4 Discrete Sliding Mode Control

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Abstract This chapter introduces discrete sliding mode controllers, including a typical discrete sliding mode controller and a kind of discrete sliding mode controller based on disturbance observer.

Keywords discrete sliding mode control, disturbance observer, stability analysis

4.1 Discrete Sliding Mode Controller Design and Analysis

4.1.1 System Description

Consider the following uncertain system

$$\boldsymbol{x}(k+1) = (\boldsymbol{A} + \Delta\boldsymbol{A})\boldsymbol{x}(k) + \boldsymbol{B}u(k) + \boldsymbol{f}(k) \tag{4.1}$$

where x is system state, $\boldsymbol{A} \in \mathbf{R}^{2\times 2}$ and $\Delta\boldsymbol{A} \in \mathbf{R}^{2\times 2}$ are matrix, $\boldsymbol{B} \in \mathbf{R}^{2\times 1}$ is a vector, $u \in \mathbf{R}$ is control input, $f \in \mathbf{R}^{2\times 1}$ is a vector, $\boldsymbol{B} = [0 \ \ b]^{\mathrm{T}}$, $b > 0$.

The uncertain term $\Delta\boldsymbol{A}$ and the perturbation term $\boldsymbol{f}(\boldsymbol{k})$ satisfy the classical matching conditions, i.e.

$$\Delta\boldsymbol{A} = \boldsymbol{B}\tilde{A}, \quad \boldsymbol{f} = \boldsymbol{B}\tilde{\boldsymbol{f}} \tag{4.2}$$

J. Liu et al., *Advanced Sliding Mode Control for Mechanical Systems*

Then, the system (4.1) can be described as

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}[\boldsymbol{u}(k) + \boldsymbol{d}(k)] \tag{4.3}$$

where $d(k) = \tilde{\boldsymbol{A}}\boldsymbol{x}(k) + \tilde{\boldsymbol{f}}(k)$.

4.1.2 Controller Design and Analysis

The controller is designed as

$$u(k) = (\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B})^{-1}(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}_{\mathrm{d}}(k+1) - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}(k) + qs(k) - \eta\,\mathrm{sgn}(s(k))) \tag{4.4}$$

where η , q , c are positive constant values, c must be Hurwitz, $\boldsymbol{C} = [c \quad 1]^{\mathrm{T}}$, $0 < q < 1$, $|d| < D$, $\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D < \eta$.

Stability analysis is given as follows: If the ideal position signal is $x_{\mathrm{d}}(k)$ then the tracking error is $e(k) = x(k) - x_{\mathrm{d}}(k)$, then

$$\begin{aligned}
&s(k+1) \\
&= \boldsymbol{C}^{\mathrm{T}}e(k+1) \\
&= \boldsymbol{C}^{\mathrm{T}}x(k+1) - \boldsymbol{C}^{\mathrm{T}}x_{\mathrm{d}}(k+1) \\
&= \boldsymbol{C}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}u(k) + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}_{\mathrm{d}}(k+1) \\
&= \boldsymbol{C}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}_{\mathrm{d}}(k+1) - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}(k) + qs(k) - \eta\,\mathrm{sgn}(s(k)) \\
&\quad + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}_{\mathrm{d}}(k+1) \\
&= qs(k) - \eta\,\mathrm{sgn}(s(k)) + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k)
\end{aligned} \tag{4.5}$$

Since $|\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k)| < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D < \eta$, then $-\eta < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) < \eta$, $-\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D$, and then we have $\eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) > 0$, $-\eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) < 0$, $\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) > 0$ and $-\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) < 0$.

Four conditions are analyzed as follows:

(1) When $s(k) \geqslant \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \eta$, we have

Consider $s(k) > 0$, $0 < q < 1$, $-\eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) < 0$, $\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) > 0$, then

$$s(k+1) - s(k) = (q-1)s(k) - \eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) < 0$$

$$\begin{aligned}
s(k+1) + s(k) &= (q+1)s(k) - \eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) \geqslant (q+1)(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \eta) - \eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) \\
&= q(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \eta) + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) > 0
\end{aligned}$$

Then,

$$s(k+1)^2 < s(k)^2$$

(2) When $0 < s(k) < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \eta$, we have

$$\begin{aligned} s(k+1) &= qs(k) - \eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) < q(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \eta) - \eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) \\ &< q(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \eta) < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \eta \end{aligned}$$

$$s(k+1) = qs(k) - \eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) > -\eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) > -\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D - \eta$$

Then,

$$|s(k+1)| < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \eta$$

(3) When $-\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D - \eta < s(k) < 0$, we have

$$\begin{aligned} s(k+1) &= qs(k) + \eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) > s(k) + \eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) \\ &> -\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D - \eta + \eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) > -\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D - \eta \end{aligned}$$

$$s(k+1) = qs(k) + \eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) < \eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \eta$$

Then,

$$|s(k+1)| < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \eta$$

(4) When $s(k) \leqslant -\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D - \eta < 0$, we have

$$s(k+1) - s(k) = (q-1)s(k) + \eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) > 0$$

$$\begin{aligned} s(k+1) + s(k) &= (q+1)s(k) + \eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) < s(k) + \eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) \\ &\leqslant -\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D - \eta + \eta + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) = -\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) < 0 \end{aligned}$$

Then,

$$s(k+1)^2 < s(k)^2$$

From the above analysis we conclude as follows:

$$\text{When } |s(k)| \geqslant \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \eta,\ \ s(k+1)^2 < s(k)^2 \tag{4.6}$$

$$\text{When } |s(k)| < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \eta,\ \ |s(k+1)| < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \eta \tag{4.7}$$

From Eqs. (4.6) and (4.7), since $\eta > \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D$, $s(k)$ converge to $\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}D + \eta$. Therefore, to increase convergence performance, a disturbance observer is required to be designed.

4.1.3 Simulation Example

Consider the plant

$$G(s) = \frac{133}{s^2 + 25s}$$

The sampling time is chosen as 0.001 s. Considering disturbance, the discrete system can be written as

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}(u(k) + d(k))$$

where $\boldsymbol{A} = \begin{bmatrix} 1 & 0.001 \\ 0 & 0.9753 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 0.0001 \\ 0.1314 \end{bmatrix}$, and $d(k)$ is disturbance.

Using the control law Eq. (4.4), and assuming the disturbance as $d(k) = 1.5\sin t$, choosing the ideal position signal as $x_{\mathrm{d}}(k) = \sin t$, and designing $\boldsymbol{C}^{\mathrm{T}} = [15 \quad 1]$, $q = 0.80$, $D = 1.5$. The initial state is $[0.15 \quad 0]$. The term $x_{\mathrm{d}}(k+1)$ can be received by extrapolation method. The simulation results are shown in Fig. 4.1 – Fig. 4.3.

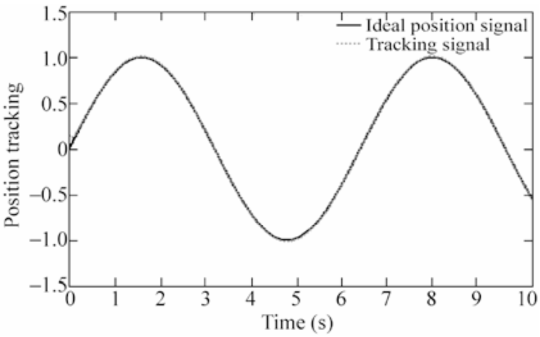

Figure 4.1 Sine signal tracking

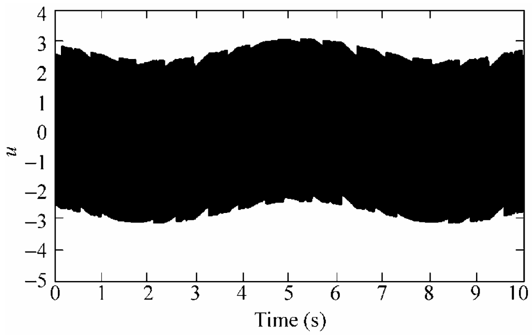

Figure 4.2 Control input

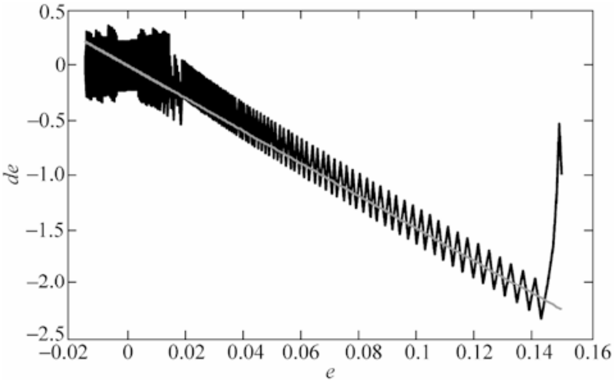

Figure 4.3 Phase trajectory

Simulation programs: chap4_1.m

```
%VSS controller based on decoupled disturbance compensator
clear all;
close all;

ts=0.001;
a=25;
b=133;
sys=tf(b,[1,a,0]);
dsys=c2d(sys,ts,'z');
[num,den]=tfdata(dsys,'v');

A=[0,1;0,-a];
B=[0;b];
C=[1,0];
D=0;
%Change transfer function to discrete position equation
[A1,B1,C1,D1]=c2dm(A,B,C,D,ts,'z');
A=A1;
b=B1;
c=15;
Ce=[c,1];
q=0.80;            %0<q<1

d_up=1.5;
eq=Ce*b*d_up+0.10;  %eq>abs(Ce*b*m/g);0<eq/fai<q<1

x_1=[0.15;0];
s_1=0;
u_1=0;
d_1=0;ed_1=0;
r_1=0;r_2=0;dr_1=0;

for k=1:1:10000
time(k)=k*ts;
```

```
d(k)=1.5*sin(k*ts);

x=A*x_1+b*(u_1+d(k));

r(k)=sin(k*ts);
%Using Waitui method
   dr(k)=(r(k)-r_1)/ts;
   dr_1=(r_1-r_2)/ts;
   r1(k)=2*r(k)-r_1;
   dr1(k)=2*dr(k)-dr_1;

   xd=[r(k);dr(k)];
   xd1=[r1(k);dr1(k)];

    e(k)=x(1)-r(k);
    de(k)=x(2)-dr(k);
    s(k)=c*e(k)+de(k);

    u(k)=inv(Ce*b)*(Ce*xd1-Ce*A*x+q*s(k)-eq*sign(s(k)));

    r_2=r_1;r_1=r(k);
    dr_1=dr(k);

    x_1=x;
    s_1=s(k);

    x1(k)=x(1);
    x2(k)=x(2);
    u_1=u(k);
end
figure(1);
plot(time,r,'k',time,x1,'r:','linewidth',2);
xlabel('time(s)');ylabel('Position tracking');
legend('Ideal position signal','tracking signal');
figure(2);
plot(time,u,'k','linewidth',2);
xlabel('time(s)');ylabel('u');
figure(3);
plot(e,de,'k',e,-Ce(1)*e,'r','linewidth',2);
xlabel('e');ylabel('de');
```

4.2 Discrete Sliding Mode Control Based on Disturbance Observer

4.2.1 System Description

Consider the uncertain discrete system as follow:

$$\boldsymbol{x}(k+1)=\boldsymbol{A}\boldsymbol{x}(k)+\boldsymbol{B}(u(k)+d(k)) \tag{4.8}$$

where x is system state, $\boldsymbol{A} \in \mathbf{R}^{2\times 2}$ is a matrix, $\boldsymbol{B} \in \mathbf{R}^{2\times 1}$ is a vector, $u \in \mathbf{R}^{2\times 1}$ is control input, $\boldsymbol{B} = [0 \quad b]^{\mathrm{T}}, b > 0, d \in \mathbf{R}$ is the disturbance.

Let the desired input command be $\boldsymbol{x}_{\mathrm{d}}(k)$, and the tracking error be $\boldsymbol{e}(k) = \boldsymbol{x}(k) - \boldsymbol{x}_{\mathrm{d}}(k)$. The sliding variable is designed as

$$s(k) = \boldsymbol{C}\boldsymbol{e}(k) \tag{4.9}$$

where $\boldsymbol{C} = [c \quad 1],\ c > 0$.

4.2.2 Discrete Sliding Mode Control Based on Disturbance Observer

In this section, we introduce a typical sliding mode controller base on disturbance observer, which was proposed by Eun et al[1].

For Eq. (4.8), the sliding mode controller consists of the sliding mode control element and the disturbance compensation. The controller proposed by Eun et al. as[1]

$$u(k) = u_{\mathrm{s}}(k) + u_{\mathrm{c}}(k) \tag{4.10}$$

where

$$u_{\mathrm{s}}(k) = (\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B})^{-1}(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}_{\mathrm{d}}(k+1) - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}(k) + qs(k) - \eta\,\mathrm{sgn}(s(k)))$$

$$u_{\mathrm{c}}(k) = -\hat{d}(k)$$

The disturbance observer was proposed by Eun et al. as:

$$\hat{d}(k) = \hat{d}(k-1) + (\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B})^{-1} g(s(k) - qs(k-1) + \eta\,\mathrm{sgn}(s(k-1))) \tag{4.11}$$

where $\tilde{d}(k) = d(k) - \hat{d}(k)$, η, q, and g are positive constants.

From Eqs. (4.8) and (4.10), we have

$$\begin{aligned}
s(k+1) &= \boldsymbol{C}^{\mathrm{T}}\boldsymbol{e}(k+1) = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}(k+1) - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}_{\mathrm{d}}(k+1) \\
&= \boldsymbol{C}^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}u(k) + \boldsymbol{B}d(k)) - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}_{\mathrm{d}}(k+1) \\
&= \boldsymbol{C}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}(k) + (\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}_{\mathrm{d}}(k+1) - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}(k) + qs(k) \\
&\quad - \eta\,\mathrm{sgn}(s(k))) - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\hat{d}(k) + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}d(k) - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}_{\mathrm{d}}(k+1) \\
&= qs(k) - \eta\,\mathrm{sgn}(s(k)) + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\tilde{d}(k)
\end{aligned} \tag{4.12}$$

From Eqs. (4.11) and (4.12), we can get

$$\begin{aligned}
\tilde{d}(k+1) &= d(k+1) - \hat{d}(k+1) \\
&= d(k+1) - \hat{d}(k) - (\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B})^{-1} g(s(k+1) - qs(k) + \eta\,\mathrm{sgn}(s(k))) \\
&= d(k+1) - d(k) + \tilde{d}(k) - (\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B})^{-1} g(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B})\tilde{d}(k) \\
&= d(k+1) - d(k) + (1-g)\tilde{d}(k)
\end{aligned} \tag{4.13}$$

4.2.3 Convergent Analysis of Disturbance Observer

Theorem 1 proposed by Eun et al. as follows.

Theorem 1[1]: For the disturbance observer Eq. (4.11) there exists a positive constant *m*, if $|d(k+1)-d(k)|<m$ then k_0 exists, and when $k>k_0$ then $\tilde{d}(k)<m/g$ is satisfied where $0<g<1$.

Proof:

$\tilde{d}(k)$ can be decomposed as

$$\tilde{d}(k)=\tilde{d}_1(k)+\tilde{d}_2(k)$$

Let $\tilde{d}_1(0)=0,$ we can get $\tilde{d}_2(0)=\tilde{d}(0),$ and because

$$\tilde{d}(k+1)=\tilde{d}_1(k+1)+\tilde{d}_2(k+1)$$

We let

$$\tilde{d}_1(k+1)=(1-g)\tilde{d}_1(k)+d(k+1)-d(k) \tag{4.14}$$

From Eq. (4.13), we have

$$\tilde{d}_2(k+1)=(1-g)\tilde{d}_2(k) \tag{4.15}$$

Inductive method is used to prove the theorem. Firstly, we prove $\tilde{d}_1(k)<m/g.$

(1) When $k=0,$ we get: $\tilde{d}_1(0)=0<m/g.$

(2) Suppose $|\tilde{d}_1(k)|<m/g,$ and from Eq. (4.14)and $0<g<1,$ we can get when $k+1,$

$$|\tilde{d}_1(k+1)|\leqslant(1-g)|\tilde{d}_1(k)|+|d(k+1)-d(k)|<(1-g)\frac{m}{g}+m=\frac{m}{g}$$

is satisfied. From the above two equations, we have

$$|\tilde{d}_1(k)|<m/g,\quad k\geqslant 0$$

From Eq. (4.15) and $0<1-g<1,$ we get

$$\tilde{d}_2(k+1)=(1-g)\tilde{d}_2(k)\leqslant(1-g)|\tilde{d}_2(k)|<|\tilde{d}_2(k)|$$

Therefore, $\tilde{d}_2(k)$ is decreasing. And if there exists $k_0',$ when $k>k_0',$ then $\tilde{d}_2(k)$ is arbitrary small.

From the analysis above, we find that there exists $k_0,$ when $k>k_0,$ such that

$$|\tilde{d}(k)|=|\tilde{d}_1(k)+\tilde{d}_2(k)|\leqslant|\tilde{d}_1(k)|+|\tilde{d}_2(k)|<\frac{m}{g}$$

4.2.4 Stability Analysis

Theorem 2 proposed by Eun et al. as follows.

Theorem 2[1]: For controller Eq. (4.10), the system is stable if the following conditions are satisfied:

(1) $0 < q < 1,\ \ 0 < g < 1$;

(2) There exists a positive constant m, $|d(k+1) - d(k)| < m$;

(3) $0 < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\dfrac{m}{g} < \eta$.

Proof: Let $v(k) = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\tilde{d}(k)$, we have

$$|v(k)| < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g} < \eta,\ \text{i.e.} -\eta < v(k) < \eta,\ \ -\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g} < v(k) < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g}$$

Equation (4.12)can be written as

$$s(k+1) = qs(k) - \eta\,\mathrm{sgn}(s(k)) + v(k)$$

The following four cases are discussed:

(1) When $s(k) \geqslant \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\dfrac{m}{g} + \eta > 0$, we have

$$s(k+1) - s(k) = (q-1)s(k) - \eta + v(k) < 0$$

$$s(k+1) + s(k) = (q+1)s(k) - \eta + v(k) \geqslant (q+1)\left(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g} + \eta\right) - \eta + v(k)$$

$$= q\left(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g} + \eta\right) + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g} + v(k) > 0$$

Therefore,

$$s(k+1)^2 < s(k)^2$$

(2) When $s(k) \leqslant -\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\dfrac{m}{g} - \eta < 0$, we have

$$s(k+1) - s(k) = (q-1)s(k) + \eta + v(k) > 0$$

$$s(k+1) + s(k) = (q+1)s(k) + \eta + v(k) < s(k) + \eta + v(k)$$

$$\leqslant -\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g} - \eta + \eta + v(k) = -\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g} + v(k) < 0$$

Therefore,

$$s(k+1)^2 < s(k)^2$$

(3) When $0 < s(k) < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\dfrac{m}{g}+\eta,$ we have

$$s(k+1) = qs(k) - \eta + v(k) < q\left(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g}+\eta\right) - \eta + v(k)$$
$$< q\left(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g}+\eta\right) < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g}+\eta$$

$$s(k+1) = qs(k) - \eta + v(k) > -\eta + v(k) > -\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g} - \eta$$

Therefore,

$$|s(k+1)| < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g}+\eta$$

(4) When $-\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\dfrac{m}{g}-\eta < s(k) < 0,$ we have

$$s(k+1) = qs(k) + \eta + v(k) > s(k) + \eta + v(k)$$
$$> -\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g} - \eta + \eta + v(k) > -\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g} - \eta$$

$$s(k+1) = qs(k) + \eta + v(k) < \eta + v(k) < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g}+\eta$$

Therefore,

$$|s(k+1)| < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g}+\eta$$

From the above analysis the following conclusions can be obtained.

$$\text{When } |s(k)| \geqslant \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g}+\eta, \quad s(k+1)^2 < s(k)^2 \tag{4.16}$$

$$\text{When } |s(k)| < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g}+\eta, \quad |s(k+1)| < \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g}+\eta \tag{4.17}$$

The disturbance $d(t)$ is supposed to be continuous. If the sampling time is sufficiently small, then $|d(k+1)-d(k)| < m$ can be guaranteed and m is sufficiently small. If m and g are selected such that $\frac{m}{g} \ll 1$, and because $\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g} < \eta$, η is selected sufficiently small and make $\boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g} + \eta \ll 1$. Therefore, the convergence of $s(k+1)$ can be realized.

4.2.5 Simulation Example

Consider the plant as follows:

$$G(s) = \frac{133}{s^2 + 25s}$$

The sampling time is 0.001 s. The discretization equation of the plant is after factoring disturbance is

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\boldsymbol{x}(k) + \boldsymbol{B}(u(k) + d(k))$$

where $\boldsymbol{A} = \begin{bmatrix} 1 & 0.001 \\ 0 & 0.9753 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 0.0001 \\ 0.1314 \end{bmatrix}$, $d(k)$ is disturbance, and $d(k) = 1.5\sin(2\pi t)$.

Let the desired command be $x_{\mathrm{d}}(k) = \sin t$, and use the control law (4.10). Therefore, according to linear extrapolation method, we get $x_{\mathrm{d}}(k+1) = 2x_{\mathrm{d}}(k) - x_{\mathrm{d}}(k-1)$. The controller parameters are $\boldsymbol{C}^{\mathrm{T}} = [15 \quad 1]$, $q = 0.80$, $g = 0.95$, $m = 0.01$, $\eta = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}\frac{m}{g} + 0.001$. The initial state vector is $[0.5 \quad 0]$. The simulation results are shown in Fig. 4.4 – Fig. 4.7.

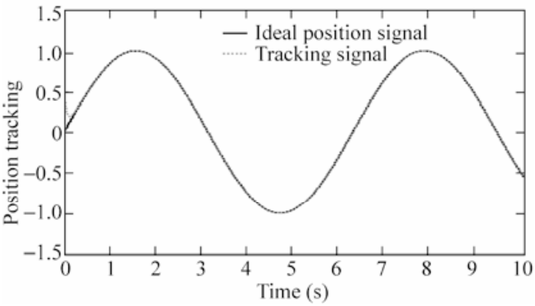

Figure 4.4 Sine tracking

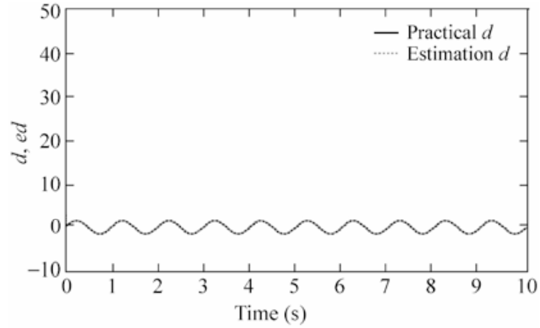

Figure 4.5 Observation of disturbance

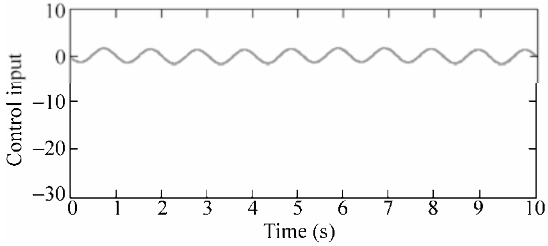

Figure 4.6 Control input

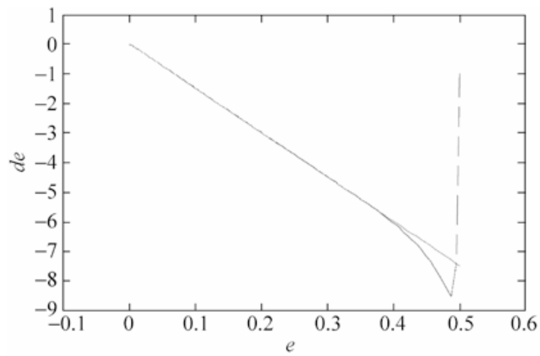

Figure 4.7 Phase trajectory

Simulation program:

chap4_2.m

```
%SMC controller based on decoupled disturbance compensator
clear all;
close all;

ts=0.001;
a=25;b=133;
sys=tf(b,[1,a,0]);
```

```
dsys=c2d(sys,ts,'z');
[num,den]=tfdata(dsys,'v');

A0=[0,1;0,-a];
B0=[0;b];
C0=[1,0];
D0=0;
%Change transfer function to discrete position xiteuation
[A1,B1,C1,D1]=c2dm(A0,B0,C0,D0,ts,'z');
A=A1;
B=B1;
c=15;
C=[c,1];
q=0.80;             %0<q<1
g=0.95;

m=0.010;            %m>abs(d(k+1)-d(k))

xite=C*B*m/g+0.0010;  %xite>abs(C*B*m/g);0<xite/fai<q<1

x_1=[0.5;0];
s_1=0;
u_1=0;
d_1=0;ed_1=0;
xd_1=0;xd_2=0;dxd_1=0;

for k=1:1:10000
time(k)=k*ts;

d(k)=1.5*sin(2*pi*k*ts);
d_1=d(k);

x=A*x_1+B*(u_1+d(k));

xd(k)=sin(k*ts);

   dxd(k)=(xd(k)-xd_1)/ts;
   dxd_1=(xd_1-xd_2)/ts;
   xd1(k)=2*xd(k)-xd_1; %Using Waitui method
   dxd1(k)=2*dxd(k)-dxd_1;
   Xd=[xd(k);dxd(k)];
   Xd1=[xd1(k);dxd1(k)];

    e(k)=x(1)-Xd(1);
    de(k)=x(2)-Xd(2);
    s(k)=C*(x-Xd);

   ed(k)=ed_1+inv(C*B)*g*(s(k)-q*s_1+xite*sign(s_1));

   u(k)=-ed(k)+inv(C*B)*(C*Xd1-C*A*x+q*s(k)-xite*sign(s(k)));
```

```
    xd_2=xd_1;xd_1=xd(k);
    dxd_1=dxd(k);

     ed_1=ed(k);
     x_1=x;
     s_1=s(k);

     x1(k)=x(1);
     x2(k)=x(2);
     u_1=u(k);
end
figure(1);
plot(time,xd,'k',time,x1,'r:','linewidth',2);
xlabel('time(s)');ylabel('Position tracking');
legend('Ideal position signal','tracking signal');
figure(2);
plot(time,d,'k',time,ed,'r:','linewidth',2);
xlabel('time(s)');ylabel('d,ed');
legend('Practical d','Estimation d');
figure(3);
plot(time,u,'r','linewidth',2);
xlabel('time(s)');ylabel('Control input');
figure(4);
plot(e,de,'b',e,-C(1)*e,'r');
xlabel('e');ylabel('de');
```

Reference

[1] Yongsoon Eun, Jung-Ho Kim, Kwangsoo Kim, Dong-ll Cho, Discrete-time variable structure controller with a decoupled disturbance compensator and its application to a CNC servomechanism, IEEE Transactions on Control Systems Technology, 1999, 7(4): 414 – 423