0,1). \tag{9.45}$$

Proof. Suppose that $\mathrm{Sol}^w(\omega,\mathrm{VP}_2)$ is nonempty and bounded. Since $\mathrm{Sol}^w(\omega,\mathrm{VP}_2)=\mathrm{Sol}^w(\mathrm{AVVI}(\omega,K))$ by virtue of (9.43), the set $\mathrm{Sol}^w(\mathrm{AVVI}(\omega,K))$ is also nonempty and bounded. Therefore, given any $\alpha>0$, by Theorems 9.4 and 9.5 we can find constants $\varepsilon>0,\ \rho>0$ such that if (9.4) is fulfilled, with $\widetilde{M}_i$ $(i=1,\dots,m)$ being symmetric positive semidefinite matrices, then the set $\mathrm{Sol}(\mathrm{AVVI}(\widetilde{\omega},K))$ is nonempty and the inclusions (9.5) and (9.6) are valid. Recalling that $\mathrm{Sol}(\widetilde{\omega},\mathrm{VP}_2)\subseteq \mathrm{Sol}^w(\widetilde{\omega},\mathrm{VP}_2)$, we can easily combine these properties with the inclusion

$$\mathrm{Sol}(\widetilde{\omega},\mathrm{VP}_2) \supseteq \mathrm{Sol}(\mathrm{AVVI}(\widetilde{\omega},K))$$

and the equality

$$\mathrm{Sol}^w(\widetilde{\omega},\mathrm{VP}_2)=\mathrm{Sol}^w(\mathrm{AVVI}(\widetilde{\omega},K)),$$

which are just the realizations of the inclusion and the equality in (9.43) when $\widetilde{\omega}$ plays the role of ω, to obtain the nonemptyness of $\mathrm{Sol}(\widetilde{\omega},\mathrm{VP}_2)$ and the estimates (9.44), (9.45), as desired. □

There are some difficulties in studying connectedness of the Pareto solution set $\mathrm{Sol}(\omega,\mathrm{VP}_2)$ via the monotone AVVI model and the scalarization method. The reason is that we know that $\mathrm{Sol}(\mathrm{VI})_\xi\subseteq\mathrm{Sol}(\omega,\mathrm{VP}_2)$ for all $\xi\in\mathrm{ri}\,\Sigma$, but for $\xi\in\Sigma\setminus(\mathrm{ri}\,\Sigma)$ with $\mathrm{Sol}(\mathrm{VI})_\xi\cap\mathrm{Sol}(\omega,\mathrm{VP}_2)\neq\emptyset$ we cannot say definitely that the whole convex set $\mathrm{Sol}(\mathrm{VI})_\xi$ lies in $\mathrm{Sol}(\omega,\mathrm{VP}_2)$, or not. It is well known that connectedness is a global property of a topological space; and eliminating a subset, even just one point, from the space, may destroy its connectedness.

Hirschberger [9, Corollary 6.2] has proved that the *image of the Pareto solution set* (called also the *efficient frontier*) of a quasiconvex quadratic vector optimization problem defined on convex polyhedral convex set, is connected, if one of the component of the vector objective function is strongly quasiconvex. We refer to [9] for the related notions and results. To our knowledge, apart from the next theorem, not much has been known about the connectedness of $\mathrm{Sol}(\omega, \mathrm{VP}_2)$.

Theorem 9.18. (Benoist [1, Theorem 5.1]) *If K is compact, then* $\mathrm{Sol}(\omega, \mathrm{VP}_2)$ *is nonempty and connected.*

Theorem 9.18 is a very special corollary of Benoist's theorem [1, Theorem 5.1] which can be stated as follows *"The Pareto solution set of a continuous strictly quasiconvex vector minimization problem on a nonempty compact convex set in $\mathbb{R}^n$ is nonempty and connected."* By definition, a function $\psi : D \to \mathbb{R}$, with $D \subseteq \mathbb{R}^n$ being a convex set, is said to be *strictly quasiconvex* on D if for any $x, u \in D$ and any $t \in (0,1)$ it holds

$$\psi((1-t)x + tu) \leq \max\{\psi(x), \psi(u)\},$$

and the inequality is strict when $\psi(x) \neq \psi(u)$.

From the above definition it follows that *if $\psi : D \to \mathbb{R}$ is convex on D, then ψ is strictly quasiconvex on D.* Besides, Lemma 9.3 implies that *if $\psi : D \to \mathbb{R}$ is a linear fractional function* (i.e., there are vectors $a, b \in \mathbb{R}^n$ and constants $\alpha, \beta \in \mathbb{R}$ such that $b^T x + \beta \neq 0$ and $\psi(x) = (a^T x + \alpha)/(b^T x + \beta)$ for every $x \in D$), *then ψ is strictly quasiconvex on D.* Hence LFVOPs and convex QVOPs are examples of continuous strictly quasiconvex vector minimization problems.

9.5 Open Problems

This final section presents some open problems related to monotone AVVIs, LFVOPs, convex QVOPs, and strictly convex vector minimization problems (strictly convex VOPs, for brevity). Solutions to these problems can give us deeper insights to the classes of problems in question.

9.5.1 Monotone AVVIs

Question 9.2. Regarding Theorem 9.5, is the solution map $\mathrm{Sol}(\mathrm{AVVI}(\cdot, K))$ is usc at ω if the set $\mathrm{Sol}^w(\mathrm{AVVI}(\omega, K))$ is nonempty and bounded? Is the final assertion of Theorem 9.5 valid under the weaker assumption that $\mathrm{Sol}(\mathrm{AVVI}(\omega, K))$ is nonempty and bounded? How to use the scalarization method to obtain sufficient conditions for the lsc property of the solution maps $\mathrm{Sol}(\mathrm{AVVI}(\cdot, K))$ and $\mathrm{Sol}^w(\mathrm{AVVI}(\cdot, K))$?

Question 9.3. The number of connected components of the set $\mathrm{Sol}^w(\mathrm{AVVI}(\omega, K))$ (resp., of $\mathrm{Sol}(\mathrm{AVVI}(\omega, K))$) is finite? If the just mentioned number is finite, can one find an explicit upper bound for it?

(The studies of [11, 12] for the special case of monotone AVVIs related to LFVOPs suggest that m – the number of affine functions in the formulation of the AVVI problem – might be such an upper bound.)

Question 9.4. The solution set $\mathrm{Sol}^w(\mathrm{AVVI}(\omega,K))$ (resp., $\mathrm{Sol}(\mathrm{AVVI}(\omega,K))$) is contractible whenever it is nonempty and bounded?

9.5.2 LFVOPs

Question 9.5. Regarding Theorem 9.10, is the solution map $\mathrm{Sol}(\cdot,\mathrm{VP}_1)$ is nonempty, is usc at ω if $\mathrm{Sol}^w(\omega,\mathrm{VP}_1)$ is nonempty and bounded? Is the final assertion of Theorem 9.10 valid under the weaker assumption that $\mathrm{Sol}(\omega,\mathrm{VP}_1)$ is nonempty and bounded? How to use the scalarization method to obtain sufficient conditions for the lsc property of the solution maps $\mathrm{Sol}(\cdot,\mathrm{VP}_1)$ and $\mathrm{Sol}^w(\cdot,\mathrm{VP}_1)$?

Question 9.6. Is it true that the following estimates hold: $\chi(\mathrm{Sol}(\mathrm{VP}_1)) \le m$, $\chi(\mathrm{Sol}^w(\mathrm{VP}_1)) \le m$, $\chi(\mathrm{Sol}(\mathrm{VP}_1)) \le n$, and $\chi(\mathrm{Sol}^w(\mathrm{VP}_1)) \le n$?

9.5.3 Convex QVOPs

Question 9.7. Regarding Theorem 9.17, is the solution map $\mathrm{Sol}(\cdot,\mathrm{VP}_2)$ is nonempty, is usc at ω if $\mathrm{Sol}^w(\omega,\mathrm{VP}_2)$ is nonempty and bounded? Is the final assertion of Theorem 9.17 valid under the weaker assumption that $\mathrm{Sol}(\omega,\mathrm{VP}_2)$ is nonempty and bounded?

Question 9.8. There exist convex QVOPs in the form (VP_2) whose weak Pareto solution set $\mathrm{Sol}^w(\omega,\mathrm{VP}_2)$ (resp., whose Pareto solution set $\mathrm{Sol}(\omega,\mathrm{VP}_2)$) is disconnected?

Question 9.9. If $\mathrm{Sol}^w(\omega,\mathrm{VP}_2)$ is bounded, then it is connected? If $\mathrm{Sol}^w(\omega,\mathrm{VP}_2)$ is disconnected, then each connected component of the solution set is unbounded?

Question 9.10. There exist convex QVOPs in the form (VP_2) whose weak Pareto solution set $\mathrm{Sol}^w(\omega,\mathrm{VP}_2)$ (resp., whose Pareto solution set $\mathrm{Sol}(\omega,\mathrm{VP}_2)$) is path-connected, but not contractible?

(The construction of [13], which has been recalled in the third section of this chapter, may be useful for studying this question.)

9.5.4 Strictly Quasiconvex VOPs

Question 9.11. For a continuous strictly quasiconvex m-criteria minimization problem on a nonempty closed convex set in $\mathbb{R}^n$, denoted by (P), is it true that $\chi(\mathrm{Sol(P)}) \le m$, $\chi(\mathrm{Sol}^w(\mathrm{P})) \le m$, $\chi(\mathrm{Sol(P)}) \le n$, and $\chi(\mathrm{Sol}^w(\mathrm{P})) \le n$?

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Chapter 10
Levitin–Polyak Type Well-Posedness in Constrained Optimization

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10.1 Introduction

Well-posedness of unconstrained and constrained scalar optimization problems was first introduced and studied in Tykhonov [30] and Levitin and Polyak [21], respectively. Since then, various notions of well-posednesses have been defined and extensively studied (see, e.g., [5,8,25,28,32–34]). Recent studies on well-posedness of optimization problems have been extended to vector optimization problems (see, e.g., [3, 7, 12, 13, 22, 24]). The study of LP well-posedness for convex scalar optimization problems with functional constraints originates from [19]. In Sect. 10.2 of this chapter, we will introduce three types of (generalized) LP well-posedness for convex scalar optimization problems with functional constraints. Characterizations and criteria for the three types of (generalized) LP well-posedness will be derived. Relations among these three types of (generalized) LP well-posednesses will be established. We will also present convergence results for a class of penalty methods and a class of augmented Lagrangian methods under the assumption of one of the three types of LP well-posedness. In Sect. 10.3, we will introduce several types of (generalized) LP well-posedness for vector optimization problems with functional constraints. Criteria and characterizations for these types of well-posednesses will be given. Relations among these types of well-posedness will be presented. We will also carry out convergence analysis for a class of penalty methods under the assumption of a type of generalized LP well-posedness.

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10.2 Generalized Levitin–Polyak Well-Posedness in Constrained Optimization

In this section, we introduce three types of (generalized) LP well-posedness for nonconvex scalar optimization problems with functional constraints. Characterizations and criteria for the three types of (generalized) LP well-posedness are derived. Relations among these three types of (generalized) LP well-posednesses are established. We also present convergence results for a class of penalty methods and a class of augmented Lagrangian methods.

10.2.1 Preliminaries

Let (X,d_1) and (Y,d_2) be two metric spaces, $X_1 \subseteq X$, $K \subseteq Y$ be two nonempty and closed sets. Consider the following constrained optimization problem:

$$\text{(P)} \qquad \min f(x)$$
$$\text{s.t. } x \in X_1,\ g(x) \in K,$$

where $f : X \to R^1$ is a lower semicontinuous function and $g : X \to Y$ is a continuous function. Denote by X_0 the set of feasible solutions of (P), i.e.,

$$X_0 = \{x \in X_1 : g(x) \in K\}.$$

Denote by $\overline{X}$ and $\bar{v}$ the optimal solution set and the optimal value of (P), respectively. Throughout this section, we always assume that $X_0 \neq \emptyset$ and $\bar{v} > -\infty$.

Let (Z,d) be a metric space and $Z_1 \subseteq Z$. We denote by $d_{Z_1}(z) = \inf\{d(z,z') : z' \in Z_1\}$ the distance from the point z to the set Z_1.

Levitin–Polyak (LP in short) well-posedness of (P) in the usual sense (when the optimal set of (P) is not necessarily a singleton) says that, for any sequence $\{x_n\} \subseteq X_1$ satisfying (i) $d_{X_0}(x_n) \to 0$ and (ii) $f(x_n) \to \bar{v}$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \overline{X}$ such that $x_{n_k} \to \bar{x}$.

It should be noted that many optimization algorithms, such as penalty type methods, e.g., penalty function methods and augmented Lagrangian methods, terminate when the constraint is approximately satisfied, i.e., $d_K(g(\bar{x})) \leq \varepsilon$ for some $\varepsilon > 0$ sufficiently small, and $\bar{x}$ is taken as an approximate solution of problem(P). These methods may generate a sequence $\{x_n\} \subseteq X_1$ that satisfies $d_K(g(x_n)) \to 0$, not necessarily $d_{X_0}(x_n) \to 0$, as shown in the following simple example.

Example 10.1. Let $\alpha > 0$. Let $X = \mathbb{R}$, $X_1 = \mathbb{R}_+$, $K = \mathbb{R}_-$ and

$$f(x) = \begin{cases} -x^{\alpha}, & \text{if } x \in [0,1]; \\ -1/x^{\alpha}, & \text{if } x \geq 1, \end{cases}$$

$$g(x) = \begin{cases} x, & \text{if } x \in [0,1] \\ 1/x^2, & \text{if } x \geq 1. \end{cases}$$

Consider the following penalty problems:

$$(PP_\alpha(n)) \quad \min_{x \in X_1} f(x) + n\left[\max\{0, g(x)\}\right]^\alpha, \quad n \in \mathbb{N}.$$

It is easily verified that $x_n = 2^{1/\alpha} n^{1/\alpha}$ is the unique global solution to the penalized problem $(PP_\alpha(n))$ for each $n \in N$. Note that $X_0 = \{0\}$. It follows that we have $d_K(g(x_n)) = 1/(2^{2/\alpha} n^{2/\alpha}) \to 0$ while $d_{X_0}(x_n) = 2^{1/\alpha} n^{1/\alpha} \to +\infty$.

Thus, it is useful to consider sequences that satisfy $d_K(g(x_n)) \to 0$ instead of $d_{X_0}(x_n) \to 0$ as $n \to \infty$ in order to study convergence of penalty type methods.

The sequence $\{x_n\}$ satisfying (i) and (ii) above is called an LP minimizing sequence. In what follows, we introduce another two types of generalized LP well-posedness.

Definition 10.1. (P) is called *LP well-posedness* in the generalized sense if, for any sequence $\{x_n\} \subseteq X_1$ satisfying (i) $d_K(g(x_n)) \to 0$ and (ii) $f(x_n) \to \bar{v}$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \overline{X}$ such that $x_{n_k} \to \bar{x}$. The sequence $\{x_n\}$ is called a generalized *LP minimizing sequence*.

Definition 10.2. (P) is called LP well-posedness in the strongly generalized sense if, for any sequence $\{x_n\} \subseteq X_1$ satisfying (i) $d_K(g(x_n)) \to 0$; and(ii) $\limsup_{n \to +\infty} f(x_n) \leq \bar{v}$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \overline{X}$ such that $x_{n_k} \to \bar{x}$. The sequence $\{x_n\}$ is called a weakly generalized LP minimizing sequence.

Remark 10.1. (a) The study of well-posedness for optimization problems with explicit constraints dates back to [19] when the abstract set X_1 does not appear. In [19], it was assumed that X is a Banach space, Y is a Banach space ordered by a closed and convex cone with some special properties, see [19] for details. What is worth emphasizing is that [19] only studied the case when (P) is a convex program. However, it is well-known that penalty-type methods such as penalization methods, augmented Lagrangian methods are mostly developed for constrained nonconvex optimization problems. This is the main motivation of this section.

(b) The LP well-posedness in the strongly generalized sense defined above was called well-posedness in the strongly generalized sense in [19], while a weakly generalized LP minimizing sequence in the above definition is called a generalized minimizing sequence in [19].

(c) It is obvious that LP well-posedness in the strongly generalized sense implies LP well-posedness in the generalized sense because a generalized LP minimizing sequence is a weakly generalized LP minimizing one.

(d) If there exists some $\delta_0 > 0$ such that g is uniformly continuous on the set

$$\{x \in X_1 : d_{X_0}(x) \leq \delta_0\},$$

then, it is not difficult to see that LP well-posedness in the generalized sense implies LP well-posedness.

(e) Any one type of (generalized) LP well-posedness defined above implies that the optimal set $\overline{X}$ of (P) is nonempty and compact.

10.2.2 Necessary and Sufficient Conditions for (Generalized) LP Well-Posedness

In this subsection, we present some criteria and characterizations for the three types of (generalized) LP well-posedness defined in Definitions 10.1, 10.2.

Consider the following statement:

$$\Big[\overline{X} \neq \emptyset \text{ and, for any LP minimizing sequence (resp. generalized}$$
$$\text{LP minimizing sequence, weakly generalized LP minimizing sequence)}$$
$$\{x_n\}, \text{we have } d_{\overline{X}}(x_n) \to 0.\Big] \tag{10.1}$$

The proof of the following proposition is elementary and thus omitted.

Proposition 10.1. *If (P) is LP well-posed (resp. LP well-posed in the generalized sense, and LP well-posed in the strongly generalized sense), then (10.1) holds. Conversely, if (10.1) holds and $\overline{X}$ is compact, then (P) is LP well-posed (resp. LP well-posed in the generalized sense, and LP well-posed in the strongly generalized sense).*

Consider a real-valued function $c = c(t,s)$ defined for $t, s \geq 0$ sufficiently small, such that

$$c(t,s) \geq 0, \quad \forall t,s, \quad c(0,0) = 0, \tag{10.2}$$

$$s_k \to 0, t_k \geq 0, c(t_k, s_k) \to 0 \text{ imply } t_k \to 0. \tag{10.3}$$

Theorem 10.1. *If (P) is LP well-posed, then there exists a function c satisfying (10.2) and (10.3) such that*

$$|f(x) - \overline{v}| \geq c(d_{\overline{X}}(x), d_{X_0}(x)), \quad \forall x \in X_1. \tag{10.4}$$

Conversely, suppose that $\overline{X}$ is nonempty and compact, and (10.4) holds for some c satisfying (10.2) and (10.3). Then (P) is LP well-posed.

Proof. Define

$$c(t,s) = \inf\{|f(x) - \overline{v}| : x \in X_1, d_{\overline{X}}(x) = t, d_{X_0}(x) = s\}.$$

It is obvious that $c(0,0) = 0$. Moreover, if $s_n \to 0$, $t_n \geq 0$ and $c(t_n, s_n) \to 0$, then, there exists a sequence $\{x_n\} \subseteq X_1$ with

$$d_{\overline{X}}(x_n) = t_n, \tag{10.5}$$

$$d_{X_0}(x_n) = s_n \tag{10.6}$$

such that

$$|f(x_n) - \overline{v}| \to 0. \tag{10.7}$$

Note that $s_n \to 0$. (10.6) and (10.7) jointly imply that $\{x_n\}$ is an LP minimizing sequence. By Proposition 10.1, we have $t_n \to 0$. This completes the proof of the first half of the theorem. Conversely, let $\{x_n\}$ be an LP minimizing sequence. Then, by (10.4), we have

$$|f(x_n) - \overline{v}| \geq c(d_{\overline{X}}(x_n), d_{X_0}(x_n)), \quad \forall x \in X_1. \tag{10.8}$$

Let

$$t_n = d_{\overline{X}}(x_n), \quad s_n = d_{X_0}(x_n).$$

Then, $s_n \to 0$. In addition, $|f(x_n) - \overline{v}| \to 0$. These facts together with (10.8) as well as the properties of the function c imply that $t_n \to 0$. By Proposition 10.1, we see that (P) is LP well-posed. □

Theorem 10.2. *If (P) is LP well-posed in the generalized sense, then there exists a function c satisfying (10.2) and (10.3) such that*

$$|f(x) - \overline{v}| \geq c(d_{\overline{X}}(x), d_K(g(x))), \quad \forall x \in X_1. \tag{10.9}$$

Conversely, suppose that $\overline{X}$ is nonempty and compact, and (10.9) holds for some c satisfying (10.2) and (10.3). Then (P) is LP well-posed in the generalized sense.

Proof. The proof is almost the same as that of Theorem 10.1. The only difference lies in the proof of the first part of Theorem 10.1. Here we define

$$c(t,s) = \inf\{|f(x) - \overline{v}| : x \in X_1, d_{\overline{X}}(x) = t, d_K(g(x)) = s\}.$$

□

Next we give a necessary and sufficient condition in the form of Furi–Vignoli [10] to characterize the LP well-posedness in the strongly generalized sense.
Let

$$\Omega(\varepsilon) = \{x \in X_1 : f(x) \leq \overline{v} + \varepsilon, d_K(g(x)) \leq \varepsilon\}.$$

Let (X, d_1) be a complete metric space. Recall that the Kuratowski measure of noncompactness for a subset A of X is defined as

$$\alpha(A) = \inf\left\{\varepsilon > 0 : A \subseteq \bigcup_{1 \leq i \leq n} C_i, \text{ for some } C_i, \text{diam}(C_i) \leq \varepsilon\right\},$$

where $\mathrm{diam}(C_i)$ is diameter of C_i defined by

$$\mathrm{diam}(C_i) = \sup\{d_1(x_1,x_2) : x_1,x_2 \in C_i\}.$$

The next theorem can be proved analogously to [19, Theorem 5.5].

Theorem 10.3. *Let (X,d_1) be a complete metric space and f be bounded below on X_0. Then, (P) is LP well-posed in the strongly generalized sense if and only if*

$$\alpha(\Omega(\varepsilon)) \to 0, \text{ as } \varepsilon \to 0.$$

Definition 10.3. Let Z be a topological space and $Z_1 \subseteq Z$ be nonempty. Suppose that $h : Z \to \mathbb{R} \cup \{+\infty\}$ is an extended real-valued function. h is said to be *level-compact* on Z_1 if, for any $s \in \mathbb{R}$, the subset $\{z \in Z_1 : h(z) \le s\}$ is compact.

For any $\delta \ge 0$, define

$$X_1(\delta) = \{x \in X_1 : d_K(g(x)) \le \delta\}. \tag{10.10}$$

The following proposition gives sufficient conditions that guarantee LP well-posedness in the strongly generalized sense.

Proposition 10.2. *Let one of the following conditions hold.*

(i) There exists $\delta_0 > 0$ such that $X_1(\delta_0)$ is compact.
(ii) f is level-compact on X_1.
(iii) X is a finite dimensional normed space and

$$\lim_{x\in X_1, \|x\|\to+\infty} \max\{f(x), d_K(g(x))\} = +\infty; \tag{10.11}$$

(iv) There exists $\delta_0 > 0$ such that f is level-compact on $X_1(\delta_0)$.

Then, (P) is LP well-posed in the strongly generalized sense.

Proof. Let $\{x_n\} \subseteq X_1$ be a weakly generalized LP minimizing sequence. Then,

$$\limsup_{n\to+\infty} f(x_n) \le \bar{v}, \tag{10.12}$$

$$d_K(g(x_n)) \to 0. \tag{10.13}$$

(i) Elementary. It is obvious that condition (ii) implies (iv). Now we show that (iii) implies (iv). Indeed, we need only to show that for any $s \in \mathbb{R}$ and any $\delta > 0$, the set

$$A = \{x \in X_1(\delta) : f(x) \le s\}$$

is bounded since X is a finite dimensional space. Suppose to the contrary that there exist $\delta > 0$, $s > 0$ and $\{x'_n\} \subseteq X_1(\delta)$ such that

$$\|x'_n\| \to +\infty \text{ and } f(x'_n) \le s.$$

By $\{x_n'\} \subseteq X_1(\delta)$, we have $\{x_n'\} \subseteq X_1$ and

$$d_K(g(x_n')) \leq \delta.$$

As a result,

$$\max\{f(x_n'), d_K(g(x_n'))\} \leq \max\{s, \delta\},$$

contradicting (10.11).

Thus, we need only to prove that if (iv) holds, then (P) is LP well-posed in the strongly generalized sense. By (10.13), it is apparent that we can assume without loss of generality that $\{x_n\} \subseteq X_1(\delta_0)$. By (10.12), we can assume without loss of generality that

$$\{x_n\} \subseteq \{x \in X_1 : f(x) \leq \bar{v} + 1\}.$$

By the level-compactness of f on $X_1(\delta_0)$, we deduce that there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\bar{x} \in X_1$ such that $x_{n_k} \to \bar{x}$. It is obvious from (10.13) that $\bar{x} \in X_0$. Furthermore, from (10.12), we deduce that $f(\bar{x}) \leq \bar{v}$. So we have $f(\bar{x}) = \bar{v}$. That is, $\bar{x} \in \overline{X}$. Hence, (P) is LP well-posed in the strongly generalized sense. □

Now we consider the case when Y is a normed space, K is a closed and convex cone with nonempty interior int K. Arbitrarily fix an $e \in \text{int}\, K$. Let $t \geq 0$ and consider the following perturbed problem of (P):

$$(P_t) \quad \min f(x)$$
$$\text{s.t. } x \in X_1,\ g(x) \in K - te. \tag{10.14}$$

Let

$$X_2(t) = \{x \in X_1 : g(x) \in K - te\}. \tag{10.15}$$

Proposition 10.3. *Let one of the following conditions hold.*

(i) There exists $t_0 > 0$ such that $X_2(t_0)$ is compact.
(ii) f is level-compact on X_1.
(iii) X is a finite dimensional normed space and

$$\lim_{x \in X_1, \|x\| \to +\infty} max\{f(x), d_K(g(x))\} = +\infty;$$

(iv) There exists $t_0 > 0$ such that f is level-compact on $X_2(t_0)$.

Then, (P) is LP well-posed in the strongly generalized sense.

Proof. Similar to that of Proposition 10.2. □

Now we make the following assumption.

Assumption 10.1 *X is a finite dimensional normed space, Y is a normed space, $X_1 \subseteq X$ is a nonempty, closed and convex set, $K \subseteq Y$ is a closed and convex cone with nonempty interior int$K$ and $e \in \text{int}K$, f and g are continuous on X_1, f is a* *convex function on* X_1 *and* g *is* K*-concave on* X_1 *(namely, for any* $x_1, x_2 \in X_1$ *and any* $\theta \in (0,1)$*, there holds that* $g(\theta x_1 + (1-\theta)x_2) - \theta g(x_1) - (1-\theta)g(x_2) \in K$*).*

It is obvious that under Assumption 10.1, (P) is a convex program.

The next lemma can be proved similarly to that of ([17], Proposition 2.4).

Lemma 10.1. *Let Assumption 10.1 hold. Then the following two statements are equivalent.*

(a) The optimal set $\overline{X}$ *of (P) is nonempty and compact.*
(b) For any $t \geq 0$*,* f *is level-compact on the set* $X_2(t)$*.*

Theorem 10.4. *Let Assumption 10.1 hold. Then, (P) is LP well-posed in the strongly generalized sense if and only if the optimal set* $\overline{X}$ *of (P) is nonempty and compact.*

Proof. The sufficiency part follows directly from Lemma 10.1 and Proposition 10.3 while the necessity part is obvious by Remark 10.1. □

The next two lemmas will be used to derive the following theorem.

Lemma 10.2 ([1]). *Let* (Z,d) *be a complete metric space and* $h : Z \to \mathbb{R} \cup \{+\infty\}$ *be lower semicontinuous and bounded below. Let* $\varepsilon > 0$*. Suppose that* $z_0 \in Z$ *satisfies* $h(z_0) \leq \inf\{h(z) : z \in Z\} + \varepsilon$*. Then, there exists* $z_\varepsilon \in Z$ *such that*

(a) $h(z_\varepsilon) \leq h(z_0)$
(b) $d(z_\varepsilon, z_0) \leq \sqrt{\varepsilon}$
(c) $h(z_\varepsilon) < h(z) + \sqrt{\varepsilon} d(z, z_\varepsilon), \forall z \in Z \backslash \{z_\varepsilon\}$.

Lemma 10.3. *Let* Y *be a normed space and* $K \subseteq Y$ *be a closed and convex cone with* $\operatorname{int} K \neq \emptyset$ *and* $e \in \operatorname{int} K$*. Suppose that* $\{y_n\} \subseteq Y$*. Then,* $d_K(y_n) \to 0$ *if and only if there exists a sequence* $\{t_n\} \subseteq \mathbb{R}_+$ *with* $t_n \to 0$ *such that* $y_n \in K - t_n e$*.*

Proof. Necessity. From $d_K(y_n) \to 0$, we have $\{u_n\} \subseteq K$ such that $\|y_n - u_n\| \to 0$. Let $y'_n = y_n - u_n$. Then $\|y'_n\| \to 0$. Let $t_n = \sqrt{\|y'_n\|}$. Then, $\{t_n\} \subseteq \mathbb{R}_+$, $t_n \to 0$ and $y'_n / t_n \to 0$. Since $e \in \operatorname{int} K$, it follows that $e + y'_n / t_n \in K$ when n is sufficiently large. Consequently, $y'_n \in K - t_n e$. Hence, $y_n = u_n + y'_n \in K - t_n e$.
Sufficiency. As $y_n \in K - t_n e$, we have $y_n + t_n e \in K$. Thus,

$$d_K(y_n) \leq \|y_n - (y_n + t_n e)\| = t_n \|e\|.$$

Hence, $d_K(y_n) \to 0$. □

Suppose that K is a cone. We denote by K^* the positive polar cone of K, i.e.,

$$K^* = \{\mu \in Y^* : \mu(u) \geq 0, \forall u \in K\}.$$

Theorem 10.5. *Assume that* X *is a Banach space,* Y *is a normed space,* $X_1 \subseteq X$ *is nonempty, closed and convex.* $K \subseteq Y$ *is a closed and convex cone with* $\operatorname{int} K \neq \emptyset$ *and* $e \in \operatorname{int} K$*. Suppose that* $f : X \to \mathbb{R}$ *is convex and continuously differentiable on* X_1*,* $g : X \to Y$ *is* K*-concave and continuously differentiable on* X_1*. Let Slater* *constraint qualification for (P) hold: there exists* $x_0 \in X_1$ *such that* $g(x_0) \in \text{int } K$. *Assume that the optimal set* $\overline{X}$ *of (P) is nonempty. Further assume that, for any sequences* $\{x_n\} \subseteq X_1$ *and* $\{\mu_n\} \subseteq K^*$ *satisfying*

(i) $\lim_{n\to+\infty} d_K(g(x_n)) = 0$
(ii) There exists a subsequence $\{\mu_{n_k}\}$ *such that* $\mu_{n_k} = 0, \forall k$ *or* $\lim_{n\to+\infty} \mu_n(g(x_n))/\|\mu_n\| = 0$
(iii) $\lim_{n\to+\infty} d_{(-N_{X_1}(x_n))}(\bigtriangledown f(x_n) - \mu_n(\bigtriangledown g(x_n)) = 0$, *where* $N_{X_1}(x_n)$ *is the normal cone of* X_1 *at* x_n,

there exists a convergent subsequence of $\{x_n\}$. *Then, (P) is LP well-posed in the strongly generalized sense.*

Proof. Suppose that $\bar{x} \in \overline{X}$. Since Slater constraint qualification holds, we have $\overline{\mu} \in K^*$ such that

$$f(\bar{x}) \leq f(x) - \overline{\mu}(g(x)), \quad \forall x \in X_1 \tag{10.16}$$

and

$$\overline{\mu}(g(\bar{x})) = 0. \tag{10.17}$$

Let $\{x_n\} \subseteq X_1$ be a weakly generalized LP minimizing sequence for (P). Then, by Lemma 10.3,

$$\limsup_{n\to+\infty} f(x_n) \leq \bar{v} \tag{10.18}$$

and

$$g(x_n) \in K - t_n e \tag{10.19}$$

for some $\{t_n\} \subseteq \mathbb{R}_+$ with $t_n \to 0$. From (10.16), we have

$$f(\bar{x}) \leq f(x) - \overline{\mu}(g(x)), \quad \forall x \in X_2(t_n).$$

Note that

$$-\overline{\mu}(g(x)) \leq t_n \overline{\mu}(e), \quad \forall x \in X_2(t_n).$$

Thus,

$$f(\bar{x}) \leq f(x) + t_n \overline{\mu}(e), \quad \forall x \in X_2(t_n). \tag{10.20}$$

Hence,

$$\inf_{x \in X_2(t_n)} f(x) > -\infty. \tag{10.21}$$

The combination of (10.19) and (10.20) gives us

$$f(\bar{x}) \leq f(x_n) + t_n \overline{\mu}(e).$$

Consequently,

$$f(\bar{x}) \leq \liminf_{n\to+\infty} f(x_n).$$

This together with (10.18) yields

$$\lim_{n\to+\infty} f(x_n) = f(\overline{x}). \tag{10.22}$$

This combined with (10.20) implies that there exists $\varepsilon_n \to 0^+$ such that

$$f(x_n) \le f(x) + \varepsilon_n, \quad \forall x \in X_2(t_n).$$

Note that $X_2(t_n) \subseteq X$ is nonempty and closed. $(X_2(t_n), \|\cdot\|)$ can be seen as a complete (metric) subspace of X. Applying Lemma 10.2, we obtain

$$x_n' \in X_2(t_n) \tag{10.23}$$

such that

$$\|x_n - x_n'\| \le \sqrt{\varepsilon_n} \tag{10.24}$$

and

$$f(x_n') \le f(x) + \sqrt{\varepsilon_n}\|x - x_n'\|, \quad \forall x \in X_2(t_n). \tag{10.25}$$

Note that Slater constraint qualification also holds for the following constrained optimization problem:

$$(P_n) \qquad \min\ f(x) + \sqrt{\varepsilon_n}\|x - x_n'\|$$
$$\text{s.t. } x \in X_1,\ g(x) \in K - t_n e,$$

and by (10.25), x_n' is an optimal solution of (P_n). Hence, there exists $\mu_n \in K^*$ such that

$$0 \in \bigtriangledown f(x_n') - \mu_n(\bigtriangledown g(x_n')) + \sqrt{\varepsilon_n} B^* + N_{X_1}(x_n') \tag{10.26}$$

and

$$\mu_n(g(x_n') + t_n e) = \mu_n(g(x_n')) + t_n \mu_n(e) = 0, \tag{10.27}$$

where B^* is the closed unit ball of $\overline{X}$. Equation (10.26) implies that

$$\lim_{n\to+\infty} d_{(-N_{X_1}(x_n'))}(\bigtriangledown f(x_n') - \mu_n(\bigtriangledown g(x_n'))) = 0. \tag{10.28}$$

From (10.27), we see that if there does not exist a subsequence $\{\mu_{n_k}\}$ such that $\mu_{n_k} = 0, \forall k$, then

$$\lim_{n\to+\infty} \mu_n(g(x_n))/\|\mu_n\| = 0. \tag{10.29}$$

The combination of (10.24), (10.28) and (10.29) implies that $\{x_n'\}$ and $\{\mu_n\}$ satisfy conditions (i)–(iii) of the theorem. Thus, $\{x_n'\}$ has a subsequence $\{x_{n_k}'\}$ which converges to some $\overline{x}' \in X_0$. From (10.24), we deduce that $x_{n_k} \to \overline{x}' \in X_0$. This combined with (10.22) implies $\overline{x}' \in \overline{X}$. Hence, (P) is LP well-posed in the strongly generalized sense. □

Remark 10.2. Conditions (i)–(iii) can be seen as the well-known Palais–Smale condition (C) [1] in the case of constrained optimization.

10.2.3 Relations Among Three Types of (Generalized) LP Well-Posedness

Simple relationships among the three types of LP well-posedness were mentioned in Remark 10.1. Now we investigate further relationships among them.

The proof of next theorem is elementary and is omitted.

Theorem 10.6. *Suppose that there exist $\delta > 0$, $\alpha > 0$ and $c > 0$ such that*

$$d_{X_0}(x) \leq c d_K^{\alpha}(g(x)), \quad \forall x \in X_1(\delta), \tag{10.30}$$

where $X_1(\delta)$ is defined by (10.10). If (P) is LP well-posed, then (P) is LP well-posed in the generalized sense.

Remark 10.3. Equation (10.30) is an error bound condition for the set X_0 in terms of the residual function

$$r(x) = d_K(g(x)), \quad \forall x \in X_1.$$

When $X = \mathbb{R}$, $Y = \mathbb{R}^m$, $X_1 = X$ and $X_0 \neq \emptyset$, by [27, Theorem 5], (10.30) holds if and only if , for any $y \in \mathbb{R}^m$ with $\|y\| \leq \delta$,

$$\Psi(y) \subseteq \Psi(0) + c\|y\|^{\alpha} B,$$

where

$$\Psi(y) = \{x \in \mathbb{R} : g(x) \in K + y\}, \quad y \in \mathbb{R}^m$$

and B is the closed unit ball of Y. Sufficient conditions guaranteeing (10.30) were given in numerous papers on error bounds for systems of inequalities and metric regularity of set-valued maps (when (10.30) holds locally with $\alpha = 1$) in finite and infinite dimensional spaces (see, e.g., [6, 9, 21] and the references therein).

Definition 10.4 ([4]). Let W be a topological space and $F : W \to 2^X$ be a set-valued map. F is said to be upper Hausdorff semicontinuous (u.H.c. in short) at $w \in W$ if, for any $\varepsilon > 0$, there exists a neighbourhood U of w such that $F(U) \subseteq B(F(w), \varepsilon)$, where, for $Z \subseteq X$ and $r > 0$,

$$B(Z, r) = \{x \in X : d_Z(x) \leq r\}.$$

Definition 10.5 ([1]). Let W be a topological space and $F : W \to 2^X$ be a set-valued map. F is said to be upper semicontinuous in the Berge's sense (u.s.c. in short) at $w \in W$ if, for any neighbourhood Ω of $F(w)$, there exists a neighbourhood U of w such that $F(U) \subseteq \Omega$.

It is obvious that the notion of u.s.c. (in Berge's sense) is stronger than u.H.c.

Clearly, $X_1(\delta)$ given by (10.10) can be seen as a set-valued map from $\mathbb{R}_+$ to X. The next two theorems use conditions similar to those for the general stability results presented in Sect. 3 of [4], where the uniform continuity of the objective function around the feasible set and the u.H.c. of the perturbation set-valued map were considered.

Theorem 10.7. *Assume that the set-valued map $X_1(\delta)$ defined by (10.10) is u.H.c. at $\mathbf{0} \in \mathbb{R}^l_+$. If (P) is LP well-posed, then (P) is LP well-posed in the generalized sense.*

Proof. Let $\{x_n\} \subseteq X_1$ be a generalized LP minimizing sequence. That is,

$$f(x_n) \to \bar{v}, \tag{10.31}$$

$$d_K(g(x_n)) \to 0. \tag{10.32}$$

(10.32), together with the u.H.c. of $X_1(\delta)$ at $\mathbf{0}$, implies that $d_{X_0}(x_n) \to 0$. This fact combined with (10.31) implies that $\{x_n\}$ is an LP minimizing sequence. Thus, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \overline{X}$ such that $x_{n_k} \to \bar{x}$. Hence, (P) is LP well-posed in the generalized sense. □

Theorem 10.8. *Assume that there exists $\varepsilon_0 > 0$ such that f is uniformly continuous on $B(X_0, \varepsilon_0)$ and the set-valued map $X_1(\delta)$ is u.H.c. at $\mathbf{0}$. If (P) is LP well-posed, then it is LP well-posed in the strongly generalized sense.*

Proof. Let $\{x_n\}$ be a weakly generalized LP minimizing sequence. That is,

$$\limsup_{n\to+\infty} f(x_n) \le \bar{v}, \tag{10.33}$$

$$d_K(g(x_n)) \to 0. \tag{10.34}$$

Note that $X_1(\delta)$ is u.H.c. at $\mathbf{0}$. This fact together with (10.34) implies that $d_{X_0}(x_n) \to 0$. Note that f is uniformly continuous on $B(X_0, \varepsilon_0)$. It follows that

$$\liminf_{n\to+\infty} f(x_n) \ge \bar{v}. \tag{10.35}$$

The combination of (10.33) and (10.35) yields that

$$f(x_n) \to \bar{v}.$$

Hence, $\{x_n\}$ is an LP minimizing sequence. Thus, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \overline{X}$ such that $x_{n_k} \to \bar{x}$. So, (P) is LP well-posed in the strongly generalized sense. □

Let $\delta \ge 0$. Consider the perturbed problem of (P):

$$(P_\delta) \qquad \min \ f(x)$$
$$\text{s.t.} \quad x \in X_1, \ d_K(g(x)) \le \delta.$$

Denote by $v_1(\delta)$ the optimal value of (P_δ). Clearly, $v_1(0) = \bar{v}$.

Theorem 10.9. *Consider problems (P) and (P_δ). Suppose that (P) is LP well-posed in the generalized sense and*

$$\liminf_{\delta\to 0^+} v_1(\delta) = \bar{v}. \tag{10.36}$$

Then, (P) is LP well-posed in the strongly generalized sense.

Proof. Let $\{x_n\} \subseteq X_1$ be a weakly generalized LP minimizing sequence. Then,

$$\limsup_{n\to+\infty} f(x_n) \le \bar{v} \tag{10.37}$$

and

$$\lim_{n\to+\infty} d_K(g(x_n)) = 0.$$

Let $\delta_n = d_K(g(x_n))$. Then, x_n is feasible for (P_{δ_n}). Thus,

$$v_1(\delta_n) \le f(x_n).$$

Passing to the lower limit, we get

$$\liminf_{n\to+\infty} v_1(\delta_n) \le \liminf_{n\to+\infty} f(x_n).$$

This together with (10.37) and (10.36) yields

$$\lim_{n\to+\infty} f(x_n) = \bar{v}.$$

It follows that $\{x_n\}$ is a generalized LP minimizing sequence. Thus, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \overline{X}$ such that $x_{n_k} \to \bar{x}$. So, (P) is LP well-posed in the strongly generalized sense. □

Remark 10.4. If the set-valued map $X_1(\delta)$ defined by (10.10) is u.s.c. at $\mathbf{0} \in \mathbb{R}^1_+$, by [2, Theorem 4.2.3 (1)], (10.36) holds. In this case, the generalized LP well-posedness of (P) implies the strongly generalized LP well-posedness of (P).

Now let Y be a normed space and $y \in Y$. Consider the following perturbed problem of (P):

$$(P_y) \qquad \min f(x)$$
$$\text{s.t. } x \in X_1,\ g(x) \in K + y.$$

Denote by

$$X_3(y) = \{x \in X_1 : g(x) \in K + y\} \tag{10.38}$$

the feasible set of (P_y) and $v_3(y)$ the optimal value of (P_y). Here we note that if $X_3(y) = \emptyset$, we set $v_3(y) = +\infty$. It is obvious that $X_3(y)$ can be seen as a set-valued map from Y to X. Corresponding to Theorems 10.7–10.9 respectively, we have the following theorems.

Theorem 10.10. *Assume that Y is a normed space and that the set-valued map $X_3(y)$ is u.H.c. at $\mathbf{0} \in Y$. If (P) is LP well-posed, then (P) is LP well-posed in the generalized sense.*

Theorem 10.11. *Assume that Y is a normed space and that there exists $\varepsilon_0 > 0$ such that f is uniformly continuous on $B(X_0, \varepsilon_0)$ and the set-valued map $X_3(y)$ is u.H.c. at $\mathbf{0} \in Y$. If (P) is LP well-posed, then it is LP well-posed in the strongly generalized sense.*

Theorem 10.12. *Assume that Y is a normed space. Consider problems (P) and (P_y). Suppose that (P) is LP well-posed in the generalized sense and*

$$\liminf_{y \to 0} v_3(y) = \bar{v}. \tag{10.39}$$

Then, (P) is LP well-posed in the strongly generalized sense.

Similar to Remark 10.4, when the set-valued map X_3 is u.s.c. at $\mathbf{0} \in Y$, then (10.39) holds. Thus, the generalized LP well-posedness of (P) implies its strongly generalized LP well-posedness.

In the special case when K is a closed and convex cone with nonempty interior int K. Arbitrarily fix an $e \in \text{int}\, K$. It is obvious that $X_2(t)$ defined by (10.15) can be seen as a set-valued map from $\mathbb{R}^1_+$ to X. Denote by $v_2(t)$ the optimal value of (P_t).

Theorem 10.13. *Assume that K is a closed and convex cone with nonempty interior int$K$ and that the set-valued map $X_2(t)$ is u.H.c. at $\mathbf{0} \in \mathbb{R}^l$. If (P) is LP well-posed, then (P) is LP well-posed in the generalized sense.*

Theorem 10.14. *Assume that K is a closed and convex cone with nonempty interior* int K *and that there exists $\varepsilon_0 > 0$ such that f is uniformly continuous on $B(X_0, \varepsilon_0)$ and the set-valued map $X_2(t)$ is u.H.c. at $\mathbf{0} \in \mathbb{R}^l$. If (P) is LP well-posed, then it is LP well-posed in the strongly generalized sense.*

Theorem 10.15. *Assume that K is a closed and convex cone with nonempty interior* int K. *Consider problems (P) and (P_t). Suppose that (P) is LP well-posed in the generalized sense and*

$$\liminf_{t \to 0^+} v_2(t) = \bar{v}. \tag{10.40}$$

Then, (P) is LP well-posed in the strongly generalized sense.

Again, as noted in Remark 10.4, when the set-valued map X_2 is u.s.c. at $0 \in \mathbb{R}_+$, then (10.39) holds. Thus, the generalized LP well-posedness of (P) implies its strongly generalized LP well-posedness.

10.2.4 Applications to Penalty-Type Methods

In this subsection, we consider the convergence of a class of penalty methods and a class of augmented Lagrangian methods under the assumption of strongly generalized LP well-posedness of (P).

10.2.4.1 Penalty Methods

Let $\alpha > 0$. Consider the following penalty problem:

$$(PP_\alpha(r)) \quad \min_{x \in X_1} f(x) + r d_K^\alpha(g(x)), \quad r > 0.$$

Denote by $v_4(r)$ the optimal value of $(PP_\alpha(r))$. It is clear that

$$v_4(r) \le \bar{v}, \quad \forall r > 0. \tag{10.41}$$

Remark 10.5. When $\alpha \in (0,1)$, $X = \mathbb{R}$, $Y = \mathbb{R}^m$, $K = \mathbb{R}_-^{m_1} \times \{0_{m-m_1}\}$, where $m \ge m_1$ and 0_{m-m_1} is the origin of the space $\mathbb{R}^{m-m_1}$, this class of penalty functions was applied to the study of mathematical programs with equilibrium constraints [26]. Necessary and sufficient conditions for the exact penalization of this class of penalty functions were derived in [15]. This class of penalty methods was also applied to mathematical programs with complementarity constraints [31] and nonlinear semidefinite programs [18]. An important advantage of this class of penalty methods is that it requires weaker conditions to guarantee its exact penalization property than the usual l_1 penalty function method (see [26]).

Theorem 10.16. *Let $0 < r_n \to +\infty$. Consider problems (P) and $(PP_\alpha(r_n))$. Assume that there exist $\bar{r} > 0$ and $m_0 \in \mathbb{R}^1$ such that*

$$f(x) + \bar{r} d_K^\alpha(g(x)) \ge m_0, \quad \forall x \in X_1. \tag{10.42}$$

Let $0 < \varepsilon_n \to 0$. Suppose that each $x_n \in X_1$ satisfies

$$f(x_n) + r_n d_K^\alpha(g(x_n)) \le v_4(r_n) + \varepsilon_n. \tag{10.43}$$

Further assume that (P) is LP well-posed in the strongly generalized sense. Then, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \overline{X}$ such that $x_{n_k} \to \bar{x}$.

Proof. From (10.41) and (10.43), we have

$$f(x_n) \le \bar{v} + \varepsilon_n.$$

Thus,

$$\limsup_{n \to +\infty} f(x_n) \le \bar{v}. \tag{10.44}$$

Moreover, from (10.41) to (10.43), we deduce that

$$f(x_n) + \bar{r} d_K^{\alpha}(g(x_n)) + (r_n - \bar{r}) d_K^{\alpha}(g(x_n)) \leq \bar{v} + \varepsilon_n.$$

Thus,

$$m_0 + (r_n - \bar{r}) d_K^{\alpha}(g(x_n)) \leq \bar{v} + \varepsilon_n,$$

implying

$$d_K(g(x_n)) \leq \left[\frac{\bar{v} + \varepsilon_n - m_0}{r_n - \bar{r}} \right]^{1/\alpha}.$$

Passing to the limit, we get

$$\lim_{n \to +\infty} d_K(g(x_n)) = 0. \tag{10.45}$$

It follows from (10.44) and (10.45) that $\{x_n\}$ is a weakly generalized LP minimizing sequence. Hence, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \overline{X}$ such that $x_{n_k} \to \bar{x}$. □

10.2.4.2 Augmented Lagrangian Methods

Let (X, d_1) be a metric space, $Y = \mathbb{R}^m$ and $K \subseteq Y$ be a nonempty, closed and convex set. Let $\sigma : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ be an augmenting function, namely, it is a lower semicontinuous, convex function satisfying

$$\min_{y \in \mathbb{R}^m} \sigma(y) = 0 \text{ and } \sigma \text{ attains its unique minimum at } y = 0.$$

Following [29, Example 11.46], we define the dualizing parametrization function by setting $X = X_1$ and $\theta = \delta_K$:

$$\bar{f}(x, u) = f(x) + \delta_{X_1}(x) + \delta_K(g(x) + u),$$

where δ_A is the indicator function of a subset A of a space Z, i.e.,

$$\delta_A(a) = \begin{cases} 0, & \text{if } a \in A, \\ +\infty, & \text{if } a \in Z \backslash A. \end{cases}$$

Constructing the augmented Lagrangian as in [29, Definition 11.55], we obtain the augmented Lagrangian:

$$\bar{l}(x, y, r) = \inf_{u \in \mathbb{R}^m} \left\{ \bar{f}(x, u) + r\sigma(u) - \langle y, u \rangle \right\}, x \in X, y \in \mathbb{R}^m, r > 0.$$

The augmented Lagrangian problem is

$$(ALP(y, r)) \quad \min_{x \in X} \bar{l}(x, y, r), \quad y \in \mathbb{R}^m, r > 0.$$

Denote by $v_5(y,r)$ the optimal value of $(ALP(y,r))$.

We have the following result.

Theorem 10.17. *Let $\{y_n\} \subseteq R^m$ be bounded and $0 < r_n \to +\infty$. Consider (P) and $(ALP(y_n,r_n))$. Assume that there exist $(\bar{y},\bar{r}) \in \mathbb{R}^m \times (0,+\infty)$ and $m_0 \in \mathbb{R}^1$ such that*

$$\bar{l}(x,\bar{y},\bar{r}) \geq m_0, \quad \forall x \in X. \tag{10.46}$$

Let $0 < \varepsilon_n \to 0$. Suppose that each x_n satisfies

$$\bar{l}(x_n,y_n,r_n) \leq v_5(y_n,r_n) + \varepsilon_n, \tag{10.47}$$

$v_5(y_n,r_n) > -\infty, \forall n$, and (P) is LP well-posed in the strongly generalized sense. Then, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\bar{x} \in \overline{X}$ such that $x_{n_k} \to \bar{x}$.

Proof. By the definition of $\bar{l}(x,y,r)$, it is easy to see that

$$\bar{l}(x,y,r) = f(x), \quad \forall x \in X_0.$$

It follows that

$$v_5(y,r) \leq \bar{v}, \quad \forall y \in \mathbb{R}^m, r > 0.$$

Thus,

$$v_5(y_n,r_n) \leq \bar{v}, \quad \forall n. \tag{10.48}$$

By the definition of $\bar{l}(x_n,y_n,r_n)$ and (10.47), $\{x_n\} \subseteq X_1$ and there exists $\{u_n\} \subseteq R^m$ satisfying

$$g(x_n) + u_n \in K, \forall n \tag{10.49}$$

such that

$$f(x_n) + r_n\sigma(u_n) - \langle y_n, u_n\rangle \leq v_5(y_n,r_n) + 2\varepsilon_n. \tag{10.50}$$

This combined with (10.46) and (10.48) implies that

$$(r_n - \bar{r})\sigma(u_n) - \langle y_n - \bar{y}, u_n\rangle \leq \bar{v} + 2\varepsilon_n - m_0. \tag{10.51}$$

We assert that $\{u_n\}$ is bounded. Otherwise, we assume without loss of generality that $\|u_n\| \to +\infty$. Since the lower semicontinuous and convex function σ has a unique minimum, by [11, Proposition 3.2.5 in IV] and [29, Corollary 3.27], $\liminf_{n\to+\infty} \sigma(u_n)/\|u_n\| > 0$. As $\{y_n\}$ is bounded, (10.51) cannot hold. So, $\{u_n\}$ should be bounded. Assume without loss of generality that $u_n \to u_0$. We deduce from (10.51) that

$$\sigma(u_0) \leq \liminf_{n\to+\infty} \sigma(u_n) = 0.$$

It follows that $u_0 = 0$. We deduce from (10.48) and (10.50) that

$$f(x_n) - \langle y_n, u_n\rangle \leq \bar{v} + 2\varepsilon_n.$$

Passing to the limit, we get

$$\limsup_{n\to+\infty} f(x_n) \le \overline{v}.$$

From (10.49) and the fact that $u_n \to 0$, we obtain

$$\lim_{n\to+\infty} d_K(g(x_n)) = 0.$$

Thus, $\{x_n\}$ is a weakly generalized LP minimizing sequence. Hence, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and some $\overline{x} \in \overline{X}$ such that $x_{n_k} \to \overline{x}$. □

10.3 Levitin–Polyak Well-Posedness of Constrained Vector Optimization Problems

In this section, we consider LP type well-posedness for a general constrained vector optimization problem. We introduce several types of (generalized) LP well-posedness. Criteria and characterizations for these types of well-posednesses are given. Relations among these types of well-posedness are investigated. Finally, we consider convergence of a class of penalty methods under the assumption of a type of generalized Levitin–Polyak well-posedness.

10.3.1 Preliminaries

Let (X,d_1) and (Z,d_2) be two metric spaces. Let Y be a normed space ordered by a closed and convex cone C with nonempty interior int C, i.e., $\forall y_1,y_2 \in Y$, $y_1 \le_C y_2$ if and only if $y_2 - y_1 \in C$. Arbitrarily fix an $e \in \text{int}\, C$. Let $X_1 \subseteq X$ and $K \subseteq Z$ be two nonempty and closed sets. Consider the following constrained vector optimization problem:

$$\text{(VP)} \quad \inf f(x)$$
$$\text{s.t. } x \in X_1,\ g(x) \in K,$$

where $f : X \to Y$ and $g : X \to Z$ are continuous functions.

Denote by X_0 the set of feasible solutions of (VP), i.e.,

$$X_0 = \{x \in X_1 : g(x) \in K\}.$$

Throughout the paper, we always assume that $X_0 \neq \emptyset$.

Denote by $\overline{X}$ the set of weakly efficient solutions of (VP), namely, for any $\overline{x} \in \overline{X}$,

(i) $\overline{x} \in X_0$.
(ii) For any $x \in X_0$, $f(x) - f(\overline{x}) \notin -\text{int}\, C$.

We denote by V the set of infimal points of (VP). That is, $v \in V$ if and only if

(i) There exists no $x \in X_0$ such that $f(x) - v \in \operatorname{int} C$
(ii) There exists a sequence $\{x_k\} \subseteq X_0$ such that $f(x_k) \to v$

Throughout the paper, we always assume that $V \neq \emptyset$. Let (P,d) be a metric space and $P_1 \subseteq P$. We denote by $d_{P_1}(p) = \inf\{d(p,p') : p' \in P_1\}$ the distance from the point p to the set P_1.

Define

$$\xi(y) = \min\{t : y \leq_C te\}, \quad \forall y \in Y.$$

It is known from [23] that ξ is continuous, homogenous, (strictly) monotone (i.e., $\xi(y_1) \leq \xi(y_2)$ if $y_2 - y_1 \in C$ and $\xi(y_1) < \xi(y_2)$ if $y_2 - y_1 \in \operatorname{int} C$) and convex.

Many optimization methods for (VP) may generate a sequence $\{x_k\} \subseteq X_1$ such that $d_{X_0}(x_k) \to 0$.

Penalty type methods for (VP) (and its special cases, e.g., $Y = \mathbb{R}$, $C = \mathbb{R}^l_+$), such as penalty function methods (see, e.g., [17]) and augmented Lagrangian methods (see, e.g., [14]) may generate a sequence $\{x_k\} \subseteq X_1$ such that $d_K(g(x_k)) \to 0$, but $d_{X_0}(x_k) \not\to 0$.

In this section, we will study such sequences under additional conditions. This study should be useful to the study of convergence of some optimization methods for (VP) as will be seen in Sect. 10.3.4.

In what follows, we will introduce several notions of Levitin–Polyak well-posedness and generalized Levitin–Polyak well-posedness for (VP).

Definition 10.6.

- (VP) is said to be type I Levitin–Polyak (LP in short) well-posed if $\overline{X} \neq \emptyset$ and, for any $\{x_k\}$ satisfying

$$d_{X_0}(x_k) \to 0 \tag{10.52}$$

  and

$$d_V(f(x_k)) \to 0, \tag{10.53}$$

  there exist a subsequence $\{x_{k_j}\}$ and an $x^* \in \overline{X}$ such that

$$\lim_{j \to +\infty} x_{k_j} = x^*.$$

- (VP) is said to be type I LP well-posed in the generalized sense if $\overline{X} \neq \emptyset$ and, for any $\{x_k\}$ satisfying

$$d_K(g(x_k)) \to 0 \tag{10.54}$$

  and (10.53),

  there exist a subsequence $\{x_{k_j}\}$ and an $x^* \in \overline{X}$ such that

$$\lim_{j \to +\infty} x_{k_j} = x^*.$$

The sequence satisfying (10.52) and (10.53) is called a type I LP minimizing one while the sequence satisfying (10.54) and (10.53) is called a type I generalized LP minimizing one.

Definition 10.7.

- (VP) is said to be type II LP well-posed if $\overline{X} \neq \emptyset$ and, for any $\{x_k\}$ satisfying (10.52) and

$$f(x_k) \leq_C v_k + \varepsilon_k e \text{ for some } \{v_k\} \subseteq V \text{ and some } 0 < \varepsilon_k \to 0, \tag{10.55}$$

  there exist a subsequence $\{x_{k_j}\}$ and an $x^* \in \overline{X}$ such that

$$\lim_{j \to +\infty} x_{k_j} = x^*.$$

- (VP) is said to be type II LP well-posed in the generalized sense if $\overline{X} \neq \emptyset$ and, for any $\{x_k\}$ meeting (10.54) and (10.55), then there exist a subsequence $\{x_{k_j}\}$ and an $x^* \in \overline{X}$ such that

$$\lim_{j \to +\infty} x_{k_j} = x^*.$$

The sequence satisfying (10.52) and (10.55) is called a type II LP minimizing one while the sequence satisfying (10.54) and (10.55) is called a type II generalized LP minimizing one.

Definition 10.8.

- (VP) is said to be type III LP well-posed if $\overline{X} \neq \emptyset$ and, for any $\{x_k\}$ satisfying (10.52) and

$$\liminf_{k \to +\infty} \left\{ \inf_{v \in V} \xi(v - f(x_k)) \right\} \geq 0, \tag{10.56}$$

  there exist a subsequence $\{x_{k_j}\}$ and an $x^* \in \overline{X}$ such that

$$\lim_{j \to +\infty} x_{k_j} = x^*.$$

- (VP) is said to be type III LP well-posed in the generalized sense if $\overline{X} \neq \emptyset$ and for any $\{x_k\}$ meeting (10.54) and (10.56), then there exist a subsequence $\{x_{k_j}\}$ and an $x^* \in \overline{X}$ such that

$$\lim_{j \to +\infty} x_{k_j} = x^*.$$

The sequence satisfying (10.52) and (10.56) is called a type III LP minimizing one while the sequence satisfying (10.54) and (10.56) is called a generalized LP minimizing one.

Remark 10.6. (a) The definitions of type I (condition (10.53)), type II (condition (10.55)) and type III (condition (10.56)) (generalized) LP minimizing sequence were motivated by Definitions 2.3–2.5 of [12].

(b) It is easy to see that a type I (generalized) LP minimizing sequence is a type II generalized LP minimizing sequence and that a type II (generalized) LP minimizing sequence is a type III (generalized) LP minimizing sequence. Thus, the type III (generalized) LP well-posedness implies the type II (generalized) LP well-posedness and the type II (generalized) LP well-posedness implies the type I (generalized) LP well-posedness.

(c) Any type of (generalized) well-posedness implies that the set $\overline{X}$ of weakly efficient solutions of (VP) is nonempty and compact.

(d) When $Y = \mathbb{R}^1$, $C = \mathbb{R}^1_+$, type I (generalized) LP well-posedness coincides with type II (generalized) LP well-posedness, type I (II) LP well-posedness is just the LP well-posedness in [16] while type I (II) generalized LP well-posedness is the generalized LP well-posedness defined in [16], and type III generalized LP well-posedness is just the strongly generalized LP well-posedness in [16].

10.3.2 *Criteria and Characterizations for (Generalized) LP Well-Posedness*

In this subsection, we give necessary and sufficient conditions for the various types of (generalized) LP well-posedness defined in Definitions 10.6–10.8.

Consider the following statement:

$$\begin{aligned}&\big[\overline{X} \neq \emptyset \text{ and, for any type I (resp. type II, type III,}\\ &\text{generalized type I, generalized type II, generalized type III)}\\ &\text{LP minimizing sequence } \{x_k\},\text{ we have } d_{\overline{X}}(x_k) \to 0.\big]\end{aligned} \tag{10.57}$$

First we have the following result, whose proof is elementary and thus omitted.

Proposition 10.4. *If (VP) is type I (resp. type II, type III, generalized type I, generalized type II, generalized type III) LP well-posed, then (10.57) holds. Conversely, if (10.57) holds and $\overline{X}$ is compact, then (VP) is type I (resp. type II, type III, generalized type I, generalized type II, generalized type III) LP well-posed.*

Now consider a real-valued function $c = c(t,s)$ defined for $t,s \geq 0$ sufficiently small, such that

$$c(t,s) \geq 0, \quad \forall t,s, \quad c(0,0) = 0, \tag{10.58}$$

$$s_k \to 0, t_k \geq 0, c(t_k,s_k) \to 0 \text{ imply } t_k \to 0. \tag{10.59}$$

Theorem 10.18. *If (VP) is type I LP well-posed, then there exists a function c satisfying (10.58) and (10.59) such that*

$$d_V(f(x)) \geq c(d_{\overline{X}}(x), d_{X_0}(x)), \quad \forall x \in X_1. \tag{10.60}$$

Conversely, suppose that $\overline{X}$ is nonempty and compact, and (10.60) holds for some c satisfying (10.58) and (10.59). Then (VP) is type I LP well-posed.

Proof. Define

$$c(t,s) = \inf\{d_V(f(x)) : x \in X_1, d_{\overline{X}}(x) = t, d_{X_0}(x) = s\}.$$

It is obvious that $c(t,s) \geq 0, \forall s,t$ and $c(0,0) = 0$. Moreover, if $s_k \to 0$, $t_k \geq 0$ and $c(t_k, s_k) \to 0$, then, there exists a sequence $\{x_k\} \subseteq X_1$ with

$$d_{\overline{X}}(x_k) = t_k, \tag{10.61}$$

$$d_{X_0}(x_k) = s_k \tag{10.62}$$

such that

$$d_V(f(x_k)) \to 0. \tag{10.63}$$

Note that $s_k \to 0$. This fact together with (10.62) and (10.63) implies that $\{x_k\}$ is a type I LP minimizing sequence. By Proposition 10.4, we have $t_k \to 0$. This completes the proof of the first part of the theorem. Conversely, let $\{x_k\}$ be a type I LP minimizing sequence. Then, by (10.60), we have

$$d_V(f(x_k)) \geq c(d_{\overline{X}}(x_k), d_{X_0}(x_k)), \forall k. \tag{10.64}$$

Let

$$t_k = d_{\overline{X}}(x_k), \quad s_k = d_{X_0}(x_k).$$

Then, $s_k \to 0$. In addition, $d_V(f(x_k)) \to 0$. These facts together with (10.64) as well as the properties of the function c imply that $t_k \to 0$. By Proposition 10.4, we see that (VP) is type I LP well-posed. □

Theorem 10.19. *If (VP) is type I LP well-posed in the generalized sense, then there exists a function c satisfying (10.58) and (10.59) such that*

$$d_V(f(x)) \geq c(d_{\overline{X}}(x), d_K(g(x))), \quad \forall x \in X_1. \tag{10.65}$$

Conversely, suppose that $\overline{X}$ is nonempty and compact, and (10.65) holds for some c satisfying (10.58) and (10.59). Then (VP) is type I LP well-posed in the generalized sense.

Proof. The proof is almost the same as that of Theorem 10.18. The only difference lies in the proof of the first part of Theorem 10.18. Here we define

$$c(t,s) = \inf\{d_V(f(x)) : x \in X_1, d_{\overline{X}}(x) = t, d_K(g(x)) = s\}.$$ □

Recall that Furi–Vignoli [10] characterized well-posedness of optimization problems (defined in a complete metric space (X,d_1)) by making use of the Kuratowski measure of noncompactness of a subset A of X defined by

$$\alpha(A) = \inf\left\{\varepsilon > 0 : A \subseteq \bigcup_{1\le i\le n} C_i, \text{ for some } C_i, \operatorname{diam}(C_i) \le \varepsilon\right\},$$

where $\operatorname{diam}(C_i)$ is the diameter of C_i defined by

$$\operatorname{diam}(C_i) = \sup\{d_1(x_1,x_2) : x_1,x_2 \in C_i\}.$$

Given two nonempty subsets A and B of X, define the excess of set A to set B by

$$e(A,B) = \sup\{d_B(a) : a \in A\}.$$

The Hausdorff distance between A and B is defined as

$$\operatorname{haus}(A,B) = \max\{e(A,B), e(B,A)\}.$$

Next we give Furi–Vignoli type characterizations for the various (generalized) LP well-posednesses.

Let, for each $\varepsilon > 0$,

$$T_1^1(\varepsilon) = \{x \in X_1 : d_V(f(x)) \le \varepsilon, d_{X_0}(x) \le \varepsilon\}.$$

Theorem 10.20. *Let (X,d_1) be a complete metric space and $V \neq \emptyset$. Then, (VP) is type I LP well-posed if and only if*

$$\lim_{\varepsilon\to 0} \alpha(T_1^1(\varepsilon)) = 0. \tag{10.66}$$

Proof. First, we show that for each $\varepsilon > 0$, $T_1^1(\varepsilon)$ is nonempty and closed. The nonemptiness of $T_1^1(\varepsilon)$ follows from the fact that $V \neq \emptyset$. Let $\{x_k\} \subseteq T_1^1(\varepsilon)$ and $x_k \to \bar{x}$. Then

$$d_V(f(x_k)) \le \varepsilon \tag{10.67}$$

and

$$d_{X_0}(x_k) \le \varepsilon. \tag{10.68}$$

From (10.68), we have

$$d_{X_0}(\bar{x}) \le \varepsilon. \tag{10.69}$$

By the continuity of f and (10.67), we obtain

$$d_V(f(\bar{x})) \le \varepsilon. \tag{10.70}$$

The combination of (10.69) and (10.70) shows that $\bar{x} \in T_1^1(\varepsilon)$. Thus, $T_1^1(\varepsilon)$ is closed.

Second, we show that

$$\overline{X} = \bigcap_{\varepsilon>0} T_1^1(\varepsilon). \tag{10.71}$$

It is obvious that $\overline{X} \subseteq \bigcap_{\varepsilon>0} T_1^1(\varepsilon)$. Now suppose that $\varepsilon_k \to 0$ and $\overline{x} \in \bigcap_{k=1}^{\infty} T_1^1(\varepsilon_k)$. Then,

$$d_V(f(\overline{x})) \leq \varepsilon_k, \forall k \tag{10.72}$$

and

$$d_{X_0}(\overline{x}) \leq \varepsilon_k, \forall k. \tag{10.73}$$

By (10.72), we have $f(\overline{x}) \in V$. By (10.73), we have $\overline{x} \in X_0$. Hence, $\overline{x} \in \overline{X}$.

Now we assume that (10.66) holds. Clearly, $T_1^1(\cdot)$ is increasing with $\varepsilon > 0$. By the Kuratowski theorem [20, p. 318], we have

$$\text{haus}(T_1^1(\varepsilon), T_1^1) \to 0 \text{ as } \varepsilon \to 0, \tag{10.74}$$

where

$$T_1^1 = \bigcap_{\varepsilon>0} T_1^1(\varepsilon)$$

is nonempty and compact.

Let $\{x_k\}$ be a type I LP minimizing sequence. Then, by taking a subsequence, we can find a decreasing sequence $\varepsilon_k \to 0$ such that $d_V(f(x_k)) \leq \varepsilon_k$ and $d_{X_0}(x_k) \leq \varepsilon_k$. Thus, $x_k \in T_1^1(\varepsilon_k)$. It follows from (10.71) and (10.74) that $d_{\overline{X}}(x_k) \to 0$. By Proposition 10.4, (VP) is type I LP well-posed.

Conversely, let (VP) be type I LP well-posed. Consider the excess

$$q(\varepsilon) = e(T_1^1(\varepsilon), \overline{X}), \varepsilon > 0.$$

We show that $q(\varepsilon) \to 0$ as $\varepsilon \to 0$. If not, there exist $\delta > 0$, $\varepsilon_k \to 0$, $x_k \in T_1^1(\varepsilon_k)$ such that

$$d_{\overline{X}}(x_k) \geq \delta, \forall k,$$

contradicting the type I LP well-posedness of (VP). Thus, $q(\varepsilon) \to 0$ as $\varepsilon \to 0$. Note that

$$T_1^1(\varepsilon) \subseteq \{x \in X_1 : d_{\overline{X}}(x) \leq q(\varepsilon)\}.$$

It follows that

$$\alpha(T_1^1(\varepsilon)) \leq 2q(\varepsilon)$$

since $\alpha(\overline{X}) = 0$. Consequently, (10.66) holds. The proof is complete. □

Consider

$$T_1^2(\varepsilon) = \{x \in X_1 : d_V(f(x)) \leq \varepsilon, d_K(g(x)) \leq \varepsilon\}.$$

The following theorem can be proved analogously to Theorem 10.20.

Theorem 10.21. *Let* (X,d_1) *be a complete metric space and* $V \neq \emptyset$*. Then, (VP) is type I LP well-posed in the generalized sense if and only if*

$$\lim_{\varepsilon \to 0} \alpha(T_1^2(\varepsilon)) = 0. \tag{10.75}$$

Define

$$T_2^1(\varepsilon) = \{x \in X_1 : d_{X_0}(x) \leq \varepsilon, f(x) \leq_C v + \varepsilon e \text{ for some } v \in V\}.$$

Theorem 10.22. *Let* (X,d_1) *be a complete metric space and* $V \neq \emptyset$*. Then, (VP) is type II LP well-posed if and only if*

$$\lim_{\varepsilon \to 0} \alpha(T_2^1(\varepsilon)) = 0. \tag{10.76}$$

Proof. It is obvious from $V \neq \emptyset$ that $T_2^1(\varepsilon) \neq \emptyset, \forall \varepsilon > 0$. Thus, cl $T_2^1(\varepsilon)$ is nonempty and closed. Of course, cl $T_2^1(\cdot)$ is increasing with ε. Now we show that

$$\overline{X} = \bigcap_{\varepsilon > 0} cl T_2^1(\varepsilon). \tag{10.77}$$

Obviously, $\overline{X} \subseteq \cap_{\varepsilon>0}$cl $T_2^1(\varepsilon)$. Let $\bar{x} \in \bigcap_{\varepsilon>0}$cl $T_2^1(\varepsilon)$ and $\varepsilon_k \downarrow 0$. By $\bar{x} \in \bigcap_{k=1}^{\infty}$ cl $T_2^1(\varepsilon_k)$, for each k, there exist $x_{k,j} \in X_1$ and $v_{k,j} \in V$ such that

$$f(x_{k,j}) \leq_C v_{k,j} + \varepsilon_k e, \tag{10.78}$$

$$x_{k,j} \to \bar{x} \tag{10.79}$$

and

$$d_{X_0}(x_{k,j}) \leq \varepsilon_k \Rightarrow d_{X_0}(\bar{x}) \leq \varepsilon_k. \tag{10.80}$$

From (10.78) and (10.79) and the continuity of f, we have that for each k, there exists $j(k)$ such that

$$f(\bar{x}) \leq_C v_{k,j(k)} + 2\varepsilon_k e. \tag{10.81}$$

Suppose to the contrary that $\bar{x} \notin \overline{X}$. Then there exist $x_0 \in X_0$ and $\delta > 0$ such that

$$f(x_0) \leq_C f(\bar{x}) - \delta e. \tag{10.82}$$

From (10.81) and (10.82), we have

$$\begin{aligned} f(x_0) &\leq_C v_{k,j(k)} + 2\varepsilon_k e - \delta e \\ &= v_{k,j(k)} - (\delta - 2\varepsilon_k)e. \end{aligned} \tag{10.83}$$

Since $\varepsilon_k \downarrow 0$, $\delta - 2\varepsilon_k \geq \delta/2$ when k is sufficiently large. Thus, (10.83) contradicts the fact that $v_{k,k(j)} \in V$ when k is sufficiently large. Hence, there holds $\bar{x} \in \overline{X}$. Thus, (10.77) is proved.

Now assume that (10.75) holds. Then

$$\alpha(\text{cl}\, T_2^1(\varepsilon)) = \alpha(T_2^1(\varepsilon)) \to 0 \text{ as } \varepsilon \to 0.$$

By the Kuratowski theorem, it follows that

$$\text{haus}(\text{cl}\, T_2^1(\varepsilon), T_2^1) \to 0 \text{ as } \varepsilon \to 0, \tag{10.84}$$

where

$$T_2^1 = \bigcap_{\varepsilon>0} clT_2^1(\varepsilon)$$

is nonempty and compact. Let $\{x_k\}$ be a type II LP minimizing sequence. Then, by taking a subsequence, we can find a decreasing sequence $\varepsilon_k \to 0$ and a sequence $\{v_k\} \subseteq V$ such that

$$f(x_k) \leq_C v_k + \varepsilon_k e, \tag{10.85}$$

$$d_{X_0}(x_k) \leq \varepsilon_k. \tag{10.86}$$

From (10.85) and (10.86), we see that $x_k \in T_2^1(\varepsilon_k)$. It follows from (10.77) and (10.84) that $d_{\overline{X}}(x_k) \to 0$. By Proposition 10.4 and the compactness of $\overline{X}$, we deduce that (VP) is type II LP well-posed. The proof of the second part of the theorem is similar to that of the second part of Theorem 10.20. □

Let

$$T_2^2(\varepsilon) = \{x \in X_1 : d_K(g(x)) \leq \varepsilon, f(x) \leq_C v + \varepsilon e \text{ for some } v \in V\}.$$

The next theorem can be proved analogously to Theorem 10.22.

Theorem 10.23. *Let (X, d_1) be a complete metric space and $V \neq \emptyset$. Then, (VP) is type II LP well-posed in the generalized sense if and only if*

$$\lim_{\varepsilon \to 0} \alpha(T_2^2(\varepsilon)) = 0.$$

Definition 10.9. (VP) is said to be inf-externally stable if for each $x_0 \in X_0$, there exists $v_0 \in V$ such that $v_0 \leq_C f(x_0)$.

Define

$$T_3^1(\varepsilon) = \left\{x \in X_1 : \inf_{v \in V} \xi(v - f(x)) \geq -\varepsilon, d_{X_0}(x) \leq \varepsilon\right\}.$$

Theorem 10.24. *Let (X, d_1) be a complete metric space and $V \neq \emptyset$. Suppose that (VP) is inf-externally stable. Then, (VP) is type III LP well-posed if and only if*

$$\lim_{\varepsilon \to 0} \alpha(T_3^1(\varepsilon)) = 0.$$

Proof. First, we show that $T_3^1(\varepsilon)$ is nonempty and closed for any $\varepsilon > 0$. The nonemptiness of $T_3^1(\varepsilon)$ follows from the fact that $V \neq \emptyset$. Now let $\{x_k\} \subseteq T_3^1(\varepsilon)$ and $x_k \to \bar{x}$. Then,

$$\inf_{v \in V} \xi(v - f(x_k)) \geq -\varepsilon, \tag{10.87}$$

$$d_{X_0}(x_k) \leq \varepsilon. \tag{10.88}$$

Note that the continuity of f implies that the function $\inf_{v \in V} \xi(v - f(\cdot))$ is upper semicontinuous. Taking the upper limit in (10.87), we have

$$\inf_{v \in V} \xi(v - f(\bar{x})) \geq -\varepsilon. \tag{10.89}$$

Taking the limit in (10.88), we obtain

$$d_{X_0}(\bar{x}) \leq \varepsilon. \tag{10.90}$$

The combination of (10.89) and (10.90) yields $\bar{x} \in T_3^1(\varepsilon)$. Hence, $T_3^1(\varepsilon)$ is closed.

Second, we show that

$$\overline{X} = \bigcap_{\varepsilon > 0} T_3^1(\varepsilon). \tag{10.91}$$

Obviously, $\overline{X} \subseteq \cap_{\varepsilon > 0} T_3^1(\varepsilon)$. Now let $\bar{x} \in \cap_{\varepsilon > 0} T_3^1(\varepsilon)$ and $\varepsilon_k \downarrow 0$. Then

$$\inf_{v \in V} \xi(v - f(\bar{x})) \geq -\varepsilon_k, \tag{10.92}$$

$$d_{X_0}(\bar{x}) \leq \varepsilon_k. \tag{10.93}$$

From (10.93), we have $\bar{x} \in X_0$. From (10.92), we have

$$\xi(v - f(\bar{x})) \geq 0, \forall v \in V. \tag{10.94}$$

Suppose to the contrary that there exist $x_0 \in X_0$ and $\delta > 0$ such that

$$f(x_0) - f(\bar{x}) \leq_C -\delta e. \tag{10.95}$$

By the inf-external stability of (VP), there exists $v_0 \in V$ such that $v_0 \leq_C f(x_0)$. This together with (10.95) implies that

$$\xi(v_0 - f(\bar{x})) \leq -\delta,$$

contradicting (10.94). Thus, (10.91) is proved. Clearly, $T_3^1(\cdot)$ is increasing with $\varepsilon > 0$. By the Kuratowski theorem, we have

$$\mathrm{haus}(T_3^1(\varepsilon), T_3^1) \to 0 \text{ as } \varepsilon \to 0, \tag{10.96}$$

where

$$T_3^1 = \bigcap_{\varepsilon>0} T_3^1(\varepsilon)$$

is nonempty and compact.

Let $\{x_k\}$ be a type III LP minimizing sequence. Then, by taking a subsequence, we can find a decreasing sequence $\varepsilon_k \to 0$ such that

$$\inf_{v\in V} \xi(v - f(x_k)) \geq -\varepsilon_k,$$

$$d_{X_0}(x_k) \leq \varepsilon_k.$$

Thus, $x_k \in T_3^1(\varepsilon_k)$. By (10.91) and (10.96), we see that $d_{\overline{X}}(x_k) \to 0$. By Proposition 10.4, (VP) is type III LP well-posed. The second part of the theorem can be proved similarly to that of Theorem 10.20. The proof is complete. □

Define

$$T_3^2(\varepsilon) = \{x \in X_1 : \inf_{v\in V} \xi(v - f(x)) \geq -\varepsilon, d_K(g(x)) \leq \varepsilon\}.$$

The following theorem can be proved analogously to Theorem 10.24.

Theorem 10.25. *Let (X, d_1) be a complete metric space and $V \neq \emptyset$. Suppose that (VP) is inf-externally stable. Then, (VP) is type III LP well-posed in the generalized sense if and only if*

$$\lim_{\varepsilon\to 0} \alpha(T_3^2(\varepsilon)) = 0.$$

Next proposition gives sufficient conditions for the type III (generalized) LP well-posedness.

Proposition 10.5. *(i) Assume that there exists $\delta > 0$ such that*

$$X_1(\delta) = \{x \in X_1 : d_{X_0}(x) \leq \delta\} \tag{10.97}$$

is compact. Then, (VP) is type III LP well-posed.

(ii) Assume that there exists $\delta > 0$ such that

$$X_2(\delta) = \{x \in X_1 : d_K(g(x)) \leq \delta\} \tag{10.98}$$

is compact.

Then, (VP) is type III LP well-posed in the generalized sense.

Proof. We prove only (i) and (ii) can be similarly proved.

Let $\{x_k\}$ be a type III LP minimizing sequence. Then

$$\liminf_{k\to+\infty}\{\inf_{v\in V} \xi(v - f(x_k))\} \geq 0, \tag{10.99}$$

$$d_{X_0}(x_k) \to 0. \tag{10.100}$$

(10.100) implies that $x_k \in X_1(\delta)$ when $k \geq k_0$ for some $k_0 > 0$. By the compactness of $X_1(\delta)$, there exists a subsequence $\{x_{k_j}\}$ and $\bar{x} \in X_1(\delta)$ such that $x_{k_j} \to \bar{x}$. This together with (10.100) implies that $\bar{x} \in X_0$. Moreover, from (10.99), we have

$$\xi(v - f(\bar{x})) \geq 0, \quad \forall v \in V. \tag{10.101}$$

Suppose to the contrary that $\bar{x} \notin \overline{X}$. Then, there exists $x_0 \in X_0$ such that

$$f(x_0) - f(\bar{x}) \in -\text{int}\, C. \tag{10.102}$$

Note that $X_0 \subseteq X_1(\delta)$ is nonempty and compact and f is continuous. Consequently, there exist $v_0 \in V$ such that

$$v_0 \leq_C f(x_0). \tag{10.103}$$

The combination of (10.101)–(10.103) leads to a contradiction. Hence, $\bar{x} \in \overline{X}$ and the proof is complete. □

Now we consider the special case when X is a finite dimensional normed space, $Y = \mathbb{R}^l$, $C = \mathbb{R}^l_+$, $e = (1, \cdots, 1) \in \mathbb{R}^l$, $\xi(y) = \max\{y_i : i = 1, \cdots, l\}, \forall y \in Y$.

Definition 10.10. Let X be a finite dimensional normed space, $X_2 \subseteq X$ be nonempty and $f_0 : X_2 \to \mathbb{R}$. f_0 is said to be level-bounded on X_2 if, for each $t \in \mathbb{R}$, the set $\{x \in X_2 : f_0(x) \leq t\}$ is bounded.

Proposition 10.6. *Assume that X is a finite dimensional space, $Y = \mathbb{R}$, $C = \mathbb{R}_+$. Further assume that one of the following conditions holds:*

(i) For each $i \in \{1, \cdots, l\}$, f_i is level-bounded on X_1
(ii) There exists $\delta > 0$ such that for each $i \in \{1, \cdots, l\}$, f_i is level-bounded on $X_1(\delta)$, where $X_1(\delta)$ is defined by (10.97)
(iii) For each $i \in \{1, \cdots, l\}$,

$$\lim_{x \in X_1, \|x\| \to +\infty} \max\{f_i(x), d_{X_0}(x)\} = +\infty. \tag{10.104}$$

Then, (VP) is type III LP well-posed.

Proof. Clearly, (i)⇒(iii)⇒(ii). So we need only to prove that if (ii) holds, then (VP) is type III LP well-posed. Let $\{x_k\}$ be a type III LP minimizing sequence. Then (10.99) and (10.100) hold. (10.100) implies that $x_k \in X_1(\delta), \forall k \geq k_0$ for some $k_0 > 0$. (10.99) implies that there exists $0 < \varepsilon_k \to 0$ such that

$$\xi(v - f(x_k)) \geq -\varepsilon_k, \forall v \in V. \tag{10.105}$$

We assert that $\{x_k\}$ is bounded. Otherwise, assume without loss of generality that $\|x_k\| \to +\infty$. Then, by the level-boundedness of each f_i on $X_1(\delta)$, we have

$$\lim_{k \to +\infty} f_i(x_k) = +\infty.$$

It follows that (10.105) cannot hold. Thus, there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ and $\bar{x} \in X_1$ such that $x_{k_j} \to \bar{x}$. This together with (10.100) implies that $\bar{x} \in X_0$. Now we show that $\bar{x} \in \overline{X}$. Otherwise, there exist $x_0 \in X_0$ and $\delta_0 > 0$ such that

$$f_i(x_0) \leq f_i(\bar{x}), i = 1, \cdots, l. \tag{10.106}$$

It is obvious that the set

$$A = \{x \in X_0 : f_i(x) \leq f_i(x_0), i = 1, \cdots, l\}$$

is nonempty and compact. Note that $x_0 \in A$. It follows that there exists $\bar{x} \in A$ such that $f(x) - f(\bar{x}) \notin -C \setminus \{\mathbf{0}\}, \forall x \in A$. It is easily verified that $\bar{x} \in \overline{X}$. Moreover, by $\bar{x} \in A$, we have

$$f(\bar{x}) \leq_C f(x_0).$$

This together with (10.106) implies that

$$f(\bar{x}) \leq_C f(\bar{x}) - \delta_0 e.$$

From $x_{k_j} \to \bar{x}$ and the continuity of f on X_1, we have

$$f(\bar{x}) \leq_C f(x_{k_j}) - \delta/2e$$

when j is large enough, contradicting (10.105). The proof is complete. □

Similarly, we can prove the next result.

Proposition 10.7. *Assume that X is a finite dimensional space, $Y = \mathbb{R}$, $C = \mathbb{R}_+$. Further assume that one of the following conditions holds:*

(i) For each $i \in \{1, \cdots, l\}$, f_i is level-bounded on X_1
(ii) There exists $\delta > 0$ such that for each $i \in \{1, \cdots, l\}$, f_i is level-bounded on $X_2(\delta)$, where $X_2(\delta)$ is defined by (10.98)
(iii) For each $i \in \{1, \cdots, l\}$,

$$\lim_{x \in X_1, \|x\| \to +\infty} \max\{f_i(x), d_K(g(x))\} = +\infty. \tag{10.107}$$

Then, (VP) is type III LP well-posed in the generalized sense.

Now we consider the case when Z is a normed space and K is a closed and convex cone with nonempty interior int K and let $e' \in \text{int } K$. Let $t \geq 0$ and denote

$$X_3(t) = \{x \in X_1 : g(x) \in K - te'\}. \tag{10.108}$$

Proposition 10.8. *Let Z be a normed space and K a closed and convex cone with nonempty interior* int K *and let $e' \in$* int K. *If there exists $t_0 > 0$ such that $X_3(t_0)$ is compact, then (VP) is type III LP well-posed in the generalized sense.*

Proof. According to (ii) of Proposition 10.5, we need only to show that there exists $\delta_0 > 0$ such that $X_2(\delta_0)$ is compact. To this purpose, we need only to show that there exists $\delta_0 > 0$ such that $X_2(\delta_0) \subseteq X_3(t_0)$. Suppose to the contrary that there exists $0 < \delta_k \to 0$ and $x_k \in X_2(\delta_k)$ such that $x_k \notin X_3(t_0)$. That is,

$$d_K(g(x_k)) \le \delta_k, \tag{10.109}$$

$$g(x_k) \notin K - t_0 e'. \tag{10.110}$$

Define

$$\eta(z) = \min\{t \in \mathbb{R}^1 : z \in -K + te'\}, \forall z \in Z.$$

It is obvious that the function η has the same properties as the function ξ. From (10.110), we get

$$\eta(-g(x_k)) \ge t_0, \forall k. \tag{10.111}$$

From (10.109), we deduce that there exists $w_k \in K$ such that $\|g(x_k) - w_k\| \to 0$. Let $z_k = w_k - g(x_k) \to 0$. Then, $-g(x_k) = z_k - w_k$, implying $\eta(-g(x_k)) \le \eta(z_k) \to 0$, contradicting (10.111). The proof is complete. □

Proposition 10.9. *Assume that X is a finite dimensional space, $Y = \mathbb{R}^l$, $C = \mathbb{R}^l_+$, $e = (1, \cdots, 1) \in \mathbb{R}^l$. Let Z be a normed space and K a closed and convex cone with nonempty interior* int K *and let $e' \in$* int K. *Further assume that one of the following conditions holds:*

(i) For each $i \in \{1, \cdots, l\}$, f_i is level-bounded on X_1
(ii) There exists $t_0 > 0$ such that for each $i \in \{1, \cdots, l\}$, f_i is level-bounded on $X_3(t_0)$
(iii) for each $i \in \{1, \cdots, l\}$, (10.107) holds

Then, (VP) is type III LP well-posed in the generalized sense.

Proof. It is easy to show that (i)⇒(iii)⇒(ii). Similar to the proof of Proposition 10.8, we can show that (ii) implies that there exists $\delta_0 > 0$ such that for each $i \in \{1, \cdots, l\}$, f_i is level-bounded on $X_2(\delta_0)$. By (ii) of Proposition 10.7, (VP) is type III LP well-posed in the generalized sense. □

Now we make the following assumption.

Assumption 10.2 *X is a finite dimensional normed space, $X_1 \subseteq X$ is a nonempty, closed and convex set, $Y = \mathbb{R}^l$, $C = \mathbb{R}^l_+$, $e = (1, \cdots, 1) \in \mathbb{R}^l$, Z is a normed space $K \subseteq Z$ is a closed and convex cone with nonempty interior* int K *and $e' \in$* int K, *each f_i $(i = 1, \cdots, l)$ is convex on X_1 and g is K-concave on X_1 (namely, for any $x_1, x_2 \in X_1$ and any $\theta \in (0,1)$, there holds that $g(\theta x_1 + (1-\theta)x_2) - \theta g(x_1) - (1-\theta)g(x_2) \in K$).*

It is obvious that under Assumption 10.2, (VP) is a convex vector program.

The next result was obtained in [17, Theorem 2.1].

Lemma 10.4. *Let Assumption 10.2 hold. Then the following statements are equivalent:*

(a) *The optimal set* $\overline{X}$ *of (VP) is nonempty and compact*
(b) *For each* $i \in \{1, \cdots, l\}$*, for any* $t \geq 0$*,* f_i *is level-bounded on the set* $X_3(t)$ *defined by (10.108).*

Theorem 10.26. *Let Assumption 10.2 hold. Then, (VP) is type III LP well-posed in the generalized sense if and only if the optimal set* $\overline{X}$ *of (VP) is nonempty and compact.*

Proof. The sufficiency part follows directly from Lemma 10.4 and Proposition 10.9, while the necessity part is obvious by (ii) of Remark 10.6. □

Lemma 10.5. *Let Assumption 10.2 hold. Then the following statements are equivalent:*

(a) *The optimal set* $\overline{X}$ *of (VP) is nonempty and compact*
(b) *For each* $i \in \{1, \cdots, l\}$*, for any* $\delta \geq 0$*,* f_i *is level-bounded on the set* $X_1(\delta)$ *defined by (10.97).*

Proof. It is clear that problem (VP) is equivalent to the following vector optimization problem

$$\text{(VP')} \qquad \inf f(x)$$
$$\text{s.t. } d_{X_0}(x) \leq 0.$$

By Assumption 10.2, X_0 is nonempty and convex. It follows that $d_{X_0}(\cdot)$ is a continuous and convex function. Applying Lemma 10.4 by setting $g(x) = d_{X_0}(x), \forall x \in X_1$, $Z = \mathbb{R}^1$ and $K = \mathbb{R}^1_+$, we see that $\overline{X}$ is nonempty and compact if and only if each f_i is level-bounded on $X_1(\delta), \forall \delta \geq 0, i \in \{1, \cdots, l\}$. □

The following theorem follows immediately from (ii) of Proposition 10.6 and Lemma 10.5.

Theorem 10.27. *Let Assumption 10.2 hold. Then, (VP) is type III LP well-posed if and only if the optimal set* $\overline{X}$ *of (VP) is nonempty and compact.*

Remark 10.7. By Theorems 10.26 and 10.27 as well as (i) of Remark 10.6, if Assumption 10.2 holds, then any type of (generalized) LP well-posednesses is equivalent to the fact that the set $\overline{X}$ is nonempty and compact.

10.3.3 Relations Among Various Types of (Generalized) LP Well-Posedness

Simple relationships among the (generalized) LP well-posednesses were mentioned in (ii) of Remark 10.6. Under Assumption 10.2, the equivalence of all the six types of (generalized) LP well-posednesses was noted in Remark 10.7. In this section, we investigate further relationships among them.

Theorem 10.28. *Suppose that there exist $\delta > 0$, $\alpha > 0$ and $c > 0$ such that*

$$d_{X_0}(x) \le c d_K^{\alpha}(g(x)), \quad \forall x \in X_2(\delta),$$

where $X_2(\delta)$ is defined by (10.98). If (VP) is type I (resp. type II, type III) LP well-posed, then (VP) is type I (resp. type II, type III) LP well-posed in the generalized sense.

Proof. The proof is elementary. □

It is clear that $X_2(\delta)$ given by (10.98) can be seen as a set-valued map from $\mathbb{R}^1_+$ to X. Thus, we have the following theorem.

Theorem 10.29. *Assume that the set-valued map $X_2(\delta)$ defined by (10.98) is u.H.c. at $\mathbf{0} \in \mathbb{R}^l_+$. If (VP) is type I (resp. type II, type III) LP well-posed, then (VP) is type I (resp. type II, type III) LP well-posed in the generalized sense.*

Proof. We prove only the type I case, the other two cases can be similarly proved. Let $\{x_k\} \subseteq X_1$ be a type I generalized LP minimizing sequence. That is,

$$d_V(f(x_k)) \to 0, \tag{10.112}$$

$$d_K(g(x_k)) \to 0. \tag{10.113}$$

(10.113), together with the u.H.c. of $X_2(\delta)$ at 0, implies that $d_{X_0}(x_k) \to 0$. This fact combined with (10.112) implies that $\{x_k\}$ is a type I LP minimizing sequence. Thus, there exist a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ and some $\bar{x} \in \overline{X}$ such that $x_{k_j} \to \bar{x}$. Hence, (VP) is type I LP well-posed in the generalized sense. □

Now we consider the case when Z is a normed space.

Remark 10.8. Let Z be a normed space and $\{x_k\} \subseteq X_1$. Then, $d_K(g(x_k)) \to 0$ if and only if there exists $\{z_k\} \subseteq Z$ with $z_k \to 0$ such that $g(x_k) \in K + z_k, \forall k$.

Proof. Necessity. From $d_K(g(x_k)) \to 0$, we deduce that there exists $\{u_k\} \subseteq K$ such that

$$\|g(x_k) - u_k\| \to 0.$$

Let $z_k = g(x_k) - u_k$. Then, $z_k \to 0$ and $g(x_k) \in K + z_k$.

Sufficiency. Since $g(x_k) - z_k \in K$,

$$d_K(g(x_k)) \le \|g(x_k) - (g(x_k) - z_k)\| = \|z_k\| \to 0.$$ □

Let

$$X_4(z) = \{x \in X_1 : g(x) \in K + z\}, \forall z \in Z. \tag{10.114}$$

Clearly, $X_4(z)$ can seen as a set-valued map from Z to X.

Corresponding to Theorem 10.29, we have the following result.

Theorem 10.30. *Assume that the set-valued map $X_4(z)$ defined by (10.114) is u.H.c. at $\mathbf{0} \in Z$. If (VP) is type I (resp. type II, type III) LP well-posed, then (VP) is type I (resp. type II, type III) LP well-posed in the generalized sense.*

In the special case when K is a closed and convex cone with nonempty interior int K and $e' \in \text{int } K$. We consider $X_3(t)$ defined by (10.108) as a set-valued map from $\mathbb{R}^l_+$ to X. We have the next result.

Theorem 10.31. *Assume that the set-valued map $X_3(t)$ defined by (10.108) is u.H.c. at $\mathbf{0} \in \mathbb{R}^l_+$. If (VP) is type I (resp. type II, type III) LP well-posed, then (VP) is type I (resp. type II, type III) LP well-posed in the generalized sense.*

Theorem 10.32. *Assume that there exists $\delta_0 > 0$ such that g is uniformly continuous on the set $X_1(\delta_0)$ defined by (10.97). If (VP) is type I (resp. type II, type III) LP well-posed in the generalized sense, then (VP) is type I (resp. type II, type III) LP well-posed.*

Proof. We prove only the type I case. Suppose that $\{x_k\} \subseteq X_1$ is a type I LP minimizing sequence. That is,

$$d_V(f(x_k)) \to 0, \tag{10.115}$$

$$d_{X_0}(x_k) \to 0. \tag{10.116}$$

By (10.116), we have $d_{X_0}(x_k) \le \delta_0$ when $k \ge k_0$ for some $k_0 > 0$. By the uniform continuity of g on $X_1(\delta_0)$, $d_K(g(x_k)) \to 0$. This together with (10.115) implies that $\{x_k\}$ is a type I generalized LP minimizing sequence. Thus, there exist a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ and some $\bar{x} \in \overline{X}$ such that $x_{k_j} \to \bar{x}$. Hence, (VP) is type I LP well-posed. □

10.3.4 Application to a Class of Penalty Methods

In this subsection, we consider the convergence of a class of penalty methods under the assumption of type III generalized LP well-posedness of (VP).

Let $\alpha > 0$ and $e \in \text{int } C$. Consider the following penalty problem for (VP):

$$(VPP_\alpha(r)) \quad \inf_{x \in X_1} f(x) + r d_K^\alpha(g(x))e, \quad r > 0.$$

Remark 10.9. This class of penalty methods was studied in, e.g., [17].

Theorem 10.33. *Let $0 < r_n \to +\infty$. Consider problems (VP) and $(VPP_\alpha(r_k))$. Assume that there exist $\bar{r} > 0$ and $m_0 \in \mathbb{R}^1$ such that*

$$f(x) + \bar{r} d_K^\alpha(g(x))e \ge_C m_0 e, \quad \forall x \in X_1. \tag{10.117}$$

Let $0 < \varepsilon_k \to 0$. Suppose that each $x_k \in X_1$ satisfies

$$f(x) + r_k d_K^\alpha(g(x))e - f(x_k) - r_k d_K^\alpha(g(x_k))e + \varepsilon_k e \notin -\text{int } C, \quad \forall x \in X_1. \tag{10.118}$$

Further assume that (VP) is type III LP well-posed in the generalized sense. Then, there exist a subsequence $\{x_{k_j}\}$ *of* $\{x_k\}$ *and some* $\bar{x} \in \overline{X}$ *such that* $x_{k_j} \to \bar{x}$*. Moreover, each limit point of* $\{x_k\}$ *belongs to* $\overline{X}$*.*

Proof. Let $x_0 \in X_0$. From (10.118), we deduce that

$$f(x_0) - f(x_k) - r_k d_K^\alpha(g(x_k))e + \varepsilon_k e \notin -\text{int}\, C. \tag{10.119}$$

The combination of (10.117) and (10.119) yields

$$f(x_0) - m_0 e - (r_k - \bar{r}) d_K^\alpha(g(x_k))e + \varepsilon_k e \notin -\text{int}\, C,$$

implying

$$\xi(f(x_0)) - m_0 - (r_k - \bar{r}) d_K^\alpha(g(x_k)) + \varepsilon_k \geq 0,$$

namely,

$$d_K(g(x_k)) \leq \left[\frac{\xi(f(x_0)) + \varepsilon_k - m_0}{r_k - \bar{r}}\right]^{1/\alpha}.$$

Hence,

$$\lim_{k \to +\infty} d_K(g(x_k)) = 0. \tag{10.120}$$

Moreover, from (10.119), we have

$$f(x_0) - f(x_k) + \varepsilon_k e \notin -\text{int}\, C.$$

By the arbitrariness of $x_0 \in X_0$, this further implies that

$$v - f(x_k) + \varepsilon_k e \notin -\text{int}\, C, \forall v \in V.$$

Therefore,

$$\xi(v - f(x_k)) + \varepsilon_k \geq 0, \forall v \in V.$$

Hence,

$$\liminf_{k \to +\infty} \{\inf_{v \in V} \xi(v - f(x_k))\} \geq 0. \tag{10.121}$$

By (10.120) and (10.121), $\{x_k\}$ is a type III generalized LP minimizing sequence. Since (VP) is type III LP well-posed in the generalized sense, there exist a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ and some $\bar{x} \in \overline{X}$ such that $x_{k_j} \to \bar{x}$. Finally, suppose that $\bar{x}$ is a limit point of $\{x_k\}$. Then, there exists a subsequence $\{x_{k_j}\}$ such that $x_{k_j} \to \bar{x}$. It is obvious that $\{x_{k_j}\}$ is also a type III generalized LP minimizing sequence. By the type III generalized LP well-posedness of (VP), there exist a subsequence $\{x_{k_{j_l}}\}$ and some $\bar{x}' \in \overline{X}$ such that $x_{k_{j_l}} \to \bar{x}'$. On the other hand, we have $x_{k_{j_l}} \to \bar{x}$. It follows that $\bar{x} = \bar{x}'$. Hence, $\bar{x} \in \overline{X}$. The proof is complete. □

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Chapter 11
Vector Variational Principles for Set-Valued Functions

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11.1 Introduction

Deriving existence results and necessary conditions for approximate solutions of nonlinear optimization problems under week assumptions is an interesting and modern field in optimization theory. It is of interest to show corresponding results for optimization problems without any convexity and compactness assumptions. Ekeland's variational principle is a very deep assertion about the existence of an exact solution of a slightly perturbed optimization problem in a neighborhood of an approximate solution of the original problem. The importance of Ekeland's variational principle in nonlinear analysis is well known. Especially, this assertion is very useful for deriving necessary conditions under certain differentiability assumptions. In optimal control Ekeland's principle can be used in order to prove an ε-maximum principle in the sense of Pontryagin and in approximation theory for deriving ε-Kolmogorov conditions.

Below we recall a versatile variant.

Proposition 11.1 (Ekeland's Variational Principle [21, 22]). *Let (X,d) be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a proper, lower semicontinuous function bounded below. Consider $\varepsilon > 0$ and $x_0 \in X$ such that $f(x_0) \leq \inf f + \varepsilon$. Then for every $\lambda > 0$ there exists $\bar{x} \in \operatorname{dom} f$ such that*

$$f(\bar{x}) + \lambda^{-1}\varepsilon d(\bar{x}, x_0) \leq f(x_0), \qquad d(\bar{x}, x_0) \leq \lambda, \tag{11.1}$$

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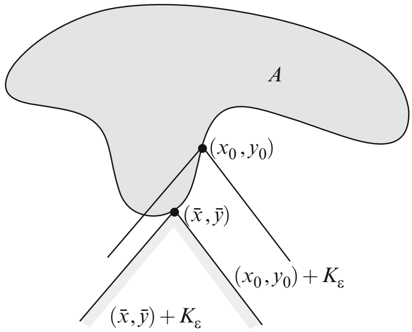

Fig. 11.1 Minimal point $(\bar{x}, \bar{y})$ of a set A with respect to K_ε

and

$$f(\bar{x}) < f(x) + \lambda^{-1}\varepsilon d(\bar{x}, x) \quad \forall x \in X \setminus \{\bar{x}\}. \tag{11.2}$$

This means that for $\lambda, \varepsilon > 0$ and x_0 an ε-approximate solution of the minimization problem

$$f(x) \to \min \text{ s.t. } x \in X, \tag{11.3}$$

there exists a new point $\bar{x}$ that is not worse than x_0 and belongs to a λ-neighborhood of x_0, and especially, $\bar{x}$ satisfies the variational inequality (11.2). Relation (11.2) says, in fact, that $\bar{x}$ minimizes globally $f + \lambda^{-1}\varepsilon d(\bar{x}, \cdot)$, which is nothing else than a Lipschitz perturbation of f (for "smooth" principles, see [11]). Note that $\lambda = \sqrt{\varepsilon}$ gives a useful compromise in Proposition 11.1. For applications see Sect. 11.5 and, e.g., [24, 25, 58, 61, 62].

There are several statements that are equivalent to Ekeland's variational principle (EVP); see, e.g., [1, 2, 5, 12–16, 27, 29–31, 33, 34, 38, 52–54].

Phelps [54] introduced for $\varepsilon > 0$ the following closed convex cone K_ε in $X \times \mathbb{R}$, where X is a Banach space:

$$K_\varepsilon := \{(x, r) \in X \times \mathbb{R} \mid \varepsilon||x|| \leq -r\} \tag{11.4}$$

(see Fig. 11.1). Sometimes the cone K_ε is called a *Phelps cone*. Phelps has shown the existence of minimal points of a set $\mathscr{A} \subseteq X \times \mathbb{R}$ with respect to K_ε under a closedness assumption (H) and a boundedness assumption (B) concerning $\mathscr{A}$.

Proposition 11.2 (Phelps Minimal-Point Theorem [53, 54]). *Let X be a Banach space and $\mathscr{A} (\neq \emptyset) \subseteq X \times \mathbb{R}$. Assume*

(H) $\mathscr{A}$ *is closed*
(B) $\inf\{r \in \mathbb{R} \mid (x, r) \in \mathscr{A}\} = 0$

Suppose $\varepsilon > 0$. Then, for any point $(x_0, r_0) \in \mathscr{A}$ there exists a point $(\bar{x}, \bar{r}) \in \mathscr{A}$ such that:

(a) $(\bar{x}, \bar{r}) \in \mathscr{A} \cap ((x_0, r_0) + K_\varepsilon)$
(b) $\{(\bar{x}, \bar{r})\} = \mathscr{A} \cap ((\bar{x}, \bar{r}) + K_\varepsilon)$

Remark 11.1. The assertion (a) in Proposition 11.2 can be considered as a *domination property* and assertion (b) describes a *minimal point* $(\bar{x},\bar{r})$ of $\mathscr{A}$ with respect to K_ε.

In Phelps [53] and [54] it is shown that Ekeland's variational principle (Proposition 11.1) is a conclusion of a minimal-point theorem (Proposition 11.2) setting $\mathscr{A} = \operatorname{epi} f$ in Proposition 11.2. We will present extensions of Phelps minimal-point theorem to general product spaces and corresponding variational principles. The aim of this chapter is to give an overview on existing minimal-point theorems and variational principles of Ekeland's type for set-valued and vector-valued objective functions. In order to show such assertions a main tool is the application of a certain scalarization technique. In the following section we will discuss scalarizing functionals and their properties.

11.2 Preliminaries

Let us recall some notions and notation for sets and functions defined on locally convex spaces. So let (X,τ) be a locally convex space and $A \subseteq X$. By $\operatorname{cl} A$ (or $\operatorname{cl}_\tau A$ or $\overline{A}$ or $\overline{A}^\tau$), $\operatorname{int} A$ and $\operatorname{bd} A$ we denote the closure (with respect to τ when we want to emphasize the topology), the interior and the boundary of A; moreover $\operatorname{conv} A$ is the convex hull of A and $\overline{\operatorname{conv}} A := \operatorname{cl}(\operatorname{conv} A)$. As usual, for $A, B \subseteq X$, $a \in X$, $\Gamma \subseteq \mathbb{R}$ and $\alpha \in \mathbb{R}$ we set

$$A + B := \{a + b \mid a \in A,\ b \in B\}, \quad a + B := \{a\} + B,$$
$$\Gamma A := \{\gamma a \mid \gamma \in \Gamma,\ a \in A\}, \quad \Gamma a := \Gamma\{a\}, \quad \alpha A := \{\alpha\}A, \quad -A := (-1)A.$$

The recession cone of the nonempty set $A \subseteq X$ is the set

$$A_\infty := \{u \in X \mid x + tu \in A\ \forall x \in A,\ \forall t \in \mathbb{R}_+\}.$$

It follows easily that A_∞ is a convex cone; A_∞ is also closed when A is closed. If A is a closed convex set then $A_\infty = \cap_{t\in\mathbb{P}} t(A-a)$, where $\mathbb{P} :=]0,+\infty[$ and $a \in A$ (A_∞ does not depend on $a \in A$). Moreover, the indicator function associated to the set $A \subseteq X$ is the function $\iota_A : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ defined by $\iota_A(x) := 0$ for $x \in A$ and $\iota_A(x) := \infty$ for $x \in X \setminus A$, where $\infty := +\infty$. A cone $K \subseteq X$ is called pointed if $K \cap (-K) = \{\mathbf{0}\}$.

Let $f : X \to \overline{\mathbb{R}}$; the domain and the epigraph of f are defined by

$$\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\}, \quad \operatorname{epi} f := \{(x,t) \in X \times \mathbb{R} \mid f(x) \le t\}.$$

The function f is said to be *convex* if $\operatorname{epi} f$ is a convex set, and f is said to be *proper* if $\operatorname{dom} f \ne \emptyset$ and f does not take the value $-\infty$. Of course, f is *lower semicontinuous* if $\operatorname{epi} f$ is closed. The class of lower semi-continuous (lsc for short) proper convex functions on X will be denoted by $\Gamma(X)$. Let $B \subseteq X$; $f : X \to \overline{\mathbb{R}}$ is called *B-monotone* if $x_2 - x_1 \in B \Rightarrow \varphi(x_1) \leq \varphi(x_2)$. Furthermore, f is called *strictly B-monotone* if $x_2 - x_1 \in B \setminus \{\mathbf{0}\} \Rightarrow \varphi(x_1) < \varphi(x_2)$.

We consider a proper closed convex cone $K \subseteq Y$ and $k^0 \in K \setminus (-K)$. As usual, we denote

$$K^+ := \{y^* \in Y^* \mid y^*(k) \geq 0 \; \forall k \in K\},$$
$$K^\# := \{y^* \in Y^* \mid y^*(k) > 0 \; \forall k \in K \setminus \{\mathbf{0}\}\}$$

the positive dual cone of the convex cone $K \subseteq Y$ and the quasi interior of K^+, respectively.

In Sect. 11.3 we show several properties of scalarizing functionals. Motivated by papers on the field of economics, especially production theory (cf. Luenberger [50]) we assume that the sets A and K verify the *free-disposal condition* $A - K = A$ included in assumption $(A1)$ introduced in Sect. 11.3.2; for Lipschitz properties of φ_{A,k^0} (see (11.5) for its definition) we need the *strong free-disposal condition* $A - (K \setminus \{0\}) = \operatorname{int} A$, which is a part of assumption $(A2)$. The main results concerning Lipschitz properties are given in Sect. 11.3.4 under assumption $(A1)$: First, without convexity assumptions for the closed set $A \subseteq Y$ we prove that φ_{A,k^0} is Lipschitz on Y under the (stronger) assumption $k^0 \in \operatorname{int} K$ (Theorem 11.4); then, assuming that A is a convex set with nonempty interior and $k^0 \notin A_\infty$ we show that φ_{A,k^0} is locally Lipschitz on $\operatorname{int}(\operatorname{dom} \varphi_A) = \mathbb{R}k^0 + \operatorname{int} A$ (Proposition 11.5). Moreover, without assuming the convexity of A and without the assumption $k^0 \in \operatorname{int} K$ we give a characterization of Lipschitz continuity of φ_{A,k^0} on a neighbourhood of $y_0 \in Y$ using the notion of epi-Lipschitz set introduced by Rockafellar [55] (Theorem 11.5). In Sect. 11.3.5 we provide formulas for the conjugate and the subdifferential of φ_{A,k^0} when A is convex. Using the properties of the scalarizing functionals we present in Sect. 11.4 minimal-point theorems and corresponding variational principles. As an application of the Lipschitz properties of φ_{A,k^0}, we establish necessary conditions for properly efficient solutions of a vector optimization problem in terms of the Mordukhovich subdifferential in Sect. 11.5.2. Taking into account the fact that the conditions in the definition of properly efficient elements are related to the strong free disposal condition in $(A2)$ we get in Theorem 11.15 useful properties for the scalarizing functional φ_{A,k^0} as well as for the Mordukhovich subdifferential of the scalarized objective function.

11.3 Nonlinear Scalarization Functions

In order to show minimal-point theorems and corresponding variational principles in Sect. 11.4 we use a scalarization method by means of certain nonlinear functionals. In this section we discuss useful properties of these functionals (cf. Göpfert et al. [32] and Tammer and Zălinescu [63]).

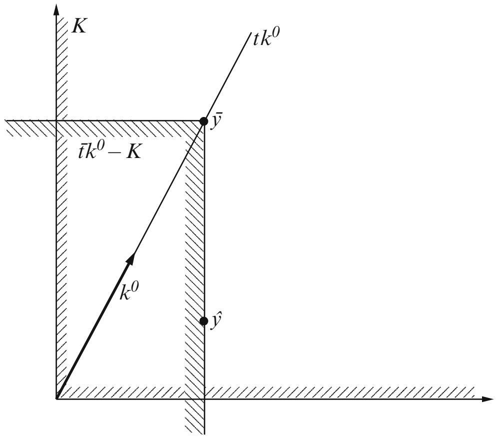

Fig. 11.2 Level sets of the function φ_{A,k^0} from (11.5), where $A = -K = -\mathbb{R}^2_+$ and $k^0 \in \operatorname{int} K$ holds

11.3.1 Construction of Scalarizing Functionals

Having a nonempty subset A of a real linear space Y and an element $k^0 \neq \mathbf{0}$ of Y, Gerstewitz (Tammer) and Iwanow [28] introduced the function (see Fig. 11.2)

$$\varphi_A := \varphi_{A,k^0} : Y \to \overline{\mathbb{R}}, \quad \varphi_{A,k^0}(y) := \inf\{t \in \mathbb{R} \mid y \in tk^0 + A\}, \tag{11.5}$$

where, as usual, $\inf \emptyset := \infty$ (and $\sup \emptyset := -\infty$); we use also the convention $(+\infty) + (-\infty) := +\infty$.

This function was used by Chr. Tammer and her collaborators, as well as by D.T. Luc etc., mainly for scalarization of vector optimization problems. Luenberger [50, Definition 4.1] considered

$$\sigma(g;y) := \inf\{\xi \in \mathbb{R} \mid y - \xi g \in \mathscr{Y}\},$$

the corresponding function being called the shortage function associated to the production possibility set $\mathscr{Y} \subseteq \mathbb{R}^m$ and $g \in \mathbb{R}^m_+ \setminus \{\mathbf{0}\}$. The case when $g = (1,\ldots,1)$ was introduced earlier by Bonnisseau and Cornet [10]. A similar function is introduced in [50, Definition 2.1] under the name of benefit function.

More recently such a function was considered in the context of mathematical finance beginning with Artzner et. al. [3]; see Heyde [42] and Hamel [39] for more historical facts. Under the name of topical function such functions were studied by Singer and his collaborators (see [59]). We discuss many important properties of φ_{A,k^0} in Sect. 11.3.2. Moreover, we study local continuity properties in Sect. 11.3.4. Very recently Bonnisseau and Crettez [4] obtained local Lipschitz properties for φ_{A,k^0} (called Luenberger shortage function in [4]) in a very special case, more general results are given by Tammer and Zălinescu [63]. Of course, φ_{A,k^0} is a continuous sublinear functional if A is a proper closed convex cone and $k^0 \in \operatorname{int} A$ (cf. Corollary 11.2) and so φ_{A,k^0} is Lipschitz continuous. Such Lipschitz properties of φ_{A,k^0} are of interest also in the case when $A \subseteq Y$ is an arbitrary (convex) set and the interior of the usual ordering cone in Y is empty like in mathematical finance where the acceptance sets are in function spaces as L_p and the corresponding risk measures are formulated by means of φ_{A,k^0} (see e.g. Föllmer and Schied [26]).

11.3.2 Properties of Scalarization Functions

Throughout this section Y is a separated locally convex space and Y^* is its topological dual, $K \subseteq Y$ is a proper closed convex cone, $k^0 \in K \setminus (-K)$ and $A \subseteq Y$ is a nonempty set. The cone K determines the order $\leq_K$ on Y defined by $y_1 \leq_K y_2$ if $y_2 - y_1 \in K$.

Furthermore, we assume that A satisfies the following condition (see also [4]):

(A1) A is closed, satisfies the free-disposal assumption $A - K = A$, and $A \neq Y$.

We shall use also the (stronger) condition:

(A2) A is closed, satisfies the strong free-disposal assumption $A - (K \setminus \{\mathbf{0}\}) = \operatorname{int} A$, and $A \neq Y$.

Because $A - K = A \cup (A - (K \setminus \{\mathbf{0}\}))$, we have that $(A2) \Rightarrow (A1)$. Moreover, the condition $A - (K \setminus \{\mathbf{0}\}) = \operatorname{int} A$ is equivalent to $A - (K \setminus \{\mathbf{0}\}) \subseteq \operatorname{int} A$.

Remark 11.2. Assume that the nonempty set A satisfies assumption $(A2)$. Then K is pointed, that is, $K \cap (-K) = \{\mathbf{0}\}$, and $A - \mathbb{P}k^0 \subseteq \operatorname{int} A$ for $k^0 \in K \setminus \{\mathbf{0}\}$.

The last assertion is obvious. For the first one, assume that $k \in K \cap (-K) \setminus \{\mathbf{0}\}$. Take $a \in \operatorname{bd} A$ $(\subseteq A)$; such an a exists because $A \neq Y$. Then $a' := a - k \in \operatorname{int} A \subseteq A$, and so $a = a' - (-k) \in \operatorname{int} A$, a contradiction.

Remark 11.3. When A satisfies condition (A1) or (A2) with respect to K and $k^0 \in K \setminus (-K)$ then A satisfies condition (A1) or (A2), respectively, with respect to $\mathbb{R}_+ k^0$. In fact in many situations it is sufficient to take $K = \mathbb{R}_+ k^0$ for some $k^0 \in Y \setminus \{\mathbf{0}\}$. In such a situation (A1) [respectively (A2)] means that A is a closed proper subset of Y and $A - \mathbb{R}_+ k^0 = A$ [respectively $A - \mathbb{P}k^0 \subseteq \operatorname{int} A$].

The free-disposal condition $A = A - K$ shows that $K \subseteq -A_\infty$. As observed above A_∞ is also closed because A is closed. Hence $-A_\infty$ is the largest closed convex cone K verifying the free-disposal assumption $A = A - K$.

The aim of this section is to find a suitable functional $\varphi : Y \to \mathbb{R}$ and conditions such that two given nonempty subsets A and H of Y can be separated by φ.

To $A \subseteq Y$ satisfying (A1) and $k^0 \in K \setminus (-K)$ we associate the function φ_{A,k^0} defined in (11.5). We consider the set

$$A' := \{(y,t) \in Y \times \mathbb{R} \mid y \in tk^0 + A\}.$$

The assumption on A shows that A' is of *epigraph type*, i.e. if $(y,t) \in A'$ and $t' \geq t$, then $(y,t') \in A'$. Indeed, if $y \in tk^0 + A$ and $t' \geq t$, since

$$tk^0 + A = t'k^0 + A - (t'-t)k^0 \subseteq t'k^0 + A,$$

(because of (A1)) we obtain that $(y,t') \in A'$. Also observe that $A' = T^{-1}(A)$, where $T : Y \times \mathbb{R} \to Y$ is the continuous linear operator defined by $T(y,t) := tk^0 + y$. So, if A is closed (convex, cone), then A' is closed (convex, cone). Obviously, the domain of φ_A is the set $\mathbb{R}k^0 + A$ and $A' \subseteq \operatorname{epi} \varphi_A \subseteq \operatorname{cl} A'$ (because A' is of epigraph type), from which it follows that $A' = \operatorname{epi} \varphi_A$ if A is closed, and so φ_A is a lower semicontinuous function.

In the next results we collect several useful properties of φ_A (compare Göpfert et al. [32]).

Theorem 11.1. *Assume that $K \subseteq Y$ is a proper closed convex cone, $k^0 \in K \setminus (-K)$ and $A \subseteq Y$ is a nonempty set. Furthermore, suppose*

(A1) A is closed, satisfies the free-disposal assumption $A - K = A$, and $A \neq Y$.

Then φ_A (defined in (11.5)) is lsc, $\operatorname{dom} \varphi_A = \mathbb{R}k^0 + A$,

$$\{y \in Y \mid \varphi_A(y) \leq \lambda\} = \lambda k^0 + A \quad \forall \lambda \in \mathbb{R}, \tag{11.6}$$

and

$$\varphi_A(y + \lambda k^0) = \varphi_A(y) + \lambda \quad \forall y \in Y,\ \forall \lambda \in \mathbb{R}. \tag{11.7}$$

Moreover,

(a) φ_A is convex if and only if A is convex; $\varphi_A(\lambda y) = \lambda \varphi_A(y)$ for all $\lambda > 0$ and $y \in Y$ if and only if A is a cone.
(b) φ_A is proper if and only if A does not contain lines parallel to k^0, i.e.,

$$\forall y \in Y,\ \exists t \in \mathbb{R} \ :\ y + tk^0 \notin A. \tag{11.8}$$

(c) φ_A is finite-valued if and only if A does not contain lines parallel to k^0 and

$$\mathbb{R}k^0 + A = Y. \tag{11.9}$$

(d) Let $B \subseteq Y$; φ_A is B-monotone if and only if $A - B \subseteq A$.
(e) φ_A is subadditive if and only if $A + A \subseteq A$.

Proof. We have already observed that $\operatorname{dom}\varphi_A = \mathbb{R}k^0 + A$ and φ_A is lsc when A is closed. From the definition of φ_A the inclusion $\supseteq$ in (11.6) is obvious, while the converse inclusion is immediate, taking into account the closedness of A. Formula (11.7) follows easily from (11.6).

(a) Since the operator T defined above is onto and $\operatorname{epi}\varphi_A = T^{-1}(A)$, we have that $\operatorname{epi}\varphi_A$ is convex (cone) if and only if $A = T(\operatorname{epi}\varphi_A)$ is so. The conclusion follows.

(b) We have

$$\varphi_A(y) = -\infty \Leftrightarrow y \in tk^0 + A \ \forall t \in \mathbb{R} \Leftrightarrow \{y + tk^0 \mid t \in \mathbb{R}\} \subseteq A.$$

The conclusion follows.

(c) The conclusion follows from (b) and the fact that $\operatorname{dom}\varphi_A = \mathbb{R}k^0 + A$.

(d) Suppose first that $A - B \subseteq A$ and take $y_1, y_2 \in Y$ with $y_2 - y_1 \in B$. Let $t \in \mathbb{R}$ be such that $y_2 \in tk^0 + A$. Then $y_1 \in y_2 - B \subseteq tk^0 + (A - B) \subseteq tk^0 + A$, and so $\varphi_A(y_1) \le t$. Hence $\varphi_A(y_1) \le \varphi_A(y_2)$. Assume now that φ_A is B-monotone and take $y \in A$ and $b \in B$. From (11.6) we have that $\varphi_A(y) \le 0$. Since $y - (y - b) \in B$, we obtain that $\varphi_A(y - b) \le \varphi_A(y) \le 0$, and so, using again (11.6), we obtain that $y - b \in A$.

(e) Suppose first that $A + A \subseteq A$ and take $y_1, y_2 \in Y$. Let $t_i \in \mathbb{R}$ be such that $y_i \in t_ik^0 + A$ for $i \in \{1, 2\}$. Then $y_1 + y_2 \in (t_1 + t_2)k^0 + (A + A) \subseteq (t_1 + t_2)k^0 + A$, and so $\varphi_A(y_1 + y_2) \le t_1 + t_2$. It follows that $\varphi_A(y_1 + y_2) \le \varphi_A(y_1) + \varphi_A(y_2)$. Assume now that φ_A is subadditive and take $y_1, y_2 \in A$. From (11.6) we have that $\varphi_A(y_1), \varphi_A(y_2) \le 0$. Since φ_A is subadditive, we obtain that $\varphi_A(y_1 + y_2) \le \varphi_A(y_1) + \varphi_A(y_2) \le 0$, and so, using again (11.6), we obtain that $y_1 + y_2 \in A$. □

Remark 11.4. From Theorem 11.1 we get under assumption (A1) that φ_A is lower semicontinuous,

$$A = \{y \in Y \mid \varphi_A(y) \le 0\}, \quad \operatorname{int} A \subseteq \{y \in Y \mid \varphi_A(y) < 0\}, \tag{11.10}$$

and so

$$\operatorname{bd} A = A \setminus \operatorname{int} A \supseteq \{y \in Y \mid \varphi_A(y) = 0\}. \tag{11.11}$$

In general the inclusion in (11.11) is strict.

Example 11.1. Consider $K := \mathbb{R}^2_+$, $k^0 := (1, 0)$ and

$$A := (]-\infty, 0] \times]-\infty, 0]) \cup ([0, \infty[\times]-\infty, -1]).$$

Then $\varphi_A(u, v) = -\infty$ for $v \le -1$, $\varphi_A(u, v) = u$ for $v \in (-1, 0]$ and $\varphi_A(u, v) = \infty$ for $v > 0$. In particular, $\varphi_A(0, -1) = -\infty$ and $(0, -1) \in \operatorname{bd} A$ (see Fig. 11.3).

Theorem 11.2. *Assume that $K \subseteq Y$ is a proper closed convex cone, $k^0 \in K \setminus (-K)$ and $A \subseteq Y$ is a nonempty set. Furthermore, suppose*

(A2) A is closed, satisfies the strong free-disposal assumption $A - (K \setminus \{\mathbf{0}\}) = \operatorname{int} A$, and $A \neq Y$.

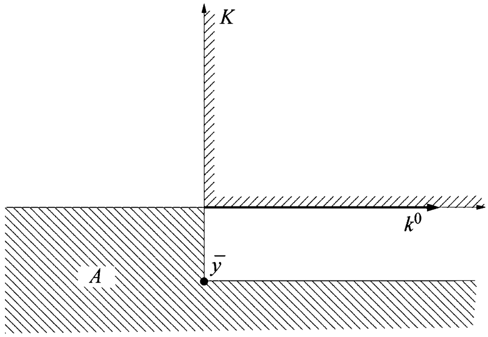

Fig. 11.3 $\bar{y} \in \operatorname{bd} A$ with $\varphi_A(\bar{y}) = -\infty$ in Example 11.1

Then (a), (b), (c) from Theorem 11.1 holds, and moreover

(f) φ_A *is continuous and*

$$\{y \in Y \mid \varphi_A(y) < \lambda\} = \lambda k^0 + \operatorname{int} A, \quad \forall \lambda \in \mathbb{R}, \tag{11.12}$$

$$\{y \in Y \mid \varphi_A(y) = \lambda\} = \lambda k^0 + \operatorname{bd} A, \quad \forall \lambda \in \mathbb{R}. \tag{11.13}$$

(g) If φ_A *is proper, then*

$$\varphi_A \text{ is } B\text{-monotone} \Leftrightarrow A - B \subseteq A \Leftrightarrow \operatorname{bd} A - B \subseteq A.$$

Moreover, if φ_A *is finite-valued, then*

$$\varphi_A \text{ strictly } B\text{-monotone} \Leftrightarrow A - (B \setminus \{\mathbf{0}\}) \subseteq \operatorname{int} A \Leftrightarrow \operatorname{bd} A - (B \setminus \{\mathbf{0}\}) \subseteq \operatorname{int} A.$$

(h) Assume that φ_A *is proper; then*

$$\varphi_A \text{ is subadditive} \Leftrightarrow A + A \subseteq A \Leftrightarrow \operatorname{bd} A + \operatorname{bd} A \subseteq A.$$

Proof. Suppose now that (A2) holds.

(f) Let $\lambda \in \mathbb{R}$ and take $y \in \lambda k^0 + \operatorname{int} A$. Since $y - \lambda k^0 \in \operatorname{int} A$, there exists $\varepsilon > 0$ such that $y - \lambda k^0 + \varepsilon k^0 \in A$. Therefore $\varphi_A(y) \leq \lambda - \varepsilon < \lambda$, which shows that the inclusion $\supseteq$ always holds in (11.12). Let $\lambda \in \mathbb{R}$ and $y \in Y$ be such that $\varphi_A(y) < \lambda$. There exists $t \in \mathbb{R}$, $t < \lambda$, such that $y \in tk^0 + A$. It follows with (A2) that $y \in \lambda k^0 + A - (\lambda - t)k^0 \subseteq \lambda k^0 + \operatorname{int} A$. Therefore (11.12) holds, and so φ_A is upper semicontinuous. Because φ_A is also lower semicontinuous, we have that φ_A is continuous. From (11.6) and (11.12) we obtain immediately that (11.13) holds.

(g) Let us prove the second part, the first one being similar to that of (and partially proved in) (d). So, let φ_A be finite-valued.

Assume that φ_A is strictly B-monotone and take $y \in A$ and $b \in -B \setminus \{\mathbf{0}\}$. From (11.6) we have that $\varphi_A(y) \leq 0$, and so, by hypothesis, $\varphi_A(y-b) < 0$. Using (11.12) we obtain that $y-b \in \operatorname{int} A$. Assume now that $\operatorname{bd} A - (B \setminus \{\mathbf{0}\}) \subseteq \operatorname{int} A$. Consider $y_1, y_2 \in Y$ with $y_2 - y_1 \in B \setminus \{\mathbf{0}\}$. From (11.13) we have that $y_2 \in \varphi_A(y_2)k^0 + \operatorname{bd} A$, and so $y_1 \in \varphi_A(y_2)k^0 - (\operatorname{bd} A + (B \setminus \{\mathbf{0}\})) \subseteq \varphi_A(y_2)k^0 + \operatorname{int} A$. From (11.12) we obtain that $\varphi_A(y_1) < \varphi_A(y_2)$. The remaining implication is obvious.

(h) Let φ_A be proper. One has to prove $\operatorname{bd} A + \operatorname{bd} A \subseteq A \Rightarrow \varphi_A$ is subadditive. Consider $y_1, y_2 \in Y$. If $\{y_1, y_2\} \not\subseteq \operatorname{dom} \varphi_A$, there is nothing to prove; hence let $y_1, y_2 \in \operatorname{dom} \varphi_A$. Then, by (11.13), $y_i \in \varphi_A(y_i)k^0 + \operatorname{bd} A$ for $i \in \{1,2\}$, and so $y_1 + y_2 \in (\varphi_A(y_1) + \varphi_A(y_2))k^0 + (\operatorname{bd} A + \operatorname{bd} A) \subseteq (\varphi_A(y_1) + \varphi_A(y_2))k^0 + A$. Therefore $\varphi_A(y_1 + y_2) \leq \varphi_A(y_1) + \varphi_A(y_2)$. □

When $k^0 \in \operatorname{int} K$ we get an additional important property of φ_A (see also Theorem 11.4).

Corollary 11.1. *Assume that $K \subseteq Y$ is a proper closed convex cone, $k^0 \in \operatorname{int} K$ and $A \subseteq Y$ satisfies condition (A1). Then φ_A is finite-valued and continuous.*

Proof. Because $k^0 \in \operatorname{int} K$ we have that $\mathbb{R}k^0 + K = Y$. From Theorem 11.1 (c) it follows that

$$\operatorname{dom} \varphi_A = A + \mathbb{R}k^0 = A - K + \mathbb{R}k^0 = A + Y = Y.$$

Assuming that φ_A is not proper, from Theorem 11.1 (c) we get $y + \mathbb{R}k^0 \subseteq A$ for some $y \in Y$. Then $Y = y + \mathbb{R}k^0 - K \subseteq A - K = A$, a contradiction. Hence φ_A is finite-valued.

Moreover, we have that $A - \mathbb{P}k^0 \subseteq A - \operatorname{int} K \subseteq \operatorname{int}(A-K) = \operatorname{int} A$. Applying Theorem 11.2 (f) for K replaced by $\mathbb{R}_+ k^0$ we obtain that φ_A is continuous. □

From the preceding results we get the following particular case.

Corollary 11.2. *Let $K \subseteq Y$ be a proper closed convex cone and $k^0 \in -\operatorname{int} K$. Then*

$$\varphi_K : Y \to \mathbb{R}, \qquad \varphi_K(y) := \inf\{t \in \mathbb{R} \mid y \in tk^0 + K\}$$

is a well-defined continuous sublinear function such that for every $\lambda \in \mathbb{R}$,

$$\{y \in Y \mid \varphi_K(y) \leq \lambda\} = \lambda k^0 + K, \quad \{y \in Y \mid \varphi_K(y) < \lambda\} = \lambda k^0 + \operatorname{int} K.$$

Moreover, φ_K is strictly $(-\operatorname{int} K)$-monotone.

Proof. The assertions follow using Theorem 11.2 and Corollary 11.1 applied for $A := K$ and K replaced by $-K$. For the last part note that $K + \operatorname{int} K = \operatorname{int} K$. □

Now all preliminaries are done, and we can prove the following nonconvex separation theorem.

Theorem 11.3 (Non-convex Separation Theorem). *Let $A \subseteq Y$ be a closed proper set with nonempty interior, $H \subseteq Y$ a nonempty set such that $H \cap \operatorname{int} A = \emptyset$. Let $K \subseteq Y$ be a proper closed convex cone and $k^0 \in \operatorname{int} K$. Furthermore, assume*

(A2) A is closed, satisfies the strong free-disposal assumption $A - (K \setminus \{\mathbf{0}\}) = \operatorname{int} A$, and $A \neq Y$.

Then φ_A defined by (11.5) is a finite-valued continuous function such that

$$\varphi_A(x) \geq 0 > \varphi_A(y) \quad \forall x \in H,\ \forall y \in \operatorname{int} A; \tag{11.14}$$

moreover, $\varphi_A(x) > 0$ for every $x \in \operatorname{int} H$.

Proof. By Corollary 11.1 φ_A is a finite-valued continuous function. By Theorem 11.2 (f) we have that $\operatorname{int} A = \{y \in Y \mid \varphi_A(y) < 0\}$, and so (11.14) obviously holds.

Take $y \in \operatorname{int} H$; then there exists $t > 0$ such that $y - tk^0 \in H$. From (11.7) and (11.12) we obtain that $0 \leq \varphi_A(y - tk^0) = \varphi_A(y) - t$, whence $\varphi_A(y) > 0$. □

Of course, if we impose additional conditions on A, we have additional properties of the separating functional φ_A (see Theorems 11.1 and 11.2).

11.3.3 Continuity Properties

If A is a proper closed subset of Y (hence $\emptyset \neq A \neq Y$) and $A - \mathbb{P}k^0 \subseteq \operatorname{int} A$, applying Theorem 11.2 for $K := \mathbb{R}_+ k^0$ we obtain that φ_A is continuous (on Y) and (11.13) holds. In the next result we characterize the continuity of φ_A at a point $y_0 \in Y$ (compare Tammer and Zălinescu [63]).

Proposition 11.3. *Assume that $K \subseteq Y$ is a proper closed convex cone, $k^0 \in K \setminus (-K)$ and $A \subseteq Y$ is a nonempty set satisfying condition (A1). Then the function φ_A is (upper semi-) continuous at $y_0 \in Y$ if and only if $y_0 -]\varphi_A(y_0), \infty[\cdot k^0 \subseteq \operatorname{int} A$.*

Proof. If $\varphi_A(y_0) = \infty$ it is clear that φ_A is upper semicontinuous at y_0 and the inclusion holds. So let $\varphi_A(y_0) < \infty$.

Assume first that φ_A is upper semicontinuous at y_0. Let $\lambda \in]\varphi_A(y_0), \infty[$. Then there exists a neighbourhood V of y_0 such that $\varphi_A(y) < \lambda$ for every $y \in V$. It follows that for $y \in V$ we have $y \in \lambda k^0 + A$, that is, $V \subseteq \lambda k^0 + A$. Hence $y_0 \in \lambda k^0 + \operatorname{int} A$, whence $y_0 - \lambda k^0 \in \operatorname{int} A$.

Assume now that $y_0 -]\varphi_A(y_0), \infty[\cdot k^0 \subseteq \operatorname{int} A$ and take $\varphi_A(y) < \lambda < \infty$. Then, by our hypothesis, $V := \lambda k^0 + A$ is a neighbourhood of y_0 and from the definition of φ_A we have that $\varphi_A(y) \leq \lambda$ for every $y \in V$. Hence φ_A is upper semicontinuous at y_0. □

Corollary 11.3. *Under the hypotheses of Proposition 11.3 assume that φ_A is continuous at $y_0 \in \operatorname{bd} A$. Then $\varphi_A(y_0) = 0$.*

Proof. Of course, $\varphi_A(y_0) \leq 0$. If $\varphi_A(y_0) < 0$, from the preceding proposition we obtain the contradiction $y_0 = y_0 - 0k^0 \in \operatorname{int} A$. □

11.3.4 Lipschitz Properties

The primary goal of this section is to study local Lipschitz properties of the functional φ_{A,k^0} under as weak as possible assumptions concerning the subset $A \subseteq Y$ and $k^0 \in Y$ (compare Tammer and Zălinescu [63]).

When A is a convex set, as noticed above, φ_A is convex. In such a situation from the continuity of φ_A at a point in the interior of its domain one obtains the local Lipschitz continuity of φ_A on the interior of its domain (if the function is proper). Moreover, when $A = -K$ and $k^0 \in \operatorname{int} K$ then (it is well known that) φ_A is a continuous sublinear function, and so φ_A is Lipschitz continuous.

Recently in the case $Y = \mathbb{R}^m$ and for $K = \mathbb{R}^m_+$ Bonnisseau–Crettez [4] obtained the Lipschitz continuity of φ_A around a point $y \in \operatorname{bd} A$ when $-k^0$ is in the interior of the Clarke tangent cone of A at y. The (global) Lipschitz continuity of φ_A can be related to a result of Gorokhovik–Gorokhovik [35] established in normed vector spaces as we shall see in the sequel.

Theorem 11.4. *Assume that $K \subseteq Y$ is a proper closed convex cone, $k^0 \in K \setminus (-K)$ and $A \subseteq Y$ is a nonempty set satisfying condition (A1).*

(a) One has

$$\varphi_A(y) \leq \varphi_A(y') + \varphi_{-K}(y - y') \quad \forall y, y' \in Y. \tag{11.15}$$

(b) If $k^0 \in \operatorname{int} K$ then φ_A is finite-valued and Lipschitz on Y.

Proof. (a) By Theorem 11.1 (applied for A and $A := -K$, respectively) we have that φ_A and φ_{-K} are lower semicontinuous functions, φ_{-K} being sublinear and proper.

Let $y, y' \in Y$. If $\varphi_A(y') = +\infty$ or $\varphi_{-K}(y - y') = +\infty$ it is nothing to prove. In the contrary case let $t, s \in \mathbb{R}$ be such that $y - y' \in tk^0 - K$ and $y' \in sk^0 + A$. Then, taking into account assumption $(A1)$

$$y \in tk^0 - K + sk^0 + A = (t+s)k^0 + (A - K) = (t+s)k^0 + A.$$

It follows that $\varphi_A(y) \leq t + s$. Passing to infimum with respect to t and s satisfying the preceding relations we get (11.15).

(b) Assume that $k^0 \in \operatorname{int} K$. Let $V \subseteq Y$ be a symmetric closed and convex neighbourhood of 0 such that $k^0 + V \subseteq K$ and let $p_V : Y \to \mathbb{R}$ be the Minkowski functional associated to V; then p_V is a continuous seminorm and $V = \{y \in Y \mid p_V(y) \leq 1\}$. Let $y \in Y$ and $t > 0$ such that $y \in tV$. Then $t^{-1}y \in V \subseteq k^0 - K$, whence $y \in tk^0 - K$. Hence $\varphi_{-K}(y) \leq t$. Therefore, $\varphi_{-K}(y) \leq p_V(y)$. This inequality confirms that ($\mathbb{R}k^0 - K =$) $\operatorname{dom}\varphi_{-K} = Y$. Moreover, since φ_{-K} is sublinear we get $\varphi_{-K}(y) \leq \varphi_{-K}(y') + p_V(y-y')$ and so

$$\left|\varphi_{-K}(y) - \varphi_{-K}(y')\right| \leq p_V(y-y') \quad \forall y, y' \in Y, \tag{11.16}$$

that is, φ_{-K} is Lipschitz.

By Corollary 11.1 we have that φ_A is finite-valued (and continuous). From (11.15) we have that $\varphi_A(y) - \varphi_A(y') \leq \varphi_{-K}(y-y') \leq p_V(y-y')$, whence (interchanging y and y')

$$\left|\varphi_A(y) - \varphi_A(y')\right| \leq p_V(y-y') \quad \forall y, y' \in Y. \tag{11.17}$$

Hence φ_A is Lipschitz continuous (on Y). □

Note that the condition $A - (K \setminus \{\mathbf{0}\}) \subseteq \operatorname{int} A$ does not imply that φ_A is proper.

Example 11.2. Take $A := \{(x,y) \in \mathbb{R}^2 \mid y \geq -|x|^{-1}\}$, with the convention $0^{-1} := \infty$, and $K := \mathbb{R}_+ k^0$ with $k^0 := (0,-1)$. Then $A - (K \setminus \{0\}) = \operatorname{int} A$ and $\varphi_A(0,1) = -\infty$.

Note that, with our notation, [4, Proposition 7] asserts that φ_{A,k^0} is finite and locally Lipschitz provided $Y = \mathbb{R}^n$, $K = \mathbb{R}^n_+$ and $k^0 \in \operatorname{int} K$, which is much less than the conclusion of Theorem 11.4 (ii).

Of course, in the conditions of Theorem 11.4 (ii) we have that $-k^0 \in \operatorname{int} A_\infty$ because $K \subseteq -A_\infty$. In fact we have also a converse of Theorem 11.4 (ii).

Proposition 11.4. *Assume that $K \subseteq Y$ is a proper closed convex cone, $k^0 \in K \setminus (-K)$ and $A \subseteq Y$ is a nonempty set satisfying condition (A1). If φ_A is finite-valued and Lipschitz then $-k^0 \in \operatorname{int} A_\infty$.*

Proof. By hypothesis there exists a closed convex and symmetric neighbourhood V of $\mathbf{0}$ such that (11.17) holds. We have that $A = \{y \in Y \mid \varphi_A(y) \leq 0\}$. Let $y \in A$, $v \in V$ and $\alpha \geq 0$. Then

$$\varphi_A(y + \alpha(v - k^0)) \leq \varphi_A(y + \alpha v) - \alpha \leq \varphi_A(y) + \alpha p_V(v) - \alpha \leq 0$$

because $V = \{y \in Y \mid p_V(y) \leq 1\}$. Hence $V - k^0 \subseteq A_\infty$, which shows that $-k^0 \in \operatorname{int} A_\infty$. □

Corollary 11.4. *Under the assumptions of Proposition 11.4, the function φ_A is finite-valued and Lipschitz if and only if $-k^0 \in \operatorname{int} A_\infty$.*

Proof. The necessity is given by Proposition 11.4. Assume that $-k^0 \in \operatorname{int} A_\infty$. Taking $K := -A_\infty$, using Theorem 11.4 (b) we obtain that φ_A is finite-valued and Lipschitz. □

If $\operatorname{int} K \neq \emptyset$ and $k^0 \notin \operatorname{int} K$, φ_{-K} is not finite-valued, and so it is not Lipschitz. One may ask if the restriction of φ_{-K} at its domain is Lipschitz. The next examples show that both situations are possible.

Example 11.3. Take $K = \mathbb{R}^2_+$ and $k^0 = (1,0)$. We have that $\varphi_{-K}(y_1,y_2) = y_1$ for $y_2 \le 0$, $\varphi_{-K}(y_1,y_2) = \infty$ for $y_2 > 0$, and so $\varphi_{-K}|_{\operatorname{dom}\varphi_{-K}}$ is Lipschitz.

Example 11.4. Take $K := \{(u,v,w) \in \mathbb{R}^3 \mid v,w \ge 0,\ u^2 \le vw\}$ and $k^0 := (0,0,1)$; then

$$\varphi_{-K}(x,y,z) = \begin{cases} \infty & \text{if } y > 0 \text{ or } [y = 0 \text{ and } x \ne 0], \\ z & \text{if } x = y = 0, \\ z - x^2/y & \text{if } y < 0. \end{cases}$$

It is clear that the restriction of φ_{-K} at its domain is not continuous at $(0,0,0) \in \operatorname{dom}\varphi_{-K}$ and the restriction of φ_{-K} at the interior of its domain is not Lipschitz. However, φ_{-K} is locally Lipschitz on the interior of its domain.

The last property mentioned in the previous example is a general one for φ_A when A is convex.

Proposition 11.5. *Let A be a proper closed subset of Y and $k^0 \in Y \setminus \{\mathbf{0}\}$ be such that $A - \mathbb{R}_+ k^0 = A$. If A is convex, has nonempty interior, and does not contain any line parallel with k^0 (or equivalently $k^0 \notin A_\infty$), then φ_A is locally Lipschitz on* $\operatorname{int}(\operatorname{dom}\varphi_A) = \mathbb{R}k^0 + \operatorname{int}A$.

Proof. Because A does not contain any line parallel with k^0, φ_A is proper (see Theorem 11.1 taking into account assumption $(A1)$). We know that $\operatorname{dom}\varphi_A = \mathbb{R}k^0 + A$, and so $\operatorname{int}(\operatorname{dom}\varphi_A) = \operatorname{int}(\mathbb{R}k^0 + A) = \mathbb{R}k^0 + \operatorname{int}A$ (see, e.g., [67, Exercise 1.4]). On the other hand it is clear that $A \subseteq \{y \in Y \mid \varphi_A(y) \le 0\}$. Since $\operatorname{int}A \ne \emptyset$, we have that φ_A is bounded above on a neighbourhood of a point, and so φ_A is locally Lipschitz on $\operatorname{int}(\operatorname{dom}\varphi_A) = \mathbb{R}k^0 + \operatorname{int}A$ (see e.g. [67, Corollary 2.2.13]). □

We have seen in Theorem 11.4 that φ_A is Lipschitz even if A is not convex when $k^0 \in \operatorname{int}K$. So, in the sequel we are interested by the case in which A is not convex, $k^0 \notin \operatorname{int}K$ and A does not contain any line parallel with k^0.

Note that for A not convex and $y \in \operatorname{int}(\operatorname{dom}\varphi_A)$ we can have situations in which φ_A is not continuous at y or φ_A is continuous but not Lipschitz around y.

Example 11.5. Take $K := \mathbb{R}^2_+$, $k^0 := (1,0)$ and

$$A_1 := (]-\infty,0] \times]-\infty,1]) \cup ([0,1] \times]-\infty,0])$$
$$A_2 := \{(a,b) \mid a \in]0,\infty[,\ b \le -a^2\} \cup (]-\infty,0] \times]-\infty,1]).$$

Then

$$\varphi_{A_1,k^0}(u,v) = \begin{cases} \infty & \text{if } v > 1, \\ u & \text{if } 0 < v \le 1, \\ u-1 & \text{if } v \le 0, \end{cases} \qquad \varphi_{A_2,k^0}(u,v) = \begin{cases} \infty & \text{if } v > 1, \\ u & \text{if } 0 < v \le 1, \\ u - \sqrt{-v} & \text{if } v \le 0. \end{cases}$$

It is clear that $(0,0) \in \operatorname{int}(\operatorname{dom}\varphi_{A_1})$ but φ_{A_1} is not continuous at $(0,0)$, and $(0,0) \in \operatorname{int}(\operatorname{dom}\varphi_{A_2})$, φ_{A_2} is continuous at $(0,0)$ but φ_{A_2} is not Lipschitz at $(0,0)$.

In what concerns the Lipschitz continuity of φ_A around a point $y \in \operatorname{dom} \varphi_A$ in finite dimensional spaces this can be obtained using the notion of epi-lipschitzianity of a set as introduced by Rockafellar [55] (see also [56]). We extend this notion in our context. We say that the set $A \subseteq Y$ is epi-Lipschitz at $\overline{y} \in A$ in the direction $v \in Y \setminus \{\mathbf{0}\}$ if there exist $\varepsilon > 0$ and a (closed convex symmetric) neighbourhood V_0 of $\mathbf{0}$ in Y such that

$$\forall y \in (\overline{y} + V_0) \cap A,\ \forall w \in v + V_0,\ \forall \lambda \in [0, \varepsilon]:\ y + \lambda w \in A. \tag{11.18}$$

Note that (11.18) holds for $v = \mathbf{0}$ if and only if $\overline{y} \in \operatorname{int} A$. Moreover, if $\overline{y} \in \operatorname{int} A$ then A is epi-Lipschitz at $\overline{y} \in A$ in any direction.

Theorem 11.5. *Let A be a proper closed subset of Y and $k^0 \in Y \setminus \{\mathbf{0}\}$ be such that $A - \mathbb{R}_+ k^0 = A$. Assume that $y_0 \in Y$ is such that $\varphi_A(y_0) \in \mathbb{R}$. Then φ_A is finite and Lipschitz on a neighbourhood of y_0 if and only if A is epi-Lipschitz at $\overline{y} := y_0 - \varphi_A(y_0) k^0$ in the direction $-k^0$.*

Proof. Using (11.7) we get $\varphi_A(\overline{y}) = 0$. Recall also that $A = \{y \in Y \mid \varphi_A(y) \leq 0\}$ and the finite values of φ_A are attained (because A is closed).

Assume that there exist a closed convex symmetric neighbourhood V of $\mathbf{0}$ in Y and $p : Y \to \mathbb{R}$ a continuous seminorm such that φ_A is finite on $y_0 + V$ and $|\varphi_A(y) - \varphi_A(y')| \leq p(y - y')$ for all $y, y' \in y_0 + V$. Taking into account (11.7), we have that φ_A is finite on $\overline{y} + V$ and

$$\left|\varphi_A(y) - \varphi_A(y')\right| \leq p(y - y') \quad \forall y, y' \in \overline{y} + V.$$

Take $V_0 := \{y \in \frac{1}{3} V \mid p(y) \leq 1\}$ and $\varepsilon \in \,]0, 1]$ such that $\varepsilon k^0 \in V_0$. Let us show that (11.18) holds with v replaced by $-k^0$. For this take $y \in (\overline{y} + V_0) \cap A$, $w \in -k^0 + V_0$ and $\lambda \in [0, \varepsilon]$. Then $y - \lambda k^0 - \overline{y} \in V_0 + V_0 \subseteq V$ and $y + \lambda w - \overline{y} = y - \lambda k^0 - \overline{y} + \lambda(w + k^0) \in V_0 + V_0 + V_0 \subseteq V$, and so

$$\begin{aligned} \varphi_A(y + \lambda w) &\leq \varphi_A(y - \lambda k^0) + p(\lambda(w + k^0)) = \varphi_A(y) - \lambda + \lambda p(w + k^0) \\ &\leq \lambda(p(w + k^0) - 1) \leq 0. \end{aligned}$$

Hence $y + \lambda w \in A$.

Assume now that (11.18) holds with v replaced by $-k^0$. Let $r \in \,]0, \varepsilon]$ be such that $2r(1 + p(k^0)) < 1$, where $p := p_{V_0}$. Of course, $\{y \mid p(y) \leq \lambda\} = \lambda V_0$ for every $\lambda > 0$ and if $p(y) = 0$ then $y \in \lambda V_0$ for every $\lambda > 0$. Set

$$M := \{y \in \overline{y} + rV_0 \mid |\varphi_A(y)| \leq p(y - \overline{y})\};$$

of course, $\overline{y} \in M$. We claim that $M = \overline{y} + rV_0$. Consider $y \in M$, $w \in V_0$ and $\lambda \in [0, r]$. Setting $y' := y - \varphi_A(y) k^0 \in A$, we have that $\varphi_A(y') = 0$ and

$$p(y' - \overline{y}) \leq p(y - \overline{y}) + |\varphi_A(y)| \cdot p(k^0) \leq r\left(1 + p(k^0)\right) < \frac{1}{2} \leq 1, \tag{11.19}$$

and so, by (11.18), $y' + \lambda(w - k^0) \in A$; hence $\varphi_A(y' + \lambda w) \leq \lambda$.

Take $v \in rV_0$. On one hand one has

$$\varphi_A(y' + v) = \varphi_A\left(y' + p(v) \cdot \frac{1}{p(v)} v\right) \leq p(v)$$

if $p(v) > 0$, and $\varphi_A(y' + v) = \varphi_A(y' + \lambda(\lambda^{-1}v)) \leq \lambda$ for every $\lambda \in]0, r]$, whence $\varphi_A(y' + v) \leq 0 = p(v)$. Therefore, $\varphi_A(y' + v) \leq p(v)$.

On the other hand, assume that $\varphi_A(y' + v) < -p(v)$. Because $2r(1 + p(k^0)) < 1$, there exists $t > 0$ such that $r + (t + r)p(k^0) \leq 1/2$ and $\varphi_A(y' + v) < -p(v) - t =: t' < 0$. It follows that $y' + v - t'k^0 \in A$. Moreover, taking into account (11.19),

$$p(y' + v - t'k^0 - \overline{y}) \leq p(y' - \overline{y}) + p(v) + (t + p(v))p(k^0) \leq 1/2 + r + (t + r)p(k^0) \leq 1,$$

and so $y' + v - t'k^0 \in (\overline{y} + V_0) \cap A$. Using (11.18), if $p(v) > 0$ then

$$y' + tk^0 = y' - \left(t' + p(v)\right) k^0 = y' + v - t'k^0 + p(v)\left(-k^0 - \frac{1}{p(v)} v\right) \in A,$$

while if $p(v) = 0$ then

$$y' + (1 - \gamma)tk^0 = y' + v - t'k^0 + \gamma t\left(-k^0 - (\gamma t)^{-1} v\right) \in A$$

for $\gamma := \min\{\frac{1}{2}, \varepsilon t^{-1}\}$. We get the contradiction $0 = \varphi_A(y') \leq -t < 0$ in the first case and $0 = \varphi_A(y') \leq -t(1 - \gamma) < 0$ in the second case. Hence $\varphi_A(y' + v) \in \mathbb{R}$ and $|\varphi_A(y' + v) - \varphi_A(y')| \leq p(v)$ for every $v \in rV_0$, or equivalently,

$$\varphi_A(y + v) \in \mathbb{R}, \quad |\varphi_A(y + v) - \varphi_A(y)| \leq p(v) \quad \forall v \in rV_0. \tag{11.20}$$

When $y := \overline{y} \in M$, from (11.20) we get $\overline{y} + rV_0 \subseteq M$, and so $M = \overline{y} + rV_0$ as claimed. Moreover, if $y, y' \in \overline{y} + \frac{1}{2} rV_0$, then $y \in M$ and $y' = y + v$ for some $v \in rV_0$; using again (11.20) we have that $|\varphi_A(y') - \varphi_A(y)| \leq p(y' - y)$. The conclusion follows. □

The next result is similar to Corollary 11.3.

Corollary 11.5. *Let A be a proper closed subset of Y and $k^0 \in Y \setminus \{\mathbf{0}\}$ be such that $A - \mathbb{R}_+ k^0 = A$. Consider $\overline{y} \in \operatorname{bd} A$. If A is epi-Lipschitz at $\overline{y}$ in the direction $-k^0$ then $\varphi_A(\overline{y}) = 0$.*

Proof. Consider $\varepsilon \in]0, 1[$ and V_0 provided by (11.18) with $v := -k^0$. Assume that $\varphi_A(\overline{y}) \neq 0$. Then there exists $t > 0$ such that $tp_{V_0}(k^0) \leq \varepsilon$ and $y := \overline{y} + tk^0 \in A$. Taking $\lambda := t$ in (11.18) we obtain that $y + t(-k^0 + V_0) = \overline{y} + tV_0 \subseteq A$, contradicting the fact that $\overline{y} \in \operatorname{bd} A$. □

Corollary 11.6. *Let A be a proper closed subset of Y and $k^0 \in Y \setminus \{\mathbf{0}\}$ be such that $A - \mathbb{R}_+ k^0 = A$. Assume that $\dim Y < \infty$ and $\overline{y} \in \operatorname{bd} A$. Then φ_A is finite and Lipschitz on a neighbourhood of $\overline{y}$ if and only if $-k^0 \in \operatorname{int} T_{Cl}(A, \overline{y})$, where $T_{Cl}(A, \overline{y})$ is the Clarke tangent cone of A at $\overline{y}$.*

Proof. By [56, Theorem 2I], $-k^0 \in \operatorname{int} T_{Cl}(A,\overline{y})$ if and only if A is epi-Lipschitz at $\overline{y}$ in the direction $-k^0$. The conclusion follows from Corollary 11.3, Theorem 11.5 and Corollary 11.5. □

The fact that φ_A is Lipschitz on a neighbourhood of $\overline{y}$ under the condition $-k^0 \in \operatorname{int} T_{Cl}(A,\overline{y})$ is obtained in [4, Proposition 6] in the case $Y = \mathbb{R}^m$ (and $K = \mathbb{R}^m_+$).

Consider $y^* \in Y^*$ such that $\langle k^0, y^* \rangle \neq 0$, $H := \ker y^*$ and take

$$\varphi_0 : H \to \overline{\mathbb{R}}, \quad \varphi_0(z) := \varphi_A(z),$$

that is, $\varphi_0 = \varphi_A|_H$. Since φ_A is lsc, so is φ_0. Then any $y \in Y$ can be written uniquely as $z - tk^0$ with $z \in H$ and $t \in \mathbb{R}$. So, by (11.7), $\varphi_A(y) = \varphi_A(z - tk^0) = \varphi_0(z) - t$. Using (11.10) we obtain that $A = \{z - tk^0 \mid (z,t) \in \operatorname{epi} \varphi_0\}$. Conversely, if $g : H \to \overline{\mathbb{R}}$ is a lsc function and $A := \{z - tk^0 \mid (z,t) \in \operatorname{epi} g\}$, then A is a closed set with $A - \mathbb{R}_+ k^0 = A$ and $\varphi_0 = g$. Therefore, the closed set A with the property $A - \mathbb{R}_+ k^0 = A$ is uniquely determined by a lsc function $\varphi_0 : H \to \overline{\mathbb{R}}$. Moreover, for $\overline{y} = \overline{z} - \overline{t}k^0$ we have that φ_A is finite (resp. continuous) at $\overline{y}$ if and only if φ_0 is finite (resp. continuous) at $\overline{z}$. Moreover, because $Y = H + \mathbb{R}k^0$ and the sum is topological (that is, the projection onto H parallel to $\mathbb{R}k^0$ is continuous), we have that φ_A is finite and Lipschitz continuous on a neighbourhood of $\overline{y}$ if and only if φ_0 is finite and Lipschitz continuous on a neighbourhood of $\overline{z}$. Similarly, φ_A is finite and Lipschitz continuous if and only if φ_0 is finite and Lipschitz continuous.

Note that for Y a normed vector space in [35] one says that A is (globally) epi-Lipschitz in the direction $e \in Y \setminus \{\mathbf{0}\}$ if there exist a closed linear subspace H of codimension 1 with $e \notin H$ and a Lipschitz function $g : H \to \mathbb{R}$ such that $A = \{y + \alpha e \mid y \in H,\ \alpha \in \mathbb{R},\ g(y) \leq \alpha\}$; A is epi-Lipschitz if there exists $e \in Y \setminus \{\mathbf{0}\}$ such that A is epi-Lipschitz in the direction e. The main result of [35] asserts that the proper closed set $A \subseteq Y$ is epi-Lipschitz in the direction e if and only if $e \in \operatorname{int} A_\infty$, and so $A \subseteq Y$ is epi-Lipschitz if and only if $\operatorname{int} A_\infty \neq \emptyset$.

The discussion above shows that not only the main theorem of [35] can be obtained from Corollary 11.4, but this one extends the main theorem of [35] to locally convex spaces.

11.3.5 The Formula for the Conjugate and Subdifferential of φ_A for A Convex

The results of this section (less the second part of Corollary 11.7) were established in several papers; we give the proofs for reader's convenience. The formula for the conjugate of φ_A is derived by Hamel [40, Theorem 3] and can be related also to [57, Theorem 3] and [60, Theorem 2.2]. Results concerning the subdifferential of φ_A are given in [17, Theorem 2.2, Lemma 2.1]. Another proof of these assertions using the formula for the conjugates is presented in Hamel [40, Corollary 12]. In the statements below we use some usual notation from convex analysis. So, having X a separated locally convex space with topological dual X^* and $f : X \to \overline{\mathbb{R}}$, the conjugate of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by $f^*(x^*) := \sup\{x^*(x) - f(x) \mid x \in X\}$ and its subdifferential at $x \in X$ with $f(x) \in \mathbb{R}$ is the set $\partial f(x) := \{x^* \in X^* \mid x^*(x' - x) \le f(x') - f(x)\ \forall x' \in X\}$; $\partial f(x) := \emptyset$ if $f(x) \notin \mathbb{R}$. Having a set $A \subseteq X$, the indicator of A is the function $\iota_A : X \to \overline{\mathbb{R}}$ defined by $\iota_A(x) := 0$ for $x \in A$ and $\iota_A(x) := \infty$ for $x \in X \setminus A$, while the support of A is the function $\sigma_A := (\iota_A)^*$. When A is nonempty the domain of σ_A is a convex cone which is called the barrier cone of A and is denoted by $\mathrm{bar} A$. Moreover, the normal cone of A at $a \in A$ is the set $N(A,a) := \partial \iota_A(a)$.

Proposition 11.6. *Let A be a proper closed subset of Y and $k^0 \in Y \setminus \{\mathbf{0}\}$ be such that $A - \mathbb{R}_+ k^0 = A$. Assume that A is convex and $k^0 \notin A_\infty$. Then $\varphi_A \in \Gamma(Y)$, that is, φ_A is a proper lsc convex function,*

$$\varphi_A^*(y^*) = \begin{cases} \sigma_A(y^*) & \text{if } y^* \in \mathrm{bar} A,\ y^*(k^0) = 1, \\ \infty & \text{otherwise,} \end{cases} \tag{11.21}$$

and $\partial \varphi_A(y) \subseteq \{y^* \in \mathrm{bar} A \mid y^*(k^0) = 1\} \subseteq \{y^* \in K^+ \mid y^*(k^0) = 1\}$ for every $y \in Y$.

Proof. From [32, Theorem 2.3.1]) we have that $\varphi_A \in \Gamma(Y)$. Consider $y^* \in Y^*$. Then

$$\begin{aligned} \varphi_A^*(y^*) &= \sup\{y^*(y) - \varphi_A(y) \mid y \in Y\} \\ &= \sup\left\{y^*(y) - t \mid y \in Y,\ t \in \mathbb{R},\ y \in tk^0 + A\right\} \\ &= \sup\left\{y^*(tk^0 + a) - t \mid y \in Y,\ t \in \mathbb{R},\ a \in A\right\} \\ &= \sup\{y^*(a) \mid a \in A\} + \sup\{t(y^*(k^0) - 1) \mid t \in \mathbb{R}\}. \end{aligned}$$

Hence (11.21) holds.

Since $\partial f(y) \subseteq \mathrm{dom} f^*$ for every proper function $f : Y \to \overline{\mathbb{R}}$ and every $y \in Y$, the first estimate for $\partial \varphi_A(y)$ follows. Moreover, because $A = A - K$ we have $\sigma_A = \sigma_{A-K} = \sigma_A + \sigma_{-K} = \sigma_A + \iota_{K^+}$, and so $\mathrm{bar} A \subseteq K^+$. □

The estimate $\mathrm{bar} A \subseteq K^+$ becomes more precise when $K = -A_\infty$; in fact one has $(A_\infty)^+ = -\mathrm{cl}_{w^*}(\mathrm{bar} A)$. Indeed, from [67, Exercise 2.23] we have that $\iota_{A_\infty} = (\iota_A)_\infty = \sigma_{\mathrm{dom}\,\iota_A^*} = \sigma_{\mathrm{dom}\,\sigma_A}$, whence $\iota_{-(A_\infty)^+} = (\iota_{A_\infty})^* = \iota_{\mathrm{cl}_{w^*}(\mathrm{dom}\,\sigma_A)}$, and so $\mathrm{cl}_{w^*}(\mathrm{dom}\,\sigma_A) = -(A_\infty)^+$.

Using Proposition 11.6 one deduces the expression of $\partial \varphi_A$ (see also [17, Theorem 2.2] for Y a normed vector space).

Corollary 11.7. *Assume that A is convex and $k^0 \notin A_\infty$. Then for all $\bar{y} \in Y$ one has*

$$\partial \varphi_A(\bar{y}) = \{y^* \in \mathrm{bar} A \mid y^*(k^0) = 1,\ y^*(\bar{y}) - \varphi_A(\bar{y}) \ge y^*(y)\ \forall y \in A\}. \tag{11.22}$$

Moreover, if (A2) holds then $\partial \varphi_A(y) \subseteq K^\#$ for every $y \in Y$.

Proof. Fix $\bar{y} \in Y$. If $\bar{y} \notin \operatorname{dom} \varphi_A$ then both sets in (11.22) are empty. Let $\bar{y} \in \operatorname{dom} \varphi_A$. Then, of course, $\bar{y} - \varphi_A(\bar{y})k^0 \in A$. If $y^* \in \partial \varphi_A(\bar{y})$ then $\varphi_A(\bar{y}) + \varphi_A^*(y^*) = y^*(\bar{y})$. Taking into account (11.21) we obtain that

$$y^* \in \operatorname{bar} A,\ y^*(k^0) = 1 \text{ and } y^*(\bar{y}) - \varphi_A(\bar{y}) \geq y^*(y)\ \forall y \in A, \tag{11.23}$$

that is, the inclusion $\subseteq$ holds in (11.22). Conversely, if $y^* \in Y^*$ is such that (11.23) holds, since $\bar{y} - \varphi_A(\bar{y})k^0 \in A$ and $y^*(k^0) = 1$, we obtain that $y^*\left(\bar{y} - \varphi_A(\bar{y})k^0\right) = \sigma_A(y^*)$, which shows that $\varphi_A(\bar{y}) + \varphi_A^*(y^*) = y^*(\bar{y})$. Hence $y^* \in \partial \varphi_A(\bar{y})$. Therefore, (11.22) holds.

Assume now that $(A2)$ holds, that is, $A - (K \setminus \{0\}) \subseteq \operatorname{int} A$, and take $y^* \in \partial \varphi_A(\bar{y})$. Hence $\bar{y} \in \operatorname{dom} \varphi_A$. Consider $k \in K \setminus \{0\}$. Since $(\bar{y} - k) - \bar{y} = -k \in -(K \setminus \{0\})$, by Theorem 11.4 (iv), we have that $y^*(-k) \leq \varphi_A(\bar{y} - k) - \varphi_A(\bar{y}) < 0$, that is, $y^*(k) > 0$. Therefore, $y^* \in K^\#$. □

11.4 Minimal-Point Theorems and Corresponding Variational Principles

11.4.1 Introduction

The celebrated Ekeland variational principle [21] (see Proposition 11.1) has many equivalent formulations and generalizations.

The aim of this section is to show general minimal-point theorems and corresponding variational principles. In Proposition 11.2 an existence result for minimal points of a set $\mathscr{A}$ with respect to the cone K_ε defined by (11.4) is presented. Taking into account (11.4) we get

$$(x_1 - x_2, r_1 - r_2) \in K_\varepsilon \quad \Longleftrightarrow \quad \varepsilon ||x_1 - x_2|| \leq -(r_1 - r_2).$$

This means

$$r_2 \geq r_1 + \varepsilon ||x_1 - x_2||. \tag{11.24}$$

Quite rapidly after the publication of the Ekeland variational principle (EVP) in 1974 there were formulated extensions to functions $f : (X, d) \to Y$, where Y is a real (topological) vector space. A systematization of such results was done in [34] (see also [32]), where instead of a function f it was considered a subset of $X \times Y$; said differently, it was considered a multifunction from X to Y. In [32] we have shown minimal-point theorems in product spaces $X \times Y$ with respect to a relation

$$(x_1, y_1) \preceq_{k^0} (x_2, y_2) \iff y_2 \in y_1 + d(x_1, x_2)k^0 + K, \tag{11.25}$$

where K is the convex ordering cone in Y and $k^0 \in K \setminus \{\mathbf{0}\}$. This is an extension of the binary relation defined by (11.24) to product spaces $X \times Y$. Very recently the term $d(x_1,x_2)k^0$ in (11.25) was replaced by $d(x,x')H$ with H a bounded convex subset of K (see [8]) or by $F(x,x') \subseteq K$, F being a so called K-metric (see [36]); in both papers one deals with functions $f : X \to Y$.

In order to formulate general minimal-point theorems in this section we replace $d(x_1,x_2)k^0$ in (11.25) by a set-valued map F with certain properties (compare Tammer and Zălinescu [64]).

It is worth mentioning that a weaker result than a full (= *authentic*) *minimal-point theorem* gives an EVP, as shown in this section. Such a weaker result is called *not authentic minimal-point theorem*.

In this section we present new results with proofs very similar to the corresponding ones in [34], which have as particular cases most part of the existing EVPs, or they are very close to them. Moreover, we use the same approach to get extensions of EVPs of Isac–Tammer and Ha types, as well as extensions of EVPs for bi-functions.

In the sequel (X,d) is a complete metric space, Y is a real topological vector space, Y^* is its topological dual, and $K \subseteq Y$ is a proper convex cone.

If Y is just a real linear space we endow it with the finest locally convex topology, that is, the topology defined by all the seminorms on Y.

As in [6] and [7], we say that $E \subseteq Y$ is *quasi bounded* (from below) if there exists a bounded set $B \subseteq Y$ such that $E \subseteq B + K$; as in [36], we say that E is *K-bounded* (by scalarization) if $y^*(E)$ is bounded from below for every $y^* \in K^+$. It is clear that any quasi bounded set is K-bounded.

Let $F : X \times X \rightrightarrows K$ satisfy the conditions:

(F1) $\mathbf{0} \in F(x,x)$ for all $x \in X$
(F2) $F(x_1,x_2) + F(x_2,x_3) \subseteq F(x_1,x_3) + K$ for all $x_1,x_2,x_3 \in X$

Using F we introduce a preorder on $X \times Y$, denoted by $\preceq_F$, in the following manner:

$$(x_1,y_1) \preceq_F (x_2,y_2) \iff y_2 \in y_1 + F(x_1,x_2) + K. \tag{11.26}$$

Indeed, $\preceq_F$ is reflexive by (F1). If $(x_1,y_1) \preceq_F (x_2,y_2)$ and $(x_2,y_2) \preceq_F (x_3,y_3)$, then

$$y_2 = y_1 + v_1 + k_1, \quad y_3 = y_2 + v_2 + k_2 \tag{11.27}$$

with $v_1 \in F(x_1,x_2)$, $v_2 \in F(x_2,x_3)$ and $k_1,k_2 \in K$. By (F2) we have that $v_1 + v_2 = v_3 + k_3$ for some $v_3 \in F(x_1,x_3)$ and $k_3 \in K$, and so

$$y_3 = y_1 + v_1 + k_1 + v_2 + k_2 = y_1 + v_3 + k_1 + k_2 + k_3 \in y_1 + F(x_1,x_3) + K;$$

hence $(x_1,y_1) \preceq_F (x_3,y_3)$, and so $\preceq_F$ is transitive. Of course,

$$(x_1,y_1) \preceq_F (x_2,y_2) \Rightarrow y_1 \leq_K y_2; \tag{11.28}$$

moreover, by (F1), we have that

$$(x,y_1) \preceq_F (x,y_2) \iff y_2 \in y_1 + K \iff y_1 \leq_K y_2. \tag{11.29}$$

Besides conditions (F1) and (F2) we shall assume to be true the condition

(F3) There exists $z^* \in K^+$ such that

$$\eta(\delta) := \inf\left\{z^*(v) \mid v \in \cup_{d(x,x')\geq\delta} F(x,x')\right\} > 0 \quad \forall\delta > 0. \tag{11.30}$$

Clearly, by (F3) we have that $0 \notin \mathrm{cl\ conv} F(x,x')$ for $x \neq x'$.
A sufficient condition for (11.30) is

$$\inf_{z\in F(x,x')} z^*(z) \geq d(x,x') \quad \forall x,x' \in X. \tag{11.31}$$

If (11.31) holds then

$$(x_1,y_1) \preceq_F (x_2,y_2) \Rightarrow d(x_1,x_2) \leq z^*(y_2) - z^*(y_1). \tag{11.32}$$

Indeed, since $F(x_1,x_2) \subseteq K$, from (11.28) we get first that $y_1 \leq_K y_2$; then from (11.27)

$$z^*(y_2) = z^*(y_1) + z^*(v_1) + z^*(k_1) \geq z^*(y_1) + \inf_{v\in F(x_1,x_2)} z^*(v) \geq z^*(y_1) + d(x_1,x_2),$$

and so (11.32) holds.
Using (11.32) we obtain that

$$[(x_1,y_1) \preceq_F (x_2,y_2),\ (x_2,y_2) \preceq_F (x_1,y_1)] \Rightarrow [x_1 = x_2,\ z^*(y_1) = z^*(y_2)]; \tag{11.33}$$

moreover, if $z^* \in K^\#$ then $\preceq_F$ is anti-symmetric, and so $\preceq_F$ is a partial order.
For F satisfying conditions (F1)–(F3), z^* being that from (F3), we introduce the order relation $\preceq_{F,z^*}$ on $X \times Y$ by

$$(x_1,y_1) \preceq_{F,z^*} (x_2,y_2) \iff \begin{cases} (x_1,y_1) = (x_2,y_2) \text{ or} \\ (x_1,y_1) \preceq_F (x_2,y_2) \text{ and } z^*(y_1) < z^*(y_2). \end{cases} \tag{11.34}$$

It is easy to verify that $\preceq_{F,z^*}$ is reflexive, transitive, and antisymmetric.

11.4.2 Minimal Points in Product Spaces

We take X,Y,K,F as above, that is, F satisfies conditions (F1)–(F3), z^* being that from (F3).

Consider a nonempty set $\mathscr{A} \subseteq X \times Y$. In the sequel we shall use the condition (H1) on $\mathscr{A}$, where $\mathbb{N}$ is the set of nonnegative integers; moreover, we set $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

The next theorem is the main result of this section.

Theorem 11.6 (Minimal-Point Theorem with Respect to $\preceq_{F,z^*}$). *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subseteq Y$ is a proper convex cone. Let $F : X \times X \rightrightarrows K$ satisfy conditions (F1)–(F3) and $\mathscr{A} \subseteq X \times K$ satisfy the condition*

(H1) For every $\preceq_F$-decreasing sequence $((x_n, y_n)) \subseteq \mathscr{A}$ with $x_n \to x \in X$ there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $(x,y) \preceq_F (x_n, y_n)$ for every $n \in \mathbb{N}$.

Furthermore, suppose that

(B1) z^* (from (F3)) is bounded from below on $\operatorname{Pr}_Y(\mathscr{A})$.

Then for every $(x_0, y_0) \in \mathscr{A}$ there exists an element $(\bar{x}, \bar{y})$ of $\mathscr{A}$ such that:

(a) $(\bar{x}, \bar{y}) \preceq_{F,z^*} (x_0, y_0)$
(b) $(\bar{x}, \bar{y})$ is a minimal element of $\mathscr{A}$ with respect to $\preceq_{F,z^*}$

Proof. Let

$$\alpha := \inf\{z^*(y) \mid \exists x \in X : (x,y) \in \mathscr{A},\ (x,y) \preceq_{F,z^*} (x_0, y_0)\} \in \mathbb{R}.$$

Let us denote by $\mathscr{A}(x,y)$ the set of those $(x', y') \in \mathscr{A}$ with $(x', y') \preceq_{F,z^*} (x,y)$. We construct a sequence $((x_n, y_n))_{n \geq 0} \subseteq \mathscr{A}$ as follows: Having $(x_n, y_n) \in \mathscr{A}$, we take $(x_{n+1}, y_{n+1}) \in \mathscr{A}(x_n, y_n)$ such that

$$z^*(y_{n+1}) \leq \inf\{z^*(y) \mid (x,y) \in \mathscr{A}(x_n, y_n)\} + 1/(n+1).$$

Of course, $((x_n, y_n))$ is $\preceq_{F,z^*}$-decreasing. It follows that $(y_n)_{n \geq 0}$ is $\leq_K$-decreasing, and so the sequence $(z^*(y_n))_{n \geq 0}$ is nonincreasing and bounded from below; hence $\gamma := \lim z^*(y_n) \in \mathbb{R}$.

If $\mathscr{A}(x_{n_0}, y_{n_0})$ is a singleton (that is, $\{(x_{n_0}, y_{n_0})\}$) for some $n_0 \in \mathbb{N}$, then clearly $(\bar{x}, \bar{y}) := (x_{n_0}, y_{n_0})$ is the desired element. In the contrary case the sequence $(z^*(y_n))$ is (strictly) decreasing; moreover, $\gamma < z^*(y_n)$ for every $n \in \mathbb{N}$.

Assume that (x_n) is not a Cauchy sequence. Then there exist $\delta > 0$ and the sequences (n_k), (p_k) from $\mathbb{N}^*$ such that $n_k \to \infty$ and $d(x_{n_k}, x_{n_k+p_k}) \geq \delta$ for every k. Since $(x_{n_k+p_k}, y_{n_k+p_k}) \preceq_{F,z^*} (x_{n_k}, y_{n_k})$ we obtain that

$$z^*(y_{n_k}) - z^*(y_{n_k+p_k}) \geq \inf\left\{z^*(v) \mid v \in F(x_{n_k+p_k}, x_{n_k})\right\} \geq \eta(\delta) \quad \forall k \in \mathbb{N}.$$

Since $\eta(\delta) > 0$ and $(z^*(y_n))$ is convergent, this is a contradiction. Therefore, (x_n) is a Cauchy sequence in the complete metric space (X,d), and so (x_n) converges to some $\bar{x} \in X$. Since $((x_n, y_n))$ is $\preceq_F$-decreasing, by (H1) there exists some $\bar{y} \in Y$ such that $(\bar{x}, \bar{y}) \in \mathscr{A}$ and $(\bar{x}, \bar{y}) \preceq_F (x_n, y_n)$ for every $n \in \mathbb{N}$. It follows that $z^*(\bar{y}) \leq$

$\lim z^*(y_n)$, and so $z^*(\bar{y}) < z^*(y_n)$ for every $n \in \mathbb{N}$. Therefore $(\bar{x},\bar{y}) \preceq_{F,z^*} (x_n,y_n)$ for every $n \in \mathbb{N}$. Let $(x',y') \in \mathscr{A}$ be such that $(x',y') \preceq_{F,z^*} (\bar{x},\bar{y})$. Since $(\bar{x},\bar{y}) \preceq_{F,z^*} (x_n,y_n)$, we have that $(x',y') \preceq_{F,z^*} (x_n,y_n)$ for every $n \in \mathbb{N}$. It follows that

$$0 \le z^*(\bar{y}) - z^*(y') \le z^*(y_n) - z^*(y') \le 1/n \quad \forall n \ge 1,$$

whence $z^*(y') = z^*(\bar{y})$. By the definition of $\preceq_{F,z^*}$ we obtain that $(x',y') = (\bar{x},\bar{y})$. □

As seen from the proof, for a fixed $(x_0,y_0) \in \mathscr{A}$ it is sufficient that z^* be bounded from below on the set $\{y \in Y \mid \exists x \in X : (x,y) \in \mathscr{A},\ (x,y) \preceq_{F,z^*} (x_0,y_0)\}$ instead of being bounded from below on $\Pr_Y(\mathscr{A})$.

Remark 11.5. When $k^0 \in K$ and $F(x,x') := \{d(x,x')k^0\}$ we have that F satisfies conditions (F1) and (F2); moreover, if Y is a separated locally convex space and $-k^0 \notin \operatorname{cl} K$, then there exists $z^* \in K^+$ with $z^*(k^0) = 1$, and so (F3) is also satisfied (even (11.31) is satisfied). In this case condition (H1) becomes condition (H1) in [32, p. 199]. So Theorem 11.6 extends [32, Theorem 3.10.7] to this framework, using practically the same proof.

In [36] one considers for a proper pointed convex cone $D \subseteq Y$ a so called set-valued D-metric, that is, a multifunction $F : X \times X \rightrightarrows D$ satisfying the following conditions:

(i) $F(x,y) \neq \emptyset$ and $F(x,x) = \{\mathbf{0}\}$ $\forall x,y \in X$, and $\mathbf{0} \notin F(x,y)$ $\forall x \neq y$
(ii) $F(x,y) = F(y,x)$ $\forall x,y \in X$
(iii) $F(x,y) + F(y,z) \subseteq F(x,z) + D$ $\forall x,y,z \in X$

The basic supplementary assumptions on D and F are:

(S1) D is w-normal and D_F is based.
(S2) $\mathbf{0} \notin \operatorname{cl}_w \left(\cup_{d(x,y)\ge\delta} F(x,y)\right)$ $\forall \delta > 0$.

Here $K_F := \operatorname{cone}(\operatorname{conv}(\cup\{F(x,y) \mid x,y \in X\}))$ and $D_F := (K_F \setminus \{\mathbf{0}\} + D) \cup \{\mathbf{0}\}$. As observed in [36], D is w-normal iff $D^+ - D^+ = Y^*$, and D_F is based iff $D^+ \cap K_F^\# \neq \emptyset$.

Comparing with our assumptions on F, we see that (F1) is weaker than (i) because we ask just $\mathbf{0} \in F(x,x)$ for every $x \in X$, and we don't ask the symmetry condition (ii). From (F3) we obtain that (S2) is verified and that $z^* \in K_F^\#$ and so $(K_F \setminus \{\mathbf{0}\} + K) \cup \{\mathbf{0}\}$ is based, but we don't need either K be w-normal or even K be pointed.

Another possible choice for F, considered also in [36], is $F(x,x') := d(x,x')H$ with $H \subseteq K \setminus \{\mathbf{0}\}$ a nonempty set such that $H + K$ is convex. Clearly (F1), (i), and (ii) are satisfied (for $D = K$). From the convexity of $H + K$ we obtain easily that (F2) (and (iii)) holds. When Y is a separated locally convex space condition (F3) is equivalent to $0 \notin \operatorname{cl}(H + K)$. In order to have that (S1) holds one needs $K^+ - K^+ = Y^*$ and the existence of $z^* \in K^+$ with $z^*(v) > 0$ for every $v \in H$, while for (S2) one needs $\mathbf{0} \notin \operatorname{cl}_w H$ (see [36, Lemma 5.9 (d)]); of course, if $H = H + K$, the last condition is equivalent to $\mathbf{0} \notin \mathrm{cl}(H+K)$. So it seems that our condition (F3) is more convenient than (S1) and (S2).

For H as above, that is, $H \subseteq K$ is a nonempty set such that $H+K$ is convex and $\mathbf{0} \notin \mathrm{cl}(H+K)$, we consider $F_H(x,x') := d(x,x')H$ for $x,x' \in X$, and we set

$$\preceq_H := \preceq_{F_H};$$

moreover, if $z^* \in K^+$ is such that $\inf z^*(H) > 0$ we set

$$\preceq_{H,z^*} := \preceq_{F_H,z^*}.$$

An immediate consequence of the preceding theorem is the next result.

Corollary 11.8 (Minimal-Point Theorem with Respect to $\preceq_{H,z^*}$). *Assume that (X,d) is a complete metric space, Y is a real topological vector space, $K \subseteq Y$ is a proper convex cone and $\mathscr{A} \subseteq X \times Y$ satisfies:*

(H1) For every $\preceq_H$-decreasing sequence $((x_n,y_n)) \subseteq \mathscr{A}$ with $x_n \to x \in X$ there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $(x,y) \preceq_H (x_n,y_n)$ for every $n \in \mathbb{N}$.

Suppose that there exists $z^* \in K^+$ such that $\inf z^*(H) > 0$ and

(B1) $\inf z^*(\mathrm{Pr}_Y(\mathscr{A})) > -\infty$.

Then for every $(x_0,y_0) \in \mathscr{A}$ there exists $(\bar{x},\bar{y}) \in \mathscr{A}$ such that:

(a) $(\bar{x},\bar{y}) \preceq_{H,z^*} (x_0,y_0)$.
(b) $(\bar{x},\bar{y})$ is a minimal element of $\mathscr{A}$ with respect to $\preceq_{H,z^*}$.

A condition related to (H1) is the next one.

(H2) For every sequence $((x_n,y_n)) \subseteq \mathscr{A}$ with $x_n \to x \in X$ and (y_n) $\leq_K$-decreasing there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $y \leq_K y_n$ for every $n \in \mathbb{N}$.

Remark 11.6. Note that (H2) holds if $\mathscr{A}$ is closed with $\mathrm{Pr}_Y(\mathscr{A}) \subseteq y_0 + K$ for some $y_0 \in Y$ and every $\leq_K$-decreasing sequence in K is convergent (i.e., K is a sequentially Daniell cone). In fact, instead of asking that $\mathscr{A}$ is closed we may assume that

$$\forall ((x_n,y_n))_{n\geq 1} \subseteq \mathscr{A} : [x_n \to x,\ y_n \to y,\ (y_n) \text{ is } \leq_K \text{-decreasing } \Rightarrow (x,y) \in \mathscr{A}].$$

Remark 11.7. Note that (H1) is verified whenever $\mathscr{A}$ satisfies (H2) and

$$\forall u \in X,\ \forall X \supseteq (x_n) \to x \in X \ : \ \bigcap_{n\in\mathbb{N}} (F(x_n,u)+K) \subseteq F(x,u)+K.$$

Indeed, let $((x_n,y_n)) \subseteq \mathscr{A}$ be $\preceq_F$-decreasing with $x_n \to x$. It is obvious that (y_n) is $\leq_K$-decreasing. By (H2), there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $y \leq_K y_n$ for every $n \in \mathbb{N}$. It follows that

$$y_n \in y_{n+p} + F(x_{n+p},x_n) + K \subseteq y + F(x_{n+p},x_n) + K \quad \forall n,p \in \mathbb{N}.$$

Fix n; then $y_n - y \in F(x_{n+p}, x_n) + K$ for every $p \in \mathbb{N}$, and so, by our hypothesis, $y_n - y \in F(x, x_n) + K$ because $\lim_{p\to\infty} x_{n+p} = x$. Therefore, $(x,y) \preceq_F (x_n, y_n)$.

Remark 11.8. In the case $F = F_H$, (H1) is verified whenever $\mathscr{A}$ satisfies (H2) and $H+K$ is closed.

Indeed, let $((x_n,y_n)) \subseteq \mathscr{A}$ be a $\preceq_H$-decreasing sequence with $x_n \to x$. It is obvious that (y_n) is $\leq_K$-decreasing. By (H2), there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $y \leq_K y_n$ for every $n \in \mathbb{N}$.

Fix n. If $x_n = x$ then clearly $(x,y) = (x_n,y) \preceq_H (x_n,y_n)$. Else, because $d(x_{n+p},x_n) \to d(x,x_n) > 0$ for $p \to \infty$, we get $d(x_{n+p},x_n) > 0$ for sufficiently large p, and so

$$y_n \in y_{n+p} + d(x_{n+p},x_n)H + K \subseteq y + d(x_{n+p},x_n)H + K = y + d(x_{n+p},x_n)(H+K)$$

for sufficiently large p. Since $H+K$ is closed we obtain that

$$y_n \in y + d(x_n,x)(H+K) = y + d(x_n,x)H + K,$$

that is, $(x,y) \preceq_H (x_n,y_n)$.

Another condition to be added to (H2) in order to have (H1) is suggested by the hypotheses of [8, Theorem 4.1]. Recall that a set $C \subseteq Y$ is cs-complete (see [67, p. 9]) if for all sequences $(\lambda_n)_{n\geq 1} \subseteq [0,\infty)$ and $(y_n)_{n\geq 1} \subseteq C$ such that $\sum_{n\geq 1} \lambda_n = 1$ and the sequence $(\sum_{m=1}^{n} \lambda_m y_m)_{n\geq 1}$ is Cauchy, the series $\sum_{n\geq 1} \lambda_n y_n$ is convergent and its sum belongs to C. One says that $C \subseteq Y$ is cs-closed if the sum of the series $\sum_{n\geq 1} \lambda_n y_n$ belongs to C whenever $\sum_{n\geq 1} \lambda_n y_n$ is convergent and $(y_n) \subseteq C$, $(\lambda_n)_{n\geq 1} \subseteq [0,\infty)$ and $\sum_{n\geq 1} \lambda_n = 1$. Of course, any cs-complete set is cs-closed; if Y is complete then the converse is true. Moreover, notice that any cs-closed set is convex.

Note that the sequence $(\sum_{m=1}^{n} \lambda_m y_m)_{n\geq 1}$ is Cauchy whenever $(\lambda_n)_{n\geq 1} \subseteq [0,\infty)$ is such that the series $\sum_{n\geq 1} \lambda_n$ is convergent and $(y_n)_{n\geq 1} \subseteq Y$ is such that $\operatorname{conv}\{y_n \mid n \geq 1\}$ is bounded; of course, if Y is a locally convex space then $B \subseteq Y$ is bounded iff $\operatorname{conv} B$ is bounded. Indeed, let $(\lambda_n)_{n\geq 1} \subseteq [0,\infty)$ with $\sum_{n\geq 1} \lambda_n$ convergent and $(y_n)_{n\geq 1} \subseteq Y$ with $B := \operatorname{conv}\{y_n \mid n \geq 1\}$ bounded. Fix $V \subseteq Y$ a balanced neighborhood of 0. Because B is bounded, there exists $\alpha > 0$ such that $B \subseteq \alpha V$. Since the series $\sum_{n\geq 1} \lambda_n$ is convergent there exists $n_0 \geq 1$ such that $\sum_{k=n}^{n+p} \lambda_k \leq \alpha^{-1}$ for all $n,p \in \mathbb{N}$ with $n \geq n_0$. Then for such n,p and some $b_{n,p} \in B$ we have

$$\sum_{k=n}^{n+p} \lambda_k y_k = \left(\sum_{k=n}^{n+p} \lambda_k\right) b_{n,p} \in [0,\alpha^{-1}]B \subseteq [0,\alpha^{-1}]\alpha V = V.$$

Proposition 11.7. *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subseteq Y$ is a proper closed convex cone. Furthermore, suppose that $H \subseteq K$ is a nonempty cs-complete bounded set with $\mathbf{0} \notin \operatorname{cl}(H+K)$. If $\mathscr{A}$ satisfies (H2) then $\mathscr{A}$ satisfies (H1), too.*

Proof. Let $((x_n,y_n))_{n\geq 1} \subseteq \mathscr{A}$ be a $\preceq_H$-decreasing sequence with $x_n \to x$. It follows that (y_n) is $\leq_K$-decreasing. By (H2), there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $y \leq_K y_n$ for every $n \in \mathbb{N}$.

Because $((x_n,y_n))_{n\geq 1}$ is $\preceq_H$-decreasing we have that

$$y_n = y_{n+1} + d(x_n,x_{n+1})h_n + k_n \tag{11.35}$$

with $h_n \in H$ and $k_n \in K$ for $n \geq 1$. If $x_n = x_{\overline{n}}$ for $n \geq \overline{n} \geq 1$ we take $x := x_{\overline{n}}$; then $(x,y) \preceq_H (x_n,y_n)$ for every $n \in \mathbb{N}$. Indeed, for $n \leq \overline{n}$ we have that $(x_{\overline{n}},y_{\overline{n}}) \preceq_H (x_n,y_n)$; because $y \leq_K y_{\overline{n}}$, by (11.29) we get $(x,y) \preceq_H (x,y_{\overline{n}}) = (x_{\overline{n}},y_{\overline{n}})$, and so $(x,y) \preceq (x_n,y_n)$. If $n > \overline{n}$, using again (11.29), we have $(x,y) = (x_n,y) \preceq_H (x_n,y_n)$.

Assume that (x_n) is not constant for large n. Fix $n \geq 1$. From (11.35), for $p \geq 0$, we have

$$\begin{aligned} y_n &= y_{n+p+1} + \sum_{l=n}^{n+p} d(x_l,x_{l+1})h_l + \sum_{l=n}^{n+p} k_l = y_{n+p+1} + \left(\sum_{l=n}^{n+p} d(x_l,x_{l+1})\right) h_{n,p} + \sum_{l=n}^{n+p} k_l \\ &= y + k'_{n,p} + \left(\sum_{l=n}^{n+p} d(x_l,x_{l+1})\right) h_{n,p} \end{aligned} \tag{11.36}$$

for some $h_{n,p} \in H$ and $k'_{n,p} \in K$. Assuming that $\sum_{l\geq n} d(x_l,x_{l+1}) = \infty$, from

$$\left(\sum_{l=n}^{n+p} d(x_l,x_{l+1})\right)^{-1} (y_n - y) = h_{n,p} + \left(\sum_{l=n}^{n+p} d(x_l,x_{l+1})\right)^{-1} k'_{n,p} \in H + K,$$

we get the contradiction $\mathbf{0} \in \operatorname{cl}(H+K)$ taking the limit for $p \to \infty$. Hence $0 < \mu := \sum_{l\geq n} d(x_l,x_{l+1}) < \infty$. Set $\lambda_l := \mu^{-1} d(x_l,x_{l+1})$ for $l \geq n$. Since H is cs-complete and $\operatorname{conv}\{h_l \mid l \geq n\}$ $(\subseteq H)$ is bounded we obtain that the series $\sum_{l\geq n} \lambda_l h_l$ is convergent and its sum $\overline{h}_n$ belongs to H. It follows that $\sum_{l\geq n} d(x_l,x_{l+1})h_l = \mu \overline{h}_n$, and so

$$\overline{k}_n := \lim_{p\to\infty} k'_p = y_n - y - \mu \overline{h}_n \in K$$

because K is closed. Since $d(x_n,x_{n+p}) \leq \sum_{l=n}^{n+p-1} d(x_l,x_{l+1})$, we obtain that $d(x_n,x) \leq \mu$, and so

$$y_n = y + d(x_n,x)\overline{h}_n + \overline{k}_n + (\mu - d(x_n,x))\,\overline{h}_n \in y + d(x_n,x)H + K.$$

Hence $(x,y) \preceq_H (x_n,y_n)$ for every $n \in \mathbb{N}$. $\square$

The most part of vector EVP type results are established for Y a separated locally convex space. However, there are topological vector spaces Y whose topological dual reduce to $\{\mathbf{0}\}$. In such a case it is not possible to find z^* satisfying the conditions of Corollary 11.8. In [6, Theorem 1], in the case H is a singleton, the authors consider such a situation.

Theorem 11.7 (Not Authentic Minimal-Point Theorem with Respect to $\preceq_H$). *Assume that (X,d) is a complete metric space, Y is a real topological vector space. Let $K \subseteq Y$ be a proper closed convex cone and $H \subseteq K$ be a nonempty cs-complete bounded set with $\mathbf{0} \notin \mathrm{cl}(H+K)$. Suppose that $\mathscr{A} \subseteq X \times Y$ satisfies*

(H2) For every sequence $((x_n,y_n)) \subseteq \mathscr{A}$ with $x_n \to x \in X$ and (y_n) $\leq_K$-decreasing there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $y \leq_K y_n$ for every $n \in \mathbb{N}$

and

(B2) $\mathrm{Pr}_Y(\mathscr{A})$ is quasi bounded.

Then for every $(x_0,y_0) \in \mathscr{A}$ there exists $(\bar{x},\bar{y}) \in \mathscr{A}$ such that:

(a) $(\bar{x},\bar{y}) \preceq_H (x_0,y_0)$
(b) $(x,y) \in \mathscr{A}$, $(x,y) \preceq_H (\bar{x},\bar{y})$ imply $x = \bar{x}$

Proof. First observe that $\mathscr{A}$ satisfies condition (H1) by Proposition 11.7. Moreover, because $\mathrm{Pr}_Y(\mathscr{A})$ is quasi bounded, there exists a bounded set $B \subseteq Y$ such that $\mathrm{Pr}_Y(\mathscr{A}) \subseteq B + K$.

Note that for $(x,y) \in \mathscr{A}$ the set $\mathrm{Pr}_X(\mathscr{A}(x,y))$ is bounded, where $\mathscr{A}(x,y) := \{(x',y') \in \mathscr{A} \mid (x',y') \preceq_H (x,y)\}$. In the contrary case there exists a sequence $((x_n,y_n))_{n\geq 1} \subseteq \mathscr{A}(x,y)$ with $d(x_n,x) \to \infty$. Hence $y = y_n + d(x_n,x)h_n + k_n = b_n + d(x_n,x)h_n + k_n'$ with $h_n \in H$, $b_n \in B$, $k_n, k_n' \in K$. It follows that $d(x_n,x)^{-1}(y - b_n) \in H + K$, whence the contradiction $0 \in \mathrm{cl}(H+K)$.

Let us construct a sequence $((x_n,y_n))_{n\geq 0} \subseteq \mathscr{A}$ in the following way: Having $(x_n,y_n) \in \mathscr{A}$, where $n \in \mathbb{N}$, because $D_n := \mathrm{Pr}_X(\mathscr{A}(x_n,y_n))$ is bounded, there exists $(x_{n+1},y_{n+1}) \in \mathscr{A}(x_n,y_n)$ such that

$$d(x_{n+1},x_n) \geq \frac{1}{2}\sup\{d(x,x_n) \mid x \in D_n\} \geq \tfrac{1}{4}\,\mathrm{diam}\, D_n.$$

We obtain in this way the sequence $((x_n,y_n))_{n\geq 0} \subseteq \mathscr{A}$, which is $\preceq_H$-decreasing. Since $\mathscr{A}(x_{n+1},y_{n+1}) \subseteq \mathscr{A}(x_n,y_n)$, we have that $D_{n+1} \subseteq D_n$ for every $n \in \mathbb{N}$. Of course, $x_n \in D_n$. Let us show that $\mathrm{diam}\, D_n \to 0$. In the contrary case there exists $\delta > 0$ such that $\mathrm{diam}\, D_n \geq 4\delta$, and so $d(x_{n+1},x_n) \geq \delta$ for every $n \in \mathbb{N}$. As in the proof of Proposition 11.7, for every $p \in \mathbb{N}$, we obtain that

$$\begin{aligned} y_0 &= y_{p+1} + \left(\sum_{l=0}^{p} d(x_l,x_{l+1})\right) h_p + \sum_{l=0}^{p} k_l = b_p + \left(\sum_{l=0}^{p} d(x_l,x_{l+1})\right) h_p + k_p' \\ &= b_p + (p+1)\delta h_p + k_p'', \end{aligned}$$

where $h_p \in H$, $b_p \in B$, $k_l, k_p', k_p'' \in K$. It follows that $[(p+1)\delta]^{-1}(y_0 - b_p) \in H + K$ for every $p \in \mathbb{N}$. Since (b_p) is bounded we obtain the contradiction $0 \in \mathrm{cl}(H+K)$. Thus we have that the sequence $(\mathrm{cl}\, D_n)$ is a decreasing sequence of nonempty closed subsets of the complete metric space (X,d), whose diameters tend to 0. By Cantor's theorem, $\bigcap_{n\in\mathbb{N}} \operatorname{cl} D_n = \{\overline{x}\}$ for some $\overline{x} \in X$. Of course, $x_n \to \overline{x}$. Since $((x_n,y_n)) \subseteq \mathscr{A}$ is a $\preceq_H$-decreasing sequence, from (H1) we get an $\overline{y} \in Y$ such that $(\overline{x},\overline{y}) \preceq_H (x_n,y_n)$ for every $n \in \mathbb{N}$; $(\overline{x},\overline{y})$ is the desired element. Indeed, $(\overline{x},\overline{y}) \preceq_H (x_0,y_0)$. Let $(x',y') \in \mathscr{A}(\overline{x},\overline{y})$. It follows that $(x',y') \in \mathscr{A}(x_n,y_n)$, and so $x' \in D_n \subseteq \operatorname{cl} D_n$ for every n. Thus $x' = \overline{x}$. □

If Y is a separated locally convex space, the preceding result follows immediately from Corollary 11.8.

Of course, the set $\mathscr{A} \subseteq X \times Y$ can be viewed as the graph of a multifunction $\Gamma : X \rightrightarrows Y$; then $\operatorname{Pr}_X(\mathscr{A}) = \operatorname{dom}\Gamma$ and $\operatorname{Pr}_Y(\mathscr{A}) = \operatorname{Im}\Gamma$. In [6] one assumes that Γ is *level-closed*, that is,

$$\begin{aligned} L(b) &:= \{x \in X \mid \exists y \in \Gamma(x) : y \leq_K b\} = \{x \in X \mid b \in \Gamma(x) + K\} \\ &= \{x \in X \mid \Gamma(x) \cap (b-K) \neq \emptyset\} \end{aligned}$$

is closed for every $b \in Y$.

For the nonempty set $E \subseteq Y$ let us set

$$\operatorname{BMMin} E := \{\overline{y} \in E \mid E \cap (\overline{y} - K) = \{\overline{y}\}\}$$

(see [7, (1.2)]); note that this set is different of the usual set

$$\operatorname{Min} E := \{\overline{y} \in E \mid E \cap (\overline{y} - K) \subseteq \overline{y} + K\},$$

but they coincide if K is pointed. As in [7, Definition 3.2], we say that $\Gamma : X \rightrightarrows Y$ satisfies the *limiting monotonicity condition* at $\overline{x} \in \operatorname{dom}\Gamma$ if for every sequence $((x_n,y_n))_{n\geq 1} \subseteq \operatorname{gph}\Gamma$ with (x_n) converging to $\overline{x}$ and (y_n) being $\leq_K$-decreasing, there exists $\overline{y} \in \operatorname{BMMin}\Gamma(\overline{x})$ such that $\overline{y} \leq y_n$ for every $n \geq 1$. As observed in [7], if Γ satisfies the limiting monotonicity condition at $\overline{x} \in \operatorname{dom}\Gamma$ then $\Gamma(\overline{x}) \subseteq \operatorname{BMMin}\Gamma(\overline{x}) + K$, that is, $\Gamma(\overline{x})$ satisfies the domination property.

In [7, Proposition 3.3], in the case Y a Banach space, there are mentioned sufficient conditions in order that Γ satisfy the limiting monotonicity condition at $\overline{x} \in \operatorname{dom}\Gamma$.

When X and Y are Banach spaces and H is a singleton the next result is practically [7, Theorem 3.5].

Corollary 11.9 (Not Authentic Minimal-Point Theorem with Respect to $\preceq_H$). *Assume that (X,d) is a complete metric space, Y is a real topological vector space. Let $K \subseteq Y$ be a proper closed convex cone and $H \subseteq K$ be a nonempty cs-complete bounded set with $\mathbf{0} \notin \operatorname{cl}(H+K)$. Suppose that:*

(H3) $\Gamma : X \rightrightarrows Y$ *is level-closed, satisfies the limiting monotonicity condition on* $\operatorname{dom}\Gamma$

(B3) $\operatorname{Im}\Gamma$ *is quasi-bounded*

Then for every $(x_0, y_0) \in \operatorname{gph}\Gamma$ *there exist* $\bar{x} \in \operatorname{dom}\Gamma$ *and* $\bar{y} \in \operatorname{BMMin}\Gamma(\bar{x})$ *such that:*

(a) $(\bar{x}, \bar{y}) \preceq_H (x_0, y_0)$
(b) $(x, y) \in \operatorname{gph}\Gamma$ $(x, y) \preceq_H (\bar{x}, \bar{y})$ *imply* $x = \bar{x}$.

Proof. In order to apply Theorem 11.7 for $\mathscr{A} := \operatorname{gph}\Gamma$ we have only to show that $\mathscr{A}$ verifies condition (H2). For this consider the sequence $((x_n, y_n))_{n\geq 1} \subseteq \mathscr{A}$ such that (y_n) is $\leq_K$-decreasing and $x_n \to \bar{x}$. Clearly, $x_n \in L(y_1)$ for every n; since Γ is level-closed, we have that $\bar{x} \in L(y_1) \subseteq \operatorname{dom}\Gamma$. Since Γ satisfies the limiting monotonicity condition at $\bar{x}$, we find $\bar{y} \in \operatorname{BMMin}\Gamma(\bar{x}) \subseteq \Gamma(\bar{x})$ such that $\bar{y} \leq y_n$ for every n. Hence (H2) holds. By Theorem 11.7 there exists $(x, y) \in \mathscr{A}$ such that $(x, y) \preceq_H (x_0, y_0)$ and $(x', y') \in \operatorname{gph}\Gamma$, $(x', y') \preceq_H (x, y)$ imply $x' = x$. Set $\bar{x} := x$ and take $\bar{y} \in \operatorname{BMMin}\Gamma(\bar{x})$ such that $\bar{y} \leq_K y$. By (11.29) we have that $(\bar{x}, \bar{y}) \preceq_H (x_0, y_0)$. Let now $(x', y') \in \operatorname{gph}\Gamma = \mathscr{A}$ with $(x', y') \preceq_H (\bar{x}, \bar{y})$. Since $(\bar{x}, \bar{y}) = (x, \bar{y}) \preceq_H (x, y)$, we have that $(x', y') \preceq_H (x, y)$, and so $x' = x = \bar{x}$. The proof is complete. □

In the case when H is a singleton the next result is practically [6, Theorem 1] under the supplementary hypothesis that $\operatorname{Min}\Gamma(x)$ is compact for every $x \in X$; it seems that this condition has to be added in order that [6, Theorem 1] be true.

Corollary 11.10 (Not Authentic Minimal-Point Theorem with Respect to $\preceq_H$). *Assume that* (X, d) *is a complete metric space,* Y *is a real topological vector space. Let* $K \subseteq Y$ *be a proper closed convex cone and* $H \subseteq K$ *be a nonempty cs-complete bounded set with* $\mathbf{0} \notin \operatorname{cl}(H + K)$. *Suppose that:*

(H4) $\Gamma : X \rightrightarrows Y$ *is level-closed,* $\operatorname{Min}\Gamma(x)$ *is compact and* $\Gamma(x) \subseteq K + \operatorname{Min}\Gamma(x)$ *for every* $x \in \operatorname{dom}\Gamma$
(B3) $\operatorname{Im}\Gamma$ *is quasi-bounded*

Then for every $(x_0, y_0) \in \operatorname{gph}\Gamma$ *there exist* $\bar{x} \in \operatorname{dom}\Gamma$ *and* $\bar{y} \in \operatorname{Min}\Gamma(\bar{x})$ *such that:*

(a) $(\bar{x}, \bar{y}) \preceq_H (x_0, y_0)$
(b) $(x, y) \in \operatorname{gph}\Gamma$*,* $(x, y) \preceq_H (\bar{x}, \bar{y})$ *imply* $x = \bar{x}$

Proof. In order to apply Theorem 11.7 for $\mathscr{A} := \operatorname{gph}\Gamma$ we have only to show that $\mathscr{A}$ verifies condition (H2). For this consider the sequence $((x_n, y_n))_{n\geq 1} \subseteq \mathscr{A}$ such that (y_n) is $\leq_K$-decreasing and $x_n \to \bar{x}$. As in the proof of the preceding corollary, $\bar{x} \in L(y_n)$ for every $n \in \mathbb{N}$. Because $\Gamma(\bar{x}) \subseteq K + \operatorname{Min}\Gamma(\bar{x})$, for every $n \in \mathbb{N}$ there exists $y'_n \in \operatorname{Min}\Gamma(\bar{x})$ such that $y'_n \leq y_n$. Because $\operatorname{Min}\Gamma(\bar{x})$ is compact, (y'_n) has a subnet $(y'_{\psi(i)})_{i\in I}$ converging to some $\bar{y} \in \operatorname{Min}\Gamma(\bar{x})$; here $\psi : (I, \succeq) \to \mathbb{N}$ is such that for every n there exists $i_n \in I$ with $\psi(i) \geq n$ for $i \succeq i_n$. Hence $y'_{\psi(i)} \leq y_{\psi(i)} \leq y_n$ for $i \succeq i_n$, whence $\bar{y} \leq y_n$ because K is closed. Therefore, (H2) holds. By Theorem 11.7, for $(x_0, y_0) \in \operatorname{gph}\Gamma$, there exists $(x, y) \in \mathscr{A}$ such that $(x, y) \preceq_H (x_0, y_0)$ and $(x', y') \in \operatorname{gph}\Gamma$, $(x', y') \preceq_H (x, y)$ imply $x' = x$. Set $\bar{x} := x$ and take $\bar{y} \in \operatorname{Min}\Gamma(\bar{x})$ such that $\bar{y} \leq_K y$. As in the proof of Corollary 11.9 we find that $(\bar{x}, \bar{y})$ is the desired element. The proof is complete. □

11.4.3 Minimal-Point Theorems of Isac–Tammer's Type

Besides $F : X \times X \rightrightarrows K$ considered in the preceding section we consider also $F' : Y \times Y \rightrightarrows K$ satisfying conditions (F1) and F(2), that is, $0 \in F'(y,y)$ for all $y \in Y$ and $F'(y_1,y_2) + F'(y_2,y_3) \subseteq F'(y_1,y_3) + K$ for all $y_1,y_2,y_3 \in Y$. Then $\Phi : Z \times Z \rightrightarrows K$ with $Z := X \times Y$, defined by $\Phi((x_1,y_1),(x_2,y_2)) := F(x_1,x_2) + F'(y_1,y_2)$, satisfies conditions (F1) and (F2), too. As in Sect. 11.4.1 we obtain that the relation $\preccurlyeq_{F,F'}$ defined by

$$(x_1,y_1) \preccurlyeq_{F,F'} (x_2,y_2) \Longleftrightarrow y_2 \in y_1 + F(x_1,x_2) + F'(y_1,y_2) + K$$

is reflexive and transitive. Moreover, for $x,x_1,x_2 \in X$ and $y_1,y_2 \in Y$ we have

$$(x_1,y_1) \preccurlyeq_{F,F'} (x_2,y_2) \Longrightarrow (x_1,y_1) \preccurlyeq_F (x_2,y_2) \Longrightarrow y_1 \leq_K y_2,$$

$$(x,y_1) \preccurlyeq_{F,F'} (x,y_2) \Longleftrightarrow y_1 \leq_K y_2.$$

As in the preceding section, for F satisfying (F1)–(F3), F' satisfying (F1), (F2) and z^* from (F3) we define the partial order $\preceq_{F,F',z^*}$ by

$$(x_1,y_1) \preceq_{F,F',z^*} (x_2,y_2) \iff \begin{cases} (x_1,y_1) = (x_2,y_2) \text{ or} \\ (x_1,y_1) \preceq_{F,F'} (x_2,y_2) \text{ and } z^*(y_1) < z^*(y_2). \end{cases}$$

Theorem 11.8 (Minimal-Point Theorem with Respect to $\preceq_{F,F',z^*}$). *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subseteq Y$ is a proper convex cone. Let $F : X \times X \rightrightarrows K$ satisfy conditions (F1)–(F3), let $F' : Y \times Y \rightrightarrows K$ satisfy (F1) and (F2), and let $\mathscr{A} \subseteq X \times Y$ satisfy the condition*

(H1b) For every $\preccurlyeq_{F,F'}$-decreasing sequence $((x_n,y_n)) \subseteq \mathscr{A}$ with $x_n \to x \in X$ there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $(x,y) \preccurlyeq_{F,F'} (x_n,y_n)$ for every $n \in \mathbb{N}$.

Suppose that

(B1) z^* (from (F3)) is bounded from below on $\mathrm{Pr}_Y(\mathscr{A})$.

Then for every $(x_0,y_0) \in \mathscr{A}$ there exists an element $(\bar{x},\bar{y}) \in \mathscr{A}$ such that:

(a) $(\bar{x},\bar{y}) \preceq_{F,F',z^*} (x_0,y_0)$.
(b) $(\bar{x},\bar{y})$ is a minimal element of $\mathscr{A}$ with respect to $\preceq_{F,F',z^*}$.

Proof. It is easy to verify that $\preceq_{F,F',z^*}$ is reflexive, transitive and antisymmetric. To get the conclusion one follows the lines of the proof of Theorem 11.6. □

Clearly, taking $F' = 0$ in Theorem 11.8 we get Theorem 11.6. As mentioned after the proof of Theorem 11.6, this extends significantly [32, Theorem 3.10.7], keeping practically the same proof. We ask ourselves if [32, Theorem 3.10.15] could be extended to this framework, taking into account that the boundedness condition on $\mathscr{A}$ in [32, Theorem 3.10.15] is much less restrictive. In [32, Theorem 3.10.15] we used a functional φ_A (defined by (11.5) and in (11.38) below) in order to prove the minimal-point theorem. Because an element k^0 does not impose itself naturally, and we need a stronger condition on the functional φ_A even if $k^0 \in K \setminus \{\mathbf{0}\} \subseteq \operatorname{int} C$, we consider an abstract K-monotone functional φ to which we impose some conditions φ_A has already.

Theorem 11.9 (Not Authentic Minimal-Point Theorem with Respect to $\preceq_{F,F'}$). *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subseteq Y$ is a proper convex cone. Let $F : X \times X \rightrightarrows K$ satisfy conditions (F1)–(F3), let $F' : Y \times Y \rightrightarrows K$ satisfy (F1) and (F2), and let $\mathscr{A} \subseteq X \times Y$ satisfy the condition*

(H1b) For every $\preccurlyeq_{F,F'}$-decreasing sequence $((x_n, y_n)) \subseteq \mathscr{A}$ with $x_n \to x \in X$ there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $(x,y) \preccurlyeq_{F,F'} (x_n, y_n)$ for every $n \in \mathbb{N}$.

Assume that there exists a functional $\varphi : Y \to \overline{\mathbb{R}}$ such that

(F4) $\quad (x_1, y_1) \preceq_{F,F'} (x_2, y_2) \Longrightarrow \varphi(y_1) + d(x_1, x_2) \le \varphi(y_2).$

Furthermore, suppose

(B4) φ is bounded below on $\operatorname{Pr}_Y(\mathscr{A})$.

Then for every point $(x_0, y_0) \in \mathscr{A}$ with $\varphi(y_0) \in \mathbb{R}$, there exists $(\bar{x}, \bar{y}) \in \mathscr{A}$ such that:

(a) $(\bar{x}, \bar{y}) \preceq_{F,F'} (x_0, y_0)$
(b) $(x', y') \in \mathscr{A}$, $(x', y') \preceq_{F,F'} (\bar{x}, \bar{y})$ imply $x' = \bar{x}$ (not authentic minimal point with respect to $\preceq_{F,F'}$)

Moreover, if φ is strictly K-monotone on $\operatorname{Pr}_Y(\mathscr{A})$, that is, $y_1, y_2 \in \operatorname{Pr}_Y(\mathscr{A})$, $y_2 - y_1 \in K \setminus \{\mathbf{0}\} \Longrightarrow \varphi(y_1) < \varphi(y_2)$, then

(b') $(\bar{x}, \bar{y})$ is a minimal point of $\mathscr{A}$ with respect to $\preceq_{F,F'}$ (minimal point with respect to $\preceq_{F,F'}$).

Proof. First note that from (F4) we have that φ is K-monotone. Let us construct a sequence $((x_n, y_n))_{n\ge 0} \subseteq \mathscr{A}$ as follows: Having $(x_n, y_n) \in \mathscr{A}$, we take $(x_{n+1}, y_{n+1}) \in \mathscr{A}$, $(x_{n+1}, y_{n+1}) \preceq_{F,F'} (x_n, y_n)$, such that

$$\varphi(y_{n+1}) \le \inf\{\varphi(y) \mid (x,y) \in \mathscr{A},\ (x,y) \preceq_{F,F'} (x_n, y_n)\} + 1/(n+1). \tag{11.37}$$

Of course, the sequence $((x_n, y_n))$ is $\preceq_{F,F'}$-decreasing, and so (y_n) $(\subseteq \operatorname{Pr}_Y(\mathscr{A}))$ is K-decreasing. It follows that the sequence $(\varphi(y_n))$ is non-increasing and bounded from below, hence convergent in $\mathbb{R}$. Because

$$(x_{n+p}, y_{n+p}) \preceq_{F,F'} (x_n, y_n) \preceq_{F,F'} (x_{n-1}, y_{n-1}),$$

using (F4) and (11.37) we get

$$d(x_{n+p}, x_n) \le \varphi(y_n) - \varphi(y_{n+p}) \le 1/n \quad \forall n, p \in \mathbb{N}^*.$$

It follows that (x_n) is a Cauchy sequence in the complete metric space (X,d), and so (x_n) is convergent to some $\bar{x} \in X$.

By (H1b) there exists $\bar{y} \in Y$ such that $(\bar{x},\bar{y}) \in \mathscr{A}$ and $(\bar{x},\bar{y}) \preceq_{F,F'} (x_n,y_n)$ for every $n \in \mathbb{N}$. Let us show that $(\bar{x},\bar{y})$ is the desired element. Indeed, $(\bar{x},\bar{y}) \preceq_{F,F'} (x_0,y_0)$. Suppose that $(x',y') \in \mathscr{A}$ is such that $(x',y') \preceq_{F,F'} (\bar{x},\bar{y})$ ($\preceq_{F,F'} (x_n,y_n)$ for every $n \in \mathbb{N}$). Thus $\varphi(y') + d(x',\bar{x}) \leq \varphi(\bar{y})$ by (F4), whence

$$d(x',\bar{x}) \leq \varphi(\bar{y}) - \varphi(y') \leq \varphi(y_n) - \varphi(y') \leq 1/n \quad \forall n \geq 1.$$

It follows that $d(x',\bar{x}) = \varphi(\bar{y}) - \varphi(y') = 0$. Hence $x' = \bar{x}$.

Assuming that φ is strictly K-monotone, because $y' \leq_K \bar{y}$ and $\varphi(\bar{y}) - \varphi(y') = 0$, we have necessarily $y' = \bar{y}$. Hence $(\bar{x},\bar{y})$ is a minimal point with respect to $\preceq_{F,F'}$. □

Note that if $C \subseteq Y$ is a proper closed convex cone such that $C - (K \setminus \{\mathbf{0}\}) = \operatorname{int} C$ and $k^0 \in K \setminus \{\mathbf{0}\}$ (see assumption (A2)), the functional $\varphi_C : Y \to \mathbb{R}$ defined by (see (11.5))

$$\varphi_C(y) := \inf\left\{t \in \mathbb{R} \mid y \in tk^0 + C\right\} \tag{11.38}$$

is a strictly K-monotone continuous sublinear functional (see Theorem 11.2). Moreover, if the condition

(B') $\quad \operatorname{Pr}_Y(\mathscr{A}) \cap (\tilde{y} - \operatorname{int} C) = \emptyset$ for some $\tilde{y} \in Y$

holds, then $\varphi := \varphi_C$ is bounded from below on $\operatorname{Pr}_Y(\mathscr{A})$, i.e., (B4) holds. Indeed, by Theorem 11.2, we have that $\varphi(y) + \varphi(-\tilde{y}) \geq \varphi(y - \tilde{y}) \geq 0$ for $y \in \operatorname{Pr}_Y(\mathscr{A})$, whence $\varphi(y) \geq -\varphi(-\tilde{y})$ for $y \in \operatorname{Pr}_Y(\mathscr{A})$.

Another example for a function φ is that defined by

$$\varphi(y) := \varphi_{K,k^0}(y - \hat{y}), \tag{11.39}$$

where K is a proper convex cone, $k^0 \in K \setminus \{\mathbf{0}\}$, and $\hat{y} \in Y$ is such that

(B") $\quad y_0 - \hat{y} \in \mathbb{R}k^0 - K, \quad \operatorname{Pr}_Y(\mathscr{A}) \cap (\hat{y} - K) = \emptyset.$

Then φ is K-monotone, $\varphi(y_0) < \infty$ and $\varphi(y) \geq 0$ for every $y \in \operatorname{Pr}_Y(\mathscr{A})$, i.e., (B4) holds.

For both of these functions in (11.38) and (11.39) we have to impose condition (F4) in order to be used in Theorem 11.9.

Remark 11.9. Using the function $\varphi = \varphi_{K,k^0}(\cdot - \hat{y})$ (defined by (11.39)) in Theorem 11.9 we can derive [41, Theorem 4.2] taking $F(x_1,x_2) := \{d(x_1,x_2)k^0\}$ and $F'(y_1,y_2) := \{\varepsilon \|y_1 - y_2\| k^0\}$ when Y is a Banach space; note that, at its turn, [41, Theorem 4.2] extends [46, Theorem 8].

11.4.4 Ekeland's Variational Principles of Ha's Type

The previous EVP type results correspond to Pareto optimality. Ha [37] established an EVP type result which corresponds to Kuroiwa optimality. The next result is an extension of this type of result. For its proof we use [65, Theorem 3.1] or [41, Theorem 2.2].

Theorem 11.10 (Variational Principle). *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subseteq Y$ is a proper convex cone. Let $F : X \times X \rightrightarrows K$ satisfy conditions (F1)–(F3) and $\Gamma : X \rightrightarrows Y$ be such that*

(H5) $\{x \in X \mid \Gamma(u) \subseteq \Gamma(x) + F(x,u) + K\}$ is closed for every $u \in X$.

Moreover, if

(B5) z^* (from (F3)) is bounded below on $\Gamma(X)$,

then for every $x_0 \in \operatorname{dom}\Gamma$ there exists $\overline{x} \in X$ such that:

(a) $\Gamma(x_0) \subseteq \Gamma(\overline{x}) + F(\overline{x},x_0) + K$
(b) $\Gamma(\overline{x}) \subseteq \Gamma(x) + F(x,\overline{x}) + K$ implies $x = \overline{x}$

Proof. Let us consider the relation $\preccurlyeq$ on X defined by $x' \preccurlyeq x$ if $\Gamma(x) \subseteq \Gamma(x') + F(x',x) + K$. By our hypotheses we have that $S(x) := \{x' \in X \mid x' \preccurlyeq x\}$ is closed for every $x \in X$. Note that for $x \in X \setminus \operatorname{dom}\Gamma$ we have that $S(x) = X$, while for $x \in \operatorname{dom}\Gamma$ we have that $S(x) \subseteq \operatorname{dom}\Gamma$. The relation $\preccurlyeq$ is reflexive and transitive. The reflexivity of $\preccurlyeq$ is obvious. Let $x' \preccurlyeq x$ and $x'' \preccurlyeq x'$. Then $\Gamma(x) \subseteq \Gamma(x') + F(x',x) + K$ and $\Gamma(x') \subseteq \Gamma(x'') + F(x'',x') + K$. Using (F2) we get

$$\Gamma(x) \subseteq \Gamma(x'') + F(x'',x') + K + F(x',x) + K \subseteq \Gamma(x'') + F(x'',x) + K,$$

that is, $x'' \preccurlyeq x$. Consider

$$\varphi : X \to \overline{\mathbb{R}}, \quad \varphi(x) := \inf z^*(\Gamma(x)),$$

with the usual convention $\inf\emptyset := +\infty$. Clearly, $\varphi(x) \geq m := \inf z^*(\Gamma(X)) > -\infty$. Moreover, if $x' \preccurlyeq x \in \operatorname{dom}\Gamma$ then $z^*(\Gamma(x)) \subseteq z^*(\Gamma(x')) + z^*(F(x',x)) + z^*(K)$, whence $\varphi(x) \geq \varphi(x') + \inf z^*(F(x',x)) \geq \varphi(x')$.

Fix $x_0 \in \operatorname{dom}\Gamma$. The conclusion of the theorem asserts that there exists $\overline{x} \in X$ such that $\overline{x} \in S(x_0)$ and $S(\overline{x}) = \{\overline{x}\}$. To get this conclusion we apply [41, Theorem 2.2] or [65, Theorem 3.1]. Because (X,d) is complete and $S(x)$ is closed for every $x \in X$, we may (and we do) assume that $\operatorname{dom}\Gamma = X$ (otherwise we replace X by $S(x_0)$). In order to apply [41, Theorem 2.2] we have to show that $d(x_n,x_{n+1}) \to 0$ provided $(x_n)_{n\geq 1} \subseteq X$ is $\preccurlyeq$-decreasing. In the contrary case there exist $\delta > 0$ and $(n_p)_{p\geq 1} \subseteq \mathbb{N}^*$ an increasing sequence such that $d(x_{n_p},x_{n_p+1}) \geq \delta$ for every $p \geq 1$. Then, as seen above, $\varphi(x_n) \geq \varphi(x_{n+1}) + \inf z^*(F(x_{n+1},x_n))$, and so

$$\varphi(x_{n_1}) \geq \varphi\left(x_{n_p+1}\right) + \sum_{l=n_1}^{n_p} \inf z^*(F(x_{l+1},x_l)) \geq m + p \cdot \eta(\delta)$$

with $\eta(\delta) > 0$ from (F3). Letting $p \to \infty$ we get a contradiction. Hence $d(x_n, x_{n+1}) \to 0$. The conclusion follows. □

Note that instead of assuming $S(u)$ to be closed for every $u \in X$ it is sufficient to have that $S(u)$ is $\preccurlyeq$-lower closed, that is, for every $\preccurlyeq$-decreasing sequence $(x_n) \subseteq S(u)$ with $x_n \to x$ we have that $x \in S(u)$. Moreover, instead of using [41, Theorem 2.2] it is possible to give a slightly longer direct proof similar to that of Theorem 11.6 (and using φ instead of z^* in the construction of (x_n)).

Remark 11.10. Taking Y to be a separated locally convex space, $K \subseteq Y$ a pointed closed convex cone and $F(x,x') := \{d(x,x')k^0\}$ with $k^0 \in K \setminus \{0\}$, we can deduce [37, Theorem 3.1]. For this assume that $\Gamma(X)$ is quasi bounded, $\Gamma(x) + K$ is closed for every $x \in X$ and Γ is level-closed (or K-lsc with the terminology from [37]). Since clearly z^* is bounded from below on $\operatorname{Im}\Gamma$, in order to apply the preceding theorem we need to have that $S(u)$ is closed for every $u \in X$; this is done in [37, Lemmma 3.2]. Below we provide another proof for the closedness of $S(u)$.

First, if $x \notin L(b)$ then there exists $\delta > 0$ such that $B(x,\delta) \cap L(b + \delta k^0) = \emptyset$. Indeed, because $x \notin L(b)$ we have that $b \notin \Gamma(x) + K$, and so $b + \delta' k^0 \notin \Gamma(x) + K$, that is, $x \notin L(b + \delta' k^0)$, for some $\delta' > 0$ (since $\Gamma(x) + K$ is closed). Because $L(b + \delta' k^0)$ is closed, there exists $\delta \in (0, \delta']$ such that $B(x,\delta) \cap L(b + \delta' k^0) = \emptyset$, and so $B(x,\delta) \cap L(b + \delta k^0) = \emptyset$.

Fix $u \in X$ and take $x \in X \setminus S(u)$, that is, $\Gamma(u) \not\subseteq \Gamma(x) + d(x,u)k^0 + K$. Then there exists $y \in \Gamma(u)$ with $b := y - d(x,u)k^0 \notin \Gamma(x) + K$. By the argument above there exists $\delta' > 0$ such that $B(x,\delta') \cap L(b + \delta' k^0) = \emptyset$, that is, $y - d(x,u)k^0 + \delta' k^0 \notin \Gamma(x') + K$ for every $x' \in B(x,\delta')$. Taking $\delta \in (0,\delta']$ sufficiently small we have that $d(x',u) \geq d(x,u) - \delta'$ for $x' \in B(x,\delta)$, and so $y \notin \Gamma(x') + d(x',u)k^0 + K$ for every $x' \in B(x,\delta)$, that is, $B(x,\delta) \cap S(u) = \emptyset$.

If we assume that $\Gamma(x_0) \not\subseteq \Gamma(x) + k^0 + K$ for every $x \in X$, then $\bar{x}$ provided by the preceding theorem satisfies $d(\bar{x}, x_0) < 1$. Indeed, in the contrary case, because $\Gamma(x_0) \subseteq \Gamma(\bar{x}) + d(\bar{x},x_0)k^0 + K$ and $d(\bar{x},x_0)k^0 + K \subseteq k^0 + K$, we get the contradiction $\Gamma(x_0) \subseteq \Gamma(\bar{x}) + k^0 + K$. Replacing k^0 by εk^0 and d by $\lambda^{-1} d$ for some $\varepsilon, \lambda > 0$ we obtain exactly the statement of [37, Theorem 3.1].

In the case in which Y is just a topological vector space we have the following version of the preceding theorem under conditions similar to those in Theorem 11.7.

Theorem 11.11 (Variational Principle). *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subseteq Y$ is a proper closed convex cone. Let $H \subseteq K$ be a nonempty cs-complete bounded set with $\mathbf{0} \notin \operatorname{cl}(H + K)$, and $\Gamma : X \rightrightarrows Y$. If*

(H6) $\{x \in X \mid \Gamma(u) \subseteq \Gamma(x) + d(x,u)H + K\}$ *is closed for every* $u \in X$.
(B6) $\Gamma(X)$ *is quasi bounded.*

then for every $x_0 \in \operatorname{dom}\Gamma$ *there exists* $\bar{x} \in X$ *such that:*

(a) $\Gamma(x_0) \subseteq \Gamma(\bar{x}) + d(\bar{x},x_0)H + K$
(b) $\Gamma(\bar{x}) \subseteq \Gamma(x) + d(x,\bar{x})H + K$ *implies* $x = \bar{x}$

Proof. Let $B \subseteq Y$ be a bounded set such that $\Gamma(X) \subseteq B+K$.

Consider $F(x,x') := d(x,x')H$ for $x,x' \in X$. As seen before, F satisfies conditions (F1) and (F2), and so the relation $\preccurlyeq$ defined in the proof of Theorem 11.10 is reflexive and transitive; moreover, by our hypotheses, $S(x) := \{x' \in X \mid x' \preccurlyeq x\}$ is closed for every $x \in X$. As in the proof of Theorem 11.10 we may (and do) assume that $X = \operatorname{dom}\Gamma$ and it is sufficient to show that $d(x_n,x_{n+1}) \to 0$ provided $(x_n)_{n\geq 1} \subseteq X$ is $\preccurlyeq$-decreasing. In the contrary case there exist $\delta > 0$ and $(n_p)_{p\geq 1} \subseteq \mathbb{N}^*$ an increasing sequence such that $d(x_{n_p},x_{n_p+1}) \geq \delta$ for every $p \geq 1$.

Fixing $y_1 \in \Gamma(x_1)$, inductively we find the sequences $(y_n)_{n\geq 0} \subseteq Y$, $(h_n)_{n\geq 0} \subseteq H$ and $(k_n)_{n\geq 0} \subseteq K$ such that $y_n = y_{n+1} + d(x_n,x_{n+1})h_n + k_n$ for every $n \geq 1$. Using the convexity of H, and the facts that $H \subseteq K$ and $\Gamma(X) \subseteq B+K$, for $p \in \mathbb{N}$ we get $h'_p \in H$, $b_p \in B$ and $k'_p, k''_p \in K$ such that

$$y_1 = y_{n_{p+1}} + \sum_{l=1}^{n_p} d(x_l,x_{l+1})h_l + \sum_{l=1}^{n_p} k_l = b_p + \delta(h_{n_1} + \ldots + h_{n_p}) + k'_p = b_p + p\delta h'_p + k''_p.$$

It follows that $(p\delta)^{-1}(y_1 - b_p) \in H + K$ for every $p \geq 1$. Since (b_p) is bounded we obtain the contradiction $0 \in \operatorname{cl}(H+K)$. The conclusion follows. □

Again, instead of assuming that $S(u)$ is closed for every $u \in X$, it is sufficient to assume that $S(u)$ is $\preccurlyeq$-lower closed for $u \in X$. A slightly longer direct proof, similar to that of Theorem 11.7, is possible. Also Theorem 11.11 covers [37, Theorem 3.1].

11.4.5 Ekeland's Variational Principle for Bi-Multifunctions

In [9] Bianchi, Kassay and Pini obtained an EVP type result for vector functions of two variables; previously such results were obtained by Isac [45] and Li et al. [49]. The next result extends [9, Theorem 1] in two directions: d is replaced by F satisfying (F1)–(F3) and instead of a single-valued function $f : X \times X \to Y$ we take a multi-valued one. For its proof we use again [65, Theorem 3.1] or [41, Theorem 2.2].

Theorem 11.12. *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subseteq Y$ is a proper convex cone. Let $F : X \times X \rightrightarrows K$ satisfy conditions (F1)–(F3). Assume that $G : X \times X \rightrightarrows Y$ has the properties:*

(i) $\mathbf{0} \in G(x,x)$ *for every* $x \in X$
(ii) $G(x_1,x_2) + G(x_2,x_3) \subseteq G(x_1,x_3) + K$ *for all* $x_1,x_2,x_3 \in X$

If

(H7) $\{x' \in X \mid [G(x,x') + F(x,x')] \cap (-K) \neq \emptyset\}$ *is closed for every* $x \in X$
(B7) z^* *(from (F3)) is bounded below on the set* $\operatorname{Im} G(x,\cdot)$ *for every* $x \in X$

then for every $x_0 \in X$ *there exists* $\overline{x} \in X$ *such that:*

(a) $[G(x_0,\overline{x}) + F(x_0,\overline{x})] \cap (-K) \neq \emptyset$
(b) $[G(\overline{x},x) + F(\overline{x},x)] \cap (-K) \neq \emptyset$ *implies* $x = \overline{x}$

Proof. Let us consider the relation $\preccurlyeq$ on X defined by

$$x \preccurlyeq x' \iff \left[G(x',x) + F(x',x)\right] \cap (-K) \neq \emptyset.$$

Then $\preccurlyeq$ is reflexive and transitive. The reflexivity is immediate from (i) and (F1). Assume that $x \preccurlyeq x'$ and $x' \preccurlyeq x''$. Then $-k \in G(x',x) + F(x',x)$ and $-k' \in G(x'',x') + F(x'',x')$ with $k,k' \in K$. Hence, by (ii) and (F2),

$$-k - k' \in G(x',x) + F(x',x) + G(x'',x') + F(x'',x') \subseteq G(x'',x) + K + F(x'',x) + K,$$

whence $[G(x'',x) + F(x'',x)] \cap (-K) \neq \emptyset$, that is, $x \preccurlyeq x''$.

Setting $S(x) := \{x' \in X \mid x' \preccurlyeq x\}$, by (H7) we have that $S(x)$ is closed for every $x \in X$. We have to show that for $(x_n)_{n\geq 1} \subseteq X$ a $\preccurlyeq$-decreasing sequence one has $d(x_n,x_{n+1}) \to 0$. In the contrary case there exist an increasing sequence $(n_l)_{l\geq 1} \subseteq \mathbb{N}$ and $\delta > 0$ such that $d(x_{n_l},x_{n_l+1}) \geq \delta$ for every $l \geq 1$. Because (x_n) is $\preccurlyeq$-decreasing, we have that $-k_n \in G(x_n,x_{n+1}) + F(x_n,x_{n+1})$ for some $k_n \in K$ and every $n \geq 1$. Then

$$-k_1 - \ldots - k_n \in G(x_1,x_{n+1}) + F(x_1,x_2) + \ldots + F(x_n,x_{n+1}) + K,$$

and so

$$\inf z^*\left(\operatorname{Im} G(x_1,\cdot)\right) + \inf z^*\left(F(x_1,x_2)\right) + \ldots + \inf z^*\left(F(x_n,x_{n+1})\right) \leq 0 \quad \forall n \geq 1.$$

Since $\inf z^*\left(F(x_n,x_{n+1})\right) \geq 0$ for every $n \geq 1$ and $\inf z^*\left(F(x_{n_l},x_{n_l+1})\right) \geq \eta(\delta) > 0$ for every $l \geq 1$, taking $n := n_p$ with $p \geq 1$, we obtain that

$$p\eta(\delta) \leq -\inf z^*\left(\operatorname{Im} G(x_1,\cdot)\right) \text{ for every } p \geq 1.$$

This yields the contradiction $\eta(\delta) \leq 0$. Hence $d(x_n,x_{n+1}) \to 0$. Applying [41, Theorem 2.2] we get some $\overline{x} \in S(x_0)$ with $S(\overline{x}) = \{\overline{x}\}$, that is, our conclusion holds. □

Remark 11.11. If we need the conclusion only for a fixed (given) point $x_0 \in X$, we may replace condition (B7) by the fact that z^* (from (F3)) is bounded below on the set $\operatorname{Im} G(x_0,\cdot)$.

Indeed, $X_0 := S(x_0)$ is closed by (H7), and so (X_0,d) is complete. If $x \in X_0$ then $-k \in G(x_0,x) + F(x_0,x) \subseteq G(x_0,x) + K$ for some $k \in K$, and so $-k' \in G(x_0,x)$ for some $k' \in K$. It follows that $-k' + G(x,u) \subseteq G(x_0,x) + G(x,u) \subseteq G(x_0,u) + K$, whence $G(x,u) \subseteq G(x_0,u) + K$ for every $u \in X$. Hence condition (B7) is verified on X_0, and so the conclusion of the theorem holds for x_0.

Remark 11.12. For $F(x,x') := \{d(x,x')k^0\}$ with $k^0 \in K \setminus \{\mathbf{0}\}$ and G single-valued, using Theorem 11.12 and the preceding remark one obtains [45, Theorem 8] and [49, Theorem 3]; in [45] K is normal and closed, while in [49] $k^0 \in \operatorname{int} K$.

Note that condition (H7) in the preceding theorem holds when G is compact-valued, $G(u,\cdot)$ is level-closed, K is closed and $F(x,x') := \{d(x,x')k^0\}$ for some $k^0 \in K$. Indeed, assume that $-k_n \in G(u,x_n) + d(x_n,u)k^0$ for every $n \geq 1$, where $k_n \in K$. Take $\varepsilon > 0$. Then there exists $n_\varepsilon \geq 1$ such that $d(x_n,u) \geq d(x,u) - \varepsilon =: \gamma_\varepsilon$ for every $n \geq n_\varepsilon$. Then for such n we have that $G(u,x_n) \cap \left(-\gamma_\varepsilon k^0 - K\right) \neq \emptyset$, whence $G(u,x) \cap \left(-\gamma_\varepsilon k^0 - K\right) \neq \emptyset$. Hence there exists $y_\varepsilon \in G(u,x)$ such that $y_\varepsilon + \gamma_\varepsilon k^0 \in -K$. Since $G(u,x)$ is compact, $(y_\varepsilon)_{\varepsilon > 0}$ has a subnet converging to $y \in G(x,u)$. Since $\lim_{\varepsilon \to 0} \gamma_\varepsilon = d(x,u)$ and K is closed, we obtain that $y + d(x,u)k^0 \in -K$.

If Y is a separated locally convex space then we may assume that G is weakly compact-valued instead of being compact-valued.

When G is single-valued and $F(x,x') := \{d(x,x')k^0\}$ with $k^0 \in K$, where K is closed and $z^*(k^0) = 1$, the preceding theorem reduces to [9, Theorem 1].

11.4.6 EVP Type Results

The framework is the same as in the previous sections. We want to apply the preceding results to obtain vectorial EVPs. To envisage functions defined on subsets of X we add to Y an element ∞ not belonging to the space Y, obtaining thus the space $Y^\bullet := Y \cup \{\infty\}$. We consider that $y \leq_K \infty$ for all $y \in Y$. Consider now the function $f : X \to Y^\bullet$. As usual, the domain of f is $\operatorname{dom} f = \{x \in X \mid f(x) \neq \infty\}$; the epigraph of f is $\operatorname{epi} f = \{(x,y) \in X \times Y \mid f(x) \leq_K y\}$; the graph of f is $\operatorname{gph} f = \{(x, f(x)) \mid x \in \operatorname{epi} f\}$. Of course, f is proper if $\operatorname{dom} f \neq \emptyset$. For $y^* \in K^+$ we set $(y^* \circ f)(x) := +\infty$ for $x \in X \setminus \operatorname{dom} f$.

Theorem 11.13. *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subseteq Y$ is a proper convex cone. Let $F : X \times X \rightrightarrows K$ satisfy the conditions (F1)–(F3) and let $f : X \to Y^\bullet$ be proper. Assume that*

(H8) For every sequence $(x_n) \subseteq \operatorname{dom} f$ with $x_n \to x \in X$ and $f(x_n) \in f(x_{n+1}) + F(x_{n+1},x_n) + K$ for every $n \in \mathbb{N}$ one has $f(x_n) \in f(x) + F(x,x_n) + K$ for every $n \in \mathbb{N}$

(B8) $z^* \circ f$ (with z^* from (F3)) is bounded from below

Then for every $x_0 \in \operatorname{dom} f$ there exists $\overline{x} \in \operatorname{dom} f$ such that:

(a) $f(x_0) \in f(\overline{x}) + F(\overline{x},x_0) + K$

(b) $\forall x \in \operatorname{dom} f \ : \ f(\overline{x}) \in f(x) + F(\overline{x},x) + K \Rightarrow x = \overline{x}$

Proof. Consider $\mathscr{A} := \operatorname{gph} f := \{(x, f(x)) \mid x \in \operatorname{dom} f\}$. Condition (H8) says nothing than (H1) is verified. Applying Theorem 11.6 we get the conclusion. □

As for Theorem 11.6, in the above theorem we may assume that z^* is bounded from below on the set

$$B_0 := \{f(x) \mid x \in \operatorname{dom} f,\ f(x_0) \in f(x) + F(x,x_0) + K\}.$$

The preceding theorem is very close to [36, Theorem 3.8] for $\gamma = 1$, which is stated for F and K satisfying conditions (i), (ii), (iii), (S1), (S2) and $f : S \to Y$ (with $S \subseteq X$ a nonempty closed set) satisfying the conditions

(S3) Let us denote $A_x^{\gamma F} := \{z \in X \mid (f(z) + \gamma F(z,x)) \cap (f(x) - K) \neq \emptyset\}$ for $x \in S$. For each $x \in S$ and $(z_n) \subseteq A_x^{\gamma F}, z_n \to z$ such that $f(z_n) \leq f(z_m)$ for $n > m$, it follows that $z \in A_x^{\gamma F}$.
(S4) The set $(f(S) - f(x_0)) \cap (-D_F)$ is K-bounded.

Because S is closed one may assume that $S = X$ and $\operatorname{dom} f = X$. Observe that (S4) implies that $y^*(B_0)$ is bounded from below for every $y^* \in K^+$, and so $z^*(B_0)$ is bounded from below. Let us prove that (S3) implies (H8) (for $\gamma = 1$). Consider $(x_n) \subseteq X = \operatorname{dom} f$ with $x_n \to x \in X$ and $f(x_n) \in f(x_{n+1}) + F(x_{n+1},x_n) + K$ for every $n \in \mathbb{N}$. Clearly, for a fixed $\overline{n} \in \mathbb{N}$ we have that $(x_n)_{n \geq \overline{n}} \subseteq A_{x_{\overline{n}}}^{1F}$ and $f(x_n) \leq f(x_m)$ for $n \geq m \geq \overline{n}$. By (S3) we have that $x \in A_{x_{\overline{n}}}^{1F}$, that is, $f(x_{\overline{n}}) \in f(x) + F(x,x_{\overline{n}}) + K$. Hence (H8) holds.

In the case in which $F(x,x') = d(x,x')H$ for some $H \subseteq K$ the condition (H8) becomes

(H9) For every sequence $(x_n) \subseteq \operatorname{dom} f$ with $x_n \to x \in X$ and $f(x_n) \in f(x_{n+1}) + d(x_{n+1},x_n)H + K$ for every $n \in \mathbb{N}$ one has $f(x_n) \in f(x) + d(x,x_n)H + K$ for every $n \in \mathbb{N}$.

In the case $H := \{k^0\}$ condition (H9) is nothing else than condition (E1) in [41].

Using Theorem 11.13 and Proposition 11.7 we have the following variant of the preceding result.

Theorem 11.14. *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subseteq Y$ is a proper closed convex cone. Let $f : X \to Y^\bullet$ be a proper function and $H \subseteq K$ be a nonempty cs-complete bounded set with $\mathbf{0} \notin \operatorname{cl}(H+K)$. If*

(H10) For every sequence $(x_n) \subseteq \operatorname{dom} f$ such that $x_n \to x \in X$ and $(f(x_n))$ is $\leq_K$-decreasing one has $f(x) \leq_K f(x_n)$ for every $n \in \mathbb{N}$.
(B10) $f(\operatorname{dom} f)$ is quasi bounded.

hold, then for every $x_0 \in \operatorname{dom} f$ there exists $\overline{x} \in \operatorname{dom} f$ such that:

(a) $(f(x_0) - K) \cap (f(\overline{x}) + d(\overline{x},x_0)H) \neq \emptyset$
(b) $(f(\overline{x}) - K) \cap (f(x) + d(\overline{x},x)H) = \emptyset \quad \forall x \in \operatorname{dom} f \setminus \{\overline{x}\}$

Proof. Since condition (H10) is exactly condition (H1) for $\mathscr{A} := \operatorname{gph} f$ and $F = F_H$, in order to have the conclusion of the theorem it is sufficient to show that (H2) is verified for this situation; then just use Proposition 11.7 and Theorem 11.13.

Let $((x_n, y_n)) \subseteq \operatorname{gph} f$ be such that $x_n \to x \in X$ and (y_n) is $\leq_K$-decreasing. Hence $y_n = f(x_n)$ for every n. By (H10) we have that $y := f(x) \leq_K f(x_n) = y_n$ for every $n \in \mathbb{N}$ and, of course, $(x, f(x)) \in \operatorname{gph} f$. The proof is complete. □

Remark 11.13. Taking H to be complete, convex and bounded, then H is cs-complete. In this case we obtain the main result in [8], that is, [8, Theorem 4.1].

Note that the closed convex subsets as well as the open convex subsets of a separated locally convex space are cs-closed; moreover, all the convex subsets of finite dimensional normed spaces are cs-closed (hence cs-complete).

Remark 11.14. Taking $H := \{k^0\}$ in the preceding theorem one obtains practically [32, Corollary 3.10.6]; there K is assumed to be closed in the direction k^0, the present condition (H10) being condition (H4) in [32, Corollary 3.10.6].

Remark 11.15. Similar results can be stated using Theorems 11.8 and 11.9. When specializing to $F(x_1,x_2) = \{d(x_1,x_2)k^0\}$ and $F'(y_1,y_2) = \{\varepsilon \|y_1 - y_2\| k^0\}$ one recovers [41, Corollary 3.1] and [41, Theorem 4.2].

11.5 Applications in Vector Optimization

11.5.1 Solution Concepts

Consider the vector minimization problem (VP) given as

$$V - \min f(x), \quad \text{s.t.} \quad x \in S,$$

where X and Y are separated locally convex spaces, $\{\mathbf{0}\} \neq K \subseteq Y$ is a closed convex cone (which induces the partial order $\leq_K$ on Y), $f : X \to Y$ and $S \subseteq X$. As in the preceding sections $k^0 \in K \setminus (-K)$ is fixed. The solution concepts for the vector optimization problem (VP) are described in the next definition.

Definition 11.1.

- The element $y_0 \in F \subseteq Y$ is said to be a *minimal point* of F with respect to K if $F \cap (y_0 - K) \subseteq y_0 + K$. The set of minimal points of F with respect to K is denoted by $\operatorname{Eff}(F,K)$. An element $x_0 \in S$ is called an *efficient solution* of (VP) if $f(x_0) \in \operatorname{Eff}(f(S),K)$.
- The element $y_0 \in F$ is said to be a *properly minimal point* of F w.r.t. K if there is a closed convex set $A \subseteq Y$ with $\mathbf{0} \in \operatorname{bd} A$ and $A - (K \setminus \{\mathbf{0}\}) \subseteq \operatorname{int} A$ such that $F \cap (y_0 + \operatorname{int} A) = \emptyset$. An element $x_0 \in S$ is called a *properly efficient* solution for (VP) if $f(x_0)$ is a properly minimal point of $f(S)$.
- The element $y_0 \in F$ is said to be a *weakly minimal point* of F if $\operatorname{int} K \neq \emptyset$ and $F \cap (y_0 - \operatorname{int} K) = \emptyset$. The set of weakly minimal points of F is denoted by $w\operatorname{Eff}(F,K)$. An element $x_0 \in S$ is a *weakly efficient solution* of (VP) if $f(x_0) \in w\operatorname{Eff}(f(S),K)$.

Note first that from the very definition of weakly minimal points of F one has

$$w\,\mathrm{Eff}(F,K) = F \setminus (F + \mathrm{int}\,K); \tag{11.40}$$

then observe that the set A appearing in the definition of a properly minimal point verifies Assumption $(A2)$. Moreover, note that what is called here a properly minimal point of F w.r.t. K is said to be an E-minimal element of F in [66] and was introduced by Iwanow and Nehse in [47] for $K = \mathbb{R}^n_+$ and Gerstewitz and Iwanow [28] for the general case.

Lemma 11.1. *Let $x_0 \in S$.*

(a) If x_0 is a properly efficient solution of (VP) and $A \subseteq Y$ is the set provided by Definition 11.1 then x_0 is a solution of the scalar minimization problem

$$\min\ \varphi_{A,k^0}(f(x) - f(x_0)) \quad s.t. \quad x \in S, \tag{11.41}$$

where $k^0 \in K \setminus \{0\}$.

(b) If x_0 is a weakly efficient solution of (VP) and $k^0 \in \mathrm{int}\,K$, then x_0 is a solution of problem (11.41) with $A := -K$.

Proof. In both cases we have that $f(S) \cap (f(x_0) + \mathrm{int}\,A) = \emptyset$. Moreover, because $0 \in \mathrm{bd}\,A$, we have that $\varphi_A(0) = 0$. Assuming that $\varphi_{A,k^0}(f(x) - f(x_0)) < \varphi_{A,k^0}(f(x_0) - f(x_0)) = 0$, we get the contradiction $f(x) - f(x_0) \in \mathrm{int}\,A$. □

11.5.2 Necessary Optimality Conditions in Vector Optimization

We consider vector optimization problems on Asplund spaces without convexity assumptions. Recall that a Banach space X is said to be an Asplund space (cf. Phelps [53, Definition 1.22]) if every continuous convex function defined on a nonempty open convex subset D of X is Fréchet differentiable at each point of some G_δ subset of D. It is known that the Banach spaces with separable dual and the reflexive Banach spaces are Asplund spaces. So c_0 and ℓ^p, $L^p[0,1]$ for $1 < p < \infty$ are Asplund spaces, but ℓ^1 is not an Asplund space.

Under the assumption that the objective function is locally Lipschitz we derive Lagrangian necessary conditions on the basis of Mordukhovich subdifferential using the Lipschitz continuity properties of φ_A discussed in Sect. 11.3.4. In the following we provide necessary conditions for properly efficient solutions of a vector optimization problem that are related to the strong free-disposal assumption in $(A2)$.

In order to present our results concerning the existence of Lagrange multipliers, we work with the Mordukhovich subdifferential ∂_M and normal cone N_M (denoted ∂ and N in [51]). One says ([51, Definition 3.25]) that a function $f : X \to Y$ is strictly Lipschitz at $\bar{x}$ if f is Lipschitz on a neighbourhood of $\bar{x}$ and there exists a neighbourhood U of the origin in X such that the sequence $(t_k^{-1}(f(x_k+t_ku)-f(x_k)))_{k\in\mathbb{N}}$ contains a (norm) convergent subsequence whenever $u\in U$, $x_k\to\bar{x}$ and $t_k\downarrow 0$. It is clear that this notion reduces to local Lipschitz continuity if Y is finite dimensional.

Remark 11.16. The function f is strictly Lipschitz at $\bar{x}$ if and only if the sequence $(\|u_n\|^{-1}[f(x_n+u_n)-f(x_n)])$ has a norm converging subsequence whenever $(x_n)\subseteq X$ converges to $\bar{x}$, $(u_n)\subseteq X\setminus\{0\}$ converges to 0 and the sequence $(\|u_n\|^{-1}u_n)$ converges in X.

For more details regarding the class of strictly Lipschitz mappings (with values in infinite dimensional spaces) see [51, Sect. 3.1.3].

We need the following calculus rules from [51] (see [51, Theorem 3.36] and [51, Corollary 3.43]) for proving one of our main results.

Lemma 11.2. *Assume that X and Y are Asplund spaces.*

(a) If $f_1,f_2:X\to\overline{\mathbb{R}}$ are proper functions and there exists a neighbourhood U of $\bar{x}\in\operatorname{dom}f_1\cap\operatorname{dom}f_2$ such that f_1 is Lipschitz on U and f_2 is lsc on U, then

$$\partial_M(f_1+f_2)(\bar{x})\subseteq\partial_M f_1(\bar{x})+\partial_M f_2(\bar{x}).$$

(b) If $f:X\to Y$ is strictly Lipschitz at $\bar{x}$ and $\varphi:Y\to\overline{\mathbb{R}}$ is finite and Lipschitz on a neighbourhood of $f(\bar{x})$, then

$$\partial_M(\varphi\circ f)(\bar{x})\subseteq\bigcup\{\partial_M(y^*\circ f)(\bar{x})\mid y^*\in\partial_M\varphi(f(\bar{x}))\}.$$

In the next result we provide necessary optimality conditions for properly efficient solutions of problem (VP).

Theorem 11.15. *Assume that X and Y are Asplund spaces, $f:X\to Y$ is strictly Lipschitz, S is a closed subset of X and $x_0\in S$. If x_0 is a properly efficient solution of (VP) then there exists $v^*\in K^\#$ such that*

$$0\in\partial_M(v^*\circ f)(x_0)+N_M(S,x_0). \tag{11.42}$$

Moreover, if f is strictly differentiable at x_0 then

$$(f'(x_0))^*v^*\in -N_M(S,x_0). \tag{11.43}$$

Proof. Assume that x_0 is a properly efficient solution for (VP). By Lemma 11.1, x_0 is a solution of the problem (11.41), or equivalently x_0 is a minimum point of

$$h:X\to\overline{\mathbb{R}},\quad h(x):=\varphi_A(f(x)-f(x_0))+\iota_S(x),$$

where $A\subseteq Y$ is a closed convex set such that $\mathbf{0}\in\operatorname{bd}A$ and $A-(K\setminus\{\mathbf{0}\})\subseteq\operatorname{int}A$. As seen in Remark 11.2 (ii), $\operatorname{dom}\varphi_A$ is open because $k^0\in K\setminus\{\mathbf{0}\}$ and

$\varphi_A(0) = 0$ because $0 \in \operatorname{bd} A$. Note that $k^0 \notin A_\infty$; otherwise we get the contradiction $0 = (0 + k^0) - k^0 \in A - \mathbb{P}k^0 \subseteq \operatorname{int} A$. Since $\mathbf{0} \in A \subseteq \operatorname{dom} \varphi_A$, by Proposition 11.5 we have that φ_A is convex and Lipschitz on a neighbourhood of 0. It follows that $0 \in \partial_M h(x_0)$ (see [51, Proposition 1.114]). Since f is strictly Lipschitz and φ_A is Lipschitz on a neighbourhood of 0, the function $x \mapsto \varphi_A(f(x) - f(x_0))$ is Lipschitz on a neighbourhood of x_0. Moreover, since S is a closed subset of X we have that ι_S is a proper lower-semicontinuous function. Using both parts of Lemma 11.2 we have that

$$0 \in \partial_M (v^* \circ f)(x_0) + N_M(S, x_0)$$

for some $v^* \in \partial_M \varphi_A(0) = \partial \varphi_A(0)$, φ_A being convex and finite and Lipschitz on a neighbourhood of 0. From Corollary 11.7 we have that $v^* \in K^\#$ and $\langle k^0, v^* \rangle = 1$, and so $v^* \neq 0$. If f is strictly differentiable at x_0 then $\partial_M f(x_0) = \{f'(x_0)\}$, and so the last conclusion follows. □

For weakly efficient solutions of (VP) we have the following result.

Theorem 11.16. *Assume that X and Y are Asplund spaces, $f : X \to Y$ is strictly Lipschitz, S is a closed subset of X and $x_0 \in S$. If x_0 is a weakly efficient solution of (VP) then there exists $v^* \in K^+ \setminus \{\mathbf{0}\}$ such that (11.42) holds. Moreover, if f is strictly differentiable at x_0 then (11.43) holds.*

Proof. If $\operatorname{int} K \neq \emptyset$ and x_0 is a weakly efficient solution for (VP), by Lemma 11.1 we have that x_0 is a minimum point of h for $A := -K$ and $k^0 \in \operatorname{int} K$. This time φ_A is Lipschitz and sublinear. The rest of the proof is similar. □

Remark 11.17. If X is an Asplund space and $g : X \to \overline{\mathbb{R}}$ is finite and Lipschitz on a neighbourhood of $x_0 \in S \subseteq X$ with S closed, the following well-known relations

$$\partial_{Cl} g(x_0) = \overline{\operatorname{conv}}^{w^*} \partial_M g(x_0) \text{ and } N_{Cl}(S, x_0) = \overline{\operatorname{conv}}^{w^*} N_M(S, x_0)$$

hold (see [51, Theorem 3.57]), where $\partial_{Cl} g(x_0)$ and $N_{Cl}(S, x_0)$ represent the Clarke's subdifferential of g at x_0 and the Clarke's normal cone of S at x_0, respectively. In the hypotheses of Theorem 11.15 from (11.42) we get the necessary optimality condition

$$\exists v^* \in K^+ \setminus \{0\} \;:\; 0 \in \partial_{Cl}(v^* \circ f)(x_0) + N_{Cl}(S, x_0) \tag{11.44}$$

in terms of the Clarke's subdifferential and normal cone. However, the optimality condition given by (11.42) is sharper than the condition given by (11.44).

Remark 11.18. Note that Theorems 11.15 and 11.16 remain valid when the Mordukhovich subdifferential ∂_M is replaced by any subdifferential ∂ which verifies conditions (H1)–(H4) in [17]. In such a situation Theorem 11.16 corresponds to Lagrangian necessary condition for weakly efficient solutions in [17, Theorem 3.1] (compare also [20, Theorem 3.2] for the case $\dim Y < \infty$). In Theorem 11.15 we have established the result for properly efficient solutions without assuming $\operatorname{int} K \neq \emptyset$.

Another application envisage fuzzy necessary optimality conditions for approximate minimizers of a Lipschitz vectorial function (compare Durea and Tammer [17]). First, we need a definition.

Definition 11.2. If $\alpha > 0$ and $k^0 \in \operatorname{int} K$, a point $x_0 \in S$ is said to be (α, k^0)-*efficient solution* of (VP) if $(f(S) - f(x_0)) \cap (-\alpha k^0 - K) = \emptyset$.

Of course, every weakly efficient solution of (VP) is a (α, k^0)-efficient solution for every $\alpha > 0$ and $k^0 \in \operatorname{int} K$, but the converse is false, in general.

We introduce now the concept of abstract subdifferential (see, e.g. [43]; see also [19] for a theory of subdifferentials for vector-valued functions). Let $\mathscr{X}$ be a class of Banach spaces which contains the class of finite dimensional normed vector spaces. By an abstract subdifferential ∂ we mean a map which associates to every lsc function $h : X \in \mathscr{X} \to \overline{\mathbb{R}}$ and to every $x \in X$ a (possible empty) subset $\partial h(x) \subseteq X^*$; $\partial h(x) = \emptyset$ if $f(x) \notin \mathbb{R}$. Let $X, Y \in \mathscr{X}$ and denote by $\mathscr{F}(X,Y)$ a class of functions acting between X and Y having the property that by composition at left with a lsc function from Y to $\overline{\mathbb{R}}$ the resulting function is still lsc. In the sequel we shall work in every specific case with some of the next properties of the abstract subdifferential ∂.

(C1) If $h : X \to \overline{\mathbb{R}}$ is a proper lsc convex function then $\partial h(x)$ coincides with the Fenchel subdifferential.

(C2) If $x \in X$ is a local minimum point for the lsc function h and $h(x) \in \mathbb{R}$ then $0 \in \partial h(x)$.

Note that (C1) and (C2) are very natural requirements for any subdifferential.

The counterparts of "exact calculus rules" are the far more general "fuzzy calculus rules".

(C3) If $X \in \mathscr{X}$, $\varphi : X \to \mathbb{R}$ is a locally Lipschitz functions and $x \in \operatorname{dom} h$, then

$$\partial(h+\varphi)(x) \subseteq \|\cdot\|^* - \limsup_{y \to_h x,\, z \to x} (\partial h(y) + \partial \varphi(z)),$$

(C4) If $\varphi : Y \to \mathbb{R}$ is locally Lipschitz and $\psi \in \mathscr{F}(X,Y)$, then for every x,

$$\partial(\varphi \circ \psi)(x) \subseteq \|\cdot\|^* - \limsup_{u \to_\psi x,\, v \to \psi(x)} \bigcup_{u^* \in \partial \varphi(v)} \partial(u^* \circ \psi)(u).$$

where the following notations are used:

1. $u \to_h x$ means that $u \to x$ and $h(u) \to h(x)$; note that if h is continuous, then $u \to_h x$ is equivalent with $u \to x$.
2. $x^* \in \|\cdot\|^* - \limsup_{u \to x} \partial h(u)$ means that for every $\varepsilon > 0$ there exist x_ε and x_ε^* such that $x_\varepsilon^* \in \partial h(x_\varepsilon)$ and $\|x_\varepsilon - x\| < \varepsilon$, $\|x_\varepsilon^* - x^*\| < \varepsilon$; the notation $x^* \in \|\cdot\|^* - \limsup_{u \to_h x} \partial h(u)$ has a similar interpretation and is equivalent with $x^* \in \|\cdot\|^* - \limsup_{u \to x} \partial h(u)$ provided that h is continuous.

The property (C3) is called fuzzy sum rule and a space X on which such a property holds is called trustworthiness space for the subdifferential ∂. For example, for the Fréchet subdifferential the trustworthiness spaces are the Asplund spaces (see [23]). This rule is also satisfied (see [48, pp. 41], [18, 44] and the references therein) by:

- The proximal subdifferential when $\mathscr{X}$ is the class of Hilbert spaces.
- The Fréchet subdifferential of viscosity when $\mathscr{X}$ is the class of Banach spaces which admit a C^1 Lipschitz bump function.
- The β-subdifferential of viscosity when $\mathscr{X}$ is the class of Banach spaces which admit a β-differentiable bump function.

The next result goes back to Durea and Tammer [17].

Theorem 11.17. *Let $X,Y \in \mathscr{X}$, let $f \in \mathscr{F}(X,Y)$ be locally Lipschitz and let S be a closed subset of X. Let $x_0 \in S$ be a weakly efficient solution of (VP). Then for every $k^0 \in \operatorname{int} K$ and $\varepsilon > 0$ there exist $u \in B(x_0,\varepsilon)$, $z \in B(x_0,\varepsilon/2) \cap S$ and $u^* \in K^+$ with $u^*(k^0) = 1$ such that*

$$0 \in \partial(u^* \circ f)(u) + N_\partial(S,z) + B(0,\varepsilon),$$

provided that ∂ satisfies conditions (C1), (C2), (C3), (C4). Moreover, for some $x \in B(x_0,\varepsilon/2)$ and $v \in B(f(x)-f(x_0),\varepsilon/2)$ we have that $u^*(v) = \varphi(v)$.

Proof. Let us consider $\varepsilon > 0$ and the functional φ given by (11.5) corresponding to a fixed $k^0 \in \operatorname{int} K$. We have that

$$f(x_0) \in w\mathrm{Eff}(f(S),K)$$

which means that

$$0 \in w\mathrm{Eff}(f(S)-f(x_0),K).$$

Thus, $\varphi(0) = 0$ and $\varphi(f(S)-f(x_0)) \geq 0$, whence x_0 is a minimum point for $(\varphi \circ g) + \iota_S$, where g is defined by $g(x) = f(x) - f(x_0)$. From (C2) we get

$$0 \in \partial(\varphi \circ g + \iota_S)(x_0)$$

and from (C3) (φ is Lipschitz, g is locally Lipschitz and ι_S is lsc because S is closed), there exist $x \in B(x_0,\varepsilon/2)$, $z \in B(x_0,\varepsilon/2) \cap S$, $p^* \in \partial(\varphi \circ g)(x)$, and $q^* \in N_\partial(S,z)$ such that

$$\|p^* + q^*\| < \varepsilon/2.$$

Since $p^* \in \partial(\varphi \circ g)(x)$, by (C4) there exist $u_1 \in B(x,\varepsilon/3) \subseteq B(x_0,5\varepsilon/6)$, $v \in B(g(x),\varepsilon/2)$, $u^* \in \partial\varphi(v)$ and $v_1^* \in \partial(u^* \circ g)(u_1)$ such that

$$\|v_1^* - p^*\| < \varepsilon/2.$$

It follows that

$$\|v_1^* + q^*\| = \|v_1^* - p^* + p^* + q^*\| < 5\varepsilon/6.$$

This means that

$$0 \in \partial(u^* \circ g)(u_1) + N_\partial(S,z) + B(0, 5\varepsilon/6).$$

But

$$\partial(u^* \circ g)(u_1) = \partial(u^* \circ (f(\cdot) - f(x_0)))(u_1) = \partial(u^* \circ f)(u_1)$$

because the function $u \mapsto -u^*(f(x_0))$ is constant (in particular convex). Applying (C3) we find $u \in B(u_1, \varepsilon/6) \subseteq B(x_0, \varepsilon)$ and $v^* \in \partial(u^* \circ f)(u)$ such that

$$\|v_1^* - v^*\| < \varepsilon/6.$$

We deduce that

$$0 \in \partial(u^* \circ f)(u) + N_\partial(S,z) + B(0,\varepsilon).$$

The assertions concerning u^* follow from Corollary 11.7 and this completes the proof. □

Concerning (α, k^0)-efficient solutions of (VP) we have the following result (compare Durea and Tammer [17]).

Theorem 11.18. *Assume that S is a closed subset of X and f is a λ-Lipschitz function. Let $x_0 \in S$ be an (α, k^0)-efficient solution of (VP). Then for every $e \in \operatorname{int} K$ and $\varepsilon > 0$, there exist $u \in B(x_0, \sqrt{\alpha} + \varepsilon)$, $z \in B(x_0, \sqrt{\alpha} + \varepsilon/2) \cap S$, $u^* \in K^+$ with $u^*(e) = 1$ and $x^* \in X^*$ with $\|x^*\| \leq 1$ such that*

$$0 \in \partial(u^* \circ f)(u) + \sqrt{\alpha} u^*(k^0) x^* + N_\partial(S,z) + B(0,\varepsilon),$$

provided that ∂ satisfies conditions (C1), (C2), (C3), (C4). Moreover, for some $x \in B(x_0, \sqrt{\alpha} + \varepsilon/2)$ and $v \in B(f(x) - f(x_0), \lambda\sqrt{\alpha} + \varepsilon)$ one has $u^*(v) = \varphi(v)$.

Proof. Since the function f is Lipschitz, it is continuous as well, and since S is a closed set in the Banach space X, S is a complete metric space with respect to the metric given by the norm. Thus, it is easy to see that we are in the conditions of the vectorial variant of Ekeland principle given in Theorem 11.13. Applying this result we get an element $\bar{x} \in S$ such that $\|\bar{x} - x_0\| < \sqrt{\alpha}$ and having the property that it is minimal element (whence weak minimal as well) over S for the function h defined by

$$h(x) := f(x) + \sqrt{\alpha}\, \|x - \bar{x}\|\, k^0.$$

Let $\varepsilon > 0$. One can apply now Theorem 11.17 for ε replaced $\delta \in]0, \varepsilon/2[$ with $\delta(1 + \sqrt{\alpha}\left\|k^0\right\|\delta) < 2\varepsilon$. Accordingly, we can find $\bar{u} \in B(\bar{x}, \delta) \subseteq B(x_0, \sqrt{\alpha} + \delta)$, $x \in B(\bar{x}, \delta/2) \subseteq B(x_0, \sqrt{\alpha} + \delta/2)$, $v \in B(h(x) - h(\bar{x}), \delta/2)$, $z \in B(\bar{x}, \delta/2) \cap S \subseteq B(x_0, \sqrt{\alpha} + \delta/2) \cap S$ and $u^* \in \partial\varphi(v)$ such that

$$0 \in \partial(u^* \circ h)(\bar{u}) + N_\partial(S,z) + B(0,\delta). \tag{11.45}$$

Let us take the element $\overline{x}^* \in \partial(u^* \circ h)(\overline{u})$ involved in (11.45). Since

$$\partial(u^* \circ h)(\overline{u}) = \partial(u^* \circ (f(\cdot) + \sqrt{\alpha}\, \|\cdot - \overline{x}\|\, k^0))(\overline{u}),$$

by use of (C3) and (C1), there exist $u \in B(\overline{u}, \delta) \subseteq B(x_0, \sqrt{\alpha} + 2\delta)$ and $u' \in B(\overline{u}, \delta)$ such that

$$\overline{x}^* \in \partial(u^* \circ f)(u) + \sqrt{\alpha} u^*(k^0)\partial(\|\cdot - \overline{x}\|)(u') + B(0, \delta). \tag{11.46}$$

By the calculation rule for the subdifferential of the norm and combining relations (11.45) and (11.46) it follows that there exists $x^* \in X^*$ with $\|x^*\| \leq 1$ such that

$$0 \in \partial(u^* \circ f)(u) + \sqrt{\alpha} u^*(k^0) x^* + N_\partial(S, z) + B(0, 2\delta).$$

Since $2\delta < \varepsilon$, it remains only to prove the estimation about the ball containing v. We can write

$$\begin{aligned} \|v - (f(x) - f(x_0))\| &\leq \|v - (h(x) - h(\overline{x}))\| + \|(h(x) - h(\overline{x})) - (f(x) - f(x_0))\| \\ &\leq \delta/2 + \left\|\sqrt{\alpha} k^0 \, \|x - \overline{x}\| - f(\overline{x}) + f(x_0))\right\| \\ &\leq \delta/2 + \sqrt{\alpha} \left\|k^0\right\| \delta/2 + \lambda\sqrt{\alpha} \\ &< \lambda\sqrt{\alpha} + \varepsilon, \end{aligned}$$

where for the last inequality we used the assumptions made on δ. The proof is complete. □

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Chapter 12
The Fermat Rule and Lagrange Multiplier Rule for Various Efficient Solutions of Set-Valued Optimization Problems Expressed in Terms of Coderivatives

Truong Xuan Duc Ha

12.1 Introduction

Recently, the interest toward set-valued optimization problems, i.e., optimization problems with set-valued objective and/or constrain maps, has grown up. Exploiting advanced tools of variational analysis and set-valued analysis, the authors obtained a number of (necessary and/or sufficient) optimality conditions expressed in terms of generalized derivatives such as contingent derivative [2, 13, 62, 66], first-order derivative [14, 15], second-order derivative [28, 40], epiderivative and generalized epiderivative [10, 22, 24, 39, 43, 44], subgradient/subdifferential [4, 5, 39, 67] and coderivatives [2, 4, 5, 18, 27–29, 33, 39, 72, 73].

In this chapter, we summarize several versions of the Fermat rule and the Lagrange multiplier rule for various efficient solutions of set-valued optimization problems, which are expressed in terms of coderivatives in the senses of Fréchet, Ioffe, Clarke and Mordukhovich and provide illustrating examples.

First, we recall fuzzy and exact versions containing *necessary conditions* for Pareto efficient solutions established by Zheng and Ng [72, 73], and Bao and Mordukhovich [5] with the help of the extremal principle of variational analysis and its modifications. We provide an example showing that these necessary conditions do not become sufficient even under convexity additional assumptions and in finite-dimensional setting.

Second, we obtain in a unified scheme the Fermat rule and the Lagrange multiplier rule containing *both necessary and sufficient conditions* (the sufficient conditions require additional convexity assumptions) for strongly efficient solutions,

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weakly efficient solutions, positive properly efficient solutions, Hurwicz properly efficient solutions, Henig global properly efficient solutions, Henig properly efficient solutions, super efficient solutions and Benson properly efficient solutions. Our new unified scalarization approach to the study of these solutions develops the one proposed in the author's recent works [27–29] and is based on the fact that the mentioned solutions can be characterized as solutions to appropriately chosen scalar optimization problems with the objective functions being linear functionals or the Hiriart-Urruty signed distance function or a Minkowski-type function.

This chapter is organized as follows. In Sect. 12.2, we recall some tools from variational analysis such as the concepts of normal cone, subdifferential, coderivative, metric regularity, pseudo-Lipschizity and the extremal principle. In Sect. 12.3, we recall the concepts of efficient points of sets and efficient solutions of set-valued optimization problems and we discuss the mentioned above unified scalarization approach. Section 12.4 is devoted to the Fermat rule and Sect. 12.5 is devoted to the Lagrange multiplier rule.

12.2 Some Tools from Variational Analysis

For the convenience of the reader we repeat the relevant material from [1, 7, 8, 12, 18, 35–38, 47, 50–59, 63, 64, 68] without proofs, thus making our exposition self-contained. Namely, we recall the concepts of normal cone, subdifferential and coderivative in the senses of Fréchet, Ioffe, Clarke and Mordukhovich, sequential normal compactness, pseudo-Lipschitzity, metric regularity, the Clarke penalization and the extremal principle.

Throughout the chapter, unless otherwise stated, X, Y and Z are Banach spaces with their dual X^*, Y^* and Z^*, respectively. We denote by $\langle\cdot,\cdot\rangle$ the dual pairing between a space and its dual and use the same notation $\|.\|$ for the norms in all these spaces. The closed unit ball and the open unit ball in any space, say X, are denoted by $I\!B_X$ and $\mathring{I\!B}_X$; we omit the subscript X when no confusion occurs. For a nonempty set $A \subset X$, int$A$ and cl$A$ stand for the interior and closure of A, cl* stands for weak-star (weak*) closure of A and cone$A := \{ta : t \in \mathbb{R}_+, a \in A\}$, where $\mathbb{R}_+ = [0,\infty[$. Further, $d(x;A)$ is the distance from x to A and $\chi_A(x)$ is the indicator function associated to A, i.e., $\chi_A(x) = 0$ if $x \in A$ and $\chi_A(x) = \infty$ otherwise. Recall that a Banach space is *Asplund* if each of its separable subspace has a separable dual. This class of spaces has been comprehensively investigated in geometric theory of Banach spaces and has been largely employed in variational analysis; see, e.g. [54, 55]. Examples of Asplund spaces are the Banach spaces $\mathbb{R}^n$, $L^p_{[0,1]}$ and l^p $(1 < p < \infty)$.

Assume that $g : X \mapsto \mathbb{R}\cup\{\infty\}$ is a function and $F : X \mapsto 2^Y$ is a set-valued map (for the sake of convenience we assume that $F(x)$ is nonempty for all $x \in X$). The domain, epigraph of g and the graph of F are the sets dom$g = \{x \in X \mid g \text{ is finite at } x\}$, epi$g = \{(x,t) \in X \times \mathbb{R} \mid g(x) \le t\}$ and gr$F = \{(x,y) \in X \times Y \mid$

$y \in F(x)\}$, respectively. In what follows, we say that g is *Lipschitz (of rank L) on some set* $A \subseteq X$ if

$$|g(x) - g(x')| \leq L\|x - x'\|$$

for all x, x' in A and g is *Lipschitz near* $\overline{x} \in X$ (or *locally Lipschitz*) if it is Lipschitz on some closed ball centered at $\overline{x}$.

12.2.1 Normal Cone, Subdifferential and Coderivative in the Senses of Fréchet, Ioffe, Clarke and Mordukhovich

In this subsection, we recall the construction for defining normal cones, subdifferentials and coderivatives in the senses of Fréchet, Ioffe, Clarke and Mordukhovich and then we list some main facts about these concepts. Note that the subdifferentials in the senses of Clarke and Ioffe have been defined firstly for a locally Lipschitz function, then the corresponding normal cones have been defined through the subdifferentials of the distance function and finally, these subdifferentials have been extended to a lower semicontinuous (l.s.c.) function through the normal cones to its epigraph, see [12, 37]. Meanwhile, the normal cones in the senses of Fréchet and Mordukhovich have been defined first and the corresponding concepts of subdifferential for a l.s.c. function have been defined later through the normal cones to its epigraph.

Recall that given a locally Lipschitz function g and $x \in \text{dom}g$, the *Ioffe approximate subdifferential* of g at x [35–37] is the set

$$\partial_A g(x) = \bigcap_{L \in \mathscr{F}} \limsup_{(\varepsilon, y) \to (0^+, x)} \partial_\varepsilon^- g_{y+L}(y),$$

where $\mathscr{F}$ is the collection of all finite dimensional subspaces of X, $g_{y+L}(u) = g(u)$ if $u \in y + L$ and $g_{y+L}(u) = +\infty$ otherwise, for $\varepsilon \geq 0$

$$\partial_\varepsilon^- g_{y+L}(y) = \{x^* \in X^* \mid x^*(v) \leq \varepsilon\|v\| + \\ + \liminf_{t \to 0^+} t^{-1}[g_{y+L}(y + tv) - g_{y+L}(y)], \forall v \in X\}$$

and the *Clarke generalized subdifferential* of g at x [12] is the set

$$\partial_C g(x) = \{x^* \in X^* \mid x^*(v) \leq g^0(x; v),\ \forall v \in X\},$$

where $g^0(x; v)$ is the generalized directional derivative of g at x in the direction v

$$g^0(x; v) = \limsup_{y \to x,\ t \to 0^+} \frac{g(y + tv) - g(y)}{t}.$$

Let Ω be a nonempty subset of X different from X and $x \in \mathrm{cl}\Omega$. The *Ioffe approximate normal cone* to Ω at x [35–37] is given by

$$N_A(x;\Omega) = \bigcup_{\lambda>0} \lambda \partial_A d(x;\Omega)$$

and the *Clarke normal cone* to Ω at x [12] is given by

$$N_C(x;\Omega) = \mathrm{cl}^* \left(\bigcup_{\lambda>0} \lambda \partial_C d(x;\Omega) \right).$$

To define the Mordukhovich normal cone, recall that given a scalar $\varepsilon > 0$, the set of ε*-normals* to Ω at x is given by

$$\hat{N}_\varepsilon(x;\Omega) = \left\{ x^* \in X^* \;\middle|\; \limsup_{x' \xrightarrow{\Omega} x} \frac{x^*(x'-x)}{\|x'-x\|} \leq \varepsilon \right\}. \tag{12.1}$$

When $\varepsilon = 0$, the elements of (12.1) are called *Fréchet normals* and their collection, denoted by $N_F(x;\Omega)$, is the *Fréchet normal cone* to Ω at x; in other words,

$$N_F(x;\Omega) = \left\{ x^* \in X^* \;\middle|\; \limsup_{x' \xrightarrow{\Omega} x} \frac{x^*(x'-x)}{\|x'-x\|} \leq 0 \right\}. \tag{12.2}$$

The *Mordukhovich normal cone* to Ω at x [50–52, 54, 55] is defined by

$$N_M(x;\Omega) = \limsup_{x' \xrightarrow{\Omega} x,\ \varepsilon \to 0^+} \hat{N}_\varepsilon(x';\Omega), \tag{12.3}$$

where the limit in the right-hand side means the sequential Kuratowski–Painlevé upper limit with respect to the norm topology in X and the weak-star ω^* topology in X^*, $x' \xrightarrow{\Omega} x$ refers to all sequences converging to x which remain in Ω. When X is Asplund, (12.3) takes the following simple form

$$N_M(x;\Omega) = \limsup_{x' \xrightarrow{\Omega} x} N_F(x';\Omega). \tag{12.4}$$

Now assume that the function g is lower semicontinuous and the set-valued map F is closed (i.e., its graph is a closed set). Denote the subdifferentials of g in the senses of Fréchet, Ioffe, Clarke and Mordukhovich respectively by ∂_F, ∂_A, ∂_C, ∂_M and the coderivatives of F in the senses of Fréchet, Ioffe, Clarke and Mordukhovich respectively by D_F^*,D_A^*, D_C^*, D_M^*. For the sake of convenience, we make the *convention* that, unless otherwise specified, the same notations N, ∂g and D^*F are used for the normal cones, the subdifferentials and the coderivatives in the above senses or in the sense of convex analysis (in the convex case) and that while concerning with the Mordukhovich normal cone and related to it concepts, we restrict ourselves to the Asplund space setting, where they enjoy full calculus, see [50–58]. The subdifferential of g and the coderivative of F are defined through the corresponding normal cone as follows

$$\partial g(x) = \{x^* \in X^* \mid (x^*, -1) \in N((x, g(x)); \mathrm{epi} g)\}$$

for any $x \in \mathrm{dom} g$ and

$$D^* F(x,y)(y^*) = \{x^* \in X^* \mid (x^*, -y^*) \in N((x,y); \mathrm{gr} F)\}$$

for any $(x,y) \in \mathrm{gr}F$ and $y^* \in Y^*$. When F is single-valued, we write $D^*F(x)$ instead of $D^*F(x,y)$.

Remark 12.1. (a) Note that in [50] Mordukhovich introduced the notion of coderivatives of a set-valued map regardless of the normal cone used. After he suggested this approach to differentiability of maps, we may consider different specific coderivatives for set-valued maps generated by different normal cones to their graphs. The Mordukhovich coderivative and the Mordukhovich normal cone are also called limiting Fréchet coderivative and limiting Fréchet normal cone. The Mordukhovich coderivative related to a normal cone in a finite dimensional space was introduced in [50] and was extended to Banach spaces in [47]. The Mordukhovich coderivative has been further developed to its full and comprehensive calculus and has been used in the study of optimal control, differential inclusions, scalar and vector optimization, economics..., see [54,55]. Coderivative for single- or set-valued maps has been proven to be the right tool for formulation of optimality conditions, see [2–5,16,18,27–29,54,55,59,72,73] and the references therein.

(b) As noted in [54, p. 143], Fréchet had nothing to do with the Fréchet normals, normal cone, subdifferential and coderivative, and we keep these names to emphasize parallels with the classical differentiation, where the Fréchet derivative is the basic tool of nonlinear analysis.

(c) The coderivative in the sense of convex analysis, i.e. coderivative for a set-valued map with a convex graph was considered in [2].

(d) The approximate normal cone and the approximate subdifferential for a l.s.c. function presented above is termed as the G-nucleus of the G-normal cone and the G-nucleus of the G-subdifferential in [36]. For the calculus of the approximate coderivative see [37, 38].

(e) We mention that Clarke never introduced nor used any coderivative concept for either set-valued or single-valued maps, but the coderivative generated by the Clarke normal cone in the scheme of [51] as above has been used under the name “Clarke’s coderivative” in [52].

Next, we recall some results that will be used in the sequel, see [12, 35–37, 50–52, 54]. We begin with main properties and relations of the normal cones from which one can easily derive corresponding results for coderivatives.

Proposition 12.1. *Assume that Ω is a nonempty closed subset of X and $x \in \Omega$. Then*

(a) $N_F(x;\Omega) \subseteq N_M(x;\Omega) \subseteq N_A(x;\Omega) \subseteq N_C(x;\Omega)$.
(b) $N_M(x;\Omega) = N_A(x;\Omega)$ *in finite-dimensional spaces.*
(c) $N(x;\Omega) = \mathbb{R}_+\partial d(x;\Omega)$ *when the normal cone and the subdifferential are understood in the sense of Fréchet, Ioffe or Mordukhovich and* $N(x;\Omega) = \mathrm{cl}^*\mathbb{R}_+\partial d(x;\Omega)$ *when the normal cone and the subdifferential are understood in the sense of Clarke.*
(d) The normal cones in the senses of Fréchet and Clarke are convex; if Ω is convex, then all the considered normal cones reduce to the normal cone of convex analysis, i.e. to the set

$$\{x^* \in X^* \mid x^*(x' - x) \leq 0,\ \forall\, x' \in \Omega\}.$$

Some useful properties of the subdifferentials are collected in the following proposition.

Proposition 12.2. *Assume that* $g : X \mapsto \mathbb{R} \cup \{\infty\}$ *is a l.s.c. function. Then*

(a) If g is strictly differentiable near x then $\partial g(x) = \{g'(x)\}$.
(b) If g is Lipschitz near x, then for any scalar t, one has $\partial_C(tg)(x) = t\partial_C g(x)$.
(c) If g is convex and Lipschitz near x, then the above subdifferentials reduce to the subdifferential of convex analysis, i.e.,

$$\partial g(x) = \{x^* \in X^* \mid x^*(x' - x) \leq g(x') - g(x),\ \forall\, x' \in \mathrm{dom} g\}.$$

(d) If $g(x') \geq g(x)$ for all x' in a neighborhood of $x \in dom g$, then $\mathbf{0} \in \partial g(x)$.
(e) (sum rule) Assume that $h : X \mapsto R \cup \{+\infty\}$ *is Lipschitz near* $x \in \mathrm{dom} g \cap \mathrm{dom} h$, *then*

$$\partial(g + h)(x) \subseteq \partial g(x) + \partial h(x)$$

and the equality holds if at least one function is strictly differentiable near x.

Let us illustrate the above concepts by some examples.

Example 12.1. (a) Let $X = \mathbb{R}^2$. For $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq -|x_1|\}$ we have

$$\begin{aligned} N_C((0,0);\Omega) &= \{(x_1, x_2) \mid x_2 \leq -|x_1|\} \\ N_M((0,0);\Omega) &= \{(x_1, x_2) \mid x_2 = -|x_1|\} \\ N_A((0,0);\Omega) &= N_M((0,0);\Omega) \\ N_F((0,0);\Omega) &= \{(0,0)\}. \end{aligned}$$

(b) Let $X = \mathbb{R}^2$. For $\Omega := \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 = |x_1|\}$ we have

$$\begin{aligned} N_C((0,0);\Omega) &= \mathbb{R}^2 \\ N_M((0,0);\Omega) &= \{(x_1,x_2) \mid x_2 \le -|x_1|\} \cup \{(x_1,x_2) \mid x_2 = |x_1|\} \\ N_A((0,0);\Omega) &= N_M((0,0);\Omega) \\ N_F((0,0);\Omega) &= \{(x_1,x_2) \mid x_2 \le -|x_1|\}. \end{aligned}$$

(c) Let $X = \mathbb{R}$ and $g : \mathbb{R} \mapsto \mathbb{R}$ with $g(x) = -|x|$. Then it follows from (a) that

$$\begin{aligned} \partial_C g(0) &= [-1,1] \\ \partial_M g(0) &= \{-1,1\} \\ \partial_A g(0) &= \partial_M g(0) \\ \partial_F g(0) &= \emptyset. \end{aligned}$$

(d) Let $X = \mathbb{R}$, $Y = \mathbb{R}$ and $F : \mathbb{R} \mapsto 2^{\mathbb{R}}$ is defined by

$$F(x) = \begin{cases} |x| & \text{if } |x| \le 1 \\ \{x,x\} & \text{otherwise.} \end{cases}$$

Then it follows from (b) that

$$\begin{aligned} D_C^* F(0,0)(y) &= \mathbb{R} \ \ \forall y \in \mathbb{R} \\ D_M^* F(0,0)(y) &= \begin{cases} [-y,y] & \text{if } y > 0 \\ \{-y,y\} & \text{if } y \le 0 \end{cases} \\ D_A^* F(0,0)(y) &= D_M^* F(0,0)(y) \ \ \forall y \in \mathbb{R} \\ D_F^* F(0,0)(y) &= \begin{cases} [-y,y] & \text{if } y \ge 0 \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned}$$

(e) Let X be an arbitrary Banach space and $g, h : X \mapsto \mathbb{R}$ are defined by $g(x) = \|x\|$ and $h(x) = -\|x\|$. Observe that the function g is convex. We have

$$\partial g(\mathbf{0}) = I\!B_{X^*}$$

and

$$\begin{aligned} \partial_C h(\mathbf{0}) &= I\!B_{X^*} \\ \partial_M h(\mathbf{0}) &= \{x^* \in X^* \mid \|x^*\| = 1\} \\ \partial_A h(\mathbf{0}) &= \partial_M h(0) \\ \partial_F h(\mathbf{0}) &= \{\mathbf{0}\}. \end{aligned}$$

Our arguments for obtaining necessary optimality conditions for set-valued optimization problems are based on the following results on necessary optimality conditions of a scalar constrained optimization problem involving either the Clarke subdifferential/normal cone in the Banach space setting or the Fréchet subdifferential/normal cone in the Asplund space setting. For Proposition 12.4, also see [8, 56] for the details.

Proposition 12.3 ([12, p.52, Corollary]). *Let X be a Banach space, $g : X \mapsto \mathbb{R}$ be a locally Lipschitz function and Ω be a closed subset of X. Suppose that g attains its minimum over Ω at $\bar{x} \in \Omega$. Then*

$$\mathbf{0} \in \partial_C g(\bar{x}) + N_C(\bar{x};\Omega).$$

Proposition 12.4 ([19]). *Let X be an Asplund space, $g : X \mapsto \mathbb{R}$ be a locally Lipschitz function and Ω be a closed subset of X. Suppose that g attains its minimum over Ω at $\bar{x} \in \Omega$. Then for any $\varepsilon > 0$ there exist $x_\varepsilon \in \bar{x} + \varepsilon I\!B_X$ and $u_\varepsilon \in \Omega \cap (\bar{x} + \varepsilon I\!B_X)$ such that*

$$\mathbf{0} \in \partial_F g(x_\varepsilon) + N_F(u_\varepsilon;\Omega) + \varepsilon I\!B_{X^*}.$$

We conclude the subsection with the Clarke penalization.

Proposition 12.5 ([12, Proposition 2.4.3]). *Let X be a Banach space, Ω be a subset of X and $f : \Omega \mapsto \mathbb{R}$ be a Lipschitz function of rank L on Ω. Let $\bar{x}$ belong to a set $U \subseteq \Omega$ and suppose that f attains a minimum over U at $\bar{x}$. Then for any $L' \geq L$, the function $\overline{f}(x) := f(x) + L'd(x;U)$ attains a minimum over U at $\bar{x}$. If $L' > L$ and U is closed, then any other point minimizing $\overline{f}$ over Ω must also lie in U.*

12.2.2 Sequential Normal Compactness, Pseudo-Lipschitzity and Metric Regularity

Optimality conditions with Lagrange–Kuhn–Tucker multipliers for set-valued optimization problems, similar to optimization problems with single-valued objectives, require *qualification assumptions*, which are often expressed in terms of sequential normal compactness, pseudo-Lipschitzity and metric regularity. This subsection is devoted to these concepts. We refer the reader to the monographs [54, 55] for details on these concepts as well as for their history. Throughout this subsection, let X and Y be Banach spaces (unless otherwise stated), $\Omega \subseteq X$ be a nonempty set, F be a set-valued map from X to Y and $(\bar{x}, \bar{y}) \in \mathrm{gr}F$.

To deal with the case of Pareto efficient solutions, Bao and Mordukhovich [5] and Zheng and Ng [72, 73] exploited certain *normal compactness* properties of sets and maps, which are *automatic in finite dimensions* while are unavoidably needed in infinite-dimensional spaces due to the natural lack of compactness therein. These local properties of sets in Banach spaces and in Asplund spaces ensure the equivalence between the weak* and norm convergence to zero of the ε-normals (12.1) and the Fréchet normals (12.2), respectively, in dual spaces. Recall [54] that

Ω is said to be *sequentially normally compact* (SNC) at $\bar{x} \in \Omega$ if for any sequence $\{(\varepsilon_i, x_i, x_i^*)\}$:

$$[\varepsilon_i \to 0^+,\ x_i \xrightarrow{\Omega} \bar{x},\ x_i^* \in \hat{N}_{\varepsilon_i}(x_i;\Omega) \text{ and } x_i^* \xrightarrow{w^*} \mathbf{0}] \text{ implies } [\|x_i^*\| \to 0]$$

and when X is Asplund, Ω is *sequentially normally compact* (SNC) at $\bar{x} \in \Omega$ if for any sequence $\{(x_i, x_i^*)\}$:

$$[x_i \xrightarrow{\Omega} \bar{x},\ x_i^* \in N_F(x_i;\Omega) \text{ and } x_i^* \xrightarrow{w^*} \mathbf{0}] \text{ implies } [\|x_i^*\| \to 0].$$

Note that the concepts of sequential normal compactness for sets was introduced by Mordukhovich and Shao in [58]. It is easy to see that every nonempty set in a finite-dimensional space is SNC at each of its points. To consider the infinite-dimensional case, we need some notions. Recall that the affine hull of Ω, denoted by aff Ω, is the smallest affine set containing Ω. The codimension of cl affΩ is defined as the dimension of the quotien space $X/(\text{cl aff}\Omega - x)$ for any $x \in \Omega$ (this dimension does not depend upon the choice of $x \in \Omega$). The relative interior of Ω is the interior of Ω w.r.t. cl affΩ. It has been established [54] that any SNC set must be finite-codimensional i.e. the SNC property in infinite-dimensional spaces may hold only for sufficiently "large" sets, and this condition is a characterization of the SNC property for convex sets with nonempty relative interior. We refer the reader to [54] for details and sufficient conditions ensuring the fulfillment of the SNC property of sets which are related to a kind of Lipschizian behavior of Ω *around* the point in question such as epi-Lipschitzian and compactly epi-Lipschitzian properties. Recall that a closed set Ω in X is said to be *epi-Lipschitz* at x [7] if there exist a neighborhood V of x, a nonempty open set U and $\lambda > 0$ such that

$$\Omega \cap V + (0,\lambda)U \subseteq \Omega$$

(any non-zero vector in U is said to be *hypertangent* to Ω at x) and it is said to be *epi-Lipschitz-like* at x [7] if there exist $\lambda > 0$, a neighborhood V of x and a convex set U with its polar U^0 being weak*-locally compact such that

$$\Omega \cap V + (0,\lambda)U \subseteq \Omega.$$

As mentioned in [54, p.30–31], the epi-Lipschitzian property of the closed set Ω means that Ω is locally homeomorphic to the epigraph of a Lipschitz continuous function; hence the terminology. Moreover, it has been established that if Ω is convex, then it is epi-Lipschitz at x iff int$\Omega \neq \emptyset$ [54, Proposition 1.25] and that if Ω is epi-Lipschitz at x then it is SNC at this point [54, Proposition 1.26].

The concepts of sequential normal compactness of a set naturally induces the corresponding property for the set-valued map F; namely, F is said to be *sequentially normally compact* (SNC) at $(\bar{x},\bar{y})$ if its graph is SNC at this point [57,58]. However, the case of maps allows us to consider also a weaker (less restrictive) property, introduced in [54, 55] under the name "partial sequential normal compactness" (PSNC), that exploits different convergences in domain and range spaces. Note that in contrast to Mordukhovich–Shao's definition of the PSNC property for the map F which involves either the set of ε-normals $\hat{N}_\varepsilon$ (in Banach space setting) or the Fréchet normal cone N_F (in Asplund space setting), its modified version introduced by Zheng and Ng in [72] involves the Clarke coderivative D_C^* and is in fact a counterpart of Mordukhovich–Shao's PSNC of the map F^{-1}. For the reader's convenience, we reformulate different concepts of PSNC property considered by Zheng and Ng in [72, 73] so that they fit into a unified frame with Mordukhovich–Shao's PSNC property; moreover, we specify additionally the normal cones involved. Namely, the set-valued map F is said to be *partially sequentially normally compact* (PSNC) w.r.t. the normal cone N at $(\bar{x},\bar{y})$ if for any sequence $\{(x_i,y_i,x_i^*,y_i^*)\}$:

$$\begin{gathered}[(x_i,y_i)\overset{\mathrm{gr}F}{\to}(\bar{x},\bar{y}),\ (x_i^*,y_i^*)\in N((x_i,y_i);\mathrm{gr}F),\ y_i^*\to\mathbf{0}\text{ and }x_i^*\overset{w^*}{\to}\mathbf{0}]\\ \text{implies}\quad[\|x_i^*\|\to 0],\end{gathered}\tag{12.5}$$

where N is either of the Fréchet normal cone or the Clarke normal cone and X, Y are either Banach spaces or Asplund spaces depending on the case under consideration. As mentioned in [5], the PSNC property is automatically implied by *robust Lipschitzian behavior* of set-valued and single-valued maps; in particular, when F is pseudo-Lipschitz around $(\bar{x},\bar{y})$. Zheng and Ng noted that by using similar arguments as in [53, 56, 57], one can show that the implication (12.5) holds if gr$F$ is epi-Lipschitz-like at $(\bar{x},\bar{y})$. Recall that F is *pseudo-Lipschitz* [1] around $(\bar{x},\bar{y})$ with some modulus $\gamma>0$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$F(x)\cap V\subseteq F(x')+\gamma\|x-x'\|\mathbb{B}_Y,\ \ \forall x,x'\in U.\tag{12.6}$$

Mordukhovich proposed the term "Lipschitz-like" or "Aubin property" for this property, see [54]. Pseudo-Lipschitzity property is fundamental in Nonlinear Analysis and Variational Analysis; it is in fact equivalent to the two other underlying properties for the inverse map F^{-1} known as linear openness/covering and metric regularity around $(\bar{x},\bar{y})$. Thibault established [68] that if F is pseudo-Lipschitz, then

$$d(y;F(x))\le(\gamma+1)d((x,y);\mathrm{gr}G)\tag{12.7}$$

for (x,y) near $(\bar{x},\bar{y})$. Rockafellar proved [64] that F is pseudo-Lipschitz around $(\bar{x},\bar{y})$ with the modulus γ iff there exists $r>0$ such that for all $x,x'\in\bar{x}+r\mathbb{B}_X$ and $y,y'\in\bar{y}+r\mathbb{B}_Y$

$$|d(y;F(x))-d(y';F(x'))|\le\gamma\|x-x'\|+\|y-y'\|.\tag{12.8}$$

The following property of a pseudo-Lipschitz set-valued map plays an important role in the forthcoming study of necessary conditions.

Proposition 12.6 ([18, Proposition 2.6]). *Suppose that a set-valued map* $G_1 : X \mapsto 2^Y$ *is* γ*-Lipschitz at* $(\bar{x},\bar{y}) \in \mathrm{gr}G_1$. *Then for any set-valued map* $G_2 : X \mapsto 2^Z$ *with* $\bar{z} \in G_2(\bar{x})$ *and for*

$$\Lambda := \{(x,y,z) \in X \times Y \times Z \mid y \in G_1(x),\ z \in G_2(x)\}$$

one has for (x,y,z) *near* $(\bar{x},\bar{y},\bar{z})$

$$d((x,y,z);\Lambda) \leq (1+\gamma)[d((x,y);\mathrm{gr}G_1) + d((x,z);\mathrm{gr}G_2)].$$

Proof. Fix $r > 0$ given by (12.6)–(12.8) applied to the map G_1, let

$$V := (\bar{x},\bar{y},\bar{z}) + \frac{r}{3}\mathbb{B}_{X\times Y\times Z}$$

and $(x,y,z) \in V$. Then for any $(a,b) \in ((\bar{x},\bar{z}) + r\mathbb{B}_{X\times Z}) \cap \mathrm{gr}G_2$ and any $c \in G_1(a)$, we have

$$d((x,y,z);\Lambda) \leq \|x-a\| + \|y-c\| + \|z-b\|$$

and hence,

$$d((x,y,z);\Lambda) \leq \|x-a\| + \|z-b\| + d(y;G_1(a)).$$

So (12.8) ensures that

$$d((x,y,z);\Lambda) \leq (1+\gamma)\|x-a\| + \|z-b\| + d(y;G_1(x))$$

and hence by (12.7)

$$\begin{aligned} d((x,y,z);\Lambda) &\leq (1+\gamma)d(x,z);\mathrm{gr}G_2 \cap ((\bar{x},\bar{z}) + r\mathbb{B}_{X\times Z})) \\ &\quad +(1+\gamma)d((x,y);\mathrm{gr}G_1) \\ &\leq (1+\gamma)[d((x,y);\mathrm{gr}G_1) + d((x,z);\mathrm{gr}G_2)]. \end{aligned}$$

□

Further, it is stated in [12, Corollary, p.58] that if Ω possesses hypertangents at $\bar{x} \in \Omega$ (in particular, when Ω is epi-Lipschitz at $\bar{x}$), then the set-valued map $N_C(\cdot;\Omega)$ is closed at $\bar{x}$ in the following sense: for any sequence $\{(x_i,x_i^*)\}$:

$$[x_i \to \bar{x}, x_i^* \in N_C(x_i,\Omega), x_i^* \xrightarrow{w^*} x^*] \text{ implies } [x_i^* \in N_C(x_i;\Omega)]. \tag{12.9}$$

Motivated by this fact, Zheng and Ng [72] say that a set Ω is *normally closed* at $\bar{x} \in \Omega$ if (12.9) holds and a closed set-valued map F is *normally closed* at $(\bar{x},\bar{y}) \in \mathrm{gr}F$ if its graph is normally closed at $(\bar{x},\bar{y})$. As noted in [72, 73], Ω is normally closed at $\bar{x} \in \Omega$ if either Ω is convex or x is a smooth boundary point of Ω in the sense that there exist a neighborhood V of x and a continuously Fréchet differentiable function g such that $g'(x) \neq 0$ and $V \cap \Omega = V \cap \{x' \in X \mid g(x') \leq 0\}$.

El Abdouni and Thibault [18] proposed to express the qualification assumptions in the form of the metric regularity of a set-valued map with respect to another one. Namely, given two set-valued maps G_1 and G_2 from X into Z and $(\bar{x},\bar{z}) \in \mathrm{gr}G_1 \cap \mathrm{gr}G_2$, we say that G_1 is *metric regular* around $(\bar{x},\bar{z})$ relatively to G_2 if there exist $\gamma \geq 0$ and $r > 0$ such that

$$d((x,z);\mathrm{gr}G_1 \cap \mathrm{gr}G_2) \leq \gamma d(x;G_1(x)) \tag{12.10}$$

for all $(x,z) \in (\bar{x}+r\mathbb{B}_X) \times (\bar{z}+r\mathbb{B}_Z) \cap \mathrm{gr}G_2$. Now, let $G : X \mapsto 2^Z$ be a set-valued map, and S and D be two subsets of X and Z. Considering the particular case $G_1 = G$ and $\mathrm{gr}G_2 = S \times D$, we say that G is *metrically regular relative* to $S \times D$ around $(\bar{x},\bar{z})$ [18] if there exist $\gamma \geq 0$ and $r > 0$ such that

$$d((x,z);S \times D \cap \mathrm{gr}G) \leq \gamma d(z;G(x)) \tag{12.11}$$

for all $(x,z) \in ((\bar{x}+r\mathbb{B}_X) \times (\bar{z}+r\mathbb{B}_Z)) \cap (S \times D)$.

12.2.3 Extremal Principle

Let us recall an important principle in Variational Analysis, called *extremal principle* [54], which can be viewed as a variational counterpart of the convex separation principle in nonconvex setting. The extremal principle provides necessary conditions for local extremal points of set systems in terms of generalized normals to nonconvex sets with no use of tangential approximations and convex separation.

Let Ω_1 and Ω_2 be nonempty sets in a Banach space X. Recall that a point $\bar{x} \in \Omega_1 \cap \Omega_2$ is *locally extremal* for the set system $\{\Omega_1, \Omega_2\}$ if there is a neighborhood U of $\bar{x}$ such that for any $\varepsilon > 0$ one can find $a \in \varepsilon\mathbb{B}_X$ with

$$\Omega_1 \cap (\Omega_2 + a) \cap V = \emptyset. \tag{12.12}$$

The Extremal Principle [54] *Let Ω_1 and Ω_2 be nonempty sets in an Asplund space X. Let $\bar{x}$ be a local extremal point of the set system $\{\Omega_1, \Omega_2\}$, where Ω_1 and Ω_2 are locally closed around $\bar{x}$. Then for every $\varepsilon > 0$ there are*

$$x_i \in \Omega_i \cap (\bar{x}+\varepsilon\mathbb{B}_X) \text{ and } x_i^* \in N_M(x_i;\Omega_i), \quad i = 1,2,$$

satisfying the relations

$$1-\varepsilon \leq \|x_1^*\| + \|x_2^*\| \leq 1+\varepsilon, \ \|x_1^* + x_2^*\| \leq \varepsilon.$$

Zheng and Ng established several modifications of the extremal principle, see [72, Lemmas 2.2 and 2.2'] and [73, Lemma 2.1]. They noted that [72, Lemma 2.2'] in general implies the extremal principle and Lemma 2.1 recaptures the extended extremal principle due to Mordukhovich et al. [59]. Below we recall the first two lemmas.

Lemma 12.1 ([72, Lemmas 2.2 and 2.2']). *Let A and B be closed subsets of a Banach space Y with $A \cap B = \emptyset$. Let $a \in A$, $b \in B$ and $\varepsilon > 0$ be such that $\|a-b\| \leq d(A,B)+\varepsilon^2$, where $d(A,B) := \inf\{\|v_1 - v_2\| \mid v_1 \in A, \text{ and } v_2 \in B\}$.*

(a) There exists $a_\varepsilon \in A$, $b_\varepsilon \in B$, $a^*_\varepsilon \in N_C(a_\varepsilon;A)+\varepsilon I\!B_{Y^*}$ and $b^*_\varepsilon \in N_C(b_\varepsilon;B)+\varepsilon I\!B_{Y^*}$ with $\|a^*_\varepsilon\| = \|b^*_\varepsilon\| = 1$ such that

$$a^*_\varepsilon + b^*_\varepsilon = \mathbf{0} \text{ and } \|a_\varepsilon - a\| + \|b_\varepsilon - b\| \leq \varepsilon.$$

(b) Assume that Y is Asplund. Then there exists $a_\varepsilon \in A$, $b_\varepsilon \in B$, $a^*_\varepsilon \in N_F(a_\varepsilon;A)+2\varepsilon I\!B_{Y^*}$ and $b^*_\varepsilon \in N_F(b_\varepsilon;B)+2\varepsilon I\!B_{Y^*}$ with $\|a^*_\varepsilon\| = \|b^*_\varepsilon\| = 1$ such that

$$a^*_\varepsilon + b^*_\varepsilon = 0 \text{ and } \|a_\varepsilon - a\| + \|b_\varepsilon - b\| < 2\varepsilon.$$

Proof. (a) Define $f\colon Y \times Y \mapsto \mathbb{R} \cup \{\infty\}$ by

$$f(u,v) := \delta_{A\times B}(u,v) + \|u-v\|, \forall (u,v) \in Y \times Y.$$

Then $\inf\{f(u,v) \mid (u,v) \in Y \times Y\} = d(A,B)$ and so, by assumption

$$f(a,b) \leq \inf\{f(u,v) \mid (u,v) \in Y \times Y\} + \varepsilon^2.$$

Equipping $Y \times Y$ with the norm $\|(u,v)\| = \|u\| + \|v\|$, by the Ekeland variational principle there exist $a_\varepsilon \in A$, $b_\varepsilon \in B$ such that

$$\|a_\varepsilon - a\| + \|b_\varepsilon - b\| \leq \varepsilon \tag{12.13}$$

and

$$f(a_\varepsilon, b_\varepsilon) \leq f(u,v) + \varepsilon(\|u - a_\varepsilon\| + \|v - b_\varepsilon\|), \ \forall (u,v) \in Y \times Y.$$

Letting

$$g(u,v) := \|u-v\| + +\varepsilon(\|u - a_\varepsilon\| + \|v - b_\varepsilon\|), \ \forall (u,v) \in Y \times Y,$$

this implies that $g(u,v)$ attains its minimum over $A \times B$ at $(a_\varepsilon, b_\varepsilon)$. It follows from Proposition 12.3 that

$$(\mathbf{0},\mathbf{0}) \in \partial_C g(a_\varepsilon, b_\varepsilon) + N_C((a_\varepsilon, b_\varepsilon); A \times B). \tag{12.14}$$

Let $h(u,v) := \|u-v\|$ and $T(u,v) = u - v$ for any $(u,v) \in Y \times Y$. It follows from [12, Theorem 2.3.10] that $\partial_C h(a_\varepsilon, b_\varepsilon) = T^*[\partial_C(\|\cdot\|)(a_\varepsilon - b_\varepsilon)]$, where T^*

is the conjugate operator of the bounded linear operator T. Noting that $T^*(y^*) = (y^*, -y^*)$ for any $y^* \in Y^*$, $a_\varepsilon - b_\varepsilon \neq \mathbf{0}$ (since $A \cap B = \emptyset$ and $(a_\varepsilon, b_\varepsilon) \in A \times B$) and

$$\partial_C(\|\cdot\|)(a_\varepsilon - b_\varepsilon) = \{y^* \in Y^* \mid \|y^*\| = 1 \text{ and } \langle y^*, a_\varepsilon - b_\varepsilon \rangle = \|a_\varepsilon - b_\varepsilon\|\}$$

the subdifferential of the convex function $h(u,v)$ at $(a_\varepsilon, b_\varepsilon)$ is equal to the set

$$U := \{(y^*, -y^*) \in Y^* \times Y^* \mid \|y^*\| = 1 \text{ and } \langle y^*, a_\varepsilon - b_\varepsilon \rangle = \|a_\varepsilon - b_\varepsilon\|\}.$$

Hence

$$\partial_C g(a_\varepsilon, b_\varepsilon) \subseteq U + \varepsilon I\!B_{Y^*} \times \varepsilon I\!B_{Y^*}.$$

Since $N_C((a_\varepsilon, b_\varepsilon); A \times B) = N_C(a_\varepsilon; A) \times N_C(b_\varepsilon; B)$, it follows from (12.14) that there is $y^* \in Y^*$ with $\|y^*\| = 1$ such that

$$(\mathbf{0}, \mathbf{0}) \in (y^*, -y^*) + \varepsilon(I\!B_{Y^*} \times I\!B_{Y^*}) + N_C(a_\varepsilon; A) \times N_C(b_\varepsilon; B).$$

Note then that

$$-y^* \in \varepsilon I\!B_{Y^*} + N_C(a_\varepsilon; A) \text{ and } y^* \in \varepsilon I\!B_{Y^*} + N_C(b_\varepsilon; B).$$

Together with (12.13), the lemma is established by letting $a_\varepsilon^* = -y^*$ and $b_\varepsilon^* = y^*$.

(b) By the same argument as in the proof of (a) but applying Proposition 12.4 in place of Proposition 12.3. □

12.3 Efficient Points of a Set and Efficient Solutions of Set-Valued Optimization Problems

This section is devoted to various concepts of efficient points of a set and efficient solutions of set-valued optimization problems. We recall these concepts and present a unified scalarization approach to the study of several kinds of efficient points/solutions.

12.3.1 Definitions of Efficient Points of a Set

In the pioneering papers [17, 60] Edgeworth and Pareto presented the concept of an efficient point for a set just began a new branch in optimization – vector optimization. As some efficient points exhibit certain abnormal properties, various concepts of properly efficient points have been introduced in order to eliminate these points. The original concept was introduced by Kuhn and Tucker and modified by Geoffrion and later it was formulated in a more general framework by Benson, Borwein, Borwein, Zhuang, Hartley, Henig, Hurwicz, et al.

Throughout the chapter, let $K \subset Y$ be a nonempty closed pointed convex cone with apex at zero (pointedness means $K \cap (-K) = \{\mathbf{0}\}$). A convex set $\Theta \subset Y$ is called a base for K if $\mathbf{0} \notin \mathrm{cl}\Theta$ and $K = \{t\theta : t \in R_+, \theta \in \Theta\}$. When Θ is bounded, we say that K has a bounded base. Denote

$$K^+ = \{y^* \in Y^* \mid y^*(k) \geq 0,\ \forall k \in K\}$$

and

$$K^{+i} = \{y^* \in Y^* \mid y^*(k) > 0,\ \forall k \in K \setminus \{\mathbf{0}\}\}.$$

It is known that K has a base iff $K^{+i} \neq \emptyset$ and K has a bounded base iff $\mathrm{int}K^+ \neq \emptyset$ [46]. Below we provide examples of cones in some classical Banach spaces.

Example 12.2. Let K be the nonnegative orthant in one of the classical Banach spaces $\mathbb{R}^n$, $C_{[0,1]}$, $L^p_{[0,1]}$ and l^p $(1 \leq p < \infty)$. It is known that the nonnegative orthants in the Banach spaces $\mathbb{R}^n$, $C_{[0,1]}$, $L^p_{[0,1]}$ and l^p $(1 \leq p < \infty)$ have bases, the nonnegative orthants in $\mathbb{R}^n$, $L^1_{[0,1]}$, l^1 have bounded bases and only the nonnegative orthants in the Banach spaces $\mathbb{R}^n$ and $C_{[0,1]}$ have nonempty interior, see [46]. For instance, the set

$$\Theta := \left\{ \varphi \in L^p_{[0,1]} \mid \int_0^1 \varphi(t)dt = 1, \varphi(t) \geq 0 \text{ a.e. on } [0,1] \right\}$$

is a base for K being the nonnegative orthant in $L^p_{[0,1]}$ $(1 \leq p < \infty)$ and the functional $y^* \in Y^*$ given by $y^*(\varphi) := \int_0^1 \varphi(t)dt$ belongs to K^{+i}. Moreover, this base is bounded and $y^* \in \mathrm{int}K^+$ for $p = 1$.

Throughout this section, let A be a nonempty subset of Y and $\overline{a} \in A$. In this paper, we consider the following concepts of efficiency in vector optimization.

Definition 12.1.

- $\overline{a}$ is said to be a Pareto efficient point of A if

$$(A - \overline{a}) \cap (-K \setminus \{\mathbf{0}\}) = \emptyset.$$

- Supposing that int $K \neq \emptyset$, $\overline{a}$ is said to be a weakly efficient point of A if

$$(A - \overline{a}) \cap (-\mathrm{int}K) = \emptyset.$$

- $\overline{a}$ is said to be a strongly (or ideal) efficient point of A if

$$A - \overline{a} \subseteq K.$$

- Supposing that $K^{+i} \neq \emptyset$, $\overline{a}$ is said to be a positive properly efficient point of A if there exists $\varphi \in K^{+i}$ such that

$$\varphi(a) \geq \varphi(\overline{a}),\ \forall\, a \in A.$$

- $\overline{a}$ is said to be a Hurwicz properly efficient point of A if

$$\text{clconvcone}[(A - \overline{a}) \cup K] \cap (-K) = \{\mathbf{0}\};$$

- $\overline{a}$ is said to be a Henig global properly efficient point of A if there exists a convex cone C with apex at zero with $K \setminus \{\mathbf{0}\} \subseteq \text{int}C$ such that

$$(A - \overline{a}) \cap (-\text{int}C) = \emptyset.$$

- Supposing that K has a base Θ, $\overline{a}$ is said to be a Henig properly efficient point of A if there is a scalar $\varepsilon > 0$ such that

$$\text{clcone}(A - \overline{a}) \cap (-\text{clcone}(\Theta + \varepsilon I\!B)) = \{\mathbf{0}\}.$$

- $\overline{a}$ is said to be a super efficient point of A if there is a scalar $\rho > 0$ such that

$$\text{clcone}(A - \overline{a}) \cap (I\!B - K) \subseteq \rho I\!B.$$

- $\overline{a}$ is said to be a Benson properly efficient point of A if

$$\text{clcone}[(A - \overline{a}) + K] \cap (-K) = \{\mathbf{0}\}.$$

The sets of efficient points in Definition 12.1 are denoted by $Min(A)$, $WMin(A)$, $SMin(A)$, $Pos(A)$, $Hu(A)$, $GHe(A)$, $He(A)$, $SE(A)$ and $Be(A)$, respectively.

We refer the reader to [41, 42, 48] for the concepts of Pareto efficiency, weak efficiency and strong efficiency. Note that positive proper efficiency and Hurwicz proper efficiency have been introduced in [34], Benson proper efficiency has been presented in [6], Henig proper efficiency and Henig global proper efficiency have been presented in [30] and super efficiency has been introduced in [9]. The above definition of Henig properly efficient points can be found in [9,74], see also [70,71]. One uses also the notation $He(A,K)$ which is taken from [70, 71] while a slightly notation $He(A,\Theta)$ emphasizing the base Θ is used in other references.

Remark 12.2. One visible disadvantage of weakly efficient points is that they can be considered only when the ordering cone has nonempty interior and we know that the nonnegative orthants in most classical Banach spaces do not satisfy this condition, see Example 12.2. Motivated by this fact, Bao and Mordukhovich [5] introduced and studied enhanced notions of *relative Pareto points/minimizers* that are defined via several kinds of relative interiors of ordering cones and occupy intermediate positions between the classical notions of Pareto efficiency and weak efficiency. Due to the lack of space, we do not discuss these interesting notions here.

Let us recall an equivalent definition of a Henig properly efficient point. Let Θ be as before a base of K. Setting

$$\delta := \inf\{\|\theta\| \mid \theta \in \Theta\} > 0,$$

for each $0 < \eta < \delta$, we can associate to K with another convex, pointed and open cone V_η, defined by

$$V_\eta = \text{cone}(\Theta + \eta \mathring{I\!B}_Y) \setminus \{\mathbf{0}\}.$$

Then $\overline{a}$ is a Henig properly efficient point of A w.r.t. Θ iff there is a scalar $\eta \in]0, \delta[$ such that

$$(A - \overline{a}) \cap (-V_\eta) = \emptyset.$$

We refer the reader to [49] for another equivalent definition of a Henig properly efficient point by means of a functional from K^{+i} and to [25] for a survey and materials on proper efficiency.

In the sequel, when speaking of weakly efficient points (respectively, positive properly efficient points) we mean that int$K$ (respectively, K^{+i}) is nonempty, when speaking of Henig properly efficient points or of K^{+i} we mean that K has a base Θ, when speaking that K has a bounded base we mean that Θ is bounded and by"properly efficient" we mean any of "positive properly efficient", "Hurwicz properly efficient" "Henig properly efficient", "Henig global properly efficient", "super efficient" and "Benson properly efficient."

We recall known relations among the above efficient points in the proposition below.

Proposition 12.7. *(a)* $SMin(A) \subseteq Min(A) \subseteq WMin(A)$.
(b) *If* $SMin(A) \neq \emptyset$, *then* $SMin(A) = Min(A) =$ *a singleton.*
(c) $Pos(A) \subseteq Hu(A)$; *if* Y *is separable, then* $Hu(A) \subseteq Pos(A)$.
(d) $Pos(A) \subseteq GHe(A)$.
(e) $SE(A) \subseteq Be(A) \subseteq Min(A)$.
(f) $SE(A) \subseteq He(A) \subseteq GHe(A) \subseteq Min(A)$ *and if* K *has a bounded base then* $SE(A) = He(A)$.
(g) *Suppose that* Y *is a separable Banach space, or* Y *is a reflexive Banach space and* K *has a base and that* $A - \overline{a}$ *is nearly* K*-subconvexlike for some* $\overline{a} \in A$. *If* $\overline{a}$ *is a Benson properly efficient point of* A *then it is a positive properly efficient point of* A.

Remark 12.3. (a) For the assertions (a)–(f), see [25, 41, 48] and for the assertion (g), see [61, Corollaries 4.2, 4.5].

(b) Recently, Benson proper efficiency has received more attention. In particular, optimality conditions for Benson properly efficient solutions were obtained under assumptions on generalized convexity of set-valued data such as convexlikeness, subconvexlikeness, near convexlikeness and near subconvexlikeness (see [61, 65, 75] for references). Near subconvexlikeness is the weakest convexity among the above four kinds of generalized convexity. Recall [75] that a nonempty set $A \subseteq Y$ is nearly K-subconvexlike if the set $\mathrm{clcone}(A+K)$ is convex.

We illustrate some of the above relations in the example below.

Example 12.3. Let $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$.

(a) Let A be a set in $\mathbb{R}^2$ given by

$$A := \{(x,y) \mid \max\{|x|,|y|\} \le 1\}.$$

Then we have

$$SMin(A) = Min(A) = \{(-1,-1)\}$$
$$WMin(A) = \{(x,y) \mid |x| \le 1, y = -1\}.$$

(b) Let A be a set in $\mathbb{R}^2$ given by

$$A := \{x,y) \mid (x+1)^2 + y^2 \le 1 \text{ or } x^2 + (y+1)^2 \le 1\}$$
$$\cup\{(x,y) \mid 0 \le x \le 1, -2 \le y \le 0\}.$$

Then we have

$$\begin{aligned}
Min(A) &= \{(x,y) \mid (x+1)^2 + y^2 = 1, -2 \le x \le -1,\\
&\quad -1 \le y \le 0\} \cup \{(x,y) \mid x^2 + (y+1)^2 = 1,\\
&\quad -1 \le x \le 0, -2 \le y \le -1\},\\
WMin(A) &= Min(A) \cup \{(x,y) \mid 0 \le x \le 1, y = -2\},\\
Pos(A) &= \{(x,y) \mid (x+1)^2 + y^2 = 1, -2 \le x \le -1 - \sqrt{2}/2,\\
&\quad -1 \le y \le 0\} \cup \{(x,y) \mid x^2 + (y+1)^2 = 1,\\
&\quad -1 \le x \le 0, -2 \le y \le -1 - \sqrt{2}/2\},\\
Hu(A) &= Pos(A),\\
GHe(A) &= Min(A) \setminus \{(-1,-1),(-2,0),(0,-2)\}\\
He(A) &= SE(A) = Be(A) = GHe(A).
\end{aligned}$$

One can check that the set $A - \overline{a}$ is nearly K-subconvexlike for $\overline{a} = (-2,0)$ and it is not nearly K-subconvexlike for $\overline{a} = (-1,-1)$.

12.3.2 Concepts of Efficient Solutions of Set-Valued Optimization Problems

In this chapter, we consider the unconstrained set-valued optimization problem (P)

$$\text{Minimize } \ F(x) \ \text{ subject to } \ x \in X$$

and the set-valued optimization problem with constraint (CP)

$$\text{Minimize } F(x) \text{ subject to } x \in \Omega \text{ and } G(x) \cap \mathscr{C} \neq \emptyset,$$

where F and G are set-valued maps from a Banach space X respectively into Banach spaces Y and Z, $\Omega \subseteq X$ and $\mathscr{C} \subseteq Z$ are nonempty sets. In what follows, by (SP) we mean one of these problems and by $\mathscr{A}$ we mean the corresponding sets of admissible solutions, i.e., $\mathscr{A} = X$ in the case of (P) and $\mathscr{A} = \{x \in \Omega \mid G(x) \cap \mathscr{C} \neq \emptyset\}$ in the case (CP). Denote

$$F(\mathscr{A}) := \cup_{x \in \mathscr{A}} F(x).$$

Various concepts of efficient points of a set in Definition 12.1 naturally induces corresponding concepts of efficient solutions of (SP). Let $\bar{x} \in \mathscr{A}$ and $(\bar{x}, \bar{y}) \in \mathrm{gr}G$.

Definition 12.2. We say that $(\bar{x}, \bar{y})$ is a local "N" efficient solution of (SP) if there exists a neighborhood U of $\bar{x}$ such that $\bar{y}$ is an "N" efficient point of $F(\mathscr{A} \cap U)$, where "N" may be Pareto, weakly, strongly, positive properly, Benson properly, Hurwicz properly, Henig global properly, Henig properly or super.

Putting $U = X$ in Definition 12.2, we get corresponding global concepts of efficient solutions of (SP).

12.3.3 A Unified Scalarization Approach to Several Kinds of Efficient Points

It is of interest to know whether some concepts of efficiencies in vector optimization have any common feature so that they can be studied in a unified scheme or not. The answer is affirmative and in this subsection, we present a unified scalarization approach to the study of all efficient points except Pareto efficient points in Definition 12.1.

Let us begin with recalling a unified geometric approach introduced recently in [28]. As mentioned by Zaffaroni [69], although the definitions of proper efficiency emphasize different aspects but they can be seen as an extension of the primitive idea that they can be geometrically described in terms of separations between the ordering cone and the considered set by means of an open convex cone or an open convex sets. A brief inquiry into the matter reveals that not only various properly efficient points in Definition 12.1 but also weakly or strongly efficient points can be described through disjointness between some set and some nonempty open (not necessarily convex) cone Q. Inspired by this fact, we presented in [28] the notion of Q-minimal point, where $Q \subset Y$ is an arbitrary nonempty *open cone* with apex at zero and different from Y.

Definition 12.3. We say that $\bar{a}$ is a *Q-minimal point* of A and write $\bar{a} \in Qmin(A)$ if

$$A \cap (\bar{a} - Q) = \emptyset$$

or, equivalently,

$$(A-\overline{a})\cap(-Q)=\emptyset.$$

Remark 12.4. Makarov and Rachkovski [49] introduced the notion of D-efficiency, i.e., efficiency w.r.t. a family D of dilating cones : $\overline{a}\in A$ is said to be a *D-minimal point* of A if there exists $C\in D$ such that

$$(A-\overline{a})\cap(-C)=\emptyset.$$

Recall that an open cone in Y is said to be a *dilating cone* (or a *dilation*) of K, or *dilating* K if it contains $K\setminus\{\mathbf{0}\}$. In contrast with D-efficiency, our concept includes not only some concepts of proper efficiency among which are these ones considered in [49] but also the concepts of strong efficiency and weak efficiency.

It turns out that all efficient points of Definition 12.1 but Pareto efficient points are in fact Q-minimal points with Q being appropriately chosen cones.

Theorem 12.1 ([28, Theorem 21.7]).

(a) $\overline{a}\in SMin(A)$ *iff* $\overline{a}\in Qmin(A)$ *with* $Q=Y\setminus(-K)$.
(b) $\overline{a}\in WMin(A)$ *iff* $\overline{a}\in Qmin(A)$ *with* $Q=\operatorname{int}K$.
(c) $\overline{a}\in Pos(A)$ *iff* $\overline{a}\in Qmin(A)$ *with* $Q=\{y\in Y\mid \varphi(y)>0\}$ *and* φ *is some functional in* K^{+i}.
(d) $\overline{a}\in Hu(A)$ *iff* $\overline{a}\in Qmin(A)$, *with* $Q=Y\setminus -\text{cl conv cone}[(A-\overline{a})\cup K]$.
(e) $\overline{a}\in GHe(A)$ *iff* $\overline{a}\in Qmin(A)$, *with* Q *being some open pointed convex cone dilating* K.
(f) $\overline{a}\in He(A)$ *iff* $\overline{a}\in Qmin(A)$ *with* $Q=V_\eta$ *and* η *is some scalar satisfying* $0<\eta<\delta$.
(g) Supposing that K *has a bounded base,* $\overline{a}\in SE(A)$ *iff* $\overline{a}\in Qmin(A)$ *with* $Q=V_\eta$ *and* η *is some scalar satisfying* $0<\eta<\delta$.

Proof. Using Definitions 12.1 and 12.3 one can easily prove the assertions (a)–(e). We prove now the assertion (f), namely, we show that $\overline{a}\in He(A)$ iff there is a scalar η with $0<\eta<\delta$ such that

$$(A-\overline{a})\cap(-V_\eta)=\emptyset. \tag{12.15}$$

Recall that by the definition, $\overline{a}\in He(A)$ iff

$$\text{cl cone}(A-\overline{a})\cap(-\text{cl cone}(\Theta+\varepsilon\mathbb{B}))=\{\mathbf{0}\}. \tag{12.16}$$

It is also known [71] that $\overline{a}\in He(A)$ iff

$$(A-\overline{a})\cap(-\overline{S}_n)=\{\mathbf{0}\} \tag{12.17}$$

for some integer $n\in N$, where $\overline{S}_n=\text{cl cone}(\Theta+\delta/(2n)\mathbb{B}_Y)$. Now, suppose that $\overline{a}\in He(A)$. Then (12.16) holds. Without lost of generality we can assume that $0<\varepsilon<\delta$. We show that (12.15) holds with $\eta=\varepsilon$. Suppose to the contrary that there is $a'\in A-\overline{a}$ such that $a'\in -V_\varepsilon$. Clearly, $a'\in\text{cl cone}(A-\overline{a})\cap(-\text{cl cone}(\Theta+\varepsilon\mathbb{B}_Y))$.

On the other hand, as $0 < \eta = \varepsilon < \delta$ and by the definition of δ, $\mathbf{0} \notin V_\varepsilon$. Hence $a' \neq \mathbf{0}$. This is a contradiction to (12.16). Next, suppose that (12.15) holds for some η. Let n be an integer satisfying $n-1 > \delta/(2\eta)$ or $\delta/(2n-2) < \eta$. By (12.15) we have

$$(A-\overline{a}) \cap (-V_{\delta/(2n-2)}) \subseteq (A-\overline{a}) \cap (-V_\eta) = \emptyset.$$

Then $(A-\overline{a}) \cap (-V_{\delta/(2n-2)} \cup \{0\}) = \{\mathbf{0}\}$. On the other hand, [71, Lemma 2.1] states that if $(A-\overline{a}) \cap (-V_{\delta/(2n-2)} \cup \{\mathbf{0}\}) = \{\mathbf{0}\}$, then $(A-\overline{a}) \cap (-\overline{S}_n) = \{\mathbf{0}\}$. Thus, (12.17) holds and therefore, $\overline{a} \in He(A)$, as it was to be shown. To complete the proof note that the assertion (g) follows from the just proved one and the assertion (f) in Proposition 12.7. □

Remark 12.5. (a) The assertion (f) in Theorem 12.1 is inspired by the definition of Henig properly efficient point for sets in locally convex spaces given by Gong in [21].
(b) One can easily see from (12.15) that any Henig properly efficient point is Henig global properly efficient point.

Based on Theorem 12.1, a *unified geometric approach* has been presented in [28]: one first studies Q-minimal points/solutions and then derives from results obtained for Q-minimal points/solutions similar ones for strongly/weakly/properly efficient points/solutions. In particular, there have been established scalar characterization by the Hiriart-Urruty signed distance function and optimality conditions in the forms of the Lagrange claim (involving first- and second-order radial derivatives), the Fermat rule and the Lagrange multiplier rule (involving coderivatives) for these solutions.

Now, let us proceed to our *unified scalarization approach*. In vector optimization, scalarization mean to transform a vector optimization problem into a single-objective optimization problem. The important role of scalarization is well-known as it allows to exploit widely developed techniques of scalar optimization. Various scalarizing functions have been used to characterize different kinds of efficient points and it turns out that all the strongly/weakly/properly efficient points we are considering can be characterized by some functions with nice properties. Motivated by this fact, we present the following concepts.

Definition 12.4. We say that $\overline{a}$ is an *s-efficient point* of A if there exist a function $s : Y \mapsto \mathbb{R}$ (called a scalarizing function) such that $\overline{a}$ is a minimizer of the function $s(.-\overline{a})$ over A; in other words, one has

$$s(a-\overline{a}) \geq s(\mathbf{0}), \quad \forall a \in A. \tag{12.18}$$

Observe that the concept of Q-minimal point may not be applicable for instance to a strongly efficient point when the ordering cone K is not closed but even then a strongly efficient point is s-efficient, see Proposition 12.8. Similarly, we can define the concept of s-efficient solutions as follows.

Definition 12.5. We say that $(\bar{x}, \bar{y})$ is a *local s-efficient solution* of (SP) if there exist a functional $s : Y \mapsto \mathbb{R}$, a neighborhood U of $\bar{x}$ such that $\bar{y}$ is a minimizer of the function $s(. - \bar{y})$ over $F(\mathscr{A} \cap U)$; in other words, one has

$$s(y - \bar{y}) \geq s(\mathbf{0}), \quad \forall y \in F(\mathscr{A} \cap U). \tag{12.19}$$

Putting $U = X$ in Definition 12.5, we get the corresponding concept of global s-efficient solutions of (SP).

Remark 12.6. A concept similar to the one in Definition 12.5 but with a linear functional $s \in K^* \setminus \{\mathbf{0}\}$ was introduced in [22].

The are various scalarizing functions but in this paper, we will be mainly concerned with the ones of the following types:

(a) A linear function from K^+ or K^{+i}.
(b) The Hiriart-Urruty signed distance function.
(c) A Minkowski-type functional.

and we refer the reader to [4, 33] for scalarization for super efficiency, to [45] for an overview on recent results on scalarization and to [41, 42, 48] for other results and references on this theme.

a. Scalarization by linear functionals from K^+ or K^{+i}

The simplest s-efficient points are listed in the following.

Proposition 12.8. *(a) The point $\bar{a}$ is a positive properly efficient point of A iff it is an s-efficient point of A with $s = \varphi$ for some $\varphi \in K^{+i}$.*
(b) The point $\bar{a}$ is a strongly efficient point of A iff it is an s-efficient point of A with $s = \varphi$ for all $\varphi \in K^+$.

Proof. (a) It is immediate from the definition.

(b) “Only if” part: By the definition, $a - \bar{a} \in K$ for all $a \in A$. Hence, for every fixed $\varphi \in K^+$ we have $\varphi(a - \bar{a}) \geq 0 = \varphi(\mathbf{0})$ for all $a \in A$, which means that (12.18) holds.

“If” part: Suppose to the contrary that $\bar{a}$ is an s-efficient point of A with $s = \varphi$ for all $\varphi \in\in K^+$, i.e. (12.18) holds for all $\varphi \in K^+$ but $\bar{a} \notin SMin(A)$. Then one can find $\hat{a} \in A$ such that $\hat{a} - \bar{a} \notin K$. By a separation theorem, there exist a functional $\varphi \in Y^*$ and a scalar $\alpha \in \mathbb{R}$ such that

$$\varphi(\hat{a} - \bar{a}) < \alpha \leq \varphi(k), \quad \forall\, k \in K.$$

Since $\mathbf{0} \in K$, we have $\alpha \leq 0$ and therefore $\varphi(\hat{a} - \bar{a}) < 0$, a contradiction to the assumption that (12.18) holds for $s = \varphi$ for all $\varphi \in K^+$. □

Remark 12.7. The assertion (b) of Proposition 12.8 is motivated by [29, Lemma 4.8 (ii)].

Corollary 12.1. *Suppose that Y is a separable Banach space, or Y is a reflexive Banach space and K has a base and that $A-\overline{a}$ is nearly K-subconvexlike for some $\overline{a}\in A$. If $\overline{a}$ is a Benson proper efficient point of A, then it is an s-efficient point of A with $s=\varphi$ for some $\varphi\in K^{+i}$.*

Proof. An immediate consequence of Proposition 12.7 (g) and Proposition 12.8 (a). □

Proposition 12.8 together with the following result will be used for establishing that necessary conditions in the Fermat rule and Lagrange multiplier rule for weakly efficient points become sufficient under additional convexity assumptions.

Proposition 12.9. *Supposing that* $\operatorname{int}K\neq\emptyset$*, if $\overline{a}$ is an s-efficient point of A for some $s=\varphi\in K^{+}\setminus\{\mathbf{0}\}$, then $\overline{a}$ is a weakly efficient point of A.*

Proof. Suppose to the contrary that (12.18) holds with such some $s=\varphi\in K^{+}\setminus\{\mathbf{0}\}$ but $\overline{a}\notin WMin(A)$. Then there exists $\hat{a}\in A$ such that $\hat{a}-\overline{a}\in-\operatorname{int}K$. As $\varphi\neq\mathbf{0}$, there exists $v\in Y$ such that $\varphi(v)<0$. Since $\hat{a}-\overline{a}\in-\operatorname{int}K$, there exists a scalar $\rho>0$ such that $-\hat{a}+\overline{a}+\rho v\in K$. As $\varphi\in K^{+}\setminus\{\mathbf{0}\}$, we have $\varphi(-\hat{a}+\overline{a}+\rho v)\geq 0$. Hence, we obtain $\varphi(\hat{a}-\overline{a})\leq\rho\varphi(v)<0$, a contradiction to (12.18). □

Remark 12.8. Proposition 12.9 is motivated by [29, Lemma 4.8(i)].

b. Scalarization by the Hiriart-Urruty signed distance function

In [31], Hiriart-Urruty introduced a so called signed distance function Δ_U associated to a nonempty set U in Y as follows:

$$\Delta_U(y):=d(y;U)-d(y;Y\setminus U).$$

This function possesses nice properties, especially when U has nonempty interior, and has been used for scalarization in vector optimization in several works [11, 23, 26, 27, 29].

The following result shows that a Q-minimal point can be characterized by the Hiriart-Urruty signed distance function and, therefore, is s-efficient.

Proposition 12.10 ([28, Proposition 21.10]). *$\overline{a}$ is a Q-minimal point of A iff it is an s-efficient point of A with $s=\Delta_{-Q}$.*

Proof. Note that, in this case, (12.18) takes the form

$$\Delta_{-Q}(a-\overline{a})\geq\Delta_{-Q}(\mathbf{0})=0,\quad\forall a\in A. \tag{12.20}$$

Observe first that the origin belongs to the boundary of the set $-Q$ and $\Delta_{-Q}(\mathbf{0})=0$. Now, let $\overline{a}\in Qmin(A)$. By the definition, $(A-\overline{a})\subseteq Y\setminus(-Q)$. Consequently, $\Delta_{-Q}(a-\overline{a})\geq 0$ for all $a\in A$, i.e. (12.20) holds. Next, suppose that (12.20) holds. If $\overline{a}\notin Qmin(A)$ then there is $a'\in A$ such that $a'-\overline{a}\in-Q$. As the set $-Q$ is open, $d(a'-\overline{a};Y\setminus(-Q))>0$ and therefore, $\Delta_{-Q}(a'-\overline{a})<0$, a contradiction to (12.20). Thus, $\overline{a}\in Qmin(A)$. □

We derive now two scalar characterizations of some efficient points/solutions in Definitions 12.1 and 12.5 as s-efficient points/solutions with s being the Hiriart-Urruty signed distance function. Such characterizations play an important role in establishing the Fermat rule and the Lagrange multiplier rule. The first result on scalar characterization for efficient points reads as follows.

Theorem 12.2. *(a)* *$\overline{a}$ is a weakly efficient point of A iff it is an s-efficient point of A with $s = \Delta_{-Q}$ and $Q = \mathrm{int}K$.*

(b) *$\overline{a}$ is a Hurwicz properly efficient point of A iff it is an s-efficient point of A with $s = \Delta_{-Q}$ and $Q = Y \setminus -\mathrm{cl\ conv\ cone}[(A-\overline{a}) \cup K]$.*

(c) *$\overline{a}$ is a Henig global properly efficient point of A iff it is an s-efficient point of A with $s = \Delta_{-Q}$ and $Q = C$, C being an open pointed convex cone dilating K.*

(d) *$\overline{a}$ is a Henig properly efficient point of A iff it is an s-efficient point of A with $s = \Delta_{-Q}$ and $Q = V_\eta$ for some scalar $\eta \in]0, \delta[$.*

(e) *supposing that K has a bounded base, $\overline{a}$ is a super efficient point of A iff it is an s-efficient point of A with $s = \Delta_{-Q}$ and $Q = V_\eta$ for some scalar $\eta \in]0, \delta[$.*

Proof. A consequence of Theorem 12.1 and Proposition 12.10. □

Remark 12.9. We refer the reader to the recent work [45] for a new scalar characterization for Benson properly efficient points by functionals from the so called augmented dual cone.

The second result on scalar characterization for efficient solutions reads as follows. Note that the assertions in Theorem 12.3 hold true for both the global and local concepts of solutions.

Theorem 12.3. *(a)* *$(\overline{x}, \overline{y})$ is a positive properly efficient solution of (SP) iff then it is an s-efficient solution of (SP) with $s = \varphi$ for some $\varphi \in K^{+i}$.*

(b) *$(\overline{x}, \overline{y})$ is a weakly efficient solution of (SP) iff it is an s-efficient solution of (SP) with $s = \Delta_{-\mathrm{int}K}$.*

(c) *$(\overline{x}, \overline{y})$ is a strongly efficient solution of (SP) iff it is an s-efficient solution of (SP) with $s = \varphi$ for all $\varphi \in K^+$.*

(d) *r $(\overline{x}, \overline{y})$ is a Henig global properly efficient solution of (SP) iff it is an s-efficient solution of (SP) with $s = \Delta_{-C}$ with C being a dilating pointed convex cone C of K.*

(e) *$(\overline{x}, \overline{y})$ is a Henig properly efficient solution of (SP) iff it is an s-efficient solution of (SP) with $s = \Delta_{-V_\eta}$ with $0 < \eta < \delta$.*

(f) *supposing that K has a bounded base, $(\overline{x}, \overline{y})$ is a super efficient solution of (SP) iff it is an s-efficient solution of (SP) with $s = \Delta_{-V_\eta}$ with $0 < \eta < \delta$.*

Proof. A consequence of Definitions 12.2 and 12.5, Proposition 12.8 and Theorem 12.2. □

Based on Theorem 12.2, we propose the following *unified scheme* for obtaining the Fermat rule and the Lagrange multiplier rule for strongly/weakly/properly efficient solutions in Definition 12.1: one first obtains these rules for s-efficient solutions and then derives similar results for other efficient points.

We collect some known properties of the function Δ_U that will be used in the sequel in the proposition below.

Proposition 12.11 ([31]). *(a) Δ_U is Lipschitz of rank 1 on Y.*
(b) $\Delta_{Y\setminus U} = -\Delta_U$.
(c) Δ_U is convex if U is convex and Δ_U is concave if U is reverse convex, i.e. $U = Y\setminus V$ with V being convex.
(d) $\Delta_U(y) < 0$ iff $y \in$int U, $\Delta_U(y) = 0$ iff $y \in$ bd U and $\Delta_U(y) > 0$ iff $y \in Y\setminus$int U.
(e) Suppose that U is convex and has a nonempty interior, and $\mathbf{0} \in$bd$U$. Then

$$\partial \Delta_U(\mathbf{0}) \subseteq N(\mathbf{0};U)\setminus\{\mathbf{0}\}.$$

The following properties of the subdifferential of $\Delta_U(\mathbf{0})$ play an important role in formulating optimality conditions.

Proposition 12.12 ([28, Proposition 21.11]). *(a) $\partial\Delta_{-\mathrm{int}K}(\mathbf{0}) \subseteq K^+\setminus\{\mathbf{0}\}$.*
(b) assuming that A has a nonempty interior, we have

$$\partial\Delta_{Y\setminus \text{cl conv cone}[(A-\bar{a})\cup K]}(\mathbf{0}) \subseteq K^+\setminus\{\mathbf{0}\}.$$

(c) Let C be an open convex cone dilating K. Then

$$\partial\Delta_{-C}(\mathbf{0}) \subseteq K^{+i}.$$

(d) For $\eta \in]0,\delta[$, we have

$$\partial\Delta_{-V_\eta}(\mathbf{0}) \subseteq \{y^* \in K^{+i} \mid y^*(\theta) \geq \eta,\ \forall\, \theta\in\Theta\}.$$

(e) supposing that Θ is bounded, we have $\partial\Delta_{-K_\eta}(\mathbf{0}) \subseteq$ intK^+.

Proof. (a) Apply Proposition 12.11 (e) to $U = -intK$ and take account of $N(\mathbf{0};-intK) = K^+$.

(b) As the Fréchet normal cone, the Ioffe approximate normal cone and the Mordukhovich normal cone are contained in the Clarke normal cone (see Proposition 12.1(a)), the Ioffe approximate subdifferential and the Mordukhovich subdifferential are contained in the Clarke subdifferential. Therefore, it suffices to show that

$$\partial_C\Delta_{Y\setminus \text{cl conv cone}[(A-\bar{a})\cup K]}(\mathbf{0}) \subseteq K^+\setminus\{\mathbf{0}\}.$$

For the simplicity, we denote $Q = Y\setminus -V$, where $V = \text{cl conv cone}[(A-\bar{a})\cup K]$. Note that V is a closed convex cone with a nonempty interior and $K\subseteq V$. We have to show that

$$\partial_C\Delta_{-Q}(\mathbf{0}) \subseteq K^+\setminus\{\mathbf{0}\}.$$

By Proposition 12.11 (b), we have $\Delta_{-Q}(\mathbf{0}) = -\Delta_{Y\setminus -V}(\mathbf{0}) = -\Delta_V(\mathbf{0})$. The properties of the Clarke subdifferential and the subdifferential of convex analysis (see Proposition 12.2 (b), (c)) yield

$$\partial_C(-\Delta_V)(\mathbf{0}) = -\partial_C\Delta_V(\mathbf{0}) = -\partial\Delta_V(\mathbf{0}),$$

where $\partial\Delta_V(\mathbf{0})$ means the subdifferential in the sense of convex analysis of the convex set V. Applying Proposition 12.11 (e) to the closed convex cone V which has a nonempty interior gives

$$-\partial\Delta_V(\mathbf{0}) \subseteq -N(\mathbf{0};V)\setminus\{\mathbf{0}\}.$$

Further, since $K\subseteq V$ we get $-N(\mathbf{0},V)\subseteq K^+$. Therefore, we get

$$-\partial\Delta_V(\mathbf{0}) \subseteq K^+\setminus\{\mathbf{0}\},$$

which yields $\partial_C\Delta_{-Q}(\mathbf{0}) \subseteq K^+\setminus\{\mathbf{0}\}$.

(c) Apply Proposition 12.11 (e) to C we get

$$\partial\Delta_{-C}(\mathbf{0}) \subseteq N(\mathbf{0};-C)\setminus\{\mathbf{0}\}.$$

Now take $y^*\in\partial\Delta_{-C}(\mathbf{0})$ and $k\in K\setminus\{\mathbf{0}\}\subseteq C$. We have to show that $y^*(k)>0$. As $y^*\neq\mathbf{0}$, there is $y\in Y$ such that $y^*(y)>0$. On the other hand, as C is open, and $k\in C$, there exist a scalar $t>0$ such that $-k+ty\in -C$. Hence, $y^*(-k+ty)\leq 0$ and $y^*(k)\geq ty^*(y)>0$. Thus, $y^*\in K^{+i}$.

(d) Let $y^*\in\partial\Delta_{-K_\eta}(\mathbf{0})$. For any $\theta\in\Theta$, we have $-(\theta+\eta\mathbb{B})\subseteq -K_\eta$. Hence,

$$d_{Y\setminus(-K_\eta)}(-\theta)\geq\eta.$$

The definition of the convex subdifferential yields

$$\begin{aligned} y^*(-\theta) &\leq \Delta_{-K_\eta}(-\theta)-\Delta_{-K_\eta}(\mathbf{0}) = d_{-K_\eta}(-\theta)-d_{Y\setminus(-K_\eta)}(-\theta)\\ &= -d_{Y\setminus(-K_\eta)}(-\theta)\leq -\eta. \end{aligned}$$

It follows then that

$$y^*(\theta)\geq\eta \quad \text{for all } \theta\in\Theta. \tag{12.21}$$

Now, let $k\in K\setminus\{\mathbf{0}\}$. As Θ is a base of K, there exist a scalar $t>0$ and $\theta\in\Theta$ such that $k=t\theta$. Then (12.21) yields $y^*(k)=y^*(t\theta)=ty^*(\theta)>0$. Therefore, $y^*\in K^{+i}$.

(e) Suppose that Θ is bounded and denote

$$\overline{\delta} = \sup\{\|\theta\| \mid \theta\in\Theta\}<\infty.$$

Let $y^* \in \partial \Delta_{-V_\eta}(\mathbf{0})$. It is known that $y^* \in \text{int}K^+$ iff y^* is uniformly positive on K in the sense that there exists a scalar $\alpha > 0$ such that $y^*(k) \geq \alpha \|k\|$ for all $k \in K \setminus \{\mathbf{0}\}$. Let $k \in K \setminus \{\mathbf{0}\}$ be an arbitrary vector. As Θ is a base of K, there exist a scalar $t > 0$ and $\theta \in \Theta$ such that $k = t\theta$. Since $-\theta + \eta\mathbb{B} \subseteq -V_\eta$, it follows that

$$-k + t\eta\mathbb{B} = t(-\theta + \eta\mathbb{B}) \subseteq -V_\eta,$$

i.e. the open ball centered at $-k$ with the radius $t\eta$ is contained in $-V_\eta$. Therefore,

$$d(-k; Y \setminus (-V_\eta)) \geq t\eta.$$

On the other hand, $t\eta = (\|k\|/\|\theta\|)\eta \geq (\eta/\overline{\delta})\|k\|$. Hence,

$$d(-k; Y \setminus (-V_\eta)) \geq \frac{\eta}{\overline{\delta}}\|k\|.$$

Clearly, $\Delta_{-V_\eta}(\mathbf{0}) = 0$. As $y^* \in \partial \Delta_{-V_\eta}(\mathbf{0})$, the definition of the subdifferential of convex analysis yields

$$\begin{aligned}\langle y^*, -k\rangle &\leq \Delta_{-V_\eta}(-k) - \Delta_{-V_\eta}(\mathbf{0}) = d(-k; -V_\eta) - d(-k; Y \setminus (-V_\eta)) \\ &= -d(-k; Y \setminus (-V_\eta)) \leq -\tfrac{\eta}{\overline{\delta}}\|k\|\end{aligned}$$

or $\langle y^*, k\rangle \geq (\eta/\overline{\delta})\|k\|$. This means that y^* is uniformly positive on K, or $y^* \in \text{int}K^+$.

□

c. Scalarization by a Minkowski-type function

In [41, 42] Jahn showed that if $\overline{a}$ is a weakly efficient point of the set A, then for every element $\hat{a} \in \overline{a} - \text{int}K$, $\overline{a}$ is a minimizer over A of the functional $\|. - \hat{a}\|_{\hat{a}}$, where $\|.\|_{\hat{a}}$ is the seminorm defined in Y through the Minkowski functional

$$\|y\|_{\hat{a}} := \inf\{\lambda > 0 \mid \lambda^{-1} y \in (\hat{a} - \overline{a} + K) \cap (\overline{a} - \hat{a} - K)\} \quad \text{for all } y \in Y;$$

more precisely, one has

$$1 = \|\overline{a} - \hat{a}\|_{\hat{a}} \leq \|a - \hat{a}\|_{\hat{a}} \quad \text{for all } a \in A.$$

Thus, a weakly efficient point of A is s-efficient with $s = \|. - \hat{a}\|_{\hat{a}}$ for every $\hat{a} \in \overline{a} - \text{int}K$. Inspired by this result, El Abdouni and Thibault used this seminorm in [18] for characterizing weakly efficient points in the case when K is just assumed to be a convex set rather than a cone. Gerth(Tammer) and Weidner considered such a general case earlier in [20] exploiting another scalarizing functional $\xi_a := \xi_{a,k} : Y \to \mathbb{R} \cup \{-\infty\}$ given by

$$\xi_a(y) := \inf\{\lambda \in \mathbb{R} \mid y \in \lambda k + a - K\},$$

where $a \in Y$ is an arbitrary vector and $k \in \text{int}K$ is a fixed vector. Note that the mentioned above Minkowski-type functions as well as their modifications possess nice properties such as monotonicity, Lipschizity and they have been proven to be useful tools in scalarization for vector optimization problems.

12.4 The Fermat Rule

This section is devoted to the Fermat rule for the following unconstrained set-valued optimization problem (P)

$$\text{Minimize } \ F(x) \ \text{ subject to } \ x \in X$$

12.4.1 The Fermat Rule for Pareto Efficient Solutions

In the case of Pareto efficient solutions, we have both fuzzy and exact versions of the Fermat rule.

Zheng and Ng obtained the following fuzzy version of the Fermat rule for Pareto efficient solutions of (P) in Banach space setting and in term of the Clarke coderivative.

Theorem 12.4 ([72, Theorem 3.1]). *Let X and Y be Banach spaces. Assume that F has a closed graph. If $(\bar{x},\bar{y})$ is a local Pareto efficient solution of (P), then for any $\varepsilon > 0$ there exist $x_\varepsilon \in \bar{x} + \varepsilon I\!B_X$, $y_\varepsilon \in F(x_\varepsilon) \cap (\bar{y} + \varepsilon I\!B_Y)$ such that the following inclusion holds*

$$\mathbf{0} \in D_C^* F(x_\varepsilon, y_\varepsilon)(y^* + \varepsilon I\!B_{Y^*}) + \varepsilon I\!B_{X^*} \tag{12.22}$$

with some $y^* \in K^+$ and $\|y^*\| = 1$.

Proof. We will prove the following equivalent form of the result: there exist a sequence $\{(x_i, y_i)\}$ in gr$F$ and a sequence $\{y_i^*\}$ in K^+ with $\|y_i^*\| = 1$ for all i such that $(x_i, y_i) \to (\bar{x}, \bar{y})$ and

$$d((\mathbf{0}, -y_i^*); N_C((x_n, y_n); \text{gr}F)) \to 0. \tag{12.23}$$

By assumption there exists $\tau > 0$ such that $\bar{y} \in Min(F(\bar{x} + \tau I\!B_X))$. Let

$$A := \{(x,y) \in \text{gr}F \mid x \in \bar{x} + \tau I\!B_X\}$$

and take $k_0 \in K$ with $\|k_0\| = 1$. For simplicity, let $B_i := \bar{y} - (1/i^2)k_0 - K$. We claim that for all natural numbers i large enough,

$$A \cap (X \times B_i) = \emptyset. \tag{12.24}$$

Indeed if it is not the case, then there exists $y' \in F(\bar{x}+\tau I\!B_X)$ such that $\bar{y}-(1/i^2)$ $k_0 - y' \in K$, contradicting $\bar{y} \in Min(F(\bar{x}+\tau I\!B_X))$. Hence (12.24) holds. By Lemma 12.1, applied to $a=(\bar{x},\bar{y})$ and $b=(\bar{x},\bar{y}-(1/i^2)k_0)$, there exist

$$(x_i,y_i) \in A, (u_i,v_i) \in X \times B_i,$$

$$(x_i^*,y_i^*) \in N_C((x_i,y_i);A)+\frac{1}{i}(I\!B_{X^*} \times I\!B_{Y^*}) \tag{12.25}$$

and

$$(u_i^*,v_i^*) \in N_C((u_i,v_i);X \times B_i)+\frac{1}{i}(I\!B_{X^*} \times I\!B_{Y^*})$$

with $\|(x_i^*,y_i^*)\| = \|(u_i^*,v_i^*)\| = 1$ such that $(x_i^*,y_i^*)+(u_i^*,v_i^*) = \mathbf{0}$

$$\|(x_i,y_i)-(\bar{x},\bar{y})\| \le \frac{1}{i} \text{ and } \|(u_i,v_i)-(\bar{x},\bar{y}-\frac{1}{i^2}k_0)\| \le \frac{1}{i}.$$

Then by the well known relation on normal cones

$$N_C((u_i,v_i);X \times B_i)) = \{\mathbf{0}\} \times N_C(v_i;B_i) \subseteq \{\mathbf{0}\} \times K^+,$$

there exist $r_i \in [1-1/i, 1+1/i]$ and $y_i^* \in K^+$ with $\|y_i^*\| = 1$ such that

$$(u_i^*,v_i^*) \in r_i(\mathbf{0},y_i^*)+\frac{1}{i}(I\!B_{X^*} \times I\!B_{Y^*}),$$

namely

$$-(x_i^*,y_i^*) \in r_i(\mathbf{0},y_i^*)+\frac{1}{i}(I\!B_{X^*} \times I\!B_{Y^*}).$$

This and (12.25) imply that

$$\begin{aligned}(\mathbf{0},-y_i^*) &\in \frac{1}{r_i}((x_i^*,y_i^*)+\frac{1}{ir_i}(I\!B_{X^*} \times I\!B_{Y^*}) \\ &\subseteq N_C((x_i,y_i);A)+\frac{2}{ir_i}(I\!B_{X^*} \times I\!B_{Y^*}) \\ &= N_C((x_i,y_i);\mathrm{gr}F)+\frac{2}{ir_i}(I\!B_{X^*} \times I\!B_{Y^*})\end{aligned}$$

where the last equality holds because $A = \mathrm{gr}F \cap ((\bar{x}+\tau I\!B_X) \times Y)$ and $(\bar{x}+\tau I\!B_X) \times Y$ is a neighborhood of (x_i,y_i) for i large enough. Thus (12.23) holds. □

In Asplund space setting, Theorem 12.4 can be strengthened to the following theorem in which the Clarke coderivateve is replaced by the Fréchet coderivative .

Theorem 12.5 ([72, Theorem 4.1]). *Let X and Y be Asplund spaces. Assume that F has a closed graph. If $(\bar{x},\bar{y})$ is a local Pareto efficient solution of (P), then for any $\varepsilon > 0$ there exist $x_\varepsilon \in \bar{x} + \varepsilon I\!B_X$, $y_\varepsilon \in F(x_\varepsilon) \cap (\bar{y} + \varepsilon I\!B_Y)$ such that the following inclusion holds*

$$\mathbf{0} \in D_F^* F(x_\varepsilon, y_\varepsilon)(y^* + \varepsilon I\!B_{Y^*}) + \varepsilon I\!B_{X^*} \tag{12.26}$$

with some $y^* \in K^+$ and $\|y^*\| = 1$.

Proof. By the same argument as in the proof of Theorem 12.4 with Proposition 12.4 used in place of Proposition 12.3. □

Remark 12.10. ([72, Remark on p. 83]) From the proof of Theorem 12.4, one can see that if $(\bar{x},\bar{y})$ is a local Pareto efficient solution of (P), then it is a local extremal point of the system $\{\mathrm{gr}F, \bar{y} - K\}$ (see [18]). Thus one can also prove Theorem 12.5 by using the extremal principle instead of Lemma 12.1.

The following example shows that $\varepsilon > 0$ in Theorem 12.4 cannot be replaced by $\varepsilon = 0$.

Example 12.4 ([72, Example 3.1]). Let X be an infinite dimensional separable space and $\{x_i\}$ be a countable dense subset of X with each $x_i \neq \mathbf{0}$. Let

$$U = \left\{ \frac{-x_i}{i\|x_i\|},\ i \in I\!N \right\} \text{ and } A = \mathrm{cl\,conv}(U \cap -U).$$

Then A is a compact subset of X and $A = -A$. Moreover, it is easy to verify that

$$X = \mathrm{cl}(\mathrm{span}(A)) \text{ and } \mathrm{span}(A) = \cup_{i=1}^{\infty} iA, \tag{12.27}$$

where $\mathrm{span}(A)$ denotes the linear subspace of X generated by A. By Baire Category Theorem, it follows that $X \neq \mathrm{span}(A)$. Let $F : X \mapsto 2^X$ be defined by $F(x) = \{x\}$ if $x \in A$ and $F(x) = \emptyset$ otherwise. Then the graph of F is a compact convex subset of $X \times X$. Take $e \in X \setminus \mathrm{span}(A)$ and consider the ordering cone K defined by $K := \{te \mid t \geq 0\}$. By the choice of e, it is easy to check that $(\mathbf{0},\mathbf{0})$ is a global Pareto efficient solution of (P). We claim that

$$\mathbf{0} \notin D_C^* F(\mathbf{0},\mathbf{0})(y^*), \ \ \forall y^* \in X^* \setminus \{\mathbf{0}\}. \tag{12.28}$$

Indeed let $y^* \in X^*$ satisfy $\mathbf{0} \in D_C^* F(\mathbf{0},\mathbf{0})(y^*)$. By definition and convexity of F, one has $\langle y^*, y\rangle \leq 0$ for all $y \in A$. It follows from (12.27) that $\langle y^*, x\rangle \leq 0$ for all $x \in X$ and hence $y^* = \mathbf{0}$. This shows that (12.28) holds.

Let us proceed to *exact versions* of the Fermat rule for Pareto efficient solutions. We need the following kind of compactness on the dual of the ordering cone.

Definition 12.6. [72, Definition 3.1] A closed convex cone K of Y is said to be *dually compact* if there exists a compact subset C of Y such that

$$K^+ \subseteq \mathscr{W}(C) := \{y^* \in Y^* \mid \|y^*\| \leq \sup\{\langle y^*, y\rangle \mid y \in C\}\}. \tag{12.29}$$

Examples of dually compact cones are provided in the following.

Proposition 12.13 ([72]). *K is dually compact if one of the following conditions holds:*

(i) *Y finite dimensional.*
(ii) $\mathrm{int}K \neq \emptyset$.

Proof. In the case (i) it suffices to take $C = \mathbb{B}_Y$. In the case (ii) let $k \in \mathrm{int}K$. Then $k + \tau\mathbb{B}_Y \subseteq K$ for some $\tau > 0$; thus, for any $k^* \in K^+$.

$$0 \leq \inf\{\langle k^*, y\rangle \mid y \in k + \tau\mathbb{B}_Y\} = \langle k^*, k\rangle - \tau\|k^*\|$$

and so $\|k^*\| \leq \langle k^*, k/\tau\rangle$. Therefore,

$$k \in \mathrm{int}K \Rightarrow K^+ \subseteq \mathscr{W}(\{rk\}) \text{ for some } r > 0.$$

□

The following example shows that the dual compactness is weaker than the condition $\mathrm{int}K \neq \emptyset$.

Example 12.5 ([72]). Let $Y = \mathbb{R}^2$, $K = \{0\} \times \mathbb{R}_+$. As Y is finite dimensional, Proposition 12.13 implies that K is dually compact. It is clear that $\mathrm{int}K = \emptyset$.

It turns out that a dually compact cone enjoys a useful property, which is in some sense similar to the partial normal compactness.

Proposition 12.14. *If K is dually compact then for any sequence $\{y_i^*\}$:*

$$[y_i^* \in K^+,\ y_i^* \xrightarrow{w^*} \mathbf{0}] \text{ implies } [\|y_i^*\| \to 0]. \tag{12.30}$$

Proof. Firstly, we recall a result established in the proof of [72, Proposition 3.1]. Namely, let C be a subset of Y such that (12.29) holds. By compactness of C there exist $c_1, \cdots, c_m \in C$ such that $C \subseteq \cup_{j=1}^m (c_j + 1/2\mathbb{B}_Y)$. Therefore, for any $y^* \in K^+$, (12.29) implies that

$$\begin{aligned}\|y^*\| &\leq \max\{\langle y^*, y\rangle \mid y \in \cup_{j=1}^m (c_j + \tfrac{1}{2}\mathbb{B}_Y)\} \\ &= \max\{\langle y^*, c_j\rangle \mid j = 1, \cdots, m\} + \tfrac{1}{2}\|y^*\|.\end{aligned}$$

Hence, we get

$$\|y^*\| \leq 2\max\{\langle y^*, c_j\rangle \mid j = 1, \cdots, m\} \text{ for all } y^* \in K^+. \tag{12.31}$$

It is easy to see that (12.30) follows immediately from (12.31). □

Note that the relations (12.31) and (12.30) are in fact the relations (3.8) and (3.9) in [72].

We are ready now to formulate exact version of the Fermat rule for Pareto efficient solutions of (P) established by Zheng and Ng in [72].

Theorem 12.6. **([72, Theorem 3.3])** *Let X and Y be Banach spaces. Assume that F has a closed graph, that $N_C(\cdot;\mathrm{gr}F)$ is closed at $(\bar{x},\bar{y})$ (this condition is automatically satisfied if F is assumed to be a closed convex set-valued map) and that one of the following two conditions holds.*

(i) The cone K is dually compact.
(ii) The inverse map F^{-1} is PSNC at $(\bar{y},\bar{x})$ w.r.t. the Clarke normal cone

If $(\bar{x},\bar{y})$ is a local Pareto efficient solution of (P), then the following inclusion holds

$$\mathbf{0} \in D_C^* F(\bar{x},\bar{y})(y^*) \tag{12.32}$$

with some $y^* \in K^+$ and $\|y^*\| = 1$.

Proof. By Theorem 12.4 there exist a sequence $\{(x_i,y_i,x_i^*,y_i^*,k_i^*)\}$ with each $(x_i,y_i) \in \mathrm{gr}F$, $k_i^* \in K^+$, $\|k_i^*\| = 1$ and $x_i^* \in D_C^* F(x_i,y_i)(y_i^*)$ such that

$$(x_i,y_i) \to (\bar{x},\bar{y}), x_i^* \to 0 \text{ and } \|y_i^* - k_i^*\| \to 0.$$

Since the unit ball in Y^* is weak* compact, without loss of generality we can assume that $k_i^* \xrightarrow{w^*} k_0^* \in C^+$ (and hence $y_i^* \xrightarrow{w^*} k_0^*$). Since $N_C(\cdot;\mathrm{gr}F)$ is closed at $(\bar{x},\bar{y})$,

$$\mathbf{0} \in D_C^* F(\bar{x},\bar{y})(k_0^*). \tag{12.33}$$

Thus it remains to prove that $k_0^* \neq \mathbf{0}$. If (i) holds, then we must also have $k_0^* \neq \mathbf{0}$, in view of the relation (12.30) in Proposition 12.14. Further, suppose to the contrary that (ii) holds but $k_0^* = \mathbf{0}$. Then we would have $k_i^* \xrightarrow{w^*} k_0^* = \mathbf{0}$ and hence, $y_i^* \xrightarrow{w^*} \mathbf{0}$. As $(x_i^*, -y_i^*) \in N_C((x_i,y_i);\mathrm{gr}F)$, we get $(-y_i^*, x_i^*) \in N_C((y_i,x_i);\mathrm{gr}F^{-1})$ and the PSNC property of F^{-1} then implies that $\|y_i^*\| \to 0$, a contradiction to $\|y_i^*\| \to 1$. □

The following corollary is a consequence of Theorem 12.6 ((ii) is automatically satisfied thanks to the epi-Lipschitzity assumption).

Corollary 12.2 ([72, Corollary 3.1]). *Let X and Y be Banach spaces. Assume that F has a closed graph, which is epi-Lipschitz at $(\bar{x},\bar{y})$. If $(\bar{x},\bar{y})$ is a local Pareto efficient solution of (P), then the following inclusion holds*

$$\mathbf{0} \in D_C^* F(\bar{x},\bar{y})(y^*)$$

with some $y^* \in K^+$ and $\|y^*\| = 1$.

In Asplund space setting, Bao and Mordukhovich used the extremal principle to obtain the following strengthened exact version of the Fermat rule for Pareto efficient solutions of (P)

Theorem 12.7 ([5, Theorem 5.1]). *Let X and Y be Asplund spaces. Assume that F has a closed graph and that one of the following conditions holds.*

(i) The cone K is SNC at the origin.
(ii) The inverse map F^{-1} is PSNC at $(\bar{y},\bar{x})$ w.r.t. the Fréchet normal cone.

If $(\bar{x},\bar{y})$ is a local Pareto efficient solution of (P), then the following inclusion holds

$$\mathbf{0} \in D^*_M F(\bar{x},\bar{y})(y^*) \tag{12.34}$$

with some $y^* \in K^+$ and $\|y^*\| = 1$.

Proof. We show that the point $(\bar{x},\bar{y})$ is a local extremal point of some system of sets in the produce space $X \times Y$. Namely, let

$$\Omega_1 := \mathrm{gr} F,\ \Omega_2 := X \times (\bar{y} - K). \tag{12.35}$$

These sets are closed and $(\bar{x},\bar{y}) \in \Omega_1 \cap \Omega_2$. To verify the local extremality of $(\bar{x},\bar{y})$ for $\{\Omega_1,\Omega_2\}$, let us show that there is a sequence $\{v_i\} \subseteq Y$ with $v_i \to \mathbf{0}$ as $i \to \infty$ such that

$$\Omega_1 \cap (\Omega_2 + (\mathbf{0},v_i)) \cap (U \times Y) = \emptyset,\ \ i \in \mathbb{N}, \tag{12.36}$$

where U is a neighborhood of $\bar{x}$ from its local minimality. This gives the required extremality relation (12.12) with $a_i := (\mathbf{0},v_i) \in X \times Y$. Let $c \in -K \setminus \{\mathbf{0}\}$ be arbitrary vector. We construct an appropriate sequence $\{v_i\}$ in (12.36) by putting $v_i = c/i$ as $i \in \mathbb{N}$. Arguing by contradiction, suppose that (12.36) does not hold, i.e.,

$$\text{there is } (x,y) \in U \times Y \text{ with } (x,y) \in \Omega_1 \cap (\Omega_2 + (\mathbf{0},v_i)). \tag{12.37}$$

Then, by the construction of sets (12.35), we find some $(x,y,k) \in X \times Y \times Y$ such that

$$x \in U, y \in F(x) + k \text{ with } k \in K \text{ and } y \in \bar{y} - K + v_i,\ i \in \mathbb{N}.$$

This implies, by the convexity of the cone K, that

$$y - k \in F(U) \text{ and } y - k \in \bar{y} - k - K + v_i \subseteq \bar{y} - K + v_i,\ i \in \mathbb{N}. \tag{12.38}$$

By the choice of v_i, we have

$$\bar{y} - K + v_i = \bar{y} - K + \frac{c}{i} \subseteq \bar{y} - K - (K \setminus \{\mathbf{0}\}) \subseteq \bar{y} - (K \setminus \{\mathbf{0}\}),\ i \in \mathbb{N}. \tag{12.39}$$

Combining the relations in (12.37)–(12.39) we have

$$y-k \in (\bar{y}-(K\setminus\{\mathbf{0}\}))\cap F(U).$$

This is a contradiction to the fact that $(\bar{x},\bar{y})$ is a local efficient solution of F and thus justifies the local extremality of $(\bar{x},\bar{y})$ for $\{\Omega_1,\Omega_2\}$.

Equip now the space $X\times Y$ with sum norm $\|(x,y)\| = \|x\|+\|y\|$ and observe that it is Asplund as a product of Asplund spaces. Then applying *extremal principle* to the set system (Ω_1,Ω_2) in (12.35) and taking into account they particular structures and the maximum form of the dual norm in $X^*\times Y^*$, for any sequences $\varepsilon_i \to 0^+$ as $i\to\infty$ we find $\{(x_{ji},y_{ji})\}\subseteq X^*\times Y^*$ as $j=1,2$ satisfying for all $i\in I\!N$ the following relations:

$$(x_{1i},y_{1i})\in \mathrm{gr}F, (x_{2i},y_{2i})\in X\times(\bar{y}-K), \|(x_{ji},y_{ji})-(\bar{x},\bar{y})\|\le \varepsilon, \tag{12.40}$$

$$(x^*_{1i},-y^*_{1i})\in N_F((x_{1i},y_{1i});\mathrm{gr}F), \mathbf{0}=x^*_{2i}\in N_F(x_{2i};X), y^*_{2i}\in N_F(\bar{y}-y_{2i};K), \tag{12.41}$$

$$\max\{\|x^*_{1i}\|,\|y^*_{1i}+y^*_{2i}\|\}\le\varepsilon_i \tag{12.42}$$

and

$$1-\varepsilon_i\le \max\{\|x^*_{1i}\|,\|y^*_{1i}\|\}+\|y^*_{2i}\|\le 1+\varepsilon_i. \tag{12.43}$$

By (12.43), the sequence $\{(x^*_{ji},y^*_{ji})\}$ are bounded in $X^*\times Y^*$ for $j=1,2$, and hence – by the Asplund property of $X\times Y$- they contain weak* converging subsequences; see, e.g., [32, 54]. Using (12.43), we get without lost of generality that

$$\|x^*_{1i}\|\to\mathbf{0},\ y^*_{2i}\xrightarrow{w^*} -y^* \text{ as } i\to\infty, \tag{12.44}$$

where the weak* limit $y^*\in Y^*$ satisfies the inclusions

$$(\mathbf{0},-y^*)\in N_M((\bar{x},\bar{y});\mathrm{gr}F) \text{ and } -y^*\in N_M(\mathbf{0};K) \tag{12.45}$$

obtained by passing to the limit in (12.40)–(12.41) as $i\to\infty$ due to construction (12.4) of the Mordukhovich normal cone via the *sequential* outer limit (12.2) of Fréchet normals.

Next we show that $y^*\neq\mathbf{0}$ in (12.45) if either K is SNC at the origin or F^{-1} is PSNC at $(\bar{x},\bar{y})$. Assume by the contrary that $y^*=\mathbf{0}$ having then from (12.44) that

$$y^*_{1k}\xrightarrow{w^*}\mathbf{0},\ y^*_{2k}\xrightarrow{w^*}\mathbf{0} \text{ as } i\to\infty. \tag{12.46}$$

If K is SNC at the origin, then the second expression in (12.46) immediately yields that $\|y^*_{2k}\|\to 0$ and therefore $\|y^*_{1k}\|\to 0$ as $i\to\infty$ by (12.42). Combining the latter with (12.44), we thus contradict the nontriviality (12.43). Using the first inclusion in (12.41) and the convergence $\|x^*_{1i}\|\to 0$ in (12.44), we conclude from the imposed PSNC property that $\|y^*_{1i}\|\to 0$ as $i\to\infty$. This gives $\|x^*_{2i}\|\to 0$ as $i\to\infty$ and also contradicts the relation (12.43). Therefore, $y^* \neq \mathbf{0}$ in (12.45), which yields the coderivative condition (12.34) by normalization and by the definition of the coderivative. □

Remark 12.11. In [5], Bao and Mordukhovich established in a unified scheme the Fermat rule not only for Pareto efficient solutions but also for quasi relative efficient solutions, intrinsic relative efficient solutions, primary relative efficient solutions. Due to the lack of space, we do not discuss these results here. We would like to mention that Bao–Mordukhovich's results gave significant improvements over recent results concerning Pareto efficient/ weak efficient solutions. We would like to note that the two assumptions (i) and (ii) in Theorem 12.7 hold automatically in finite-dimensional setting.

As mentioned in [5], the "dual compactness" of the cone K surely implies the SNC property of K and as the PSNC property w.r.t. the Clarke normal cone or w.r.t. the Mordukhovich normal cone surely implies the PSNC property w.r.t. the Fréchet normal cone, Theorem 12.7 is a generalization of Theorem 12.6 in Asplund space setting and of the following result.

Corollary 12.3 ([72, Theorem 4.2]). *Let X and Y be Asplund spaces. Assume that F has a closed graph and that one of the following conditions holds.*

(i) The cone K is dually compact.
(ii) int$K \neq \emptyset$ or Y is finite dimensional.
(ii) The inverse map F^{-1} is PSNC at $(\bar{y}, \bar{x})$ w.r.t. the Mordukhovich normal cone.

If $(\bar{x}, \bar{y})$ is a local Pareto efficient solution of (P), then the following inclusion holds

$$\mathbf{0} \in D^*_M F(\bar{x}, \bar{y})(y^*)$$

with some $y^* \in K^+$ and $\|y^*\| = 1$.

We illustrate the above results by an example.

Example 12.6. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Let $F : \mathbb{R} \mapsto 2^{\mathbb{R}^2}$ be defined by

$$F(x) = \begin{cases} \{(u,v) \mid u^2 + v^2 \leq 1 - x^2\} & \text{if } |x| \leq 1 \\ \{(u,v) \mid u^2 + v^2 \leq 1 - x^{-2}\} & \text{if } |x| > 1. \end{cases}$$

(a) Let $\bar{x} = 0$, $\bar{y} = (-1, 0)$. One can check that

$$N(0, (-1,0)); \text{gr}F) = \{(0, (-t, 0)) \mid t \geq 0\},$$

that $(0, (-1,0))$ is a local Pareto efficient solution of (P) and (12.34) holds

$$0 \in D^*F(0, (-1,0))((1,0)),$$

with $y^* = (1,0) \in K^+ = R^2_+$, where the normal cone and coderivative are in the sense of convex analysis as the graph of F is locally convex around $(0,(-1,0))$.

(b) Let $\bar{x} = 0$, $\bar{y} = (1,0)$. One can check that

$$N((0,(1,0)); \mathrm{gr}F) = \{(0,(t,0)) \mid t \geq 0\}.$$

Therefore, $0 \notin D^*F(0,(1,0))((r_1,r_2))$ for all $y^* = (r_1,r_2) \in \mathbb{R}^2_+ \setminus \{0\}$, where the normal cone and coderivative are in the sense of convex analysis as the graph of F is locally convex around $(0,(1,0))$. Thus, (12.34) does not holds for any $y^* \in K^+$ and $(0,(1,0))$ is not a local Pareto efficient solution of (P).

The following example shows that the necessary conditions for Pareto efficient solutions in Theorems 12.4, 12.5, 12.6 and 12.7 do not become sufficient even under additional convexity assumptions in finite-dimensional setting.

Example 12.7. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $K = \mathbb{R}^2_+$. Let $F : \mathbb{R} \mapsto 2^{\mathbb{R}^2}$ be defined by

$$F(x) := \{(u,v) \mid |u| \leq 1\,,\ |v| \leq 1\}$$

all $x \in \mathbb{R}$. Let $\bar{x} = 0$, $\bar{y} = (0,-1)$. One can check that

$$N(0,(0,-1)); \mathrm{gr}F) = \{(0,(0,-t)) \mid t \geq 0\}$$

and (12.34) holds

$$0 \in D^*F((0,(0,-1))((0,1)),$$

with $y^* = (0,1) \in K^+ = \mathbb{R}^2_+$, where the normal cone and the coderivative are in the sense of convex analysis as the graph of F is convex. However, it is easy to see that $(0,(0,-1))$ is not a local Pareto efficient solution of (P) (moreover, one has that $(x,(-1,-1))$ is a strongly efficient solution of (P) for any $x \in \mathbb{R}$).

12.4.2 The Fermat Rule for Strongly, Weakly and Properly Efficient Solutions of (P)

In this subsection, we discuss exact versions of the Fermat rule obtained in a unified scalarizing scheme for strongly/ weakly/properly efficient solutions of (P), see [28]. Our techniques are motivated by the ones developed in [18] for weakly efficient solutions and is based on the concept of s-efficiency and the Clarke penalization. In this subsection, D^* stands for the coderivatives in the senses of Ioffe, Clarke and Mordukhovich and X and Y are assumed to be Asplund when the coderivative is understood in the sense of Mordukhovich.

Firstly, we prove the Fermat rule for s-efficient solution of (P).

Theorem 12.8. *Assume that X and Y are Banach spaces, F has a closed graph and $(\bar{x},\bar{y})$ is a local s-efficient solution of (P) for some functional $s\colon Y \mapsto \mathbb{R}$. Assume further that s is Lipschitz on $F(U)$, where U is the neighborhood of $\bar{x}$ as in Definition 12.5. Then the following inclusion holds*

$$\mathbf{0} \in D^*F(\bar{x},\bar{y})(y^*) \tag{12.47}$$

with some $y^* \in \partial s(\mathbf{0})$.

Proof. By the definition, the functional $s(.-\bar{y})$ attains its minimum over $F(U)$ at $\bar{y}$. We get that $(\bar{x},\bar{y})$ is a minimizer of the functional $(x,y) \mapsto s(y-\bar{y})$ on $(U\times Y)\cap \mathrm{gr}F$. Applying the Clarke penalization, see Proposition 12.5, we have that $(\bar{x},\bar{y})$ is a minimizer of the functional $(x,y) \mapsto s(y-\bar{y}) + Ld((x,y),(U\times Y)\cap \mathrm{gr}F)$ on $X\times Y$, where L is the Lipschitz constant of s on $F(U)$. Therefore, Proposition 12.2 implies

$$\begin{aligned}(\mathbf{0},\mathbf{0}) &\in \partial[s(\cdot-\bar{y}) + Ld(\cdot,(U\times Y)\cap \mathrm{gr}F)](\bar{x},\bar{y})\\ &\subseteq \{\mathbf{0}\}\times \partial s(\mathbf{0}) + L\partial d(\cdot,(U\times Y)\cap \mathrm{gr}F)](\bar{x},\bar{y})\\ &= \{\mathbf{0}\}\times \partial s(\mathbf{0}) + N((\bar{x},\bar{y});(U\times Y)\cap \mathrm{gr}F)\\ &= \{\mathbf{0}\}\times \partial s(\mathbf{0}) + N((\bar{x},\bar{y});\mathrm{gr}F)\end{aligned}$$

(recall that $N((\bar{x},\bar{y});(U\times Y)\cap\mathrm{gr}F) = N((\bar{x},\bar{y});\mathrm{gr}F)$). Hence, there exists $y^* \in \partial s(\mathbf{0})$ with $(\mathbf{0},-y^*) \in N((\bar{x},\bar{y});\mathrm{gr}F)$. This implies $\mathbf{0} \in D^*F(\bar{x},\bar{y})(y^*)$. □

We derive now the Fermat rule for (local/global) strongly/ weakly/properly efficient solutions of (P).

Theorem 12.9. *Assume that X and Y are Banach spaces and F has a closed graph.*

(a) If $(\bar{x},\bar{y})$ is a strongly efficient solution of (P), then (12.47) holds for all $y^* \in K^+$.
(b) If $(\bar{x},\bar{y})$ is a weakly efficient solution of (P) or $(\bar{x},\bar{y})$ is a Hurwicz properly efficient solution of (P) and the image $F(X)$ has a nonempty interior, then (12.47) holds for some $y^* \in K^+\setminus\{\mathbf{0}\}$.
(c) If $(\bar{x},\bar{y})$ is a Henig global properly efficient solution (in particular, if $(\bar{x},\bar{y})$ is a positively properly efficient solution) of (P), then (12.47) holds for some $y^* \in K^{+i}$.
(d) If $(\bar{x},\bar{y})$ is a Henig properly efficient solution (in particular, if $(\bar{x},\bar{y})$ is a super efficient solution) of (P), then (12.47) holds for some $y^* \in K^{+i}$ satisfying $\inf_{\theta\in\Theta} y^*(\theta) > 0$.
(e) If $(\bar{x},\bar{y})$ is a super efficient solution of (P) and K has a bounded base, then (12.47) holds for some $y^* \in \mathrm{int}K^+$.

Proof. (a) By Proposition 12.8(ii), $(\bar{x},\bar{y})$ is an s-efficient solution of (P) with $s=\varphi$ for all $\varphi \in K^+$. The assertion then follows from Theorem 12.8 and the fact that $\partial s(\mathbf{0}) = \{\varphi\}$.

(b) By Theorem 12.2(a), (b), $(\bar{x},\bar{y})$ is an s-efficient solution of (P) with $s = \Delta_{-Q}$, where $Q = \text{int}K$ when $(\bar{x},\bar{y})$ is a weakly efficient solution of (P) and $Q = Y \setminus -\text{cl conv cone}[(A-\bar{a}) \cup K]$ when $(\bar{x},\bar{y})$ is a Hurwicz properly efficient solution of (P). The assertion then follows from Theorem 12.8 and the fact that in these cases, Proposition 12.12 (a), (b) give $\partial s(\mathbf{0}) = \partial \Delta_{-Q}(\mathbf{0}) \subseteq K^+ \setminus \{\mathbf{0}\}$.

(c) If $(\bar{x},\bar{y})$ is a Henig global properly efficient solution then by Theorem 12.2 (c), it is s-efficient solution of (P) with $s = \Delta_{-Q}$ and $Q = C$, C being an open pointed convex cone dilating K. The assertion (c) then follows from Theorem 12.8 and the fact that in this case, Proposition 12.12 (c) gives $\partial s(\mathbf{0}) = \partial \Delta_{-Q}(\mathbf{0}) \subseteq K^{+i}$. If $(\bar{x},\bar{y})$ is a positively properly efficient solution of (P), then by Proposition 12.7(e), it is a global Henig properly efficient solution and the assertion follows.

(d) Use the same argument as in (c) and take account of Propositions 12.12(d) and 12.7(f).

(e) Use the same argument as in (c) and take account of Proposition 12.12 (e). □

We can derive now the Fermat rule for Benson properly efficient solutions.

Corollary 12.4. *Assume that Y is a separable Banach space, or that Y is a reflexive Banach space and K has a base. Assume further that F has a closed graph and $F - \bar{y}$ is nearly K-subconvexlike on X, i.e., cl cone $(F(X) - \bar{y} + K)$ is convex. If $(\bar{x},\bar{y})$ is a Benson properly efficient solution of (P), then (12.47) holds for some $y^* \in K^{+i}$.*

Proof. The assertions follow from Proposition 12.7(g) and Theorem 12.9(c). □

Remark 12.12. Note that the Fermat rule has been formulated firstly in terms of coderivative of convex analysis for strongly efficient solutions under the additional assumption that F is convex in [2]. The case with weakly efficient solutions was considered in [5, 18], and the case with Henig properly efficient solutions and super efficient solutions in [4, 33] while the case with Hurwicz properly efficient solutions, Henig global properly efficient solutions, positive properly efficient solutions and Benson properly efficient solutions was studied in [28] with the help of the unified approach based on the concept of Q-minimal points.

When K is not assumed to have a bounded base, Huang obtain the following version of the Fermat rule for super efficient solutions in terms of the Clarke coderivative.

Theorem 12.10 ([33, Corollary 3.1]). *Assume that F has a closed graph and $(\bar{x},\bar{y})$ is a local super efficient solution of (P). Then for any $u^* \in I\!B_{Y^*}$, there exists $k^* \in K^+$ with $\|k^*\| \leq M$ such that*

$$\mathbf{0} \in D_C^* F(\bar{x},\bar{y})(k^* - u^*),$$

where $M > 0$ is a constant independent of $u^* \in I\!B_{Y^*}$.

We refer the interested reader to [33] for the details on Theorem 12.10. We would like to mention that the proof of this theorem is based on a scalar characterization of super efficient solutions which is derived from the definition of the latter.

Let us illustrate Theorem 12.9 by an example.

Example 12.8. Let X, Y, K and F be as in Example 12.6.

(a) Let $\bar{x} = 0$, $\bar{y} = (-\sqrt{2}/2, -\sqrt{2}/2)$. One can check that

$$N\left(0,\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)\right);\mathrm{gr}F) = \{(0,(-t,-t)) \mid t \geq 0\}$$

that $(0,(-\sqrt{2}/2,-\sqrt{2}/2))$ is a local Henig properly efficient solution of (P) and (12.47) holds

$$0 \in D^*F\left(0,\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)\right)((1,1)),$$

with $y^* = (1,1) \in K^{+i} = \{(r_1,r_2) \mid r_1 > 0, r_2 > 0\}$, where the normal cone and the coderivative are in the sense of convex analysis as the graph of F is locally convex around $(0,(-\sqrt{2}/2,-\sqrt{2}/2))$.

(b) Let $\bar{x} = 0$, $\bar{y} = (-1,0)$ as in the case (a) in Example 12.6. As

$$N(0,(-1,0));\mathrm{gr}F) = \{(0,(-t,0)) \mid t \geq 0\},$$

one has that $0 \notin D^*F(0,(-1,0))((r_1,r_2))$ for all $y^* = (r_1,r_2) \in K^{+i}$, i.e. (12.47) does not hold. Thus, $(0,(-1,0))$ is a local Pareto efficient solution of (P) but it is not a local Henig properly efficient solution of (P).

Remark 12.13. When F has a convex graph, some necessary conditions in Theorems 12.8 and 12.9 become also sufficient. Since the proof of this fact is similar to that used for the Lagrange multiplier rule, we do not present it here.

12.5 The Lagrange Multiplier Rule

This section is devoted to the Lagrange multiplier rule for the following set-valued optimization problem with constraints (CP)

$$\text{Minimize } \; F(x) \; \text{ subject to } \; x \in \Omega \text{ and } G(x) \cap \mathscr{C} \neq \emptyset,$$

where X, Y and Z are Banach spaces unless otherwise stated, F and G are set-valued maps from X respectively into Y and Z, $\Omega \subseteq X$ and $\mathscr{C} \subseteq Z$ are nonempty closed sets. We refer the interested readers to [5, 29, 33] for the set-valued optimization problem which contains only geometric constrain $x \in \Omega$. In this section, we always assume that $\bar{x} \in \Omega$, $(\bar{x},\bar{y}) \in \mathrm{gr}F$, $(\bar{x},\bar{z}) \in \mathrm{gr}G$ and $\bar{z} \in \mathscr{C}$, i.e., $(\bar{x},\bar{z}) \in \mathrm{gr}G \cap (\Omega \times \mathscr{C})$.

12.5.1 The Lagrange Multiplier Rule for Pareto Efficient Solutions of (CP)

The following fuzzy and exact versions of the Lagrange multiplier rule are adaptation from the ones formulated in [73] for a constrained set-valued optimization problem with several constraint maps, to our problem (CP) which contains only one constraint map G. We refer the reader to [73] for the proofs which are based on Lemma 12.1. In this subsection, $\mathscr{C}$ is assumed to be a closed convex cone in Z and $\mathscr{C}^+$ denotes the positive dual of $\mathscr{C}$, i.e. $\mathscr{C}^+ = \{c^* \in Z^* \mid \langle c, c^* \rangle \geq o\}$.

The fuzzy version of the Lagrange multiplier rule for Pareto efficient solutions of (CP) reads as follows.

Theorem 12.11 ([73, Theorems 3.1 and 3.2]). *Let X, Y and Z be Banach spaces. Assume that F and G have closed graphs. If $(\bar{x}, \bar{y})$ is a local Pareto efficient solution of (CP), then one of the following assertions holds.*

(a) For any $\varepsilon > 0$ there exist $x_1, x_2 \in \bar{x} + \varepsilon I\!B_X$, $w \in \Omega \cap (\bar{x} + \varepsilon I\!B_X)$, $y \in F(x_1) \cap (\bar{y} + \varepsilon I\!B_Y)$, $z \in G(x_2) \cap (\bar{z} + \varepsilon I\!B_Z)$, and $k^* \in K^+$, $c^* \in \mathscr{C}^+$ such that

$$\|k^*\| + \|c^*\| = 1$$

and

$$\mathbf{0} \in D_C^* F(x_1, y)(k^*) + D_C^* G(x_2, z)(c^*) \cap mI\!B_{X^*} + N_C(w; \Omega) \cap mI\!B_{X^*} + \varepsilon I\!B_{X^*},$$

where $m > 0$ is a constant independent of ε.

(b) For any $\varepsilon > 0$ there exist $x_1, x_2 \in \bar{x} + \varepsilon I\!B_X$, $w \in \Omega \cap (\bar{x} + \varepsilon I\!B_X)$, $y \in F(x_1) \cap (\bar{y} + \varepsilon I\!B_Y)$, $z \in G(x_2) \cap (\bar{z} + \varepsilon I\!B_Z)$, $x_1^* \in D_C^* F(x, y)(\varepsilon I\!B_{Y^*})$, $x_2^* \in D_C^* G(x, z)(\varepsilon I\!B_{Z^*})$ and $w^* \in N_C(w; \Omega) + \varepsilon I\!B_{Y^*}$ such that

$$\|w^*\| + \|x_1^*\| + \|x_2^*\| = 1 \text{ and } w^* + x_1^* + x_2^* = 0.$$

(c) Suppose that X, Y and Z are Asplund spaces. Then we can replace the Clarke normal cone and the Clarke coderivative in the assertions (a) and (b) by the Fréchet normal cone and the Fréchet coderivative. If in additions F and G are pseudo-Lipschitz at $(\bar{x}, \bar{y})$ and $(\bar{x}, \bar{z})$, respectively, then the assertion (a) expressed in terms of the Fréchet normal cone and the Fréchet coderivative holds.

We refer the reader to [73] for the proof of Theorem 12.11, which is based on [73, Lemma 2.1]. Note that [73, Lemma 2.1] recaptures the extended extremal principle due to Mordukhovich et al. [59] and contains Lemma 12.1 as a special case.

Remark 12.14. In the special case when $G(x) = \{\mathbf{0}\}$ for all $x \in X$ and $\Omega = X$, Theorem 12.11 recaptures Theorems 12.4 and 12.5.

Imposing additionally the dual compactness on the considered cones and the partial sequential normal compactness w.r.t. the Fréchet normal cone (i.e. PSNC in the sense of Mordukhovich and Shao) or w.r.t. the Clarke normal cone, or pseudo-Lipschizity on the considered maps, Zheng and Ng obtained the following exact versions of the Lagrange multiplier rule involving the coderivatives and normal cones in the senses of Mordukhovich or Clarke.

Theorem 12.12 ([73, Theorems 4.1 and 4.2]). *Let X, Y and Z be Asplund spaces. Assume that F and G have closed graphs and that the following conditions hold.*

(i) The cones K and $\mathscr{C}$ are dually compact.
(ii) The maps F and G are PSNC w.r.t. the Fréchet normal cone at $(\bar{x},\bar{y})$ and $(\bar{x},\bar{z})$, respectively.

If $(\bar{x},\bar{y})$ is a local Pareto efficient solution of (CP), then one of the following assertions holds.

(a) There exist $k^* \in K^+$, $c^* \in \mathscr{C}^+$ such that

$$\|k^*\| + \|c^*\| = 1$$

and

$$\mathbf{0} \in D_M^* F(\bar{x},\bar{y})(k^*) + D_M^* G(\bar{x},\bar{z})(c^*) + N_M(\bar{x};\Omega). \tag{12.48}$$

(b) There exist $x_1^* \in D_M^* F(\bar{x},\bar{y})(\mathbf{0})$, $x_2^* \in D_M^* G(\bar{x},\bar{z})(\mathbf{0})$ and $w^* \in N_M(\bar{x};\Omega)$ such that

$$\|w^*\| + \|x_1^*\| + \|x_2^*\| = 1 \text{ and } w^* + x_1^* + x_2^* = \mathbf{0}.$$

If, in addition, F and G are pseudo-Lipschitz at $(\bar{x},\bar{y})$ and $(\bar{x},\bar{z})$, respectively, then the assertion (a) holds.

Theorem 12.13 ([73, Theorem 4.3]). *Let X, Y and Z be Banach spaces. Assume that the following conditions hold.*

(i) The cones K and $\mathscr{C}$ are dually compact.
(ii) The maps F and G are PSNC w.r.t. the Clarke normal cone at $(\bar{x},\bar{y})$ and $(\bar{x},\bar{z})$, respectively.
(iii) The set Ω and the maps F, G are normally closed at $\bar{x}$, $(\bar{x},\bar{y})$ and $(\bar{x},\bar{z})$, respectively.

If $(\bar{x},\bar{y})$ is a local Pareto efficient solution of (CP), then one of the following assertions holds.

(a) There exist $k^* \in K^+$, $c^* \in \mathscr{C}^+$ such that

$$\|k^*\| + \|c^*\| = 1$$

and

$$\mathbf{0} \in D^*_C F(\bar{x},\bar{y})(k^*) + D^*_C G(\bar{x},\bar{z})(c^*) + N_C(\bar{x};\Omega).$$

(b) There exist $x_1^* \in D^*_C F(\bar{x},\bar{y})(\mathbf{0})$*,* $x_2^* \in D^*_C G(\bar{x},\bar{z})(\mathbf{0})$ *and* $w^* \in N_C(\bar{x};\Omega)$ *such that*

$$\|w^*\| + \|x_1^*\| + \|x_2^*\| = 1 \text{ and } w^* + x_1^* + x_2^* = \mathbf{0}.$$

We refer the reader to [73] for the proofs of Theorems 12.12–12.13, which is based on Theorem 12.11.

Let us illustrate Theorems 12.11–12.13 by an example.

Example 12.9. Let X, Y, K and $F : \mathbb{R} \mapsto 2^{\mathbb{R}^2}$ be as in Example 12.6. Let $\Omega = [-1,1]$, $Z = \mathbb{R}^2$, $\mathscr{C} = \mathbb{R}^2_-$ and $G : \mathbb{R} \mapsto \mathbb{R}^2$ be defined as $G(x) = (x,x)$.

(a) Let $\bar{x} = 0$, $\bar{y} = (-1,0)$, $\bar{z} = (0,0)$. One can check that $(0,(-1,0))$ is a local Pareto efficient solution of (CP). Further, it is easy to see that

$$N(0,(-1,0)); \mathrm{gr}F) = \{(0,(-t,0)) \mid t \geq 0\},$$

$N(0;[-1,1]) = \{0\}, N(0;\mathscr{C}) = \mathbb{R}^2_+$ and

$$N(0,(0,0)); \mathrm{gr}G) = \{(u,(r_1,r_2)) \mid u + r_1 + r_2 = 0\},$$

(the coderivative and the normal cone are in the sense of convex analysis). Then (with $k^* = (1,0)$, $w = 0$, $z^* = (r_1,r_2) = (0,0)$, $c^* = (0,0)$) we have

$$0 \in D^*F(0,(-1,0))((1,0)) + D^*G(0,(0,0))((0,0)) + N(0;[-1,1]),$$

i.e. (12.48) holds.

(b) Let $\bar{x} = 0$, $\bar{y} = (1,0)$, $\bar{z} = (0,0)$. Then $(0,(1,0))$ is not a local Pareto efficient solution of (CP). Indeed, for any $z^* = (r_1,r_2) \in N(0;\mathscr{C}) = \mathbb{R}^2_+$ one has $D^*G(0,(0,0))((r_1,r_2)) = r_1 + r_2$. Since $N(0,(1,0)); \mathrm{gr}F) = \{(0,(t,0)) \mid t \geq 0\}$, there are no $k^* \in K^+$ and $c^* \in N(0;\mathscr{C})$ with $\|k^*\| + \|c^*\| = 1$ such that

$$0 \in D^*F(0,(1,0))(k^*) + D^*G(0,(0,0))(c^*) + N(0;[-1,1]),$$

i.e. (12.48) does not hold.

Remark 12.15. Example 12.7 (Ω=X, Z=$I\!R^2$, $\mathscr{C}$=$\{(0,0)\}$ and $G(x)$=$\{(0,0)\}$, $\forall x \in X$) shows that the necessary conditions for Pareto efficient solutions in Theorems 12.11–12.13 do not become sufficient even under additional convexity assumptions in finite-dimensional setting while the sufficient conditions stated in [73, Proposition 4.1] are stronger than the necessary ones stated in the same paper (Theorems 12.11–12.13 here) and are in fact sufficient conditions for positive properly efficient solutions.

12.5.2 The Lagrange Multiplier Rule for Weakly, Strongly, Properly Efficient Solutions of (CP)

In this subsection, we obtain in a unified scheme versions of the Lagrange multiplier rule for weakly, strongly, properly efficient solutions of (CP). Firstly, we apply the techniques of [18] to establish the Lagrange multiplier rule for s-efficient solutions of (CP) and then derive similar versions for strongly, weakly, properly efficient solutions of (CP). We also show that several obtained necessary conditions are sufficient. In this subsection, N and D^* stand for the normal cones and the coderivatives in the senses of Ioffe, Clarke and Mordukhovich, and the spaces are Asplund when the normal cone and the coderivative is in the sense of Mordukhovich.

We will need the following assumption.

Assumption (A):

(1) The sets Ω and $\mathscr{C}$ are closed.
(2) F and G are closed and pseudo-Lipschitz around $(\bar{x},\bar{y})$ and $(\bar{x},\bar{z})$ respectively.
(3) G is metrically regular around $(\bar{x},\bar{z})$ relatively to $\Omega \times \mathscr{C}$.

The Lagrange multiplier rule for s-efficient solutions of (CP) reads as follows.

Theorem 12.14. *Let Assumption (A) be satisfied and $(\bar{x},\bar{y})$ be a local s-efficient solution of (CP) for some functional $s\colon Y \mapsto \mathbb{R}$. Assume further that s is Lipschitz on $F(\mathscr{A} \cap U_{\bar{x}})$, where $U_{\bar{x}}$ is the neighborhood of $\bar{x}$ as in Definition 12.5. Then there exists $y^* \in \partial s(\mathbf{0})$ such that*

$$\mathbf{0} \in D^*F(\bar{x},\bar{y})(y^*) + D^*G(\bar{x},\bar{z})(z^*) + N(\bar{x};\Omega) \quad \text{for some } z^* \in N(\bar{z};\mathscr{C}). \tag{12.49}$$

Proof. The proof is similar to that of [18, Theorem 3.7]. Without loss of generality, we can assume that the same constant $\gamma \geq 0$ is figured in the definitions of the pseudo-Lipschitzity property of F and of G around $(\bar{x},\bar{y})$ and around $(\bar{x},\bar{z})$, respectively, and of the metric regularity of G around $(\bar{x},\bar{z})$ relatively to $\Omega \times \mathscr{C}$. By the definition, $\bar{y}$ is a s-efficient point of $F(S \cap U_{\bar{x}})$ for some neighborhood $U_{\bar{x}}$ of $\bar{x}$. Define $q : X \times Y \times Z \to \mathbb{R}$ by $q(x,y,z) := s(y-\bar{y})$ and

$$\Lambda := \{(x,y,z) \in X \times Y \times Z \mid (x,y) \in (U_{\bar{x}} \times Y) \cap \operatorname{gr} F,\ (x,z) \in (\Omega \times \mathscr{C}) \cap \operatorname{gr} G\}.$$

One can easily check that $\Lambda \subseteq U_{\bar{x}} \times Y \times Z$ and $(\bar{x},\bar{y},\bar{z})$ is a minimizer of the functional q on Λ. Suppose that s is Lipschitz of rank L on $F(U_{\bar{x}})$, then q is Lipschitz of rank L on $U_{\bar{x}} \times Y \times Z$. Then by the Clarke penalization, see Proposition 12.5, $(\bar{x},\bar{y},\bar{z})$ is a minimizer (and hence, a local minimizer) of the functional $q(\cdot) + Ld(\cdot;\Lambda)$ on $U_{\bar{x}} \times Y \times Z$. Therefore, we can assume that

$$0 = s(\mathbf{0}) + Ld((\bar{x},\bar{y},\bar{z});\Lambda) \leq s(y-\bar{y}) + Ld((x,y,z);\Lambda) \tag{12.50}$$

for (x,y,z) near $(\bar{x},\bar{y},\bar{z})$. By Proposition 12.6, we have

$$d((x,y,z);\Lambda) \leq (1+\gamma)[d((x,y);(U_{\bar{x}}\times Y)\cap \mathrm{gr}\, F)+d((x,z);(\Omega\times\mathscr{C})\cap \mathrm{gr}\, G)] \tag{12.51}$$

for (x,y,z) near $(\bar{x},\bar{y},\bar{z})$. Further, the metric regularity of G around $(\bar{x},\bar{z})$ relatively to $\Omega\times\mathscr{C}$ yields that

$$d((x,z);(X\times\mathscr{C})\cap \mathrm{gr}\, G) \leq \gamma d(z;G(x)) \tag{12.52}$$

for $(x,z)\in\Omega\times\mathscr{C}$ near $(\bar{x},\bar{z})$ and the pseudo-Lipschitz property of G around $(\bar{x},\bar{z})$ (see (12.7)) yields

$$d(z;G(x)) \leq (1+\gamma)d((x,z);\mathrm{gr}\, G) \tag{12.53}$$

for $(x,z)\in\Omega\times\mathscr{C}$ near $(\bar{x},\bar{z})$. Combining (12.50)–(12.53) gives

$$\begin{aligned} 0 &\leq s(y-\bar{y})+Ld((x,y,z);\Lambda)\\ &\leq s(y-\bar{y})+L(1+\gamma)[d((x,y);(U_{\bar{x}}\times Y)\cap \mathrm{gr}\, F)+d((x,z);(\Omega\times\mathscr{C})\cap \mathrm{gr}\, G)]\\ &\leq s(y-\bar{y})+L(1+\gamma)[d((x,y);(U_{\bar{x}}\times Y)\cap \mathrm{gr}\, F)+\gamma(1+\gamma)d((x,z);\mathrm{gr}\, G)] \end{aligned}$$

for $(x,y,z)\in\Omega\times Y\times\mathscr{C}$ near $(\bar{x},\bar{y},\bar{z})$, which means that $(\bar{x},\bar{y},\bar{z})$ is a local minimizer of the functional $\tilde{q}$ over $\Omega\times Y\times\mathscr{C}$ that is defined by $\tilde{q}\colon X\times Y\times Z\to\mathbb{R}$,

$$\tilde{q}(x,y,z):=s(y-\bar{y})+(1+\gamma)Ld((x,y);(U_{\bar{x}}\times Y)\cap \mathrm{gr}\, F)+\gamma(1+\gamma)^2Ld((x,z);\mathrm{gr}\, G).$$

Applying the Clarke penalization again, we see that for some integer $l>0$ large enough, $(\bar{x},\bar{y},\bar{z})$ is an unconstrained local minimizer of

$$(x,y,z)\mapsto s(y-\bar{y})+ld((x,y);\mathrm{gr}F)+ld((x,z);\mathrm{gr}G)+ld((x,z);\Omega\times\mathscr{C}).$$

By Proposition 12.2, $\mathbf{0}$ is in the sum of the subdifferentials, that is there exist $y_1^*\in\partial s(\mathbf{0})$,

$$(x_2^*,y_2^*)\in l\partial d((\bar{x},\bar{y});\mathrm{gr}F)\subseteq N((\bar{x},\bar{y});\mathrm{gr}F),$$
$$(x_3^*,z_3^*)\in l\partial d((\bar{x},\bar{z});\mathrm{gr}G)\subseteq N((\bar{x},\bar{z});\mathrm{gr}G)$$

and

$$(x_4^*,-z_4^*)\in l\partial d((\bar{x},\bar{z});\Omega\times\mathscr{C})\subseteq N((\bar{x},\bar{z});\Omega\times\mathscr{C})$$

such that

$$\mathbf{0}=x_2^*+x_3^*+x_4^*, \mathbf{0}=y_1^*+y_2^* \text{ and } \mathbf{0}=z_3^*+z_4^*.$$

Putting $y^*=y_1^*=-y_2^*$ and $z^*=z_4^*=-z_3^*$, we obtain

$$\mathbf{0}\in D^*F(\bar{x},\bar{y})(y^*)+D^*G(\bar{x},\bar{z})(z^*)+N(\bar{x};\Omega)$$

and (12.49) holds. □

The following Lagrange multiplier rule for several types of efficient solutions of (CP) reads as follows.

Theorem 12.15. *Let Assumption (A) be satisfied and let the notations be as in Theorem 12.14.*

(a) If $(\bar{x}, \bar{y})$ *is a strongly efficient solution of* (CP)*, then (12.49) holds for all* $y^* \in K^+ \setminus \{\mathbf{0}\}$.
(b) If $(\bar{x}, \bar{y})$ *is a weakly efficient solution of* (CP) *or* $(\bar{x}, \bar{y})$ *is a Hurwicz properly efficient solution of* (CP) *and the image* $F(\mathscr{A})$ *has a nonempty interior, then (12.49) holds for some* $y^* \in K^+ \setminus \{\mathbf{0}\}$.
(c) If $(\bar{x}, \bar{y})$ *is a Henig global properly efficient solution (in particular, a positive properly efficient solution) of* (CP)*, then (12.49) holds for some* $y^* \in K^{+i}$.
(d If $(\bar{x}, \bar{y})$ *is a Henig properly efficient solution (in particular, a super efficient solution) of* (CP)*, then (12.49) holds for some* $y^* \in K^{+i}$ *satisfying* $\inf_{\theta \in \Theta} y^*(\theta) > 0$.
(e) If $(\bar{x}, \bar{y})$ *is a super efficient solution of* (CP) *and K has a bounded base, then (12.49) holds for some* $y^* \in intK^+$.

Proof. This theorem can be proved in the same way as for Theorem 12.9 (by applying Theorem 12.14 in place of Theorem 12.8). □

Remark 12.16. (a) The above version of Lagrange multiplier rule are known for weakly efficient solutions in [18], for strongly efficient solutions or positive properly efficient solutions in [27] and for Hurwicz properly efficient solution, Henig global properly efficient solution, Henig properly efficient solution and super efficient solutions in [28].
(b) One can use the techniques of [18] further to obtain other optimality conditions for (CP) for instance in terms of coderivatives of the map (F, G), where $F, G)(x) = F(x) \times G(x)$.
(c) The case when the constraints set of (SP) contains only the geometric constraint $x \in \Omega$ was recently considered in [4, 29, 33]. Namely, the Lagrange multiplier rule involving coderivatives has been established for super efficient solutions in Asplund space settings by Bao and Mordukhovich in [4] and in Banach space settings by Huang in [33]. In [29], we applied results on the Lagrange multiplier rule for weakly, strongly, properly efficient solutions of set-valued optimization problems to the study of optimality conditions for weakly, strongly, properly efficient solutions of set-valued equilibrium problems.

We can derive now the Lagrange multiplier rule for Benson properly efficient solutions.

Corollary 12.5. *Let Assumption (A) be satisfied and let the notations be as in Theorem 12.14. Assume that Y is a separable Banach space, or that Y is a reflexive Banach space and K has a base. Assume further that* $F - \bar{y}$ *is nearly K-subconvexlike on* $\mathscr{A}$*, i.e., cl cone* $(F(\mathscr{A}) - \bar{y} + K)$ *is convex. If* $(\bar{x}, \bar{y})$ *is a Benson efficient solution of* (CP)*, then (12.49) holds for some* $y^* \in K^{+i}$.

Proof. The assertions follow from Proposition 12.7 (g), Theorem 12.15(c). □

Let us illustrate Theorem 12.15 by an example.

Example 12.10. Let X, Y, Z, K, $\mathscr{C}$, Ω, F and G be as in Example 12.9.

(a) Let $\bar{x} = 0$, $\bar{y} = (-\sqrt{2}/2,\ -\sqrt{2}/2)$, $\bar{z} = (0,0)$. One can check that $(0,(-\sqrt{2}/2,-\sqrt{2}/2))$ is a local Henig properly efficient solution of (CP) and (12.49) holds

$$0 \in D^*F\left(0,\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)\right)((1,1))+D^*G(0,(0,0))((0,0))+N(0;[-1,1]).$$

(b) Let $\bar{x}=0, \bar{y}=(-1,0), \bar{z}=(0,0)$. By an argument similar to that for the case (b) in Example 12.8 one can check that $(0,(-1,0))$ is not a local Henig properly efficient solution of (CP) because (12.49) does not hold.

Below we show that under convexity assumptions on the sets Ω and $\mathscr{C}$ as well on the graphs of the maps F and G, some necessary conditions in Theorem 12.15 become sufficient.

Theorem 12.16. *Suppose that*

(i) The sets Ω and $\mathscr{C}$ ares closed and convex.
(ii) F and G are closed and convex, i.e. their graphs are closed and convex.

Then

(a) If (12.49) holds for some $y^* \in K^{+i}$, then $(\bar{x},\bar{y})$ is a positive properly efficient solution of (CP).
(b) If (12.49) holds for all $y^* \in K^+ \setminus \{\mathbf{0}\}$, then $(\bar{x},\bar{y})$ is a strongly efficient solution of (CP).
(c) If (12.49) holds for some $y^* \in K^+ \setminus \{\mathbf{0}\}$ and $\mathrm{int}K \neq \emptyset$, then $(\bar{x},\bar{y})$ is a weakly efficient solution of (CP).

Proof. Observe first that since the graphs of F, G and the sets Ω, $\mathscr{C}$ are convex, the coderivatives and normal cones figured in (12.49) are understood in the sense of convex analysis. Suppose now that (12.49) holds for some $y^* \in K^+ \setminus \{\mathbf{0}\}$. Then one can find elements $x_1^*, x_2^* \in X^*$, and $x_3^* \in N(\bar{x};\Omega)$ such that $(x_1^*,-y^*) \in N((\bar{x},\bar{y});\mathrm{gr}\,F)$, $(x_2^*,-z^*) \in N((\bar{x},\bar{z});\mathrm{gr}\,G)$ and

$$x_1^*+x_2^*+x_3^*=\mathbf{0}. \tag{12.54}$$

According to the definition of the normal cone of convex analysis, see Proposition 12.1, we have

$$\langle (x_1^*,-y^*),(x,y)-(\bar{x},\bar{y})\rangle \le 0 \text{ for all } (x,y)\in \mathrm{gr}\,F \tag{12.55}$$

$$\langle (x_2^*,-z^*),(x,z)-(\bar{x},\bar{z})\rangle \le 0 \text{ for all } (x,z)\in \mathrm{gr}\,G, \tag{12.56}$$

$$\langle x_3^*,x-\bar{x}\rangle \le 0 \text{ for all } x\in\Omega \tag{12.57}$$

and

$$\langle z^*, z - \bar{z} \rangle \leq 0 \text{ for all } z \in \mathscr{C}. \tag{12.58}$$

Summarizing (12.55)–(12.58) and taking account of (12.54) we obtain

$$\langle y^*, y - \bar{y} \rangle \geq 0 \text{ for all } y \in F(\mathscr{A}). \tag{12.59}$$

Therefore, if (12.49) holds for some $y^* \in K^{+i} \subseteq K^+ \setminus \{\mathbf{0}\}$, then (12.59) and Proposition 12.8 (a) implies that $(\bar{x}, \bar{y})$ is a positive properly efficient solution of (CP). Next, if (12.49) holds for all $y^* \in K^+ \setminus \{\mathbf{0}\}$, then (12.59) holds for all $y^* \in K^+ \setminus \{\mathbf{0}\}$ and Proposition 12.8 (b) implies that $(\bar{x}, \bar{y})$ is a strongly efficient solution of (CP). Finally, if (12.49) holds for some $y^* \in K^+ \setminus \{\mathbf{0}\}$ and int$K \neq \emptyset$, then (12.59) holds for some $y^* \in K^+ \setminus \{\mathbf{0}\}$ and Proposition 12.9 implies that $(\bar{x}, \bar{y})$ is a weakly efficient solution of (CP). □

Remark 12.17. (a) The above sufficient conditions in the form of the Lagrange multiplier rule have been obtained for weakly efficient solutions of (CP) in [18] and for strongly efficient solutions and positive properly efficient solutions of (CP) in the first time here.

(b) In [29], we established that the necessary conditions in the form of the Lagrange multiplier rule for Henig global properly efficient solutions, Henig properly efficient solutions and super efficient solutions become sufficient under the convexity assumptions when the constraint set contains only the geometric condition $x \in \Omega$. This result can be extended to these solutions of (CP) by applying the techniques of [29].

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Chapter 13
Extended Pareto Optimality in Multiobjective Problems

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13.1 Introduction

This chapter largely discusses some major notions of optimal/efficient solutions in multiobjective optimization and studies general necessary conditions for minimal points of sets and for minimizers of constrained set-valued optimization problems with respect to extended Pareto preference relations.

Let us first recall the definitions of preferences from [42, Definition 1.4]; cf. also the classical mathematical economics books [29, 41], the recent books [16, 22, 27, 35] on vector optimization, and the references therein.

Definition 13.1 (Preference, Strict Preference, and Indifference). Let $Q \subseteq Z \times Z$ be an arbitrary subset of a product space $Z \times Z$, and let R be a binary relation on Q describing by xRy if and only if $(x,y) \in Q$ for all $x,y \in Z$.

- The relation $\prec$ defined by $[x \prec y \text{ iff } xRy \text{ and } \neg yRx]$ is called a *strict preference* on Q.
- The relation $\sim$ defined by $[x \sim y \text{ iff } xRy \text{ and } yRx]$ is called an *indifference* on Q.
- The disjoint union $R := \prec \cup \sim$ denoted by $\preceq$ is called a *preference* on Q.
- A preference is a *partial order* if and only if it is reflexive, anti-symmetric, and transitive.
- A strict preference is a *strict partial order* if and only if it is nonreflexive, anti-symmetric, and transitive.

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Note that in what follows the term "preference" stands for either reflexive or nonreflexive preference depending on the context and the notation $\prec$ or $\preceq$.

Associate a preference $\prec$ (respectively, $\preceq$) with a preference-set/level-set multifunction $P_\prec$ (respectively, $P_\preceq : Z \rightrightarrows Z$) given by

$$P_\prec(z) := \{y \in Z \mid y \prec z\}, \quad \text{(respectively, } P_\preceq(z) := \{y \in Z \mid y \preceq z\},)$$

which describes the set of points being *better* than (respectively, better than or equal to) the point in question. It is worth to emphasizing that the adjective "better" means bigger in the economics models while it stands for smaller in vector optimization.

Definition 13.2 (Preference Points). Let $\Xi \subseteq Z$ be a subset of the space Z, and let $\bar{z} \in \Xi$. We say that:

- $\bar{z}$ is a *weak preference point* with respect to $\prec$ if there is no $z \in \Xi$ such that it is preferred to $\bar{z}$ with respect to the preference $\prec$, i.e.,

$$\Xi \cap P_\prec(\bar{z}) = \emptyset.$$

- $\bar{z}$ is a *preference point* with respect to $\preceq$ if there is no $z \in \Xi \setminus \{\bar{z}\}$ such that $z \preceq \bar{z}$, i.e.,

$$\Xi \cap P_\preceq(\bar{z}) = \{\bar{z}\}.$$

Given a strict preference $\prec$, define an indifference relation $\sim$ by $[x \sim y \text{ iff } x = y]$. Then one cannot distinguish between weak preference points with respect to $\prec$ and preference points with respect to its induced preference $\preceq := \prec \cup \sim$. Observe that the preference relation $\preceq$ obtained by $P_\preceq(z) := \mathrm{cl}\, P_\prec(z)$ from the preference $\prec$ generally distinguishes two kinds of preference points given in Definition 13.2. In particular, such an operation was used in some recent publications as an attempt to weaken the transitivity property of ordering relations. For simplicity, we use the notation $P(\bar{z})$ and $\mathrm{cl}\, P(\bar{z})$ instead of $P_\prec(\bar{z})$ and $P_\preceq(\bar{z})$, respectively.

Next we recall the concept of Pareto efficiency, which surely has its root in economic equilibrium and welfare theory. Given a set of alternative allocations of goods or income for a set of individuals, a change from one allocation to another that can make at least one individual better off without making any other individual worse off is called a Pareto improvement. An allocation is *Pareto efficient/optimal* when no further Pareto improvement can be made. It is important to emphasize that both Francis Edgeworth (1845–1926) and Vilfredo Pareto (1848–1923) are credited for originally introducing the concept of noninferiority (known as Pareto minimality) in economics, and thus it should be *Edgeworth–Pareto efficiency*. The reader is referred to [43] for a biographical survey of history and developments in multiobjective optimization with more details, commentaries, and proofs.

We illustrate this concept via a simple exchange economy $\mathscr{E}$ with n customers. Let E be a commodity space, let $C_i \subseteq E$ be a consumption set of customer i, and let $\overline{w}_i$ be an initial endowment of customer i for $i = 1, \ldots, n$. The quantity $\overline{w} = \sum_{i=1}^{n} \overline{w}_i$ is the *total endowment* of the economy $\mathscr{E}$. The bundle $z = (z_1, \ldots, z_n) \in C_1 \times \cdots \times C_n$ is an *admissible* allocation of $\mathscr{E}$. It is said to be *feasible* provided that the exchange condition $\sum_{i=1}^{n} z_i = \overline{w}$ is satisfied. Denote the collection of all the feasible allocations to $\mathscr{E}$ by Ξ. Assume that each customer i has his/her own preference, denoted by $\prec_i$, reflecting his/her "feeling" or "taste" and corresponding to the preference-set multifunction $P_i : \prod_{i=1}^{n} C_i \rightrightarrows C_i$ with $P_i(z) := \{y \in C_i \mid y \prec_i z_i\}$ for all $i = 1, \ldots, n$. We also have the derived preferences $\preceq_i$ with the preference sets $\operatorname{cl}(P_i(z))$. Recall that:

- A feasible allocation $\overline{z} \in \Xi$ is a *weak Pareto optimal allocation* of $\mathscr{E}$ if there is no feasible allocation $z \in \Xi$ such that $z_i \prec \overline{z}_i$ for all $i = 1, \ldots, n$, i.e.,

$$\Xi \cap \left(\prod_{i=1}^{n} P_i(\overline{z}) \right) = \emptyset \tag{13.1}$$

- A feasible allocation $\overline{z} \in \Xi$ is a *Pareto optimal allocation* of $\mathscr{E}$ if there is no other feasible allocations $z \in \Xi \setminus \{\overline{z}\}$ such that $z_i \preceq \overline{z}_i$ for all $i = 1, \ldots, n$, i.e.,

$$\Xi \cap \left(\prod_{i=1}^{n} \operatorname{cl} P_i(\overline{z}) \right) = \{\overline{z}\} \tag{13.2}$$

Observe that by defining two product preferences $\prec := \prod_{i=1}^{n} \prec_i$ and $\preceq := \prod_{i=1}^{n} \preceq_i$ by, respectively,

$$P(z) := \prod_{i=1}^{n} P_i(z) \quad \text{and} \quad \operatorname{cl} P(z) := \prod_{i=1}^{n} \operatorname{cl} P_i(z), \tag{13.3}$$

we can see that the point $\overline{z}$ is a weak Pareto/Pareto optimal allocation of $\mathscr{E}$ if and only if it is a weak preference/preference point to Ξ with respect to $\prec$ and $\preceq$. The reader is referred to [22, Example 4.6] for a cooperative n player game in which the preference of each player is induced by a convex cone. Proceeding in this way is by far different from the utility approach widely used in the early development of mathematical economics; see, e.g., the basic economics books [29, 41]. The latter approach heavily relies on the utility description of preferences. A function $u : Z \to \mathbb{R}$ is a *utility function* for the preference $\prec$ (respectively, $\preceq$) if it is *order-preserving*, i.e., for every $y, z \in Z$ we have the implication

$$[y \prec z \quad \Longrightarrow \quad u(y) < u(z)], \quad (\text{respectively, } [y \preceq z \quad \Longrightarrow \quad u(y) \le u(z)]).$$

Assume that for each $i \in \{1, \ldots, n\}$, u_i is a utility function of $\prec_i$. It is easy to check that $\overline{z}$ is a weak Pareto optimal allocation of $\mathscr{E}$ if there is no $z \in \Xi$ such that $u_i(z) < u_i(\overline{z})$ for all $i = 1, \ldots, n$, i.e.,

$$u(\Xi) \cap (\overline{u} - \operatorname{int} \mathbb{R}^n_+) = \emptyset,$$

and that $\bar{z}$ is a Pareto optimal allocation of $\mathscr{E}$ if there is no $z \in \Xi \setminus \{\bar{z}\}$ such that $u_i(z) \le u_i(\bar{z})$ for all $i \in \{1,\ldots,n\}$ while there is some $i_0 \in \{1,\ldots,n\}$ such that $u_{i_0}(z) < u_{i_0}(\bar{z})$, i.e.,

$$u(\Xi)\cap(\bar{u}-\mathbb{R}^n_+) = \{\bar{u}\},$$

where $u : Z \to \mathbb{R}^n$ is defined by $u(z) := (u_1(z),\ldots,u_n(z))$, $u(\Xi) := \{u(z) \mid z \in \Xi\}$ is the image set to Ξ, and $\bar{u} := (u_1(\bar{z}),\ldots,u_n(\bar{z}))$. In the other words, $\bar{z}$ is a weak Pareto/Pareto optimal allocation to $\mathscr{E}$ if and only if it is a weak Pareto/Pareto minimal point to the image set $u(\Xi)$ with respect to the weak Pareto/Pareto preference generated by the positive cone of $\mathbb{R}^n$.

Vector optimization studies certain "*optimal/efficient*" elements of a nonempty subset Ξ in a partially ordered linear space Z, where the ordering relation is generated by a closed, convex, and pointed cone $\Theta \subseteq Z$. Denoting the weak Pareto ordering relation by

$$z_1 <_\Theta z_2 \quad \text{if and only if} \quad z_2 - z_1 \in \operatorname{int}\Theta \tag{13.4}$$

and the Pareto one by

$$z_1 \le_\Theta z_2 \quad \text{if and only if} \quad z_2 - z_1 \in \Theta, \tag{13.5}$$

respectively. These partial orders are known as the *weak Pareto preference* $\prec_\Theta$ with preference sets given by

$$P_\prec(z) := z - \operatorname{int}\Theta, \tag{13.6}$$

and the *Pareto preference* $\preceq_\Theta$ with

$$P_\preceq(z) := z - \Theta. \tag{13.7}$$

We say that $\bar{z} \in \Xi$ is a *weak Pareto / Pareto minimal point* to Ξ if and only if it is a *weak preference / preference* point to Ξ with respect to the preference $<_\Theta/\le_\Theta$ if and only if the optimality condition holds:

$$\Xi\cap(\bar{z}-\operatorname{int}\Theta) = \emptyset, \quad (\text{respectively, } \Xi\cap(\bar{z}-\Theta) = \{\bar{z}\}).$$

It is known that the Pareto preference $\le_\Theta$ is a partial order if and only if Θ is a closed, convex, and pointed cone. However, it is no longer assumed to be pointed in several recent papers, for example, [5,6], and thus the preference $\le_\Theta$ is not a partial order since it does not have the transitivity property.

It is important to emphasize that the most common tool used in establishing necessary optimality conditions for minimal points to sets in vector optimization is the separation theorem for convex sets. It has various extensions to nonconvex settings. One of them was proposed by Tammer and Weidner in [44].

Theorem 13.1 ([16, Theorems 2.3.1]). *Let $C \subseteq Z$ be a closed proper set in a topological vector space and $k^\circ \in \operatorname{int} C$ such that*

$$C + \mathbb{R}_+ \cdot k^\circ \subseteq C.$$

Then the function $\varphi : Z \to \mathbb{R}$ defined by

$$\varphi_{C,k^\circ}(z) := \inf\{t \in \mathbb{R} \mid z \in tk^\circ - C\}$$

is lower semicontinuous and $\varphi_{C,k^\circ}(z + tk^\circ) = \varphi_{C,k^\circ}(z) + t$ for all $z \in Z$ and $t \in \mathbb{R}$. It is continuous provided that $C + \mathbb{R}_+ \cdot k^\circ \subseteq \operatorname{int} C$. Assume further that C is a convex cone. If $\bar{z}$ is a Pareto minimal point to Ξ with respect to C, then it is a minimum of the function φ_{C,k° over Ξ.

Proof. See the aforementioned references. □

Note that we still need the convexity of the ordering cone C to convert a vector optimization problem to a scalar one; in the other words, it has not been applied to a full nonconvex optimization problem yet. See, for example, [12–15, 18]. Note further that the nonempty interiority condition $\operatorname{int} C \neq \emptyset$ is unavoidable in numerous separation results in both convex and nonconvex settings to ensure the continuity property of separation/utility functions. However, it is a troublesome in infinite-dimensional spaces, since the natural ordering cone of each Lebesgue space l^p and L^p for $1 \leq p < \infty$ has an empty relative interior, and thus an empty interior. In the convex setting it might be weakened to the nonempty quasi-interiority condition; see, e.g., [9,10,19]; the latter is automatics in separable Banach spaces for nonempty convex sets by [9, Theorem 2.8].

In contrast to the utility function approach known as scalarization, the variational approach does not require such a condition. The latter approach relies on the extremal principle, which can be seen as a variational counterpart of the separation theorem for nonconvex sets and goes back to the original publication by Mordukhovich [30, 31] and by Kruger and Mordukhovich [20], where necessary conditions were established for problems of scalar optimization as well as for some vector problems with respect to the classical Pareto utility notion. It was extended to more involved vector-valued and set-valued optimization in [1,3–5,8,35,37,46,47] and the references therein. In this chapter, we adopt this approach to study necessary conditions for broad classes of multiobjective optimization problems mainly focusing on eliminating or weakening the convexity and nonempty interiority requirements imposed on ordering cones.

It is pointed out in [43, page 22] that "much of the mathematical theory is based on maximality with respect to partial orders or partially preorders, at best. More general concepts of optimality based on preferences satisfying conditions other than merely those of reflexivity and transitivity" should be considered. It is known that a preference $\preceq$ on a finite-dimensional space Z is determined by a continuous utility function if and only if both lower-level and upper-level sets at every $z \in Z$

$$\{y \in Z \mid y \preceq z\} \quad \text{and} \quad \{y \in Z \mid z \preceq y\}$$

are closed in Z, and that the lexicographic preference is a classical example of rational preferences that are not representable by a utility function. Recall that in economics the lexicographic preference (the lexicographical order based on the order of amount of each good) describes comparative preferences, where an economic agent infinitely prefers one good X to another Y. Thus if offered several bundles of goods, the agent will choose the bundle that offers the most X, no matter how much Y there is. Only when there is a tie of Xs between bundles will the agent start comparing Ys. On $\mathbb{R}^n$ the *lexicographical preference* $\prec_{lex}$ is defined as follow. We say that $y \prec_{lex} z$ if there is an integer $j \in \{0,\ldots,m-1\}$ such that $y_i = z_i$ for $i = 1,\ldots,j$ and $y_{j+1} < z_{j+1}$ for the corresponding components of the vectors $y,z \in \mathbb{R}^n$. It is easy to check that the induced preference $\preceq_{lex}$ is a complete generalized Pareto preference induced by the convex and pointed ordering cone

$$\Theta_{lex} := \{z \in \mathbb{R}^n \mid \mathbf{0} \prec_{lex} z\} \cup \{\mathbf{0}\}. \tag{13.8}$$

Obviously this cone is not locally closed around the origin.

In more recent publications [10, 35, 37, 48] some necessary conditions are for optimization problems with a broader class of preferences containing those with or without a utility description and generalized Pareto preferences generated by closed, convex and pointed cones whose interior might be empty. Recall that a preference is said to be *closed* if it meets the following requirements:

- *Nonreflexivity:* $z \notin P(z)\ \forall\, z \in Z$
- *Local satiation:* $z \in \operatorname{cl} P(z)\ \forall\, z \in U$, where U is a neighborhood of $\bar{z}$
- *Almost transitivity:* $\forall\, u \in P(z),\ \forall v \in \operatorname{cl} P(u), v \in P(z)$

Note that the *almost transitivity* is quite natural for classical orders in optimization. Indeed, if we define $\preceq$ via the closure of preference sets of $\prec$, i.e., $P_{\preceq}(z) := \operatorname{cl} P_{\prec}(z)$, then the almost transitivity reduces to

$$[v \preceq u \quad \text{and} \quad u \prec z] \quad \Longrightarrow \quad u \prec z,$$

which is automatic for the strict order ($<$) and the usual order ($\leq$) on real numbers, since $a \leq b$ and $b < c$ imply that $a < c$, and for the weak Pareto order and the Pareto order due to the fact that $\Theta + \operatorname{int}\Theta \subseteq \operatorname{int}\Theta$ for every convex cone Θ. However, we would mention that this requirement seems to be rather restrictive even for the class of generalized Pareto preferences defined in (13.5) with respect to a closed cone $\Theta \subseteq Z$, since it forces Θ to be convex and pointed by [35, Proposition 5.56], and thus this class of closed preferences excludes, in particular, the lexicographical preference.

The primary goal of this chapter is to introduce and study a refined optimality notion unifying, in particular, those known in the literature. It is named the *extended Pareto order*, since it extends the generalized Pareto order defined via the ordering relation (13.5) by replacing the closed and convex ordering cone with an ordering set containing the origin and satisfying a rather mild requirement. Then we establish refined necessary conditions for minimal points of sets with respect to the extended

Pareto order and derive from them several types of first-order necessary conditions for various kinds of optimal solutions to set-valued optimization problems with geometric constraints and/or operator constraints.

The rest of the chapter is organized as follows. Sect. 13.2 provides some of the basic concepts and tools from variational analysis and generalized differentiation broadly used in the sequel. In Sect. 13.3 we introduce and discuss the extended Pareto optimality concept. As mentioned, the essence of such an extension is the possibility to unify known optimality notions in set and multiobjective optimization. As a direct consequence of this, what we can establish for the new notion automatically holds for the others. We obtain in this section new necessary optimality conditions for minimal points of nonempty sets by employing a variational approach based on the extremal principle. Note that we do not assume the convexity for both the set in question and the ordering set. To compensate this, we need a new property of sets called the *local asymptotic closedness condition*. It holds under the standard assumptions imposed on ordering cones, while there are also nonconvex sets enjoying this property. Some characterizations of the local asymptotic closedness property are illustrated by providing several conditions ensuring the validity of it as well as various examples.

In Sects. 13.4 and 13.5 we apply the necessary conditions for minimal points of sets developed in Sect. 13.3 to graphs of set-valued mappings. In this way we derive extended necessary conditions for optimal solutions of constrained multiobjective problems. Furthermore, we observe that the necessary conditions hold for particular types of optimal solutions including known kinds of minimizers in set-valued optimization problems with geometric, operator, and functional constraints. The results obtained are new in both finite-dimensional and infinite-dimensional spaces.

Throughout the chapter we employ the standard notation of variational analysis; cf. [34, 40]. For a Banach space X, denote its norm by $\|\cdot\|$ and consider the dual space X^* equipped with the weak* topology w^*, where $\langle\cdot,\cdot\rangle$ stands for the canonical pairing between X and X^*. Given a set-valued mapping $F\colon X \rightrightarrows X^*$, recall that

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{x^* \in X^* \Big| \exists \text{ sequences } x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in F(x_k) \text{ for all } k \in \mathbb{N}\Big\} \tag{13.9}$$

signifies the *sequential Painlevé-Kuratowski upper/outer limit* with respect to the norm topology of X and the weak* topology of X^*, where $\mathbb{N} := \{1,2,\ldots\}$. For a nonempty subset $\Omega \subseteq X$ the symbols $\operatorname{cl}\Omega$, $\operatorname{cone}\Omega$, and $\operatorname{co}\Omega$ stand for the closure, conic hull, and convex hull of Ω, respectively, while the expression $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ and $x \in \Omega$. Given a set-valued mapping $F : X \rightrightarrows Z$ with the graph

$$\operatorname{gph} F := \big\{(x,z) \in X \times Z \big|\, z \in F(x)\big\},$$

the closure mapping $\operatorname{cl} F : X \rightrightarrows Z$ of F is defined by

$$\operatorname{cl} F(x) := \big\{z \in Z \big|\, (x,z) \in \operatorname{cl}(\operatorname{gph} F)\big\} \quad \text{with} \quad \operatorname{gph}(\operatorname{cl} F) = \operatorname{cl}(\operatorname{gph} F).$$

13.2 Basic Tools of Variational Analysis

The key tool in this chapter is the extremal principle in variational analysis. Since it unconditionally holds in Asplund spaces and provides in fact a characterization of the class of Asplund spaces, throughout the chapter all Banach spaces are assumed to be Asplund unless otherwise stated. Recall that a Banach space X is *Asplund* if each of its separable subspaces has a separable dual. There are many other equivalent descriptions of the original Asplund property, which can be found, e.g., in [34, Chap. 2]. Observe, in particular, that every reflexive Banach space is Asplund.

Let us begin with reviewing constructions of *generalized differentiation* enjoying comprehensive calculus properties ("full calculus") in Asplund spaces. The reader is referred to [34] for their useful modifications in general Banach spaces.

Let $\Omega \subseteq X$ be a subset of an Asplund space, and let $\bar{x} \in \Omega$. The (basic, limiting, Mordukhovich) *normal cone* to Ω at $\bar{x}$ is defined by

$$N(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{x \to \bar{x}} \widehat{N}(x;\Omega) \tag{13.10}$$

via the sequential Painlevé–Kuratowski outer limit (13.9) of the *prenormal cones* (known also as the Fréchet or regular normal cones) to Ω at x constructed by

$$\widehat{N}(x;\Omega) := \left\{ x^* \in X^* \;\middle|\; \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u-x\rangle}{\|u-x\|} \le 0 \right\}. \tag{13.11}$$

Note that, in contrast to (13.11), the basic normal cone (13.10) is often nonconvex enjoying nevertheless full calculus, which is mainly based on the extremal principle. If $X = \mathbb{R}^n$ and if Ω is locally closed around $\bar{x}$, the normal cone (13.10) can be equivalently described as

$$N(\bar{x};\Omega) = \operatorname*{Lim\,sup}_{x \to \bar{x}} \left[\operatorname{cone}\left(x - \Pi(x;\Omega)\right)\right]$$

via the Euclidean projector $\Pi(\cdot;\Omega)$ for the set Ω; this in fact was the original definition of the normal cone in [30].

Consider now a set-valued mapping $F\colon X \rightrightarrows Z$ between Asplund spaces and let $(\bar{x},\bar{z}) \in \operatorname{gph} F$. Recall the following two coderivative constructions used in the paper:

- The *normal coderivative* $D^*_N F(\bar{x},\bar{z})\colon Z^* \rightrightarrows X^*$ of F at $(\bar{x},\bar{z})$ in direction z^* is defined via the normal cone (12.9) to the graph of F at $(\bar{x},\bar{z})$ by

$$\begin{aligned} D^*_N F(\bar{x},\bar{z})(z^*) &:= \left\{x^* \in X^* \,\middle|\, (x^*,-z^*) \in N\left((\bar{x},\bar{z});\operatorname{gph} F\right)\right\} \\ &= \Big\{x^* \in X^* \,\Big|\, \exists \text{ sequences } (x_k,z_k) \xrightarrow{\operatorname{gph} F} (\bar{x},\bar{z}) \text{ and} \\ &\qquad (x^*_k,z^*_k) \xrightarrow{w^*} (x^*,z^*) \text{ with } (x^*_k,-z^*_k) \in \widehat{N}\left((x_k,z_k);\operatorname{gph} F\right)\Big\}. \end{aligned} \tag{13.12}$$

- The *mixed coderivative* $D^*_M F(\bar{x},\bar{z})\colon Z^* \rightrightarrows X^*$ is defined by replacing the weak* convergence $z^*_k \xrightarrow{w^*} z^*$ in (13.12) with the norm convergence $z^*_k \xrightarrow{\|\cdot\|} z^*$, i.e.,

$$D^*_M F(\bar{x},\bar{z})(z^*) := \Big\{ x^* \in X^* \Big| \exists \text{ sequences } (x_k,z_k) \xrightarrow{\text{gph}\,F} (\bar{x},\bar{z}), \quad x^*_k \xrightarrow{w^*} x^*,\ z^*_k \xrightarrow{\|\cdot\|} z^* \text{ with } (x^*_k,-z^*_k) \in \widehat{N}\big((x_k,z_k);\text{gph}\,F\big) \Big\}. \tag{13.13}$$

Note that $\bar{z} = f(\bar{x})$ is always omitted in the coderivative notation if $F = f\colon X \to Z$ is single-valued. Obviously, we have the relationship

$$D^*_M F(\bar{x},\bar{z})(z^*) \subseteq D^*_N F(\bar{x},\bar{z})(z^*) \text{ for all } z^* \in Z^*, \tag{13.14}$$

where the equality holds if Z is finite-dimensional. The equality holds in (13.14) in broad classes of mappings with infinite-dimensional image spaces; the latter property is postulated in [34] as *strong coderivative normality* of the mapping F at the point $(\bar{x},\bar{z})$. In particular, if $f\colon X \to Y$ is *strictly differentiable* at $\bar{x}$ (which is automatic when it is C^1 around this point), then the equality in (13.14) holds:

$$D^*_N f(\bar{x})(z^*) = D^*_M f(\bar{x})(z^*) = \{\nabla f(\bar{x})^* z^*\} \text{ for all } z^* \in Z^*.$$

One of the most important ingredients of variational analysis in infinite dimensions, in contrast to the case of finite-dimensional spaces, is the necessity to impose some "normal compactness" properties, which allow us to perform limiting procedures of deriving nontrivial calculus rules and optimality conditions and which are automatic in finite dimensions. Let us recall some of these properties in Asplund spaces; see [34] for the corresponding ones in general Banach space settings.

Let $\Omega \subseteq X \times Z$, and let $(\bar{x},\bar{z}) \in \Omega$. We say that:

- Ω is *sequentially normally compact* (SNC) at $(\bar{x},\bar{z}) \in \Omega$ if for any sequences

$$(x_k,z_k) \xrightarrow{\Omega} (\bar{x},\bar{z}), \text{ and } (x^*_k,z^*_k) \in \widehat{N}((x_k,z_k);\Omega), \quad k \in \mathbb{N}, \tag{13.15}$$

  we have the implication $(x^*_k,z^*_k) \xrightarrow{w^*} \mathbf{0} \Longrightarrow \|(x^*_k,z^*_k)\| \to 0$ as $k \to \infty$. Observed that the product structure of the space in question plays no role in this property in contrast to its following partial modification.
- Ω is *partially sequentially normally compact* (PSNC) with respect to X at $(\bar{x},\bar{z}) \in \Omega$ if for any sequences (x_k,z_k,x^*_k,z^*_k) satisfying (13.15) we have the implication $[x^*_k \xrightarrow{w^*} \mathbf{0},\ \|z^*_k\| \to 0] \Longrightarrow \|x^*_k\| \to 0$ as $k \to \infty$.

Employing these SNC/PSNC properties to graphs of set-valued mappings $F: X \rightrightarrows Z$ at $(\bar{x},\bar{z}) \in \text{gph}\,F$, we say that:

- F is SNC at $(\bar{x},\bar{z})$ if $\text{gph}\,F$ is SNC at $(\bar{x},\bar{z})$.

- F is PSNC at $(\bar{x},\bar{z})$ if $\operatorname{gph} F$ is PSNC at $(\bar{x},\bar{z})$ with respect to X.

Note that the PSNC property of mappings holds automatically for a broad class of mappings exhibiting certain Lipschitzian behavior. Recall that a set-valued mapping $F\colon X \rightrightarrows Z$ is *Lipschitz-like* around $(\bar{x},\bar{z})$ with modulus $\ell \geq 0$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{z}$ such that

$$F(x)\cap V \subseteq F(u)+\ell\|x-u\|I\!B \text{ for all } x,u\in U,$$

where $I\!B$ stands for the closed unit ball of Z. This property, which is also known as the Aubin or pseudo-Lipschitzian property (cf. the discussions in [34, 40]) agrees with the classical local Lipschitzian behavior in the case of single-valued functions and reduces to the standard (Hausdorff) local Lipschitzian property of set-valued mappings when $V = Z$. Furthermore, it is equivalent to both *metric regularity* and *linear openness* properties of the inverse mapping F^{-1}.

We have by [34, Theorem 1.44] that

$$D^*_M F(\bar{x},\bar{z})(\mathbf{0}) = \{\mathbf{0}\} \tag{13.16}$$

for any set-valued mapping $F\colon X \rightrightarrows Z$ between Banach spaces, which is Lipschitz-like around $(\bar{x},\bar{z} \in \operatorname{gph} F)$. Furthermore, the coderivative condition (13.16) is not only *necessary* for the Lipschitz-like property of F around $(\bar{x},\bar{z})$ but also *sufficient* for this property of closed-graph mappings between Asplund spaces under the PSNC property of F; see [34, Theorem 4.10] for the full account of this coderivative characterization (known as the Mordukhovich criterion) and [32, Theorem 5.7], [40, Theorem 7.40] for the case of mappings between finite-dimensional spaces.

We now present the aforementioned *extremal principle* for two sets used in this paper; see [34, Chap. 2] for a full version. Recall that $\bar{z} \in \Xi_1 \cap \Xi_2$ is a *local extremal point* of the set system $\{\Xi_1,\Xi_2\}$ in Z if there is a neighborhood V of $\bar{z}$ and a sequence $\{a_k\} \subseteq Z$ with $\|a_k\| \to 0$ such that

$$\Xi_1 \cap (\Xi_2 + a_k) \cap V = \emptyset \quad \text{for all} \quad k \in \mathbb{N}. \tag{13.17}$$

Let $\bar{z}$ be a local extremal point to the set system $\{\Xi_1,\Xi_2\}$ in an Asplund space Z, where both sets Ξ_1 and Ξ_2 are locally closed around $\bar{z}$. We get from [34, Theorem 2.20] that the extremal system $\{\Xi_1,\Xi_2,\bar{z}\}$ satisfies the *approximate extremal principle*: for every $\varepsilon > 0$ there are $(z_1,z_2,z_1^*,z_2^*) \in Z\times Z\times Z^*\times Z^*$ such that

$$\begin{cases} z_1 \in \Xi_1 \cap (\bar{z}+\varepsilon I\!B), \quad z_2 \in \Xi_2 \cap (\bar{z}+\varepsilon I\!B),\\ z_1^* \in \widehat{N}(z_1;\Xi_1)+\varepsilon I\!B^*, \quad z_2^* \in \widehat{N}(z_2;\Xi_2)+\varepsilon I\!B^*,\\ z_1^*+z_2^* = \mathbf{0}, \quad \|z_1^*\|+\|z_2^*\| = 1. \end{cases} \tag{13.18}$$

Assuming further that either Ξ_1 or Ξ_2 is SNC at $\bar{z}$, we have the *exact extremal principle* from [34, Theorem 2.22]: there is $z^* \in X^*$ with $\|z^*\| = 1$ such that

$$z^* \in N(\bar{z};\Xi_1) \cap (-N(\bar{z};\Xi_2)). \tag{13.19}$$

The extremal principle, which is the main single tool in establishing calculus rules for normal cones to sets and coderivatives of mappings, can be viewed as a local variational counterpart of the classical convex separation in nonconvex settings. In fact, it plays a fundamental role in variational analysis similar to that played by convex separation and equivalent results in convex analysis as well as in its outgrowths and applications; see the books [34, 35] and the references therein. In this chapter we develop new applications of the extremal principle to necessary conditions for minimal points to sets and for minimizers in multiobjective optimization problems.

13.3 Necessary Optimality Conditions for Extended Pareto Minimal Points of Sets

In this section we first introduce an extension of Pareto optimality that unifies all the optimality notions known in the literature, including Pareto-type optimality and preference optimality. Then we discuss a local asymptotic closedness property of sets and establish new necessary conditions for minimal points of sets with respect to an extended Pareto order under asymptotic closedness assumptions.

Definition 13.3 (Extended Pareto Minimal or Θ-Minimal Points). Let Z be a Banach space with an ordering subset $\Theta \subseteq Z$ containing the origin, let $\emptyset \neq \Xi \subseteq Z$, and let $\bar{z} \in \Xi$. We say that:

- $\bar{z}$ is a *local extended Pareto minimal point* of Ξ with respect to Θ, or it is a *local Θ-minimal point* of Ξ for short, if there exists a neighborhood V of $\bar{z}$ such that

$$\Xi \cap (\bar{z} - \Theta) \cap V = \{\bar{z}\}. \tag{13.20}$$

- $\bar{z}$ is a *global Θ-minimal point* to Ξ if we can choose $V = Z$ in (13.20).

Remark 13.1 (On Optimality Notions). (a) The adjective "extended" is used in Definition 13.3 to distinguish this type of minimal points with generalized Pareto minimal points known in the literature.

(b) We do *not* assume that the ordering set is *locally closed* around the origin to cover important preference relations; in particular, the lexicographical ordering cone (13.8) is neither closed nor open. Furthermore, in practical applications we often use the product preferences (13.3), where a part of the component preferences are Pareto while the rest are weak Pareto. To this end, consider the product space $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$ and two order relations on real numbers $<$ and $\leq$. The mixed product preferences $\leq \times <$ and $< \times \leq$ on $\mathbb{R}^2$ can be seen as an extended Pareto preference generated by a convex and pointed cone being not locally closed around the origin. To the best of our knowledge, these kinds of preference have not been considered in multiobjective optimization.

(c) The Θ-minimality condition (13.20) is a bit different from the conventional Pareto minimality notion for non-pointed ordering cones given by

$$(\bar{z}-C)\cap\Xi\subseteq(\bar{z}+C), \tag{13.21}$$

where C is a convex ordering cone. However, it does not restrict the domain of applications due to the fact that $\bar{z}$ is a Pareto minimal point of Ξ with respect to C in the sense of (13.21) *if and only if* it is a $\widetilde{\Theta}$-minimal point to Ξ with

$$\widetilde{\Theta}:=C\cap(Z\backslash(-C))\cup\{\mathbf{0}\}.$$

Obviously, $\widetilde{\Theta}$ is pointed, but it is *not locally closed* around the origin.

(d) The introduced notion of Θ-minimality unifies all known kinds of Pareto-type optimality. Let $\Xi\subseteq Z$ be a subset of a Banach space Z ordered by a closed and convex cone $C\subseteq Z$ satisfying $C\setminus(-C)\neq\emptyset$, and let $\bar{z}\in\Xi$.

- $\bar{z}$ is a *Pareto minimal point* to Ξ (with respect to C) if

$$\Xi\cap(\bar{z}-C)=\{\bar{z}\}\quad\text{or}\quad\Xi\cap(\bar{z}-C\setminus\{\mathbf{0}\})=\emptyset. \tag{13.22}$$

- $\bar{z}$ is a *weak Pareto minimal point* to Ξ if

$$\Xi\cap(\bar{z}-\operatorname{int}C)=\emptyset\quad\text{provided that}\quad\operatorname{int}C\neq\emptyset. \tag{13.23}$$

- $\bar{z}$ is an *ideal Pareto minimal point* to Ξ if

$$\Xi\subseteq\bar{z}+C\quad\text{or}\quad\Xi\cap(\bar{x}-(Z\setminus(-C)))=\emptyset. \tag{13.24}$$

- $\bar{z}$ is a *(primary) relative minimal point* to Ξ if

$$\Xi\cap(\bar{z}-\operatorname{ri}C)=\emptyset\quad\text{provided that}\quad\operatorname{ri}C\neq\emptyset, \tag{13.25}$$

  where $\operatorname{ri}C$ is the collection of interior points of C with respect to the closed affine hull of C.

- $\bar{z}$ is a *quasi relative minimal point* to Ξ if

$$\Xi\cap(\bar{z}-\operatorname{qri}C)=\emptyset\quad\text{provided that}\quad\operatorname{qri}C\neq\emptyset, \tag{13.26}$$

  where $\operatorname{qri}C$ is the collections of those points $z\in C$ for which the set $\operatorname{cl}(\operatorname{cone}(C-z))$ is a linear subspace of Z.

Obviously, a Pareto, weak Pareto, ideal, primary relative, and quasi relative minimal point in (13.22)–(13.26) can be unified by a Θ-minimal point, where

Θ is, for each kind of minimal points, defined by

$$\begin{aligned}
&\bullet\ \text{Pareto} && \Theta := C,\\
&\bullet\ \text{Weak Pareto} && \Theta := \operatorname{int} C \cup \{\mathbf{0}\},\\
&\bullet\ \text{Ideal Pareto} && \Theta := (Z \setminus (-C)) \cup \{\mathbf{0}\},\\
&\bullet\ \text{Relative Pareto} && \Theta := \operatorname{ri} C \cup \{\mathbf{0}\},\\
&\bullet\ \text{Quasi relative Pareto} && \Theta := \operatorname{qri} C \cup \{\mathbf{0}\}.
\end{aligned} \tag{13.27}$$

The first notion in (13.22) is well recognized under the pointedness assumption of ordering cones which can be weakened to the non-subspace property, i.e., $C \setminus (-C) \neq \emptyset$; see [5, 6]. However, the vast majority of publications on multiobjective optimization, even in the simplest frameworks, concerns the weak notion in (13.23), which are much more convenient to deal with in the vein of the conventional scalarization techniques under the nonempty interiority of the ordering cone. Such a condition is essential and unavoidable whenever we would like to inherit necessary results in scalar optimization, but it is a serious restriction in both finite-dimensional and infinite-dimensional settings. To improve the situation, the relative notions in (13.25) and (13.26) become certain alternatives in finite-dimensional and reflexive Banach spaces, respectively, since the corresponding relative interiors are nonempty for convex ordering cones. Observe to this end that each weak minimal point is a primary relative minimal point, since the condition $\operatorname{int} C \neq \emptyset$ yields $\operatorname{ri} C = \operatorname{int} C$. Further, each primary relative minimal point is a quasi relative minimal point, since the condition $\operatorname{ri} C \neq \emptyset$ yields $\operatorname{qri} C = \operatorname{ri} C$. The reader can find more discussions on primary and quasi relative interiors of convex sets in Banach spaces in [5,9,10].

(e) The notion of Θ-minimality can be equivalently described via an appropriate preference. Indeed, every Θ-minimal point of a set Ξ is a preference point for this set with respect to the preference defined via the ordering relation (13.7) with the given ordering set Θ. Conversely, each preference point $\bar{z}$ of Ξ with respect to a certain preference $\preceq$ is a Θ-minimal point of Ξ, where the ordering set Θ is defined by

$$\Theta := \bar{z} - \{z \mid z \preceq \bar{z}\} = \bar{z} - P_{\preceq}(\bar{z}).$$

In this paper it is more convenient for us to deal with the Θ-optimality description to emphasize the point-based assumptions imposed on sets at the point in question in comparison with the neighborhood-based ones required for the class of closed preferences.

(f) The Θ-minimality notion is related to while different from the generalized optimality order introduced by Mordukhovich in [35, Definition 5.53] via the local extremality of sets. Given a mapping $f : X \to Z$ between Banach spaces and an ordering set $\Lambda \subseteq Z$ containing the origin, we says that $\bar{x}$ is a *locally* (f,Λ)*-optimal* solution if $\bar{z} = f(\bar{x})$ is a (global) extremal point to the set system $\{f(X \cap U), f(\bar{x}) + \Lambda\}$, i.e., there are a neighborhood U of $\bar{x}$ and a sequence $\{z_k\} \subseteq Z$ with $\|z_k\| \to 0$ as $k \to \infty$ such that

$$f(X \cap U) \cap (f(\overline{x}) + \Lambda - z_k) = \emptyset \quad \text{for all} \quad k \in \mathbb{N}. \tag{13.28}$$

Note that if $\Lambda = -C$ is a convex subcone of Z with $\mathrm{ri}\, C \neq \emptyset$, then the above optimality concept covers the conventional concept of optimality (called sometimes Slater optimality) requiring that there is no $x \in U$ with $f(x) - f(\overline{x}) \in \mathrm{ri}\, C$. This extends the notion of weak Pareto optimality/ efficiency corresponding to $f(x) \in f(\overline{x}) - \mathrm{int}\, C$ in the above relations. To reduce it to the notion in Definition 5.53, we take $z_k := \overline{z}/k$ for $k \in \mathbb{N}$ in (13.28) with some $\overline{z} \in \mathrm{ri}\, C$. The standard notion of Pareto optimality can be formulated in these terms as the absence of $x \in U$ for which $f(x) \in f(\overline{x}) - C$ and $f(\overline{x}) \neq f(x)$.

To establish necessary optimality conditions for Θ-minimal points of sets, we adopt the variational approach based on reducing a local Θ-minimal point to a local extremal point of an appropriate set system and then on applying the extremal principle. Note that the validity of the process of deriving the extremality of an appropriate set system at a minimal point from the corresponding optimality notion is justified provided that the space Z is ordered by a closed and convex cone, which is not a subspace of Z; see [1–3, 5] for more details. In this paper we show that the process is valid for Θ-minimal points of sets satisfying the following property.

Definition 13.4 (Local Asymptotic Closedness). Let $\Xi \subseteq Z$ be a subset of a Banach space Z, and let $\overline{z} \in \mathrm{cl}\, \Xi$. We say that Ξ has the *local asymptotic closedness* (LAC) property at $\overline{z}$ if there are a neighborhood V of $\overline{z}$ and a sequence $\{c_k\} \subseteq Z$ with $\|c_k\| \to 0$ satisfying

$$(\mathrm{cl}\, \Xi + c_k) \cap V \subseteq \Xi \backslash \{\overline{z}\}. \tag{13.29}$$

We obviously have that the closure $\mathrm{cl}\, \Xi$ is LAC at $\overline{z} \in \mathrm{cl}\, \Xi$ if Ξ enjoys this property. However, the converse implication does not hold. It is easy to verify this for the set $\Xi \subseteq \mathbb{R}^2$ given by

$$\Xi := \left\{ z \in \mathbb{Q}^2 \,\middle|\, z_1 \geq 0 \right\} \quad \text{and} \quad \overline{z} = (0,0) \in \mathbb{R}^2,$$

where $\mathbb{Q}$ is the collection of all rational numbers.

Note that the LAC property is independent of the local closedness property for sets. Indeed, the open set $\Xi := (-\infty, 0) \subseteq \mathbb{R}$ is LAC at 0, but the closed set $\Xi := \mathbb{R}$ is not LAC at this point. Observe also that if Ξ is LAC at $\overline{z}$, then we get from (13.29) that $\overline{z} - c_k \notin \mathrm{cl}\, \Xi$ for all k, i.e., $\overline{z}$ is a boundary point of Ξ. However, a closed set may not be LAC at its boundary point as, e.g., in the case of $\Xi = \mathbb{R}^2_+ \cup \mathbb{R}^2_-$ and $\overline{z} = (0,0)$.

Next we show that the Θ-minimality (13.20) implies the extremality property (13.17) under LAC assumptions.

Theorem 13.2 (Extremality at Θ-Minimal Points). *Let $\overline{z} \in \Xi$ be a local Θ-minimal point to Ξ. The following assertions hold:*

(a) $\bar{z}$ *is a local extremal point of the set system* $\{\Xi, (\bar{z} - \mathrm{cl}\,\Theta)\}$ *provided that* Θ *is LAC at* $\mathbf{0}$.
(b) $\bar{z}$ *is a local extremal point of the set system* $\{\mathrm{cl}\,\Xi, (\bar{z} - \Theta)\}$ *provided that* Ξ *is LAC at* $\bar{z}$.
(c) $\bar{z}$ *is a local extremal point of the set system* $\{\mathrm{cl}\,\Xi, (\bar{z} - \mathrm{cl}\,\Theta)\}$ *provided that both* Ξ *and* Θ *are LAC at* $\bar{z}$ *and* $\mathbf{0}$*, respectively.*

Proof. It is not difficult to check that the validity of the extremality condition (13.17) for the system $\{\mathrm{cl}\,\Xi, \bar{z} - \mathrm{cl}\,\Theta\}$ follows from the LAC property (13.29) and the minimality condition (13.20). For brevity, we provide the proof of assertion (c) only.

Let $\bar{z}$ be a local Θ-minimal point of Ξ. Then we have from Definition 13.3 that

$$\Xi \cap (\bar{z} - \Theta \setminus \{\mathbf{0}\}) \cap U = \emptyset$$

for a neighborhood U of $\bar{z}$. Since both sets Ξ and $\bar{z} - \Theta$ are LAC at $\bar{z}$, there exist a neighborhood V of $\bar{z}$ (without loss of generality put $V = U$) and two sequences $\{a_k\}$ and $\{c_k\}$ converging to zero as $k \to \infty$ such that

$$(\mathrm{cl}\,\Xi + a_k) \cap U \subseteq \Xi \setminus \{\bar{z}\}, \quad \text{and} \quad (\bar{z} - \mathrm{cl}\,\Theta + c_k) \cap U \subseteq (\bar{z} - \Theta) \setminus \{\bar{z}\}$$

for all $k \in \mathbb{N}$. The latter implies the relationships

$$\begin{aligned} &(\mathrm{cl}\,\Xi + a_k) \cap (\bar{z} - \mathrm{cl}\,\Theta + c_k) \cap U \\ &= ((\mathrm{cl}\,\Xi + a_k) \cap U) \cap (\mathrm{cl}\,(\bar{z} - \Theta) + c_k) \cap U) \cap U \\ &\subseteq (\Xi \setminus \{\bar{z}\}) \cap ((\bar{z} - \Theta) \setminus \{\bar{z}\}) \cap U = \emptyset, \end{aligned}$$

which surely yield the extremality condition (13.17) and thus the local extremality of the set system $\{\mathrm{cl}\,\Xi, (\bar{z} - \mathrm{cl}\,\Theta)\}$ at $\bar{z}$. This justifies assertion (c). The proofs for (a) and (b) are similar with either $\{a_k\} \equiv \mathbf{0}$ or $\{c_k\} \equiv \mathbf{0}$, respectively. □

Note that the LAC assumptions in Theorem 13.2 (c) cannot be generally dropped. To illustrate this, consider two closed sets in $\mathbb{R}^2$ given by

$$\Xi := \{(x, -x) \in \mathbb{R}^2 \mid x \in \mathbb{R}\} \quad \text{and} \quad \Theta := \mathbb{R}^2_+ \cup \mathbb{R}^2_-.$$

It is easy to check that neither Ξ nor Θ are LAC at $\mathbf{0}$, that $\mathbf{0}$ is a Θ-minimal point to Ξ, but that it is not a local extremal point to the set system $\{\Xi; \Theta\}$.

Note also that the LAC condition is only *sufficient* while not necessary to ensure the extremality property of the set system $\{\Xi; (\bar{z} - \Theta)\}$ at a point $\bar{z}$ satisfying (13.20). Indeed, there are two closed sets in $\mathbb{R}^2$ described by

$$\Xi = \Theta := \mathrm{bd}\,\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2_+ \mid x \cdot y = 0\}$$

such that none of them is LAC at $\mathbf{0}$ and that the set system $\{\Xi; -\Theta\}$ is local extremal at the origin, which is the unique Θ-minimal point of Ξ.

As follows from the proof of Theorem 13.2, the LAC assumptions imposed on the sets therein naturally appears to ensure the extremality of the set system $\{\mathrm{cl}\,\Xi, \bar{z} - \mathrm{cl}\,\Theta\}$ in the *lack of the convexity*. It is important to mention that we do not require any convexity of not only the given set Ξ but also the ordering set Θ.

Remark 13.2 (LAC Property in Welfare Economics). One of the most fundamental results of general equilibrium theory, the so-called Second Welfare Theorem for convex economies, tells us that any Pareto optimal allocation can be decentralized at price equilibria, i.e., it can be sustained by a nonzero price vector at which each consumer minimizes his/her expenditures and each firm maximizes its profit. Extensions of this fundamental result to nonconvex economies requires certain *net demand qualification* (NDQ) conditions [33, 35]; see also [21, 23–26] for related developments. We refer the reader to the recent papers [6,7] for various interconnections between such conditions and the LAC property and new relationships between welfare economics and set-valued optimization.

Now we establish new necessary optimality conditions for Θ-minimal points of general sets under LAC assumptions.

Theorem 13.3 (Necessary Conditions for Θ-Minimal Points). *Let $\Xi \subseteq Z$ be a subset in an Asplund space Z, let $\Theta \subseteq Z$ be an ordering set containing the origin, and let $\bar{z} \in \Xi$ be a local Θ-minimal point to Ξ. Assume that one of the following LAC conditions is satisfied:*

(i) Ξ is locally closed around $\bar{z}$ and Θ is LAC at $\mathbf{0}$.
(ii) Ξ is LAC at $\bar{z}$ and Θ is locally closed around $\mathbf{0}$.
(iii) Ξ and Θ are LAC at $\bar{z}$ and $\mathbf{0}$, respectively.

Then there is a nonzero dual element $z^* \in Z^*$ such that

$$z^* \in N(\bar{z}; \mathrm{cl}\,\Xi) \cap N(\mathbf{0}; \mathrm{cl}\,\Theta) \tag{13.30}$$

provided that either $\mathrm{cl}\,\Xi$ is SNC at $\bar{z}$ or $\mathrm{cl}\,\Theta$ is SNC at the origin.

Proof. Since $\bar{z}$ is a local Θ-minimal point of Ξ, it is a local extremal point of the set system $\{\mathrm{cl}\,\Xi; \bar{z} - \mathrm{cl}\,\Theta\}$ by Theorem 13.2 under each of the LAC assumptions (i)–(iii) imposed on Ξ and Θ. Employing the exact extremal principle (13.19) to the local extremal point $\bar{z}$ under the SNC assumptions made, we have

$$\mathbf{0} \neq z^* \in N(\bar{z}; \mathrm{cl}\,\Xi) \cap (-N(\bar{z}; (\bar{z} - \mathrm{cl}\,\Theta))) = N(\bar{z}; \mathrm{cl}\,\Xi) \cap N(\mathbf{0}; \mathrm{cl}\,\Theta),$$

which justifies the validity of the necessary optimality condition (13.30) and thus completes the proof of the theorem. □

The following is a useful consequence of the theorem for the case of closed sets.

Corollary 13.1. (Necessary Conditions for Θ-Minimal Points Under Closedness Assumptions). *Let Ξ, Θ, and $\bar{z}$ be as in Theorem 13.3. Assume that the sets Ξ and Θ are locally closed around $\bar{z}$ and $\mathbf{0}$, respectively, that either Ξ is SNC at $\bar{z}$ or*

Θ is SNC at $\mathbf{0}$, and that either Ξ is LAC at $\bar{z}$ or Θ is LAC at $\mathbf{0}$. Then the optimality condition (13.30), which reduces to

$$\mathbf{0} \neq z^* \in N(\bar{z};\Xi) \cap N(\mathbf{0};\Theta),$$

in this case, is necessary for the local Θ-minimality of the point $\bar{z} \in \Xi$.

Proof. It directly follows from Theorem 13.3. □

In the rest of this section we derive from Theorem 13.3 necessary conditions for Pareto-type minimal points of sets and for those given by preference relations. Prior to this, we obtain sufficient conditions for the LAC property, which are either completely new or extend those established in [7].

First let us show that *convexity implies LAC* due to the following facts obtained in [9, Lemma 3.1] and [9, Corollary 3.2]. Namely, for a convex subset $C \subseteq Z$ of a Banach space Z we have the relationships:

- If $z \in C$ and $\bar{z} \in \operatorname{qri} C$, then $tz + (1-t)\bar{z} \in \operatorname{qri} C$ for every $t \in [0,1)$.
- The set $\operatorname{qri} C$ is convex and dense in C when it is nonempty.
- Assume C is a convex cone but not a subspace, i.e., $C \setminus (-C) \neq \emptyset$. If $\bar{z} \in \operatorname{qri} C$, then $\bar{z} \notin \operatorname{bd}(\operatorname{cl} C)$. Hence $\operatorname{qri} C = \operatorname{qri}(\operatorname{cl} C)$.

To verify the last statement, we argue by contradiction and assume that $\bar{z} \in \operatorname{bd}(\operatorname{cl} C)$. Since $\operatorname{cl} C$ is closed, convex and it is not a subspace, there is a closed half space H containing the set $\operatorname{cl} C - \bar{z}$; thus $\operatorname{cl}(\operatorname{cone}(\operatorname{cl} C - \bar{z})) \subseteq H$. This implies that the set $\operatorname{cl}(\operatorname{cone}(C - \bar{z})) \subseteq \operatorname{cl}(\operatorname{cone}(\operatorname{cl} C - \bar{z}))$ is not a subspace of Z, which yields in turn that $\bar{z}$ does not belong to the quasi relative interior of C.

Proposition 13.1 (LAC Property for Convex Cones). *Let $C \subseteq Z$ be a convex subcone of a Banach space Z such that $C \setminus (-C) \neq \emptyset$. Then the following hold:*

(a) $\operatorname{qri} C$ *is LAC at* $\mathbf{0}$ *provided that* $\operatorname{qri} C \neq \emptyset$, *which is automatic for any convex set in a reflexive Banach space.*
(b) C is LAC at $\mathbf{0}$ provided that C is a closed set.

Proof. First we prove (a). Assume that $\operatorname{qri} C \neq \emptyset$ and note that the assumption $C \setminus (-C) \neq \emptyset$ implies that $\mathbf{0} \notin \operatorname{qri} C$. Pick an arbitrary nonzero element $\bar{c} \in \operatorname{qri} C$ and define the sequence $\{c_k\}$ by

$$c_k := k^{-1}\,\bar{c} \quad \text{for all} \quad k \in \mathbb{N}. \tag{13.31}$$

Obviously, $\{c_k\}$ converges to zero as $k \to \infty$ with $c_k \in \operatorname{qri} C$ for for all $k \in \mathbb{N}$. Taking into account the properties of quasi relative interiors listed above, we get

$$\operatorname{cl} C + c_k \subseteq \operatorname{cl} C + \operatorname{qri}(\operatorname{cl} C) \subseteq \operatorname{qri}(\operatorname{cl} C) = \operatorname{qri} C = \operatorname{qri} C \setminus \{\mathbf{0}\}$$

for all $k \in \mathbb{N}$, which surely verifies the validity of the LAC condition (13.29), and thus the LAC property of $\operatorname{qri} C$ at the origin.

Let us next prove (b) for a closed and convex cone (whose quasi relative interior might be empty). Pick an arbitrary element $\mathbf{0} \neq \bar{c} \in C \setminus (-C)$, which exists by the assumption $C \setminus (-C) \neq \emptyset$. Since C is a closed and convex cone, we have

$$\mathrm{cl}\, C + c_k = C + c_k \subseteq C + C = C \tag{13.32}$$

for every $k \in \mathbb{N}$. Defining c_k as in (13.31) gives us that $c_k = k^{-1}\bar{c} \in C \setminus (-C)$ with $c_k \notin -C$, i.e., $\mathbf{0} \notin C + c_k$. Combining this and (13.32), we arrive at

$$\mathrm{cl}\, C + c_k \subseteq C \setminus \{\mathbf{0}\},$$

which verifies the LAC property of C at the origin. □

Thanks to the fact that the closure of a LAC set is LAC, we derive from Proposition 13.1 that the Pareto, weak Pareto, ideal Pareto, primary relative, and quasi relative ordering cones in (13.27) all are LAC at the origin under the standard assumptions for these ordering cones.

Corollary 13.2 (LAC Property for Conventional Ordering Cones). *Let $C \subseteq Z$ be a convex cone of a Banach space such that $C \setminus (-C) \neq \emptyset$. The we have:*

(a) The Pareto ordering cone $\Theta_p := C$ is LAC at the origin provided that either C is closed or $\mathrm{qri}\, C \neq \emptyset$.
(b) The weak Pareto ordering cone $\Theta_w := \mathrm{int}\, C \cup \{\mathbf{0}\}$ is LAC at the origin provided that $\mathrm{int}\, C \neq \emptyset$.
(c) The relative Pareto ordering cone $\Theta_{ri} := \mathrm{ri}\, C \cup \{\mathbf{0}\}$ is LAC at the origin provided that $\mathrm{ri}\, C \neq \emptyset$; this assumption is automatic in finite dimensions.
(d) The quasi relative Pareto ordering cone $\Theta_q := \mathrm{qri}\, C \cup \{\mathbf{0}\}$ is LAC at the origin provided that $\mathrm{qri}\, C \neq \emptyset$; this assumption is automatic in reflexive spaces.
(e) The ideal Pareto ordering cone $\Theta_i := (Z \setminus (-C)) \cup \{\mathbf{0}\}$ is LAC at $\mathbf{0}$ provided that $\mathrm{int}\, C \neq \emptyset$.
(f) The lexicographical ordering cone Θ_{lex} is LAC at $\mathbf{0}$.

Proof. Note that assertions (a) and (d) follow directly from Proposition 13.1 and that assertion (f) is a specification of (a). It is known from [9, Theorem 2.12] that $\mathrm{ri}\, C = \mathrm{qri}\, C$ provided that $\mathrm{ri}\, C \neq \emptyset$, and that $\mathrm{int}\, C = \mathrm{qri}\, C$ provided that $\mathrm{int}\, C \neq \emptyset$. Further, it is also known from [9, Theorem 2.8] that every nonempty convex subset C of a separable Banach space Z has nonempty quasi relative interior. It is a classical result that every nonempty convex subset C of a finite-dimensional space Z has nonempty relative interior. Thus we get assertions (b) and (c) from assertion (d).

To verify finally the LAC property of Θ_i in (e), it is sufficient to show that

$$\mathrm{cl}\, \Theta + c_k = Z \setminus (-\mathrm{int}\, C) + c_k \subseteq Z \setminus (-C) = \Theta \setminus \{\mathbf{0}\} \tag{13.33}$$

for all $k \in \mathbb{N}$, where the sequence $\{c_k\}$ is defined by (13.31) with $\bar{c} \in \mathrm{int}\, C$. Arguing by contradiction, assume that the inclusion does not hold, i.e., there is $z \in Z$ satisfying the relationships

$$z \in Z \setminus (-\mathrm{int}\, C) + c_k \quad \text{and} \quad z \notin Z \setminus (-C). \tag{13.34}$$

The second relationship in (13.34) implies that $z \in -C$, which in turn yields that $z - c_k \in -\text{int}\,\Theta$, i.e., $z - c_k \notin Z \setminus (-\text{int}\,\Theta)$. This contradicts the first inclusion in (13.34) and thus justifies the validity of (13.33), that is, verifies the LAC property of the ideal Pareto ordering cone Θ_i at the origin. □

Note that the nonempty interiority condition $\text{int}\,C \neq \emptyset$ in Corollary 13.2 (v) is essential. Indeed, consider a closed and convex cone in $\mathbb{R}^2$ defined by $C := \{(x,0) \in \mathbb{R}^2 \mid x \geq 0\} \subseteq \mathbb{R}^2$. Since the closure of the ideal Pareto ordering cone Θ_i induced by this cone is the whole space, Θ_i is not LAC at any point $\bar{z} \in \mathbb{R}^2$.

The next result provides verifiable conditions ensuring the LAC property of generalized epigraphical sets.

Proposition 13.2 (LAC Property for Epigraphical Sets). *Let $C \subseteq Z$ be a closed and convex cone in a Banach space Z with $C \setminus (-C) \neq \emptyset$, and let $\Xi \subseteq Z$ be a nonempty subset of Z. Denote the* epigraphical set *associated with C by*

$$\mathscr{E}_\Xi := \Xi + C \tag{13.35}$$

and the collection of C-minimal points to Ξ by $\text{Min}\,(\Xi; C)$. *Then we have:*

(a) $\mathscr{E}_\Xi$ is LAC at $\bar{z} \in \text{Min}\,(\Xi;C)$ provided that it is locally closed around this point.
(b) $\mathscr{E}_\Xi$ is LAC at $\bar{z} \in WMin\,(\Xi;C) := \text{Min}\,(\Xi;(\text{int}\,C \cup \{\mathbf{0}\}))$ provided that $\text{int}\,C \neq \emptyset$.

Proof. First we prove (a). Without loss of generality, suppose that $\mathscr{E}_\Xi$ is a closed set; otherwise, we consider the restriction $\mathscr{E}_\Xi \cap (\bar{z} + \varepsilon I\!B)$ for some $\varepsilon > 0$. By Proposition 13.1 (b) the set C is LAC at $\mathbf{0}$ with $V = Z$. Indeed, there is a sequence $\{c_k\}$ in (13.31) with $\bar{c} \in C \setminus (-C)$ such that

$$C + c_k \subseteq C \setminus \{\mathbf{0}\} \quad \text{for all} \quad k \in \mathbb{N}. \tag{13.36}$$

Using this and the closedness property of $\mathscr{E}_\Xi$, we have $\text{cl}\,\mathscr{E}_\Xi = \mathscr{E}_\Xi = \Xi + C$ and

$$\text{cl}\,\mathscr{E}_\Xi + c_k = \Xi + C + c_k \subseteq \Xi + C \setminus \{\mathbf{0}\} \subseteq (\Xi + C) \setminus \{\bar{z}\} = \mathscr{E}_\Xi \setminus \{\bar{z}\}, \tag{13.37}$$

where the first inclusion is due to the choice of c_k and the second one holds by the minimality of $\bar{z} \in \text{Min}\,(\Xi;C)$. To check the latter, suppose that it does not hold, i.e., $\bar{z} \in \Xi + C \setminus \{\mathbf{0}\}$. Then $\bar{z}$ can be represented in the form of

$$\bar{z} = z + c \quad \text{for some} \quad z \in \Xi \text{ and } c \in C \setminus \{\mathbf{0}\}, \tag{13.38}$$

which implies that $z = \bar{z} - c \in \Xi \cap (\bar{z} - C \setminus \{\mathbf{0}\})$ and thus contradicts the C-minimality of $\bar{x}$ for Ξ. This contradiction justifies the validity of (13.37) and hence of the LAC property of $\mathscr{E}_\Xi$ at each Pareto minimal point of Ξ with respect to C.

Next we prove (b) under the nonempty interiority condition $\text{int}\,C \neq \emptyset$. It follows from Proposition 13.1 (a) that

$$C + c_k \subseteq \text{int}\,C \quad \text{for all} \quad k \in \mathbb{N}, \tag{13.39}$$

where the sequence $\{c_k\} \subseteq \operatorname{int} C$ is defined by (13.31) with $\bar{c} \in \operatorname{int} C$. Fix k and take arbitrarily $z \in \operatorname{cl} \mathscr{E}_\Xi$. The latter ensures the existence of a sequence $\{z_n\} \subseteq \mathscr{E}_\Xi$ such that $z_n \to z$ as $n \to \infty$. Without loss of generality we may assume that $z - z_n + c_k \in \operatorname{int} C$ for all $n \in \mathbb{N}$, since $c_k \in \operatorname{int} C$. This gives

$$z + c_k = z_n + (z - z_n + c_k) \in (\Xi + C) + \operatorname{int} C \subseteq \Xi + \operatorname{int} C \subseteq (\Xi + C) \setminus \{\bar{z}\} = \mathscr{E}_\Xi \setminus \{\bar{z}\},$$

where the second inclusion holds for each WMin $(\Xi; C)$. Since the point z was chosen arbitrarily in $\operatorname{cl} \mathscr{E}_\Xi$, we get

$$\operatorname{cl} \mathscr{E}_\Xi + c_k \subseteq \mathscr{E}_\Xi \setminus \{\bar{z}\},$$

which implies the LAC property of $\mathscr{E}_\Xi$ at each weak Pareto minimal point of Ξ with respect to C and thus completes the proof of the proposition. □

Observe that there is a subset Ξ in $\mathbb{R}^2$ equipped with the usual Pareto order $C = \mathbb{R}^2_+$, which does not have the LAC property at $\mathbf{0}$ while its epigraphical set (13.35) associated with C does; see, e.g.,

$$\Xi := \left\{ \left(x - 1, \frac{1}{x} - 1 \right) \Big| \; x > 0 \right\}.$$

Conversely, there is a subset Ξ of $\mathbb{R}^2$ such that it is LAC at $\mathbf{0}$, but its epigraphical set (13.35) associated with the above cone C is not; see

$$\Xi := \left\{ (x, y) \in \mathbb{R}^2 \big| \; y \le x \right\}.$$

The next result provides useful consequences of Proposition 13.2 for epigraphs of set-valued mappings between arbitrary Banach spaces.

Corollary 13.3 (LAC Property for Epigraphs of Set-Valued Mappings). *Consider a set-valued mapping $F\colon X \rightrightarrows Z$ between Banach spaces, where the image space Z is partially ordered by a closed and convex cone $C \subseteq Z$ satisfying $C \setminus (-C) \neq \emptyset$. Denote the* (generalized) epigraph *of F (with respect to C) by*

$$\operatorname{epi} F := \left\{ (x, z) \in X \times Z \big| \; z \in F(x) + C \right\}.$$

Then the following assertions hold.

(a) The epigraph of F is LAC at $(\bar{x}, \bar{z})$ provided that $\bar{z} \in \operatorname{Min}(F(\bar{x}); C)$ and that the set $\operatorname{epi} F$ is locally closed around $(\bar{x}, \bar{z})$.

(b) The epigraph of a single-valued mapping $f: X \to Z$ is LAC at $(\bar{x}, f(\bar{x}))$ provided that f is continuous around $\bar{x}$.

(c) The epigraph of an extended-real-valued function $\varphi : X \to \mathbb{R} \cup \{\infty\}$ is LAC at $(\bar{x}, \varphi(\bar{x}))$ provided that φ is lower semicontinuous around $\bar{x}$.

Proof. Assertion (a) easily follows from Proposition 13.2 (a) with $\Xi = \text{gph}\, F$ and $\Theta = \{0\} \times C$ in the product space $X \times Z$ by taking into account that

$$\text{epi}F = \text{gph}\, F + \Theta \quad \text{and} \quad [\bar{z} \in \text{Min}\,(F(\bar{x}); C) \Rightarrow (\bar{x}, \bar{z}) \in \text{Min}\,(\text{gph}\, F; \Theta)]\,.$$

Assertions (b) and (c) are two specifications of assertion (a) for single-valued mappings and extended-real-valued functions. □

Note that Corollary 13.3 is useful to check the LAC property for nonconvex sets. For example, the nonconvex cone $\mathbb{R}^2 \setminus (-\text{int}\, \mathbb{R}^2)$ is LAC at the origin, since it is homeomorphic to the epigraph of the function $\varphi(x) = -|x|$.

Remark 13.3 (On the LAC Property of Sets). Observe the following:

(a) To establish a *separation theorem* for nonconvex sets in [44], Tammer and Weidner assumed that the ordering set D in a Banach space Z contains the ray generated by $k^0 \in Z \setminus \{\mathbf{0}\}$, i.e.,

$$D + k^0 \cdot [0, \infty) \subseteq D.$$

In other words, moving $-D$ along this ray allows us to obtain an epigraph-type set

$$\mathscr{E}_D := \{(z,t) \in Z \times \mathbb{R} \mid z \in tk^0 - D\}.$$

The latter set is LAC at $(\bar{z}, \varphi(\bar{z}))$, since it is identical to the epigraph of the real-valued function defined by

$$\varphi(z) := \inf\{t \in R \mid z \in tk^0 - D\}.$$

(b) *The epi-Lipschitzian property implies the LAC property*. Recall that $\Xi \subseteq Z$ is *epi-Lipschitzian* around $\bar{z} \in \text{cl}\, \Xi$ if there are neighborhoods U and V of $\bar{z}$ and the origin, a vector $c \in Z$, and a number $\gamma > 0$ such that

$$\Omega \cap U + tV \subseteq \Omega + tc \quad \text{for all} \quad t \in (0, \gamma).$$

If Ξ is epi-Lipschitzian at $\bar{z}$ with $c \neq 0$, then by [39] it is locally homeomorphic to the epigraph of a real-valued and Lipschitz continuous function. Thus it is asymptotically closed at this point.

(c) Consider the classical sequence space l_2 and take its unit orthonormal vectors $e_k := (0, \ldots, 0, 1, 0, \ldots)$, where the k^{th} component is 1 while all the others are zeros. Define the set

$$\Xi := \text{cl}\left[\prod_{i=1}^{\infty} \left(\frac{1}{i}\mathbb{N}\right)\right]$$

and check easily the relationships

$$\prod_{i=1}^{\infty}\left(\tfrac{1}{i}\mathbb{N}\right)+\tfrac{1}{k}e_k\subseteq\prod_{i=1}^{\infty}\left(\tfrac{1}{i}\mathbb{N}\right),$$

$$\left[\prod_{i=1}^{\infty}\left(\tfrac{1}{i}\mathbb{N}\right)+\tfrac{1}{k}e_k\right]\cap\tfrac{1}{2k}I\!B=\emptyset.$$

Hence we have $\operatorname{cl}\Xi+\frac{1}{k}e_k\subseteq\Xi\setminus\{\mathbf{0}\}$ for all $k\in\mathbb{N}$, which implies in turn that the set Ξ is LAC at the origin.

Now we are ready to derive various consequences of Theorem 13.2 for minimal points of particular types.

Corollary 13.4 (Necessary Conditions for Pareto-Type Minimal Points). *Let Ξ be a subset in an Asplund space Z ordered by a convex cone $C\subseteq Z$ with $C\setminus(-C)\neq\emptyset$, and let $\bar z\in\Xi$. Assume that Ξ is either locally closed around $\bar z$ or LAC at this point. Then the condition (13.30) is necessary for $\bar z$ to be a local minimal point of Ξ in each of the following senses:*

(a) *$\bar z$ is a local Pareto minimal point of Ξ provided that C is locally closed around the origin and either Ξ is SNC at $\bar z$ or C is SNC at $\mathbf{0}$.*
(b) *$\bar z$ is a local quasi minimal point of Ξ provided that either Ξ is SNC at $\bar z$ or C is SNC at $\mathbf{0}$.*
(c) *$\bar z$ is a local relative minimal point to Ξ provided that either Ξ is SNC at $\bar z$ or the affine closure of C is finite-codimensional in Z.*
(d) *$\bar z$ is a local weak minimal point of Ξ.*

Proof. Observe first that $\bar z$ is a local Θ-minimal point of Ξ, where Θ is defined in (13.27) for each type of minimal points under consideration. Recall from [35, Theorem 1.21] that if Θ is a convex set and $\operatorname{ri}\Theta\neq\emptyset$, the SNC of Θ at the origin is equivalent to the finite-codimension of the closure of its affine hull. Recall also from [34, Proposition 1.25] and [34, Theorem 1.26] that the nonempty interiority condition for a convex cone C implies the SNC property of C at the origin. Therefore, all the assertions (a)–(d) can be unified to that $\bar z$ is a local Θ-minimal point to Ξ provided that either Ξ is SNC at $\bar z$ or Θ is SNC at the origin. This allows us to complete the proof of this corollary by employing Theorem 13.3 to the Θ-minimal point $\bar z$ of Ξ under the assumptions made. □

The next result presents refined necessary optimality conditions for weak Pareto minimal points of sets.

Corollary 13.5 (Refined Necessary Conditions for Weak Pareto Minimal Points). *Let Ξ be a subset in an Asplund space Z ordered by a convex cone $C\subseteq Z$ with $C\setminus(-C)\neq\emptyset$, and let $\bar z\in\Xi$ be a local weak Pareto minimal point to Ξ. Then the necessary condition (13.30) holds.*

Proof. By Proposition 13.2 the weak ordering cone $\Theta_w=\operatorname{int}C\cup\{\mathbf{0}\}$ is LAC at the origin. It follows from the extremality of Θ-minimal points in Theorem 13.2 (a) that the Θ-minimal point $\bar z$ of Ξ is a local minimal point of the set system $\{\Xi,\bar z-\operatorname{cl}C\}$.

Thus there is a neighborhood V of $\bar{z}$ and a sequence $\{c_k\} \subseteq \operatorname{int} C$ defined by (13.31) with $c_k \to \mathbf{0}$ as $k \to \infty$ such that

$$\Xi \cap (\bar{z} - \operatorname{cl} C - c_k) \cap V = \emptyset \quad \text{for all} \quad k \in \mathbb{N},$$

which implies in turn that

$$\operatorname{cl} \Xi \cap (\bar{z} - \operatorname{int} C - c_k) \cap V = \emptyset \quad \text{for all} \quad k \in \mathbb{N}.$$

Taking into account the choice of c_k in (13.31) with $c_k \in \operatorname{int} C$ and the fact that $\operatorname{cl} C + \frac{1}{2} c_k \subseteq \operatorname{int} C$ for any convex cone C, we get

$$\begin{aligned} \operatorname{cl} \Xi \cap (\bar{z} - \operatorname{cl} C - c_k) \cap V &= \operatorname{cl} \Xi \cap \left[\bar{z} - \left(\operatorname{cl} C + \frac{1}{2} c_k \right) - \frac{1}{2} c_k \right] \cap V \\ &\subseteq \operatorname{cl} \Xi \cap \left(\bar{z} - \operatorname{int} C - \frac{1}{2} c_k \right) \cap V = \emptyset. \end{aligned}$$

This verifies the extremality of $\bar{z}$ for the set system $\{\operatorname{cl} \Xi, \bar{z} - \operatorname{cl} C\}$ and thus completes the proof of the corollary by employing the exact extremal principle from Sect. 13.2 to this system due to the unconditional fulfillment of SNC property for convex cones Θ with nonempty interiors; see [35, Proposition 1.25]. □

Observe that the necessary conditions in Corollary 13.5 are identical to those in [5, Theorem 5.3] for constant set-valued mappings $F(x) \equiv \Xi$ associated with a closed set being new for non-closed (indeed, locally asymptotically closed) sets.

Note also that in what follows we will not explicitly state results involving relative minimal points and weak minimal points, since they are consequences of quasi minimality results due to the fact that every weak minimal point is a relative minimal point, which in turn is a quasi minimal point, and due to the known sufficient conditions for the SNC condition of a convex cone under the nonempty relative interiority condition in (c) and under the nonempty interiority one in (d) in Corollary 13.4.

Next we present new results for ideal minimal points of sets.

Corollary 13.6 (Necessary Conditions for Ideal Points). *Let $\bar{z} \in \Xi$ be a local ideal minimal point of Ξ in an Asplund Z partially ordered by an ordering cone $C \subseteq Z$. The following assertions hold:*

(a) Assume that C is convex. Then for any $z^* \in N(\mathbf{0}; C)$ we have

$$z^* \in N(\mathbf{0}; \operatorname{cl} \Xi). \tag{13.40}$$

(b) Assume that Ξ is locally closed around $\bar{z}$ that the ideal ordering cone $\Theta_i := (Z \setminus (-C)) \cup \{\mathbf{0}\}$ is asymptotically closed and SNC at $\mathbf{0}$; the latter is automatic when C is a convex cone with $\operatorname{int} C \neq \emptyset$. Then there is a nonzero dual element

$z^* \in Z^* \setminus \{\mathbf{0}\}$ satisfying

$$z^* \in N(\mathbf{0};\mathrm{cl}\,\Xi) \cap N(\mathbf{0};Z \setminus (-\mathrm{int}C)). \tag{13.41}$$

Proof. The proof of (a) can be found in standard vector optimization texts; see, in particular, [22, Theorem 5.6]. The proof of (b) follows from Theorem 13.3 by employing it to the Θ_i-minimal point $\bar{z}$ of Ξ. □

Now we deduce from Theorem 13.3 necessary conditions for general preference points including those with respect to the lexicographical preference.

Corollary 13.7 (Necessary Conditions for Preference Points). *Let $\preceq$ be a preference on an Asplund space Z, and let $\bar{z} \in \Xi$ be a local preference point for a set $\Xi \subseteq Z$. Assume that Ξ is either locally closed around $\bar{z}$ or LAC at this point and that the preference set $P(\bar{z})$ is LAC at $\bar{z}$. Suppose also that either Ξ or $\mathrm{cl}\,P(\bar{z})$ is SNC at $\bar{z}$. Then there is a dual element $z^* \in Z^* \setminus \{\mathbf{0}\}$ satisfying the inclusion*

$$z^* \in N(\bar{z};\mathrm{cl}\,\Xi) \cap (-N(\bar{z};\mathrm{cl}\,P(\bar{z}))), \tag{13.42}$$

which reduces to $(-1,0,\ldots,0) \in N(\bar{z};\mathrm{cl}\,\Xi)$ for the lexicographical preference on $\mathbb{R}^n$.

Proof. The necessary condition (13.42) follows directly from Theorem 13.3, since every preference point $\bar{z}$ of Ξ with respect to $\preceq$ is a local Θ-minimal point of Ξ with respect to $\Theta := \bar{z} - P(\bar{z})$ by Remark 13.1 (e) and also since

$$N(\mathbf{0};\mathrm{cl}\,\Theta) = N(\mathbf{0};\bar{z} - \mathrm{cl}\,P(\bar{z})) = -N(\bar{z};\mathrm{cl}\,P(\bar{z})).$$

In the case of the lexicographical preference in finite dimensions the set $\mathrm{cl}\,\Theta_{\mathrm{lex}}$ is automatically SNC. Employing the necessary condition (13.42) to $\mathrm{cl}\,\Theta_{\mathrm{lex}}$ and taking into account that $N(\mathbf{0};\mathrm{cl}\,\Theta_{lex}) = \mathbb{R}_+ \cdot (1,0,\ldots,0)$, we arrive at the desired inclusion and complete the proof of the corollary. □

Observe that the driving force for the validity of the necessary condition (13.42) for preferences is the *exact extremal principle* for sets formulated in Sect. 13.2 while not its counterpart for multifunctions from [35, Chap. 5] and its special cases used in [37, 48]. The latter results for multifunctions have some major drawbacks, namely: (1) preferences are assumed to have the almost transitivity property, and (2) the necessary conditions are formulated therein in terms of the *limiting normals to moving sets* that are in general *larger* than the basic one in (13.10).

Let us provide, e.g., two preferences having the LAC property while not satisfying the almost transitivity property. The first one is a generalized Pareto preference $\preceq_\Theta$ in (13.7) induced by a convex cone $\Theta \subseteq \mathbb{R}^2$ defined by

$$\Theta := (\mathbb{R} \times \mathbb{R}_+) \setminus (\mathbb{R}_> \times \{0\}) \quad \text{with} \quad \mathbb{R}_+ := [0,\infty) \quad \text{and} \quad \mathbb{R}_> := (0,\infty),$$

which is not closed around the origin. Since Θ is convex with $\mathrm{int}\,\Theta = \mathbb{R} \times \mathbb{R}_>$, it is LAC at $\mathbf{0}$ by Proposition 13.1. On the other hand, we can easily check by

$$(0,2) \in P(0,0) = -\Theta,\ (0,-4) \in \mathrm{cl}\,P(0,2) = \mathbb{R} \times \mathbb{R}_-,\ \text{ and }\ (0,-4) \notin P(0,0)$$

that this set does not have the almost transitivity property and thus the preference is not closed.

Let us define the second preference on $\mathbb{R}^2$ via the preference-set mapping

$$P(z) := z + \mathrm{epi}\varphi_z \quad \text{with} \quad \varphi_z(x) := \frac{|z_1| + |z_2| + 2}{|z_1| + |z_2| + 1} \cdot |x|.$$

It is obviously not a generalized Pareto preference. For any $z \in \mathbb{R}^2$ the preference set $P(z)$ is LAC at this point by Proposition 13.3, since it is homeomorphic to the epigraph of a function. Moreover, we have

$$P(0,0) = \mathrm{epi}(2|\cdot|),\ P(0,1) = (0,1) + \mathrm{epi}\left(\frac{3}{2}|\cdot|\right),\ (0,1) \in P(0,0),$$

$$(4,7) = (0,1) + \left(3, \frac{3}{2} \cdot |3| + \frac{3}{2}\right) \in P(0,1),\ \text{but}\ (4,7) = (4, 2 \cdot |4| - 1) \notin P(0,0),$$

which exclude the almost transitivity property of this preference at the origin. Thus it is not a closed preference in the sense of [35, Definition 5.55].

However, it is important to mention that there exist preferences, which are closed and induced by utility functions while not enjoying the LAC property. For example, consider the preference $\prec$ on $\mathbb{R}^2$ defined in [8] by

$$(v,t) \prec (x,y) \quad \text{if and only if} \quad u(v,t) \le u(x,y)\ \text{ and }\ (v,t) \ne (x,y),$$

where the utility function $u : \mathbb{R}^2 \to \mathbb{R}$ is given by $u(x,y) := |x| - |y|$. It is a closed preference with the following preference set at $\mathbf{0}$:

$$P(\mathbf{0}) := \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; |x| - |y| \le 0\right\} = \left\{(x,y) \in \mathbb{R}^2 \middle|\; |x| \le |y|\right\}.$$

Thus the set $P(\mathbf{0})$ is not LAC at the origin.

To deal with preferences whose preference/ordering sets are not LAC, we need to make some LAC decomposition of these sets. Given $\Theta \in Z$ with $\mathbf{0} \in \Theta$, let us say that a subset $\widetilde{\Theta}$ of Θ containing the origin is a *maximal LAC ordering set* of Θ when the following conditions hold:

- $\widetilde{\Theta}$ is LAC at $\mathbf{0}$.
- If $\widehat{\Theta}$ with $\widetilde{\Theta} \subseteq \widehat{\Theta}$ is LAC at $\mathbf{0}$, then $\widetilde{\Theta} = \widehat{\Theta}$.

Denote the collection of all the maximal LAC ordering sets of Θ by $\mathrm{MLAC}(\Theta)$ and get the next consequence of Theorem 13.3 providing refined necessary optimality conditions for Θ-minimal points of sets.

Corollary 13.8 (Refined Necessary Conditions for Θ-Minimal Points). *Let $\bar{z} \in \Xi$ be a local Θ-minimal point of Ξ in an Asplund space Z, where Ξ is either locally closed around $\bar{z}$ or LAC at this point, and where $\Theta \subseteq Z$ is an ordering set containing the origin. Then for any set $\widetilde{\Theta} \in MLAC(\Theta)$ there is $z^* \in Z^* \setminus \{\mathbf{0}\}$ such that*

$$z^* \in N(\bar{z}; \text{cl}\,\Xi) \cap N(\mathbf{0}; \text{cl}\,\widetilde{\Theta}) \tag{13.43}$$

provided that either $\text{cl}\,\Xi$ *is SNC at* $\bar{z}$ *or* $\text{cl}\,\widetilde{\Theta}$ *is SNC at the origin.*

Proof. It is obviously follows from Definition 13.3 that $\bar{z}$ is a local $\widetilde{\Theta}$-minimal point of Ξ. Applying now Theorem 13.3 to the local $\widetilde{\Theta}$-minimal point $\bar{z}$ of Ξ, we get from (13.30) the necessary condition (13.43) and thus complete the proof. □

If the ordering set Θ is LAC at the origin, then $\text{MLAC}(\Theta)$ is singleton $\{\Theta\}$, and thus Corollary 13.8 reduces to Theorem 13.3. To illustrate the improvement provided by this corollary, consider two sets in $\mathbb{R}^2$ given by

$$\Xi := \mathbb{R} \cdot (1, -1) \quad \text{and} \quad \Theta := \mathbb{R}^2_+ \cup \mathbb{R}^2_-.$$

We obviously have that the origin is a Θ-minimal point of Ξ and that neither Ξ nor Θ is LAC at $\mathbf{0}$. Hence Theorem 13.3 can not be applied to this minimal point. However, we have $\text{MLAC}(\Theta) = \{\mathbb{R}^2_+, \mathbb{R}^2_-\}$, and thus Corollary 13.8 gives us $z^* = (-1, -1)$ and $y^* = (1, 1)$ satisfying the necessary condition (13.43) due to

$$N(\mathbf{0}; \Xi) \in \mathbb{R} \cdot (1, 1), \quad N(\mathbf{0}; \mathbb{R}^2_+) = \mathbb{R}^2_-, \ \text{and } N(\mathbf{0}; \mathbb{R}^2_-) = \mathbb{R}^2_+.$$

13.4 Applications to Set-Valued Optimization

In this section we first establish relationships between various notions of optimality for problems of multiobjective optimization and Θ-minimal points of sets. These relationships combining with the results obtained in the previous section and with generalized differential calculus of variational analysis allow us to derive enhanced necessary optimality conditions in constrained problems of set-valued optimization.

The basic object of our consideration in this section is the following problem of set-valued optimization with geometric constraints:

$$\Theta\text{-minimize} \quad F(x) \quad \text{subject to} \quad x \in \Omega, \tag{13.44}$$

where $F : X \rightrightarrows Z$ is a set-valued mapping between Banach spaces, $\Omega \subseteq X$ is a nonempty set, and "Θ-minimization" is defined below.

Note that a (global) Pareto optimal solution $\bar{x}$ to a vector-valued mapping $f : X \to Z$ is defined via the minimality of the image point $\bar{z} := f(\bar{x})$ of the image set $\Xi := f(X)$. Since $\bar{z}$ is uniquely defined, this point is never mentioned in optimality notions for minimizers of single-valued mappings. When considering its local versions, the localized role of $\bar{z}$ seems to be abandoned; in fact, it is indeed not necessary under the continuity/differentiation assumptions imposed on cost mappings, while this restricts its applications in the case of discontinuous single-valued mappings. In the case of set-valued optimization the given image point $\bar{z} \in F(\bar{x})$ surely plays a very important role in local aspects of optimality.

Let $\bar{z}$ be a *local, but not global* minimal point of some set Ξ in a Banach space Z partially ordered by a closed, convex, and pointed cone $\Theta \subseteq Z$. Associate with Ξ a set-valued mapping $F : \mathbb{R} \rightrightarrows Z$ defined by

$$F(x) := \Xi \quad \text{for all} \quad x \in \mathbb{R}.$$

Then for any $\bar{x} \in \mathbb{R}$ and for any neighborhood U of it, the point $\bar{z} \in F(\bar{x})$ is not a global minimal point of the image set $F(U) = \Xi$, and thus it is not a local minimizer for F in the usual sense. In this paper we develop the *image localization* of minimizers and study the following kinds of optimal solutions to problem (13.44).

Definition 13.5 (Fully Localized Minimizers for Constrained Multiobjective Problems). Let $(\bar{x}, \bar{z}) \in \operatorname{gph} F$ with $\bar{x} \in \Omega$. Then we say that:

- $(\bar{x}, \bar{z})$ is a *fully localized minimizer* for the multiobjective problem (13.44) if there exist neighborhoods U of $\bar{x}$ and V of $\bar{z}$ such that $\bar{z}$ is a local Θ-minimal point of the image set $F(\Omega \cap U)$, i.e.,

$$F(\Omega \cap U) \cap (\bar{z} - \Theta) \cap V = \{\bar{z}\}, \tag{13.45}$$

  where the image set $F(\Omega \cap U)$ is defined as usual by

$$F(\Omega \cap U) := \bigcup \left\{ F(x) \middle| \, x \in \Omega \cap U \right\}.$$

- $(\bar{x}, \bar{z})$ is a *fully localized strong minimizer* for (13.44) if there exist neighborhoods U of $\bar{x}$ and V of $\bar{z}$ such that there is no $(x, z) \in \operatorname{cl} \operatorname{gph} F \cap (U \times V)$ with $(x, z) \neq (\bar{x}, \bar{z})$ satisfying $x \in \operatorname{cl} \Omega$ and $z \in \bar{z} - \operatorname{cl} \Theta$, i.e.,

$$\operatorname{cl} \operatorname{gph} F \cap (\operatorname{cl} \Omega \times (\bar{z} - \operatorname{cl} \Theta)) \cap (U \times V) = \{(\bar{x}, \bar{z})\}. \tag{13.46}$$

- We replace the adjective "fully" by "*partially*" in (13.45) and (13.46) if $V = Z$ above.
- We omit the adjective "fully localized" or replace it by "*global*" in (13.45) and (13.46) if $U = X$ and $V = Z$ above.

It is easy to check the implications *global* $\Longrightarrow$ *partially localized* $\Longrightarrow$ *fully localized* for the minimizers defined above and that *strong minimizer* $\Longrightarrow$ *minimizer* for the same categories in Definition 13.5. If $\Omega = X$ therein, we speak about the corresponding minimizers for the mapping F.

The notion of partially localized minimizers is conventional and known as local minimizers in literature, while the fully localized notions in (13.45) and (13.46) are new even in single-valued discontinuous objectives $F = f : X \to Z$. In the case of real-valued functions these notions reduce to standard minimizers and isolated minimizers, respectively.

The next proposition shows that there is no difference between fully localized and partially localized minimizers for single-valued and *continuous* mappings.

Proposition 13.3 (Conventional Local Minimizers of Single-Valued Continuous Mappings). *Let $f: X \to Z$ be a single-valued between Banach spaces that is continuous around a given Θ-minimizer $\bar{x} \in \Xi$. Then each fully localized minimizer is a partially localized minimizer in Definition 13.5, and thus there is no difference between partially and fully localized minimizers in this case.*

Proof. Assume that $\bar{x}$ is a fully localized minimizer for problem (13.44). Then there are neighborhoods U of $\bar{x}$ and V of $\bar{z}$ such that

$$f(\Omega \cap U) \cap (\bar{z} - \Theta) \cap V = \{\bar{z}\}. \tag{13.47}$$

Since f is continuous, the preimage set of f for the neighborhood V given by

$$f^{-1}(V) := \left\{x \in X \,\middle|\, f(x) \in V\right\}$$

is an open set. Consider a new neighborhood $\widetilde{U} := U \cap f^{-1}(V)$ and check that

$$f(\Omega \cap \widetilde{U}) \cap (\bar{z} - \Theta) = \{\bar{z}\}. \tag{13.48}$$

Arguing by contradiction, suppose (13.48) does not hold, i.e., there is some $z \neq \bar{z}$ belonging to the intersection on the left-hand side of (13.48). Thus we have

$$z \in f(\Omega \cap \widetilde{U}) \quad \text{and} \quad z \in \bar{z} - \Theta.$$

By the choice of $\widetilde{U}$, find $x \in \Omega \cap U$ with $x \in f^{-1}(V)$, i.e., $z = f(x) \in V$. This yields

$$z \in f(\Omega \cap U) \cap (\bar{z} - \Theta) \cap V,$$

which contradicts (13.47) and thus verifies (13.48) and the partially localized minimality of $\bar{x}$ in the vector optimization problem (13.44). □

Note that the conclusion of Proposition 13.3 is no longer valid for discontinuous mappings, even in scalar optimization. To illustrate this, consider an upper semicontinuous function $\varphi: \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(x) \quad := \quad \begin{cases} \ln(|\, x \,|) \text{ if } x \neq 0, \\ \quad 0 \qquad \text{if } x = 0. \end{cases}$$

We obviously have that $(0,0)$ is a fully localized minimizer for φ with $U = V = (-0.5, 0.5)$, but it is not a partially localized minimizer for this function.

To the best of our knowledge, the notion of (global and local) *strong minimizers* from Definition 13.5 has not been considered earlier in the literature on multiobjective optimization; it is inspired here by Khan's notion of strong Pareto optimal allocations for models of welfare economics. The reader is referred to [25, 26, 35] for motivations of this concept and to our recent paper [6] for some related notions and their applications to welfare economies.

Note to this end that a *fully* localized strong minimizer may not be a *partially* localized minimizer (i.e., with $V = Z$) in (13.45). Indeed, the point $(0,1)$ is a fully localized strong minimizer for the function $\varphi\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(x) \quad := \quad \begin{cases} -x & \text{if } x < 0, \\ x+1 & \text{if } x \ge 0, \end{cases}$$

but it is not a partially localized minimizer for this function.

The next result compares Θ-minimizers from Definition 13.5 with the so-called (f,Θ)-minimizers to (13.44) from [35, Definition 5.53].

Proposition 13.4 (Θ-Minimizers and (f,Θ)-Minimizers). *Let $f : X \to Z$ be a single-valued mapping between Banach spaces, let $\Theta \subseteq Z$ be an ordering subset of Z containing the origin, and let $\bar{x}$ be partially localized Θ-minimizer for f. Then $\bar{x}$ is a local $(f,-\mathrm{cl}\,\Theta)$-optimal solution to (13.44) provided that the set Θ is globally asymptotically closed at the origin, i.e., Θ is LAC at $\mathbf{0}$ with $V = Z$.*

Proof. Since $\bar{x}$ be a partially localized Θ-minimizer for f, we have

$$f(U) \cap (\bar{z} - \Theta) = \{\bar{z}\},$$

where $\bar{z} := f(\bar{x})$ and U is a neighborhood of $\bar{x}$. In addition, the global asymptotic closedness assumption of Θ at the origin ensures the existence of a sequence $\{c_k\}$ with $c_k \to \mathbf{0}$ as $k \to \infty$ such that

$$\mathrm{cl}\,\Theta + c_k \subseteq \Theta \setminus \{\mathbf{0}\} \quad \text{for all } k \in \mathbb{N}.$$

Combining the last two inclusions, we arrive at

$$f(U) \cap (\bar{z} - \mathrm{cl}\,\Theta + c_k) = \emptyset \quad \text{for all } k \in \mathbb{N},$$

which surely implies that $\bar{x}$ is locally $(f,-\mathrm{cl}\,\Theta)$-optimal solution to (13.44) and thus completes the proof of this proposition. □

Note that the conclusion of Proposition 13.4 may not hold without the LAC assumption. To illustrate this, define a mapping $f : \mathbb{R} \to \mathbb{R}^2$ and an ordering set $\Theta \subseteq \mathbb{R}^2$ by

$$f(x) := (-x, x) \quad \text{and} \quad \Theta := \mathbb{R}^2_+ \cup \mathbb{R}^2_-.$$

It is easy to check that $\mathbf{0}$ is a local Θ-minimizer for f, but it is not a locally $(f,-\Theta)$-optimal solution to this problem.

Observe also that the implication reverse to the one in Proposition 13.4 does not hold in general. Indeed, define $f:\mathbb{R}\to\mathbb{R}^2$ by $f(x)=(x,0)$ and consider the usual Pareto order $\Theta=\mathbb{R}^2_+$ on $\mathbb{R}^2$. We can see the origin of $\mathbb{R}^2$ is a local (in fact, global) $(f,-\Theta)$-optimal solution to this problems by taking $\{z_k\}$ with $z_k=(k^{-1},k^{-1})$ for $k\in\mathbb{N}$. On the other hand, it is not a local Θ-minimizer of f, since

$$f(\varepsilon I\!B)\cap(-\Theta)=(-\varepsilon,+\varepsilon)\times\{0\}\neq\{(0,0)\}\quad\text{for each}\quad\varepsilon>0.$$

The next proposition presents relationships between Θ-minimal points for the graph of F in the sense of Definition 13.3. This will allow us to derive necessary conditions for multiobjective optimization problems from the corresponding necessary conditions obtained in Sect. 13.3 for minimal points of sets.

Proposition 13.5 (Θ-Minimizers via Θ-Minimal Points). *The following hold:*

(a) The pair $(\bar{x},\bar{z})$ is a fully localized minimizer for (13.44) if and only if it is a local Θ_m-minimal point for the graph of F with Θ_m given by

$$\Theta_m:=((\bar{x}-\Omega)\times(\Theta\setminus\{\mathbf{0}\}))\cup\{(0,0)\}. \tag{13.49}$$

(b) The pair $(\bar{x},\bar{z})$ is a fully localized strong minimizer for (13.44) if and only if it is a local Θ_{sg}-minimal point for the closure of gph$F$ *with Θ_{sg} given by*

$$\Theta_{sg}:=(\bar{x}-\mathrm{cl}\,\Omega)\times\mathrm{cl}\,\Theta. \tag{13.50}$$

Proof. It follows directly from Definitions 13.5 and 13.3. □

To obtain necessary conditions for Θ-minimizers by employing Theorem 13.2 for sets with respect to the ordering sets Θ_m in (13.49) and Θ_{sg} in (13.50), we need to verify the validity of the LAC property of the sets Θ_m and Θ_{sg}. It can be done by using [6, Proposition 3.3]. For the reader's convenience we recall this result.

Proposition 13.6 (LAC Property for Cartesian Products of Sets). *Let $\bar{z}\in\mathrm{cl}\prod_{i=1}^n\Xi_i\subseteq\prod_{i=1}^n Z_i$ in the Banach space setting, let $I\subseteq\{1,\ldots,n\}$ be a nonempty index set, and let $J:=\{1,\ldots,n\}\setminus I$. Assume that the sets Ξ_i are LAC at $\bar{z}_i\in\mathrm{cl}\,\Xi_i$ for $i\in I$ while the other sets Ξ_j are locally closed around $\bar{z}_j$ for $j\in J$. Then the product set*

$$\Xi:=\prod_{i=1}^n\Xi_i \tag{13.51}$$

enjoys the LAC property at $\bar{z}$.

Proof. Without loss of generality, assume that $I=\{1,\ldots,m\}$ with some $0<m\le n$. Since for each $i\in I$ the set Ξ_i is LAC at $\bar{z}_i$, there are a neighborhood U_i of $\bar{z}_i$ and a sequence $\{c_{ki}\}$ with $\|c_{ki}\|\to 0$ as $k\to\infty$ such that for any $k\in\mathbb{N}$ we have

$$(\mathrm{cl}\,\Xi_i+c_{ki})\cap U_i\subseteq\Xi_i\setminus\{\bar{z}_i\},\quad i\in I.$$

On the other hand, by the assumed local closedness of Ξ_j around $\bar{z}_j$, for each $j \in J$ we find a neighborhood U_j of $\bar{z}_j$ such that

$$\mathrm{cl}\,\Xi_j \cap U_j \subseteq \Xi_j, \quad j \in J.$$

It is obvious that the set $U := \prod_{i\in I} U_i \times \prod_{j\in J} U_j$ is a neighborhood of $\bar{z}$ in the product of Banach spaces $Z := \prod_{i=1}^{n} Z_i$ equipped with the maximum norm. Furthermore, the sequence $\{c_k\} \subseteq Z$ defined by

$$c_k := (c_{k1}, \ldots, c_{km}, 0, \ldots, 0)$$

converges to zero as $k \to \infty$ and satisfies the conditions

$$\begin{aligned}(\mathrm{cl}\,\Xi + c_k) \cap U &= \left(\prod_{i\in I}(\mathrm{cl}\,\Xi_i + c_{ki}) \cap U_i\right) \times \left(\prod_{j\in J}(\mathrm{cl}\,\Xi_j \cap U_j)\right) \\ &\subseteq \left(\prod_{i\in I}(\Xi_i \setminus \{\bar{z}_i\})\right) \times \left(\prod_{j\in J}\Xi_j\right) \quad \subseteq \quad \left(\prod_{i=1}^{n}\Xi_i\right) \setminus \{\bar{z}\} = \Xi \setminus \{\bar{z}\},\end{aligned}$$

where the last inclusion holds due to $I \neq \emptyset$. This gives (13.29) and thus justifies the LAC property of the product set Ξ at $\bar{z}$. □

Now we are ready to establish the main results of this section containing several versions of necessary optimality conditions for fully localized minimizers of the multiobjective optimization problem (13.44).

Theorem 13.4 (Necessary Conditions for Fully Localized Minimizers in Set-Valued Optimization). *Let* $(\bar{x}, \bar{z})$ *be a fully localized minimizer to problem* (13.44)*. Assume that either one of the conditions* (i)–(vii) *below is fulfilled:*

(i) *The sets* $\mathrm{gph}\,F$ *and* Ω *are locally closed around* $(\bar{x}, \bar{z})$ *and* $\bar{x}$*, respectively, and the set* Θ *is LAC at* $\mathbf{0}$*.*
(ii) $\mathrm{gph}\,F$ *is locally closed around* $(\bar{x}, \bar{z})$*,* Ω *is LAC at* $\bar{x}$*, and* Θ *is locally closed around* $\mathbf{0}$*.*
(iii) $\mathrm{gph}\,F$ *is locally closed around* $(\bar{x}, \bar{z})$*,* Ω *is LAC at* $\bar{x}$*, and* Θ *is LAC at* $\mathbf{0}$*.*
(iv) $\mathrm{gph}\,F$ *is LAC at* $(\bar{x}, \bar{z})$*,* Ω *is locally closed around* $\bar{x}$ *and* Θ *is locally closed around* $\mathbf{0}$*.*
(v) $\mathrm{gph}\,F$ *is LAC at* $(\bar{x}, \bar{z})$*,* Ω *is locally closed around* $\bar{x}$*, and* Θ *is LAC at* $\mathbf{0}$*.*
(vi) $\mathrm{gph}\,F$ *is LAC at* $(\bar{x}, \bar{z})$*,* Ω *is LAC at* $\bar{x}$*, and* Θ *is locally closed around* $\mathbf{0}$*.*
(vii) $\mathrm{gph}\,F$*,* Ω*, and* Θ *are LAC at* $(\bar{x}, \bar{z})$*,* $\bar{x}$*, and* $\mathbf{0}$*, respectively.*

Then the following versions of optimality conditions for $(\bar{x}, \bar{z})$ *are satisfied:*

A. FUZZY VERSION. *For every* $\varepsilon > 0$*, there are points*

$$(x_1, z_1) \in \mathrm{gph}\,\mathrm{cl}\,F \cap ((\bar{x}, \bar{z}) + \varepsilon I\!B),\ x_2 \in \mathrm{cl}\,\Omega \cap (\bar{x} + \varepsilon I\!B),\ \textit{and}\ z_2 \in \mathrm{cl}\,\Theta \cap \varepsilon I\!B$$

and dual elements $(x^*,\, z^*) \in X^* \times Z^*$ *with* $\|(x^*, z^*)\| = 1$ *such that*

$$\begin{cases} (x^*, -z^*) \in \widehat{N}\big((x_1, z_1); \operatorname{gph} \operatorname{cl} F\big) + \varepsilon I\!B^*, \\ -x^* \in \widehat{N}(x_2; \operatorname{cl} \Omega) + \varepsilon B^*, \quad \text{and} \quad z^* \in -\widehat{N}(z_2; \operatorname{cl} \Theta) + \varepsilon I\!B^*. \end{cases} \tag{13.52}$$

B. FRITZ JOHN VERSION. *Assume further that either one of the following SNC conditions* (I)–(IV) *is satisfied:*

(I) *Both* $\operatorname{cl}\Omega$ *and* $\operatorname{cl}\Theta$ *are SNC at* $\bar{x}$ *and the origin, respectively.*
(II) $\operatorname{cl}\Theta$ *is SNC at the origin and* $\operatorname{cl}F$ *is PSNC at* $(\bar{x}, \bar{z})$*.*
(III) $\operatorname{cl}\Omega$ *is SNC at* $\bar{x}$ *and* $\operatorname{cl}F^{-1}$ *is PSNC at* $(\bar{z}, \bar{x})$*.*
(IV) $\operatorname{cl}F$ *is SNC at* $(\bar{x}, \bar{z})$*.*

Then there is $(x^*, z^*) \in X^* \times Z^*$ *with* $\|(x^*, z^*)\| = 1$ *such that*

$$x^* \in D^*_N \operatorname{cl} F(\bar{x}, \bar{z})(z^*), \; -x^* \in N(\bar{x}; \operatorname{cl}\Omega), \; \text{and } z^* \in -N(\mathbf{0}; \operatorname{cl}\Theta). \tag{13.53}$$

C. LAGRANGE VERSION. *Assume now that either one of the SNC conditions* (I), (II) *holds. Suppose also that the mixed qualification condition*

$$D^*_M \operatorname{cl} F(\bar{x}, \bar{z})(\mathbf{0}) \cap \big(-N(\bar{x}; \operatorname{cl}\Omega)\big) = \{\mathbf{0}\} \tag{13.54}$$

is satisfied, which is automatic when F *is Lipschitz-like around* $(\bar{x}, \bar{z})$*. Then there is* $z^* \in -N(\mathbf{0}; \operatorname{cl}\Theta)$ *with* $\|z^*\| = 1$ *such that*

$$\mathbf{0} \in D^*_N \operatorname{cl} F(\bar{x}, \bar{z})(z^*) + N(\bar{x}; \operatorname{cl}\Omega). \tag{13.55}$$

D. INVERSE VERSION. *Assume that either one of the SNC conditions* (I), (III) *holds. Suppose also that the inverse qualification condition*

$$D^*_M \operatorname{cl} F^{-1}(\bar{z}, \bar{x})(\mathbf{0}) \cap N(\mathbf{0}; \operatorname{cl}\Theta) = \{\mathbf{0}\} \tag{13.56}$$

is satisfied, which is automatic when $\operatorname{cl} F^{-1}$ *is Lipschitz-like at* $(\bar{z}, \bar{x})$*. Then there is* $x^* \in N(\bar{x}; \operatorname{cl}\Omega)$ *with* $\|x^*\| = 1$ *such that*

$$\mathbf{0} \in D^*_N \operatorname{cl} F^{-1}(\bar{z}, \bar{x})(x^*) + N(\mathbf{0}; \operatorname{cl}\Theta). \tag{13.57}$$

Proof. If follows from Proposition 13.5 that the fully localized minimizer $(\bar{x}, \bar{z})$ for problem (13.44) is a local Θ_m-minimal point of $\operatorname{gph} F$, where Θ_m is given in (13.49). Proposition 13.6 ensures that each LAC condition in (i)–(vii) implies one of the three LAC conditions (i), (ii), and (iii) in Theorem 13.3. Precisely we have the following:

- Either conditions (i), (ii), and (iii) implies that $\operatorname{gph} F$ is locally closed around $(\bar{x}, \bar{z})$ and Θ_m is LAC at $\mathbf{0}$.

- Condition (iv) implies that $\operatorname{gph} F$ is LAC at $(\bar{x},\bar{z})$ and that Θ_m is locally closed around $\mathbf{0}$.
- Either conditions (v), (vi), and (vii) implies that both sets $\operatorname{gph} F$ and Θ_m are LAC at $(\bar{x},\bar{z})$ and $\mathbf{0}$, respectively.

Applying now Theorem 13.2 to the Θ_m-minimal point $(\bar{x},\bar{z})$ of $\operatorname{gph} F$, we have that $(\bar{x},\bar{z})$ is a *local extremal point* of the set system

$$\{\operatorname{cl}\operatorname{gph} F, (\bar{x},\bar{z}) - \operatorname{cl}\Theta_m\} \equiv \{\operatorname{gph}\operatorname{cl} F, \operatorname{cl}\Omega \times (\bar{z} - \operatorname{cl}\Theta)\}. \tag{13.58}$$

First observe that the fuzzy version (I) of the theorem follows directly from the application of the approximate extremal principle (13.18) to the extremal system (13.35) at its local extremal point $(\bar{x},\bar{z})$.

To derive now the Fritz John version (II) of the theorem take $\varepsilon = k^{-1}$ for $k \in \mathbb{N}$ and find by (13.52) sequences of $\{(x_{1k},z_{1k},x_{2k},z_{2k})\}$ and $\{(x^*_{1k},z^*_{1k},x^*_{2k},z^*_{2k})\}$ with

$$\begin{cases} (x_{1k},z_{1k}) \in \operatorname{gph}\operatorname{cl} F, \quad x_{2k} \in \operatorname{cl}\Omega, \quad z_{2k} \in \bar{z} - \operatorname{cl}\Theta, \\ (x^*_{1k},z^*_{1k}) \in \widehat{N}\big((x_{1k},z_{1k}); \operatorname{gph}\operatorname{cl} F\big), \\ x^*_{2k} \in \widehat{N}(x_{2k};\operatorname{cl}\Omega), \quad z^*_{2k} \in \widehat{N}(z_{2k};\bar{z} - \operatorname{cl}\Theta), \end{cases} \tag{13.59}$$

satisfying the relationships

$$\begin{cases} \|(x_{1k},z_{1k}) - (\bar{x},\bar{z})\| \le k^{-1}, \quad \|x_{2k} - \bar{x}\| \le k^{-1}, \quad \|z_{2k} - \bar{z}\| \le k^{-1}, \\ \|(x^*_{1k},z^*_{1k}) + (x^*_{2k},z^*_{2k})\| \le 2k^{-1}, \\ 1 - k^{-1} \le \|(x^*_{1k},z^*_{1k})\| + \|(x^*_{2k},z^*_{2k})\| \le 1 + k^{-1}. \end{cases} \tag{13.60}$$

It is obvious from the last line of (13.60) that the sequences $\{(x^*_{1k},z^*_{1k})\}$ and $\{(x^*_{2k},z^*_{2k})\}$ are bounded. Invoking the weak* *sequential compactness* of bounded sets in duals to Asplund spaces gives us $x^*_1, x^*_2 \in X^*$ and $z^*_1, z^*_2 \in Z^*$ such that

$$(x^*_{1k},z^*_{1k},x^*_{1k},z^*_{2k}) \xrightarrow{w^*} (x^*_1,z^*_1,x^*_2,z^*_2) \text{ as } k \to \infty \tag{13.61}$$

Taking further into account the second line of (13.60), we have $x^* := x^*_1 = -x^*_2$ and $z^* := -z^*_1 = z^*_2$. It follows from definitions of the limiting normal cone (13.10) and the normal coderivative (13.12) that

$$\begin{cases} (x^*,-z^*) \in N\big((\bar{x},\bar{z}); \operatorname{gph}\operatorname{cl} F\big) \quad \Longleftrightarrow \quad x^* \in D^*_N \operatorname{cl} F(\bar{x},\bar{z})(z^*), \\ -x^* \in N(\bar{x};\operatorname{cl}\Omega), \quad \text{and} \quad z^* \in N(\mathbf{0};-\operatorname{cl}\Theta) = -N(\mathbf{0};\operatorname{cl}\Theta) \end{cases} \tag{13.62}$$

by passing to the limit in (13.59). This justifies the Fritz John necessary condition (13.53) in (B) except the required nontriviality condition $\|(x^*,z^*)\| = 1$, which is clearly equivalent to $(x^*,z^*) \neq \mathbf{0}$. To proceed with proving the latter, assume the contrary that $(x^*,z^*)=\mathbf{0}$ and then get from the weak convergence (13.61) of the sequences in (13.59) and each of the imposed SNC conditions (I)–(IV) that:

- The SNC properties of Θ and Ω in (I) imply that $\|x^*_{2k}\|\to 0$ and that $\|z^*_{2k}\|\to 0$, and thus $\|x^*_{1k}\|\to 0$ and $\|z^*_{1k}\|\to 0$ due to the second condition of (13.60).
- The SNC property of Θ in (II) implies that $\|z^*_{2k}\|\to 0$ and thus $\|z^*_{1k}\|\to 0$ due to the second condition of (13.60). Furthermore, the PSNC property of F in (II) implies that $\|x^*_{1k}\|\to 0$ and thus $\|x^*_{2k}\|\to 0$ due to the second condition of (13.60).
- The SNC property of Ω in (III) implies that $\|x^*_{2k}\|\to 0$ and thus $\|x^*_{1k}\|\to 0$ due to the second condition of (13.60). Furthermore, the PSNC property of F^{-1} in (III) implies that $\|z^*_{1k}\|\to 0$ and thus $\|z^*_{2k}\|\to 0$ due to the second condition of (13.60).
- The SNC property of F in (IV) implies that $\|(x^*_{1k},z^*_{1k})\|\to 0$ and thus $\|x^*_{2k}\|\to 0$ and $\|z^*_{2k}\|\to 0$ due to the second condition of (13.60).

Hence under each of the above SNC conditions we get that $\|(x^*_{1k},z^*_{1k},x^*_{2k},z^*_{2k})\|\to 0$ as $k\to\infty$, which contradicts the third relationship in (13.60). This contradiction verifies the required nontriviality condition $(x^*,z^*)\neq\mathbf{0}$ and thus completes the proof of the Fritz John version (B) of the theorem.

Next we justify the Lagrange version (C) of the theorem by further elaborating the relationships in (13.59), (13.60), and (13.61) under the additional mixed qualification condition (13.54). Arguing by contradiction, suppose that $z^*=\mathbf{0}$. Since $\mathrm{cl}\,\Theta$ is SNC at $\mathbf{0}$ in both conditions (I) and (II), it follows from (13.60) and (13.62) that $\|z^*_{2k}\|\to 0$ and thus $\|z^*_{1k}\|\to 0$ due to $\|z^*_{1k}+z^*_{2k}\|\to 0$ in (13.61). This implies that

$$(x^*_{1k},z^*_{1k})\in\widehat{N}\big((x_{1k},z_{1k});\mathrm{cl}\,\mathrm{gph}\,F\big),\quad x^*_{1k}\xrightarrow{w^*}x^*,\quad\text{and}\quad\|z^*_{1k}\|\to 0$$

as $k\to\infty$, which yields in turn that

$$x^*\in D^*_M\mathrm{cl}\,F(\bar{x},\bar{z})(\mathbf{0})\ \text{ and therefore }\ x^*\in D^*_M\mathrm{cl}\,F(\bar{x},\bar{z})(\mathbf{0})\cap\big(-N(\bar{x};\mathrm{cl}\,\Omega)\big).$$

Employing now the mixed qualification condition (13.54), we get $x^*=\mathbf{0}$ and hence $(x^*,z^*)=\mathbf{0}$. This contradicts the nontriviality condition in (13.53) and thus complete the proof of the Lagrange version (C) in the theorem.

Finally, let us justify the inverse version (D) of the necessary optimality conditions in the theorem. It is easy to observe that if $(\bar{x},\bar{z})$ is a local Θ_m-minimal point of the set $\mathrm{gph}\,F$, then $(\bar{z},\bar{x})$ is a local $\widetilde{\Theta}$-minimal point to the set $\mathrm{gph}\,F^{-1}$ with

$$\widetilde{\Theta}:=\big((\Theta\setminus\{\mathbf{0}\})\times(\bar{x}-\Omega)\big)\cup\{\mathbf{0}\},$$

i.e., $(\bar{z},\bar{x})$ is an optimal solution to the following "inverse" problem:

$$(\bar{x}-\Omega)\text{-minimize}\quad F^{-1}(z)\quad\text{subject to}\quad z\in(\bar{z}-\Theta).\tag{13.63}$$

Applying the Lagrangian necessary condition (13.55) in (C) to the inverse problem (13.63) under the inverse qualification condition (13.56) , we arrive at the necessary condition (13.57) in (D) and thus complete the proof of the theorem. □

Let us discuss some specific features and modifications of the major results obtained in Theorem 13.4.

Remark 13.4 (Lagrangian Necessary Conditions under other SNC Properties). Suppose that either the SNC assumptions in (III) or those in (IV) of Theorem 13.4 are satisfied. Then the Lagrange multiplier rule (13.55) is still valid provided that the mixed qualification condition in Theorem 13.53 is replaced by the following *normal qualification condition*:

$$D_N^* \mathrm{cl}\, F(\bar{x},\bar{z})(\mathbf{0}) \cap (-N(\bar{x};\mathrm{cl}\,\Omega)) = \{\mathbf{0}\}. \tag{13.64}$$

Indeed, we get from the Fritz John necessary condition (13.53) that

$$x^* \in D_N^* \mathrm{cl}\, F(\bar{x},\bar{z})(z^*), \quad -x^* \in N(\bar{x};\mathrm{cl}\,\Omega), \quad \text{and} \quad z^* \in -N(\mathbf{0};\mathrm{cl}\,\Theta).$$

If $z^* = \mathbf{0}$, then $x^* \in D_N^* \mathrm{cl}\, F(\bar{x},\bar{z})(\mathbf{0})$ and $-x^* \in N(\bar{x};\mathrm{cl}\,\Omega)$, which surely yield that $x^* = \mathbf{0}$ under the normal qualification condition (13.64). This contradicts the nontriviality condition $(x^*,z^*) \neq \mathbf{0}$. It is worth emphasizing that the SNC assumptions in (II) as well as the mixed qualification condition (13.54) are fulfilled provided that F is Lipschitz-like around $(\bar{x},\bar{z})$ while the validity of the normal qualification condition (13.64) requires in addition that F is *strongly coderivative normal* at $(\bar{x},\bar{z})$, i.e.,

$$D_N^* \mathrm{cl}\, F(\bar{x},\bar{z})(z^*) = D_M^* \mathrm{cl}\, F(\bar{x},\bar{z})(z^*).$$

Although the latter qualification condition is much more restrictive than the former, it can be applied to problem (13.44), where the ordering set is not SNC at $\mathbf{0}$. The reader can find the description of some important classes of strongly coderivatively normal mappings in [34, Proposition 4.9].

Remark 13.5 (Θ-Minimal Points to Sets via Θ-Minimizers to Constant Mappings). Let $\Xi \subseteq Z$ be a subset of Z, and let Θ be an ordering set in Z containing the origin. Consider the constant set-valued mapping $C_\Xi : \mathbb{R} \rightrightarrows Z$ with Ξ generated by Ξ as follows:

$$C_\Xi(x) := \Xi \quad \text{for all} \quad x \in \mathbb{R}.$$

Observe that if $\bar{z} \in \Xi$ is a local Θ-minimal point of Ξ, then for any $\bar{x} \in X$ the pair $(\bar{x},\bar{z})$ is a fully localized minimizer for the mapping C_Ξ. Note that the mapping C_Ξ is strongly coderivative normal at $(\bar{x},\bar{z})$, since $D_M^* C_\Xi(\bar{x},\bar{z})(\mathbf{0}) = D_N^* C_\Xi(\bar{x},\bar{z})(\mathbf{0}) = \{\mathbf{0}\}$. There is no difference between the mixed qualification condition (13.54) and the normal qualification condition (13.64), which both are unconditionally satisfied. Thus the Lagrange multiplier rule in Theorem 13.53 (C) holds under each of all the four SNC assumptions (I)–(IV). This allows us to deduce Theorem 13.3 from

Theorem 13.53 (C), since the fulfillment of the LAC and SNC assumptions in Theorem 13.3 ensures the validity of corresponding assumptions in Theorem 13.4, namely:

- LAC condition (i) in Theorem 13.3 implies that of (i) in Theorem 13.4
- LAC condition (ii) in Theorem 13.3 implies that of (iv) in Theorem 13.4
- LAC condition (iii) in Theorem 13.3 implies that of (vi) in Theorem 13.4
- SNC condition in $\mathrm{cl}\,\Theta$ in Theorem 13.3 implies that of (I) in Theorem 13.4
- SNC condition of $\mathrm{cl}\,\Xi$ in Theorem 13.3 implies that of (IV) in Theorem 13.4

Applying now Theorem 13.4 (C) to the constant mapping C_Ξ, we find an element $z^* \neq \mathbf{0}$ satisfying the inclusions

$$z^* \in -N(\mathbf{0};\mathrm{cl}\,\Theta) \quad \text{and} \quad \mathbf{0} \in D_N^*\mathrm{cl}\,C_\Xi(\bar{x},\bar{z})(z^*).$$

The latter inclusion yields that $(\mathbf{0},-z^*) \in N((\bar{x},\bar{z});\mathbb{R}\times \mathrm{cl}\,\Xi)$ and thus $-z^* \in N(\bar{z};\mathrm{cl}\,\Xi)$, which gives the necessary condition (13.30) in Theorem 13.3.

Next we formulate two corollaries of Theorem 13.4. The first one is a generalized Fermat rule for set-valued mappings and the other is a specification of Theorem 13.4 for problem (13.44) with Lipschitzian data.

Corollary 13.9 (Generalized Fermat Rules for Pareto-Type Minimizers of Set-Valued Mappings). *Let $F\colon X \rightrightarrows Z$ be a set-valued mapping between Asplund spaces such that its graph is locally closed around the reference point while the image space Z is partially ordered by a convex cone $C \subseteq Z$ with $C\setminus(-C) \neq \emptyset$. Assume that either C is SNC at the origin or F^{-1} is PSNC at the point in question. Then the condition*

$$\mathbf{0} \in D_N^*F(\bar{x},\bar{z})(z^*) \;\textit{ for some }\; -z^* \in N(\mathbf{0};C) \;\textit{ with }\; \|z^*\| = 1 \tag{13.65}$$

is necessary for the local optimality of $(\bar{x},\bar{z}) \in \mathrm{gph}\,F$ to the mapping F in each of the following senses:

(a) $(\bar{x},\bar{z})$ is a fully localized Pareto minimizer for the mapping F provided that C is a closed cone.
(b) $(\bar{x},\bar{z})$ is a fully localized quasi Pareto minimizer for the mapping F provided that $\mathrm{qri}\,C \neq \emptyset$.

Proof. It follows from the Fritz John multiplier rule in Theorem 13.4, Part (B) by the arguments used in the proof of the necessary conditions for Pareto-type minimal points in Corollary 13.4. □

In the case of a lower semicontinuous extended-real-value function $\varphi : X \to \overline{\mathbb{R}}$, the generalized Fermat rule in Corollary 13.9 applied to the epigraphical mapping

$$F(x) := \big\{\mu \in \mathbb{R}\big|\ \mu \geq \varphi(x)\big\}$$

reduces to the well-known result of [34, Proposition 1.114] stating that

$$0 \in D^*\varphi(\bar{x},\varphi(\bar{x}))(1) = \partial\varphi(\bar{x}),$$

where $\partial\varphi(\bar{x})$ stands for the basic/limiting subdifferential by Mordukhovich. It is easy to see that the latter stationary condition could be replaced by its counterpart via the Fréchet subdifferential.

It is important to observe that, in contrast to scalar optimization, the basic coderivative necessary optimality condition (13.65) is "*essential*" and can not be replaced by its Fréchet counterpart. This can be illustrated via the following constant set-valued mapping $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ defined by

$$F(x) \equiv \Xi \quad \text{with} \quad \Xi := \left\{(z_1,z_2) \in \mathbb{R}^2 \middle|\, 2z_1 + z_2 \geq 0 \text{ or } z_1 + 2z_2 \geq 0\right\}.$$

The pair $(0,0) \in \mathbb{R} \times \mathbb{R}^2$ is obviously a Pareto minimizer of F, and the basic coderivative optimality condition (13.65) is fulfilled for either $z^* = \left(2/\sqrt{5}, 1/\sqrt{5}\right)$ or $z^* = \left(1/\sqrt{5}, 2/\sqrt{5}\right)$ since

$$N((0,0);\text{gph}\, F) = \{0\} \times \text{cone}\,\{(-2,-1),(-1,-2)\}.$$

We can check that

$$\widehat{N}((0,0);\mathbb{R}\times\Xi) = \widehat{N}(0;\mathbb{R}) \times \widehat{N}(0;\Xi) = \{0\} \times \emptyset = \emptyset,$$

which implies that $\widehat{D}^*F(0,0)(z^*) = \emptyset$, and thus the Fréchet coderivative counterpart of (13.65) does not hold.

Now we proceed with necessary optimality conditions for set-valued optimization problems with Lipschitzian objectives.

Corollary 13.10 (Necessary Conditions for Pareto-Type Minimizers Under Lipschitzian Assumptions). *Let $F : X \rightrightarrows Z$ be a set-valued mapping between Asplund spaces with the image space Z partially ordered by a convex cone $C \subseteq Z$ with $C \setminus (-C) \neq \emptyset$. Suppose that the constraint set $\Omega \subseteq X$ is locally closed around $\bar{x}$ and that the ordering cone C is SNC at the origin. Assume also that F is Lipschitz-like around $(\bar{x},\bar{z})$, which is equivalent to the simultaneous fulfillment of the PSNC property of F at $(\bar{x},\bar{z})$ and the mixed coderivative condition*

$$D^*_M F(\bar{x},\bar{z})(\mathbf{0}) = \{\mathbf{0}\}.$$

Then there exists $-z^* \in N(\mathbf{0};C)$ with $\|z^*\| = 1$ such that

$$\mathbf{0} \in D^*_N F(\bar{x},\bar{z})(z^*) + N(\bar{x};\Omega) \tag{13.66}$$

in each of the following cases of local minimizers:

(a) $(\bar{x},\bar{z})$ *is a* FULLY LOCALIZED PARETO MINIMIZER *for problem* (13.44) *provided that* C *is a closed cone.*
(b) $(\bar{x},\bar{z})$ *is a* FULLY LOCALIZED QUASI PARETO MINIMIZER *for problem* (13.44) *provided that* $\operatorname{qri} C \neq \emptyset$.

Proof. It follow from the Lagrange multiplier rule in Theorem 13.4, Part (C) and the arguments used in the proof of the necessary conditions for Pareto-type minimal points in Corollary 13.4. The coderivative criterion for the Lipschitz-like property is taken from [34, Theorem 4.10]. □

Let us compare the results obtained in Theorem 13.4 and its corollaries with those previously known in the literature.

Remark 13.6 (Comparisons with known Necessary Conditions). All the necessary conditions in Theorem 13.4 are new at least in two regards: (1) the refined LAC assumptions are used in order to avoid the closedness and convexity properties of ordering sets/cones, and (2) the fully localized minimizers are used instead of the partially localized (known as local) minimizers. Besides these, they have other improvements discussed below.

(a) If $\Omega := X$ and if Θ is a proper, closed, and convex cone, the fuzzy necessary condition (13.52) in Part (A) reduces to [13, Theorem 3.7], where the mapping $\operatorname{cl} F$ is considered to drop the closedness assumption of $\operatorname{gph} F$ at the point under consideration provided that $\operatorname{int}\Theta \neq \emptyset$. Since the latter condition implies the SNC property of Θ at $\mathbf{0}$, the alternative SNC conditions were not studied in [13].
(b) If Θ is a closed and convex cone with $\Theta \setminus (-\Theta) \neq \emptyset$, the Lagrangian necessary condition (13.55) in Part (C) reduces to [5, Theorem 5.3] in which the SNC requirements were imposed on the restriction F_Ω of F over Ω.
(c) If Θ enjoys some dual compactness property implying the SNC condition of Θ at the origin and if F has the Lipschitz-like property around $(\bar{x},\bar{z})$, which ensures by [35, Theorem 4.10] that F is PSNC at this point and that $D^*_M F(\bar{x},\bar{z})(\mathbf{0}) = \{\mathbf{0}\}$, then the Lagrangian necessary condition (13.55) in Part (C) reduces to [47, Theorem 4.2], and [46, Theorem 4.2] with $\Omega = X$. Note also that the fuzzy counterpart of [46, Theorem 4.2] can be deduced from the fuzzy necessary condition (13.52) in Part (A) in this setting.
(d) The fuzzy necessary conditions in [14, Theorem 4.1] and the exact ones in [14, Theorem 3.1] are specifications of Theorem 13.4, Parts (A) and (C), when the cost mapping $F = f : X \to Z$ is single-valued and locally Lipschitzian and when the image space is partially ordered by a closed and convex cone $\Theta \subsetneq Z$ with $\operatorname{int}\Theta \neq \emptyset$. Note that the authors of [14] employed a scalarization approach, which is based on the separation theorem [44] for nonconvex sets stated in Sect. 13.1, and thus the nonempty interiority condition plays a vital role to implement their approach. Observe also that the necessary conditions obtained in [14] are for weak Pareto minimizers.
(e) The Lagrange multiplier rules for vector-valued and set-valued optimization problems established in [11, Theorem 4.2] and [11, Theorem 4.3] can be

deduced from the Lagrangian necessary condition (13.55) in Part (C) in the Asplund space setting. Note that paper [11] imposes the assumption that the ordering cone Θ is asymptotically compact, i.e., there is a closed neighborhood U of $\mathbf{0}$ such that $\Theta \cap U$ is a compact set. This assumption is independent of our SNC requirement on Θ at the origin. Indeed, in a Hilbert space with a complete orthonormal basis $\{e_2, e_3, \ldots\}$ the closed cone $\Theta := \mathbb{R}_+ \cdot e_1$ is asymptotically compact but not SNC. On the other hand, the subspace $\Theta := \text{span}\,\{e_1, e_2, \ldots\}$ is SNC at the origin while not asymptotically compact. Observe also that the asymptotic compactness conditions and others assumptions imposed in [11] seem to be rather restrictive; in particular, they imply that every Pareto minimizer is a proper Pareto one. Roughly speaking, the necessary optimality conditions in [11] are established for proper Pareto minimizers.

(f) The Lagrangian necessary condition (13.55) in (C) agrees with [35, Theorem 5.73 (ii)], which employs only the two SNC properties (I) and (IV). However, their alternatives (II) and (III) seem to be more efficient, since they are automatic under the Lipschitz-like property of F and F^{-1}, respectively. Note also that the necessary condition in [35, Theorem 5.73 (ii)] is established for single-valued continuous cost mappings and for closed preferences in terms of the *limiting normals to moving sets* $N_+(\bar{z}; \text{cl}\, P_z(\bar{z}))$ defined by

$$N_+\left(\bar{z}; \text{cl}\, P(\bar{z})\right) := \mathop{\text{Lim sup}}_{\substack{(z,u) \longrightarrow (\bar{z},\bar{z}) \\ (z,u) \in \text{gph}(\text{cl}\, P)}} \widehat{N}\left(u; \text{cl}\, P(z)\right).$$

Since $N(\bar{z}; \text{cl}\, P(\bar{z})) \subseteq N_+(\bar{z}; \text{cl}\, P(\bar{z}))$ is *strict* in general, our result is more efficient. To this end we would like to mention a fuzzy counterpart of [35, Theorem 5.73 (ii)] developed in [37, Proposition 5.1], which can be derived from the fuzzy necessary condition obtained in Part (A).

(g) The Fritz John necessary condition from [8, Theorem 4.1] can be seen as a counterpart of [35, Theorem 5.73 (ii)] in the case of arbitrary Banach spaces in terms of the so-called *approximate normal cone* and the corresponding derivative-like constructions developed by Ioffe. The approximate normal cone reduces to our basic one (12.9) in the case of weakly compactly generated Asplund spaces but may be larger than the latter in the general Asplund space setting; see [34, Sect. 3.2.3] and [36, Sect. 9] with the references therein. Observe also that more restrictive epi-Lipschitzian and strongly Lipschitzian properties were imposed on sets and mappings in [8]. Thus Theorem 13.4, Part (B) is a far-going extension of the corresponding results of [8] in the case of general Asplund spaces.

(h) If $F = f\colon X \to Z$ is a single-valued and continuous mapping, $\bar{x}$ is a partially localized Θ-minimal point to f over Ω, and Θ is LAC at the origin, then $\bar{x}$ is a locally $(f, -\text{cl}\,\Theta)$-optimal solution relative to Ω. Hence the optimality notion used in [35, Theorem 5.59] is more general than the Θ-optimality notion considered above in Theorem 13.4. However, [35, Theorem 5.59] requires that: (1) the ordering set $\Theta \subseteq Z$ containing the origin is locally closed around the origin, (2) either Θ is SNC at $\mathbf{0}$, or f_Ω^{-1} is PSNC at $(\bar{z}, \bar{x})$, (3) f is Lipschitz continuous around $\bar{x}$ relative to Ω, and (4) the restriction f_Ω of f to Ω is *strongly coderivative normal* at $\bar{x}$. These assumptions are significantly more demanding than those imposed in Theorem 13.4.

The next result is a counterpart of Theorem 13.4 for the case of strong minimizers.

Theorem 13.5 (Necessary Conditions for Fully Localized Strong Minimizers). *Let* $(\bar{x},\bar{z})$ *be a fully localized strong minimizer for problem* (13.44)*. Assume that one of the sets* $\operatorname{gph} F$*,* Ω*, and* Θ *is LAC at* $(\bar{x},\bar{z})$*,* $\bar{x}$*, and* $\mathbf{0}$*, respectively. Then the necessary conditions of Theorem* 13.4, Parts (A)–(D) *are satisfied.*

Proof. Note that if either $\operatorname{gph} F$, Ω, or Θ is LAC at $(\bar{x},\bar{z})$, $\bar{x}$, and $\mathbf{0}$, respectively, then either $\operatorname{gph}\operatorname{cl} F$ is LAC at $(\bar{x},\bar{z})$, or the ordering set $\Theta_{sg} = (\bar{x} - \operatorname{cl}\Omega) \times \operatorname{cl}\Theta$ defined in (13.50) is LAC at the origin. We can obtain therefore the fuzzy, Fritz John, Lagrange, and reverse necessary optimality conditions given in Theorem 13.4, Parts (A–D) by using similar arguments with just one change: replace Θ_m by Θ_{sg} therein. □

Let us conclude this section with the following two remarks.

Remark 13.7 (Applications to Welfare Economics). The obtained necessary conditions for fully localized minimizers in set-valued optimization problems allow us to derive enhanced versions of the second welfare theorem for nonconvex economies by showing that marginal prices at Pareto-type optimal allocations of nonsmooth and nonconvex economies is nothing but common multipliers in appropriate constrained optimization problems; see [6, 7] for more details.

Remark 13.8 (New Necessary Conditions for Pareto-Type Minimizers). As consequences of Theorems 13.4 and 13.5 we can derive new necessary optimality conditions for *fully localized* Pareto-type minimizers extending the corresponding results of [5, Theorem 5.3] obtained for their partially localized counterparts. To proceed, take $\Theta = C$ in Theorem 13.4 and use Corollary 13.4.

13.5 Multiobjective Optimization with Operator Constraints

In the last section of the chapter we derive necessary conditions for Θ-minimal solutions of multiobjective optimization problems with general operators constraints. The results obtained below are largely based on those in the preceding section and on well-developed calculus rules for limiting normals and coderivatives.

Consider the following multiobjective optimization problems containing the so-called *operator constraints* together with geometric ones:

$$\begin{aligned} &\Theta\text{-minimize } && F(x) \\ &\text{subject to } && G(x) \cap (-\Lambda) \neq \emptyset, \\ & && x \in \Omega, \end{aligned} \tag{13.67}$$

where $F : X \rightrightarrows Z$ and $G : X \rightrightarrows W$ are set-valued mappings between Banach spaces, Θ is an ordering set of Z with $\mathbf{0} \in \Theta$, Ω and Λ are subsets of the spaces X and W, respectively, and where the solution to (13.67) is understood in the above sense of Θ-minimality. Note that the operator constraints $G(x) \cap (-\Lambda) \neq \emptyset$ in (13.67) reduce to the *generalized inequality constraints* when $\Lambda := K$ is a closed, convex, and pointed cone of W and to the *functional inequality constraints*

$$g_i(x) \leq 0, \quad i = 1, \ldots, n, \tag{13.68}$$

when $Z = \mathbb{R}^n$, $K = \mathbb{R}^n_+$, and $G(x) = (g_1, \ldots, g_n) : X \to \mathbb{R}^n$ is a single-valued mapping. Observe also that model (13.67) covers many special classes of constrained optimization problems including, in particular, mathematical programs with equilibrium constraints; cf. [28, 35, 38]. On the other hand, it can be treated as a special case of the so-called extended equilibrium constraints studied in [2, 3].

We say that $(\bar{x}, \bar{z})$ is a *fully localized Θ-minimizer* of problem (13.67) if it is a fully localized Θ-minimizer of problem (13.44) with the set constraint

$$\widetilde{\Omega} := \left\{ x \in X \,\middle|\, G(x) \cap (-\Lambda) \neq \emptyset \text{ and } x \in \Omega \right\}.$$

In order to establish new necessary conditions in the general multiobjective optimization problem (13.67), we follow the reduction technique initially suggested by Guerraggio and Luc [17] for a particular case of (13.67), where $\Theta = C$ and $\Lambda = K$ are closed, convex, and pointed cones and where $F = f : X \to Z$ is a single-valued cost mapping. It is observed in [17] that a local Pareto minimizer for such a constrained vector optimization problem is also a minimizer for the following optimization problem with the only geometric constraint:

$$\text{minimize} \quad \widetilde{F}(x) \quad \text{subject to} \quad x \in \Omega,$$

where the new cost mapping $\widetilde{F} : X \rightrightarrows Z \times W$ between Banach spaces is defined by

$$\widetilde{F}(x) := (f(x), G(x)) + C \times K,$$

and where "minimization" is understood in the sense of Pareto partial ordering generated by the cone $C \times K$ in the product space $W := Z \times Y$. Developing this idea, we relate the original problem (13.67) to the *"product"* multiobjective problem:

$$\widetilde{\Theta}\text{-minimize} \quad \widetilde{F}(x) \quad \text{subject to} \quad x \in \Omega, \tag{13.69}$$

where the set-valued cost mapping $\widetilde{F} : X \rightrightarrows Y := Z \times W$ and the ordering set $\widetilde{\Theta}$ in the space Y are defined by

$$\widetilde{F}(x) := F(x) \times G(x) \quad \text{and} \quad \widetilde{\Theta} := \Theta \times (\Lambda + \overline{w}) \tag{13.70}$$

for the given element $\overline{w} \in G(\bar{x}) \cap (-\Lambda)$. Then we can reduce (13.67)–(13.69).

Proposition 13.7 (Reduction of Operator Constraints). *In the general Banach space setting we have the following relationships:*

(a) If $(\bar{x},\bar{z})$ is a fully localized Θ-minimizer of (13.67), then for each $\bar{w}\in G(\bar{x})\cap(-\Lambda)$ the triple $(\bar{x},\bar{z},\bar{w})\in \operatorname{gph}\widetilde{F}$ is a fully localized $\widetilde{\Theta}$-minimizer of (13.69).

(b) If $(\bar{x},\bar{z},\bar{w})\in \operatorname{gph}\widetilde{F}$ is a partially localized $\widetilde{\Theta}$-minimizer of (13.69), then $(\bar{x},\bar{z})$ is a partially localized Θ-minimizer of (13.67).

Proof. First we prove (a). Let $(\bar{x},\bar{z})$ be a fully localized Θ-minimizer of problem (13.67). Then there are neighborhoods U of $\bar{x}$, and V of $\bar{z}$ such that

$$F(x)\cap(\bar{z}-\Theta)=\{\bar{z}\}\quad\text{for all}\quad x\in\Omega\cap G^{-1}(-\Lambda)\cap U. \tag{13.71}$$

Taking any $\bar{w}\in G(\bar{x})$, we claim that $(\bar{x},\bar{z},\bar{w})\in\operatorname{gph}\widetilde{F}$ is a fully localized $\widetilde{\Theta}$-minimizer of problem (13.69) with the neighborhood $U\times V\times W$ of $(\bar{x},\bar{z},\bar{w})$, i.e.,

$$\widetilde{F}(x)\cap\Big((\bar{z},\bar{w})-\widetilde{\Theta}\Big)\cap(U\times V\times W)=\{(\bar{z},\bar{w})\}.$$

Arguing by contradiction, suppose that it does not hold and then find $(x,z,w)\in U\times V\times W$ with $(z,w)\neq(\bar{z},\bar{w})$ satisfying

$$(z,w)\in(\bar{z},\bar{w})-\Theta\times(\Lambda+\bar{w}),\quad\forall\ (z,w)\in F(x)\times G(x)\ \text{ and }\ x\in\Omega$$

due to the structures of $\widetilde{F}$ and $\widetilde{\Theta}$ in (13.70); this clearly contradicts (13.71). The contradiction obtained verifies the claim and thus completes the proof of assertion (a).

Next we justify (b). Let $(\bar{x},\bar{z},\bar{w})\in\operatorname{gph}\widetilde{F}$ be a partially localized $\widetilde{\Theta}$-minimizer of problem (13.69), i.e., there is a neighborhood U of $\bar{x}$ such that

$$\widetilde{F}(\Omega\cap U)\cap\Big((\bar{z},\bar{w})-\widetilde{\Theta}\Big)=\{(\bar{z},\bar{w})\}. \tag{13.72}$$

We show that the relation in (13.71) holds with $V=Z$. Arguing by contradiction, suppose that it does not hold and then find $x\in\Omega\cap U$, $w\in G(x)\cap(-\Lambda)\neq\emptyset$, and $z\in F(x)$ satisfying

$$z\neq\bar{z}\quad\text{and}\quad z\in(\bar{z}-\Theta),$$

which surely implies that

$$(z,w)\in(F(x)\times G(x))\cap(\bar{z}-\Theta)\times(-\Lambda)=\widetilde{F}(x)\cap\Big((\bar{z},\bar{w})-\widetilde{\Theta}\Big)$$

and thus clearly contradicts (13.72) due to $z\neq\bar{z}$. This contradiction verifies the partially localized optimality of $(\bar{x},\bar{z})$ in (13.44) and thus completes the proof of (b). □

For simplicity we assume in what follows that all the sets under consideration are locally closed around the points in question; see Theorem 13.53 for the alternative LAC assumptions. The next theorem is the main result of this section.

Theorem 13.6 (Necessary Optimality Conditions for Multiobjective Problems with Operator Constraints). *Let $(\bar{x},\bar{z})$ be a fully localized Θ-minimizer of problem (13.67) in Asplund spaces, and let $\overline{w} \in \Omega \cap G^{-1}(-\Lambda)$. Suppose that the sets Ω, Θ, Λ, $\operatorname{gph} F$, and $\operatorname{gph} G$ are locally closed around $\bar{x}$, $\mathbf{0}$, $\overline{w}$, $(\bar{x},\bar{z})$, and $(\bar{x},\overline{w})$, respectively, and that one of these sets is asymptotically closed at the corresponding point. Assume also that the two qualification conditions*

$$(D_M^*F(\bar{x},\bar{z})(\mathbf{0}) + D_M^*G(\bar{x},\bar{z})(\mathbf{0})) \cap (-N(\bar{x};\Omega)) = \{\mathbf{0}\}, \tag{13.73}$$

$$D_M^*F(\bar{x},\bar{z})(\mathbf{0}) \cap (-D_M^*G(\bar{x},\bar{z})(\mathbf{0})) = \{\mathbf{0}\} \tag{13.74}$$

are satisfied, which is automatic provided that both mappings F and G are Lipschitz-like around $(\bar{x},\bar{z})$ and $(\bar{x},\overline{w})$, respectively. Assume finally that either F is PSNC at $(\bar{x},\bar{z})$ or G is PSNC at $(\bar{x},\bar{z})$ and that one of the following SNC properties holds:

(I) Ω, Θ, and Λ are SNC at $\bar{x}$, $\mathbf{0}$, and $-\overline{w}$, respectively.
(II) Θ is SNC at $\mathbf{0}$, Λ is SNC at $\overline{w}$, and both mappings F and G are PSNC at $(\bar{x},\bar{z})$ and $(\bar{x},\overline{w})$, respectively.

Then there are $z^* \in -N(\mathbf{0};\Theta)$ and $w^* \in -N(-\overline{w};\Lambda)$ with $(x^*,w^*) \neq \mathbf{0}$ satisfying

$$\mathbf{0} \in D_N^*F(\bar{x},\bar{z})(z^*) + D_N^*G(\bar{x},\overline{w})(w^*) + N(\bar{x};\Omega). \tag{13.75}$$

Proof. Taking $(\bar{x},\bar{z})$ and $\overline{w} \in \Omega \cap G^{-1}(-\Lambda)$ from the formulation of the theorem, we get from Proposition 13.7 that $(\bar{x},\bar{z},\overline{w})$ is a fully localized $\widetilde{\Theta}$-minimizer for problem (13.69) with $\widetilde{\Theta}$ from (13.70). To apply the Lagrangian optimality condition (13.55) from Theorem 13.4 (C) to problem (13.69), we need to check all the assumptions imposed therein on the cost mapping $\widetilde{F}$, the ordering set $\widetilde{\Theta}$, and the constraint set Ω.

First compute the coderivative of $\widetilde{F}$. Observe that $\widetilde{F}$ can be expressed as a sum of two mappings $\widetilde{F}(x) = H_1(x) + H_2(x)$, where $H_1, H_2 : X \rightrightarrows Z \times W$ are defined by

$$H_1(x) := F(x) \times \{\mathbf{0}\} \quad \text{and} \quad H_2(x) := \{\mathbf{0}\} \times G(x). \tag{13.76}$$

Observe further that $H_1(x) = (\varphi \circ F)(x)$ is a composition of the inner mapping F from the cost of (13.67) and the differentiable outer mapping $\varphi : Z \to Z \times W$ given by $\varphi(z) := (z,\mathbf{0})$. Applying now the chain rules for the normal and mixed coderivatives from [34, Theorem 3.13] to this composition gives us the inclusions

$$\begin{cases} D_N^*H_1(\bar{x},\bar{z},\mathbf{0})(z^*,w^*) \subseteq D_N^*F(\bar{x},\bar{z})(z^*), \\ D_M^*H_1(\bar{x},\bar{z},\mathbf{0})(\mathbf{0}) \subseteq D_M^*F(\bar{x},\bar{z})(\mathbf{0}). \end{cases} \tag{13.77}$$

Representing similarly $H_2(x) = (\psi \circ G)(x)$ with $\psi : W \to Z \times W$ given by $\psi(w) := (\mathbf{0},w)$ and applying the coderivative chain rules mentioned above, we get

$$\begin{cases} D_N^* H_2(\bar{x}, \mathbf{0}, \bar{w})(z^*, w^*) \subseteq D_N^* G(\bar{x}, \bar{w})(w^*), \\ D_M^* H_2(\bar{x}, \mathbf{0}, \bar{w})(\mathbf{0}) \subseteq D_M^* G(\bar{x}, \bar{w})(\mathbf{0}). \end{cases} \tag{13.78}$$

To apply further the coderivative sum rules from [34, Theorem 3.10] to H_1 and H_2, we need to check the following assumptions:

(i) The mapping $S : X \times Z \times W \rightrightarrows (Z \times W)^2$ defined by

$$\begin{aligned} S(x,z,w) &:= \big\{ (z_1, w_1, z_2, w_2) \big|\ (z_1, w_1) \in H_1(x), \\ &\qquad (z_2, w_2) \in H_2(x),\ (z,w) = (z_1, w_1) + (z_2, w_2) \big\} \\ &= \big\{ (z, \mathbf{0}, \mathbf{0}, w) \big\} \end{aligned}$$

is inner semicontinuous at $(\bar{x}, \bar{z}, \bar{w})$.

(ii) Either H_1 or H_2 is PSNC at $(\bar{x}, \bar{z}, \mathbf{0})$ or $(\bar{x}, \mathbf{0}, \bar{w})$, respectively.

(iii) The pair $\{H_1, H_2\}$ satisfies the qualification condition

$$D_M^* H_1(\bar{x}, \bar{z}, \mathbf{0})(\mathbf{0}) \cap \big(-D_M^* H_2(\bar{x}, \mathbf{0}, \bar{w})(\mathbf{0})\big) = \{\mathbf{0}\}.$$

It is easy to check that condition (i) holds trivially, that condition (ii) reduces to that either F or G is PSNC at $(\bar{x}, \bar{z})$ or $(\bar{x}, \bar{w})$, respectively, and that the qualification condition (13.74) assumed in the theorem implies the one in (iii) since

$$D_M^* H_1(\bar{x}, \bar{z}, \mathbf{0})(\mathbf{0}) \cap \big(-D_M^* H_2(\bar{x}, \mathbf{0}, \bar{w})(\mathbf{0})\big) \subseteq D_M^* F(\bar{x}, \bar{z})(\mathbf{0}) \cap \big(-D_M^* G(\bar{x}, \bar{w})(\mathbf{0})\big)$$

due to the coderivative estimates obtained in (13.77) and (13.78). Thus we get from the coderivative sum rules that

$$\begin{cases} D_N^* \widetilde{F}(\bar{x}, \bar{z}, \bar{w})(z^*, w^*) \subseteq D_N^* F(\bar{x}, \bar{z})(z^*) + D_N^* G(\bar{x}, \bar{w})(w^*), \\ D_M^* \widetilde{F}(\bar{x}, \bar{z}, \bar{w})(\mathbf{0}) \subseteq D_M^* F(\bar{x}, \bar{z})(\mathbf{0}) + D_M^* G(\bar{x}, \bar{w})(\mathbf{0}). \end{cases} \tag{13.79}$$

Next we show that $\widetilde{F}$ is PSNC at $(\bar{x}, \bar{z}, \bar{w})$ provided that both F and G are PSNC at $(\bar{x}, \bar{z})$ and $(\bar{x}, \bar{w})$, respectively. Observe that the set $\operatorname{gph} \widetilde{F}$ can be written as an intersection of two sets $\Omega_1, \Omega_2 \subseteq X \times Z \times W$ defined by

$$\begin{cases} \Omega_1 := \big\{ (x,z,w) \in X \times Z \times W \big|\ (x,z) \in \operatorname{gph} F \big\}, \\ \Omega_2 := \big\{ (x,z,w) \in X \times Z \times W \big|\ (x,w) \in \operatorname{gph} G \big\} \end{cases} \tag{13.80}$$

and that all the assumptions of [34, Theorem 3.79] (PSNC property of set intersections) are fulfilled, that is, we have the following:

- Ω_1 is PSNC with respect to $X \times W$ and Ω_2 is PSNC with respect to $X \times Z$, since both F and G are PSNC at $(\bar{x}, \bar{z})$ and $(\bar{x}, \bar{w})$, respectively.
- Ω_1 is strongly PSNC with respect to W and Ω_2 is strongly PSNC with respect to Z due to the structures of sets Ω_1 and Ω_2 in (13.80).
- The set system $\{\Omega_1, \Omega_2\}$ satisfies the mixed qualification condition at $(\bar{x}, \bar{z}, \bar{w})$ with respect to $Z \times W$, i.e., for every sequences

$$\{(x_{1k}, z_{1k}, w_{1k}, x_{2k}, z_{2k}, w_{2k}, x^*_{1k}, z^*_{1k}, w^*_{1k}, x^*_{2k}, z^*_{2k}, w^*_{2k})\}$$

satisfying the limiting relationships

$$\begin{cases} (x_{1k}, z_{1k}, w_{1k}) \xrightarrow{\Omega_1} (\bar{x}, \bar{z}, \bar{w}), \quad (x_{2k}, z_{2k}, w_{2k}) \xrightarrow{\Omega_2} (\bar{x}, \bar{z}, \bar{w}), \\ (x^*_{1k}, z^*_{1k}, w^*_{1k}) \xrightarrow{w^*} (\bar{x}^*_1, \bar{z}^*_1, \bar{w}^*_1), \quad (x^*_{2k}, z^*_{2k}, w^*_{2k}) \xrightarrow{w^*} (\bar{x}^*_2, \bar{z}^*_2, \bar{w}^*_2), \\ (x^*_{1k}, z^*_{1k}, w^*_{1k}) \in \widehat{N}\left((x_{1k}, z_{1k}, w_{1k}); \Omega_1\right), \\ (x^*_{2k}, z^*_{2k}, w^*_{2k}) \in \widehat{N}\left((x_{2k}, z_{2k}, w_{2k}); \Omega_2\right), \end{cases}$$

we have the implication

$$\left[x^*_{1k} + x^*_{1k} \xrightarrow{w^*} \mathbf{0}, \quad \|z^*_{1k} + z^*_{2k}\| \to \mathbf{0}, \quad \|w^*_{1k} + w^*_{1k}\| \to \mathbf{0}\right]$$
$$\Longrightarrow \quad (\bar{x}^*_1, \bar{z}^*_1, \bar{w}^*_1) = (\bar{x}^*_2, \bar{z}^*_2, \bar{w}^*_2) = (\mathbf{0}, \mathbf{0}, \mathbf{0}).$$

Indeed, it follows from the structures of the sets Ω_1 and Ω_2 in (13.80) that $w^*_{1k} \equiv \mathbf{0}$ and $z^*_{2k} \equiv \mathbf{0}$ for all $k \in \mathbb{N}$, and thus the mixed qualification condition for the set system $\{\Omega_1, \Omega_2\}$ is equivalent to the assumed qualification condition (13.74).

Hence we get from [34, Theorem 3.79] applied to the intersection set

$$\operatorname{gph} \widetilde{F} = \Omega_1 \cap \Omega_2$$

that $\operatorname{gph} \widetilde{F}$ is PSNC at $(\bar{x}, \bar{z}, \bar{w})$ with respect to X, which means that the mapping $\widetilde{F}$ from (13.70) is PSNC at this point.

The above arguments check the fulfillment of the assumptions (I) and (II) in Theorem 13.4 (B) for problem (13.69). Observe also that under the LAC property imposed in the theorem we get by Proposition 13.6 that either one from the LAC assumptions (i) and (ii) in Theorem 13.4 is satisfied for problem (13.69). Observe finally that the qualification condition (13.73) implies the qualification condition (13.54) in Theorem 13.4 for problem (13.69), since

$$D^*_M \widetilde{F}(\bar{x}, \bar{z}, \bar{z})(\mathbf{0}) \cap (-N(\bar{x}; \Omega))$$
$$\subseteq (D^*_M F(\bar{x}, \bar{z})(\mathbf{0}) + D^*_M G(\bar{x}, \bar{z})(\mathbf{0})) \cap (-N(\bar{x}; \Omega)) = \{\mathbf{0}\}$$

due to the mixed coderivative estimate for $\widetilde{F}$ in (13.79).

Thus we ensure the validity of all the assumptions of Theorem 13.4 for problem (13.69) with the cost mapping $\widetilde{F}$, the ordering set $\widetilde{\Theta}$, and the constraint set Ω. Applying now Theorem 13.4 to the latter problem, we find dual elements $(z^*,w^*) \in Z^* \times W^*$ with $(z^*,w^*) \neq \mathbf{0}$ satisfying

$$\mathbf{0} \in D_N^* \widetilde{F}(\bar{x},\bar{z},\overline{w})(z^*,w^*) + N(\bar{x};\Omega),$$

which surely verifies the necessary optimality condition (13.75) by using the normal coderivative estimate (13.79) for $\widetilde{F}$. This completes the proof of the theorem. □

Let us discuss some particular features of the results obtained in Theorem 13.6 and their relationships with those known in the literature.

Remark 13.9 (Specific Features and Comparisons with known Results). As everywhere in this paper, the necessary optimality conditions for multiobjective problems are established in Theorem 13.6 for general ordering sets, which may not be either closed or convex cones. Let us mention some other improvements over known results for problems with operator constraints and the like:

(a) Theorem 13.6 is an extension of [35, Theorem 5.11] (iv) from scalar cost functions to set-valued mappings. Note that the latter theorem requires an addition that the auxiliary set-valued mapping $S(\cdot) := F(\cdot) \cap \Lambda$ is either inner semicontinuous at $(\bar{x},\overline{w})$ or inner semicompact at every point $\overline{w} \in S(\bar{x})$. Recall that [34, Definition 1.63] $S: X \rightrightarrows W$ S is (1) *inner semicompact* at $\bar{x}$ with $S(\bar{x}) \neq \emptyset$ if for every sequence $x_k \to \bar{x}$ as $k \to \infty$ there is a sequence $w_k \in S(x_k)$ that contains a convergent subsequence, and (2) it is *inner semicontinuous* at $(\bar{x},\overline{w}) \in \operatorname{gph} S$ if for every sequence $x_k \to \bar{x}$ there is a sequence $w_k \in S(x_k)$ converging to $\overline{w}$. We do not impose these assumptions in Theorem 13.6.

(b) Observe that the operator constraints in (13.67) can be seen as a specification of the so-called *extended equilibrium constraints*

$$\mathbf{0} \in G(x) + Q(x), \tag{13.81}$$

studied in [1,3], where both $G,Q: X \rightrightarrows W$ are set-valued mappings; this reduces to the operator constraints in (13.67) when $Q(x) \equiv \Lambda$ is a constraint mapping. In this way we obtain from [1, Theorem 3.3] and [3, Theorem 3.4] necessary optimality conditions for (13.67) that are different from those in Theorem 13.6 by constraint qualifications: instead of (13.73) and (13.74) above we use

$$\begin{cases} \left[x^* + x_G^* + x_\Omega^* = \mathbf{0}\right] \quad \Longrightarrow \quad x^* = x_G^* = x_\Omega^* = \mathbf{0}, \text{ whenever} \\ x^* \in D_M^* f(\bar{x})(\mathbf{0}), \quad x_G^* \in D_N^* G(\bar{x},\overline{w})(\mathbf{0}), \quad x_\Omega^* \in N(\bar{x};\Omega) \end{cases} \tag{13.82}$$

in [1,3]. It is easy to check that the qualification condition (13.82) implies both qualification conditions (13.73) and (13.74). Due to the presence of the normal coderivative $D_N^* G(\bar{x},\overline{w})(\mathbf{0})$ in (13.82) the latter may not hold when both F and G are Lipschitz-like around $(\bar{x},\bar{z})$ and $(\bar{x},\overline{w})$, respectively.

We conclude this paper with deriving from Theorem 13.67 necessary optimality conditions for multiobjective problems *generalized inequality constraints*, i.e., for a specification of (13.67) to the case when both Θ and Λ closed, convex, and pointed cones in the spaces Z and W, respectively; this case obviously contains standard inequality constraints given in (13.68).

Corollary 13.11 (Necessary Optimality Conditions for Multiobjective Problems with Generalized Inequality Constraints). *Let $(\bar{x},\bar{z})$ be a fully localized Pareto minimizer to problem (13.67) with $\Theta := C$ and $\Lambda := K$, and let $\overline{w} \in G(\bar{x}) \cap (-K)$. Suppose that $C \subseteq Z$ and $K \subseteq W$ are closed, convex, and pointed cones and that the sets Ω, $\operatorname{gph} F$, and $\operatorname{gph} G$ are locally closed around $\bar{x}$, $(\bar{x},\bar{z})$, and $(\bar{x},\overline{w})$, respectively. Assume further that both mappings F and G are Lipschitz-like around $(\bar{x},\bar{z})$ and $(\bar{x},\overline{w})$, respectively, and that both sets C and K are SNC at the origin; the latter SNC conditions are fulfilled if either the corresponding spaces are finite-dimensional or the corresponding ordering cones have nonempty interiors. Then there are dual elements $z^* \in -N(\mathbf{0};C)$ and $w^* \in -N(\mathbf{0};K)$ with $(z^*,w^*) \neq \mathbf{0}$ satisfying*

$$\mathbf{0} \in D_N^* F(\bar{x},\bar{z})(z^*) + D_N^* G(\bar{x},\overline{w})(w^*) + N(\bar{x};\Omega).$$

Proof. It follows from Theorem 13.6 by taking into account that:

- If F and G are Lipschitz-like around $(\bar{x},\bar{z})$ and $(\bar{x},\overline{w})$, respectively, then they are PSNC at these points satisfying $D_M^* F(\bar{x},\bar{z})(\mathbf{0}) = \{\mathbf{0}\}$ and $D_M^* G(\bar{x},\bar{z})(\mathbf{0}) = \{\mathbf{0}\}$. Thus both qualification conditions (13.73) and (13.74) are satisfied.
- Since K is a convex cone, we have $N(-\overline{w};K) \subseteq N(\mathbf{0};K)$ for any $\overline{w} \in -K$.

This completes the proof of the corollary. □

Remark 13.10 (Other Necessary Conditions in Vector Optimization with Generalized Inequality Constraints). Corollary 13.11 provides a refined coderivative version of multiplier rules for vector optimization problems with generalized inequality constraints obtained in [17, Theorem 5.1] under convexity assumptions and in [45, Theorem 5.1] under some extended convexity.

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Chapter 14
Vector Optimization and Cooperative Games

Tetsuzo Tanino

14.1 Introduction

Theory of cooperative games is quite useful in analyzing decision making situations along with multiple decision makers who can form coalitions. An ordinary cooperative game is specified by a real-valued characteristic function defined on the set of coalitions of a finite number of players. A huge number of interesting results have been obtained in this field.

On the other hand, as is explained in the other chapters of this monograph, vector optimization (multiobjective optimization) is both theoretically and practically interesting and useful. Therefore it is quite natural to consider games in which several criteria are be considered simultaneously. Several authors have developed multiobjective noncooperative games. Therefore it is worth reviewing some results connecting cooperative games and vector optimization (multiobjective optimization) in this chapter.

The most simple way of extending ordinary cooperative games to the multiobjective case is to introduce multidimensional vectors of worth in characteristic functions. This leads to the class of vector-valued games proposed in Fernández et al. [6]. If we consider more general worth in partially ordered spaces, partially ordered cooperative games are defined as in Puerto et al. [17]. Moreover, if we consider games derived from multiobjective optimization problems, the sets of Pareto optimal values are not single points but sets in the multi-dimensional space. Therefore it might be more natural to deal with cooperative games with set-valued characteristic functions (Tanino et al. [20], Nishizaki and Sakawa [14], Fernández et al. [9]). The main solution concept in those vector-valued and set-valued

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cooperative games is the core. Some definitions of the core and several sufficient conditions which guarantee the non-emptiness of the core have been investigated.

In this chapter we explain several results concerning vector-valued and set-valued cooperative games obtained in the recent years. We also refer to mutliobjective linear production games and multiobjective minimum cost spanning tree games.

The contents in this chapter are organized as follows. First we briefly review some fundamental results on the theory of cooperative games. The next section deals with the most basic vector-valued cooperative games and the core is defined in terms of scalarization of vectors. This section is based on Fernández et al. [6] and the objective space is $\mathbb{R}^n$. On the contrary, in the fourth section, more general partially ordered cooperative games are dealt with. Two solution concepts, the core and the extended Shapley value are explained based on the work by Puerto et al. [17]. Vector-valued games are extend to set-valued games in a straight forward manner and two types of core, the dominance core and the preference core of a set-valued game are defined and explained in the fifth section based on Fernández et al. [9]. In the sixth section, we discuss multiobjective games (set-valued games) with restrictions on coalitions developed by Tanino [21]. Moreover two interesting kinds of multiobjective games, linear production games and minimum cost spanning tree games, are dealt with in the following two sections respectively. The results concerning multiobjective linear production games are due to Nishizaki and Sakawa [13, 14]. Multiobjective minimum cost spanning tree games were studied by Fernández et al. [8]. In order to make this survey rather simple, we omit all the proofs of the results. They can be found in the original articles. A reader who is interested in a certain section in this chapter is requested to read the corresponding original article.

14.2 Fundamentals of Cooperative Games

14.2.1 Cooperative Games

In this section we briefly review some fundamental results concerning cooperative games, e.g., Owen [16]. A cooperative game (transferable utility game, TU-game for short) is a pair (N, v), where $N = \{1, 2, \ldots, n\}$ is a finite set of players and v is a real valued function defined on the power set of N, i.e., $v : 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. Each subset S of N is called a *coalition* and the value $v(S)$ is called the *worth* of S. Since we fix the player set N throughout this chapter, we regard a function v, called a *characteristic function*, as a game and denote by G^N the set of all games on N. Throughout this chapter we use abbreviated notations such as $v(\{i\}) = v(i)$, $S \cup \{i\} = S \cup i$, and so on. We also discriminate two set inclusive relations $S \subseteq T$ and $S \subseteq T$, where the latter is proper inclusion.

Definition 14.1. A game $v \in G^N$ is said to be

- *Monotonic* if
$$v(S) \leq v(T), \ \forall S,T \subseteq N: \ S \subseteq T, \tag{14.1}$$
- *Additive* if
$$v(S \cup T) = v(S) + v(T), \ \forall S,T \subseteq N: \ S \cap T = \emptyset \tag{14.2}$$
- *Superadditive* if
$$v(S \cup T) \geq v(S) + v(T), \ \forall S,T \subseteq N: \ S \cap T = \emptyset \tag{14.3}$$
- *Convex* if
$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T), \ \forall S,T \subseteq N \tag{14.4}$$

Given two games $v, w \in G^N$, the sum $v + w \in G^N$ of v and w is defined by

$$(v + w)(S) = v(S) + w(S), \ \forall S \subseteq N. \tag{14.5}$$

Analogously, given a game $v \in G^N$ and a scalar $\alpha \in \mathbb{R}$, the scalar multiplication αv of v by α is defined by

$$(\alpha v)(S) = \alpha v(S), \ \forall S \subseteq N. \tag{14.6}$$

Thus the set G^N forms a linear space over $\mathbb{R}$. It is clear the dimension of this space is $2^n - 1$. Usually we consider the following unanimity games as a basis of G^N.

Definition 14.2. The game u_T defined by

$$u_T(S) = \begin{cases} 1 \text{ if } S \supseteq T \\ 0 \text{ otherwise} \end{cases} \tag{14.7}$$

for each $T \subseteq N$, $T \neq \emptyset$, is called the *unanimity game*.

The set of all unanimity games $\{u_T | T \subseteq N, \ T \neq \emptyset\}$ is a basis of G^N and each $v \in G^N$ can be represented by a linear combination of u_T as

$$v = \sum_{T \subseteq N, T \neq \emptyset} \Delta_T(v) u_T \tag{14.8}$$

The coefficients $\Delta_T(v)$ are called *Harsanyi dividends* of v and given by the Möbius formula

$$\Delta_T(v) = \sum_{S \subseteq T} (-1)^{|T|-|S|} v(S), \tag{14.9}$$

where $|T|$ denotes the cardinal number of T, i.e., the number of the elements in T. If we put $\Delta_\emptyset(v) = 0$ for convenience, the dividends can be obtained by the following recursive formula

$$\Delta_T(v) = \begin{cases} 0 & \text{if } T = \emptyset \\ v(T) - \sum_{S \subsetneq T} \Delta_S(v) & \text{if } T \neq \emptyset \end{cases} \tag{14.10}$$

14.2.2 Solutions of Cooperative Games

In the cooperative game theory the most important topic is to find an appropriate rule for allocating the worth of the grand coalition among the players. Such a rule is usually called a solution of the cooperative game. The allocated profit vector is denoted by $x = (x_1, x_2, \ldots, x_n)^\top$, where x_i is the profit of the ith player. It is quite natural that this vector satisfies the efficiency

$$\sum_{i \in N} x_i = v(N). \tag{14.11}$$

The set of these vectors are often referred to as the preimputation set

$$I^*(v) = \left\{ x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N) \right\}, \tag{14.12}$$

and, clearly, we have $I^*(v) \neq \emptyset$.

Roughly speaking, two types of approach have been taken in developing solutions of TU-games. One of them is based on the "objections" of the coalitions, and the other is based on the "contributions" of the players. Typical example of the former type is the core, while that of the latter is the Shapley value.

First a payoff vector which satisfies both efficiency and individually rationality defined by

$$x_i \geq v(i) \text{ for all } i \in N \tag{14.13}$$

is called an *imputation*, and the set of all imputations of the game $v \in G^N$ is denoted by

$$I(v) = \left\{ x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N),\ x_i \geq v(i)\ \forall i \in N \right\}. \tag{14.14}$$

Definition 14.3. The core of the game is a set-valued solution, which is defined by

$$C(v) = \left\{ x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N),\ \sum_{i \in S} x_i \geq v(S),\ \forall S \subseteq N \right\} \tag{14.15}$$

for a game $v \in G^N$.

If we define the excess of the coalition S with respect to x by

$$e(S;x) = v(S) - \sum_{i \in S} x_i, \tag{14.16}$$

the core can be rewritten as follows:

$$C(v) = \{ x \in \mathbb{R}^n \mid e(N;x) = 0,\ e(S;x) \leq 0\ \forall S \subseteq N \}. \tag{14.17}$$

It is clear that the core is a convex polyhedron, since it is represented by a linear equation and $2^n - 2$ linear inequalities. In general, unfortunately, the core of a game may be empty. Hence some extended solution concepts such as the ε-core and the least core were proposed. Moreover, the lexicographic minimization of the excesses leads to the solution concept "nucleolus."

Definition 14.4. A game which has a nonempty core is called *balanced*.

Actually balancedness is specified by introducing an optimization problem for checking the emptiness of the core and its dual problem.

Another interpretation of the core is given by dominance of imputations. Let $v \in G^N$, $x, y \in I(v)$ and $S \subseteq N$. We say that x dominates y through coalition S, and denote it by $x \operatorname{dom}_S y$ if

1. $x_i > y_i$ for all $i \in S$.
2. $\sum_{i \in S} x_i \leq v(S)$.

Moreover, we say that x dominates y, and denote it by $x \operatorname{dom} y$ if there exists $S \subseteq N$ such that $x \operatorname{dom}_S y$.

Definition 14.5. For a game $v \in G^N$, its *dominance core* is defined by

$$DC(v) = \{x \in I(v) \mid \nexists y \in I(v) : \ y \operatorname{dom} x\}. \tag{14.18}$$

Proposition 14.1. *For a game $v \in G^N$, the following relationship holds:*

$$C(v) \subseteq DC(v). \tag{14.19}$$

Moreover, if the game v is superadditive, the equality holds in the above relationship.

The most famous solution in the latter type is the Shapley value, which is a point-valued solution. Let π be a permutation of the player set N, where all the players are arranged in a line and Player i has the $\pi(i)$th position. Denote by $\Pi(N)$ the set of all permutation on N. Now let

$$P(\pi, i) = \{j \in N \mid \pi(j) < \pi(j)\}, \tag{14.20}$$

Then the marginal contribution of i in the order π is defined by

$$m_i^\pi(v) = v(P(\pi, i) \cup i) - v(P(\pi, i)). \tag{14.21}$$

The marginal vector is $m^\pi(v) = (m_1^\pi(v), m_2^\pi(v), \ldots, m_n^\pi(v))^\top \in \mathbb{R}^n$. Thus the Shapley value is defined as follows:

Definition 14.6. The *Shapley value* $\phi(v) \in \mathbb{R}^n$ for a game $v \in G^N$ is defined by

$$\phi(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} m^\pi(v). \tag{14.22}$$

Another equivalent definition of the Shapley value is

$$\phi_i(v) = \sum_{T \subseteq N, T \ni i} \frac{(|T|-1)!(n-|T|)!}{n!}(v(T) - v(T \setminus i)). \tag{14.23}$$

If we use the Harsanyi dividends, the Shapley value can be represented as

$$\phi(v) = \sum_{T \subseteq N, T \neq \emptyset} \frac{\Delta_T(v)}{|T|}. \tag{14.24}$$

The Shapley value is not generally contained in the core. It is known well that in convex games this is the case.

14.3 Core Solutions in Vector-Valued Games

14.3.1 Vector-Valued Cooperative Games

In this section we deal with vector-valued cooperative games which are straightforward extensions of ordinary cooperative games. In these games, the worth of a coalition is given by a (finite dimensional) vector rather than by a scalar. Fernández et al. [6] defined them and analyzed core solution concepts. Thus the results in this section are based on their article, though some notations are different.

As in Sect. 11.2, let $N = \{1, 2, \ldots, n\}$ be a set of players. Now a characteristic function v takes values in $\mathbb{R}^m$ not in $\mathbb{R}$. Here we assume that $v(\emptyset) = \mathbf{0} \in \mathbb{R}^m$. Since we fix N in this paper, we regard v as a game and denote by vG^N the family of all the vector-valued cooperative games.

If a vector-valued game is played under the grand coalition N, the grand worth $v(N) \in \mathbb{R}^m$ will be allocated among all the players. The payoff allocated to Player i is also a vector in $\mathbb{R}^m$, which is denoted by

$$x^i = (x_1^i, x_2^i, \ldots, x_m^i)^\top. \tag{14.25}$$

Since each payoff x^i is a vector in $\mathbb{R}^m$, the allocation in a vector-valued game is represented by an $m \times n$ matrix

$$X = (x^1 \; x^2 \; \cdots \; x^n) = \begin{pmatrix} x_1^1 & x_1^2 & \cdots & x_1^n \\ x_2^1 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \ddots & \vdots \\ x_m^1 & x_m^2 & \cdots & x_m^n \end{pmatrix}. \tag{14.26}$$

For $X \in \mathbb{R}^{m \times n}$, let

$$x^S = \sum_{i \in S} x^i \in \mathbb{R}^m.$$

This sum is the overall payoff obtained by coalition S.

Let $\mathbb{R}^m$ be the m dimensional real space and $\mathbb{R}^m_+$ the nonnegative orthant in $\mathbb{R}^m$, i.e.,

$$\mathbb{R}^m_+ = \{x = (x_1, \ldots, x_p)^\top \in \mathbb{R}^m \mid x_i \geq 0,\ i = 1, \ldots, p\}.$$

For two vectors $x = (x_1, x_2, \ldots, x_m)^\top$ and $y = (y_1, y_2, \ldots, y_m)^\top$ in $\mathbb{R}^m$ we define the following inequalities

$$\begin{array}{lcl} x \leq y & \Longleftrightarrow & x_i \leq y_i,\ \forall i = 1, 2, \ldots, m \\ x \preceq y & \Longleftrightarrow & x \leq y \text{ and } x \neq y \\ x < y & \Longleftrightarrow & x_i < y_i,\ \forall i = 1, 2, \ldots, m \end{array} \tag{14.27}$$

Now we consider scalarization by a weight vector in dealing with vector-valued games. Let

$$W \subseteq \left\{ w \in \mathbb{R}^m_+ \mid \sum_{i=1}^{m} w_i = 1 \right\} \tag{14.28}$$

be a set of weight vectors, which is assumed to be a closed polyhedron with a nonempty interior. For two vectors $x, y \in \mathbb{R}^m$, we say that:

- x is at least as preferred as y according to the information set W:

$$x \geq_W y \iff w^\top x \geq w^\top y,\ \forall w \in W$$

- x is not worse than y according to the information set W:

$$x \succeq_W y \iff \exists w \in W : w^\top x \geq w^\top y.$$

Definition 14.7. Let $v \in vG^N$ and $w \in W$. The *w-weighted game* is the scalar game $v_w \in G^N$ whose characteristic function is given by $v_w(S) = w^\top v(S)$ for each coalition $S \subseteq N$.

Let

$$I^*(v) = \left\{ X = (x^1\ x^2\ \cdots\ x^n) \in \mathbb{R}^{m \times n} \mid x^N = \sum_{i \in N} x^i = v(N) \right\} \tag{14.29}$$

be the set of efficient allocations.

14.3.2 Solution Concepts with Weak Ordering

In this subsection we introduce the core based on the weak ordering $\succeq_W$. An allocation is considered not to be acceptable if it does not satisfy the following individual rationality.

Definition 14.8. An allocation $X \in I^*(v)$ of the game $v \in vG^N$ is a generalized imputation or simply an imputation if

$$x^i \succeq_W v(i),\ \forall i \in N. \tag{14.30}$$

The set of all imputations of v is denoted by $I(v;\succeq_W)$.

The collective rationality is represented by the following dominance concept.

Definition 14.9. Let $X, Y \in \mathbb{R}^{m\times n}$ and let $S \subseteq N$ be a coalition. Y dominates X through S, according to $\preceq_W$, if

$$y^S \geq_W x^S \text{ and } y^S \preceq_W v(S). \tag{14.31}$$

This relation is denoted by $Y \ \mathrm{dom}_{\succeq_W,S}\ X$.

Definition 14.10. An imputation $X \in I(v;\succeq_W)$ of the vector-valued game $v \in vG^N$ is said to be *nondominated* if for any coalition $S \subseteq N$, there does not exist an imputation $Y \in I(v;\succeq_W)$ such that $Y \ \mathrm{dom}_{\succeq_W,S}\ X$. The set of nondominated imputations is denoted by

$$NDI(v;\succeq_W) = \{X \in I(v;\succeq_W) \mid \ \nexists S \subseteq N, Y \in I(v;\succeq_W):\ Y \ \mathrm{dom}_{\succeq_W,S} X\}. \tag{14.32}$$

A refinement of nondominated imputations is given as follows.

Definition 14.11. An imputation $X \in I(v;\succeq_W)$ of the vector-valued game $v \in vG^N$ is said to be *nondominated by allocations* if for any coalition $S \subseteq N$, there does not exist an imputation $Y \in I^*(v)$ such that $Y \ \mathrm{dom}_{\succeq_W,S}\ X$. The set of nondominated imputations is denoted by

$$NDIA(v;\succeq_W) = \{X \in I(v;\succeq_W) \mid \ \nexists S \subseteq N, Y \in I^*(v):\ Y \ \mathrm{dom}_{\succeq_W,S} X\}. \tag{14.33}$$

Since $I(v;\succeq_W) \subseteq I^*(v)$, it is clear that

$$NDIA(v;\succeq_W) \subseteq NDI(v;\succeq_W). \tag{14.34}$$

Although these two sets are different in general, they may coincide under mild conditions.

Now we provide the definition of the core.

Definition 14.12. The core of the vector-valued game $v \in vG^N$ is defined as the set of allocations such that x^S is not dominated by $v(S)$ for every coalition S and is denoted by

$$C(v;\succeq_W) = \{X \in I^*(v) \mid x^S \succeq_W v(S),\ \forall S \subseteq N\}. \tag{14.35}$$

The core can be characterized alternatively by the dominance concept as in the following theorem.

Theorem 14.1. *For the vector-valued game $v \in vG^N$, the following equality holds:*

$$C(v;\succeq_W) = \{X \in I(v;\succeq_W) \mid \nexists S \subseteq N, Y \in I^*(v): \ v(S) \geq_W y^S \succeq_W x^S\}. \quad (14.36)$$

Corollary 14.1. *For the vector-valued game $v \in vG^N$, the following equality holds:*

$$NDIA(v;\succeq_W) \subseteq C(v;\succeq_W). \quad (14.37)$$

A sufficient condition for the core to be nonempty is given as follows:

Theorem 14.2. *If the w-weighted game v_w for a vector-valued game $v \in vG^N$ is balanced for some $w \in \operatorname{int} W$, then $C(v;\succeq_W) \neq \emptyset$.*

14.3.3 Solution Concepts with Strong Ordering

In this subsection we explain the case where the ordering is the stronger one.

Definition 14.13. An allocation $X \in I^*(v)$ of the game $v \in vG^N$ is a *preference imputation* if

$$x^i \geq_W v(i), \ \forall i \in N. \quad (14.38)$$

The set of all preference imputations in v will be denoted by $I(v;\geq_W)$.

It is worth noting that

$$I(v;\geq_W) \subseteq I(v;\succeq_W). \quad (14.39)$$

Definition 14.14. Let $X,Y \in \mathbb{R}^{m\times n}$ and let $S \subseteq N$ be a coalition. Y dominates X individually through S, according to $\geq_W$, if

$$y^i \geq_W x^i \ \ \forall i \in S \text{ and } y^S \preceq_W v(S). \quad (14.40)$$

This relation is denoted by $Y \operatorname{domi}_{\geq_W,S} X$. Y dominates X through S according to $\geq_W$ if

$$y^S \geq_W x^S \text{ and } y^S \leq_W v(S). \quad (14.41)$$

This relation is denoted by $Y \operatorname{dom}_{\geq_W,S} X$.

Theorem 14.3. *Let X be an allocation of the game $v \in vG^N$. The following statements are equivalent:*

(a) $\exists Y \in I^*(v)$ *such that* $Y \operatorname{domi}_{\geq_W,S} X$.
(b) $\exists Y \in I^*(v)$ *such that* $Y \operatorname{dom}_{\geq_W,S} X$.
(c) $x^S \leq_W v(S)$.

Definition 14.15. A preference imputation $X \in I(v;\geq_W)$ of the vector-valued game $v \in vG^N$ is said to be *nondominated* if no coalition $S \subseteq N$ can find another imputation $Y \in I(v;\geq_W)$ such that Y $\text{dom}_{\geq_W,S}$ X. The set of preference imputations is denoted by

$$NDI(v;\geq_W) = \{X \in I(v;\geq_W) \mid \nexists S \subseteq N, Y \in I(v;\geq_W) : Y \text{ dom}_{\geq_W,S} X\}. \quad (14.42)$$

Let W_E be the $m \times p$ matrix whose columns are the extreme points $\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_p$ of W. The w_j-weighted game $v_{\overline{w}_j} \in G^N$ is called the W-component game of $v \in vG^N$.

Theorem 14.4. *If the rows of $W_E^\top X \in \mathbb{R}^{p\times n}$ are nondominated imputations of the scalar W-component games of $v \in vG^N$, then $X \in NDI(v;\geq_W)$.*

A refinement of nondominated preference imputations is given as follows.

Definition 14.16. A preference imputation $X \in I(v;\geq_W)$ of the vector-valued game $v \in vG^N$ is said to be *nondominated by allocations* if no coalition $S \subseteq N$ can find another allocation $Y \in I^*(v)$ such that Y $\text{dom}_{\geq_W,S}$ X. The set of nondominated preference imputations is denoted by

$$NDIA(v;\geq_W) = \{X \in I(v;\geq_W) \mid \nexists S \subseteq N, Y \in I^*(v) : Y \text{ dom}_{\geq_W,S} X\}. \quad (14.43)$$

Notice that

$$NDIA(v;\geq_W) \subseteq NDI(v;\geq_W). \quad (14.44)$$

Definition 14.17. The preference core of a vector-valued game $v \in vG^N$ is the set of allocations $X \in I^*(v)$ such that

$$x^S \geq_S v(S), \ \forall S \subseteq N. \quad (14.45)$$

We will denote this set as $C(v;\geq_W)$.

Theorem 14.5. *The following equality holds:*

$$C(v;\geq_W) = \{X \in I(v;\succeq_W) \mid \nexists S \subseteq N, Y \in I^*(v) : v(S) \succeq_W y^S \geq_W x^S\}. \quad (14.46)$$

A necessary and sufficient condition for the non-emptiness of the preference core is given as follows.

Theorem 14.6. *Let v be a vector-valued game. A necessary and sufficient condition for $C(v;\geq_W)$ being nonempty is that the p scalar weighted game $v_{\overline{w}_j}$, $j = 1,2,\ldots,p$ are balanced.*

Finally we provide relationships between two cores and the set of preference imputations nondominated by allocations.

Theorem 14.7. *The following relationship holds:*

$$C(v;\geq_W) \subseteq NDIA(v;\geq_W). \quad (14.47)$$

Theorem 14.8. *The following relationship holds:*

$$NDIA(v;\geq_W) = C(v;\succeq_W) \cap I(v;\geq_W). \tag{14.48}$$

14.4 Partially Ordered Cooperative Games

14.4.1 Partially Ordered Cooperative Games

In the former section we explained cooperative games whose characteristic function takes values in $\mathbb{R}^m$. In this section we deal with more general games whose characteristic function takes values in partially ordered linear space. The results in this section are based on Puerto et al. [17].

Let Q be a linear space over the real field $\mathbb{R}$. We assume that there exists a partial order $\geq$ which satisfies reflexivity and transitivity, but not necessarily antisymmetry, defined on Q. We represent by $>$ the corresponding strict partial order and by $\sim$ the indifference relation. We require to this partial order a denseness condition:

$$y \not\geq x \implies \exists z \in Q, x \not\leq z > y, \ \forall x \neq y \in Q. \tag{14.49}$$

A partially ordered cooperative game (N,v) is a pair of a set $N = \{1,2,\dots,n\}$ of players and a map $v : 2^N \to Q$ such that $v(\emptyset) = \mathbf{0}$, where $\mathbf{0}$ is the null vector in Q. Since N is fixed in this chapter, this game is simply denoted by v in this section and the set of all those games is denoted by pG^N. It is worth noting that ordinary cooperative games, vector-valued cooperative games (Fernández et al. [6], also the previous section) and stochastic cooperative games (Fernández et al. [7] and Suijs et al. [19]) are particular cases of this formulation, just considering $Q = \mathbb{R}$ with the standard ordering $\geq$, $Q = \mathbb{R}^m$ with the component-wise order, and $Q = L^1(\mathbb{R})$ with the stochastic dominance order, respectively.

Each allocation for partially ordered games is represented as

$$X = (x^1, x^2, \dots, x^n) \in Q^n, \tag{14.50}$$

where $x^i \in Q$ $(i = 1,2,\dots,n)$ stands for the payoff of Player i. The set of allocations which satisfy the efficiency principle is denoted by

$$I^*(v) = \left\{ X \in Q^n \mid x^N = \sum_{i \in N} x^i \sim v(N) \right\}. \tag{14.51}$$

14.4.2 Core Solutions

The set of all allocations that fulfils the property that x^i is not worse than the worth $v(i)$ is called *imputation set of the game* $v \in pG^N$ and is denoted by $I(v)$, i.e.,

$$I(v) = \{X \in I^*(v) \mid x^i \not\leq v(i),\ \forall i \in N\} \tag{14.52}$$

Imposing the collective rationality to imputations leads to the following definition of the core.

Definition 14.18. The core of the partially ordered game $v \in pG^N$ is defined as the set of allocations such that $x^S = \sum_{i \in S} x^i$ is at least as preferred as $v(S)$ for every coalition S and is denoted by

$$C(v;\geq) = \{X \in I^*(v) \mid x^S \geq v(S),\ \forall S \subseteq N\}. \tag{14.53}$$

In order to characterize the core we introduce the coalition dominance.

Definition 14.19. Let $v \in pG^N$, $X, Y \in I^*(v)$ and $S \subseteq N$. Y dominates X through S according to $\not\leq$ and we will denote Y dom $_{\not\leq,S} X$ if $y^S > x^S$ and $v(S) \not\leq y^S$.

Definition 14.20. An imputation $X \in I(v)$ of the game $v \in pG^N$ is nondominated by allocations if for any coalition $S \subseteq N$ it does not exist an allocation $Y \in I^*(v)$ such that Y dom $_{\not\leq,S} X$. This set is given by

$$NDIA(v;\not\leq) = \{X \in I(v) \mid\ \nexists S \subseteq N, Y \in I^*(v), Y \not\sim X : Y \text{ dom}_{\not\leq,S} X\}. \tag{14.54}$$

Theorem 14.9. *The following relationship holds:*

$$NDIA(v;\not\leq) = C(v;\geq). \tag{14.55}$$

Now we consider a sufficient condition for non-emptiness of the core. Let u be a function $u : Q \to \mathbb{R}$ satisfying

$$x \geq y \implies u(x) \geq u(y),\ \forall x, y \in Q. \tag{14.56}$$

Let

$$C(v_u) = \{X \in I^*(v) \mid u(v(S)) \leq u(x^S),\ \forall S \subseteq N\}. \tag{14.57}$$

Theorem 14.10. *The following relationship holds:*

$$C(v,\not\leq) \subseteq C(v_u). \tag{14.58}$$

Moreover, if we assume that the partial order $\geq$ is defined by a family U of functions in the sense that

$$x \geq y \iff u(x) \geq u(y),\ \forall u \in U, \tag{14.59}$$

then

$$C(v;\not\leq) = \bigcap_{u\in U} C(v_u). \tag{14.60}$$

This result is a generalization of that in vector-valued games.

Analogous to the concept for ordinary cooperative games, convexity of the partially ordered game $v \in pG^N$ can be defined in terms of the marginal contribution d_i of Player i, which is defined as

$$d_i(S) = \begin{cases} v(S\cup i) - v(S) \text{ if } i \in S, \\ v(S) - v(S\setminus i) \text{ otherwise.} \end{cases} \tag{14.61}$$

Definition 14.21. A partially ordered game $v \in pG^N$ is said to be $\geq$-*convex* if, for any $S \subseteq T \subseteq N$,

$$d_i(T) \geq d_i(S),\ \forall i \in N. \tag{14.62}$$

Theorem 14.11. *If a partially ordered game $v \in pG^N$ is $\geq$-convex, then $C(v;\geq) \neq \emptyset$.*

14.4.3 The Extended Shapley Value

Now in this subsection we consider the extended Shapley value for partially ordered games.

Definition 14.22. The extended Shapley value of a partially ordered game $v \in pG^N$ is defined as

$$\phi_i(v) \sim \sum_{T\subseteq N, T\ni i} \frac{(|T|-1)!(n-|T|)!}{n!}(v(T) - v(T\setminus i)),\ i = 1,2,\ldots,n. \tag{14.63}$$

Puerto et al. [17] characterized the extended Shapley value in terms of the extended potential. They also gave the axiomatic characterization of the extended Shapley value.

Axioms. Let φ be a value. A1. *Dummy player*.

$$\sum_{i\in S} \varphi_i(v) \sim v(S) \tag{14.64}$$

for any S such that $v(S) = v(S\cap T)$ for all $T \subseteq N$.

A2. *Symmetry*. For any permutation $\pi \in \Pi(N)$ and $i \in N$,

$$\varphi_{\pi(v)}(\pi v) \sim \varphi_i(v), \tag{14.65}$$

where the game πv means the game defined by $\pi v(\{\pi(i_1),\ldots,\pi(i_s)\}) = v(\{i_1,\ldots,i_s\})$.

A3. *Linear-Continuity*. Let $(u^k)_{k\in\mathbf{N}}$ be a sequence of partially ordered cooperative games. Then

$$\varphi_i\left(\sum_{k\in\mathbf{N}} u^k\right) = \sum_{k\in\mathbf{N}} \varphi_i(u^k). \tag{14.66}$$

Theorem 14.12. *The extended Shapley value ϕ is the unique value defined on all the partially ordered cooperative games satisfying Axioms A1, A2 and A3.*

14.5 Core Solutions in Set-Valued Games

14.5.1 Set-Valued Games and Core Concepts

More general forms of vector-values games have been investigated by several researchers. In those games the worth of each coalition is given by a set, not by a vector, in $\mathbb{R}^m$. In this section, we explain some results obtained by Fernández et al. [9] concerning the core concepts in set-valued games. A set-valued game is a pair (N,V) or simply V, where $N = \{1,2,\dots,n\}$ is the set of players and V is a set-valued map which assigns to each coalition $S \subseteq N$ a subset $V(S) \subseteq \mathbb{R}^m$, the characteristic set of coalitions S, such that $V(\emptyset) = \{\mathbf{0}\}$. If every $V(S)$ is a singleton, the set-valued game reduces to a vector-valued game and hence it is a generalization of a vector-valued game. The set of set-valued games on N is denoted by sG^N. Some examples of set-valued games are given in Fernández et al. [9] and we deal with multiobjective linear production games and multiobjective minimum cost spanning tree games later. Another approach by Tanino et al. [20] will be introduced in the next section, together with additional studies on restrictions on coalitions.

As before, for two vectors $x = (x_1, x_2, \dots, x_m)^\top$ and $y = (y_1, y_2, \dots, y_m)^\top$ in $\mathbb{R}^m$ we define the following inequalities

$$\begin{aligned} x \leq y &\iff x_i \leq y_i,\ \forall i = 1,2,\dots,m \\ x \preceq y &\iff x \leq y \text{ and } x \neq y \\ x < y &\iff x_i < y_i,\ \forall i = 1,2,\dots,m \end{aligned} \tag{14.67}$$

For $D \subseteq \mathbb{R}^m$ and the nonnegative orthant $\mathbb{R}^m_+$, let

$$D_- = D - \mathbb{R}^m_-,\ D_+ = D + \mathbb{R}^m_+. \tag{14.68}$$

Given a set-valued game $V \in sG^N$ and an $m \times n$ payoff matrix X (the meaning is the same as that in a vector-valued game), to simplify the presentation in the following, $x^S \npreceq V(S)$ means $x^S \npreceq y$, $\forall y \in V(S)$, that is there does not exist $y \in V(S)$ such that $x^S \preceq y$. Analogously $x^S \geq V(S)$ means $x^S \geq y$, $\forall y \in V(S)$.

Now a profit matrix X is said to be an allocation of the set-valued game $V \in sG^N$ if $x^N = \sum_{i\in N} x^i \in V(N)$. The set of the allocations of V is denoted by $I^*(V)$. Two types of cores can be defined as follows.

Definition 14.23. The *dominance core of a set-valued game* $V \in sG^N$ is the set

$$C(V;\not\preceq) = \{X \in I^*(V) \mid x^S \not\preceq V(S),\ \forall S \subseteq N\}. \tag{14.69}$$

Definition 14.24. The *preference core of a set-valued game* $V \in sG^N$ is the set

$$C(V;\geq) = \{X \in I^*(V) \mid x^S \geq V(S),\ \forall S \subseteq N\}. \tag{14.70}$$

It is obvious from the definitions that

$$C(V;\geq) \subseteq C(V;\not\preceq). \tag{14.71}$$

Definition 14.25. Let us consider two payoff matrices $X, Y \in \mathbb{R}^{m\times n}$ and a coalition $S \subseteq N$.

- Y dominates X through S according to $\not\preceq$, and we will denote $Y\ \mathrm{dom}_{S,\not\preceq} X$, if

$$y^S \not\preceq x^S,\ y^S \neq x^S,\ y^S \in [V(S)]_-. \tag{14.72}$$

- Y dominates X through S according to $\geq$, and we will denote $Y\ \mathrm{dom}_{S,\succeq} X$, if

$$y^S \geq x^S,\ y^S \neq x^S,\ y^S \in [V(S)]_-. \tag{14.73}$$

Definition 14.26. Two types of the set of non-dominated imputations are defined as follows:

- $NDA(V;\not\preceq) = \{X \in I^*(V) \mid\ \nexists S \subseteq N, Y \in I^*(V) : Y\ \mathrm{dom}_{S,\not\preceq} X\}$.
- $NDA(V;\geq) = \{X \in I^*(V) \mid\ \nexists S \subseteq N, Y \in I^*(V) : Y\ \mathrm{dom}_{S,\succeq} X\}$.

The following theorem shows that both cores are the sets of nondominated allocations.

Theorem 14.13. *The core sets hold the following properties:*

(a) $C(V;\geq) = NDA(V;\not\preceq)$
(b) $C(V;\not\preceq) = NDA(V;\geq)$

14.5.2 Existence Theorems

Now we give conditions that ensure non-emptiness of two types of cores. Let

$$W = \left\{ w \in \mathbb{R}^m \mid w_j > 0, j = 1,\ldots,m,\ \sum_{j=1}^{m} w_k = 1 \right\}. \tag{14.74}$$

For a set-valued game $V \in cG^N$, we define the scalar game $v_w \in G^N$ as

$$v_w(\emptyset) = \mathbf{0},\ v_w(S) = \max_{y \in [V(S)]_-} w^\top y. \tag{14.75}$$

The following theorem provides a sufficient condition for the non-emptiness of the dominance core.

Theorem 14.14. *The core $C(V; \not\leq)$ of the set-valued game $V \in sG^N$ is nonempty if and only if there exists $w \in W$ such that the scalar game $v_w \in G^N$ is balanced and it satisfies $v_w(N) \neq \mathbf{0}$.*

Associated with a coalition S in the set-valued game $V \in sG^N$, we consider m different scalar optimization problems:

$$(P_S(j)) \quad \begin{array}{l} \text{maximize } y_j \\ \text{subject to } \ y \in [V(S)]_-. \end{array} \tag{14.76}$$

Let us denote by $z^*(S,j)$ the optimal value of the above problem and by $z^*(S)$ the m-dimensional vector $z^*(S) = (z^*(S,1), z^*(S,2), \ldots, z^*(S,m))^\top$. Notice that for a fixed coalition S if an allocation X of the set-valued game $V \in sG^N$ satisfies $x^S \geq V(S)$ then $x^S \geq z^*(S)$ and conversely.

For each $z = (z_1, z_2, \ldots, z_m)^\top \in V(N)$, we introduce the scalar j-component game v_j^z defined as follows:

$$v_j^z(\emptyset) = 0,\ v_j^z(S) = z^*(S,j), \forall S \subseteq N,\ v_j^z(N) = z_j. \tag{14.77}$$

A necessary and sufficient condition for the non-emptiness of the preference core is given in the next theorem.

Theorem 14.15. *The preference core $C(V; \geq)$ of the set-valued game $V \in sG^N$ is nonempty if and only if there exists at least one $z \in V(N)$ such that all the scalar j-component games v_j^z are balanced.*

14.6 Multiobjective Games with Restrictions on Coalitions

In this section, we consider a multiobjective cooperative game with restrictions on coalitions. We define the restricted game of the original game and discuss its properties, namely inheritance of superadditivity and convexity. We also study the core of the restricted game. The results in this section are based on Tanino [21].

14.6.1 Maximum and Minimum of a Set in $\mathbb{R}^m$

Let $\mathbb{R}^m$ be the m dimensional real space and $\mathbb{R}^m_+$ the nonnegative orthant in $\mathbb{R}^m$, i.e.,

$$\mathbb{R}^m_+ = \{x = (x_1,\dots,x_p) \in \mathbb{R}^m \mid x_i \geq 0,\ i = 1,\dots,p\}.$$

We define the sets Y_+, Y_{++}, Y_-, and Y_{--} for a set $Y \subseteq \mathbb{R}^m$ as follows:

$$\begin{aligned} Y_+ &= Y + \mathbb{R}^m_+,\ \ Y_{++} = Y + (\mathbb{R}^m_+ \setminus \{\mathbf{0}\}) \\ Y_- &= Y - \mathbb{R}^m_+,\ \ Y_{--} = Y - (\mathbb{R}^m_+ \setminus \{\mathbf{0}\}), \end{aligned}$$

where $(0,\dots,0)^\top \in \mathbb{R}^m$ is also denoted by $\mathbf{0}$. In terms of these notations, we can define the minimum and maximum of a set in $\mathbb{R}^m$ as follows.

Definition 14.27. For a set $Y \subseteq \mathbb{R}^m$, the minimum and maximum of Y are defined by

$$\begin{aligned} \text{Min}\, Y &= \{y \in Y \mid (Y-y) \cap (-\mathbb{R}^m_+) = \{\mathbf{0}\}\} = Y \setminus Y_{++} \\ \text{Max}\, Y &= \{y \in Y \mid (Y-y) \cap \mathbb{R}^m_+ = \{\mathbf{0}\}\} = Y \setminus Y_{--}, \end{aligned}$$

respectively.

A particular type of sets in $\mathbb{R}^m$ satisfies the condition that the minimum or the maximum of a set coincides with the set itself.

Definition 14.28. A set $Y \subseteq \mathbb{R}^m$ is said to be *thin* (with respect to $\mathbb{R}^m_+$) if one of the following equivalent conditions is satisfied:

$$\begin{aligned} &1)\ Y = \text{Min}\, Y \\ &2)\ Y = \text{Max}\, Y \\ &3)\ Y_+ \setminus Y = Y_{++} \\ &4)\ Y_- \setminus Y = Y_{--} \end{aligned}$$

Remark 14.1. For any $Y \subseteq \mathbb{R}^m$, the sets Min Y and Max Y are obviously thin with respect to $\mathbb{R}^m_+$.

14.6.2 Multiobjective Cooperative Games

A multiobjective cooperative game is specified by a subset of $\mathbb{R}^m$ [9, 14, 20] and therefore it is a set-valued game. Thus a multiobjective cooperative game (MO-game for short) is a pair (N,V), where V is a set-valued mapping from 2^N to $\mathbb{R}^m$, i.e. $V(S) \subseteq \mathbb{R}^m$ for any $S \subseteq N$. We assume that $V(\emptyset) = \{\mathbf{0}\}$ and that $V(S)$ is nonempty, compact and thin for any $S \subseteq N$ throughout this section. Thus a multiobjective cooperative game is the same as a set-valued game except these assumptions.

The second condition implies that the multidimensional worth $V(S)$ of S is Pareto efficient in the MO-game. Namely there is no Pareto ordering between two points in $V(S)$. If y is contained in $V(S)_-$, then it should not be contained in $V(S)$.

In practical situations a number of important cooperative games arise from optimization problems (See Curiel [4] and Borm [3] for example). Those optimization problems are linear production programming problems, assignment problems, minimum cost spanning tree problems, and so on. They can be extended to multiobjective problems and therefore we can obtain multiobjective cooperative games arising from them. For example, Nishizaki and Sakawa discussed multiobjective linear production programming games in detail [13]. Since solving a multiobjective optimization problem leads to the Pareto efficient set in the objective space, which is regarded as the worth in a multiobjective cooperative game, it is quite natural that this set is thin.

Definition 14.29. An MO-game (N,V) is said to be *superadditive* if

$$V(S)+V(T) \subseteq V(S\cup T)_-, \text{ for all } S,T \subseteq N,\ S\cap T = \emptyset.$$

Remark 14.2. From the above definition, if an MO-game (N,V) is superadditive, then for any $S_k \subseteq N\ (k \in K)$ such that $S_k \cap S_{k'} = \emptyset$ for $k \neq k'$, $\sum_{k\in K} V(S_k) \subseteq V(\bigcup_{k\in K} S_k)_-$.

Definition 14.30. An MO-game (N,V) is said to be *convex* if

$$V(S)+V(T) \subseteq [V(S\cup T)+V(S\cap T)]_-, \text{ for all } S,T \subseteq N.$$

It is obvious that convexity is a stronger requirement than superadditiveity.

14.6.3 Restricted Multiobjective Cooperative Games by Partition Systems

In fundamental cooperative games and also in MO-games, it is assumed that an arbitrary subset S of N can form a coalition, i.e., every S is feasible or admissible. In practical situations, however, this assumption is not necessarily valid. Some coalitions may not be feasible because of physical or ideological reasons. Those situations are dealt with by introducing the concept of feasible coalition system [1]. A set system is a pair $(N,\mathscr{F})$, with $\mathscr{F} \subseteq 2^N$. The sets belonging to $\mathscr{F}$ are called feasible coalitions. For any $S \subseteq N$, maximal feasible subsets of S are called components of S. In many cases we impose appropriate combinatorial structures on $(N,\mathscr{F})$.

Definition 14.31 ([1]). A partition system is a set system satisfying

(i) $\emptyset \in \mathscr{F}$, and $\{i\} \in \mathscr{F}$ for every $i \in N$
(ii) for all $S \subseteq N$, the components of S, denoted by $\Pi_{\mathscr{F}}(S) = \{T_1,\dots T_l\}$ form a partition of S

Proposition 14.2 ([1]). *A set system $(N, \mathscr{F})$ which satisfies the first condition of the above definition is a partition system if and only if $S, T \in \mathscr{F}$ and $S \cap T \neq \emptyset$ imply $S \cup T \in \mathscr{F}$.*

A typical example of a partition system is the communication structure by Myerson [12] or Slikker and van den Nouweland [18].

Definition 14.32. Let (N, V) be an MO-game and let $(N, \mathscr{F})$ be a partition system. The $\mathscr{F}$-restricted game $(N, V^{\mathscr{F}})$, is defined by

$$V^{\mathscr{F}}(S) = \text{Max} \sum_{T \in \Pi_{\mathscr{F}}(S)} V(T),$$

where $\Pi_{\mathscr{F}}(S)$ is the collection of the components of $S \subseteq N$.

Remark 14.3. Since $V(T)$ is compact for any $T \subseteq N$, $V^{\mathscr{F}}(S)$ is also compact and thin. If $S \in \mathscr{F}$, then $\Pi_{\mathscr{F}}(S) = \{S\}$ and hence $V^{\mathscr{F}}(S) = V(S)$.

Lemma 14.1. *Let $(N, \mathscr{F})$ be a partition system, $S, T \subseteq N$ with $S \cap T = \emptyset$,*

$$\Pi_{\mathscr{F}}(S) = \{S_k\}_{k \in K},\ \Pi_{\mathscr{F}}(T) = \{T_l\}_{l \in L},\ \text{and}\ \Pi_{\mathscr{F}}(S \cup T) = \{U_m\}_{m \in M}.$$

Then $\{S_k\}_{k \in K} \cup \{T_l\}_{l \in L}$ is a subpartition of $\{U_m\}_{m \in M}$.

Due to this lemma we can prove the following theorem which shows the inheritance of superadditivity of the original game to the $\mathscr{F}$-restricted game.

Theorem 14.16. *Let (N, V) be a superadditive MO-game and $(N, \mathscr{F})$ be a partition system. Then the $\mathscr{F}$-restricted game $(N, V^{\mathscr{F}})$ is also superadditive.*

14.6.4 Inheritance of Convexity

In this section we consider a more special type of feasible coalition systems called intersecting systems, and prove the inheritance of convexity to the restricted games by intersecting systems.

Definition 14.33. A partition system $(N, \mathscr{F})$ is called an *intersecting system* if for all $S, T \in \mathscr{F}$ with $S \cap T \neq \emptyset$ we have $S \cap T \in \mathscr{F}$.

Remark 14.4. In Bilbao [1], a set system $(N, \mathscr{F})$ is called an intersecting family if for all $S, T \in \mathscr{F}$ with $S \cap T \neq \emptyset$ we have $S \cap T \in \mathscr{F}$ and $S \cup T \in \mathscr{F}$. Therefore an intersecting system is an intersecting family satisfying the first condition, $\emptyset \in \mathscr{F}$ and $\{i\} \in \mathscr{F}$, of the partition system.

Theorem 14.17. *Let (N, V) be a convex MO-game and $(N, \mathscr{F})$ be an intersecting system. Then the restricted game $(N, V^{\mathscr{F}})$ is also convex.*

14.6.5 The Core of Restricted Games

In a cooperative game, an allocation scheme of the profit among the players is regarded as a solution of the game. For an MO-game, this allocation is described by an $m \times n$ matrix $X = (x^1 \ \dots \ x^n)$, where each x^i $(i = 1, \dots, n)$ is an m dimensional vector representing a payoff vector received by player i.

The core is a fundamental solution concept not only in cooperative games, but also in MO-games [9,13,20]. It is characterized by two types of requirements: group rationality and coalition rationality.

Definition 14.34. The *core of an MO-game* (N,V) is defined by

$$C(V) = \left\{ x \in \mathbb{R}^{m \times n} \mid \sum_{i \in N} x^i \in V(N),\ \sum_{i \in S} x^i \in V(S)_+ \text{ for all } S \subseteq N \right\}. \tag{14.78}$$

Then we can obtain the following characterization of the core of the $\mathscr{F}$-restricted game for a partition system $(N, \mathscr{F})$.

Theorem 14.18. *Let (N,V) be an MO-game and let $(N,\mathscr{F})$ be a partition system such that $V(N) = V^{\mathscr{F}}(N)$, which is true when $N \in \mathscr{F}$. Then*

$$C(V^{\mathscr{F}}) \subseteq \left\{ x \in \mathbb{R}^{nm} \mid \sum_{i \in N} x^i \in V(N), \sum_{i \in S} x^i \in V(S)_+ \textit{ for all } \ S \in \mathscr{F} \right\}$$

Moreover, if $\sum_{T \in \Pi_{\mathscr{F}}(S)} V(T)$ is thin for any $S \subseteq N$, then the equality holds in the above relation, and therefore $C(V) \subseteq C(V^{\mathscr{F}})$.

14.7 Multiobjective Linear Production Games

14.7.1 Linear Production Games

This section is devoted to linear production (programming) games studied by Nishizaki and Sakawa [13, 14].

A linear production programming problem is a typical example of linear programming problems. A factory produces p types of products using r kinds of resources (materials). In order to produce one unit of the ith product, a_{ji} units of the jth resource is required. The total amount of the available jth resource is b_j $(j = 1, 2, \dots, r)$. Each unit of the ith product brings the profit c_i. Now the factory plans to produce x_i units of the ith product so that the total profit is maximized under the constraints of the available resources. This problem can be formulated as a linear programming problem

$$\begin{aligned} &\text{maximize } c^\top x \\ &\text{subject to } Ax \leq b \\ &\qquad\qquad\quad x \geq \mathbf{0} \end{aligned} \tag{14.79}$$

where $x = (x_1, x_2, \ldots, x_p)^\top$ is the production vector, $c = (c_1, c_2, \ldots, c_p)^\top$ is the profit coefficient vector, and $b = (b_1, b_2, \ldots, b_r)^\top$ is the resource amount vector. The $r \times p$ matrix $A = (a_{ij})$ is the resource-product matrix.

A cooperative game can be derived from this problem (Owen [15]). Now suppose that there are n factories and the resource-product matrix is common to all the players. Each factory k has the amount $b(k)$ of the resource. Therefore, if some set of the factories $S \subseteq N = \{1, 2, \ldots, n\}$ cooperates, then the total amount of the resource is

$$b(S) = \sum_{k \in S} b(k) \tag{14.80}$$

and this coalition get the profit

$$\begin{aligned} &\text{maximize } c^\top x \\ &\text{subject to } Ax \leq b(S) \\ &\qquad\qquad\quad x \geq \mathbf{0}. \end{aligned} \tag{14.81}$$

Thus we can define a cooperative game by regarding the optimal value of the above problem as the worth $v(S)$ of S. This game is called the linear production game. It has a nice property as in the following proposition.

Proposition 14.3. *A linear production game is totally balanced.*

14.7.2 Multiobjective Linear Production Games

Nishizaki and Sakawa extended linear production games to multiobjective linear production (programming) games [13] (See also Fernándes et al. [5]). In this case the profit is measured by m criteria $c_1^\top x,\ c_2^\top x,\ \ldots, c_m^\top x$. Thus we define a *multiobjective linear production programming problem*

$$\begin{aligned} &\text{maximize } Cx \\ &\text{subject to } Ax \leq b(S) \\ &\qquad\qquad\quad x \geq \mathbf{0} \end{aligned} \tag{14.82}$$

with the $m \times p$ matrix

$$C = \begin{pmatrix} c_{11} & c_{12} & \ldots & c_{1p} \\ c_{21} & c_{22} & \ldots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \ldots & c_{mp} \end{pmatrix}. \tag{14.83}$$

Let

$$T(S) = \{Cx \mid Ax \leq b(S),\, x \geq \mathbf{0}\} \tag{14.84}$$

and

$$V(S) = (\text{Max } T(S) - \mathbb{R}^m_+) \cap \mathbb{R}^m_+. \tag{14.85}$$

Nishizaki and Sakawa called it a multiobjective linear production (programming) game V.

They defined the set of payoff vectors satisfying the individual rationality as

$$IR(V) = \{X \in \mathbb{R}^{m \times n} \mid x^i \notin V(i) \setminus \text{Max } V(i),\, \forall i \in N\} \tag{14.86}$$

and the set of payoff vectors satisfying the collective rationality as

$$GR(V) = \{X \in \mathbb{R}^{m \times n} \mid x^N \in \text{Max } V(N)\}, \tag{14.87}$$

where $x^N = \sum_{i \in N} x^i$. The set of all imputations, which is a set of payoff vectors satisfying both the individual rationality and the collective rationality, is defined as

$$I(V) = \{X \in \mathbb{R}^{m \times n}_+ \mid X \in IR(V) \cap GR(V)\} \tag{14.88}$$

For $S \subseteq N$ and $X, Y \in I(V)$, we say that X dominates Y through S if $x^i > y^i$ for all $i \in S$ and $x^S \in V(S)$. Let X dom$_S$ Y denote that X dominates Y through S. We say that X dominates Y if there is any coalition $S \subseteq N$ such that X dom$_S$ Y. The dominance core of V is defined by

$$DC(V) = \{X \in I(V) \mid \nexists S \subseteq N, Y \in I(V) : Y \text{ dom}_S X\}. \tag{14.89}$$

The stable set is defined by

$$SO(V) = \{X \in GR(V) \mid x^S \notin V(S) \setminus \text{Max } V(S),\, \forall S \subseteq N\} \tag{14.90}$$

Now some properties of the multiobjective linear production game will be explained according to Nishizaki and Sakawa [13, 14].

Theorem 14.19. *The multiobjective linear production game V has superadditivity property*

$$V(S) + V(T) \subseteq V(S \cup T) \textit{ for } S, T \subseteq N, S \cap T = \emptyset. \tag{14.91}$$

It follows from this theorem that $DC(V) = SO(V)$ in the multiobjective linear production game.

The concept of balancedness was defined by van den Nouweland et al. in multi-commodity games. We apply this concept to the linear production game. A game is said to be balanced if, for each balanced map $\lambda : 2^N \to \mathbb{R}_+$ such that

$$\sum_{S \subseteq N, S \ni i} \lambda(S) = 1,\, \forall i \in N, \tag{14.92}$$

we have

$$\sum_{S \subseteq N} \lambda(S) V(S) \subseteq V(N). \tag{14.93}$$

Theorem 14.20. *The multiobjective linear production game V is balanced.*

14.7.3 Solutions of the Multiobjective Linear Production Games

In ordinary single-objective cooperative games, an excess function $e(S;x)$ plays a vital role in defining solution concepts such as the core, the least core and the nucleolus. It can be extended to multiobjective games. It is proper for the excess function $E(S;X)$ to satisfy the following conditions:

(1) If $X, Y \in \mathbb{R}^{m\times n}$ satisfy $\sum_{i\in S} x_k^i = \sum_{i\in S} y_k^i$ for every $k = 1,2,\ldots,m$, then $E(S,X) = E(S;Y)$.

(2) If $X, Y \in \mathbb{R}^{m\times n}$ satisfy $\sum_{i\in S} x_k^i < \sum_{i\in S} y_k^i$ for every $k = 1,2,\ldots,m$, then $E(S,X) > E(S;Y)$.

(3) $E(S;X)$ is jointly continuous with respect to X and $V(S)$.

In Nishizaki and Sakawa [13, 14] four kinds of excess function were proposed.

1. Distance between a payoff vector $x^S = \sum_{i\in S} x^i$ and the set Max $V(S)$

$$E_1(S;X) = \begin{cases} \min\limits_{y\in V(S)} \max\limits_{k=1,\ldots,m} (y_k - x_k^S) \text{ if } x^S \notin V(S) \\ \max\limits_{y\in V(S)} \min\limits_{k=1,\ldots,m} (x_k^S - v_k) \text{ if } x^S \in V(S) \end{cases} = \min_{y\in V(S)} \max_{k=1,\ldots,m} (y_k - x_k^S) \tag{14.94}$$

   and equivalently

$$E_1(S;X) = \max\{\varepsilon \in \mathbb{R} \mid x^S + \varepsilon e \in V(S)\}, \tag{14.95}$$

   where e is m-dimensional vector every component of which is a unit

2. Distance between a payoff vector x^S and an ideal point $\hat{y}^S = (\max\limits_{y\in V(S)} y_1, \ldots, \max\limits_{y\in V(S)} y_m)^\top$ of $V(S)$

$$E_2(S;X) = \min_{k=1,\ldots,m} (\hat{y}_k^S - x_k^S) \tag{14.96}$$

3. Distance between a payoff vector x^S and a hyperplane $h^S(z, \hat{y}^S) = 0$, which is constructed by assessing a reference point $\hat{y}^S$ in Max $V(S)$

$$E_3(S;X) = \max\{\varepsilon \in \mathbb{R} \mid x^S + \varepsilon e \in \overline{V}^S\}, \tag{14.97}$$

where $\overline{V}^S = \{z \in \mathbb{R}^m_+ \mid \sum_{k=1,\ldots,m} \hat{y}^S_k z_k \le \sum_{k=1,\ldots,m} (\hat{y}^S_k)^2\}$. Equivalently, it can be represented by

$$E_3(S;X) = \frac{\sum_{k=1}^{m} (\hat{y}^S_k)^2 - \sum_{k=1}^{m} x^S_k \hat{y}^S_k}{\sum_{k=1}^{m} \hat{y}^S_k} \tag{14.98}$$

4. Distance based on the augmented Tchebyshev metric between a payoff vector x^S and a reference point $\hat{y}^S$ in Max $V(S)$

$$E_4(S;X) = \min_{k=1,\ldots,m} (\hat{y}_k - x^S_k) + \alpha \sum_{k=1,\ldots,m} (\hat{y}_k - x^S_k) \tag{14.99}$$

Let $E(S;X)$ be an excess function of S with respect to X. The core can be defined by using the excess function as follows:

$$C(V) = \{X \in GR(V) \mid E(S;X) \le 0,\ \forall S \subseteq N\}. \tag{14.100}$$

Furthermore, for a given set of payoff matrices $\mathscr{X}$, we define the ε-core and the least core over the set $\mathscr{X}$ in a general way:

$$C_\varepsilon(V) = \{X \in \mathscr{X} \mid E(S;X) \le \varepsilon,\ \forall S \subseteq N\} \tag{14.101}$$

$$LC(V) = \{X \in \mathscr{X} \mid \max_{S\subseteq N} E(S;X) \le \max_{S\subseteq N} E(S;Y),\ \forall Y \in \mathscr{X}\}. \tag{14.102}$$

Nishizaki and Sakawa [13, 14] also defined the nucleolus as a set of payoff matrices minimizing the excess function in lexicographic order in a manner similar to the definition of the nucleolus in an ordinary cooperative game. For a given payoff matrix X, we define the 2^n-dimensional vector $\theta(X)$ as the vector whose components are the excess $E(S;X)$ of 2^n subset $S \subseteq N$ arranged in decreasing order. Then, for a given set of payoff matrices $\mathscr{X}$ such as $\mathscr{X} = IR(V) \cap GR(V)$ or $\mathscr{X} = GR(V)$, the nucleolus of multiobjective linear production game over the set $\mathscr{X}$ can be defined as

$$N(V,\mathscr{X}) = \{X \in \mathscr{X} \mid \theta(X) \le_L \theta(Y),\ \forall Y \in \mathscr{X}\}, \tag{14.103}$$

where $\le_L$ means "smaller than or equal to" in lexicographic order. It can be proved that $N(V,\mathscr{X})$ is not empty if $E(S;X)$ is continuous jointly in X and V, and if $\mathscr{X}$ is compact.

In Nishizaki and Sakawa [13, 14], these solution concepts are studied more in detail for each excess function $E_i(S;X)$ $(i = 1,2,3,4)$.

14.8 Multiobjective Minimum Cost Spanning Tree Games

14.8.1 Minimum Cost Spanning Tree Games

In this section we focus on minimum cost spanning tree games developed by Fernández et al. [8].

Minimum cost spanning tree problems are fundamental optimization problems. A *minimum cost spanning tree situation (mcsts)* is specified by (N_0, C). Here $N = \{1, 2, \ldots, n\}$ is a set of agents who are willing to connected as cheap as possible to a source (supplier of a service) denoted by 0 and $N_0 = \{0\} \cup N$. On the other hand, $C = (c_{ij})_{i,j \in N_0}$ is an $(n+1) \times (n+1)$ cost matrix, in which each element $c_{ij} \geq 0$, $c_{ii} = 0$ is the cost of direct link between i and j for $i, j \in N_0$. We should note that each subset $S \subseteq N$ induces the mcsts (S_0, C).

A *network* T over N_0 is a subset of $\{(i, j) \mid i, j \in N_0, i \neq j\}$. The elements of T are called *arcs*. Each arc (i, j) is undirected, i.e., $(i, j) = (j, i)$ and $c_{ij} = c_{ji}$. The network induced by T over S is given by $T_S = \{(i, j) \mid i, j \in S\}$.

A *spanning tree* is a network such that there is a unique path from i to 0 for all $i \in N$. A tree is represented as $T = \{(i^0, i)\}$, where i^0 is the first agent (or source) in the unique path in T from i to 0.

The *cost* of a network T is given by $c(N_0, C, T) = \sum_{(i,j) \in T} c_{ij} = c(T)$. A *minimum cost spanning tree (mt)* for (N_0, C) is a tree T such that

$$c(T) = \min\{c(T') \mid T' \text{ is a spanning tree}\}. \tag{14.104}$$

The cost associated with any mt T in (N_0, C) is denoted by $m(N_0, C)$. The minimum cost of an induced mcstp (S_0, C) is denoted by $m(S_0, C)$. This is the minimal cost of connecting all agents from S to the source 0, using only connections between elements of $S \cup 0$. Bird [2] introduced the TU-game (N, v_C) for each mcsts (N_0, C) by

$$v_C(S) = m(S_0, C), \;\; S \subseteq N. \tag{14.105}$$

On the other hand, as a milder version of the above case, the minimal cost $\tilde{v}_C(S)$ is defined as the minimal cost of connecting all agents from S to the source 0 using a tree that may include some nodes inhabited by the agents outside S. It is obvious that

$$\tilde{v}_C(S) \leq v_C(S). \tag{14.106}$$

14.8.2 Multiobjective Minimum Cost Spanning Tree Games

Fernández et al. [8] dealt with a multiobjective minimum cost spanning tree game (mmcst-game, for short) in which the set of agents is $N = \{1, 2, \ldots, n\}$ and the source is 0. The cost of each edge (i,j) is not a scalar, but a vector $c_{ij} = (c^1_{ij}, c^2_{ij}, \ldots, c^m_{ij})^\top$. Thus we can define the set-valued game V as follows:

1. $V(\emptyset) = \{\mathbf{0}\}$
2. For each nonempty coalition $S \subseteq N$,

$$V(S) = \mathrm{Min}\left\{ \sum_{(i,j)\in T} c_{ij} \mid T \text{ is a spanning tree for } S_0 \right\} \tag{14.107}$$

where the spanning tree for S_0 must contain S_0 but it may also contain some additional nodes. We should note that this game is not a ordinary profit game, but a cost game.

In this section we explain the results by Fernández et al. [8]. As before, an allocation consists of a matrix $X = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^{m\times n}$ whose ith column represents the payoffs of player i for each criteria. The sum $x^S = \sum_{i\in S} x^i$ is the overall cost allocated to the coalition S. The matrix X is an allocation of the game V if

$$x^N = \sum_{i\in N} x^i \in V(N). \tag{14.108}$$

The set of all allocations of the game V is denoted by $I^*(V)$.

14.8.3 Core Concepts in Multiobjective Minimum Cost Spanning Tree Games

It is reasonable to think that coalitions only accept allocations if they pay less than any of the worths given by the characteristic set. We again denote by $x^S \leq V(S)$ that $x^S \leq y$ for all $y \in V(S)$. Since the game is a cost game we should rewritten the core concepts.

Definition 14.35. The *preference core* of an mmcst-game V is the set

$$C(V;\leq) = \{X \in I^*(V) \mid x^S \leq V(S),\ \forall S \subseteq N\}. \tag{14.109}$$

We also define the following m scalar games.

Definition 14.36. The *scalar l-component minimum cost spanning tree game* $(j = 1, 2, \ldots, m)$ associated to $z \in \mathbb{R}^m$ is a game $v^z_j \in G^N$ defined by

1. $v^z_l(\emptyset) = \mathbf{0}$.
2. For each nonempty coalition $S \subseteq N$,

$$v^z_l(S) = \min\{ \sum_{(i,j)\in T} c^l_{ij} \mid T \text{ is a spanning tree for } S_0\}. \tag{14.110}$$

3. $v^z_l(N) = z_l$.

For each nonempty coalition $S \subseteq N$, $v^z(S)$ can be obtained by solving the problem

$$\begin{aligned} &\text{minimize } y_l \\ &\text{subject to } y \in V(S). \end{aligned} \tag{14.111}$$

Notice that for a fixed coalition S, if an allocation X of the mmcst-game verifies $x^S \leq V(S)$ then $x^S \leq z^*(S)$, where $z^*(S) = (v_1^z(S), v_2^z(S), \ldots, v_m^z(S))^\top$ denote the m-dimensional vector whose components are, respectively, are the solutions of the above problems. Conversely, if $x^S \leq z^*(S)$ then $x^S \leq V(S)$.

A necessary and sufficient condition for the non-emptiness of the preference core is given in the next theorem.

Theorem 14.21. *The preference core of the mmcst-game is nonempty if and only there exists at least one $z \in V(N)$ such that all the scalar l-component games v_l^z are balanced.*

In scalar mcst-game there exists a simple rule called Bird rule (Bird [2]) to allocate costs among the players. This rule can be extended to the mmcst-game by allocating to each player the cost vector of the edge incident upon it on the unique path between 0 and the player's node, in the corresponding Pareto-minimum cost spanning tree. Unfortunately, however, extended Bird's cost allocation scheme is not, in general, a way to obtain allocations in the preference core.

Now suppose that each coalition S will not accept to pay a total cost greater than any of the guaranteed costs in $V(S)$. This will be denoted by $x^S \not\geq V(S)$ and means that there does not exist $y \in V(S)$ such that $x^S \geq y$ and $x^S \neq y$.

Definition 14.37. The *dominance core of the mmcst-game V* is the set

$$C(V; \not\geq) = \{X \in I^*(V) \mid x^S \not\geq V(S),\ \forall S \subseteq N\}. \tag{14.112}$$

Bird's cost allocation scheme always leads to an element in the scalar core. In the following result shows that any vectorial Bird's cost allocation belongs to the dominance core.

Theorem 14.22. *Let T be a Pareto-minimum cost spanning tree in the multiobjective minimum cost spanning tree problem. Then the corresponding vectorial Bird's cost allocation is in the dominance core.*

Apart from Bird's cost allocations, there are many other allocations in the dominance core. In order to find a condition that permits to divide among the players a total cost $y \in V(N)$ accordingly with a given strictly increasing linear utility function u, we will define the following scalar game $v_u \in G^N$

$$v_u(\emptyset) = 0,\ v_u(S) = \min_{y \in V(S)} u(y),\ \forall S \subseteq N, S \neq \emptyset. \tag{14.113}$$

Using Bird's rule in the scalar game v_u, we can construct dominance core allocations for some $y \in V(N)$.

let $b = (b^1, b^2, \ldots, b^n)^\top$ be the Bird's allocation of the game v_u. This vector allows us to give a proportional allocation of $y \in V(N)$ in the dominance core.

Theorem 14.23. *If $v_u(N) = u(y)$ for $y \in V(N)$, then the proportional allocation $X = (x^1\ x^2\ \ldots\ x^n)$ defined by*

$$x^i = \frac{b^i}{u(y)},\ \forall i \in N \tag{14.114}$$

belongs to the dominance core $C(V; \not\leq)$ of the mmcst-game V.

14.9 Conclusion

In this chapter we have surveyed several aspects of relationships between vector optimization and cooperative games, mainly focusing on vector-valued or set-valued cooperative games. In those games, the core is a main solution concept and several studied have been made. We have also discussed two important classes of multiobjective cooperative games, multiobjective linear production games and multiobjective minimum cost spanning tree games.

Since multiobjective cooperative games are not so easy to deal with, there remain several important problems which should be solved. As was explained in Sect. 14.2, the Shapley value is also a very important solution concept in cooperative games. However, this concept has not been fully studied in multiobjective cooperative games (even in vector-valued games). In multi-criteria simple games, the Shapley–Shubik index has been studied recently, for example, Monroy and Fernández [10, 11], and therefore some new development concerning the Shapley value will be expected in a few years. The concepts of the unanimity games and the dividends have not been discussed either. Thus we should expect more and more effort in the future in this field.

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