Chapter 5
The Structure of Unsteady NS–F Equations at Large Reynolds Numbers

Numerous papers are devoted to investigations of NS–F equations in the case of large Reynolds numbers, and many references are cited in my *Theory and Applications of Viscous Fluid Flows* [47]. Curiously, the case of unsteady NS–F equations for large Reynolds numbers, when Re tends to infinity, has been very poorly considered. However, in Chap. 4 of [47] (Sect. 4.3) there is a presentation of the asymptotic structure of unsteady state NS–F equations.

Below, in this Chap. 5, I again give a presentation of this structure – the reason being that such unsteady NS–F equations structure, at Re $\uparrow\propto$, very well illustrates the importance of the various limiting processes related to fixed time and space, and forms a guideline in the elaboration of our mathematics for the RAMA.

On the other hand, such a structure illuminates the various consequences of the singular nature of the Prandtl (1904) concept of the boundary layer. A strong singularity for NS–F equations opens (it seems to me) new perspectives for the resolution of various paradoxes encountered in unsteady compressible fluid flow theory. It is surprising that this close initial time singularity of the Prandtl boundary layer, was for so long ignored.

5.1 Introduction

Fluid dynamicians and applied mathematicians have always found fluid dynamics to be a rich and interesting field for investigations, because the basic system of partial derivative equations for a Newtonian fluid – the so-called Navier–Stokes–Fourier equations (NS–F) equations – have a great capacity for producing various particular fluid flow models.

In particular, a large class of such fluid flow models is closely linked with the analysis of a dimensionless form of the NS–F system, and more specifically with the large Reynolds numbers (Re $\uparrow\propto$) for a compressible, weakly viscous, and heat-conducting fluid flow.

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Indeed, in technologically and geophysically relevant fluid flows, Re is usually quite large. In 1904 Prandtl took into account this fact and derived (in an ad hoc manner) his well-known 'Prandtl Boundary-Layer (BL) equations'. Surprisingly, it seems that the case of an unsteady fluid flow, when initial conditions are prescribed, has not been carefully considered, and this BL Prandtl concept has become default and become singular near $t = 0$! Only in 1980, in a short note [98], was I the (apparently) first to show that:

> The limiting form (at $Re \uparrow \infty$) of the unsteady NS–F equations, near $t = 0$ and in the vicinity of a wall ($z = 0$), bounding below the compressible, viscous and heat-conducting fluid flow, is identified – in place of Prandtl BL equations – rather with the equations of the Rayleigh compressible problem, considered first in 1951 [99] by Howarth.

In Chap. 5 I fully intend to present a more deep and careful investigation of this singular problem, which not only has an obvious theoretical interest, but also seemly a practical one. For the NS–F unsteady full equations, we have considered, as a typical working case, an "emergency situation": namely, a sudden rise in temperature locally on the wall at initial time $t = 0$ – which is a possible application of our new four regions structure for the NS–F unsteady equations!

5.2 The Emergence of the Four Regions as a Consequence of the Singular Nature of BL Equations Near t = 0

The main feature of the well-known Prandtl 1904 BL concept in a thin region near the wall ($z = 0$), in an aerodynamics problem for high Reynolds number, is linked with the strong simplification of the equation of motion for the vertical component w of the velocity $\mathbf{u} = (\mathbf{v}; w)$.

In steady and unsteady, in compressible viscous, in heat-conducting, and in incompressible viscous fluid flows, this Prandtl BL concept in all cases produces a very degenerate limit equation, when $Re \uparrow \infty$, for w!

When we work with dimensionless quantities, then, for the variation of the pressure in the direction normal to the (horizontal) wall, $z = 0$, relative to vertical coordinate, z, we obtain (when the gravity force is not taken into account):

$$\partial p/\partial z = 0. \tag{5.1}$$

and in particular, the partial derivative in time for the component w of the velocity disappears in a BL system of equations!

If this failure seems not to have serious consequences in the usual case of steady or unsteady incompressible viscous fluid flows, conversely, this is not the case for an unsteady compressible viscous and heat-conducting fluid flow. In such a case, as an unfortunate consequence of (5.1), we have a new "four regions" structure for NS–F equations governing these fluid flows at high Reynolds numbers.

This new structure of unsteady NS–F equations seems, in particular, very significant for the various applications linked with the heat emergency situations (sudden rise of a thermal source) – explosions, fires, failures of oil and gas pipelines, and so on – in a local domain on the wall in contact with the fluid.

Such a "four regions" structure, at $\mathrm{Re} \uparrow \infty$, is linked with four limiting processes in full unsteady NS–F equations and replaces the two classical regions, "Euler–Prandtl" and regular coupling, linked with the following two limiting processes (the horizontal coordinates x and y being fixed):

$$\mathrm{Lim}^{\mathrm{E}} = [\varepsilon \downarrow 0, \text{ with t and z fixed}]. \tag{5.2a}$$

and

$$\mathrm{Lim}^{\mathrm{Pr}} = [\varepsilon \downarrow 0, \text{ with t and } \zeta = z/\varepsilon \text{ fixed}], \tag{5.2b}$$

As a consequence of the strong degeneracy linked with (5.1), in the unsteady case, the limiting process (5.2b) is singular near $t = 0$. Therefore, because the partial time derivative of the vertical component of the velocity is absent in BL equations, we do not have the possibility of taking into account the corresponding data which is prescribed for full unsteady NS–F equations.

It is necessary to consider a third limiting process, inner in time – a so-called "acoustic" limiting process:

$$\mathrm{Lim}^{\mathrm{Ac}} = [\varepsilon \downarrow 0, \text{ with } \tau = t/\varepsilon \text{ and } \zeta = z/\varepsilon \text{ fixed}], \tag{5.3}$$

An acoustic problem must be considered in a third "acoustic region" close to initial time, and then (if possible) a matching with the boundary-layer (BL) region far from the region of the acoustics. Indeed, the consideration, in this acoustic region, of an unsteady adjustment problem, when $\tau \to \infty$, presents the possibility (in principle) of prescribing the correct initial conditions for unsteady BL equations significant only far of the initial time in the "Prandtl BL region".

But a new problem emerges because in this third non-viscous, near the initial time acoustic region, we do not have the possibility of taking into account the thermal condition on the wall $z = 0$, prescribed for NS–F equations in the framework of the heat emergency problem.

As a consequence, it is necessary to consider a fourth limiting process simultaneously near $t = 0$ and $z = 0$:

$$\mathrm{Lim}^{\mathrm{Ra}} = [\varepsilon \downarrow 0, \text{ with } \theta = t/\varepsilon^2 \text{ and } \eta = z/\varepsilon^2 \text{ fixed}], \tag{5.4}$$

In such a case, in this corner small fourth region, we derive the unsteady one-dimensional NS–F equations governing the compressible Rayleigh problem, with the corresponding thermal condition on $z = 0$ (Fig. 5.1).

Precisely, the above (related with the limiting process (5.4)) compressible viscous and heat-conducting Rayleigh problem presents the possibility of producing an answer to following question: What is the significant problem, governing the unsteady compressible viscous and heat-conducting fluid flow, emerging as a result of a suddenly rise in temperature locally on the wall, z = 0, at time, t = 0?

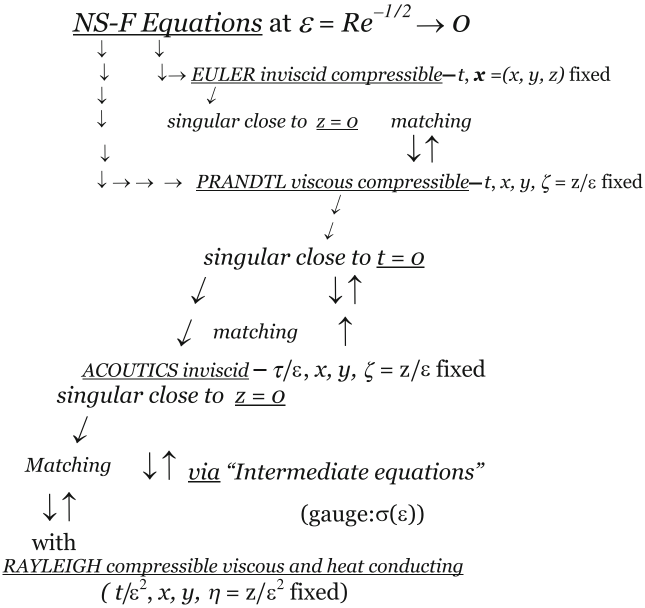

Fig. 5.1 Four regions structure of NS–F unsteady equations at large Reynolds number

In Sect. 5.3 the formulation of the starting unsteady NS–F problem is given, while Sect. 5.4 is devoted to the derivation of the corresponding four problems linked with the above four limiting processes – (5.2a, 5.2b), (5.3), and (5.4). Section 5.5 concerns the problem of the unsteady adjustment (via the Rayleigh problem significant in the fourth region) to Prandtl BL evolution, in time, which is significant in the second region close to wall z = 0, but far from the third and fourth regions near the initial time t = 0. In Sect. 5.6 some conclusions are presented.

5.3 Formulation of the Unsteady NS–F Problem

We consider the atmospheric dry air as a thermally perfect, viscous, and heat-conducting gas, with constant dissipative coefficients. In such a case, the dimensionless NS–F equations are written (for dimensionless functions, **v**, w, p, ρ, and T) in the following form – all functions, variables, and coefficients being non-dimensional:

$$\partial\rho/\partial t + \mathbf{D}\cdot(\rho\mathbf{v}) + \partial(\rho w)/\partial z = 0, \tag{5.5}$$

$$\rho[\partial \mathbf{v}/\partial t + (\mathbf{v}\,.\,\mathbf{D})\mathbf{v} + w\partial \mathbf{v}/\partial z] + (1/\gamma M^2)\mathbf{D}p = \varepsilon^2\{\Delta \mathbf{v} + (1/3)\,\mathbf{D}\,[\mathbf{D}\,.\,\mathbf{v} + \partial w/\partial z]\}, \tag{5.6}$$

$$\rho[\partial w/\partial t + (\mathbf{v}\,.\,\mathbf{D})w + w\,\partial w/\partial z] + (1/\gamma M^2)\partial p/\partial z = \varepsilon^2\{\Delta w + (1/3)\partial/\partial z[\mathbf{D}\,.\,\mathbf{v} + \partial w/\partial z], \tag{5.7}$$

$$\rho[\partial T/\partial t + (\mathbf{v}\,.\,\mathbf{D})T + w\,\partial T/\partial z] + (\gamma - 1)p\,[\mathbf{D}\,.\,\mathbf{v} + \partial w/\partial z] = (\gamma/Pr)\,\varepsilon^2\Delta T + \gamma(\gamma-1)\varepsilon^2 M^2\{\Phi - (2/3)[\mathbf{D}\,.\,\mathbf{v} + \partial w/\partial z]^2\}, \tag{5.8}$$

with

$$p = \rho T, \tag{5.9}$$

where the horizontal (relative to coordinates (x, y)) velocity vector is $\mathbf{v} = (u, v)$, and viscous dissipation is written in the following form:

$$\Phi = [\partial u/\partial z + \partial w/\partial x]^2 + [\partial v/\partial z + \partial w/\partial y]^2 + [\partial u/\partial y + \partial v/\partial x]^2 + 2\left[(\partial u/\partial x)^2 + (\partial v/\partial y)^2 + (\partial w/\partial z)^2\right], \tag{5.10}$$

and $\Delta = \mathbf{D}^2 + \partial^2/\partial z^2$ with $\mathbf{D} = (\partial/\partial x,\ \partial/\partial y)$.

For the above evolution equations (5.5)–(5.8) we write. as initial conditions at initial time $t = 0$:

$$t^- \geq 0 : \mathbf{v} = 0,\ w = 0,\ \rho = 1 \text{ and } T = 1. \tag{5.11}$$

At the horizontal solid wall, $z = 0$, we assume:

$$z = 0 : \mathbf{v} = w = 0 \tag{5.12a}$$

and

$$T = \Theta(t/\beta, P),\ \text{when } t^+ \geq 0,\ \Theta(t/\beta, P) \equiv 0,\ \text{when } t^- \leq 0, \tag{5.12b}$$

where

$$\beta \equiv t^*/t^0 \ll 1, \tag{5.13}$$

is a ratio of two time scales: t^*, a short time scale, in comparison to characteristic evolution time scale t°, which appears in Strouhal number S in (3.3).

In condition (5.12a, 5.12b), the dimensionless function $\Theta(t/\beta, P)$ is used to simulate an emergency of a thermal spot at $t^+ \geq 0$, P being a point on a local domain on the wall, $P \subset D$, for which the reference length scale L° is a diameter for this time-dependent domain $D = D(t)$.

In the above four dimensionless NS–F equations (5.5)–(5.8), we have assumed that (S is the Strouhal number):

$$S \equiv 1 \Rightarrow U^0 = L^0 / t^0,$$

and in the fact (t' is the time with dimension):

$$\Theta(t/\beta,\ P) \equiv \Theta(t'/t^*,\ P).$$

The characteristic time $t^\circ = L^\circ/U^\circ$ is a "long" time and characterizes the evolution of fluid flow after the thermal spot emergency, while the characteristic "short" – small – time, $t^* \ll t^\circ$, is linked just with this emerging thermal spot short (time) interval.

The above formulated NS–F initial-boundary value problem, (5.5)–(5.13), is a very complicated mathematical problem, and the rigorous proof concerning its well-posedness is obviously an intractable question!

In fact, our objective below is rather to analyze the specific structure of this problem, when ε tends to zero – for large Reynolds number − and consider the relations, via matching, between the four particular fluid flows regions discussed in Sect. 5.2.

The existence of these four fluid flows regions, at large Reynolds number, shows that we have the possibility of considering this above NS–F system of equations as a "puzzle!" A challenging, but difficult, approach is to "deconstruct"[1] this puzzle, relative to limiting values − vanishing or infinity − of various reference parameters, Re, M, or Pr..., in order to unify − by a RAMA process – the set of various, ill-assorted, partial approximate system of equations, customarily used in classical fluid dynamics.

We observe that from our above formulated mathematical–physical (5.11)–(5.13) problem for the NS–F equations (5.5)–(5.8) with (5.9), it follows that we have the possibility of first investigating not only the initial stage of the motion, emerging as a cause and effect of a sudden rise in temperature locally on the wall $z = 0$ at $t = 0$ in a corner fourth region, but also the evolution of this "thermal accident" in second, Prandtl, and first, Euler, BL viscous and inviscid Eulerian regions, by two matching processes.

[1] In fact, the system of NS–F equations, despite its dimensionless form, do not have a fixed meaning, even if the various reference parameters, in NS–F equations, give a good idea for an investigation in this way! A real meaning is "created", each time, in the act of the RAM Approach, relative to a precise parameter (high or low), via the derivation of a consistent simplified model, this process is just a deconstruction ("à la Derrida") of NS–F system of equations.

The considered physical case is a typical free problem independent of any given external flow. This case can be generalized for the atmosphere, where gravity plays an obvious and important role in emerging convective local motion. For such an atmospheric case, a typical example is a forest fire, which is actually a very bad accident which causes a large amount of damage, and is considered to be an "environmental disaster"!

5.4 Derivation of the Corresponding Four Model Problems

Below we consider the derivation of approximate leading-order equations for the corresponding four limiting processes (5.2a, 5.2b), (5.3), and (5.4), discussed in Sect. 5.2, and linked with the limiting process $\varepsilon \downarrow 0$, from our full NS–F problem: (5.5)–(5.9), with (5.10), and (5.11)–(5.13) – our main objective being a consistent obtention of these four particular systems of equations related to the four limiting processes (5.2a, 5.2b)–(5.4).

5.4.1 Euler–Prandtl Regular Coupling

When we consider the first, Euler, limiting process (5.2a), Lim^{E}, at $\varepsilon \downarrow 0$, with t and z fixed, from NS–F, evolution equations (5.5)–(5.8), with (5.9), we obtain a strong degeneracy (close to the horizontal solid wall $z = 0$), which leads to a Euler system of equations for the leading-order functions, $\mathbf{v}_E,\ w_E,\ p_E,\ T_E$ and ρ_E, in the following Euler asymptotic expansion associated with (5.2a):

$$(\mathbf{v}, w) = (\mathbf{v}_E, w_E) + \varepsilon\left(\mathbf{v}_E{}^1, w_E{}^1\right) + \ldots, \tag{5.14a}$$

$$(p, T, \rho) = (p_E, T_E, \rho_E) + \varepsilon(p_E{}^1, T_E{}^1, \rho_E{}^1) + \ldots, \tag{5.14b}$$

where the Eulerian terms (with "$_E$" as subscript) are dependent on t, x, y, and z. We then obtain the following system of Euler inviscid compressible, adiabatic, equations:

$$\partial\rho_E/\partial t + \mathbf{D}.(\rho_E \mathbf{v}_E) + \partial(\rho_E w_E)/\partial z = 0; \tag{5.15a}$$

$$\rho_E[\partial\mathbf{v}_E/\partial t + (\mathbf{v}_E.\mathbf{D})\mathbf{v}_E + w_E\partial\mathbf{v}_E/\partial z] + (1/\gamma M^2)\mathbf{D}p_E = 0; \tag{5.15b}$$

$$\rho_E[\partial w_E/\partial t + (\mathbf{v}_E.\mathbf{D})w_E + w_E\partial w_E/\partial z] + (1/\gamma M^2)\partial p_E/\partial z = 0, \tag{5.15c}$$

$$\rho_E[\partial T_E/\partial t + (\mathbf{v}_E.\mathbf{D})T_E + w_E \partial T_E/\partial z] + (\gamma - 1)p_E[\mathbf{D}.\mathbf{v}_E + \partial w_E/\partial z] = 0, \tag{5.15d}$$

with

$$p_E = \rho_E T_E. \tag{5.15e}$$

Because all the dissipative terms, in the right-hand side of NS–F equations (5.6)–(5.8), are absent in limiting, leading-order, Euler equations (5.15a–5.15d) (see, for instance [100], and [37], Chap. 9), these Euler equations cannot be valid in the vicinity of the wall $z = 0$.

The significant equations valid near $z = 0$, which replace the above Euler equations (5.15a–5.15e), are derived when we introduce, in place of z, an inner vertical coordinate, significant in the vicinity of $z = 0$: namely,

$$\zeta = z/\varepsilon, \tag{5.16a}$$

when we consider the second, Prandtl, limiting process (5.2b), $\mathrm{Lim}^{\mathrm{Pr}}$ at $\varepsilon \downarrow 0$, with t and ζ fixed.

In such a case, again, from full NS–F equations (5.5)–(5.8), with (5.9), for the leading-order functions $\mathbf{v}_{\mathrm{Pr}}, p_{\mathrm{Pr}}, T_{\mathrm{Pr}}, \rho_{\mathrm{Pr}}$, and second−order vertical component of the velocity, ${w_{\mathrm{Pr}}}^1$, we consider the following asymptotic, *à la* Prandtl, expansion associated with (5.2b):

$$(\mathbf{v}, w) = (\mathbf{v}_{\mathrm{Pr}}, 0) + \varepsilon\left({\mathbf{v}_{\mathrm{Pr}}}^1, {w_{\mathrm{Pr}}}^1\right) + \ldots, \tag{5.17a}$$

$$(p, T, \rho) = (p_{\mathrm{Pr}}, T_{\mathrm{Pr}}, \rho_{\mathrm{Pr}}) + \varepsilon({p_{\mathrm{Pr}}}^1, {T_{\mathrm{Pr}}}^1, {\rho_{\mathrm{Pr}}}^1) + \ldots, \tag{5.17b}$$

where the Prandtl (BL) terms (with "$_{\mathrm{Pr}}$" as subscript) are dependent on t, x, y, and ζ. Then, with (5.2b) and (5.17a, 5.17b), we derive the Prandtl BL unsteady equations (see Stewartson [101], and the more recent book by Oleinik and Samokhin [102]):

$$\partial \rho_{\mathrm{Pr}}/\partial t + \mathbf{D}.(\rho_{\mathrm{Pr}}\mathbf{v}_{\mathrm{Pr}}) + \partial(\rho_{\mathrm{Pr}}{w_{\mathrm{Pr}}}^1)/\partial\zeta = 0; \tag{5.18a}$$

$$\begin{aligned}\rho_{\mathrm{Pr}}[\partial \mathbf{v}_{\mathrm{Pr}}/\partial t + (\mathbf{v}_{\mathrm{Pr}}.\mathbf{D})\mathbf{v}_{\mathrm{Pr}} + {w_{\mathrm{Pr}}}^1 \partial \mathbf{v}_{\mathrm{Pr}}/\partial\zeta]&\\ + (1/\gamma M^2)\mathbf{D}p_{\mathrm{Pr}} = \partial^2 \mathbf{v}_{\mathrm{Pr}}/\partial\zeta^2;&\end{aligned} \tag{5.18b}$$

$$\partial p_{\mathrm{Pr}}/\partial\zeta = 0; \tag{5.18c}$$

$$\begin{aligned}\rho_{\mathrm{Pr}}\left[\partial T_{\mathrm{Pr}}/\partial t + (\mathbf{v}_{\mathrm{Pr}}.\mathbf{D})T_{\mathrm{Pr}} + {w_{\mathrm{Pr}}}^1 \partial T_{\mathrm{Pr}}/\partial\zeta\right]&\\ + (\gamma - 1)p_{\mathrm{Pr}}[\mathbf{D}.\mathbf{v}_{\mathrm{Pr}} + \partial {w_{\mathrm{Pr}}}^1/\partial\zeta]&\\ = (\gamma/\mathrm{Pr})\partial^2 T_{\mathrm{Pr}}/\partial\zeta^2 + \gamma(\gamma - 1)M^2|\partial \mathbf{v}_{\mathrm{Pr}}/\partial\zeta|^2,&\end{aligned} \tag{5.18d}$$

with

$$p_{Pr} = \rho_{Pr} T_{Pr}. \tag{5.18e}$$

It is well known that both the above system of equations – outer, Euler (5.15a–5.15e), and inner, Prandtl (5.18a–5.18e) – are related to the following classical matching relation (discussed in Sect. 6.4.2):

$$\lim_{\zeta \uparrow \infty} \left[\mathrm{Lim}^{Pr}\right] = \lim_{z \downarrow 0} \left[\mathrm{Lim}^{E}\right], \tag{5.19a}$$

and, as a first consequence of (5.19a), we obtain for the Euler outer system of Eqs. 5.15a–5.15e the following single (slip!) condition:

$$w_E = 0 \text{ at } z = 0. \tag{5.19b}$$

Then, as a second consequence of (5.19a), we see that from the strong degenerated equation (5.18c), in the Pranddtl system of Eqs. 5.18a–5.18e, we have the possibility of relating the constant value of p_{Pr}, with respect to vertical BL coordinate, ζ, with the value of p_E at $z = 0$:

$$p_{Pr}(t, x, y) \equiv p_E(t, x, y, 0) = p_{E,0}(t, x, y)$$

and

$$\mathbf{D} p_{Pr} \equiv (\gamma M^2) \rho_{E,0} [\partial w_{E,0}/\partial t + (\mathbf{v}_{E,0}.\mathbf{D}) w_{E,0}]. \tag{5.19c}$$

Now, concerning the thermal condition (5.12), with the parameter β given by (5.13), we are obliged to assume that (when t is fixed, in Lim^{Pr}, far of the initial time):

$$\beta = \beta(\varepsilon) \downarrow 0 \text{ with } \varepsilon \downarrow 0, \tag{5.20a}$$

and we have only the possibility of writing, for the BL equations (5.18a–5.18e), the following conditions:

$$\begin{aligned} &\text{on } \zeta = 0 : \mathbf{v}_{Pr} = 0, {w_{Pr}}^1 = 0, \\ &T_{Pr} = \Theta(\infty, P), \text{ at } t > 0 \text{ fixed.} \end{aligned} \tag{5.20b}$$

Obviously, in the framework of the Euler–Prandtl regular coupling we do not have the possibility of taking into account the “thermal accident” linked with the “temperature emergency at initial time on the wall z = 0.”

On the other hand, the above Prandtl system of Eqs. 5.18a–5.18d, with (5.18e), due to the reduced BL equation (5.18c), must be considered as a system of two equations for $\mathbf{v}_{Pr}$ and T_{Pr}, while the Prandtl vertical velocity ${w_{Pr}}^1$ must be computed through the following relation:

$$ {w_{Pr}}^1 = \left(T_{Pr}/p_{E,0}\right)\int_0^{\zeta}\{[\partial/\partial t + (\mathbf{v}_{Pr}.\mathbf{D})]\rho_{Pr} + \rho_{Pr}(\mathbf{D}.\mathbf{v}_{Pr})\}\mathrm{d}\,\zeta, \tag{5.20c} $$

since ${w_{Pr}}^1 = 0$ on $\zeta = 0$, according to the second condition in (5.20b), and the (matching) relation:

$$ \lim_{\zeta\uparrow\infty}\left[{w_{Pr}}^1\right] = {w_{E,0}}^1, \tag{5.20d} $$

is, in fact, a regular coupling condition with the second-order linearized Euler equations for the terms with "${}_E{}^1$" proportional to ε, in Euler asymptotic expansion (5.14a, 5.14b).

However, the problem of two initial conditions for the two unsteady Prandtl equations, (5.18b) and (5.18d), for $\mathbf{v}_{Pr}$ and T_{Pr}, is more subtle, and is a direct consequence of the change of the nature of Prandtl equations relative to the incomplete parabolic character of NS–F equations (see [59, 60, 67]).

In fact, for $\mathbf{v}_{Pr}$ and T_{Pr} we have a system of two hyperbolic–parabolic equations [102]:

$$ \rho_{Pr} D_{Pr}\mathbf{v}_{Pr}/Dt - \partial^2\mathbf{v}_{Pr}/\partial\zeta^2 = \boldsymbol{F}, \tag{5.21a} $$

$$ \rho_{Pr} D_{Pr} T_{Pr}/Dt - (\gamma/Pr)\partial^2 T_{Pr}/\partial\zeta^2 = G, \tag{5.21b} $$

where $D_{Pr}/Dt = \partial/\partial t + \mathbf{v}_{Pr}.\mathbf{D}$, and the right-hand side $\boldsymbol{F}$ and G are a collection of terms with the first-order derivatives relative to $\mathbf{D}$ and ζ.

The continuity equation in the Prandtl system (5.18a–5.18d) is, in fact, an equation determining ${w_{Pr}}^1$, due to (5.20c), and in place of (5.18e) the relation:

$$ \rho_{Pr} = p_{E,0}/T_{Pr}, \tag{5.21c} $$

determines the density ρ_{Pr}.

Without loss, the generality the hyperbolic–parabolic character of the system of Eqs. 5.21a, 5.21b is related to the transmission of the information in the planes $\zeta =$ const, along the trajectories linked with the derivative operator $D_{Pr}/Dt = \partial/\partial t + \mathbf{v}_{Pr}.\mathbf{D}$, supporting the hyperbolicity – this information being instantaneously diffused by vertical coordinate ζ on each normal direction to the wall $\zeta = 0$, at each moment t (which just characterizes the "parabolicity").

We observe also, that the domain of the dependence, for a fixed moment, of the point on the wall has an angular form, but in the unsteady case the precise form of this domain is not easy definable.

The above brief discussion shows explicitly that there is obviously a change in the mathematical character of the fluid dynamics equations, when we pass from a fourth order in time (four partial derivatives in time) unsteady NS–F system of Eqs. 5.5–5.8, to an unsteady Prandtl reduced system of two Eqs. 5.21a, 5.21b. This strong modification leads to a singular nature of the system (5.21a, 5.21b) near the initial time $t = 0$ – this singular nature of the Prandtl boundary-layer concept for the unsteady case being (curiously) ignored up to 1980 (See, for instance, a recent (1994) discussion by Van Dyke: *Nineteenth-century roots of the boundary-layer idea* [103]).

More precisely, it is necessary to prescribe in the framework of Prandtl BL equations (for instance for Eqs. 5.21a, 5.21b) only two initial conditions at $t = 0$. But unfortunately, the initial data for $\mathbf{v}_{Pr}$ and T_{Pr}, at time $t = 0$ (designated by: $\mathbf{v}^0{}_{Pr}$ and $T^0{}_{Pr}$), are certainly different from the initial conditions (in particular, (5.11)) for the full NS–F system of Eqs. 5.5–5.8.

Indeed, the main question is the following. Since two, NS–F, initial conditions are lost during the Prandtl limiting process (5.2b), how are the (unknown?) initial conditions:

$$\mathbf{v}_{Pr} = \mathbf{v}^0{}_{Pr} \text{ and } T_{Pr} = T^0{}_{Pr}, \text{ at } t = 0 \tag{5.22}$$

for the unsteady Prandtl BL equations (5.18b) and (5.18d), and how are the data $\mathbf{v}^{\mathbf{0}}{}_{Pr}$ and $T^{\mathbf{0}}{}_{Pr}$ linked with the initial data in conditions prescribed at the start (at $t = 0$) for the unsteady NS–F equations. The answer is strongly related to the obtention, from the full unsteady NS–F equations, of a particular system of equations valid near initial time and written relative to a short time. In fact, we have three short times:

$$\tau = t/\varepsilon, \quad \theta = t/\varepsilon^2, \text{ and } \sigma = t/\beta, \tag{5.23}$$

and the choice of data $\mathbf{v}^{\mathbf{0}}{}_{Pr}$ and $T^{\mathbf{0}}{}_{Pr}$, is realized via an unsteady adjustment problem, when the adequate(?) short time tends to infinity in unsteady adjustment equations valid near initial time!

5.4.2 Acoustic and Rayleigh Problems Near the Initial Time $t = 0$

First, with the limiting process (5.3),

$$\text{Lim}^{Ac}, \text{ when } \varepsilon \downarrow 0, \text{ with } \tau = t/\varepsilon \text{ and } \zeta = z/\varepsilon \text{ fixed},$$

from the NS–F equations (5.5)–(5.8), with (5.9), for the leading-order functions, $\mathbf{v}_{Ac}, w_{Ac}, p_{Ac}, T_{Ac}, \text{and } \rho_{Ac}$, in the following asymptotic acoustic expansions, associated with (5.3):

$$(\mathbf{v}, \mathrm{w}) = (\mathbf{v}_{\mathrm{Ac}}, \mathrm{w}_{\mathrm{Ac}}) + \varepsilon\left(\mathbf{v}_{\mathrm{Ac}}{}^{1}, \mathrm{w}_{\mathrm{Ac}}{}^{1}\right) + \ldots, \tag{5.24a}$$

$$(\mathrm{p},\ \mathrm{T}, \rho) = (\mathrm{p}_{\mathrm{Ac}}, \mathrm{T}_{\mathrm{Ac}}, \rho_{\mathrm{Ac}}) + \varepsilon(\mathrm{p}_{\mathrm{Ac}}{}^{1}, \mathrm{T}_{\mathrm{Ac}}{}^{1}, \rho_{\mathrm{Ac}}{}^{1}) + \ldots, \tag{5.24b}$$

where the acoustic terms (with "$_{\mathrm{Ac}}$" as subscript) are dependent on τ, x, y, and ζ, we derive the following, compressible, non-viscous, adiabatic, unsteady, one-dimensional, (τ, ζ), equations:

$$\partial\rho_{\mathrm{Ac}}/\partial\tau + \partial(\rho_{\mathrm{Ac}}\mathrm{w}_{\mathrm{Ac}})/\partial\zeta = 0; \tag{5.25a}$$

$$\rho_{\mathrm{Ac}}[\partial\mathrm{w}_{\mathrm{Ac}}/\partial\tau + \mathrm{w}_{\mathrm{Ac}}\partial\mathrm{w}_{\mathrm{Ac}}/\partial\zeta] + (1/\gamma\mathrm{M}^2)\partial\mathrm{p}_{\mathrm{Ac}}/\partial\zeta = 0; \tag{5.25b}$$

$$\rho_{\mathrm{Ac}}[\partial\mathrm{T}_{\mathrm{Ac}}/\partial\mathrm{t}\ + \mathrm{w}_{\mathrm{Ac}}\partial\mathrm{T}_{\mathrm{Ac}}/\partial\zeta] + (\gamma - 1)\mathrm{p}_{\mathrm{Ac}}\partial\mathrm{w}_{\mathrm{Ac}}/\partial\zeta] = 0, \tag{5.25c}$$

$$\mathrm{p}_{\mathrm{Ac}} = \rho_{\mathrm{Ac}}\mathrm{T}_{\mathrm{Ac}}. \tag{5.25d}$$

and also the following transport equation for $\mathbf{v}_{\mathrm{Ac}}$:

$$\partial\mathbf{v}_{\mathrm{Ac}}/\partial\tau + \mathrm{w}_{\mathrm{Ac}}\partial\mathbf{v}_{\mathrm{Ac}}/\partial\zeta = 0. \tag{5.26}$$

The system of Eqs. 5.25a–5.25d – valid simultaneously close to initial time, $\tau = 0$, and in a thin layer in the vicinity of the wall, $\zeta = 0$ – are identical to the usual equations for one-dimensional vertical unsteady motion in (non-viscous, adiabatic) gas dynamics.

Once w_{Ac} has been obtained, through the solution of the system (5.25a–5.25d) with the proper initial conditions (that is, the initial conditions, (5.11), for starting NS–F equations), single (slip) boundary condition:

$$\mathrm{w}_{\mathrm{Ac}} = 0 \text{ on } \ \zeta = 0, \quad \tau > 0, \tag{5.27a}$$

and matching condition (in time)

$$\mathrm{Lim}_{\tau\uparrow\infty}\mathrm{w}_{\mathrm{Ac}} = \mathrm{w}_{\mathrm{Pr}|\mathrm{t}=0} = 0, \tag{5.27b}$$

we may use the transport equation (5.26) in order to compute $\mathbf{v}_{\mathrm{Ac}}$.

Unfortunately, with the above non-viscous adiabatic system of Eqs. (5.25a–5.25d), and transport equation (5.26), we do not have the possibility of taking into account our main emergency thermal effect (via the thermal spot $\Theta(\mathrm{t}/\beta, P)$ on the wall), since the inviscid (non-viscous, adiabatic) system (5.25a–5.25d) and Eq. 5.26 are not valid close to the wall, where the conditions (5.12a, 5.12b), with (5.13), are prescribed.

The Eq. 5.26 for $\mathbf{v}_{\mathrm{Ac}}$ shows that

$$\mathbf{v}^{0}{}_{\mathrm{Pr}} = \ \mathrm{Lim}_{\tau\uparrow\infty}\mathbf{v}_{\mathrm{Ac}}, \tag{5.27c}$$

from matching, and it seems (in (5.22)) that we can assume (for our particular case), as a value for $\mathbf{v}^0{}_{Pr}$, zero; but this is certainly not the case for $T^0{}_{Pr}$.

Again, near initial time and close to the wall, where we have the conditions (5.12a, 5.12b), according to Zeyounian [98], it is necessary to consider the Rayleigh limiting process (see (5.4)):

$$\mathrm{Lim}^{Ra},\ \text{when}\ \varepsilon \downarrow 0,\ \text{with}\ \ \theta = t/\varepsilon^2\ \text{and}\ \ \eta = z/\varepsilon^2\ \text{fixed}, \tag{5.28a}$$

with

$$(\mathbf{v},\ w) = (\mathbf{v}_{Ra}, w_{Ra}) + \varepsilon\left(\mathbf{v}_{Ra}{}^1, w_{Ra}{}^1\right) + \ldots, \tag{5.28b}$$

$$(p, T, \rho) = (p_{Ra}, T_{Ra}, \rho_{Ra}) + \varepsilon(p_{Ra}{}^1, T_{Ra}{}^1, \rho_{Ra}{}^1) + \ldots, \tag{5.28c}$$

where the Rayleigh terms, in (5.28a, 5.28b), with "$_{Ra}$" as subscript, are dependent on $\theta,\ x,\ y$, and η.

In such a case, from full NS–F unsteady equations (5.5)–(5.8), (5.9), for leading-order functions, $\mathbf{v}_{Ra}, w_{Ra}, p_{Ra}, T_{Ra}$, and ρ_{RA}, we derive below the Rayleigh equations (5.29) and (5.30a–5.30d) used in the compressible Rayleigh problem, which are, in fact, the one-dimensional reduced form of the full NS–F equations valid in a corner region near initial time $\theta = 0$, and close to the wall $\eta = 0$. Namely:

$$\rho_{Ra}[\partial \mathbf{v}_{Ra}/\partial\theta + w_{Ra}\partial \mathbf{v}_{Ra}/\partial\eta] = \partial^2 \mathbf{v}_{Ra}/\partial\eta^2, \tag{5.29}$$

$$\partial\rho_{Ra}/\partial\theta + \partial(\rho_{Ra} w_{Ra})/\partial\eta = 0; \tag{5.30a}$$

$$\begin{aligned}&\rho_{Ra}[\partial w_{Ra}/\partial\theta + w_{Ra}\partial w_{Ra}/\partial\eta] + (1/\gamma M^2)\partial p_{Ra}/\partial\eta\\ &\quad = (4/3)\partial^2 w_{Ra}/\partial\eta^2;\end{aligned} \tag{5.30b}$$

$$\begin{aligned}&\rho_{Ra}[\partial T_{Ra}/\partial\theta + w_{Ra}\partial T_{Ra}/\partial\eta] + (\gamma - 1)p_{Ra}\partial w_{Ra}/\partial\eta]\\ &\quad = (\gamma/Pr)\partial^2 T_{Ra}/\partial\eta^2 + \gamma(\gamma-1)M^2\{|\partial \mathbf{v}_{Ra}/\partial\eta|^2\\ &\quad + (4/3)|\partial w_{Ra}/\partial\eta|^2\},\end{aligned} \tag{5.30c}$$

$$p_{Ra} = \rho_{Ra} T_{Ra}, \tag{5.30d}$$

These above Rayleigh equations, (5.29) and (5.30a–5.30d), are applied in [99] for the Rayleigh compressible problem by Howarth in 1951, but in the case of an infinite flat horizontal plate (submerged in a viscous and heat-conducting and originally quiescent fluid) which is impulsively started moving in its own plane with a constant velocity.

In fact, from 0ur above RAM Approach I can now affirm that in a corner region (θ, η), which is significant for the small time near initial time and in thin layer close to wall, at the leading-order for large Reynolds number, the above compressible Rayleigh equations, (5.29) and (5.30a–5.30d), for a viscous and heat-conducting fluid flow, consistently replace the full unsteady NS–F equations.

Both these unsteady systems, ((5.29) and (5.30a–5.30d), and (5.25a–5.25d) and (5.26)), both valid near the initial time, are related amongst themselves by the following matching relations:

$$\mathrm{Lim}_{\eta\uparrow\infty} \mathrm{w}_{\mathrm{Ra}} = \mathrm{w}_{\mathrm{Ac}|\zeta=o}, \tag{5.31a}$$

$$\mathrm{Lim}_{\theta\uparrow\infty}[\mathbf{v}_{\mathrm{Ra}}, \mathrm{w}_{\mathrm{Ra}}, \rho_{\mathrm{Ra}}, \mathrm{T}_{\mathrm{Ra}}] = [\mathbf{v}_{\mathrm{Ac}}, \mathrm{w}_{\mathrm{Ac}}, \rho_{\mathrm{Ac}}, \mathrm{T}_{\mathrm{Ac}}]_{|\tau=0}. \tag{5.31b}$$

The reader can find in Antontsev et al. [104], Chap. 2, some mathematically rigorous results concerning the above, *à la* Rayleigh, equations (5.29) and (5.30a–5.30d); and see also the review paper by Solonnikov and Kazhykhov [105].

Finally, if we assume that in a thermal spot

$$\Theta(\mathrm{t}/\beta, P), \ \text{with} \ \ \beta \ll 1,$$

β defined by (5.13), is equal to ε^2, then, for the emergency of the "temperature accident" we have the possibility of taking into account all starting initial (5.11) and wall (5.12a, b) conditions, in the framework of an initial-boundary values Rayleigh problem.

We therefore write, for Eqs. 5.29−5.30d, the following initial conditions:

$$\theta^- \le 0 : \mathbf{v}_{\mathrm{Ra}} = 0, \ \mathrm{w}_{\mathrm{Ra}} = 0, \ \rho_{\mathrm{Ra}} = 1 \text{ and } \mathrm{T}_{\mathrm{Ra}} = 1, \tag{5.32a}$$

and, at the horizontal solid wall, $\eta = 0$, we assume:

$$\eta = 0 : \mathbf{v}_{\mathrm{Ra}} = \mathrm{w}_{\mathrm{Ra}} = 0 \text{ and } \mathrm{T}_{\mathrm{Ra}} = \Theta(\theta, P), \ \theta^+ \ge 0. \tag{5.32b}$$

The above "starting problem", (5.29)−(5.30a–5.30d) with (5.32a, 5.32b), is a typical problem for various "emergency−temperature–accident phenomena" which develop when $\theta^+ \ge 0$.

5.5 Adjustment Processes Towards the Prandtl BL Evolution Problem

If we want to take into account the sudden heat emergency, at the time $\theta^+ \ge 0$, in a local domain, $P \subset D$, on the wall $\eta = 0$, then it now seems justifiable that the main working problem is just the above compressible, viscous, and heat-conducting Rayleigh problem ((5.29), (5.30a–5.30d), (5.32a, 5.32b)).

This Rayleigh problem is valid, simultaneously, near the initial time and close to wall – in a small corner (fourth) region – with a "physical size" of order $(\nu^{\circ}/U^{\circ 2})$ relative to time and (ν°/U°) relative to vertical coordinate – these time and length scales being exactly those used by Howarth [99].

5.5.1 *Adjustment Process Via the Acoustics/Gas Dynamics Equations*

Once the above Rayleigh problem – (5.29), (5.30a–5.30d), and (5.32a, 5.32b) – is solved (numerically), then we have the possibility (first) of prescribing, by matching relations (5.31a, 5.31b) to Eqs. 5.25a–5.25d, with (5.26), of gas dynamics – significant in the third inviscid region near the time, $\tau = 0$, and characterized by τ and ζ – the consistent conditions at $\tau = 0$ and $\zeta = 0$.

As a consequence of this above matching, it seems that we can expect that in conditions (at $\tau = 0$ and $\zeta = 0$) for Eqs. 5.25a–5.25d with (5.26) of gas dynamics, the influence of the wall condition for the temperature is taken into account, but only via the limit value $\Theta(\propto, P)$.

We observe that in the wall, $\zeta = 0$ at $t > 0$, condition (5.20b) for the Prandtl BL equation (5.18a–5.18e), this same function (independent of time?) $\Theta(\propto, P)$ is also present.

The acoustic/gas dynamics equations (5.25a–5.25d) with (5.26), with these initial conditions (5.31b) and single boundary condition (5.31a), for w_{AC} at $\zeta = 0$, which take into account the (partial?) influence of thermal spot (but independent of time function $\Theta(\propto, P)$), present the possibility of considering, for $\tau \to \infty$, an unsteady adjustment inviscid problem for the initialization of the Prandtl BL equations. As a typical example, see, for instance, our paper co-authored with Guiraud [106], which determines, in particular, the initial data $T^0_{Pr.}$[2]

When both $\mathbf{v}_{Pr}{}^0$ and $T_{Pr}{}^0$ are known, as a result of the above unsteady adjustment inviscid problem, then later, via the initial-boundary value BL problem, significant in the second Prandtl, (t, ζ) BL region, we have the opportunity to investigate the quasi-steady evolution of the "temperature accident" arising from the Rayleigh corner fourth region.

[2] In [106], with Guiraud, we have formulated, for the "primitive Kibel equations" (see Sect. 9.2) – which are derived from the hydrostatic approximation to the Euler equations for non-viscous and adiabatic motion – a problem analogous to the one that was considered by Rossby (1938) concerning the quasigeostrophic approximation (a problem which is now well known as the adjustment to geostrophy). The major conclusion of our "adjustment to hydrostatic balance" is that the initial conditions for the primitive equations may be derived from a full set of initial conditions, for the full Euler equations, where in these Eulerian initial conditions the initial data need not fit the hydrostatic balance. This obtention of initial conditions for primitive equations is realized by solving the associated one-dimensional unsteady adjustment problem of vertical motion to hydrostatic balance.

Curiously, however, this unsteady inviscid adjustment scenario, from gas dynamics to BL, does not seem to be the only one possible. Indeed, a more detailed analysis (see Sect. 5.5.2 below) shows the existence of a fifth matching region between Rayleigh (fourth) and Prandtl (second) regions – the existence of such an intermediate fifth region ensuring matching between the Rayleigh corner and Prandtl BL regions.

5.5.2 Adjustment Process Via the Rayleigh Equations

This fifth intermediate matching region appears when we investigate, first, the far behaviour of the Rayleigh equations (5.29) and (5.30a–5.30d), for large values of θ and η. In the Rayleigh corner, fourth region, with Prandtl variables,

$$t = \varepsilon^2\,\theta \;\text{ and }\; \zeta = \varepsilon\eta \text{ as } \varepsilon \downarrow 0, \tag{5.33a}$$

we can introduce new intermediate variables, t* and z*:

$$\theta = t^*/\kappa(\varepsilon) \text{ and } \eta = z^*/\surd\kappa(\varepsilon),\; 0 < \kappa(\varepsilon) \downarrow 0 \text{ with } \varepsilon \downarrow 0, \tag{5.33b}$$

where $\kappa(\varepsilon)$ is an arbitrary gauge, and t* and z*, the new intermediate variables, are fixed when ε↓ 0, in such a way that both Rayleigh variables θ and η tend to infinity.

In this intermediate matching fifth region we have, as leading-order functions:

$$[\mathbf{v}_{\text{Int}}, w^*_{\text{Int}}, \rho_{\text{Int}}, T_{\text{Int}}] = \text{Lim}_{\kappa(\varepsilon)\downarrow 0}[\mathbf{v}_{\text{Ra}}, w_{\text{Ra}}/\surd\kappa(\varepsilon), \rho_{\text{Ra}}, T_{\text{Ra}}], \tag{5.33c}$$

where all intermediate functions (with subscript "$_{\text{Int}}$") are dependent on time–space variables, t*, z*, and x, y.

When we take into account that the (unknown) gauge $\kappa(\varepsilon)$ is certainly in an order between ε^2 and ε, such that we derive from the Rayleigh equations, (5.29) and (5.30a–5.30d), due to (5.33b, 5.33c), the following intermediate-matching model equations for $\mathbf{v}_{\text{Int}}$, w^*_{Int}, ρ_{Int}, p_{Int}, and T_{Int}:

$$\partial\rho_{\text{Int}}/\partial t^* + \partial(\rho_{\text{Int}} w^*_{\text{Int}})/\partial z^* = 0; \tag{5.34a}$$

$$\partial p_{\text{Int}}/\partial z^* = 0; \tag{5.34b}$$

$$\begin{aligned}\rho_{\text{Int}}\left[\partial T_{\text{Int}}/\partial t^* + w^*_{\text{Int}}\partial T_{\text{Int}}/\partial z^*\right] + (\gamma-1)p_{\text{Int}}\partial w^*_{\text{Int}}/\partial z^* \\ = (\gamma/\text{Pr})\partial^2 T_{\text{Int}}/\partial z^{*2} + \gamma\,(\gamma-1)M^2|\partial\mathbf{v}_{\text{Int}}/\partial z^*|^2,\end{aligned} \tag{5.34c}$$

$$\rho_{\text{Int}}[\partial\mathbf{v}_{\text{Int}}/\partial t^* + w_{*\text{Int}}\partial\mathbf{v}_{\text{Int}}/\partial z^*] = \partial^2\mathbf{v}_{\text{Int}}/\partial z^{*\mathbf{2}}; \tag{5.34d}$$

with

$$p_{Int} = \rho_{Int} T_{Int}. \tag{5.34e}$$

Obviously, the above intermediate matching model equations, (5.34a–5.34e), first pointed out in our short note [98], are those derived when we carry out, on the Rayleigh equations (5.29) and (5.30a–5.30d), the usual approximations of the classical Prandtl boundary-layer theory.

In particular, as in BL region, the pressure p_{Int} is independent of z* and is determined by matching:

$$\lim_{z^* \uparrow \infty}[\mathrm{Lim}^{Int}] = \lim_{\eta \downarrow 0}[\mathrm{Lim}^{Ra}], \tag{5.35a}$$

where

$$\mathrm{Lim}^{Int} = [\varepsilon \downarrow 0,\ \text{with } t^* = t/\sigma(\varepsilon),\ z^* = z/\varepsilon\sqrt{\sigma(\varepsilon)},\ \text{fixed}], \tag{5.35b}$$

the intermediate variables (t*, z*) being directly related to the starting (in NS–F equations) variables (t, z).

The intermediate gauge $\sigma(\varepsilon)$, in Lim^{Int}, (5.35b), is linked with the above gauge $\kappa(\varepsilon)$ by the following relation:

$$\kappa(\varepsilon)\,\sigma(\varepsilon) = \varepsilon^2. \tag{5.36}$$

More precisely, this compatibility relation (5.36) is a direct consequence of the investigation of the behaviour of the Prandtl BL equations (5.18a–5.18e), when t and ζ both tend to zero, towards the intermediate fifth region.

Indeed, if we write (again with t* and z* fixed):

$$t = \sigma(\varepsilon)t^*,\quad \zeta = \sqrt{\sigma(\varepsilon)}z^*,\ 0 < \sigma(\varepsilon) \downarrow 0 \text{ with } \varepsilon \downarrow 0, \tag{5.37a}$$

and if

$$[\mathbf{v}_{Int}, w^*_{Int}, \rho_{Int}, T_{Int}] = \mathrm{Lim}_{\sigma(\varepsilon)\downarrow 0}[\mathbf{v}_{Pr}, \sqrt{\sigma(\varepsilon)}w_{Pr}, \rho_{Pr}, T_{Pr}], \tag{5.37b}$$

then again we derive the same above intermediate matching model equations (5.34a–5.34d) with (5.34e), but from (5.18a–5.18e). The relation (5.36) is, in fact, a consequence of the compatibility between (5.33b, 5.33c) and (5.37a, 5.37b).

Unfortunately, the precise localization of this intermediate matching region (characterized by the gauge $\sigma(\varepsilon)$), between the Rayleigh and Prandtl regions, does not seem possible at this stage of asymptotic analysis, and more careful (second-order?) investigations are obviously necessary.

Finally, we observe that if on the one hand, when $\sigma(\varepsilon) = \varepsilon^2$, then $t^* = \theta$ and $z^* = \zeta/\varepsilon = z/\varepsilon^2 = \eta$, then we recover the Rayleigh region; and if on the other hand, when $\sigma(\varepsilon) = \varepsilon^0 \equiv 1$, then $t^* = t$ and $z^* = \zeta$, and we recover the Prandtl region.

The existence of such an intermediate region is a striking indication that it seems possible (as a conjecture) to directly match the Rayleigh and Prandtl equations by an adjustment problem, via the intermediate equations (5.34a–5.34e), when the intermediate time $t^* \downarrow\propto$, using the matching condition:

$$\lim_{t^* \downarrow\propto}[\mathbf{v}_{\mathrm{Int}}, \mathrm{T}_{\mathrm{Int}}] = \left[\mathbf{v}^0{}_{\mathrm{Pr}}, \mathrm{T}^0{}_{\mathrm{Pr}}\right]. \tag{5.38}$$

This presents the possibility of obtaining, consistently, the associated initial data, $\mathbf{v}_{\mathrm{Pr}}{}^0$ and $\mathrm{T}_{\mathrm{Pr}}{}^0$, in (5.22), for Prandtl unsteady equations (5.18b) and (5.18d) for $\mathbf{v}_{\mathrm{Pr}}$ and T_{Pr}.

5.6 Some Conclusions

First, it is clear that the above problem of matching, (5.38), deserves careful consideration and may be an interesting numerical/computational problem.

We then observe that it is possible to considerably simplify the matching problem between the Rayleigh and Prandtl equations, if we assume that the Mach number M, in Rayleigh equations, is a small parameter, and assume, for this, that in the wall the thermal condition (5.32b) can be written in the following form:

$$\Theta(\theta, P) = 1 + \Lambda_0 \mathrm{M}^2 \Sigma(\theta, P), \tag{5.39}$$

where $\Lambda_0 = \mathrm{O}(1)$, and $\Sigma(\theta, P)$ replace thermal spot $\Theta(\theta, P)$.

In such a case, the solution of the Rayleigh problem is also expanded relative to a low Mach number, $\mathrm{M} \ll 1$ (as in Howarth's paper [99]). But here we do not proceed further.

A third remark concerns the fact that further investigations are necessary for a complete understanding of the above intriguing five-regions structure, which is very interesting, because it is unusual and does not have an obvious clear interpretation! But the above new five-regions (four regions plus the intermediate region) structure of NS–F equations, at large Reynolds number, as a consequence of the singular nature of the unsteady Prandtl BL equations near the initial time, do not restrict investigations to emergency phenomena, and have fundamental importance in the RAM Approach of NS–F equations.

I think that from this detailed further re-examination of boundary-layer Prandtl theory, it is now possible to resolve some singularities arising in various unsteady boundary-layer problems (see, for instance, Stewartson [101]).

A final remark concerns the pedagogical interest of such partition of NS–F equations, in five regions, for large Reynolds number fluid flows, and this RAM Approach presents the possibility of deriving a new logical interpretation of Euler, Prandtl, Rayleigh, acoustic/gas dynamics, and intermediate equations, as five significant and particular models of full NS–F unsteady equations for Newtonian fluid flow at large Reynolds number.

Again we observe that not only Prandtl in 1904, but also (it seems) Blasius and Schlichting (in Germany), Lagerstrom, Cole, and Kaplun (at Caltech), Van Dyke (at Stanford), Stewartson and Smith (in England), and Germain (in France), did not realize that indeed the concept of boundary-layer, which is an extension to long-waves approximation in the case of a viscous fluid flow, is singular, in the case of an unsteady fluid flow, near initial time, where initial data are prescribed in a well-posed initial-boundary value problem.