Chapter 7
The RAM Approach in Aerodynamics

7.1 Derivation of a Through-Flow Model Problem for Fluid Flow in an Axial Compressor

First, in Sect. 7.1.1, we again consider the simple case examined in Veuillot's thesis [140] devoted to turbomachinery fluid flow, simulated in Sect. 6.3.1; then, in Sect. 7.1.2, the more sophisticated G–Z RAM Approach [141].

It is obvious that the following asymptotic theory of axial flow through a turbine, which is likely be of considerable interest to specialists, is a fascinating application of a complicated engineering problem using the RAM Approach with the basic large parameter being the number of turbine blades per rotor.

7.1.1 The Veuillot Approach

We starting from the system of non-dimensional equations (6.18a)–(6.18c), for u*, w*, and $\Gamma^* = rv^*$, and also the relations (6.17a)–(6.17c) relying, u*, w, and v*, with the functions ψ^* and Θ.

First, we expand the functions u*, w*, Γ^*, and ψ^*, relative to our main small parameter ε:

$$(u^*, w^*, \Gamma^*, \psi^*) = (u^*{}_o, w^*{}_o, \Gamma^*_o, \psi^*{}_o) + \varepsilon\,(u^*_1, w^*_1, \Gamma^*_1, \psi^*_1) + \ldots \tag{7.1}$$

and at zeroth-order we derive the following leading-order approximate equations for the functions $u^*_o = (1/r\Delta)\,\partial\psi^*_o/\partial z, w^*_o = -(1/r\Delta)\,\partial\psi^*_o/\partial r$, $\Gamma^*_o = r^2(u^*_o\partial\Theta/\partial r + w^*{}_o\,\partial\Theta/\partial z)$: namely,

$$\partial u^*_o/\partial\chi + \ldots.. + (\partial\Theta/\partial r)\,\partial\Gamma^*_o/\partial\chi = 0 \tag{7.2a}$$

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$$\partial w_o^*/\partial\chi + \cdots + (\partial\Theta/\partial z)\,\partial\Gamma_o^*/\partial\chi = 0, \tag{7.2b}$$

$$(\partial\Theta/\partial z)\;\partial u_o^*/\partial\chi - (\partial\Theta/\partial r)\partial w_o^*/\partial\chi + \cdots = 0. \tag{7.2c}$$

But these three equations are not independent, since the determinant of coefficients for $\partial u_o^*/\partial\chi$, $\partial w_o^*/\partial\chi$, and $\partial\Gamma_o^*/\partial\chi$ is zero! On the other hand, from them we easily derive the following two relations:

$$\partial\Gamma_o^*/\partial\chi = r^2\,[(\partial\Theta/\partial r)\partial u_o^*/\partial\chi + (\partial\Theta/\partial z)\,\partial w_o^*/\partial\chi] \tag{7.3a}$$

$$(\partial\Gamma_o^*/\partial\chi)\,\{1 + r^2[(\partial\Theta/\partial r)^2 + (\partial\Theta/\partial z)^2]\} = 0. \tag{7.3b}$$

Because:

$$1 + r^2[(\partial\Theta/\partial r)^2 + (\partial\Theta/\partial z)^2] \neq 0,\;\; \partial\Gamma_o^*/\partial\chi \equiv 0,$$

and as a consequence:

$$\partial\Gamma_o^*/\partial\chi = 0,\;\; \partial u_o^*/\partial\chi = 0, \partial w_o^*/\partial\chi = 0 \tag{7.4}$$

Therefore: when ε tends to zero, and as a consequence of a uniform and constant steady flow far of the row, in the leading-order the approximate limiting through-flow in the row of an axial compressor (turbo-machine) is independent of the short (micro)-scale χ.

Therefore, it is necessary to consider in the starting Eqs. 6.18a–6.18c the next order (terms proportional to ε!) In a such case we obtain, for $u^*{}_1$, $w^*{}_1$, and Γ_1^*, the following three equations:

$$(1/r)\partial\Gamma_o^*/\partial r = (1/r\Delta)[\partial u_1^*/\partial\chi + (\partial\Theta/\partial r)\,\partial\Gamma_1^*/\partial\chi]; \tag{7.5a}$$

$$\begin{aligned}\partial u_o^*/\partial z - \partial w_o^*/\partial r = (1/\Delta)[\,(\partial\Theta/\partial z)\;\partial u^*{}_1/\partial\chi \\ -(\partial\Theta/\partial r)\;\partial w^*{}_1/\partial\chi];\end{aligned} \tag{7.5b}$$

$$(1/r)\,\partial\Gamma_o^*/\partial z = (1/r\Delta)[\partial w_1^*/\partial\chi + (\partial\Theta/\partial z)\partial\Gamma_1^*/\partial\chi]. \tag{7.5c}$$

From these equations, by elimination of the first-order functions, $u^*{}_1$, $w^*{}_1$, and Γ_1^*, we obtain as a compatibility relation:

$$\partial u_o^*/\partial z - \partial w_o^*/\partial r = (\partial\Theta/\partial z)\;\partial\Gamma_o^*/\partial r - (\partial\Theta/\partial r)\;\partial\Gamma_o^*/\partial z$$

and with

$$u_o^* = (1/r\Delta)\,\partial\,\psi_o^*/\partial z, w_o^* = -(1/r\Delta)\,\partial\,\psi_o^*/\partial r$$

we derive an equation for the function $\psi^*_o = \Lambda_o(r, z)$, which characterizes the limiting through-flow in the row. Namely:

$$\partial/\partial r\{(1/r\Delta)\partial\Lambda_o/\partial r\} + \partial/\partial z\{(1/r\Delta)\partial\Lambda_o/\partial z\} = (\partial\Theta/\partial z)\partial\Gamma^*_o/\partial r - (\partial\Theta/\partial r)\,\partial\Gamma^*_o/\partial z. \tag{7.6}$$

This model equation for the function Λ_o is our first main rational and entirely consistent result with the RAM Approach.

Again, due to being far upstream of the row, we have a uniform and constant steady incompressible($\rho_o =$ constant) fluid flow: $(1/2)\rho_o u^2 + p = \text{constant}$. Then the jump,

$$|[p]| \equiv p_{\chi=+1/2} - p_{\chi=-1/2} \tag{7.7a}$$

of the pressure, from blade to blade, gives

$$|[p]| = -\varepsilon\,\rho_o\{u^*{}_o|[u^*_1]| + w^*{}_o|[w^*_1]| + (1/r^2)\Gamma^*_o|[\Gamma^*_1]|\} + O(\varepsilon^2) \tag{7.7b}$$

where |[p]| is a quantity of the order ε.

But, according to Eqs. 7.5a–7.5c, obviously $u^*{}_1$, $w^*{}_1$, and Γ^*_1 are linear functions of the microscale structure χ.

As a consequence of (7.7b), we have to write:

$$\mathrm{Lim}_{\varepsilon\to 0}(-p/\varepsilon\rho_o) = u^*_o\partial u^*_1/\partial\chi + w^*_o\partial w^*_1/\partial\chi + (1/r^2)\Gamma^*_o\partial\Gamma^*_1/\partial\chi = \Pi_o. \tag{7.8}$$

From (7.5a–7.5c), we now eliminate the terms with the χ – derivatives in the (7.8) relation, and express the function Π_o, in (7.8) simply by:

$$\Pi_o = \Delta[u^*_o\partial\Gamma^*_o/\partial r + w^*_o\partial\Gamma^*_o/\partial z] \tag{7.9a}$$

in the row.

Outside the row, p remains continuous, even in the presence of wakes – which are, in the considered Eulerian fluid flow, only vortex sheets (contact discontinuity surfaces).

Finally, taking into account the periodicity in χ, *outside the row*, we derive in place of (7.9a) the following relation:

$$|[p]| = 0 \Rightarrow \Pi_o = 0 \tag{7.9b}$$

and

$$u^*_o\partial\Gamma^*_o/\partial r + w^*_o\partial\Gamma^*_o/\partial z = 0 \Rightarrow \Gamma^*_o = \Gamma^*_o(\Lambda_o). \tag{7.9c}$$

As a consequence:

$$\text{In the whole outside region upstream ot the row}: \ \Gamma^*_o = 0 \tag{7.10a}$$

$$\text{In downstream region outside of the row}: \ \Gamma^*_o = \Gamma^*_o(\Lambda_o) \tag{7.10b}$$

From the above we can formulate the following main results relative to a homogenized through-flow.

The velocity vector $\mathbf{U}^*o = (u^*_o, \ w^*_o, \ v^*_o)$ of the homogenized through-flow is

$$\mathbf{U}^*_o = (1/\Delta)[\nabla(\theta - \Theta) \wedge \nabla\Lambda_o], \tag{7.11}$$

such that the streamlines of the through-flow are obtained by the crossing of *median surfaces*,

$$\theta = \Theta(r, \ z) + \text{ constant}$$

in the inter-blade row–channel, with the *cylindrical surfaces*, resulting from the rotation around of the z-axis of the turbo-machine of meridian streamline surfaces

$$\Lambda_o = \text{ constant}$$

For this through-flow we have for the function $\Lambda_o(r, \ z)$, (see Eq. 7.6):

$$\begin{aligned} &\partial/\partial r\{(1/r\Delta)\partial\Lambda_o/\partial r\} + \partial/\partial z\{(1/r\Delta)\partial L_o/\partial z\} \\ &= (\partial\Theta/\partial z)\,\partial\Gamma^*_o/\partial r - (\partial\Theta/\partial r)\,\partial\Gamma^*_o/\partial z, \end{aligned} \tag{7.12a}$$

with, as conditions (if we use (7.11) for the composante $v^*{}_o$):

$$\Gamma^*_o = 0, \text{ upstream of the row,} \tag{7.12b}$$

$$\Gamma^*_o = (r/\Delta) \ [\partial\Theta/\partial r) \ \partial\Lambda_o/\partial z \text{ - } (\partial\Theta/\partial z)\,\partial\Lambda_o/\partial r], \text{in the row,} \tag{7.12c}$$

$$\Gamma^*_o = \Gamma^*_o(\Lambda_o), \text{ downsream of the row.} \tag{7.12d}$$

This axially symmetric through-flow model, which is dependent only on coordinates r and z, introduces a fictitious force:

$$\boldsymbol{F} = (\Pi_o/\Delta)\nabla(\ \theta - \Theta), \tag{7.12e}$$

which simulates the action of the blades in the row on the turbomachinery flow.

The force $\boldsymbol{F}$, given by (7.12e), is a memory term (a trace) which − via homogenization – replaces (simulates) the (vanishing) effect of the blades in the row.

The first numerical applications of the above through-flow model in a turbomachine blade row was realized by Veuillot [140] at the ONERA and see also his [142] paper.

7.1.2 *The G–Z Approach*

In a more sophisticated general case, Guiraud and Zeytounian [141] consider in 1971 at the beginning, in the cylindrical coordinates r, θ z, the following starting Eulerian incompressible equation written in the matrix form:

$$\partial T/\partial t + \partial R/\partial r + \partial Z/\partial z + (1/r)\partial S/\partial \theta + H/r = 0, \tag{7.13}$$

and make the change of coordinates from (t, r, θ, z) to (t, r, z, χ) as shown, in (7.14a):

$$\theta = \Theta(t, r, z) + 2\pi\varepsilon(k + \chi), \tag{7.14a}$$

with the idea in mind that through-flow will be independent of χ whereas r will appear as a parameter for cascade flow.

Without any approximation the flow has to be periodic in χ, and we enforce this by:

$$U_k(t, r, z, \varepsilon; \chi + 1) = U_{k+1}(t, r, z, \varepsilon; \chi), \tag{7.14b}$$

$$U_{k+N}(t, r, z, \varepsilon; \chi) = U_k(t, r, z, \varepsilon; \chi), \tag{7.14c}$$

using for convenience the index k which runs from 1 to N ($\varepsilon = 1/N << 1$), the number of blades in a row, and accordingly we assume that χ is between zero and one.

We expand, formally, U_k as powers of ε, but we obviously need two such expansions, because it is clear that the model through-flow in row is invalid near the locus of the leading/trailing edges of the row.

The first one is a kind of outer expansion (as in (7.1) above):

$$U_k = U_{k,o} + \varepsilon U_{k,1} + \ldots., \tag{7.15a}$$

and will fail near both ends of the row where (two) inner expansions

$$U_k(t, r, z = h(r) + \varepsilon\zeta, \varepsilon; \chi) = U^*_{k,0} + \varepsilon U^*_{k,1} + \ldots, \tag{7.15b}$$

are needed. In (7.15b), $z = h(r)$ is the locus of the leading (or trailing) edge of a row.

When the change of coordinates(from θ to χ) is made in the basic matrix form Eq. 7.13 of the inviscid and incompressible Eulerian fluid flow, we obtain (7.16a, b):

$$\partial \Gamma_k / \partial \chi + 2\, \pi\, \varepsilon\, r\, L_k = 0 \tag{7.16a}$$

with

$$L_k \equiv \partial T_k / \partial t\, + \partial R_k / \partial r\, + \partial Z_k / \partial z\, + H_k / r \tag{7.16b}$$

and we observe that in matrix column Γ_k the following two parameters are present:

$$\lambda = \omega^\circ D / w^\propto,\ \mu = D / w^\propto t^\circ \tag{7.16c}$$

and we have introduced

$$\gamma_k = v_k - r\, u_k\, \partial \Theta / \partial r - r\, w_k\, \partial \Theta / \partial z - r \partial \Theta / \partial t \tag{7.16d}$$

In (7.16c), t° is a reference time, D is the diameter of the row, ω° is the reference value of the angular velocity (ω) of the row, and $w^\propto$ is the upstream uniform axial velocity.

Two facts should be stressed at the outset. First, if we assume axially symmetric flow, $\partial / \partial \chi = 0$, we obtain $L_k = 0$, and it may be checked that $L_k = 0$ is the matrix form of axially symmetric through-flow. Second, if we use, by "brute force" ⇒ $\varepsilon = 0$ in (7.16a), we do not obtain the equations of axially symmetric flow but, rather, the highly degenerate equation $\partial \Gamma_k / \partial \chi = 0$! This is somewhat strange, but is not unexpected.

The way in which χ has been defined substantiates that: when ε is small, variations in the χ direction are magnified by 1/ ε, in comparison with variations in t, and in the r, or z direction.

Now, substituting the basic outer expansion (7.15a) in Eq. 7.16a we derive a hierarchy of equations. But here we write only first two:

$$\begin{aligned} &\partial \Gamma_{k,0} / \partial \chi = 0, \\ &\partial \Gamma_{k,1} / \partial \chi + 2\, \pi\, r\, L_{k,0} = 0, \end{aligned} \tag{7.17a}$$

which consist of equations to be solved in turn.

We choose, as appropriate to the present problem, the solution of

$$\partial \Gamma_{k,0} / \partial \chi = 0 \tag{7.17b}$$

for which $u_{k,0}$, $w_{k,0}$, $v_{k,0}$, and $p_{k,0}$ are all independent of χ.

At this step we do not know the way in which these functions depend on t, r, and z! Now, if we use the second equation of two equations (7.17a) in order to compute

$u_{k,1}, \ldots$ and so on, we encounter a compatibility condition arising from periodicity, which forces $L_{k,0}$ to be zero!

We have thus obtained a through-flow, axially symmetric theory. The interesting point is that we may go a step further and produce a through-flow theory to order ε inclusively! For this, it is first necessary to define the channel between two consecutive blades:

$$\chi_e \leq \chi \leq \chi_i, \text{ and we define}$$

$$\Delta(r, z) = \chi_i - \chi_e$$

and we then introduce an average, < >, and a jump, [ ], operation, thus:

$$<U> = (1/\Delta) \int U d\chi, \text{ integration from } \chi_e \text{ to } \chi_i \tag{7.18a}$$

and

$$[U] = U\chi_i - U\chi_e. \tag{7.18b}$$

Now, if we think of the pressure for U (in (7.18b)), then the bracketed [p] may be viewed as: the pressure difference between the two sides of one and the same blade.

Below, the various equations shows the basic results of the G–Z RAM Approach.

Up to first order in ε, the average of velocity and pressure

$$<\mathbf{V}^{(1)}> = \mathbf{V}_{k,0} + \varepsilon<\mathbf{V}_{k,1}>, <p^{(1)}> = p_{k,0} + \varepsilon<p_{k,1}> \tag{7.19a}$$

satisfies, with an error of order ε^2, axially symmetric through-flow equations:

$$\begin{aligned} &\mathrm{Div}(\Delta<\mathbf{V}^{(1)}>) = O(\varepsilon);\\ &\partial<\mathbf{V}^{(1)}>/\partial t + \{\mathrm{rot}<\mathbf{V}^{(1)}> + 2\Omega\mathbf{e}_z\} \wedge <\mathbf{V}^{(1)}> + \nabla I^{(1)},\\ &= \boldsymbol{F}^{(1)} + O(\varepsilon^2), \end{aligned} \tag{7.19b}$$

where

$$\begin{aligned} I^{(1)} &= <p^{(1)}> + (1/2)|<\mathbf{V}^{(1)}>|^2 - (1/2)\Omega^2 r^2; \Omega = \lambda\,\omega,\\ \Pi^{(1)} &= (1/2\,\pi)[<p_{k,1}> + \varepsilon<p_{k,2}>],\\ \Sigma &= S + 2\,\pi\,\varepsilon[(1/2)(\chi_i + \chi_e)],\ S = \Theta - \theta, \end{aligned} \tag{7.19c}$$

with

$$\partial\Sigma/\partial t + <\mathbf{V}^{(1)}>.\nabla\Sigma = O(\varepsilon^2) \tag{7.19d}$$

$$\boldsymbol{F}^{(1)} \equiv (1/\Delta)\Pi^{(1)}\nabla\Sigma \Rightarrow \boldsymbol{F}^{(1)}\,.\,\mathrm{rot}\,\boldsymbol{F}^{(1)} = 0. \tag{7.19e}$$

Two points again need to be stressed. First, the breadth of the channel from blade to blade, set as

$$\Delta(\mathrm{r},\ \mathrm{s})$$

enters in the continuity equation in an obvious way. Second, in the momentum equation there is a source term,

$$(1/\Delta)\,\Pi^{(1)}\nabla\Sigma\ \equiv\ \boldsymbol{F}^{(1)}$$

which is proportional to the jump in pressure and is orthogonal to $\Sigma =$ constant – a surface which is just in the middle of the channel. This force has long been known in through-flow theory; but in fact, via a very subtle ad hoc consideration (first published, it seems, by Chung-Hua Wu, in NACA TN 2288 (1951); see also Wu [143] paper).

The force $\boldsymbol{F}^{(1)}$ occurs from redistribution (homogenization) of forces acted on the flow by the blades of the row.

The G–Z [141] derivation given above is illuminating with regard to the error involved in the approximation. To order one there is a dependency on χ which may be computed once the through-flow is known.

7.1.3 Transmission Conditions, Local Solution at the Leading/Trailing Edges, and Matching

The above through-flow model in axial turbomachine is invalid near the locus of leading/trailing edges of a row.

According to G–Z theory [144], a local asymptotic analysis is performed by considering the inner expansions (7.15b) and rewriting the starting matrix equation (7.13). We obtain:

$$\partial\Gamma^*_{\mathrm{k}}/\partial\chi + 2\pi\mathrm{r}\partial\mathrm{N}^*_{\mathrm{k}}/\partial\zeta + 2\pi\varepsilon r M^*_{\mathrm{k}} = 0 \tag{7.20a}$$

with

$$\mathrm{N}^*_{\mathrm{k}} = \mathrm{Z}^*_{\mathrm{k}} - (\mathrm{dh}/\mathrm{dr})R^*_{\mathrm{k}} \tag{7.20b}$$

$$M^*_{\mathrm{k}} = \partial\mathrm{T}^*_{\mathrm{k}}/\partial\mathrm{t}\ + \partial\mathrm{R_k}^*/\partial\mathrm{r}\ +\ \mathrm{H}^*_{\mathrm{k}}/\mathrm{r} \tag{7.20c}$$

and to zeroth order we obtain:

$$\partial\Gamma^*_{\mathrm{k},0}/\partial\chi + 2\pi\mathrm{r}\partial\mathrm{N}^*_{\mathrm{k},0}/\partial\zeta = 0, \tag{7.21}$$

which is, in fact, the equations of cascade flow – but the configuration is that of semi-infinite cascade flow. In [144] a detailed analysis of (7.21) is performed, adapted to a local frame linked with the curve:

$$\Gamma : \{z = h(r); \theta = \Theta(r, h(r))\}$$

The semi-infinite cascade flow fills the gap between external (outside the row), force-free, axially symmetric through-flow, and internal (in row) through-flow with the source term $\boldsymbol{F}^{(1)}$. Matching provides transmission conditions between these two disconnected through-flows.

The necessity of such conditions appears readily as soon as any numerical treatment of the whole through-flow in a two-row stage is attempted – from upstream to downstream (infinity) of this two-row stage!

To zeroth order these transmission conditions are rather simple – and, indeed, obvious on physical grounds: They mean that mass flow is conserved, as well as the component of momentum parallel to the leading or trailing edge.

Local analysis has also been carried out by Guiraud and Zeytounian [144], to first order, without a simple interpretation of the (rather complicated) result linked with the transmission conditions!

We can consider the singular regions near the entry and exit of the row as planes of discontinuity, if we impose the associated transmission conditions.

7.1.4 Some Complements

We now turn, briefly, to various cases concerning my work devoted with Guiraud during 1969–1978, to turbomachinery fluid flows.

After the axial flow in a turbomachine, with the approximation of ideal incompressible flow, has been analyzed by using an asymptotic method, assuming that the blades are infinitely near one another – [141], and in a companion paper [144] – a local study reveals the nature of the flow in their neighbourhood and leads to a system of transmission conditions, because the partial differential equations of the through-flow (in three different regions: upstream of a row, in a row, and downstream of the row) must be supplemented by them in order to produce a well-posed problem for the whole of the turbomachine (from upstream of the row to downstream of this row) – an application of the concept of multiple scales was considered.

Namely, in [145] an asymptotic theory for the flow in an axial compressor was considered, with the aim of devising a coupling process between the so-called meridian through-flow and the flow around cascades. Again the small parameter ε is the inverse of the (supposed $\gg 1$) number of blades per row and/or number of stages. As a matter of fact, the cascade flow is treated as a small perturbation of the through-flow, and has to be computed, locally, as the two-dimensional unsteady flow around an array of couples of cascades alternately fixed and in motion. The array is constructed by developing on a plane the section of the compressor by a circular cylinder, and continuing, by periodicity, the pair of cascades so obtained, at each location. The coupling between through-flow and cascade flow is part of the analysis. It happens, incidentally, that the equations of through-flow are obtained through an averaging process, completed on a domain of periodicity of the array of cascades flow, while the through-flow appears locally as an unperturbed flow for the linearized problem defining the cascade flow. The 3D nature of the complete flow is built in by the coupling itself, as is visualized by the occurrence of source terms in each of the two sets of equations describing through-flow and cascade flow. This paper [145] is aimed at producing a preliminary answer to the question of: how to devise, as rationally as possible, a way of describing the familiar scheme of cascade flow within the computation of a mean through-flow. The main conclusion is that the concept of cascade flow should be revisited and reassessed as one of unsteady flow around an array of cascades.

In 1978, I published, with Guiraud, a fourth paper [146], entitled "Cascade and through-flow theories as inner and outer expansions". In this, a technique of matched asymptotic expansions is used in order to combine two kinds of approximation. Through-flow theory forms the basis for an outer expansion, while cascade theory forms the basis for an inner one, and matching provides boundary conditions for both flows. It appears that for the downstream through-flow, a technique of multiple scales is necessary – at least in the vibrating case (vibrations induced by harmonic vibrations of the blades) – in order to deal with the unsteady wakes generated by the vibrating blades, and slowly modulated downstream by the steady part of the through-flow.

Although there are very many good papers on the theory of turbomachine flow, we have not found any attempt analogous to that described above in [145] and [146], and it seems difficult to comment on the relation of the work to be presented with others. Concerning the two papers [141] and [144], we observe that in Sirotkin's paper [147] only some results are similar to ours, but the main difference is in the approach, which is less systematic.

Our objective, with Guiraud, concerning the papers [141] and [144–146] was very modest, and we did not solve any problems nor present any results!

What we have proposed in our above-mentioned papers devoted to a rational asymptotic description of turbomachine flows in the framework of a RAM Approach, may be stated as follows.

Considering incompressible non-viscous fluid flow through a one-row machine, assuming that there is a great number of blades, and, in [146], that the corresponding cascade has a chord-to-spacing ratio of order one, we want to show that the first few terms of an asymptotic representation of the 3D flow may be guessed as having the form of an inner and outer multiple-scale expansion.

We confirm our guess – as is usually done with problems not amenable to mathematically rigorous analysis – by an *internal consistency argument*: we show that each term of the expansion, up to the order considered, may be computed by solving well-set problems. We show the rational asymptotic process by which these problems may be extracted from the definition of the original 3D problem, which is a typical RAM Approach.

For *engineering applications* it would have been very useful to find, as partial problems, cascade flow theory as well as through-flow theory across a thick row. Unfortunately, we have been unable to find any asymptotic process leading to such a scheme. As a matter of fact, the obvious way to do so leads to only two significant degeneracies. One is the through-flow of [141] and [144], which leaves no room for cascade flow, and the other is the one considered in [146], which leads to cascade flow but leaves no room for through-flow, including a thick row.

This conclusion inevitably leads to some deception, because there is no way to embed Wu's [143] technique within an asymptotic rational framework.

7.2 The Flow Within a Cavity Which is Changing Its Shape and Volume with Time: Low Mach Number Limiting Case

Concerning the aerodynamics applications it is necessary to distinguish between confined and unconfined flows and, especially, to elucidate the role of the Strouhal number S (unsteadiness).

When we consider the low Mach number case, two distinguished limiting processes emerge. One of them leads, from the full unsteady NS–F equations, to the model of incompressible flow (Navier equations) and, in the case of confined configuration provides a dynamic interpretation of thermostatics. The other one is at the root of acoustics.

The first limiting process (incompressible) corresponds roughly to

$$\mathrm{M} \to 0 \text{ with S fixed}$$

while the second limiting process (of acoustics) corresponds to

$$\mathrm{M} \to 0 \text{ with SM } = \mathrm{O}(1).$$

Curiously enough, the acoustic model enters into the scene even in the situation which is apparently ruled by incompressible aerodynamics.

This occurrence is due to non-uniformities: a spatial one near infinity in the case of unconfined flow, and a temporal one for small time (in particular near time $= 0$, where the initial data are given) and also for high-frequency oscillations in the case of confined flows.

In [85], the influence of these high-frequency oscillations was taken into account by Zeytounian and Guiraud via a judicious multiple-scales technique, but with an infinity of short acoustic scales! Here, below, we consider, as a physical situation, the low Mach number flow within a cavity which is changing its shape and volume with time. Such a problem presents industrial interest in the case of the compression phase flow in an internal combustion engine (see our short notes in [86]).

We show how a limiting process corresponding to the Mach number going to zero leads to an incompressible unsteady model flow, provided that acoustic waves are averaged out over a great number of periods. This scheme may even describe the case of a gas with a purely temporal variation of density due to substantial changes of volume.

7.2.1 Formulation of the Inviscid Problem

We start from the Euler compressible dimensionless equations:

$$\begin{aligned} &\mathrm{D}\rho/\mathrm{Dt} + \rho\nabla.\mathbf{u} = 0 \\ &\mathrm{D}\mathbf{u}/\mathrm{Dt} + (1/\gamma \mathrm{M}^2)\nabla \mathrm{p} = 0 \\ &\mathrm{D}S/\mathrm{Dt} = 0 \\ &\mathrm{p} = \rho^{\gamma}\exp S \end{aligned} \tag{7.22a}$$

where $\mathrm{D}/\mathrm{Dt} = \partial/\partial\mathrm{t} + \mathbf{u}.\nabla$, with standard notation.

The Strouhal number in the first three of these equations is assumed equal to one, such that the reference time is $\mathrm{t}^\circ = \mathrm{L}^\circ/\mathrm{U}^\circ$, with L° a typical length scale of the cavity Ω(t), and U° a characteristic velocity related to the motion of the wall $\partial\Omega(\mathrm{t}) \equiv \Sigma(\mathrm{t})$.

As boundary condition we write the slip condition:

$$(\mathbf{u}.\mathbf{n})|_{\Sigma(\mathrm{t})} \equiv \mathrm{w}|_{\Sigma(\mathrm{t})} = W(\mathrm{t}, \mathbf{P}), \mathbf{P} \in \Sigma(\mathrm{t}) \tag{7.22b}$$

where the (data) velocity $W(\mathrm{t}, \boldsymbol{P})$ characterizes the normal displacement of the wall Σ(t), and **n** is the unit vector normal to this wall, directed inside Ω (t) – ***P*** being the position point vector on the wall Σ(t).

But it is also necessary to take into account the conservation of the global mass m° of the cavity (a bounded domain with L° as a diameter). In the dimensionless reduced form we have:

$$\rho = 1/\mathrm{V}(\mathrm{t});\ \mathrm{V}(\mathrm{t}) = \rho^\circ|\Omega(\mathrm{t})|/m^\circ, \mathrm{V}(\mathrm{t}=0) = 1 \tag{7.22c}$$

where $|\Omega$ (t)$|$ is the volume of the cavity – a known function of time t.

As initial conditions we write:

$$\mathrm{t} = 0 : \mathbf{u} = 0,\ \mathrm{p} = \rho = 1 \text{ and } \mathrm{S} = 0 \tag{7.22d}$$

More precisely, we consider the case when the motion of the wall, Σ(t), is started impulsively from rest, and in a such case,

$$W(\mathrm{t},\boldsymbol{P}) = H(\mathrm{t})\mathrm{W}_{\Sigma}(\boldsymbol{P}),\ \text{all along}\ \Sigma(\mathrm{t}). \tag{7.23a}$$

In this condition, the function $H(\mathrm{t})$ is the Heaviside (or unit) function, such that

$$\mathrm{Lim}_{\mathrm{t}\to 0^+} H(\mathrm{t}) \equiv 1,\ \text{but initially}: H(\mathrm{t}\ = 0^-) \equiv 0. \tag{7.23b}$$

7.2.2 *The Persistence of Acoustic Oscillations*

Asymptotics analysis and the RAM Approach in the formulated problem above is not easy task, mainly due to the persistence of acoustic oscillations in the cavity emerging for $\mathrm{t}\ = 0^+$.

Therefore, if the Mach number, M, is sufficiently small, the zeroth order approximation leads to the thermostatic isentropic evolution of the gas within the cavity as a whole. Superimposed onto them we have acoustic oscillations which remain undamped as long as viscosity is neglected – when ones takes it into account the Euler equations (7.22a), in place of full unsteady NS–F equations.

When the NS–F equations are considered (in place of Euler equations (7.22a)), then a rather longer time $\mathrm{O}(\mathrm{Re}^{1/2})$ is necessary in order to damp out the oscillations – but a much longer time is necessary in order that the heat exchanges can take place. As a matter of fact, this time is O(Re)!

Indeed, some new features occur (as a consequence of the unsteadiness of the compressible fluid flow) when one deals with internal aerodynamics. The first concerns the leading term in the expansion of pressure which is function of time instead of being a constant. We write:

$$\mathbf{u} = \mathbf{u}_0 + \ \mathrm{M}\mathbf{u}^* + \mathrm{M}^2(\mathrm{u}^{**} + \mathbf{u}_2) + \ldots, \tag{7.24a}$$

$$\mathrm{p} = \mathrm{P}_0(\mathrm{t}) + \mathrm{M}^2(\mathrm{p}^* + \mathrm{p}_2) + \mathrm{M}^3\mathrm{p}^{**} + \mathrm{M}^4(\mathrm{p}^{***} + \mathrm{p}_4) + \ldots \tag{7.24b}$$

and an expansion similar to the one for p is valid for density ρ.

At the leading order, one finds that $\rho_0(\mathrm{t})$ or $\mathrm{P}_0(\mathrm{t})$ belongs to a family of adiabatic thermostatic evolution of the gas in the cavity (container-bounded domain), and are determined from the overall conservation of mass. That is, they do not depend on position either. Furthermore:

$$\mathrm{P}_0(\mathrm{t}) = [\ \rho_0(\mathrm{t})]^{\gamma}\ \text{and}\ \rho_0(\mathrm{t}) = 1/\mathrm{V}(\mathrm{t}). \tag{7.25}$$

The pair ($\mathbf{u}_0$, p_2) belongs to a so-called “quasi-incompressible” model, but, as a second peculiar feature, we find that it is perturbed by the pair $(\mathbf{u}^*,\ \mathrm{p}^*)$, which consists in acoustic oscillations generated during the setting-up of the motion. Therefore, as demonstrated below, we have:

$$\mathbf{u}^* = -\Sigma_{n\geq 1} B_n(t)\sin[\phi_n(t)/M] U_n(t, \mathbf{x}), \tag{7.26a}$$

$$\rho^* = \rho_0(t)[\rho_0/P_0(t)]^{1/2} \Sigma_{n\geq 1} B_n(t)\cos[\phi_n(t)/M] R_n(t, \mathbf{x}), \tag{7.26b}$$

where

$$d\phi_n(t)/dt = [P_0(t)/\rho_0(t)]^{1/2}\omega_n(t). \tag{7.26c}$$

In (7.26c), $\omega_n(t)$ is one of the acoustic frequencies corresponding to the shape of the container at time t, while the pair $\{U_n(t, \mathbf{x}), R_n(t, \mathbf{x})\}$ serves to define the normal mode of oscillation at frequency $\omega_n(t)$ normalized to:

$$\int_{\Omega(t)} [(\mathbf{U}_n)^2 + (R_n)^2] dv = 1$$

where the integral is over the bounded container.

A third peculiar feature is that the first correction due to compressibility corresponds to the pair,

$$(\mathbf{u}_2, p_4 + \omega_4(t)), \text{where } 4\omega_4(t) = \gamma\, \Sigma_{n\geq 1} {B_n}^2 |\mathbf{U}_n|^2. \tag{7.26d}$$

The B_n are readily found as function of time t by writing down conservation of acoustic energy. We then obtain, in particular:

$$[\rho_0(t)]^{1/2} Bn(t) = Bn(0). \tag{7.26e}$$

On a time $t = O(Re/M^{1/2})$ this acoustic energy is mainly damped out by viscosity and heat conduction within a Stokes-like boundary layer (see Zeytounian and Guiraud [85]).

Of course, turbulent mixing would be much more efficient, and the long time persistency of acoustic oscillations is mainly a further proof that laminar mixing is very poor!

It must be emphasized that on a laminar basis, even when the transient acoustics has been damped out, $P_0(t)$ and $\rho_0(t)$ remain adiabatically related, but a much longer time would be needed for inducing isothermal evolution.

A final feature should be pointed out. Under resonance conditions ($\mathbf{u}^*$, p^*) gains energy from the motion of the container, and their limiting amplitude is derived from a fairly complicated non-linear process which is understood only in one-dimensional situations (Chester [148], Rott [149]).

7.2.3 Derivation of an Average Continuity Equation

First we should use the time t, a slow time, and then we would bring into the solution an infinity of fast times designed to cope with the infinity of periods of free vibrations of the cavity $\Omega(t)$. Below we set U for the solution U expressed through this variety of time-scales, and in such a case we write

$$\partial U/\partial t = \partial U/\partial t \ + (1/M)DU \tag{7.27}$$

where $\partial U/\partial t$ stands for the time derivative computed when all fast times are maintained constant, while $(1/M)DU$ is the time derivative (with D, a differential operator) occurring through all the fast times.

We carry such a change into the starting Euler equations (7.22a), and then expand according to:

$$U = U_0 + M\ U_1 + M^2U_2 + \dots,\ \text{with U}\ \langle U\rangle + U^* \tag{7.28}$$

where $\langle U\rangle$ is average (over all rapid oscillations and depends only of the slow time t and space position **x**) of U, and U^* is the fluctuating (oscillating, which depends of all fast times) part of U. More precisely, the operation

$$U \Rightarrow \langle U\rangle \tag{7.29a}$$

erases all the oscillations associated with the fast times, and obviously

$$D\langle U\rangle = 0. \tag{7.29b}$$

For instance, for the fluctuating parts of $\mathbf{u}^*{}_0$ and $\rho^*{}_1$, we can write, respectively, as a (more complete) solution:

$$\mathbf{u}_0^* = \Sigma_{n\geq 1}[A_n(t)\ Cn - B_n(t)\ Sn]\boldsymbol{U}_n \tag{7.30a}$$

$$\rho_1^* = \rho_0(t)[\rho_0(t)/p_0(t)]^{1/2}\Sigma n \geq 1[A_n(t)\ Sn + B_n(t)\ Cn]R_n \tag{7.30b}$$

with

$$Cn = \cos[(1/M)\varphi n(t)]\ \text{and}\ Sn = \sin[(1/M)\varphi n(t)] \tag{7.31a}$$

$$\begin{gathered}\langle Cp\ Cq\rangle = \langle Sp\ Sq\rangle = (1/2)\delta pq,\\ \text{and}\ \delta pq = 0,\ \text{if}\ p \neq q, \delta pq = 1,\ \text{if}\ p \equiv q,\end{gathered} \tag{7.31b}$$

$$\langle Cn\rangle = 0, \langle Sn\rangle = 0, \langle Cp\ Sq\rangle = \langle Cq\ Sp\rangle \equiv 0, \tag{7.31c}$$

$$DCn\ = -(d\varphi_n(t)/dt)Sn,\ DSn = (d\varphi_n(t)/dt)Cn, \tag{7.31d}$$

where again

$$d\phi_n(t)/dt = [p_0(t)/\rho_0(t)]^{1/2}\omega_n;\ \phi_n(0) = 0. \tag{7.32}$$

In (7.30a, 7.30b) $\boldsymbol{U}_n$ and R_n are the normal modes of vibrations of $\Omega(t)$ with eigen-frequencies ω_n: namely,

$$\omega_n R_n + \nabla.\boldsymbol{U}_n = 0, -\omega_n \boldsymbol{U}_n + \nabla R_n = 0; (\boldsymbol{U}_n.\mathbf{n})_{\Sigma(t)} = 0. \tag{7.33}$$

The relation (7.32) defines the scales of the fast times in relation to the speed of sound in cavity (at the time t) and with the eigenfrequencies of the cavity at the same time.

With (7.27), from the Euler equations (7.22a) we obtain the following equations for the functions $\mathbf{u}$, ρ, p, and S:

$$\begin{aligned} &D\rho + \mathrm{M}\ (\partial\rho/\partial t\ + \mathbf{u}.\nabla\rho + \rho\nabla.\mathbf{u}) = 0 \\ &(1/\gamma)\nabla p\ +\ \mathrm{M}\rho D\mathbf{u} + \mathrm{M}^2\rho[\partial\mathbf{u}/\partial t + (\mathbf{u}.\nabla)\mathbf{u}] = 0 \\ &DS + \mathrm{M}\ (\partial S/\partial t\ + \mathbf{u}.\nabla S) = 0 \\ &p\ = \rho^\gamma \exp S; \end{aligned} \tag{7.34a}$$

with the slip condition

$$(\mathbf{u}.\mathbf{n})_{\Sigma(t)} = W(t, \boldsymbol{P}). \tag{7.34b}$$

From the expansion (7.28), at the zero-order, from the above system (7.34a) we derive:

$$Dp_0 = 0, D\rho_0 = 0, DS_0 = 0$$

which shows that ρ_0 and S_0 are independent of the fast times and, as a consequence of the equation of state, that this is also the case for p_0, which is, in fact, a function of only the slow time t:

$$p_0 = p_0(t),\ \ \rho_0 =\ \rho_0(t, \mathbf{x}), S_0 = S_0(t, \mathbf{x}). \tag{7.35a}$$

Now, at the first order, from the equation for S, we derive the equation

$$DS_1 + \partial S_0/\partial t\ + \mathbf{u}_0.\nabla S_0 = 0$$

and, since $S_0 = S_0(t, \mathbf{x})$ is independent of the fast time and $D\langle S_1\rangle = 0$, we have the following average equation for $S_0(t, \mathbf{x})$:

$$\partial S_0/\partial t\ + \langle\mathbf{u}_0\rangle.\nabla S_0 = 0. \tag{7.35b}$$

But, close to initial time ($t = 0$), when we consider the Euler equations (7.22a) written with the short time, $\tau = t/M$, in place of the slow time, t, we use the local-in-time asymptotic expansion

$$S = S_{a0} + MS_{a1} + MS_{a2} + \dots$$

where $S_{ak} = S_{ak}(\tau, \mathbf{x})$, $k = 0, 1, 2, \dots,$ and the initial condition

$$S = 0 \text{ at } \tau = 0$$

We derive

$$\partial S_{a0}/\partial\tau = 0, \partial S_{a1}/\partial\tau = 0 \Rightarrow S_{a0} = 0, S_{a1} = 0$$

and, as a consequence, from (7.35b), by continuity

$$S_0 = 0. \tag{7.35c}$$

On the other hand, with (7.35c) we obtain

$$p_0(t) = [\rho_0(t)]^{\gamma} \tag{7.35d}$$

the function $\rho_0(t)$ being determined by the relation

$$\int_{\Omega(t)} \rho_0(t)dv = \rho_0(t)\int_{\Omega(t)} dv \Rightarrow \rho_0(t)|\Omega(t)| = m^{\circ}$$

where $|\Omega(t)|$ is the volume of the cavity, such that

$$d|\Omega(t)|/dt = -\int_{\Sigma(t)} W(t, \boldsymbol{P})\, ds \tag{7.35e}$$

and m° (= const) is the whole mass of the cavity, and, according to the initial condition for the density we have $|\Omega(0)| \equiv m^{\circ}$.

If, in particular, we assume that $\rho_0(t) \equiv 1$ (and as a consequence $p_0(t) \equiv 1$ also) then $|\Omega(t)| \equiv m^{\circ} \equiv$const. Obviously, this is not the case in the various applications!

At the first order, from the first three equations of (7.34a), with the above results, we derive the following two equations:

$$D\rho_1 + d\rho_0/dt + \rho_0\nabla.\mathbf{u}_0 = 0, \tag{7.36a}$$

$$(1/\gamma)\nabla p_1 + \rho_0 D\mathbf{u}_0 = 0, \tag{7.36b}$$

with, from (7.34b),

$$(\mathbf{u}_0.\mathbf{n})_{\Sigma(\mathrm{t})} = W(\mathrm{t}, \boldsymbol{P}). \tag{7.36c}$$

Since, $D\langle \rho_1 \rangle = 0$, from (7.36a) we derive an average (zero-order) continuity equation:

$$(1/\rho_0)\mathrm{d}\rho_0/\mathrm{dt} + \nabla.\langle \mathbf{u}_0 \rangle = 0. \tag{7.36d}$$

with (from (7.36c))

$$(\langle \mathbf{u}_0 \rangle.\mathbf{n})_{\Sigma(\mathrm{t})} = W(\mathrm{t}, \boldsymbol{P}). \tag{7.36e}$$

From (7.36b), since $D\langle \mathbf{u_0} \rangle = 0$, we also have:

$$\nabla\langle \mathrm{p}_1 \rangle = 0 \Rightarrow \langle \mathrm{p}_1 \rangle = 0; \mathrm{p}_1 \equiv \mathrm{p}_1^*. \tag{7.36f}$$

7.2.4 *Solution for the Fluctuations* $\mathbf{u}_0^*$ *and* ρ_1^*

For the fluctuations we derive, from Eqs. 7.36a, 7.36b with 7.36c, the following acoustic-type equations with slip condition:

$$D\,\rho_1^* + \rho_0 \nabla.\mathbf{u}_0^* = 0, \tag{7.37a}$$

$$(1/\gamma)\nabla \mathrm{p}_1^* + \rho_0 D\mathbf{u}_0^* = 0, \tag{7.37b}$$

with

$$\left(\mathbf{u}_0^*.\mathbf{n}\right)_{\Sigma(\mathrm{t})} = 0. \tag{7.37c}$$

Concerning the equation for the specific entropy, we have (because $S_0 = 0$):

$$DS_1^* = 0 \Rightarrow S_1^* = 0, \tag{7.37d}$$

and as a consequence, from the equation of state, for the fluctuation of the pressure, we derive

$$\mathrm{p}_1^* = \gamma(\mathrm{p}_0/\mathrm{r}_0)\,\rho_1^*. \tag{7.37e}$$

As a consequence of (7.36f) we also have

$$\langle \rho_1 \rangle = 0 \Rightarrow \rho_1 \equiv \rho_1^* \tag{7.37f}$$

The solution of the two equations for $\mathbf{u}_0^*$ and (ρ_1^*/ρ_0), obtained from (7.37a, 7.37b) when we use (7.37c), is given by (7.30a, 7.30b).

Indeed, if we use the solutions (7.30a, 7.30b) in Eqs. 7.37a and 7.37b, then:

$$D(\rho_1^*/\rho_0) + \nabla.\mathbf{u}_0^* = \Sigma_{n\geq 1}[\mathrm{A_n(t)\ Cn}$$

$$- \mathrm{B_n(t)\ Sn}]\{[\rho_0(t)/p_0(t)]^{1/2}(\mathrm{d}\varphi_n(t)/\mathrm{dt})R_n + \nabla.\boldsymbol{U}_n\}$$

and

$$(p_0/\rho_0)\nabla(\rho_1^*/\rho_0) + D\mathbf{u}_0^* = \Sigma_{n\geq 1}[\mathrm{A_n(t)\ Sn}$$

$$+ \mathrm{B_n(t)\ Cn}]\{[p_0(t)/\rho_0(t)]^{1/2}\nabla R_n - (\mathrm{d}\varphi_n(t)/\mathrm{dt})\boldsymbol{U}_n\}$$

and using (7.31a–7.31d) and (7.32) we determine that the right-hand side of the above equations are quite zero.

We observe also that the eigenfunctions (the normal modes of vibrations of $\Omega(t)$ with eigenfrequencies ω_n), $\boldsymbol{U}_n$ and R_n, are normalized according to:

$$\int_{\Omega_{(t)}} [(\boldsymbol{U}_n)^2 + (R_n)^2]\ \mathrm{dv} = 1$$

It is now necessary to determine, from (7.34a, 7.34b), the equations for the second-order approximation, and then derive, first, a system of two equations for the amplitudes, $\mathrm{A_n(t)}$ and $\mathrm{B_n(t)}$, which present the possibility of considering the long time evolution of the rapid oscillations.

However, it is also necessary to derive an equation for the average value of $\mathbf{u}_0$, which gives, with the average continuity equation (7.36d), a system of two average equations for $\langle \mathbf{u}_0 \rangle$ and $\langle p_2 \rangle$.

7.2.5 The Second-Order Approximation

We return to system of Eqs. 7.34a with 7.34b, and consider the second-order approximation for S_2, p_2, and ρ_2. First, we obtain:

$$DS_2 + \partial\langle S_1 \rangle/\partial t + [\langle \mathbf{u}_0 \rangle.\nabla)]\langle S_1 \rangle = 0$$

but according to (7.37d), $S_1^* = 0$, and also

$$\partial\langle S_1 \rangle/\partial t + [\langle \mathbf{u}_0^* \rangle.\nabla]\langle S_1 \rangle = 0.$$

With zero initial condition at $t = 0$, we have (since $\langle S_1 \rangle = 0$):

$$S_1 \equiv 0, \text{ and then} : S_2^* = 0 \tag{7.38a}$$

For the third-order approximation we have

$$DS_3 + \partial\langle S_2 \rangle/\partial t + [\langle \mathbf{u}_0 \rangle.\nabla]\langle S_2 \rangle = 0$$

and again (according to the second relation in (7.38a)) we obtain

$$S_2 = 0 \text{ and } S_3^* = 0 \tag{7.38b}$$

Finally, from the equation of state, when we take into account that

$$S_0 = S_1 = S_2 \equiv 0$$

we derive the following relation between p_2 and ρ_2:

$$p_2 = \gamma(p_0/\rho_0)[\rho_2 + (1/2\rho_0)(\gamma - 1)(\rho_1)^2]. \tag{7.38c}$$

Now, again from the system of Eq. (7.34a), we derive two second-order equations:

$$D\rho_2 + \rho_0\nabla.\mathbf{u}_1 + \partial\rho_1/\partial t + \mathbf{u}_0.\nabla\rho_1 + \rho_1\nabla.\mathbf{u}_0 = 0 \tag{7.39a}$$

$$\nabla(p_2/\gamma\rho_0) + D\mathbf{u}_1 + (\rho_1/\rho_0)D\mathbf{u}_0 + \partial\mathbf{u}_0/\partial t + (\mathbf{u}_0.\nabla)\mathbf{u}_0 = 0 \tag{7.39b}$$

with

$$(\mathbf{u}_1.\mathbf{n})_{\Sigma(t)} = 0 \tag{7.39c}$$

From (7.39b) we now have, first, the possibility of deriving the following average equation for $\langle \mathbf{u}_0 \rangle$:

$$\partial\langle \mathbf{u}_0 \rangle/\partial t + \langle(\mathbf{u}_0.\nabla)\mathbf{u}_0\rangle + \nabla(\langle p_2 \rangle/\gamma\rho_0) + (1/\rho_0)\langle \rho_1 D\mathbf{u}_0 \rangle = 0. \tag{7.39d}$$

This equation is explained below.

On the other hand, using (7.38c), we write Eqs. 7.39a, 7.39b as an inhomogeneous, acoustic-type, system for ρ_2/ρ_0 and $\mathbf{u}_1$: namely,

$$\begin{gathered} D(\rho_2/\rho_0) + \nabla.\mathbf{u}_1 + G = 0, \\ (p_0/\rho_0)\nabla(\rho_2/\rho_0) + D\mathbf{u}_1 + \boldsymbol{F} = 0, \\ (\mathbf{u}_1.\mathbf{n})_{\Sigma(t)} = 0, \end{gathered} \tag{7.40a}$$

where

$$G = \partial(\rho_1/\rho_0)\partial t \ + \nabla.[(\ \rho_1/\rho_0)\mathbf{u}_0] + (1/\rho_0)[d\rho_0/dt](\rho_1/\rho_0); \qquad (7.40b)$$

$$\boldsymbol{F} = \partial\mathbf{u}_0/\partial t \ + (\mathbf{u}_0.\nabla)\mathbf{u}_0 + (\gamma - 2)(p_0/\rho_0)(\rho_1/\rho_0)\nabla(\rho_1/\rho_0). \qquad (7.40c)$$

In G and $\boldsymbol{F}$, according to (7.40b, 7.40c), we have three categories of terms:

1. The (average $\langle G\rangle$ and $\langle \boldsymbol{F}\rangle$) terms independent of the scale of the fast times.
2. The terms ($G_L$and $\boldsymbol{F}_L$) which are linearly dependent on the Cn, and Sn.
3. The terms which depend quadratically $\left(G_Q\text{and } \boldsymbol{F}_Q\right)$ on the Cn and Sn, and are proportional to $\cos[(1/M)(\varphi_p(t) \pm \varphi_q(t))]$or $\sin[(1/M)(\varphi_p(t) \pm \varphi_q(t))]$.

As a consequence we write, in system (7.40a) for the inhomogeneous terms G and $\boldsymbol{F}$, the following formal representation:

$$G = \langle G\rangle + [p_0(t)/\rho_0(t)]^{1/2}\Sigma_{n\geq 1}[G_{nC}Cn \ + G_{nS}\ Sn] + G_Q; \qquad (7.41a)$$

$$\mathbf{F} = \langle \mathbf{F}\rangle + [p_0(t)/\rho_0(t)]\Sigma_{n\geq 1}[\mathbf{F}_{nS}Sn \ + \mathbf{F}_{nC}\ Cn] + \mathbf{F}_Q, \qquad (7.41b)$$

where $\langle G\rangle$ and $\langle \mathbf{F}\rangle$ and also coefficients. $G_{nC}, G_{nS}, \mathbf{F}_{nC}, \mathbf{F}_{nS}$, are determined from (7.40b, 7.40c).

More precisely, in (7.41a, 7.41b), the terms $\langle G\rangle$ and $\langle \mathbf{F}\rangle$ indicate the terms independent of fast times, while in the $\Sigma_{n\geq 1}$ we have the terms with Cn and Sn according to (7.30a, 7.30b). On the other hand, in G_Q and $\mathbf{F}_Q$ we have the terms proportional to

$$\cos[(\varphi_p \pm \varphi_q)/M] \text{ or } \sin[(\varphi_p \pm \varphi_q)/M].$$

Below we assume that the last quadratic terms, G_Q and $\boldsymbol{F}_Q$, are not resonant triads satisfying the relation:

$$|\varphi_p(t) \pm \varphi_q(t)| = \varphi_r(t), \forall p,\ q,\ r \qquad (7.41c)$$

Thus, none of the quadratic terms can interfere with any of the terms depending linearly on the Cn and Sn.

As a consequence of the linearity of our system (7.40a), we can, in particular, write the solution for the fluctuations (ρ_2^*/ρ_0) and $\mathbf{u}_1^*$, corresponding only to the terms linearly dependent on the Cn and Sn in (7.41a, 7.41b), in the following form:

$$\rho_2^*/\rho_0 = \Sigma_{n\geq 1}[R_{nC}Cn \ + R_{nS}\ Sn] \qquad (7.42a)$$

$$\mathbf{u}_1^* = (p_0/\rho_0)^{1/2}\Sigma_{n\geq 1}[\boldsymbol{U}_{nC}Cn \ - \boldsymbol{U}_{nS}Sn] \qquad (7.42b)$$

and, for example, the amplitudes R_{nS} and $\boldsymbol{U}_{nC}$ satisfies the system:

$$\omega_{\mathrm{n}} R_{\mathrm{nS}} + \nabla.\boldsymbol{U}_{\mathrm{nC}} + G_{\mathrm{nC}} = 0$$

$$-\ \omega_{\mathrm{n}} \boldsymbol{U}_{\mathrm{nC}} + \nabla R_{\mathrm{nS}} + \boldsymbol{F}_{\mathrm{nS}} = 0 \tag{7.42c}$$

$$\boldsymbol{U}_{\mathrm{nC}}.\mathbf{n} = 0$$
$$\text{on } \Sigma(\mathrm{t}).$$

Obviously, for R_{nC} and $\boldsymbol{U}_{\mathrm{nS}}$ we obtain a similar system when in place of R_{nS}, $\boldsymbol{U}_{\mathrm{nC}}$, G_{nC} and $\boldsymbol{F}_{nS}$ we write R_{nC}, $\boldsymbol{U}_{\mathrm{nS}}$, G_{nS} and $\boldsymbol{F}_{n\mathrm{C}}$.

For the existence of a solution of both these inhomogeneous systems it is necessary to use two compatibility relations (which are, in fact, a consequence of the Fredholm alternative), respectively related to $(G_{nC,}\, \boldsymbol{F}_{nS})$ and $\left(G_{nS,}\boldsymbol{F}_{nC}\right)$, and for this the system (7.33), for the normal modes (R_{n}, $\boldsymbol{U}_{\mathrm{n}}$) of vibrations of the cavity Ω(t) with eigenfrequencies ω_{n}, must be taken into account.

Therefore, from (7.33), after an integration by parts, it follows that

$$\begin{aligned} 0 &= \int_{\Omega(\mathrm{t})} \{[\omega_{\mathrm{n}} R_{\mathrm{n}} + \nabla.\boldsymbol{U}_{\mathrm{n}}] R_{\mathrm{nC}} - [\ \omega_{\mathrm{n}} \boldsymbol{U}_{\mathrm{n}} - \nabla R_{\mathrm{n}}] \boldsymbol{U}_{\mathrm{nC}}\} \mathrm{dv} \\ &= \int_{\Omega(\mathrm{t})} \{[\omega_{\mathrm{n}} R_{\mathrm{nC}} + \nabla.\boldsymbol{U}_{\mathrm{nC}}] R_{\mathrm{n}} - [\ \omega_{\mathrm{n}} \boldsymbol{U}_{\mathrm{nC}} - \nabla R_{\mathrm{nC}}] \boldsymbol{U}_{\mathrm{n}}\} \mathrm{dv}, \end{aligned} \tag{7.43a}$$

when we also take into account the boundary $(\text{on } \partial\Omega(\mathrm{t}) = \Sigma(\mathrm{t}))$, the conditions:

$$\boldsymbol{U}_{\mathrm{nC}}.\mathbf{n} = 0, \text{ and } \boldsymbol{U}_{\mathrm{nS}}.\mathbf{n} = 0, \text{ on } \Sigma(\mathrm{t}). \tag{7.43b}$$

As a consequence, we derive the following compatibility condition for the resolvability of the above, (7.42c), inhomogeneous system:

$$\int_{\Omega(\mathrm{t})} [G_{nC} R_{\mathrm{n}} - \mathbf{F}_{\mathrm{nS}}.\boldsymbol{U}_{\mathrm{n}}] \mathrm{dv} = 0 \tag{7.44}$$

Of course, a compatibility relation similar to (7.44) is verified if we write, in place of G_{nC} and $\boldsymbol{F}_{\mathrm{nS}}$, respectively, G_{nS} and $\boldsymbol{F}_{\mathrm{nC}}$, after the use of a system similar to (7.42c) for R_{nC}, $\boldsymbol{U}_{\mathrm{nS}}$, with G_{nS} and $\boldsymbol{F}_{\mathrm{nC}}$.

7.2.6 The Average System of Equations for the Slow Variation

With the average continuity equation (7.36d) and slip condition (7.36e), for $\langle \mathbf{u}_0 \rangle$, we lack sufficient information for the determination of the slow (nearly incompressible) variation! Such information is derived from the average equation (7.39d). Again, therefore, according to solution (7.30a, 7.30b), we first obtain:

$$\langle(\mathbf{u}_0.\nabla)\mathbf{u}_0\rangle = (\langle\mathbf{u}_0\rangle.\nabla)\langle\mathbf{u}_0\rangle$$
$$+ (1/2)\Sigma_{n\geq 1}({A_n}^2 + {B_n}^2)[\boldsymbol{U}_n.\nabla]\boldsymbol{U}_n \qquad (7.45a)$$

and

$$(1/\ \rho_0)\langle\ \rho_1 D\mathbf{u}_0\rangle = -(1/2)\Sigma_{n\geq 1}({A_n}^2 + {B_n}^2)[\boldsymbol{R}_n.\nabla]\boldsymbol{R}_n \qquad (7.45b)$$

when we also make use of (7.33). From this equation we also derive the relation:

$$[\boldsymbol{U}_n.\nabla]\boldsymbol{U}_n = (1/2)|\nabla\boldsymbol{U}_n|^2$$

Finally, for $\langle\mathbf{u}_0\rangle$ we derive the following average equation of motion:

$$\partial\langle\mathbf{u}_0\rangle/\partial t\ + (\langle\mathbf{u}_0\rangle.\nabla)\langle\mathbf{u}_0\rangle + \nabla\Pi = 0 \qquad (7.46a)$$

where

$$\Pi = (\langle p_2\rangle/\gamma\rho_0) + (1/4)\Sigma_{n\geq 1}({A_n}^2 + {B_n}^2)\{|\boldsymbol{U}_n|^2 - |\boldsymbol{R}_n|^2\} \qquad (7.46b)$$

is a pseudo-pressure affected by the acoustic perturbations.

But, $\langle\mathbf{u}_0\rangle|_{t=0}$ being irrotational (according to a detailed investigation in [85], Sect. 4 and see also Sect. 7.2.7 below), and according to the average equation (7.46a), for $\langle\mathbf{u}_0\rangle$, it remains irrotational for any time $t > 0$:

$$\langle\mathbf{u}_0\rangle = \nabla\varphi. \qquad (7.47)$$

In such a case, the average continuity equation (7.36d) for $\langle\mathbf{u}_0\rangle$, with the slip condition (7.36e) on the wall $\Sigma(t)$, allows us to determine $\langle\mathbf{u}_0\rangle$ due to the following Neumann problem for potential function φ:

$$\Delta\varphi +\ d\log\rho_0(t)/dt, \qquad (7.48a)$$

with

$$(d\varphi/dn)_{\Sigma(\tau)} =\ W(t,\mathbf{P}). \qquad (7.48b)$$

In such a case, for the first term in Π given by (7.46b), we write:

$$\langle p_2\rangle/\gamma\rho_0 = -[\partial\varphi/\partial t\ + (1/2)|\nabla\varphi|^2]$$
$$- (1/4)\Sigma_{n\geq 1}({A_n}^2 + {B_n}^2)\{|\boldsymbol{U}_n|^2 - |\boldsymbol{R}_n|^2\} \qquad (7.48c)$$

which take into account, explicitly, the influence of the acoustics on the averaged pressure $\langle p_2\rangle$.

The term

$$- (1/4)\Sigma_{n\geq 1}({A_n}^2 + {B_n}^2)\{|\boldsymbol{U}_n|^2 - |\boldsymbol{R}_n|^2\},$$

in (7.48c) is a trace of the acoustics, in the model problem (7.48a, 7.48b) with (7.48c) – a sequel (a memory of the acoustic oscillations) of the application of the homogenization technique.

We observe also that as an initial condition for a $\langle \mathbf{u}_0 \rangle$, at $t = 0$, solution of the average equation (7.46a) with (7.46b), according to solution (7.30a) and the starting initial condition (7.22d), we can write:

$$\langle \mathbf{u}_0 \rangle + \Sigma_{n \geq 1} A_n(0) \boldsymbol{U}_n(0, \mathbf{x}) = 0 \text{ for t } = 0 \tag{7.48d}$$

7.2.7 *The Long Time Evolution of the Fast Oscillations*

With the above derivation of the average system of equations for slow variation we have eliminated only part of the secular terms in $\mathbf{u}_1$ and ρ_2. As a consequence it is necessary to consider in detail the system of compatibility conditions (7.44) for G_{nC} and $\mathbf{F}_{nS}$, and similarly for G_{nS} and $\mathbf{F}_{nC}$.

First, we consider G_{nC}, $\mathbf{F}_{nS}$, G_{nS} and $\mathbf{F}_{nC}$, and take into account the relations (7.40b, 7.40c), (7.41a, 7.41b) and the solution (7.30a, 7.30b), for $\mathbf{u}^*{}_0$ and $\rho^*{}_1$, with $\mathbf{u}_0 = \langle \mathbf{u}_0 \rangle + \mathbf{u}_0^*$ and $\rho_1 \equiv \rho_1^*$.

A straightforward but technically long calculation produces the following formulae:

$$G_{nC} = (\rho_0/p_0)\{(dB_n/dt)R_n + B_n[(\partial R_n/\partial t) + (d\log\rho_0/dt)R_n + \nabla.(\langle \mathbf{u}_0 \rangle R_n)]\} \tag{7.49a}$$

$$\mathbf{F}_{nS} = -(\rho_0/p_0)\{(dB_n/dt)\boldsymbol{U}_n + B_n[(\partial \boldsymbol{U}_n/\partial t) + (\boldsymbol{U}_n.\nabla)\langle \mathbf{u}_0 \rangle + (\langle \mathbf{u}_0 \rangle.\nabla)\boldsymbol{U}_n]\} \tag{7.49b}$$

$$G_{nS} = -(\rho_0/p_0)\{(dA_n/dt)R_n + A_n[(\partial R_n/\partial t) + (d\log\rho_0/dt)R_n + \nabla.(\langle \mathbf{u}_0 \rangle R_n)]\} \tag{7.49c}$$

$$\mathbf{F}_{nS} = (\rho_0/p_0)\{(dA_n/dt)\boldsymbol{U}_n + A_n[(\partial \boldsymbol{U}_n/\partial t) + (\boldsymbol{U}_n.\nabla)\langle \mathbf{u}_0 \rangle + (\langle \mathbf{u}_0 \rangle.\nabla)\boldsymbol{U}_n]\} \tag{7.49d}$$

and we observe that in (7.49a) and (7.49c), according to (7.36d):

$$\begin{aligned}\nabla.(\langle \mathbf{u}_0 \rangle R_n) &= R_n(\nabla.\langle \mathbf{u}_0 \rangle) + \langle \mathbf{u}_0 \rangle.\nabla R_n \\ &\equiv \langle \mathbf{u}_0 \rangle.\nabla R_n - (d\log\rho_0/dt)R_n\end{aligned} \tag{7.50}$$

Then, from the compatibility relation (7.44) with (7.49a, 7.49b), and from a similar (to 7.44) compatibility relation, together with (7.49c, 7.49d), we derive the two ordinary differential equations for the amplitudes $A_n(t)$ and $B_n(t)$, taking into account the normalization condition: namely,

$$dA_n/dt + \gamma_n(t)A_n = 0, \tag{7.51a}$$

$$dB_n/dt + \gamma_n(t)B_n = 0, \tag{7.51b}$$

where $$\gamma_n(t) = (1/2)\int_{D(t)} \partial/\partial t[|\boldsymbol{U}_n|^2 + |R_n|^2]dv$$
$$+ (1/2)\int_{D(t)}\{\langle \mathbf{u}_o\rangle.\nabla[|\boldsymbol{U}_n|^2 + |R_n|^2]\}dv$$
$$+ \int_{D(t)}[(\boldsymbol{U}_n.\nabla)\langle \mathbf{u}_o\rangle].\boldsymbol{U}_n dv$$

which can be rewritten as

$$\gamma_n(t) = (1/2)\mathrm{dlog}\ \rho_o/dt + \int_{D(t)}[(\boldsymbol{U}_n.\nabla)\langle \mathbf{u}_o\rangle].\boldsymbol{U}_n dv. \tag{7.51c}$$

This above relation is derived when we take into account that, respectively:

$$(1/2)\int_{D(t)} \partial/\partial t[|\boldsymbol{U}_n|^2 + |R_n|^2]dv = -(1/2)\int_{\Sigma(t)}[|\boldsymbol{U}_n|^2 + |R_n|^2]W(t, \boldsymbol{P})ds$$

due to normalization and (7.35e), and also that

$$(1/2)\int_{D(t)}\{\langle \mathbf{u}_o\rangle.\nabla[|\boldsymbol{U}_n|^2 + |R_n|^2]\}dv$$
$$= (1/2)\int_{D(t)}\nabla\{[|\boldsymbol{U}_n|^2 + |R_n|^2]\langle \mathbf{u}_0\rangle\}dv$$
$$-(1/2)\int_{D(t)}[|\boldsymbol{U}_n|^2 + |R_n|^2](\nabla.\langle \mathbf{u}_o\rangle)dv.$$

But:

$$(1/2)\int_{D(t)}\nabla[|\boldsymbol{U}_n|^2 + |R_n|^2]\langle \mathbf{u}_o\rangle\}dv$$
$$= (1/2)\int_{\Sigma(t)}[|\boldsymbol{U}_n|^2 + |R_n|^2]W(t, \boldsymbol{P})ds,$$

due to slip condition (7.36e), and

$$-(1/2)\int_{D(t)}[|\boldsymbol{U}_n|^2 + |R_n|^2](\nabla.\langle \mathbf{u}_o\rangle)dv = (1/2)\mathrm{dlog}\ \rho_o/dt$$

according to continuity equation (7.36d) and normalization condition.

At t = 0 we have, as initial conditions from (7.48d),

$$t^{+} = 0 : \Sigma_{n>1} A_n(0) \boldsymbol{U}_n(0, \mathbf{x}) = -\langle \mathbf{u}_o \rangle \tag{7.51d}$$

and

$$B_n(0) = 0, \ n = 1, 2, \ldots \tag{7.51e}$$

We derive the above initial conditions for An(t) and Bn(t) by applying the starting initial conditions (7.22d) for **u** and ρ, and this gives, first, for An the condition (7.48d), because **u** = 0 at t = 0, when we take into account the solution (7.30a) for $\mathbf{u}^*_0$ and also the decomposition (7.28) for $U = \langle U \rangle + U^*$.

The value of $B_n(0) = 0$ is related with the initial condition at t = 0, for $\rho(= 1)$, which is compatible with the leading-order solution:

$$\rho^*_0(t = 0) = 1, \text{ and } \rho^*_1(0, \vec{\mathbf{x}}) = 0.$$

Due to Eq. (7.51b) for B_n, obviously:

$$B_n(t) \equiv 0 \text{ for all t.} \tag{7.52a}$$

Concerning $A_n(0)$, its values must be derived from (7.48d/7.51d), and it depends on the value of $< \mathbf{u}_0 >$ at t = 0. On the other hand, obviously, if in condition (7.22b) $W(0, \boldsymbol{P}) = 0$, then $< \mathbf{u}_0 >$ is also zero at t = 0, and

$$A_n(0) = 0$$

which also implies that

$$A_n(t) \equiv 0 \text{ (is zero for all t)} \tag{7.52b}$$

and then the oscillations are absent!

However, *if the motion of the wall of the deformable in time cavity is started impulsively from rest (or accelerated from rest to a finite velocity in a time O(M)), then accordingly we have*:

$$W(0^-, \mathbf{P}) = 0 \ldots \text{ but} : W(0^+, \mathbf{P}) \neq 0 \tag{7.53a}$$

and the same holds for the averaged velocity, $< \mathbf{u}_0 >$.

In this case we have $A_n(0^+) \neq 0$, and as consequence:

$$A_n(t) \text{ is also non} - \text{zero, when } t \geq 0^+ \tag{7.53b}$$

7.2.8 Some Concluding Comments

The most important result we obtain is as follows. *If the motion of the wall of the deformable (in time) cavity, where the inviscid gas is confined, is started impulsively from rest, then the acoustic oscillations remain present and have a strong effect on the pressure. Therefore, this pressure would be felt by a gauge, and would not be related to the mean (averaged) motion. The same holds if the motion of the wall is accelerated from rest to a finite velocity in a time O(M).*

We again stress the necessity of building into the structure of the non-viscous solution for $U(\mathbf{u}, \rho, p, S)$, when we consider the Euler equations (7.22a), a multiplicity of times – a family of fast times − in contrast to Müller [150], Meister [151], and Ali [152].

If we deal with a *slightly viscous flow*, when the Mach number $M \ll 1$, we must start from the full unsteady NS–F equations. In such a dissipative (viscous and heat-conducting) case, we bring into the analysis a second small parameter Re^{-1}, the inverse of a (large) Reynolds, $Re \gg 1$, number, and we must then expect that the acoustic oscillations are damped out.

Unfortunately, a precise analytical (when a similarity rule between M and Re^{-1} is assumed) multiple time-scale asymptotic investigation of this damping phenomenon appears to be even more difficult problem, and raises many questions! This damping problem is considered mainly in the framework of the hypothesis (see [13], pp. 148–161):

$$Re >> 1/M \tag{7.54}$$

In Müller [150], the author provides insight into the compressible Navier–Stokes equations at low Mach number when slow flow is affected by acoustic effects in a bounded domain over a long time! As an example of an application, Müller mentions a closed piston-cylinder system in which the isentropic compression due to a slow motion is modified by acoustic waves. Müller uses only a two-time scale analysis, which is obviously insufficient for the elimination of the secular terms in derived approximate systems (as has been mentioned in Sect. 7.3.5).

The results obtained recently by Ali [152] are more interesting than those formally derived by Müller [150], in spite of the fact that in Ali's paper a two-time scale analysis is again used – the Euler equations for a compressible perfect fluid being considered on a bounded time-dependent domain $\Omega_t \in \Re^n$, where Ω_0 denotes the domain at the initial time $t = 0$. The evolution of the bounded time-dependent domain is described by a family of invertible maps:

$$\Phi_t : \Re^n \to \Re^n \tag{7.55a}$$

depending continuously on the time t, such that $\Omega_t = \Phi_t(\Omega_0)$ for all t.

This severe assumption on the domain Ω_t is, nevertheless, general enough to include a moving rigid domain, or a cylinder cut by a fixed surface and a moving surface (piston problem), or a contracting–expanding sphere (star). In the particular case of a moving rigid domain, Ali [152], p. 2023, writes:

$$\Phi_t(\mathbf{x}) = \mathbf{x} + \mathbf{c}(t). \tag{7.55b}$$

The map Φ_t has a geometric meaning and is related neither to the fluid motion nor to the Lagrangian variables. Moreover, Φ_t does not need to be globally unique, since only its restriction to a neighbourhood of the boundary $\partial\Omega_o$ characterizes the motion of the domain's boundary $\partial\Omega_t$.

From the conclusions of Ali [152], pp. 2037–2038, we mention that his analysis is not conclusive, since the theory presented is not capable of providing a full resolution of high-frequency acoustics. Nevertheless, the representation derived, in Sect. 6 of his paper, provides a hint of a partial theoretical comprehension of the acoustic modes generated by the motion of the boundary.

Obviously, the main key point is that one fast time variable is not sufficient to describe the sequence of modes produced by a generic motion of the boundary. Thus we need to extend the "Ali [152] theory" to include a family of fast time variables non-linearly related to the slow time and (eventually) to the space variables.

It is an open question whether the number of independent fast variables for each term of the asymptotic expansion should be increased with the order of the term. This extension, mentioned by Ali, has a theoretical interest in itself, and is a necessary step for the development of efficient numerical schemes for low Mach number flows in a time-dependent bounded domain (as is the case in a combustion problem). It was, in fact, discovered by J.-P. Guiraud and myself 30 years ago, in 1980, and it is formally realized in Sects. 7.2.1–7.2.6 above.

Obviously, the case when the starting equations, in place of (7.22a), are the full unsteady NS-F equations, with

$$\mathrm{Re} = O(1) \text{ and } \mathrm{Pr} = O(1) \text{ fixed, with } \mathrm{M} \to 0 \tag{7.56}$$

is a more difficult problem!

Here we mention only a partial result which concerns the derivation of the following averaged reduced system:

$$\begin{aligned} &\nabla[\langle p_2\rangle/\gamma\ \rho_o(t)] + \langle \mathbf{F}\rangle = 0 \\ &\nabla.\langle \mathbf{u}_1\rangle + \langle G\rangle = 0 \\ &\langle H\rangle - (\gamma - 1)T_w(t)\langle G\rangle = 0. \end{aligned} \tag{7.57}$$

This average system merits careful analysis. In the first equation of (7.57), in $< \mathbf{F}>$, the Reynolds number Re is present, while in the third equation, in $< H>$, the Péclet (Pé = PrRe) number is present. From this set of equations a Navier–Fourier-type nearly incompressible average system of equations has been derived (see [13], pp. 149–154).

A difficult problem is also the study of the viscous damping of the acoustic fast oscillations. Obviously, the inviscid theory developed in Sect. 7.2 do not present the possibility of investigating this damping process, and this is also the case when $\mathrm{Re} = O(1)$ fixed in the framework of a Navier–Fourier model.

On the other hand, if we deal with a slightly viscous flow (large Reynolds number, $\mathrm{Re} \gg 1$), we must start from NS–F equations, in place of the Euler equations analysed above, and bring into the analysis a second small parameter:

$$\varepsilon^2 = 1/\mathrm{Re} << 1 \tag{7.58a}$$

which is the inverse of the Reynolds number.

We must then expect that the acoustic fast oscillations are damped out. Unfortunately, a precise analysis of this damping phenomenon – which appears, for instance, when a general similarity rule

$$\varepsilon^2 = \mathrm{M}^{\beta}, \quad \beta > 0 \tag{7.58b}$$

is assumed – appears not to be an easy task, and raises many questions.

Therefore, it is first necessary to take into account an acoustic-type inhomogeneous system with a family of very slow times via a new operator ($\boldsymbol{\mathcal{D}}$):

$$\delta\, \boldsymbol{\mathcal{D}}\, \mathrm{U} \text{ in } (7.27) \tag{7.58c}$$

where it is assumed that order $\delta > \mathrm{M}$!

It appears that as a consequence of the inhomogeneity, a boundary–layer analysis is necessary, which is related with a Stokes-layer of thickness

$$\chi^2 = \varepsilon^2\, \mathrm{M}. \tag{7.58d}$$

The analysis of the Stokes-layer equations is rather complicated, but is necessary for the investigation of this damping process.

A matching condition (evaluating the flux outward from the Stokes layer) between the acoustic and Stokes-layer components of the normal velocity gives:

$$\chi = \delta \mathrm{M} \Rightarrow \delta = \chi/\mathrm{M} = [\mathrm{Re}\ \mathrm{M}]^{-1/2}. \tag{7.59}$$

However, further investigations are necessary if we want to understand *how viscous damping operates when this relation is not satisfied*, and other points meriting investigation include the *behaviour of the Rayleigh layer*.

For a deeper investigation of dissipative effects in the case of a time-dependent cavity – a problem which has practical interest in the simulation of the *starting process of a space rocket driven by a stream of gases emitted behind it when the fuel is burned inside* – it is necessary to consider the similarity rule (7.58b) for large

Reynolds, Re $>>1(\varepsilon<<1)$, numbers and low Mach, $M \ll 1$, numbers – at least during the starting (at $t^+ = 0$) short time interval.

Various interesting results relating to the above-mentioned "combustion problem" are included in our monograph [13] devoted to low Mach numbers: Chapter 1, pp. 14–15, discusses a simple model for combustion (with various references); Chap. 2, pp. 32–33, presents a brief account of the low Mach number theory applied to combustion (with references); and Sects. 3.3 and 3.4 in Chap. 3 deal with different non-viscous and heat-conducting models in a bounded time-dependent domain.

Concerning the *damping of acoustic oscillations* by viscosity, we observe that the Rayleigh-layer emerges, in the solution of the problem related with the damping phenomenon, because of the conditions on the wall in Stokes-layer equations. The investigations of the evolution of the Rayleigh-layers with time is a difficult problem! If, on the one hand, the Stokes-layer corresponds to acoustic (oscillating) eigenfunctions of the cavity, the Rayleigh-layer corresponds, on the other hand, to conditions on the wall of this cavity. Moreover, the thickness of the Stokes-layer, being given by

$$\chi = [M/Re]^{1/2} \tag{7.60}$$

is independent of the time and the behaviour of the Stokes-layer, for a large time, does not have any influence on the Stokes-layer! Concerning the Rayleigh-layer, however, its thickness grows as the square root of the time, and obviously a deeper analysis of the interaction between these two boundary-layers, when time increase to infinity, is required.

A last remark concerning the adaptation to the initial conditions in a time-dependent bounded container is that our first paper,[1] with Guiraud [85], includes some preliminary results concerning this problem:

> Only with the help of a multiple-scale technique, via an infinity of fast times (designed to cope with the infinity of period of free vibration of the bounded container), do we have the possibility of eliminating the various secular terms in derived model equations.

Unfortunately, a two-time, simple technique, with t and $\tau = t/M$, is not adequate, because such a technique does not provide the possibility of eliminating all seculars terms. The main reason is that the acoustic eigenfrequencies of the bounded container appear in the internal problem, and because the container is a function of the slow time t (the time of the boundary velocity–wall velocity related with the deformation of the container in time), these eigenfrequencies are also functions of the (slow) time τ. More precisely, when the wall, at $t^+ = 0$, is started impulsively from rest ($t^- = 0$), the limiting case

[1] This paper was subject of a communication during the 7th "Colloque d'Acoustique Aerodynamique" in Lyon (France), 4–5 November 1980 and, also, of a very fruitful discussion with D. G. Crighton during this "Lyon's colloque".

$$\mathrm{M} \to 0 \text{ with t fixed} \tag{7.61}$$

is singular near the initial time, and it is necessary to consider a local-in-time limiting case:

$$\mathrm{M} \to 0 \text{ with } \tau \text{ fixed} \tag{7.62}$$

with $\tau = \mathrm{t}/\mathrm{M}$.

In such a case, close to initial time $(\tau = 0)$, we derive the classical equations of acoustics and obtain the corresponding solution (see [13], Sect. 3.3.4) of the Chapter 3. Unfortunately, this solution of the acoustic problem does not tend to a defined limit when τ tends to infinity, which shows that matching is not possible!