Electro-Elastic Continuum Models for Electrostrictive Elastomers

R. Vertechy, G. Berselli, V. Parenti Castelli and M. Bergamasco

Abstract A continuum finite-deformation model is described for the study of the isothermal electro-elastic deformations of electrostrictive elastomers. The model comprises general balance equations of motion, electrostatics and electro-mechanical energy, along with phenomenological invariant-based constitutive relations. The model is presented in both Eulerian (spatial) and Lagrangian (material) description. Specialization of the considered model is also presented for "Dielectric Elastomers", which are a specific class of electrostrictive elastomers having dielectric properties independent of deformation.

1 Introduction

Electrostrictive elastomers are a special kind of "electroactive polymeric material" whose electrical and structural behavior is highly non-linear and strongly coupled. In essence, they are rubber-like dielectric solids that experience large finite deformations in response to applied electric fields while, at the same time, alter the existing electrostatic fields in response to the deformations undergone.

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Practical electrostrictive elastomers are polyurethanes [1–3], ferroelectric polymers [4, 5], graft elastomers [6], silicone and polyacrylate elastomers [7–11], styrene-butadiene and styrene-isoprene-styrene elastomers [11], liquid-crystal elastomers [12], interpenetrating-polymer-network elastomers and nano-structured rubber [13–15], as well as a number of other elastomeric composites employing either conducting, semiconducting or ferroelectric particles as fillers [11, 16–20]. Typical stresses and strains that can be induced in electrostrictive elastomers via electrical activation are [6, 7, 21]: 2 MPa and 11 % for polyurethanes; 54 Mpa and 7 % for ferroelectric elastomers; 4 MPa and 4 % for graft elastomers; 3 MPa and 120 % for silicone elastomers; 7 MPa and 380 % for polyacrylate elastomers; 1 MPa and 4 % for liquid-crystal elastomers. Typical rates of such electrically-induced strains are larger than 1,000 %/s for almost all electrostrictive elastomers [21].

Thanks to the peculiar electromechanical coupling, along with the intrinsic compliance, lightness, malleability, easy-manufacturability and low-cost, electrostrictive elastomers are perfectly suited for the development of novel solid-state mechatronic transducers which are more resilient, lightweight, integrated, economic and disposable than traditional devices obtainable via conventional material technologies. Practical transduction devices that can be developed by using electrostrictive elastomers are [22–27]: compliant muscle-like actuators; compact and portable Braille displays; distributed force/displacement sensors; solid-state electrical energy harvester/generators; large solid-state loud-speakers. Potential applications are in the field of machine interfaces for human assistance and entertainment, safe and low-cost robotics, as well as disposable and consumable mechatronics.

To give some example, a linear actuator prototype based on an electrostrictive elastomer is shown in Fig. 1. As depicted in Fig. 2, the device is a lozenge-shaped planar actuator featuring two electroactive sheets that are connected to the opposing sides of a four-bar mechanism having identical rigid links and tension-tape hinges. Each electroactive sheet is made of three compliant electrodes made of a carbon conductive grease, which are separated by two acrylic elastomer films

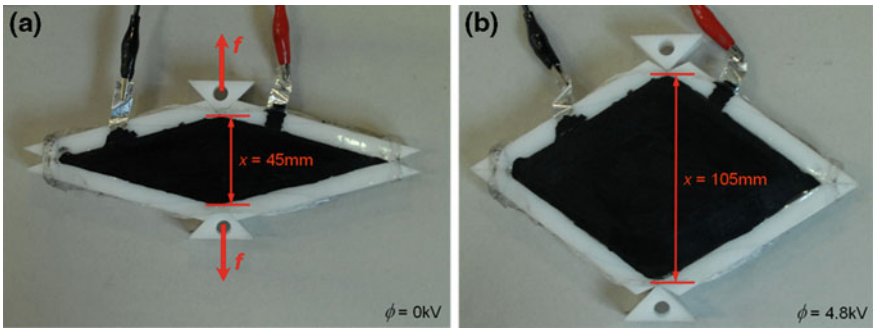

Fig. 1 Lozenge actuator prototype: **a** actuator inactive and **b** actuator active

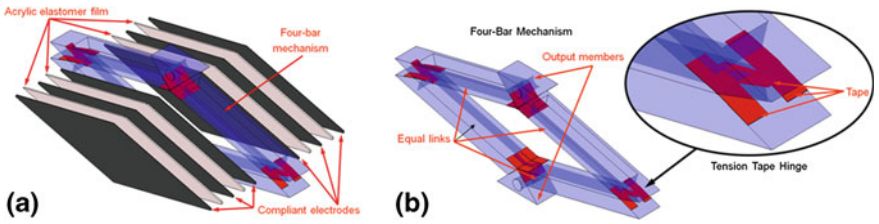

Fig. 2 Lozenge actuator CAD drawings: **a** actuator exploded view and **b** four-bar mechanism

that are bi-axially pre-stretched and connected along their boundary to the links of the four-bar mechanism. The placement of an electric potential difference (hereafter also called voltage) between the inner and the outer electrodes of the stack induces the acrylic elastomer films to shrink in thickness and to expand in area (see Fig. 1). In this device, the four-bar mechanism constrains the area expansion of the acrylic elastomer films to be uniform and enables the transmission to the film boundary of the useful mechanical work which can be performed by any external force acting on the mechanism links. Further details concerning the design and the performance of this linear actuator can be found in [28].

Typical force-length characteristic curves of this lozenge actuator are shown in Fig. 3 for different activation voltages ϕ, where the length x is the distance between the axes of the two joints of the four-bar mechanism that lie on one of the lozenge diagonals and the force f is the external equilibrating force acting on the same joint axes with direction equaling that of the same lozenge diagonal (see Fig. 1a). As shown in Fig. 3, the electrically-induced deformations occurring in electrostrictive elastomers can be quite large and rather non-linear. Owing to this complex multi-physical behavior, the design and the control of practical devices based on electrostrictive elastomers are extremely challenging problems. Intuitive approaches are usually defied, which calls for the availability of model-based engineering simulation tools.

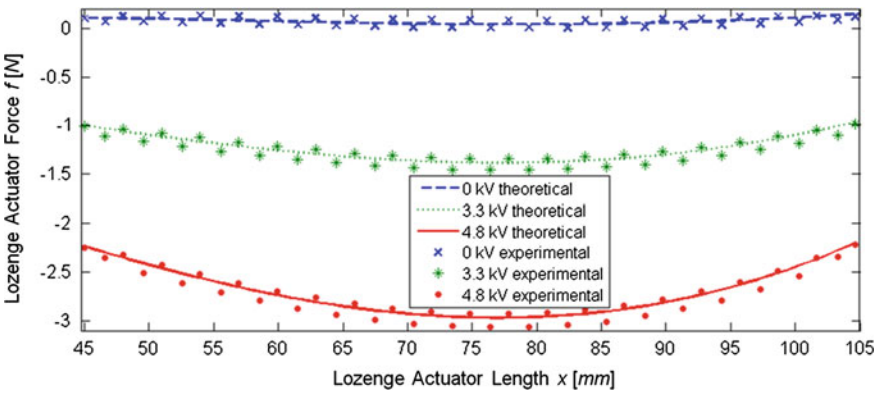

Fig. 3 Force-length curves of the lozenge actuator

The characterization of the electromechanical interaction between a dielectric body and an electric field has been a topic of intensive research since the end of the nineteen century [29–67]. Based on diverse interaction models, postulates, and solution methods, a number of continuum electro-mechanical models with dissimilar expressions for the electrically-induced force and stress-tensor have been proposed by different authors [29–67], which have been a subject of controversy for over a century [36, 50, 52, 53, 59, 64, 66, 68–71]. In particular: some authors directly postulated a particular expression for the electrically-induced body force and/or stress tensor, and simply added them to the continuum balance law of linear mechanical momentum (for instance in [34, 38, 40, 41]); other authors postulated elementary models, such as charges and dipoles, which characterize the macroscopic behavior of the polarized material (for instance in [40, 45, 49, 51]), thus deriving an expression of the electrically-induced body force and/or stress tensor from them; other authors postulated that the interaction at the microscopic scale is characterized by electrons moving in ether, thus obtaining the electrically-induced force and/or stress tensor by a statistical averaging process (for instance in [33, 47, 54]); other authors postulated a particular form of the global energy balance from which the coupled balance laws of mass, momenta (directly including the electrically-induced forces and/or stress tensor) and electricity are obtained by imposing certain invariance properties (for instance in [45, 52, 53, 55]); other authors postulated a given form of the electric enthalpy from which the coupled balance laws of linear momentum (directly including the electrically-induced forces and/or stress tensor) and electricity are obtained via some variational principle (for instance [38, 56, 63, 66, 67]).

In this context, this chapter presents a finite-deformation electro-elastic continuum model for the study of the isothermal electrically-induced deformations of electrostrictive elastomers that are conservative and isotropic. The model relies on the standard balance laws of mass, momenta (which do not explicitly include any electrically-induced force and stress tensor term) and electrostatics, and accounts for the electromechanical coupling via an appropriately chosen phenomenological constitutive relation which is consistent with the principles of thermodynamics and based on the theory of invariants. Specialization of the constitutive theory is also presented for the so called “Dielectric Elastomers” [25], which are a special kind of electrostrictive elastomers whose dielectric properties are deformation independent. The model is formulated both in the Eulerian (spatial) description, which enables for a more immediate comprehension of the electro-elastic phenomena that occur in electrostrictive elastomers, and in the Lagrangian (material) description, which is more suited for the numerical simulation of virtual prototypes of electrostrictive-elastomer-based devices via engineering tools alike finite element analysis software. For further reference, the finite element implementation of the considered model is discussed in [72], whereas its extension to the study of the thermo-electro-elastic deformations of electrostrictive elastomers is presented in [73].

2 Kinematics of Continuous Media

Consider the closed and electrically isolated system $\mathcal{B}$ depicted in Fig. 4, which comprises dielectric and conducting bodies as well as free space. Both dielectric and conducting bodies can move and deform under the action of externally applied loads of both mechanical and electrical origin. During such body motions/deformations, the physical region occupied by $\mathcal{B}$ does not remain fixed, but it moves/deforms accordingly. However, while these body motions/deformations occur: (1) no mass can enter or leave the boundary of $\mathcal{B}$; (2) energy (in the form of work or heat) can cross the boundary of $\mathcal{B}$; (3) no interaction occurs between the electrical charges that lie within $\mathcal{B}$ and those outside [i.e. the boundary of $\mathcal{B}$ either is electrically shielded from its exterior or has an infinite extent].

In this perspective, define with $\Im_l$ the known reference (laboratory) frame with respect to which the motions/deformations of $\mathcal{B}$ are measured. For any arbitrary time instant $t \geq 0$, identify the moving/deforming region occupied by an arbitrary inner subsystem of $\mathcal{B}$ with the current (deformed) volume $\Omega(t)$ and its boundary surface $\partial\Omega(t)$; $\bar{\Omega} = \Omega(t = \bar{t} \equiv 0)$ and $\partial\bar{\Omega} = \partial\Omega(t = \bar{t} \equiv 0)$ being the referential (undeformed) volume and boundary surface respectively.

Consider now a general material point P (i.e. a particle) belonging to the arbitrary subsystem of $\mathcal{B}$, and indicate with X and $x(t)$ the position vectors expressing the location (relative to the origin O of $\Im_l$) occupied by P when the system $\mathcal{B}$ is in its undeformed (i.e. $t = \bar{t} \equiv 0$) and deformed configuration respectively. Then, for any $\boldsymbol{X} \in \bar{\Omega}$ and any $t \geq 0$, the motion of $\mathcal{B}$ can be described by

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, \boldsymbol{t}), \tag{1}$$

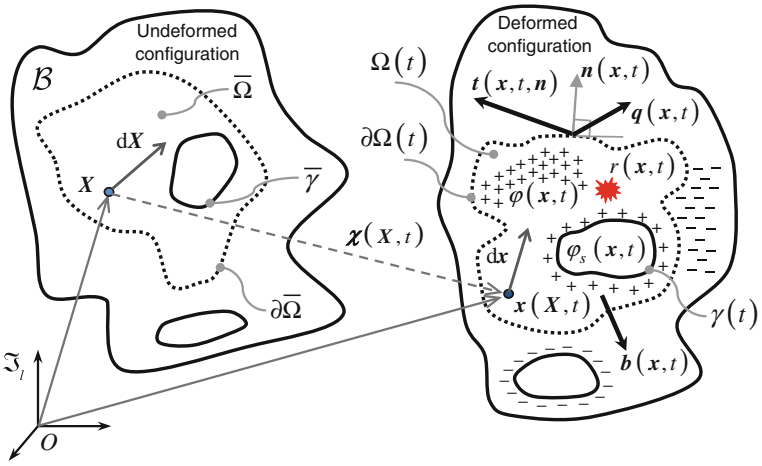

Fig. 4 Electrostrictive elastomer system: undeformed and deformed configuration

where the map $\boldsymbol{\chi}(\boldsymbol{X}, t)$ is a suitably regular and single-valued vector field (i.e. it is invertible and possesses continuous time derivatives with respect to both position and time) that carries all points P from places $\boldsymbol{X}$ (within the referential configuration $\bar{\Omega}$) to places $\boldsymbol{x}$ (in the current configuration $\Omega(t)$ at time t). By definition, the map $\boldsymbol{\chi}$ needs to satisfy the two conditions $\boldsymbol{X} = \boldsymbol{\chi}(\boldsymbol{X}, 0)$ and $\boldsymbol{X} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}, t)$.

Time differentiation of the map described by Eq. (1) provides the velocity field of the arbitrary subsystem of $\mathcal{B}$ expressed in either the Lagrangian form, $\boldsymbol{V}(\boldsymbol{X}, t)$ (i.e. the material description with respect to the coordinate $\boldsymbol{X}$ in the referential configuration), or the Eulerian form, $\boldsymbol{v}(\boldsymbol{x}, t)$ (i.e. the spatial description with respect to the coordinate $\boldsymbol{x}$ in the current configuration),

$$\boldsymbol{V}(\boldsymbol{X}, t) = \frac{\partial \boldsymbol{\chi}(\boldsymbol{X}, t)}{\partial t}, \tag{2}$$

$$\boldsymbol{v}(\boldsymbol{x}, t) \;=\; \boldsymbol{V}\big(\boldsymbol{\chi}^{-1}(\boldsymbol{x}, t), t\big). \tag{3}$$

The material gradient (i.e. the derivative with respect to the referential coordinate $\boldsymbol{X}$) and the spatial gradient (i.e. the derivative with respect to the spatial coordinate $\boldsymbol{x}$) respectively yield the deformation gradient, $\mathbf{F}(\boldsymbol{X}, t)$, and its inverse, $\mathbf{F}^{-1}(\boldsymbol{x}, t)$,

$$\mathbf{F} = \mathrm{Grad}\boldsymbol{x} = \frac{\partial \boldsymbol{\chi}(\boldsymbol{X}, t)}{\partial \boldsymbol{X}}, \tag{4}$$

$$\mathbf{F}^{-1} = \mathrm{grad}\boldsymbol{X} = \frac{\partial \boldsymbol{\chi}^{-1}(\boldsymbol{x}, t)}{\partial \boldsymbol{x}}, \tag{5}$$

which essentially provide the map

$$\mathrm{d}\boldsymbol{x} = \mathbf{F}\mathrm{d}\boldsymbol{X}, \tag{6}$$

between the undeformed line element, $\mathrm{d}\boldsymbol{X}$, and the deformed line element $\mathrm{d}\boldsymbol{x}$. Equation (6) describes the change in length and orientation (i.e. a rigid body rotation) of a physical line element when system $\mathcal{B}$ passes from the undeformed to the deformed configuration. Whenever the rigid body rotation is not of interest, one may conveniently resort to the quadratic forms

$$|\mathrm{d}\boldsymbol{x}|^2 = \mathrm{d}\boldsymbol{X} \cdot \mathbf{C}\mathrm{d}\boldsymbol{X}, \tag{7}$$

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}, \tag{8}$$

$$|\mathrm{d}\boldsymbol{X}|^2 = \mathrm{d}\boldsymbol{x} \cdot \mathbf{B}^{-1}\mathrm{d}\boldsymbol{x}, \tag{9}$$

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T, \tag{10}$$

where $\mathbf{C}$ and $\mathbf{B}$ are symmetric tensors usually known as the Green deformation tensor and the Finger deformation tensor respectively.

The determinant of the deformation gradient gives the Jacobian, $J(\boldsymbol{X}, t)$,

$$J = \det \mathbf{F} > 0, \tag{11}$$

which is also known as the volume ratio because of the relationship

$$\mathrm{dv} = J\mathrm{dV}, \tag{12}$$

between the infinitesimal undeformed volume element, dV, and the infinitesimal deformed volume element, dv.

By considering

$$\mathrm{dV} = \mathrm{d}\boldsymbol{X} \cdot \boldsymbol{N}\mathrm{dS}, \tag{13}$$

$$\mathrm{dv} = \mathrm{d}\boldsymbol{x} \cdot \boldsymbol{n}\mathrm{ds}, \tag{14}$$

where dS and ds are the infinitesimal undeformed and deformed surface elements, with $\boldsymbol{N}$ and $\boldsymbol{n}$ being the respective unit vector normals, use of Eqs. (6) and (12) gives the Nanson's formula

$$\boldsymbol{n}\mathrm{ds} = J\mathbf{F}^{-T}\boldsymbol{N}\mathrm{dS}. \tag{15}$$

Regarding the differential kinematics, time-differentiation of Eq. (4) and use of the Schwarz's theorem together with the definition given in Eq. (2) provides

$$\mathrm{Grad}\boldsymbol{V} = \dot{\mathbf{F}}, \tag{16}$$

which, because of the chain rule differentiation, makes it possible to write

$$\mathrm{grad}\boldsymbol{v} = \mathrm{Grad}\boldsymbol{V}\mathbf{F}^{-1} = \dot{\mathbf{F}}\mathbf{F}^{-1}. \tag{17}$$

Moreover, the derivative of the Jacobian with respect to the deformation gradient yields

$$\frac{\partial J}{\partial \mathbf{F}} = J\mathbf{F}^{-T}, \tag{18}$$

which, by the chain rule differentiation, makes it possible to write

$$\dot{J} = J\mathbf{F}^{-T} : \dot{\mathbf{F}} = J\mathbf{1} : \mathrm{grad}\boldsymbol{v} = J\mathrm{div}\boldsymbol{v}, \tag{19}$$

where $\mathbf{1}$ is the 3×3 identity matrix, while the operator $(:)$ indicates the double contraction of two tensors [as defined in Eq. (185)].

Finally, by time-differentiating the identity $\mathbf{F}^{-1}\mathbf{F} = \mathbf{1}$, the derivative of the inverse of the deformation gradient follows as

$$\frac{\mathrm{d}\mathbf{F}^{-1}}{\mathrm{d}t} = -\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1} = -\mathbf{F}^{-1}\mathrm{grad}\boldsymbol{v}. \tag{20}$$

3 Conservation of Mass

Under the assumption that the arbitrary inner subsystem of $\mathcal{B}$ (which has been depicted in Fig. 4) is a closed system, no mass can enter or leave the subsystem boundary $\partial\Omega(t)$. That is, the overall mass m of the volume $\Omega(t)$ must remain constant at any time instant $t \geq 0$. In mathematical terms, this requirement reads as

$$\dot{m} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega(t)} \rho \mathrm{dv} = 0, \tag{21}$$

where $\rho(x, t)$ is the spatial mass density. Equation (21) represents the Continuity Mass Equation (CME) expressed in global form and in the current configuration.

3.1 Continuity Mass Equation in Eulerian Description

Using the Reynold's transport theorem [Eq. (207)] and since $\Omega(t)$ is an arbitrary volume, Eq. (21) can also be written as

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\boldsymbol{v}\rho) = \dot{\rho} + \rho \mathrm{div}(\boldsymbol{v}) = 0, \tag{22}$$

corresponding to the local (rate) form of the CME in the Eulerian (spatial) description.

3.1.1 Continuity Mass Equation in Lagrangian Description

Resorting to Eq. (12), (21) can be rewritten in the referential configuration as

$$\dot{m} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\bar{\Omega}} \bar{\rho} \mathrm{dV} = \int_{\bar{\Omega}} \dot{\bar{\rho}} \mathrm{dV} = 0, \tag{23}$$

where $\bar{\rho}(X)$ is defined as the reference mass density

$$\bar{\rho} = J\rho. \tag{24}$$

By considering that $\bar{\Omega}$ is an arbitrary volume, Eq. (23) immediately yields

$$\dot{\bar{\rho}} = 0, \tag{25}$$

representing the local (rate) form of the CME in the Lagrangian (material) description.

4 Balance of Mechanical Momentum

In the framework of Newtonian mechanics, the motion and deformation of the bodies contained in any arbitrary closed subsystem of $\mathcal{B}$ (which has been depicted in Fig. 4) require the rate of change of the associated mechanical momentum to be equivalent to the ensemble of forces acting on the enclosed bodies themselves. In mathematical terms, given the definitions of linear momentum, $\boldsymbol{L}(t)$,

$$\boldsymbol{L}(t) = \int_{\Omega(t)} \rho \boldsymbol{v} \mathrm{dv}, \tag{26}$$

and of angular momentum, $\boldsymbol{A}_0(t)$, with respect to a general point identified by the position vector $\boldsymbol{x}_0$

$$\boldsymbol{A}_0(t) = \int_{\Omega(t)} \boldsymbol{r} \times \boldsymbol{v} \rho \mathrm{dv}, \tag{27}$$

with $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}_0$, the equivalence requirement stated above amounts to the two following momentum balance principles

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{L} = \int_{\partial\Omega(t)} \boldsymbol{t} \mathrm{ds} + \int_{\Omega(t)} \boldsymbol{f} \mathrm{dv}, \tag{28}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A}_0 = \int_{\partial\Omega(t)} \boldsymbol{r} \times \boldsymbol{t} \mathrm{ds} + \int_{\Omega(t)} \boldsymbol{r} \times \boldsymbol{f} \mathrm{dv}, \tag{29}$$

where $\boldsymbol{f}(\boldsymbol{x}, t)$ is the (spatial) body force density measured per unit of the current volume $\Omega(t)$, while $\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})$ is the Cauchy (or true) traction vector (a force measured per unit surface area defined in the deformed configuration) acting on the current surface $\partial\Omega(t)$ and having unit normal $\boldsymbol{n}(\boldsymbol{x}, t)$. In particular, Eqs. (28) and (29) respectively represent the Balance of Linear Momentum (BLM) and the Balance of Moment of Momentum (BMM), both expressed in global forms and in the current configuration.

4.1 Cauchy's First Equation of Motion in Eulerian Description

Consider the Cauchy's Stress Theorem (in the spatial description)

$$\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) = \boldsymbol{\sigma} \cdot \boldsymbol{n}, \tag{30}$$

where $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ denotes a spatial tensor field called the Cauchy (or true) stress tensor and with the scalar product between tensor $\boldsymbol{\sigma}$ and vector $\boldsymbol{n}$ being defined as in Eq. (183). Then, by employing the Reynold's transport theorem [Eq. (207)] and Eq. (22) to its left-hand side, and the Cauchy's Stress Theorem [Eq. (30)] and the divergence theorem [Eq. (204)] to the first term of its right-hand side, Eq. (26) becomes

$$\int_{\Omega(t)} (\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{f} - \rho\dot{\boldsymbol{v}})\mathrm{dv} = \boldsymbol{0}, \tag{31}$$

which represents the global form of the Cauchy's First Equation of Motion (CFEM) in the current configuration. By considering that $\Omega(t)$ is an arbitrary volume, Eq. (31) implies

$$\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{f} - \rho\dot{\boldsymbol{v}} = \boldsymbol{0}, \tag{32}$$

that corresponds to the local (rate) form of CFEM in the Eulerian description.

4.2 Cauchy's First Equation of Motion in Lagrangian Description

Resorting to the volume ratio relationship (Eq. 12), Eq. (28) can be rewritten in the referential configuration as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\bar{\Omega}} \bar{\rho}\boldsymbol{V}\mathrm{dV} = \int_{\partial\bar{\Omega}} \boldsymbol{T}\mathrm{dS} + \int_{\bar{\Omega}} \bar{\boldsymbol{f}}\mathrm{dV}, \tag{33}$$

where $\bar{\boldsymbol{f}}(\boldsymbol{X}, t)$ is defined as the reference body force

$$\bar{\boldsymbol{f}} = J\boldsymbol{f}, \tag{34}$$

whereas $\boldsymbol{T}(\boldsymbol{X}, t, \boldsymbol{N})$ represents the first Piola-Kirchhoff (or nominal) traction vector (a force measured per unit area defined in the undeformed configuration) satisfying the relation

$$\boldsymbol{t}\mathrm{ds} = \boldsymbol{T}\mathrm{dS}. \tag{35}$$

That is, by definition, the direction of the first Piola-Kirchhoff traction vector, $\boldsymbol{T}$, coincides to that of the Cauchy traction vector, $\boldsymbol{t}$.

Then, resorting to Eq. (25) on the left-hand side, and to both the Cauchy's Stress Theorem (in the material description)

$$\boldsymbol{T}(\boldsymbol{X}, t, \boldsymbol{N}) = \boldsymbol{\Pi} \cdot \boldsymbol{N} \tag{36}$$

where $\boldsymbol{\Pi}(\boldsymbol{X},\boldsymbol{t})$ denotes a material tensor field called the Nominal stress tensor, and the divergence theorem [Eq. (204)] on the first term of the right-hand side, Eq. (33) can also be written as

$$\int_{\bar{\Omega}} \left(\mathrm{Div}\boldsymbol{\Pi} + \bar{\boldsymbol{f}} - \bar{\rho}\dot{\boldsymbol{V}}\right)\mathrm{dV} = \boldsymbol{0}, \tag{37}$$

which represents the global form of the CFEM in the referential configuration. Since $\bar{\Omega}$ is an arbitrary volume, Eq. (37) entails

$$\mathrm{Div}\boldsymbol{\Pi} + \bar{\boldsymbol{f}} - \bar{\rho}\dot{\boldsymbol{V}}, \tag{38}$$

which corresponds to the local (rate) form of CFEM in the Lagrangian description.

4.3 Cauchy's Second Equation of Motion in Eulerian Description

By the Reynold's transport theorem [Eq. (207)], Eq. (22) and the equivalence $\dot{\boldsymbol{r}} = \dot{\boldsymbol{x}} - \dot{\boldsymbol{x}}_0 = \boldsymbol{v}$ (since $\dot{\boldsymbol{x}}_0 = \boldsymbol{0}$), the rate of the angular momentum $\boldsymbol{A}_0(t)$ which is required on the left-hand side of Eq. (29) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A}_0 = \int_{\Omega(t)} (\dot{\boldsymbol{r}} \times \boldsymbol{v} + \boldsymbol{r} \times \dot{\boldsymbol{v}})\rho\mathrm{dv} = \int_{\Omega(t)} \boldsymbol{r} \times \dot{\boldsymbol{v}}\rho\mathrm{dv}. \tag{39}$$

Besides, by Eq. (30) and the divergence theorem [Eq. (204)], the first term on the right-hand side of Eq. (29) becomes

$$\int_{\partial\Omega(t)} \boldsymbol{r} \times \boldsymbol{t}\mathrm{ds} = \int_{\partial\Omega(t)} \boldsymbol{r} \times (\boldsymbol{\sigma} \cdot \boldsymbol{n})\mathrm{ds} = \int_{\Omega(t)} (\boldsymbol{r} \times \mathrm{div}\boldsymbol{\sigma} + \mathcal{E} : \boldsymbol{\sigma})\mathrm{dv}, \tag{40}$$

where $\mathcal{E}$ indicates the third-order Levi-Civita (or permutation) tensor

$$\mathcal{E} = \left[\varepsilon_{ijk}\right], \tag{41}$$

with ε_{ijk} being the permutation symbol defined according to Eq. (191).

Then, use of Eqs. (39) and (40), together with Eq. (32), enables to simplify Eq. (29) as

$$\int_{\Omega(t)} \mathcal{E} : \boldsymbol{\sigma}\mathrm{dv} = \boldsymbol{0}, \tag{42}$$

representing the global form of the Cauchy's Second Equation of Motion (CSEM) in the current configuration. By considering that $\Omega(t)$ is an arbitrary volume, Eq. (42) gives

$$\mathcal{E} : \boldsymbol{\sigma} = \mathbf{0}, \tag{43}$$

or identically

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T, \tag{44}$$

which implies the symmetry of the Cauchy stress tensor $\boldsymbol{\sigma}$.

4.4 Cauchy's Second Equation of Motion in Lagrangian Description

According to the definitions introduced in the previous sections, Eq. (29) can be converted in the referential configuration as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\bar{\Omega}} \boldsymbol{r} \times \boldsymbol{V}\bar{\rho}\mathrm{dV} = \int_{\bar{\Omega}} \boldsymbol{r} \times \dot{\boldsymbol{V}}\bar{\rho}\mathrm{dV} = \int_{\partial\bar{\Omega}} \boldsymbol{r} \times \boldsymbol{T}\mathrm{dS} + \int_{\bar{\Omega}} \boldsymbol{r} \times \bar{\boldsymbol{f}}\mathrm{dV}. \tag{45}$$

Besides, owing to Eq. (36) and the divergence theorem [Eq. (204)], and by the chain-rule differentiation with $\mathrm{Grad}\boldsymbol{r} = \mathrm{Grad}(\boldsymbol{x} - x_0) = \mathbf{F}$, the first term on the right-hand side of Eq. (45) can be rewritten as

$$\int_{\partial\bar{\Omega}} \boldsymbol{r} \times \boldsymbol{T}\boldsymbol{dS} = \int_{\partial\bar{\Omega}} \boldsymbol{r} \times (\boldsymbol{\Pi} \cdot \boldsymbol{N})\boldsymbol{dS} = \int_{\bar{\Omega}} (\boldsymbol{r} \times \boldsymbol{Div}\boldsymbol{\Pi} + \mathcal{E} : (\mathbf{F}\boldsymbol{\Pi}))\mathrm{dV} \tag{46}$$

As a consequence, with the use of Eqs. (46) and (38), Eq. (45) reduces to

$$\int_{\bar{\Omega}} \mathcal{E} : (\mathbf{F}\boldsymbol{\Pi})\mathrm{dV} = \mathbf{0}, \tag{47}$$

which, under the assumption that $\bar{\Omega}$ is an arbitrary volume, gives

$$\mathcal{E} : (\mathbf{F}\boldsymbol{\Pi}) = \mathbf{0}, \tag{48}$$

or identically

$$\mathbf{F}\boldsymbol{\Pi} = \boldsymbol{\Pi}^T\mathbf{F}^T, \tag{49}$$

highlighting that differently than the Cauchy stress tensor, $\boldsymbol{\sigma}$, the Nominal stress tensor, $\boldsymbol{\Pi}$, is generally not symmetric. This is indeed confirmed by the Piola transformation

$$\boldsymbol{\Pi} = J\mathbf{F}^{-1}\boldsymbol{\sigma}, \tag{50}$$

which can be derived by combining Eqs. (35), (30) and (36), together with the Nanson's formula (Eq. 15).

5 Electrostatic Interactions

In the framework of Maxwellian electromagnetism [29], assuming that any existing magnetic field is stationary, the electrical phenomena acting between the dielectric and the conducting bodies enclosed in the system depicted in Fig. 4 can be described by the two following laws: the simplified version of Faraday's law

$$\int_{S(t)} \mathrm{rot}\boldsymbol{E} \cdot \boldsymbol{n}\mathrm{ds} = 0, \tag{51}$$

where $\boldsymbol{E}(x,t)$ is the electric field acting on any arbitrary spatial open surface $S(t)$ that belongs to $\mathcal{B}$, and the Gauss's law

$$\int_{\partial\Omega(t)-\gamma(t)} \boldsymbol{D} \cdot \boldsymbol{n}\mathrm{ds} = \int_{\Omega(t)-\gamma(t)} \varphi\mathrm{dv} + \int_{\gamma(t)} \varphi_s\mathrm{dv}, \tag{52}$$

with

$$\boldsymbol{D} = \boldsymbol{D}_0 + \boldsymbol{P}, \tag{53}$$

$$\boldsymbol{D}_0 = \varepsilon_0 \boldsymbol{E}, \tag{54}$$

where $\varepsilon_0 = 8.85\ e^{-12} F/m$ is the constant vacuum permittivity, $\boldsymbol{P}(\boldsymbol{x},t)$ and $\boldsymbol{D}(\boldsymbol{x},t)$ are the electric polarization and the electric displacement permeating any arbitrary volume $\Omega(t)$ that belongs to $\mathcal{B}$, whereas $\varphi(\boldsymbol{x},t)$ and $\varphi_s(\boldsymbol{x},t)$ are the free-charge densities per unit of spatial volume and per unit of spatial surface respectively. Specifically, φ resides within the deformed volume $\Omega(t)$ (usually in the form of injected electrons and/or ions), whereas φ_s lies on some deformed discontinuity surface $\gamma(t)$ (usually a conducting surface electrode) comprised in $\Omega(t)$.

5.1 Balance Law of Electrostatic in Eulerian Description

By considering that $S(t)$ is a general open surface, Eq. (51) immediately yields

$$\mathrm{rot}\boldsymbol{E} = 0. \tag{55}$$

That is, the Faraday's law directly implies the existence of an electric potential field, $\phi(\boldsymbol{x},t)$, such that

$$\boldsymbol{E} = -\mathrm{grad}(\phi). \tag{56}$$

Besides, using the divergence theorem [Eq. (205)] to the left-hand side of Eq. (52), the Gauss's law can be rewritten as

$$\int_{\Omega(t)-\gamma(t)} \mathrm{div}\boldsymbol{D}\mathrm{dv} + \int_{\gamma(t)} [\![\boldsymbol{D}]\!] \cdot \boldsymbol{n}\mathrm{ds} = \int_{\Omega(t)-\gamma(t)} \varphi\mathrm{dv} + \int_{\gamma(t)} \varphi_s \mathrm{ds}, \tag{57}$$

where $[\![\boldsymbol{D}]\!] \equiv (\boldsymbol{D}^+ - \boldsymbol{D}^-)$ indicates the jump of the quantity $\boldsymbol{D}$ from the positive (+) side to the negative (−) side of the discontinuity surface $\gamma(t)$. By considering that $\Omega(t)$ and $\gamma(t)$ are arbitrary volumes and surfaces, Eq. (57) immediately implies

$$\mathrm{div}\boldsymbol{D} = \varphi, \tag{58}$$

$$[\![\boldsymbol{D}]\!] \cdot \boldsymbol{n} = \varphi_s. \tag{59}$$

Equations (58) and (59) correspond to the local form and the associated boundary condition of the Gauss's law expressed in the Eulerian description.

More concisely, by considering Eqs. (53), (54), (56), (58) and (59), the electric phenomena occurring within any arbitrary subsystem of $\mathcal{B}$ (such as the one depicted in Fig. 4) can be described by the following balance law (expressed in local form and in the Eulerian description)

$$\varepsilon_0 \mathrm{div}(\mathrm{grad}\phi) = \mathrm{div}\boldsymbol{P} - \varphi, \tag{60}$$

along with the associated boundary condition

$$[\![\boldsymbol{P} - \varepsilon_0 \mathrm{grad}\phi]\!] \cdot \boldsymbol{n} = \varphi_s. \tag{61}$$

5.2 Balance Law of Electrostatic in Lagrangian Description

Considering the Faraday's law, by the Stoke's theorem [Eq. (202)] and Eq. (6), the following holds for Eq. (51)

$$\int_{S(t)} \mathrm{rot}\boldsymbol{E} \cdot \boldsymbol{n}\mathrm{ds} = \int_{L(t)} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{x} = \int_{\bar{L}} \bar{\boldsymbol{E}} \cdot \mathrm{d}\boldsymbol{X} = \int_{\bar{S}} \mathrm{Rot}\bar{\boldsymbol{E}} \cdot \boldsymbol{N}\mathrm{dS} = 0, \tag{62}$$

where $L(t)$ is the closed spatial curve bounding the open spatial surface $S(t)$, $\bar{L}$ and $\bar{S}$ are the material counterparts of $L(t)$ and $S(t)$, while $\bar{\boldsymbol{E}}(\boldsymbol{X}, t)$ is defined as the reference electric field

$$\bar{\boldsymbol{E}} = \mathbf{F}^T \boldsymbol{E}, \tag{63}$$

From Eq. (56) and through the chain-rule differentiation, Eq. (63) yields

$$\bar{\boldsymbol{E}} = -\mathrm{Grad}\bar{\phi}, \tag{64}$$

where $\bar{\phi}(\boldsymbol{X},t)=\phi(\boldsymbol{x},t)$ is the electric potential expressed with respect to the referential configuration.

Considering the Gauss's law, by Eqs. (12) and (15), Eq. (52) can be rewritten as

$$\int\limits_{\partial\bar{\Omega}-\bar{\gamma}} \bar{\boldsymbol{D}}\cdot\boldsymbol{N}\mathrm{dS}=\int\limits_{\bar{\Omega}-\bar{\gamma}} \bar{\varphi}\mathrm{dV}+\int\limits_{\bar{\gamma}} \bar{\varphi}_s\mathrm{dS}, \tag{65}$$

where $\bar{\gamma}$ is the material counterpart of the discontinuity surface $\gamma(t)$, whereas $\bar{\boldsymbol{D}}(\boldsymbol{X},t)$, $\bar{\boldsymbol{P}}(\boldsymbol{X},t)$, $\bar{\varphi}(\boldsymbol{X},t)$ and $\bar{\varphi}_s(\boldsymbol{X},t)$ are the reference electric displacement, the reference electric polarization and the reference free-charge densities respectively defined as

$$\bar{\boldsymbol{D}}=\bar{\boldsymbol{D}}_0+\bar{\boldsymbol{P}}=J\mathbf{F}^{-1}\mathbf{D}, \tag{66}$$

$$\bar{\boldsymbol{P}}=J\mathbf{F}^{-1}\boldsymbol{P}, \tag{67}$$

$$\bar{\boldsymbol{D}}_0=J\mathbf{F}^{-1}\boldsymbol{D}_0, \tag{68}$$

$$\bar{\varphi}=J\varphi, \tag{69}$$

$$\bar{\varphi}_s=\varphi_s\mathrm{ds}/\mathrm{dS}. \tag{70}$$

Then, use of the divergence theorem [Eq. (205)] to the left-hand side of Eq. (65) and since $\bar{\Omega}$ and $\bar{\gamma}$ are arbitrary volumes and surfaces, the Lagrangian counterparts of Eqs. (58) and (59) follow as

$$\mathrm{Div}\bar{\boldsymbol{D}}=\bar{\varphi}, \tag{71}$$

$$[\![\bar{\boldsymbol{D}}]\!]\cdot\boldsymbol{N}=\bar{\varphi}_s. \tag{72}$$

Moreover, from Eqs. (68), (54) and (63), the Lagrangian counterpart of Eq. (54) is

$$\bar{\boldsymbol{D}}_0=\varepsilon_0 J\mathbf{C}^{-1}\bar{\boldsymbol{E}}, \tag{73}$$

which highlights that the electrostatic analysis of a moving and deforming system of dielectric and conducting bodies contained in any arbitrary vacuum volume $\Omega(t)$ is equivalent to the study of a fixed system of dielectric and conducting bodies embedded in a media with fixed volume $\bar{\Omega}$ and characterized by an anisotropic and inhomogeneous dielectric tensor equaling $\varepsilon_0 J\mathbf{C}^{-1}$ [54].

Summarizing, by considering Eqs. (71), (72) (66), (73) and (64), the electric phenomena occurring within any arbitrary subsystem of $\mathcal{B}$ (such as the one depicted in Fig. 4) can be described by the following balance law (expressed in local form and in the Lagrangian description)

$$\varepsilon_0\mathrm{Div}\left(J\mathbf{C}^{-1}\mathrm{Grad}\bar{\phi}\right)=\mathrm{Div}\bar{\boldsymbol{P}}-\bar{\varphi}, \tag{74}$$

along with the associated boundary condition:

$$\left[\!\left[\overline{\boldsymbol{P}} - \varepsilon_0 J \mathbf{C}^{-1} \mathrm{Grad}\bar{\phi}\right]\!\right] \cdot \boldsymbol{N} = \bar{\varphi}_s. \tag{75}$$

6 Conservation of Total Energy

For any arbitrary closed subsystem of $\mathcal{B}$ (such as the one depicted in Fig. 4), energy can cross the system boundary $\partial\Omega(t)$ in the form of heat, electrical work and mechanical work. Differently than $\mathcal{B}$, the subsystem is not electrically isolated and thus interactions may exist between the electrical charges that lie within the volume $\Omega(t)$ and those outside, which certainly need to be accounted for in the balance of subsystem energy. However, since the charges outside the boundary $\partial\Omega(t)$ can always be replaced, without modifying in any way the electric potential at any interior point of $\Omega(t)$, by an equivalent single and double layer of charges that are distributed on $\partial\Omega(t)$ with surface density equaling [34, 35]

$$\hat{\varphi}_s = -\boldsymbol{D} \cdot \boldsymbol{n} = -\bar{\boldsymbol{D}} \cdot \boldsymbol{N} dS/ds, \tag{76}$$

then the energy balance of any arbitrary subsystem of $\mathcal{B}$ can be performed by considering an equivalent isolated subsystem as shown in Fig. 5. That is, in mathematical terms, the balance of electro-thermo-mechanical energy (i.e. the first law of thermodynamics) for any arbitrary subsystem of $\mathcal{B}$ which is bounded by the surface $\partial\Omega(t)$ reads as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K} + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W} + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I} = \mathcal{P}_{me} + \mathcal{P}_{el} + \mathcal{Q}, \tag{77}$$

where $\mathcal{K}(t)$, $\mathcal{W}(t)$ and $\mathcal{I}(t)$ are the kinetic, electrostatic and internal energies associated to the physical space contained within the boundary $\partial\Omega(t)$

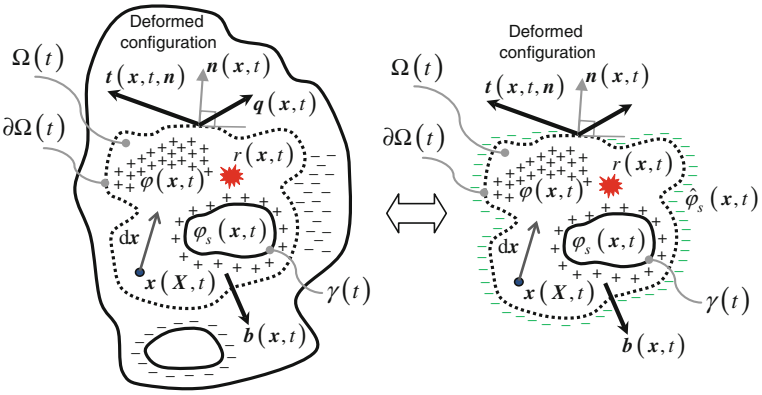

Fig. 5 Electrostrictive elastomer system: arbitrary electrically-non-isolated interior subsystem and its electrically-isolated equivalent

$$\mathcal{K} = \int_{\Omega(t)} \frac{1}{2}\rho \boldsymbol{v}^2 \mathrm{dv}, \tag{78}$$

$$\mathcal{W} = \int_{\Omega(t)-\gamma(t)} \left(\frac{1}{2}\varepsilon_0 \boldsymbol{E}^2 + \boldsymbol{E}\cdot\boldsymbol{P}\right)\mathrm{dv}, \tag{79}$$

$$\mathcal{I} = \int_{\Omega(t)-\gamma(t)} \rho U \mathrm{dv}, \tag{80}$$

$U(\boldsymbol{x},t)$ being the internal energy density associated to the bodies contained in the volume $\Omega(t)$; whereas $\mathcal{P}_{me}(t)$, $\mathcal{P}_{el}(t)$ and $\mathcal{Q}(t)$ are the external mechanical, electrical and thermal powers entering in the system from the outside of its boundary $\partial\Omega(t)$

$$\mathcal{P}_{me} = \int_{\partial\Omega(t)} \boldsymbol{t}\cdot\boldsymbol{v}\mathrm{ds} + \int_{\Omega(t)} \boldsymbol{f}\cdot\boldsymbol{v}\mathrm{dv}, \tag{81}$$

$$\mathcal{P}_{el} = \int_{\Omega(t)-\gamma(t)} \phi\frac{\mathrm{d}}{\mathrm{d}t}(\varphi\mathrm{dv}) + \int_{\gamma(t)} \phi\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_s\mathrm{ds}) + \int_{\partial\Omega(t)} \phi\frac{\mathrm{d}}{\mathrm{d}t}(\hat{\varphi}_s\mathrm{ds}), \tag{82}$$

$$\mathcal{Q} = -\int_{\partial\Omega(t)} \boldsymbol{Q}\cdot\boldsymbol{n}\mathrm{ds} + \int_{\Omega(t)} R\mathrm{dv}, \tag{83}$$

where $\boldsymbol{Q}(\boldsymbol{x},t)$ is the Cauchy heat flux vector entering the system (defined per unit area of the deformed surface $\partial\Omega(t)$). And $R(\boldsymbol{x},t)$ is the heat source density (i.e. a reservoir of heat) defined per unit volume of the deformed $\Omega(t)$.

6.1 Conservation of Total Energy in Eulerian Description

By the Reynold's transport theorem [Eq. (207)], Eqs. (19) and (22), the three terms on the left-hand side of Eq. (77) follow as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K} = \int_{\Omega(t)} \rho\dot{\boldsymbol{v}}\cdot\boldsymbol{v}\mathrm{dv}, \tag{84}$$

$$\frac{d}{dt}\mathcal{W} = \int_{\Omega(t)-\gamma(t)} \left(\varepsilon_0\boldsymbol{E}\cdot\dot{\boldsymbol{E}} + \boldsymbol{P}\cdot\dot{\boldsymbol{E}} + \boldsymbol{E}\cdot\dot{\boldsymbol{P}}\right)\mathrm{dv} + \int_{\Omega(t)-\gamma(t)} \left(\frac{1}{2}\varepsilon_0\boldsymbol{E}^2 + \boldsymbol{E}\cdot\boldsymbol{P}\right)\mathrm{div}\boldsymbol{v}\mathrm{dv}, \tag{85}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I} = \int\limits_{\Omega(t)-\gamma(t)} \rho\dot{U}\mathrm{dv}. \tag{86}$$

As for the first term on right-hand side of Eq. (77), by using Eq. (30) and the divergence theorem [Eq. (204)], and by additionally resorting to Eq. (198), (81) can also be rewritten as

$$\mathcal{P}_{me} = \int\limits_{\Omega(t)} \mathrm{div}(\sigma\boldsymbol{v})\mathrm{dv} + \int\limits_{\Omega(t)} \boldsymbol{f}\cdot\boldsymbol{v}\mathrm{dv} = \int\limits_{\Omega(t)} \boldsymbol{\sigma}^T : \mathrm{grad}\boldsymbol{v}\mathrm{dv} + \int\limits_{\Omega(t)} (\mathrm{div}\boldsymbol{\sigma}+\boldsymbol{f})\cdot\boldsymbol{v}\mathrm{dv}, \tag{87}$$

which, by considering Eqs. (77) and (32), yields

$$\mathcal{P}_{me} - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K} = \int\limits_{\Omega(t)} \boldsymbol{\sigma}^T : \mathrm{grad}\boldsymbol{v}\mathrm{dv}. \tag{88}$$

As for the second term on right-hand side of Eq. (77), using Eqs. (59), (76), (12) and (15), Eq. (82) can be rewritten as

$$\mathcal{P}_{el} = \int\limits_{\bar{\Omega}-\bar{\gamma}} \phi\frac{\mathrm{d}}{\mathrm{d}t}(\varphi J\mathrm{dV}) + \int\limits_{\bar{\gamma}} \phi\frac{\mathrm{d}}{\mathrm{d}t}\left(\llbracket\boldsymbol{D}\rrbracket\cdot J\mathbf{F}^{-T}\boldsymbol{N}\mathrm{dS}\right) - \int\limits_{\partial\bar{\Omega}} \phi\frac{\mathrm{d}}{\mathrm{d}t}\left(\boldsymbol{D}\cdot J\mathbf{F}^{-T}\boldsymbol{N}\mathrm{dS}\right), \tag{89}$$

which, upon differentiation and use of the kinematic Eqs. (19) and (20), becomes

$$\begin{aligned}\mathcal{P}_{el} = &\int\limits_{\Omega(t)-\gamma(t)} \phi[\dot{\varphi} + \varphi\mathrm{div}\boldsymbol{v}]\mathrm{dv} + \int\limits_{\gamma(t)} \left[\phi\left(\llbracket\dot{\boldsymbol{D}}\rrbracket + \mathrm{div}\boldsymbol{v}\llbracket\boldsymbol{D}\rrbracket - \mathrm{grad}\boldsymbol{v}\llbracket\boldsymbol{D}\rrbracket\right)\right]\cdot\boldsymbol{n}\mathrm{ds}\\ &- \int\limits_{\partial\Omega(t)} \left[\phi\left(\dot{\boldsymbol{D}} + \mathrm{div}\boldsymbol{v}\boldsymbol{D} - \mathrm{grad}\boldsymbol{v}\boldsymbol{D}\right)\right]\cdot\boldsymbol{n}\mathrm{ds}.\end{aligned} \tag{90}$$

Then, application of the divergence theorem [Eq. (205)] to the last two terms on the right-hand side of Eq. (90) yields

$$\mathcal{P}_{el} = \int\limits_{\Omega(t)-\gamma(t)} \phi[\dot{\varphi} + \varphi\mathrm{div}\boldsymbol{v}]\mathrm{dv} - \int\limits_{\Omega(t)-\gamma(t)} \mathrm{div}\left[\phi\left(\dot{\boldsymbol{D}} + \mathrm{div}\boldsymbol{v}\boldsymbol{D} - \mathrm{grad}\boldsymbol{v}\boldsymbol{D}\right)\right]\mathrm{dv}, \tag{91}$$

which, with the use of Eq. (199) and owing to Eq. (56), can be more conveniently rewritten as

$$\mathcal{P}_{el} = \int\limits_{\Omega(t)-\gamma(t)} \phi(\dot{\varphi} + \varphi \mathrm{div}\boldsymbol{v})\mathrm{d}v - \int\limits_{\Omega(t)-\gamma(t)} \phi \mathrm{div}\left(\dot{\boldsymbol{D}} + \mathrm{div}\boldsymbol{v}\boldsymbol{D} - \mathrm{grad}\boldsymbol{v}\boldsymbol{D}\right)\mathrm{d}v + \int\limits_{\Omega(t)-\gamma(t)} \boldsymbol{E} \cdot \left(\dot{\boldsymbol{D}} + \mathrm{div}\boldsymbol{v}\boldsymbol{D} - \mathrm{grad}\boldsymbol{v}\boldsymbol{D}\right)\mathrm{d}v. \tag{92}$$

Further, since

$$\mathrm{div}\dot{\boldsymbol{D}} = \dot{\varphi} + (\mathrm{grad}\boldsymbol{D})^T : \mathrm{grad}\boldsymbol{v}, \tag{93}$$

$$\mathrm{div}(\mathrm{div}\boldsymbol{v}\boldsymbol{D}) = \mathrm{grad}(\mathrm{div}\boldsymbol{v}) \cdot \boldsymbol{D} + \varphi \mathrm{div}\boldsymbol{v}, \tag{94}$$

$$\mathrm{div}(\mathrm{grad}\boldsymbol{v}\boldsymbol{D}) = \mathrm{grad}(\mathrm{div}\boldsymbol{v}) \cdot \boldsymbol{D} + (\mathrm{grad}\boldsymbol{D})^T : \mathrm{grad}\boldsymbol{v}, \tag{95}$$

Eq. (92) simplifies into

$$\mathcal{P}_{el} = \int\limits_{\Omega(t)-\gamma(t)} \boldsymbol{E} \cdot \left(\dot{\boldsymbol{D}} + \mathrm{div}\boldsymbol{v}\boldsymbol{D} - \mathrm{grad}\boldsymbol{v}\boldsymbol{D}\right)\mathrm{d}v, \tag{96}$$

which, by considering Eq. (85) together with the identities $\dot{\boldsymbol{D}} = \varepsilon_0 \dot{\boldsymbol{E}} + \dot{\boldsymbol{P}}$, $\mathrm{div}\boldsymbol{v} = \boldsymbol{1} : \mathrm{grad}\boldsymbol{v}$ and $\boldsymbol{E} \cdot (\mathrm{grad}\boldsymbol{v}\boldsymbol{D}) = (\boldsymbol{E} \otimes \boldsymbol{D}) : \mathrm{grad}\boldsymbol{v}$ [$\boldsymbol{E} \otimes \boldsymbol{D}$ indicating the tensor product between vectors $\boldsymbol{E}$ and $\boldsymbol{D}$ as defined in Eq. (181)], yields

$$\mathcal{P}_{el} - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W} = \int\limits_{\Omega(t)-\gamma(t)} \left[-\boldsymbol{P} \cdot \dot{\boldsymbol{E}} + \left(\frac{1}{2}\varepsilon_0 \boldsymbol{E}^2 \boldsymbol{1} - \boldsymbol{E} \otimes \boldsymbol{D}\right) : \mathrm{grad}\boldsymbol{v}\right]\mathrm{d}v. \tag{97}$$

As for the last term on right-hand side of Eq. (77), by the divergence theorem [Eq. (203)], Eq. (83) can be rewritten as

$$\mathcal{Q} = \int\limits_{\Omega(t)} (R - \mathrm{div}\boldsymbol{Q})\mathrm{d}v. \tag{98}$$

In summary, by considering Eqs. (86), (88), (97) and (98), and by assuming that no discontinuity across any surface $\gamma(t)$ exists in the variables ρ, $\boldsymbol{\sigma}$, and $\boldsymbol{Q}$, Eq. (77) reduces to

$$\int\limits_{\Omega(t)-\gamma(t)} \left[-\boldsymbol{P} \cdot \dot{\boldsymbol{E}} - \rho\dot{U} + R - \mathrm{div}\boldsymbol{Q} + \left(\boldsymbol{\sigma}^T - \boldsymbol{E} \otimes \boldsymbol{D} + \frac{1}{2}\varepsilon_0 \boldsymbol{E}^2 \boldsymbol{1}\right) : \mathrm{grad}\boldsymbol{v}\right]\mathrm{d}v = 0 \tag{99}$$

which, since $\Omega(t)$ and $\gamma(t)$ are arbitrary volumes and surfaces, yields

$$-\boldsymbol{P} \cdot \dot{\boldsymbol{E}} - \rho\dot{U} + R - \mathrm{div}\boldsymbol{Q} + \left(\boldsymbol{\sigma}^T - \boldsymbol{E} \otimes \boldsymbol{D} + \frac{1}{2}\varepsilon_0 \boldsymbol{E}^2 \boldsymbol{1}\right) : \mathrm{grad}\boldsymbol{v} = 0, \tag{100}$$

representing the local form of the First Law of Thermodynamics (FLT) for electrostrictive elastomers expressed in Eulerian description.

6.2 Conservation of Total Energy in Lagrangian Description

According to the notation introduced in the previous sections, the Lagrangian counterparts of Eqs. (78–80) are

$$\mathcal{K} = \int_{\bar{\Omega}} \frac{1}{2} \bar{\rho} \boldsymbol{V}^2 \mathrm{dV}, \tag{101}$$

$$\mathcal{W} = \int_{\bar{\Omega}-\bar{\gamma}} \left(\frac{1}{2} \bar{\boldsymbol{E}} \cdot \bar{\boldsymbol{D}}_0 + \bar{\boldsymbol{E}} \cdot \bar{\boldsymbol{P}} \right) \mathrm{dV}, \tag{102}$$

$$\mathcal{I} = \int_{\bar{\Omega}-\bar{\gamma}} \bar{\rho} \bar{U} \mathrm{dV}, \tag{103}$$

where $\bar{U}(\boldsymbol{X}, t) = U(\boldsymbol{x}, t)$ is the internal energy density expressed with respect to the referential configuration. Time derivation of Eqs. (101–103) directly yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{K} = \int_{\bar{\Omega}} \bar{\rho} \dot{\boldsymbol{V}} \cdot \boldsymbol{V} \mathrm{dV}, \tag{104}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{W} = \int_{\bar{\Omega}-\bar{\gamma}} \frac{1}{2} \left(\dot{\bar{\boldsymbol{E}}} \cdot \bar{\boldsymbol{D}}_0 + \bar{\boldsymbol{E}} \cdot \dot{\bar{\boldsymbol{D}}}_0 + \bar{\boldsymbol{P}} \cdot \dot{\bar{\boldsymbol{E}}} + \bar{\boldsymbol{E}} \cdot \dot{\bar{\boldsymbol{P}}} \right) \mathrm{dV}, \tag{105}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{I} = \int_{\bar{\Omega}-\bar{\gamma}} \bar{\rho} \dot{\bar{U}} \mathrm{dV}. \tag{106}$$

Besides, the Lagrangian counterparts of Eqs. (81)–(83) are

$$\mathcal{P}_{me} = \int_{\partial\bar{\Omega}} \boldsymbol{T} \cdot \boldsymbol{V} \mathrm{dS} + \int_{\bar{\Omega}} \bar{\boldsymbol{f}} \cdot \boldsymbol{V} \mathrm{dV}, \tag{107}$$

$$\begin{aligned} \mathcal{P}_{el} &= \int_{\bar{\Omega}-\bar{\gamma}} \bar{\phi} \frac{\mathrm{d}}{\mathrm{d}t} (\bar{\varphi} \mathrm{dV}) + \int_{\bar{\gamma}} \bar{\phi} \frac{\mathrm{d}}{\mathrm{d}t} (\bar{\varphi}_s \mathrm{dS}) - \int_{\partial\bar{\Omega}} \bar{\phi} \frac{\mathrm{d}}{\mathrm{d}t} (\bar{\boldsymbol{D}} \cdot \boldsymbol{N} \mathrm{dS}) \\ &= \int_{\bar{\Omega}-\bar{\gamma}} \bar{\phi} \dot{\bar{\varphi}} \mathrm{dV} + \int_{\bar{\gamma}} \bar{\phi} \dot{\bar{\varphi}}_s \mathrm{dS} - \int_{\partial\bar{\Omega}} \bar{\phi} \dot{\bar{\boldsymbol{D}}} \cdot \boldsymbol{N} \mathrm{dS}, \end{aligned} \tag{108}$$

$$\mathcal{Q} = -\int_{\partial\bar{\Omega}} \bar{\boldsymbol{Q}} \cdot \boldsymbol{N} \mathrm{d}S + \int_{\bar{\Omega}} \bar{R} \mathrm{d}V, \tag{109}$$

where the Piola-Kirchhoff heat flux, $\bar{\boldsymbol{Q}}(\boldsymbol{X}, t)$, which is defined per unit area of the reference surface $\partial\bar{\Omega}$, and the heat source, $\bar{R}(\boldsymbol{X}, t)$, which is defined per unit reference volume, read as

$$\bar{\boldsymbol{Q}} = J\mathbf{F}^{-1}\boldsymbol{Q}, \tag{110}$$

$$\bar{R} = JR. \tag{111}$$

Starting with $\mathcal{P}_{me}(t)$, use of Eq. (36), of the divergence theorem [Eq. (204)] and of Eq. (200), enables to write

$$\begin{aligned} \mathcal{P}_{me} &= \int_{\bar{\Omega}} \mathrm{Div}(\boldsymbol{\Pi}\boldsymbol{V}) \mathrm{d}V + \int_{\bar{\Omega}} \bar{\boldsymbol{f}} \cdot \boldsymbol{V} \mathrm{d}V \\ &= \int_{\bar{\Omega}} \boldsymbol{\Pi}^T : \mathrm{Grad}(\boldsymbol{V}) \mathrm{d}V + \int_{\bar{\Omega}} (\mathrm{Div}\boldsymbol{\Pi} + \boldsymbol{f}) \cdot \boldsymbol{V} \mathrm{d}V, \end{aligned} \tag{112}$$

which, by considering Eqs. (38) and (16), yields

$$\mathcal{P}_{me} - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K} = \int_{\bar{\Omega}} \boldsymbol{\Pi}^T : \dot{\mathbf{F}} \mathrm{d}V. \tag{113}$$

Skipping to $\mathcal{P}_{el}(t)$, considering Eqs. (71) and (72), Eq. (108) can be rewritten as

$$\mathcal{P}_{el}(t) = \int_{\bar{\Omega}-\bar{\gamma}} \bar{\phi} \mathrm{Div}\dot{\bar{\boldsymbol{D}}} \mathrm{d}V + \int_{\bar{\gamma}} \bar{\phi} [\![\dot{\bar{\boldsymbol{D}}}]\!] \cdot \boldsymbol{N} \mathrm{d}S - \int_{\partial\bar{\Omega}} \bar{\phi}\dot{\bar{\boldsymbol{D}}} \cdot \boldsymbol{N} \mathrm{d}S, \tag{114}$$

which, by applying the divergence theorem [Eq. (205)] to the second term on the right-hand side, and because of Eqs. (201) and (64), gives

$$\mathcal{P}_{el}(t) = \int_{\bar{\Omega}-\bar{\gamma}} \bar{\phi} \mathrm{Div}\dot{\bar{\boldsymbol{D}}} \mathrm{d}V - \int_{\bar{\Omega}-\bar{\gamma}} \mathrm{Div}\left(\bar{\phi}\dot{\bar{\boldsymbol{D}}}\right) \mathrm{d}V = \int_{\bar{\Omega}-\bar{\gamma}} \bar{\boldsymbol{E}} \cdot \dot{\bar{\boldsymbol{D}}} \mathrm{d}V. \tag{115}$$

From Eqs. (105) and (115), the Lagrangian counterpart of Eq. (97) is

$$\mathcal{P}_{el}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(t) = \int_{\bar{\Omega}-\bar{\gamma}} \left[-\bar{\boldsymbol{P}} \cdot \dot{\bar{\boldsymbol{E}}} + \frac{1}{2}\left(\bar{\boldsymbol{E}} \cdot \dot{\bar{\boldsymbol{D}}}_0 - \dot{\bar{\boldsymbol{E}}} \cdot \bar{\boldsymbol{D}}_0 \right) \right] \mathrm{d}V, \tag{116}$$

which, since

$$\bar{\boldsymbol{E}} \cdot \dot{\bar{\boldsymbol{D}}}_0 = \bar{\boldsymbol{E}} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon_0 J\mathbf{C}^{-1}\bar{\boldsymbol{E}}\right) = \dot{\bar{\boldsymbol{E}}} \cdot \bar{\boldsymbol{D}}_0 - \varepsilon_0 J(2\boldsymbol{E} \otimes \boldsymbol{E} - \boldsymbol{E} \cdot \boldsymbol{E}\mathbf{1})\mathbf{F}^{-T} : \dot{\mathbf{F}}, \tag{117}$$

simply reduces to

$$\mathcal{P}_{el}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}(t) = \int\limits_{\bar{\Omega}-\bar{\gamma}} \left[-\bar{\boldsymbol{P}}\cdot\dot{\bar{\boldsymbol{E}}} - \boldsymbol{\Pi}_M^T : \dot{\mathbf{F}}\right]\mathrm{dV}, \tag{118}$$

where

$$\boldsymbol{\Pi}_M = J\mathbf{F}^{-1}\boldsymbol{\sigma}_M, \tag{119}$$

$$\boldsymbol{\sigma}_M = \boldsymbol{\sigma}_M^T = \varepsilon_0\left(\boldsymbol{E}\otimes\boldsymbol{E} - \frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{E}\mathbf{1}\right), \tag{120}$$

$\boldsymbol{\sigma}_M$ being the well known Maxwell stress tensor (i.e. a true stress tensor) for the vacuum [29] expressed in the spatial description, whereas $\boldsymbol{\Pi}_M$ being its material counterpart (i.e. a nominal stress tensor).

Regarding $\mathcal{Q}(t)$, use of the divergence theorem [Eq. (203)] enables to rewrite Eq. (109) as

$$\mathcal{Q} = \int\limits_{\bar{\Omega}} (\bar{R} - \mathrm{Div}\bar{\boldsymbol{Q}})\mathrm{dV}. \tag{121}$$

In summary, by considering Eqs. (106), (113), (118) and (121), and by assuming that no discontinuity across any surface $\bar{\gamma}$ exists in the variables $\bar{\rho}$, $\boldsymbol{\Pi}$, and $\bar{\boldsymbol{Q}}$, Eq. (78) also reads as

$$\int\limits_{\bar{\Omega}-\bar{\gamma}} \left[-\bar{\boldsymbol{P}}\cdot\dot{\bar{\boldsymbol{E}}} - \bar{\rho}\dot{\bar{U}} + \bar{R} - \mathrm{Div}\bar{\boldsymbol{Q}} + \left(\boldsymbol{\Pi}^T - \boldsymbol{\Pi}_M^T\right):\dot{\mathbf{F}}\right]\mathrm{dV} = 0 \tag{122}$$

which, since $\bar{\Omega}$ and $\bar{\gamma}$ are arbitrary volumes and surfaces, yields

$$-\bar{\boldsymbol{P}}\cdot\dot{\bar{\boldsymbol{E}}} - \bar{\rho}\dot{\bar{U}} + \bar{R} - \mathrm{Div}\bar{\boldsymbol{Q}} + \left(\boldsymbol{\Pi}^T - \boldsymbol{\Pi}_M^T\right):\dot{\mathbf{F}} = 0, \tag{123}$$

representing the local form of the FLT for electrostrictive elastomers expressed in Lagrangian description.

7 Entropy Inequality Principle

In addition to Eq. (77), complete description of the energy transfer occurring across any arbitrary subsystem of $\mathcal{B}$ (such as the one depicted in Fig. 4) also requires the total entropy, which is produced during all the admissible thermo-electro-mechanical processes, to be non-negative. In mathematical terms, this can be stated by the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S} - \mathcal{H} \geq 0, \tag{124}$$

where $\mathcal{S}(t)$ and $\mathcal{H}(t)$ are the total entropy of the subsystem occupying the volume $\Omega(t)$ and the rate of entropy input entering from the subsystem boundary $\partial\Omega(t)$ respectively, and defined as

$$\mathcal{S}(t) = \int_{\Omega(t)} \rho\eta \mathrm{dv}, \tag{125}$$

$$\mathcal{H}(t) = -\int_{\partial\Omega(t)} \frac{\boldsymbol{Q}}{T}\cdot\boldsymbol{n}\mathrm{ds} + \int_{\Omega(t)} \frac{R}{T}\mathrm{dv}, \tag{126}$$

$\eta(\boldsymbol{x},t)$ and $T(\boldsymbol{x},t)$ (with $T(\boldsymbol{x},t)\geq 0$) respectively being the entropy density (per unit mass) and the absolute temperature both expressed in the current configuration.

7.1 Entropy Inequality in Eulerian Description

By applying the Reynold's transport theorem [Eq. (207)] and Eq. (22) to the time derivative of Eq. (125), and by employing the divergence theorem [Eq. (203)] to the first term on the right-hand side of Eq. (126), (124) reads as

$$\int_{\Omega(t)} \left[\rho\dot{\eta} + \mathrm{div}\left(\frac{\boldsymbol{Q}}{T}\right) - \frac{R}{T}\right]\mathrm{dv} \geq 0, \tag{127}$$

which, since $\Omega(t)$ is an arbitrary volume, yields

$$\rho\dot{\eta} + \mathrm{div}\left(\frac{\boldsymbol{Q}}{T}\right) - \frac{R}{T} \geq 0, \tag{128}$$

corresponding to the local form of the Clausius-Duhem inequality in the Eulerian description.

Resorting to Eq. (100), (128) becomes

$$\rho T\dot{\eta} - \rho\dot{U} - \boldsymbol{P}\cdot\dot{\boldsymbol{E}} + \left(\boldsymbol{\sigma}^T - \boldsymbol{E}\otimes\boldsymbol{D} + \frac{1}{2}\varepsilon_0\boldsymbol{E}^2\mathbf{1}\right) : \mathrm{grad}\boldsymbol{v} - \frac{\boldsymbol{Q}}{T}\cdot\mathrm{grad}T \geq 0, \tag{129}$$

which, by considering the following Legendre transformation (i.e. a procedure used to replace one or more variables with their conjugate counterparts [74])

$$\Psi = U - T\eta, \tag{130}$$

where $\Psi(\boldsymbol{x},t)$ is a free-energy function, can also be written as

$$\rho\eta\dot{T} + \rho\dot{\Psi} + \boldsymbol{P}\cdot\dot{\boldsymbol{E}} - \left(\boldsymbol{\sigma}^T - \boldsymbol{E}\otimes\boldsymbol{D} + \frac{1}{2}\varepsilon_0\boldsymbol{E}^2\mathbf{1}\right) : \mathrm{grad}\boldsymbol{v} + \frac{\boldsymbol{Q}}{T}\cdot\mathrm{grad}T \leq 0. \tag{131}$$

Further, by considering that heat does not flow against a temperature gradient, i.e.

$$\boldsymbol{Q} \cdot \mathrm{grad} T \leq 0, \tag{132}$$

Eq. (131) can be reduced into the stronger form

$$\rho\eta\dot{T} + \rho\dot{\Psi} + \boldsymbol{P} \cdot \dot{\boldsymbol{E}} - \left(\boldsymbol{\sigma}^T - \boldsymbol{E} \otimes \boldsymbol{D} + \frac{1}{2}\varepsilon_0 \boldsymbol{E}^2 \mathbf{1}\right) : \mathrm{grad}\boldsymbol{v} \leq 0, \tag{133}$$

which represents the local form of the Clausius-Planck Inequality (CPI) of thermodynamics for electrostictive elastomers expressed in the Eulerian formulation.

7.2 Entropy Inequality in Lagrangian Description

According to the notation introduced in previous sections, the Lagrangian counterparts of Eqs. (125) and (126) are

$$\mathcal{S}(t) = \int_{\bar{\Omega}} \bar{\rho}\bar{\eta} \mathrm{dV}, \tag{134}$$

$$\mathcal{H}(t) = -\int_{\partial\bar{\Omega}} \frac{\bar{\boldsymbol{Q}}}{\bar{T}} \cdot \boldsymbol{N} \mathrm{dS} + \int_{V\bar{\Omega}} \frac{\bar{R}}{\bar{T}} \mathrm{dV}, \tag{135}$$

$\bar{\eta}(\boldsymbol{X}, t) = \eta(\boldsymbol{x}, t)$ and $\bar{T}(\boldsymbol{X}, t) = T(\boldsymbol{x}, t)$ being the entropy density and the absolute temperature expressed with respect to the referential configuration.

By differentiating Eq. (134) and by employing the divergence theorem [Eq. (203)] to the first term on the right-hand side of Eq. (135), (124) can also be rewritten as

$$\int_{\bar{\Omega}} \left(\bar{\rho}\dot{\bar{\eta}} + \mathrm{Div}\left(\frac{\bar{\boldsymbol{Q}}}{\bar{T}}\right) - \frac{\bar{R}}{\bar{T}}\right) \mathrm{dV} \geq 0, \tag{136}$$

which, since $\bar{\Omega}$ is an arbitrary volume, yields

$$\bar{\rho}\dot{\bar{\eta}} + \mathrm{Div}\left(\frac{\bar{\boldsymbol{Q}}}{\bar{T}}\right) - \frac{\bar{R}}{\bar{T}} \geq 0, \tag{137}$$

corresponding to the local form of the Clausius-Duhem inequality in the Lagrangian description.

Resorting to Eq. (123), (137) becomes

$$\bar{T}\bar{\rho}\dot{\bar{\eta}} - \bar{\rho}\dot{\bar{U}} - \bar{\boldsymbol{P}} \cdot \dot{\bar{\boldsymbol{E}}} + \left(\boldsymbol{\Pi}^T - \boldsymbol{\Pi}_M^T\right) : \dot{\mathbf{F}} - \frac{\bar{\boldsymbol{Q}}}{\bar{T}} \mathrm{Grad}\bar{T} \geq 0, \tag{138}$$

which, by considering the following Legendre transformation

$$\bar{\Psi} = \bar{U} - \bar{T}\bar{\eta}, \tag{139}$$

where $\bar{\Psi}(\boldsymbol{X}, t) = \Psi(\boldsymbol{x}, t)$ is the same free-energy function defined in Eq. (130) but expressed with respect to the reference configuration, can also be rewritten as

$$\bar{\rho}\bar{\eta}\dot{\bar{T}} + \bar{\rho}\dot{\bar{\Psi}} + \bar{\boldsymbol{P}} \cdot \dot{\bar{\boldsymbol{E}}} - \left(\boldsymbol{\Pi}^T - \boldsymbol{\Pi}_M^T\right) : \dot{\mathbf{F}} + \frac{\bar{\boldsymbol{Q}}}{\bar{T}} \mathrm{Grad}\bar{T} \leq 0. \tag{140}$$

Furthermore, considering the Lagrangian counterpart of Eq. (132)

$$\bar{\boldsymbol{Q}} \cdot \mathrm{Grad}\bar{T} \leq 0, \tag{141}$$

Eq. (140) reduces to the stronger form

$$\bar{\rho}\bar{\eta}\dot{\bar{T}} + \bar{\rho}\dot{\bar{\Psi}} + \bar{\boldsymbol{P}} \cdot \dot{\bar{\boldsymbol{E}}} - \left(\boldsymbol{\Pi}^T - \boldsymbol{\Pi}_M^T\right) : \dot{\mathbf{F}} \leq 0, \tag{142}$$

representing the local form of the CPI of thermodynamics for electrostrictive elastomers expressed in the Lagrangian description.

8 Constitutive Equations

According to the relations described in the previous sections, complete knowledge of the thermo-electro-mechanical state of the closed and electrically isolated system depicted in Fig. 4 requires the determination of the variables ρ, $\boldsymbol{v}$ (or $\mathbf{F}$), $\boldsymbol{\sigma}$, φ, φ_s, $\boldsymbol{E}$, $\boldsymbol{P}$, $\boldsymbol{Q}$, R, T, Ψ and η [or equivalently by their Lagrangian counterparts $\bar{\rho}$, $\boldsymbol{V}$ (or $\mathbf{F}$), $\boldsymbol{\Pi}$, $\bar{\varphi}$, $\bar{\varphi}_s$, $\bar{\boldsymbol{E}}$, $\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{Q}}$, $\bar{R}$, $\bar{T}$, $\bar{\Psi}$ and $\bar{\eta}$], whose evolution is subjected to the balance Eqs. (22), (32), (44), (56), (58), (59), (100) and constrained by the inequalities (132) and (133) [or equivalently governed by the balance Eqs. (25), (38), (49), (71), (72), (123) and constrained by the inequalities (141) and (142)]. Despite the free-charge densities, φ and φ_s, and the heat source, R, are usually externally imposed (and thus known) quantities, the balance equations are not sufficient by themselves to make the problem determined. Therefore, other relationships need to be introduced.

In the derivation of the balance laws, no specification was provided regarding the nature of the substance constituting the dielectric and conducting deformable bodies comprised within the closed system $\mathcal{B}$ depicted in Fig. 4. Balance laws are indeed valid for all types of substances (e.g. gas, fluids, or solids having a variety of different properties). Of course, complete determination of the thermo-electro-mechanical problem requires the phenomenological properties of specific materials to be considered and adequately expressed in clean mathematical forms, usually known as constitutive relations. Contrarily to balance equations, constitutive relations are only aimed at modeling the important features of material

responses such as stress-strain behavior, material electrical polarizability, as well as cross-effects alike electrostriction.

Constitutive relations can be physically-based or phenomenological invariant-based. The former are mechanistically motivated relationships which come from a microstructurally-based statistical mechanics treatment of the interaction between matter and field. The latter are mathematically motivated relationships which come from a continuum mechanics treatment of the matter-field interaction.

Notwithstanding the approach used, constitutive equations require certain constants to be determined from a (preferably small) number of experiments. In the case of mechanistically motivated theories, these constants are physically-based parameters which are independent of the thermo-electro-mechanical state of the material and usually provide the constitutive model with strong predictive capabilities. Instead, in the case of mathematically motivated theories, these constants lack a direct physical connection to the underlying mechanism of matter-field interaction and, therefore, provide the constitutive model with weaker predictive capabilities and require a stability check after the constitutive equations have been fitted to the experimental data.

Though the use of physically-based approaches should be best advised because of their inherent stability and because of the enhanced predictive capabilities, since theories of this kind have not yet been developed for electrostrictive elastomers, a phenomenological invariant-based constitutive theory is described in the following [54, 57, 60, 62]. This theory should suffice in most cases whenever the resulting constitutive equations are made fit experimental data which are representative of the effective working state of the material in the practical system under investigation.

8.1 Requirements of Phenomenological Constitutive Theories

Systematic development of phenomenological constitutive equations requires the necessary satisfaction of certain physical and mathematical requirements, also called axioms, which are either verified experimentally or considered as obvious. No matter the physics under investigation, the basic axioms for the construction of consistent constitutive equations are thoroughly described in [75]. For electrostrictive elastomers with no memory effect (i.e. with no dependence on the specific history of the occurring thermo-electro-mechanical processes), the fundamental axioms to be satisfied are [54, 76]:

- *Axiom of Admissibility*: Constitutive equations must be consistent with balance laws and entropy inequalities.
- *Axiom of Causality*: Deformation, temperature and electric field of any material point belonging to an electroelastic body are self-evident and observable in any thermo-electro-mechanical behavior of the system and can be considered as

independent variables. Correspondingly, the density, the free-energy density, the entropy, the stress tensor, the electric polarization and the heat flux vector are considered as dependent variables.

- *Axiom of Neighborhood*: Constitutive relations are subjected to continuity requirements.
- *Axiom of Equipresence*: Constitutive relations should be considered as dependent on the same constitutive variables, until the contrary is deduced.
- *Axiom of Objectivity* (Principle of Material Frame Indifference): Constitutive relations need to be form-invariant under arbitrary rigid motions of the spatial frame of reference as well as under a constant shift of the origin of time. That is, the thermo-electro-mechanical properties of materials cannot depend on the motion of the observer.
- *Axiom of Time Reversal*: Entropy production must be nonnegative under time reversal.
- *Axiom of Material Invariance*: Constitutive relations must be form-invariant with respect to rigid body motions superimposed on the referential configuration as well as to microscopic time reversals representing specific material symmetry conditions.

In the following, these axioms are used to develop an appropriate constitutive theory for the isothermal electro-elastic behavior of conservative and isotropic electrostrictive elastomers. According to this restriction, the study of the system depicted in Fig. 4 simplifies since the energy balance equations [either Eq. (100) or Eq. (123)] can be neglected together with the constitutive equations for the heat flux vector (either $\boldsymbol{Q}$ or $\bar{\boldsymbol{Q}}$).

8.2 Constitutive Equations for the Isothermal Behavior of Conservative and Isotropic Electro-Elastic Solids

Owing to the axioms of causality, neighborhood and equipresence, the free-energy function, Ψ (and $\bar{\Psi}$), the entropy density, η (and $\bar{\eta}$), the stress tensor, $\boldsymbol{\sigma}$ (and $\boldsymbol{\Pi}$), and the electric polarization, $\boldsymbol{P}$ (and $\bar{\boldsymbol{P}}$), are taken as continuous functions of $\mathbf{F}$, $\boldsymbol{E}$ (or $\bar{\boldsymbol{E}}$) and T. In particular

$$\Psi = \Psi(\mathbf{F}, \boldsymbol{E}, T) = \bar{\Psi}(\mathbf{F}, \bar{\boldsymbol{E}}, T). \tag{143}$$

Then, according to Eq. (143), the CPI given by Eqs. (133) and (142) can respectively be rewritten as

$$\rho\left(\eta + \frac{\partial \Psi}{\partial T}\right)\dot{T} + \left(\boldsymbol{P} + \rho\frac{\partial \Psi}{\partial E}\right)\cdot\dot{\boldsymbol{E}} + \left[\rho\frac{\partial \Psi}{\partial \mathbf{F}}\mathbf{F}^T - \left(\boldsymbol{\sigma}^T - \boldsymbol{E}\otimes\boldsymbol{D} + \frac{1}{2}\varepsilon_0 E^2\mathbf{1}\right)\right] : \mathrm{grad}\boldsymbol{v} \leq 0, \tag{144}$$

$$\bar{\rho}\left(\bar{\eta}+\frac{\partial\bar{\Psi}}{\partial T}\right)\dot{\bar{T}}+\left(\bar{\boldsymbol{P}}+\bar{\rho}\frac{\partial\bar{\Psi}}{\partial\bar{\boldsymbol{E}}}\right)\cdot\dot{\bar{\boldsymbol{E}}}+\left(\bar{\rho}\frac{\partial\bar{\Psi}}{\partial\mathbf{F}}-\boldsymbol{\Pi}^{T}+\boldsymbol{\Pi}_{M}^{T}\right):\dot{\mathbf{F}}\leq 0, \tag{145}$$

which, for all admissible thermo-electro-mechanical processes (i.e. for the axioms of admissibility and time-reversal), yield the following identities in the Eulerian description

$$\eta(\mathbf{F},\boldsymbol{E},T)=-\frac{\partial\Psi}{\partial T}, \tag{146}$$

$$\boldsymbol{P}(\mathbf{F},\boldsymbol{E},T)=-\rho\frac{\partial\Psi}{\partial\boldsymbol{E}}, \tag{147}$$

$$\sigma^{T}(\mathbf{F},\boldsymbol{E},T)=\rho\frac{\partial\Psi}{\partial\mathbf{F}}\mathbf{F}^{T}+\boldsymbol{E}\otimes\boldsymbol{D}-\frac{1}{2}\varepsilon_{0}\boldsymbol{E}^{2}\mathbf{1}, \tag{148}$$

or equivalently

$$\sigma^{T}(\mathbf{F},\boldsymbol{E},T)=2\rho\frac{\partial\Psi}{\partial\mathbf{B}}\mathbf{B}+\boldsymbol{E}\otimes\boldsymbol{D}-\frac{1}{2}\varepsilon_{0}\boldsymbol{E}^{2}\mathbf{1}, \tag{149}$$

and the following identities in the Lagrangian description

$$\bar{\eta}=-\frac{\partial\bar{\Psi}}{\partial T}, \tag{150}$$

$$\bar{\boldsymbol{P}}=-\bar{\rho}\frac{\partial\bar{\Psi}}{\partial\bar{\boldsymbol{E}}}, \tag{151}$$

$$\boldsymbol{\Pi}^{T}=\bar{\rho}\frac{\partial\bar{\Psi}}{\partial\mathbf{F}}+\boldsymbol{\Pi}_{M}^{T}, \tag{152}$$

or equivalently

$$\boldsymbol{\Pi}^{T}=2\bar{\rho}\mathbf{F}\frac{\partial\bar{\Psi}}{\partial C}+\boldsymbol{\Pi}_{M}^{T}. \tag{153}$$

Equations (146–153) highlight that the complete solution of the isothermal electro-elastic problem only requires the definition of the free-energy function, Ψ (or identically $\bar{\Psi}$).

In this regard, for the axiom of objectivity [75, 76], the free-energy function of the considered materials needs to satisfy

$$\Psi(\mathbf{F},\boldsymbol{E},T)=\Psi(\mathbb{R}\mathbf{F},\mathbb{R}\boldsymbol{E},T), \tag{154}$$

for every proper orthogonal tensor $\mathbb{R}$ (i.e. $\mathbb{R}^{T}\mathbb{R}=1$ and $\det\mathbb{R}=+1$) representing an arbitrary rotation superimposed on the current configuration. Note that the objectivity requirement given by Eq. (154) is immediately guaranteed whenever the free-energy is an explicit function of the Green deformation tensor $\mathbf{C}$ and the reference electric field $\bar{E}$.

Besides, for isotropic electrostrictive elastomers, the axiom of material invariance also requires

$$\bar{\Psi}(\mathbf{F},\bar{\boldsymbol{E}},T)=\bar{\Psi}\big(\mathbf{F}\mathbb{R}^{T},\mathbb{R}\bar{\boldsymbol{E}},T\big), \tag{155}$$

for every proper orthogonal tensor $\mathbb{R}$ (i.e. $\mathbb{R}^{T}\mathbb{R}=\mathbf{1}$ and $\det\mathbb{R}=+1$) representing an arbitrary rotation superimposed on the reference configuration. Note that the isotropy requirement expressed by Eq. (155) is immediately guaranteed whenever the free-energy is an explicit function of the Finger deformation tensor $\mathbf{B}$ and the spatial electric field $\boldsymbol{E}$; in which case, owing to the representation theorem for invariants [77], Ψ (and $\bar{\Psi}$) admits the following irreducible representation [78–82]

$$\Psi=\Psi(I_1,I_2,I_3,I_4,I_5,I_6,T)=\bar{\Psi}(\bar{I}_1,\bar{I}_2,\bar{I}_3,\bar{I}_4,\bar{I}_5,\bar{I}_6,T) \tag{156}$$

in terms of the minimal set of invariants

$$I_1=\mathrm{tr}\mathbf{B}=\bar{I}_1=\mathrm{tr}\mathbf{C} \tag{157}$$

$$I_2=\frac{1}{2}\Big[(\mathrm{tr}\mathbf{B})^2-\mathrm{tr}\big(\mathbf{B}^2\big)\Big]=\bar{I}_2=\frac{1}{2}\Big[(\mathrm{tr}\mathbf{C})^2-\mathrm{tr}\big(\mathbf{C}^2\big)\Big] \tag{158}$$

$$I_3=\det\mathbf{B}=\bar{I}_3=\det\mathbf{C}=J^2 \tag{159}$$

$$I_4=\boldsymbol{E}\cdot\boldsymbol{E}=\bar{I}_4=\bar{\boldsymbol{E}}\cdot\big(\mathbf{C}^{-1}\bar{\boldsymbol{E}}\big) \tag{160}$$

$$I_5=\boldsymbol{E}\cdot(\mathbf{B}\boldsymbol{E})=\bar{I}_5=\bar{\boldsymbol{E}}\cdot\bar{\boldsymbol{E}} \tag{161}$$

$$I_6=\boldsymbol{E}\cdot\big(\mathbf{B}^2\boldsymbol{E}\big)=\bar{I}_6=\bar{\boldsymbol{E}}\cdot(\mathbf{C}\bar{\boldsymbol{E}}) \tag{162}$$

Of course this irreducible representation is also objective since Eq. (156), with Eqs. (157–162), immediately satisfies Eq. (154).

Then, by assuming a free-energy function in the form of Eqs. (156)–(162), and with the definition

$$a_i=\partial\Psi/\partial I_i=\partial\bar{\Psi}/\partial\bar{I}_i,\quad\text{for } i=1,\ldots,6 \tag{163}$$

the constitutive laws for the electric polarization and the stress tensor reduce to

$$\boldsymbol{P}=-2\rho\big(a_4\boldsymbol{E}+a_5\mathbf{B}\boldsymbol{E}+a_6\mathbf{B}^2\boldsymbol{E}\big) \tag{164}$$

$$\begin{aligned}\boldsymbol{\sigma}^{T}=&\,2\rho\big[(a_1+a_2I_1)\mathbf{B}-a_2\mathbf{B}^2+a_3I_3\mathbf{1}\big]\\&+\left[(\varepsilon_0-2\rho a_4)\boldsymbol{E}\otimes\boldsymbol{E}-\frac{1}{2}\varepsilon_0I_4\mathbf{1}+2\rho a_6(\mathbf{B}\boldsymbol{E})\otimes(\mathbf{B}\boldsymbol{E})\right]\end{aligned} \tag{165}$$

for the Eulerian description, and

$$\bar{\boldsymbol{P}}=-2\bar{\rho}\big(a_4\mathbf{C}^{-1}\bar{\boldsymbol{E}}+a_5\bar{\boldsymbol{E}}+a_6\mathbf{C}\bar{\boldsymbol{E}}\big), \tag{166}$$

$$\Pi^T = 2\bar{\rho}\mathbf{F}\Big[(a_1 + a_2 I_1)\mathbf{1} - a_2\mathbf{C} + a_3 I_3 \mathbf{C}^{-1} + \left(\frac{\varepsilon_0}{2\bar{\rho}}\sqrt{I_3} - a_4\right)\left(\mathbf{C}^{-1}\bar{\boldsymbol{E}}\right) \otimes \left(\mathbf{C}^{-1}\bar{\boldsymbol{E}}\right) - \frac{\varepsilon_0}{4\bar{\rho}}\sqrt{I_3} I_4 \mathbf{C}^{-1} + a_6 \bar{\boldsymbol{E}} \otimes \bar{\boldsymbol{E}}\Big] \tag{167}$$

for the Lagrangian description. Of course, Eqs. (164) and (166) identically satisfy Eq. (67), whereas Eqs. (165) and (167) identically satisfy Eq. (50).

8.3 Reduced Constitutive Equations for Dielectric Elastomers

Among the class of electrostrictive rubber-like solids, Dielectric Elastomers are a special type of elastic insulating material whose spatial electric polarization, $\boldsymbol{P}$, only depends on the spatial electric field, $\boldsymbol{E}$(and not on the deformation gradient $\mathbf{F}$). Consequently, in mathematical terms, Dielectric Elastomers can be characterized by a free-energy function with the form

$$\Psi = \Psi_{hyp}(I_1, I_2, I_3, T) - \frac{(\varepsilon_a - \varepsilon_0)}{2\bar{\rho}} I_4 \sqrt{I_3}, \tag{168}$$

where $\Psi_{hyp}(I_1, I_2, I_3, T)$ is some hyperelastic strain-energy function (i.e. a free-energy only depending on deformation), while the material parameter ε_a is the absolute permittivity of the dielectric elastomer. Note that the absolute permittivity may depend on the isothermal temperature of the material, i.e. $\varepsilon_a = \varepsilon_a(T)$ (see for instance in [73]).

Then, substitution of Eq. (168) into Eqs. (164–167), provides the following constitutive equations for general Dielectric Elastomers in isothermal conditions

$$\boldsymbol{P} = (\varepsilon_a - \varepsilon_0)\boldsymbol{E}, \tag{169}$$

$$\boldsymbol{\sigma}^T = 2\rho\left[(a_1' + a_2' I_1)\mathbf{B} - a_2'\mathbf{B}^2 + a_3' I_3 \mathbf{1}\right] + \left[\varepsilon_a \boldsymbol{E} \otimes \boldsymbol{E} - \frac{1}{2}\varepsilon_a \boldsymbol{E}^2 \mathbf{1}\right], \tag{170}$$

for the Eulerian description, whereas

$$\bar{\boldsymbol{P}} = J(\varepsilon_a - \varepsilon_0)\mathbf{C}^{-1}\bar{\boldsymbol{E}}, \tag{171}$$

$$\boldsymbol{\Pi}^T = 2\bar{\rho}\mathbf{F}\Big[(a_1' + a_2' I_1)\mathbf{1} - a_2'\mathbf{C} + a_3' I_3 \mathbf{C}^{-1} + \frac{\varepsilon_a}{2\bar{\rho}}\sqrt{I_3}\left(\mathbf{C}^{-1}\bar{\mathbf{E}}\right) \otimes \left(\mathbf{C}^{-1}\bar{\mathbf{E}}\right) - \frac{\varepsilon_a}{4\bar{\rho}}\sqrt{I_3}\bar{\mathbf{E}} \cdot \left(\mathbf{C}^{-1}\bar{\mathbf{E}}\right)\mathbf{C}^{-1}\Big] \tag{172}$$

for the Lagrangian description. In Eqs. (170)–(172), the quantities a_i' read as

$$a_i' = \partial \Psi_{hyp} / \partial I_i, \quad \text{for } i = 1, \ldots, 3. \tag{173}$$

Furthermore, for typical Dielectric Elastomers that are incompressible (i.e. $I_3 = J = 1$) and with deviatoric deformation response which is well described by a Yeoh's hyperelastic model [83], the strain-energy function can be assumed as

$$\Psi_{hyp}(I_1, I_2, I_3, T) = \Big[c_1(I_1 - 3) + c_2(I_1 - 3)^2 + c_3(I_1 - 3)^3 - p\big(\sqrt{I_3} - 1\big)\Big] \Big/ \bar{\rho}, \tag{174}$$

where c_1, c_2 and c_3 are material parameters only depending on the temperature of the material (see for instance in [73]), while p is an hydrostatic pressure which can only be determined from the equilibrium equations and the associated boundary conditions. Accordingly, use of Eq. (174) into Eqs. (170) and (172), respectively provides the following constitutive equations for the Cauchy's and the Nominal stress tensors of incompressible Dielectric Elastomers in isothermal conditions

$$\boldsymbol{\sigma}^T = -p\mathbf{1} + 2\Big[c_1 + 2c_2(I_1 - 3) + 3c_3(I_1 - 3)^2\Big]\mathbf{B} + \varepsilon_a \boldsymbol{E} \otimes \boldsymbol{E} - \frac{1}{2}\varepsilon_a \boldsymbol{E}^2 \mathbf{1}, \tag{175}$$

$$\begin{aligned} \boldsymbol{\Pi}^T = \mathbf{F}\Big[-p\mathbf{C}^{-1} + 2\Big[c_1 + 2c_2(I_1 - 3) + 3c_3(I_1 - 3)^2\Big]\mathbf{1} + \varepsilon_a\big(\mathbf{C}^{-1}\bar{\boldsymbol{E}}\big) \\ \otimes \big(\mathbf{C}^{-1}\bar{\boldsymbol{E}}\big) - \frac{\varepsilon_a}{2}\bar{\boldsymbol{E}} \cdot \big(\mathbf{C}^{-1}\bar{\boldsymbol{E}}\big)\mathbf{C}^{-1}\Big], \end{aligned} \tag{176}$$

which, by masking the hydrostatic electrically-induced terms into the unknown pressure p, identically read as

$$\sigma^T = -p\mathbf{1} + 2\Big[c_1 + 2c_2(I_1 - 3) + 3c_3(I_1 - 3)^2\Big]\mathbf{B} + \varepsilon_a \boldsymbol{E} \otimes \boldsymbol{E}, \tag{177}$$

$$\Pi^T = \mathbf{F}\Big[-p\mathbf{C}^{-1} + 2\Big[c_1 + 2c_2(I_1 - 3) + 3c_3(I_1 - 3)^2\Big]\mathbf{1} + \varepsilon_a\big(\mathbf{C}^{-1}\bar{\boldsymbol{E}}\big) \otimes \big(\mathbf{C}^{-1}\bar{\boldsymbol{E}}\big)\Big]. \tag{178}$$

9 Conclusions

This chapter presented a fully-coupled electromechanical continuum model for the study of the isothermal electro-elastic large deformations of general electrostrictive elastomers which are conservative and isotropic. The model comprises the standard equilibrium equations of both electrostatics and finite elasticity, along with their associated boundary conditions, and a specific electro-hyperelastic constitutive equation which accounts for all the coupling effects that arise from the interaction between polarizable elastic bodies and electrostatic fields. As a special case of this electromechanical model, a specific electro-elastic continuum model for Dielectric Elastomer materials has been also deduced. To enable for both

immediate understanding on the underlying electro-mechanical coupling and easy implementation in numerical multi-physics simulation environments, the model has been formulated both in the Eulerian (spatial) description and in the Lagrangian (material) description.

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Appendix A. Mathematical Operators

This appendix defines the mathematical operators that have been employed throughout this chapter.

Consider the 3×1 vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, and the 3×3 matrices, $\mathbf{A}$ and $\mathbf{B}$,

$$\boldsymbol{a} = [a_i] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \boldsymbol{b} = [b_i] = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \mathbf{A} = \left[A_{ij}\right] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \mathbf{B} = \left[B_{ij}\right] = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \tag{179}$$

The scalar product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, yielding a scalar quantity, is defined as

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{l=1}^{3} a_l b_l. \tag{180}$$

The tensor product (or the dyad) of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, yielding a 3×3 matrix, is defined as

$$\boldsymbol{a} \otimes \boldsymbol{b} = \left[(\boldsymbol{a} \otimes \boldsymbol{b})_{ij}\right] = \left[a_i b_j\right]. \tag{181}$$

The product between a matrix $\mathbf{A}$ and a vector $\boldsymbol{b}$, yielding a 3×1 vector, is defined as

$$\mathbf{A}\boldsymbol{b} = \left[(\mathbf{A}\boldsymbol{b})_i\right] = \left[\sum_{l=1}^{3} a_{il} b_l\right]. \tag{182}$$

The scalar product between a matrix $\mathbf{A}$ and a vector $\boldsymbol{b}$, yielding a 3×1 vector, is defined as

$$\mathbf{A}\cdot\boldsymbol{b}=\left[(\mathbf{A}\cdot\boldsymbol{b})_i\right]=\left[\sum_{l=1}^{3}a_{li}b_l\right]. \tag{183}$$

Note that Eqs. (182) and (183) differ in the matrix index used for the summation.

The product between matrices $\mathbf{A}$ and $\mathbf{B}$, yielding a 3×3 matrix, is defined as

$$\mathbf{AB}=\left[(\mathbf{AB})_{ij}\right]=\left[\sum_{l=1}^{3}A_{il}B_{lj}\right]. \tag{184}$$

The double contraction between matrices $\mathbf{A}$ and $\mathbf{B}$, yielding a scalar, is defined as

$$\mathbf{A}:\mathbf{B}=\sum_{m=1}^{3}\sum_{l=1}^{3}A_{lm}B_{lm}. \tag{185}$$

Given a third-order tensor $\mathcal{A}=\left[\mathcal{A}_{ijk}\right]$, the double contraction of $\mathcal{A}$ and a matrix $\mathbf{B}$, yielding a 3×1 vector, is defined as

$$\mathcal{A}:\mathbf{B}=\left[(\mathcal{A}:\mathbf{B})_i\right]=\sum_{l=1}^{3}\sum_{m=1}^{3}\mathcal{A}_{ilm}B_{lm}. \tag{186}$$

Given the spatial position coordinate $\boldsymbol{x}=[\,x_1\quad x_2\quad x_3\,]^T$, the spatial gradient (grad) of a scalar quantity $\phi=\phi(x,t)$, t being the time variable, is defined as

$$\mathrm{grad}\phi=\left[(\mathrm{grad}\phi)_i\right]=[\partial\phi/\partial x_i]; \tag{187}$$

whereas the spatial gradient (grad), divergence (div) and rotation (rot) of a vector $\boldsymbol{a}=\boldsymbol{a}(\boldsymbol{x},t)$, yielding 3×3 matrix, a scalar and a 3×1 vector respectively, are defined as

$$\mathrm{grad}\boldsymbol{a}=\left[(\mathrm{grad}\boldsymbol{a})_{ij}\right]=\left[\partial a_i/\partial x_j\right], \tag{188}$$

$$\mathrm{div}\boldsymbol{a}=\sum_{l=1}^{3}\partial a_l/\partial x_l, \tag{189}$$

$$\mathrm{rot}\boldsymbol{a}=\left[(\mathrm{rot}\boldsymbol{a})_i\right]=\left[\sum_{l=1}^{3}\sum_{m=1}^{3}\varepsilon_{ilm}\partial a_m/\partial x_l\right], \tag{190}$$

where ε_{ijk} is the permutation symbol such that

$$\varepsilon_{ijk}=\begin{cases}1, & \text{for even permutation of } (i,j,k),\ (\text{i.e. } 123,\ 231,\ 312)\\ -1, & \text{for odd permutation of } (i,j,k),\ (\text{i.e. } 132,\ 213,\ 321)\\ 0, & \text{if there is a repeated index}\end{cases}. \tag{191}$$

Besides, the spatial divergence (div) of a matrix $\mathbf{A} = \mathbf{A}(\boldsymbol{x}, t)$ is defined as

$$\mathrm{div}\mathbf{A} = \left[(\mathrm{div}\mathbf{A})_i\right] = \sum_{l=1}^{3} \partial A_{li}/\partial x_l. \tag{192}$$

Given the material position coordinate $\boldsymbol{X} = [\,X_1 \quad X_2 \quad X_3\,]^T$, the material gradient (Grad) of a scalar quantity $\phi = \phi(\boldsymbol{X}, t)$ is defined as

$$\mathrm{Grad}\phi = \left[(\mathrm{Grad}\phi)_i\right] = [\partial\phi/\partial X_i]; \tag{193}$$

whereas the material gradient (Grad), divergence (Div) and rotation (Rot) of a vector $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{X}, t)$ are respectively defined as

$$\mathrm{Grad}\boldsymbol{a} = \left[(\mathrm{Grad}\boldsymbol{a})_{ij}\right] = \left[\partial a_i/\partial X_j\right], \tag{194}$$

$$\mathrm{Div}\boldsymbol{a} = \sum_{l=1}^{3} \partial a_l/\partial X_l, \tag{195}$$

$$\mathrm{Rot}\boldsymbol{a} = \left[(\mathrm{Rot}\boldsymbol{a})_i\right] = \left[\sum_{l=1}^{3}\sum_{m=1}^{3} \varepsilon_{ilm}\partial a_m/\partial X_l\right]. \tag{196}$$

Besides, the material divergence (Div) of a matrix $\mathbf{A} = \mathbf{A}(\boldsymbol{X}, t)$ is defined as

$$\mathrm{Div}\mathbf{A} = \left[(\mathrm{div}\mathbf{A})_i\right] = \sum_{l=1}^{3} \partial A_{li}/\partial X_l. \tag{197}$$

With regard to the divergence of the products between vector $\boldsymbol{a}$ and either matrix $\mathbf{A}$ or scalar ϕ

$$\mathrm{div}(\mathbf{A}\boldsymbol{a}) = \mathrm{div}\mathbf{A} \cdot \boldsymbol{a} + \mathbf{A}^T : \mathrm{grad}\boldsymbol{a}, \tag{198}$$

$$\mathrm{div}(\phi\boldsymbol{a}) = \phi\mathrm{div}\boldsymbol{a} + \mathrm{grad}\phi \cdot \boldsymbol{a}, \tag{199}$$

$$\mathrm{Div}(\mathbf{A}\boldsymbol{a}) = \mathrm{Div}\mathbf{A} \cdot \boldsymbol{a} + \mathbf{A}^T : \mathrm{Grad}\boldsymbol{a}, \tag{200}$$

$$\mathrm{Div}(\phi\boldsymbol{a}) = \phi\mathrm{Div}\boldsymbol{a} + \mathrm{Grad}\phi \cdot \boldsymbol{a}. \tag{201}$$

Appendix B. Fundamental Mathematical Theorems

This appendix summarizes the fundamental mathematical theorems that have been employed throughout this chapter.

For any open surface $S(t)$ with bounding closed curve $L(t)$, the Stokes' theorem states

$$\int_{S(t)} \mathrm{rot}\boldsymbol{a} \cdot \boldsymbol{n}\mathrm{d}s = \int_{L(t)} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{x}. \tag{202}$$

where ds is the infinitesimal surface belonging to $S(t)$ and with unit normal $\boldsymbol{n}$, whereas d$\boldsymbol{x}$ is the infinitesimal line element belonging to $L(t)$.

For any volume $V(t)$ with bounding closed surface $\partial V(t)$, the Gauss' divergence theorem states

$$\int_{V(t)} \mathrm{div}\boldsymbol{a}\mathrm{d}v = \int_{\partial V} \boldsymbol{a} \cdot \boldsymbol{n}\mathrm{d}s, \tag{203}$$

$$\int_{V(t)} \mathrm{div}\mathbf{A}\mathrm{d}v = \int_{\partial V} \mathbf{A} \cdot \boldsymbol{n}\mathrm{d}s, \tag{204}$$

where dv is the infinitesimal volume belonging to $V(t)$, whereas ds is the infinitesimal surface belonging to $\partial V(t)$ and with unit normal $\boldsymbol{n}$. In the presence of a discontinuity surface $\gamma(t)$, within volume $V(t)$, across which some vector $\boldsymbol{a}$ and tensor $\mathbf{A}$ admit non-continuous values, the Gauss' divergence theorem states

$$\int_{V(t)-\gamma(t)} \mathrm{div}\boldsymbol{a}\mathrm{d}v + \int_{\partial\gamma(t)} [\![\boldsymbol{a}]\!] \cdot \boldsymbol{n}\mathrm{d}s = \int_{\partial V-\gamma(t)} \boldsymbol{a} \cdot \boldsymbol{n}\mathrm{d}s, \tag{205}$$

$$\int_{V(t)-\gamma(t)} \mathrm{div}\mathbf{A}\mathrm{d}v + \int_{\gamma(t)} [\![\boldsymbol{A}]\!] \cdot \boldsymbol{n}\mathrm{d}s = \int_{\partial V-\gamma(t)} \mathbf{A} \cdot \boldsymbol{n}\mathrm{d}s, \tag{206}$$

where $[\![\boldsymbol{a}]\!] \equiv \boldsymbol{a}^+ - \boldsymbol{a}^-$ and $[[\mathbf{A}]] \equiv \mathbf{A}^+ - \mathbf{A}^-$ indicate the jumps of vector $\boldsymbol{a}$ and matrix $\mathbf{A}$ from the positive (+) side to the negative (−) side of the discontinuity.

For any given scalar quantity ϕ ($\boldsymbol{x}$,t), the Reynolds' transport theorem states

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \phi\mathrm{d}v = \int_{V(t)} \left[\dot{\phi} + \phi\mathrm{div}\boldsymbol{v}\right]\mathrm{d}v = \int_{V(t)} \left[\frac{\partial\phi}{\partial t} + \mathrm{div}(\phi\boldsymbol{v})\right]\mathrm{d}v. \tag{207}$$

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